1. Math 240B (Driver) Final Exam: Wednesday 04/17/04

Directions: No open notes or books on this exam. However you may quote results from the notes and the homework problems. Clearly explain and justify your steps, i.e. indicate the "substantial" theorems that you are using in solving the problem. The test has 7 problems. Problems 1-6 are each worth 10 points while problem 7 is worth 20 points.

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1. Find the value of the following iterated integral.

$$\int_{[0,\infty)} \left(\int_{[0,\infty)} 2x\sqrt{y} \exp(-x^2\sqrt{y} - y) dy \right) dx.$$

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2. If $u \in C\left(\mathbb{R}^d, \mathbb{R}\right)$ satisfies the mean value property, i.e.

$$u(x) = \frac{1}{\sigma\left(S^{d-1}\right)} \int_{S^{d-1}} u(x + r\omega)\, d\sigma(\omega) \;\; \forall\, x \in \mathbb{R}^d \text{ and } r \geq 0,$$

then $u \in C^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$. **Hint:** show $u = u * \phi$ where $\phi(x) := g\left(|x|^2\right)$ and $g \in C_c^\infty(\mathbb{R}, [0,\infty))$ is chosen so that $\int_{\mathbb{R}^d} \phi(x)\, dx = 1$.

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3. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) = 1$ and $f : X \to \mathbb{C} \setminus \{0\}$ is a measurable function. Show

$$\frac{1}{\int_X |f|^2 \, d\mu} \leq \int_X \frac{1}{|f|^2} d\mu.$$

Hint: the inequality if equivalent to showing

$$1 = \mu(X) \leq \int_X \frac{1}{|f|^2} d\mu \cdot \int_X |f|^2 \, d\mu.$$

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4. Suppose $k \in C\left([0,1]^2, \mathbb{R}\right)$ and for $f \in C([0,1], \mathbb{R})$, let

$$Kf(x) := \int_0^1 k(x,y) f(y)\, dy \text{ for all } x \in [0,1].$$

Show K is a compact operator on $\left(C([0,1], \mathbb{R}), \|\cdot\|_\infty\right)$.

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5. Let $f \in L^1([1,\infty), m)$ where m is Lebesgue measure. Suppose that

$$\int_1^\infty f(x)x^{-n}dm(x) = 0 \tag{1}$$

for all $n = 2, 4, 6, 8, \ldots$. Show that $f = 0$ a.e.

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6. Let $(H, \langle\cdot|\cdot\rangle)$ be a Hilbert space and suppose that $\{P_n\}_{n=1}^\infty$ is a sequence of orthogonal projection operators on H such that $P_n(H) \subset P_{n+1}(H)$ for all n. Let $M := \cup_{n=1}^\infty P_n(H)$ (a subspace of H) and let P denote orthogonal projection onto $\overline{M}$.

Show $\lim_{n\to\infty} P_n x = Px$ for all $x \in H$.

Hint: first prove the result for $x \in M^\perp$, then for $x \in M$ and then for $x \in \overline{M}$.

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7 (On pointwise convergence of Fourier series)**.** Let

$$d\lambda(\theta) := \frac{d\theta}{2\pi}, \ \ H := L^2\left([-\pi,\pi], d\lambda\right),$$

$$\langle u|v\rangle = \int_{-\pi}^{\pi} u(\theta)\,\bar{v}(\theta)\,d\lambda(\theta)$$

and $e_k(\theta) := e^{ik\theta}$. As we **already** know, $\{e_k : k \in \mathbb{Z}\}$ is an orthonormal basis for the Hilbert space, H.

Suppose that $u \in C^1(\mathbb{R}, \mathbb{C})$ and u is 2π - periodic, i.e.

$$u(\theta + 2\pi) = u(\theta) \text{ for all } \theta \in \mathbb{R}.$$

Show:

(1) $\langle u'|e_k\rangle = ik\langle u|e_k\rangle$ where $u'(\theta) := \frac{d}{d\theta}u(\theta)$.

(2) Making use of item 1., show

$$\sum_{k\in\mathbb{Z}} |\langle u|e_k\rangle| < \infty.$$

(3) Verify that the sum,

$$s(\theta) := \sum_{k\in\mathbb{Z}} \langle u|e_k\rangle e^{ik\theta},$$

is convergent for every θ and that the function $s(\theta)$ so defined in continuous.

(4) Prove that

$$u(\theta) = s(\theta) = \sum_{k\in\mathbb{Z}} \langle u|e_k\rangle e^{ik\theta} \text{ for } \textbf{all } \theta \in \mathbb{R}.$$

The solutions below are brief. Your solutions should contain a little more detail. I will just outline the key points.

