Solution to 1. Let $x_k \uparrow 1$ as $k \to \infty$, then by the monotone convergence theorem,

$$0 = \lim_{k\to\infty} \sqrt{1-x_k} = 1 - \lim_{k\to\infty} \sum_{n=1}^{\infty} \alpha_n x_k^n \overset{\text{M.C.T.}}{=} 1 - \sum_{n=1}^{\infty} \lim_{k\to\infty} \alpha_n x_k^n$$
$$= 1 - \sum_{n=1}^{\infty} \alpha_n.$$

■

Solution to 2.

(1) $|Tf| = |f(1)| \le \|f\|_\infty$ with equality when f is constant, therefore, $\|T\| = 1$.

(2)
$$|Tf| \le \int_0^2 \left| \left(\sin \frac{\pi}{2}\tau\right) f(\tau) \right| d\tau$$
$$\le \|f\|_\infty \int_0^2 \sin \frac{\pi}{2}\tau d\tau = \|f\|_\infty \left(-\frac{2}{\pi}\cos\frac{\pi}{2}\tau\right)\Big|_0^2 = \frac{4}{\pi}\|f\|_\infty$$

and equality occurs when $f = 1$, therefore $\|T\| = \frac{4}{\pi}$.

■

Solution to 3.

(1) For any $x \in X$ there exists $x_n \in A$ such that $x = \lim_{n\to\infty} x_n$. Since F and G are continuous, it follows that
$$F(x) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} G(x_n) = G(x).$$

(2) Let $f(x) = \frac{1}{x-\pi}$. This is continuous on $\mathbb{R}\setminus\{\pi\}$ and hence on $\mathbb{Q}$ but not on $\mathbb{R}$.

■

Solution to 4. Given $x \in X$, there exists $x_n \in A$ such that $x = \lim_{n\to\infty} x_n$. Now because f is uniformly continuous, $d(f(x_n), f(x_m)) \to 0$ as $m, n \to \infty$. Since Y is complete, it follows that $\lim_{n\to\infty} f(x_n) =: F(x)$ exists. If $\xi_n \in A$ such that $x = \lim_{n\to\infty} \xi_n$, then by uniform continuity again, $\lim_{n\to\infty} d(f(x_n), f(\xi_n)) = 0$ so that $F(x)$ is well defined. Finally to see that F is continuous, let $\varepsilon > 0$ and $\delta > 0$ be as in the definition of uniform continuity. Suppose $x, z \in X$ with $\rho(x,z) < \delta$ and $x_n, z_n \in A$ with $x_n \to x$ and $z_n \to z$. Since $\lim_{n\to\infty} \rho(x_n, z_n) = \rho(x,z) < \delta$, eventually $\rho(x_n, z_n) < \delta$. Therefore,
$$d(f(x_n), f(z_n)) < \varepsilon \text{ for a.a. } n$$
and by passing to the limit we learn that
$$d(F(x), F(y)) \le \varepsilon.$$

■

Solution to 5.

(1) Using the fundamental theorem of calculus and basic properties of the Riemann integral,

$$\left\|\frac{f_n(t+h)-f_n(t)}{h}\right\| = \left\|\frac{1}{h}\int_t^{t+h}\dot{f}_n(\tau)\,d\tau\right\| \le \left|\frac{1}{h}\int_t^{t+h}\left\|\dot{f}_n(\tau)\right\|d\tau\right| \le \left|\frac{1}{h}\int_t^{t+h}a_n d\tau\right| = a_n.$$

(2) Since $\sum_{n=1}^{\infty} f_n(t)$ and $\sum_{n=1}^{\infty}\dot{f}_n(t)$ are absolutely convergent, $F(t)$ and the proposed formula for $\dot{F}(t)$ are well defined. For $h \neq 0$,

$$\left\|\frac{F(t+h)-F(t)}{h} - \sum_{n=1}^{\infty}\dot{f}_n(t)\right\| = \left\|\sum_{n=1}^{\infty}\left[\frac{f_n(t+h)-f_n(t)}{h} - \dot{f}_n(t)\right]\right\| \le \sum_{n=1}^{\infty}\left\|\frac{f_n(t+h)-f_n(t)}{h} - \dot{f}_n(t)\right\|.$$

By the triangle inequality and item 1.,

$$\sup_{t,h}\left\|\frac{f_n(t+h)-f_n(t)}{h} - \dot{f}_n(t)\right\| \le 2a_n$$

and therefore by the D.C.T. (along with the usual sequence argument) and the continuity of the norm,

$$\lim_{h\to 0}\sum_{n=1}^{\infty}\left\|\frac{f_n(t+h)-f_n(t)}{h} - \dot{f}_n(t)\right\| \overset{\text{D.C.T.}}{=} \sum_{n=1}^{\infty}\lim_{h\to 0}\left\|\frac{f_n(t+h)-f_n(t)}{h} - \dot{f}_n(t)\right\|$$

$$= \sum_{n=1}^{\infty}\left\|\lim_{h\to 0}\frac{f_n(t+h)-f_n(t)}{h} - \dot{f}_n(t)\right\| = 0$$

form which it follows that

$$\lim_{h\to 0}\left\|\frac{F(t+h)-F(t)}{h} - \sum_{n=1}^{\infty}\dot{f}_n(t)\right\| = 0,$$

i.e. $F(t)$ is differtiable and $\dot{F}(t) = \sum_{n=1}^{\infty}\dot{f}_n(t)$.

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