1. Math 240B (Driver) Midterm: Monday 02/18/04

Directions: Please work **alone** on this test. The test is **due** on Monday, 02/23/04 in class. All problems have equal value. Clearly explain and justify your steps, i.e. indicate the "substantial" theorems that you are using in solving the problem.

I would **suggest** you first take the test without notes or books with a time limit of 1.5 hours. You may then write a second draft of your solutions to be handed in to me. For the second draft you may consult Folland or the lecture notes. If you like, I would be happy to comment on your first draft as well.

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1. Let $(X, \mathcal{M}, \mu)$ be a $\sigma-$ finite measure space, $f : X \to [0, \infty)$ be a measurable function and $p \in (0, \infty)$. Show

$$\int_X f^p d\mu = p \int_0^\infty t^{p-1} \mu(f > t)\, dt.$$

Hint: write $\mu(f > t) := \mu(\{x \in X : f(x) > t\})$ as the integral of a simple function.

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2. Let $(X, \mathcal{M}, \mu)$ be a $\sigma-$ finite measure space, $p \in (1, \infty)$, $q = \frac{p}{p-1}$, $f \in L^q(\mu)$ and $k \in L^p(X \times X, \mathcal{M} \otimes \mathcal{M}, \mu \otimes \mu)$. Show:

(1) For $\mu-$ a.e. x,

$$M_x := \int_X |k(x, y) f(y)|\, d\mu(y) < \infty.$$

Now define

$$Kf(x) := \begin{cases} \int_X k(x, y) f(y)\, d\mu(y) & \text{if } \quad M_x < \infty \\ 0 & \text{if } \quad M_x = \infty. \end{cases}$$

(2) Show

$$\|Kf\|_{L^p(\mu)} \le \|k\|_{L^p(\mu \otimes \mu)} \|f\|_{L^q(\mu)}.$$

Hint: Think about part (2) before doing part (1).

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3. Let m be Lebesgue measure on $\mathbb{R}^d$. Determine which of the following functions on $\mathbb{R}^d$ are Lebesgue integrable:

$$1. \quad f(x) = \frac{e^{-|x|}}{|x|^d} \qquad 2. \quad f(x) = \frac{x_1 e^{-|x|}}{|x|^d}, \qquad 3. \quad f(x) = \frac{\sin^2(x_1)}{|x|^{d+1}},$$

where $x = (x_1, \ldots, x_d)$ and $|x| := \sqrt{x_1^2 + \cdots + x_d^2}$. Please justify your answer.

(**Extra credit:** what about the function $f(x) = \frac{x_1}{|x|^d}$.)

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4. Let X be a topological space. Prove X is connected if it is path connected.

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5. Show every second countable topological space (X, τ) is separable. Show the converse is not true by showing

$$X := \mathbb{R} \text{ with } \tau := \{\emptyset\} \cup \{V \subset \mathbb{R} : 0 \in V\}$$

is a separable, first countable but not a second countable topological space.

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6. Prove the following statement. If $f \in C([0, \pi/2], \mathbb{R})$ is a function such that

$$\int_0^{\pi/2} f(x) [\cos(x)]^n \, dx = 0 \text{ for all } n = 1, 2, \ldots$$

then $f \equiv 0$.

2. Test 1 Solutions: Math 240B (Driver) Midterm: Friday 02/13/04

Solution to 1. Tonelli's theorem and the fundamental theorem of calculus justifies the following computation,

$$\begin{aligned} p\int_0^\infty t^{p-1}\mu(f>t)\,dt &= p\int_0^\infty t^{p-1}\left(\int_X 1_{f>t}d\mu\right)dt \\ &= \int_X d\mu\; p\int_0^\infty t^{p-1}\left(1_{f>t}\right)dt \\ &= \int_X d\mu\; p\int_0^f t^{p-1}dt = \int_X d\mu\; t^p|_0^f = \int_X f^p d\mu. \end{aligned}$$