Solution to 1. Since the integrand is positive Tonelli's theorem implies,

$$\begin{aligned}
&\int_{[0,\infty)}\left(\int_{[0,\infty)} 2x\sqrt{y}\exp(-x^2\sqrt{y}-y)dy\right)dx \\
&\quad=\int_{[0,\infty)}\left(\int_{[0,\infty)} 2x\sqrt{y}\exp(-x^2\sqrt{y}-y)dx\right)dy \\
&\quad=\int_{[0,\infty)}\left(\int_{[0,\infty)} \left[-\frac{d}{dx}\exp(-x^2\sqrt{y}-y)\right]dx\right)dy \\
&\quad=\int_{[0,\infty)} e^{-y}dy=1.
\end{aligned}$$

I have also implicitly used the monotone convergence theorem and the fundamental theorem of calculus. ■

Solution to 2. Here we use abstract polar coordinates to find

$$\begin{aligned}
u*\phi(x)&=\int_{\mathbb{R}^d} u(x-y)\,\phi(y)\,dy=\int_{\mathbb{R}^d} u(x+y)\,\phi(-y)\,dy \\
&=\int_{\mathbb{R}^d} u(x+y)\,\phi(y)\,dy \\
&=\int_0^\infty dr\int_{S^{d-1}} u(x+r\omega)\,g\left(r^2\right)r^{d-1}d\sigma(\omega) \\
&=\sigma\left(S^{d-1}\right)u(x)\int g\left(r^2\right)r^{d-1}dr=u(x)
\end{aligned}$$

as desired. This shows u is smooth since we have seen that $D^\alpha(u*\phi)=u*D^\alpha\phi$. ■

Solution to 3. Integrating $1=|f|\cdot|f^{-1}|$ and using Hölder's inequality with $p=q=2$, gives

$$1=\left(\int_X |f|\cdot\left|f^{-1}\right|d\mu\right)^2\le\int_X |f|^2\,d\mu\cdot\int_X\frac{1}{|f|^2}d\mu.$$

Alternatively you could use Jensen's inequality with $\phi(x)=x^{-2}$. ■

Solution to 4. By the dominated convergence theorem, $Kf\in C([0,1],\mathbb{R})$ for all $f\in C([0,1],\mathbb{R})$ and by the basic properties of the integral K is linear. Moreover, K is bounded, because

$$|Kf(x)|\le\int_0^1|k(x,y)|\,|f(y)|\,dy\le\|k\|_\infty\,\|f\|_\infty$$

which shows that K is a bounded operator. By uniform continuity of k, for every $\varepsilon>0$ there exists a $\delta>0$ such that $|k(x,y)-k(x',y')|<\varepsilon$

if $|(x,y)-(x',y')|<\delta$. Therefore if $\|f\|_\infty \le 1$ and $|x-x'|<\delta$ we have

$$|Kf(x)-Kf(x')| \le \int_0^1 |k(x,y)-k(x',y)|\,|f(y)|\,dy \le \varepsilon$$

which shows $\{Kf : \|f\|_\infty \le 1\}$ is pointwise bounded and equicontinuous. Therefore by the Ascoli-Arzela Theorem, $\{Kf : \|f\|_\infty \le 1\}$ is precompact in $C([0,1],\mathbb{R})$ and hence K is a compact operator. ■

Solution to 5. If Eq. (1) holds, then by the Stone Weierstrass theorem (notice that x^{-2} separates points and is positive on $[0,\infty)$) and DCT,

$$\int_1^\infty f(x)g(x)\,dm(x) = 0 \tag{2}$$

for all $g \in C_0([1,\infty))$ and in particular of all $g \in C_c([1,\infty))$. We now give 3 methods to finish the problem.

(1) Just refer to a theorem in the notes to see $f \equiv 0$. We give proofs of this statement in 2. and 3. below.

(2) Let $d\mu(x) := |f(x)|\,dm(x)$. Then μ is a finite measure and we know $C_c([1,\infty))$ is dense in $L^1(\mu)$. Choose $g_n \in C_c([1,\infty))$ such that $g_n \to \overline{\mathrm{sgn}(f)}$ in $L^1(\mu)$. Then

$$\begin{aligned}\left|\int_1^\infty |f|\,dm - \int_1^\infty g_n \mathrm{sgn}(f)d\mu\right| &= \left|\int_1^\infty 1d\mu - \int_1^\infty g_n\mathrm{sgn}(f)d\mu\right| \\ &\le \int_1^\infty |1-g_n\mathrm{sgn}(f)|\,d\mu \\ &= \int_1^\infty \left|\left[\overline{\mathrm{sgn}(f)}-g_n\right]\mathrm{sgn}(f)\right| d\mu \\ &\le \int_1^\infty \left|\overline{\mathrm{sgn}(f)}-g_n\right| d\mu \to 0.\end{aligned}$$

Since

$$\int_1^\infty g_n\mathrm{sgn}(f)d\mu = \int_1^\infty g_n\mathrm{sgn}(f)\,|f|\,dm = \int_1^\infty g_n f dm = 0,$$

it follows that $\int_1^\infty |f|\,dm = 0$ and hence $f=0$ a.e..

(3) Let $\mathcal{H}$ denote the bounded measurable functions, g, on $[1,\infty)$ such that Eq. (2) holds. Then it is easily seen that $\mathcal{H}$ is a vector space closed under bounded convergence (use DCT) and contains $C_0([1,\infty))$. So by the multiplicative system theorem, $\mathcal{H}$ consists of all bounded $\sigma(C_0([1,\infty))) = \mathcal{B}_{[0,\infty)}$ measurable functions. Now take $g = \overline{\mathrm{sgn}(f)}$ in Eq. (2) to lean that

$$\int_1^\infty |f|\,dm = \int_1^\infty f(x)\overline{\mathrm{sgn}(f)}dm(x) = 0.$$

Remark: A couple of people started this problem by making the change of variables $x = u^{-1}$ in Eq. (2) and in this way tried to reduce the problem to one on $[0,1]$. This will work as well, the proof would go similarly except now one only needs the compact version of the Stone W. theorem. ■

Solution to 6. If $x \in M^{\perp}$, then $x \in P_n(H)^{\perp}$ for all n and therefore, $P_n x = 0$ for all n. If $x \in M$, we have $P_n x = x$ for all sufficiently large n and therefore, $\lim_{n\to\infty} P_n x = x = Px$. If $x \in \bar{M}$ and $y \in M$, then

$$\begin{aligned}\|P_n x - x\| &\leq \|P_n x - P_n y\| + \|P_n y - y\| + \|y - x\| \\ &\leq 2\|y - x\| + \|P_n y - y\|\end{aligned}$$

and therefore

$$\limsup_{n\to\infty} \|P_n x - x\| \leq 2\|y - x\| + \limsup_{n\to\infty} \|P_n y - y\| = 2\|y - x\|$$

and the latter term goes to zero as $y \to x$. Hence $\lim_{n\to\infty} P_n x = 0 = Px$.

For general $x \in H$, we write $x = y + z$ with $y \in \bar{M}$ and $z \in M^{\perp}$, then

$$P_n x = P_n y + P_n z = P_n y \to y = Px \text{ as } n \to \infty.$$

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Solution to 7.

(1) By integration by parts,

$$\begin{aligned}\langle u'|e_k\rangle &= \int_{-\pi}^{\pi} u'(\theta)\, e^{-ik\theta} d\theta = -\int_{-\pi}^{\pi} u(\theta) \frac{d}{d\theta} e^{-ik\theta} d\theta \\ &= ik\langle u|e_k\rangle.\end{aligned}$$

There are no boundary terms since all functions involved are 2π -periodic.

(2) From item 1. and Hölder's inequality with $p = q = 2$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}\setminus\{0\}} |\langle u|e_k\rangle| &= \sum_{k\in\mathbb{Z}\setminus\{0\}} \left|\frac{1}{k}\langle u'|e_k\rangle\right| \leq \sqrt{\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{k^2}} \sqrt{\sum_{k\in\mathbb{Z}\setminus\{0\}} |\langle u'|e_k\rangle|^2} \\ &= \sqrt{\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{k^2}}\, \|u'\|_2 < \infty.\end{aligned}$$

(3) The sum, $s(\theta) := \sum_{k\in\mathbb{Z}} \langle u|e_k\rangle e^{ik\theta}$, is absolutely convergence since

$$\sum_{k\in\mathbb{Z}} \left|\langle u|e_k\rangle e^{ik\theta}\right| = \sum_{k\in\mathbb{Z}} |\langle u|e_k\rangle| < \infty.$$

It follows by the dominated convergence theorem that $s(\theta)$ is continuous.

(4) By Fourier analysis, we know that $s_N := \sum_{|k|\leq N} \langle u|e_k\rangle e_k$ converges to u in L^2 and hence by passing to a subsequence, we may assume the convergence holds for a.e. θ as well. This combined with item 3. shows $s(\theta) = u(\theta)$ for a.e. θ. By continuity it follows that $s \equiv u$.

Alternatively: Since

$$\sum_{k\in\mathbb{Z}}\int_{-\pi}^{\pi}\left|\langle u|e_k\rangle e^{ik\theta}e^{-in\theta}\right|d\lambda\left(\theta\right)=\sum_{k\in\mathbb{Z}}\left|\langle u|e_k\rangle\right|<\infty$$

it follows that

$$\langle s|e_n\rangle=\int_{-\pi}^{\pi}\sum_{k\in\mathbb{Z}}\langle u|e_k\rangle e^{ik\theta}e^{-in\theta}d\lambda\left(\theta\right)=\langle u|e_n\rangle$$

and hence $(s-u)$ is perpendicular to the orthonormal basis $\{e_k : k\in\mathbb{Z}\}$ and therefore $s-u=0$ in $L^2\left(d\lambda\right)$, i.e. $s=u$ a.e. Since both $s-u$ is continuous, it must be zero.

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