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Solution to 2. By Hölder's inequality,

$$\int_X |k(x,y)f(y)|\,d\mu(y) \le \|k(x,\cdot)\|_p\|f\|_q$$

and hence

$$\begin{aligned} &\int_X d\mu(x)\left[\int_X |k(x,y)f(y)|\,d\mu(y)\right]^p \\ &\qquad \le \int_X d\mu(x)\,\|k(x,\cdot)\|_p^p\|f\|_q^p = \|k\|^p_{L^p(\mu\otimes\mu)}\|f\|_q^p. \end{aligned}$$

Thus it follows that

$$\int_X |k(x,y)f(y)|\,d\mu(y) < \infty \text{ for a.e. } x.$$

Since $|Kf(x)| \le \int_X |k(x,y)f(y)|\,d\mu(y)$, we have also proved

$$\|Kf\|_p \le \|k\|_{L^p(\mu\otimes\mu)}\|f\|_q.$$

Here is a more general result. By Minikoski's inequality for integrals and Hölder's inequality,

$$\begin{aligned} \|Kf\|_p = \left\|\int_X k(\cdot,y)f(y)\,d\mu(y)\right\|_p &\le \int_X \|k(\cdot,y)\|_p|f(y)|\,d\mu(y) \\ &\le \left[\int_X \|k(\cdot,y)\|_p^a\,d\mu(y)\right]^{1/a}\|f\|_b. \end{aligned}$$

Here we have

$$\left[\int_X \|k(\cdot,y)\|_p^a\,d\mu(y)\right]^{1/a} = \left[\int_X\left[\int_X |k(x,y)|^p\,d\mu(x)\right]^{a/p}d\mu(y)\right]^{1/a}.$$

as a special case if $a=p$ so that $b=q$ we have

$$\begin{aligned} \|Kf\|_p &\le \left[\int_X\left[\int_X |k(x,y)|^p\,d\mu(x)\right]d\mu(y)\right]^{1/p}\|f\|_q \\ &= \|k\|_{L^p(\mu\otimes\mu)}\|f\|_q. \end{aligned}$$

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 Solution to 3. Let m be Lebesgue measure on $\mathbb{R}^d$.

(1)
$$\int_{\mathbb{R}^d} \frac{e^{-|x|}}{|x|^d} dx = \sigma\left(S^{d-1}\right) \int_0^\infty \frac{e^{-r}}{r^d} r^{d-1} dr = \sigma\left(S^{d-1}\right) \int_0^\infty \frac{e^{-r}}{r} dr = \infty$$
since $\frac{1}{r}$ is not integrable near 0.

(2)
$$\int_{\mathbb{R}^d} \frac{|x_1| e^{-|x|}}{|x|^d} dx \le \int_{\mathbb{R}^d} \frac{|x| e^{-|x|}}{|x|^d} dx = \sigma\left(S^{d-1}\right) \int_0^\infty e^{-r} dr = \sigma\left(S^{d-1}\right) < \infty.$$

(3) Since $\sin(x) = \int_0^x \cos(y)\, dy$, we have $|\sin(x)| \le |x| \wedge 1$. Therefore,
$$\begin{aligned}\int_{\mathbb{R}^d} \frac{\sin^2(x_1)}{|x|^{d+1}} dx &\le \int_{\mathbb{R}^d} \frac{|x|^2 \wedge 1}{|x|^{d+1}} dx \\ &= \sigma\left(S^{d-1}\right) \left[\int_0^1 \frac{1}{r^{d-1}} r^{d-1} dr + \int_1^\infty \frac{1}{r^{d+1}} r^{d-1} dr\right] \\ &= \sigma\left(S^{d-1}\right) [1+1] < \infty.\end{aligned}$$

(4) Extra credit;
$$\begin{aligned}\int_{\mathbb{R}^d} \frac{|x_1|}{|x|^d} dx &= \int_0^\infty dr\ r^{d-1} \int_{S^{d-1}} \frac{r\,|\omega_1|}{r^d} d\sigma(\omega) \\ &= C \int_0^\infty dr = \infty\end{aligned}$$
wherein we have used
$$C = \int_{S^{d-1}} |\omega_1|\, d\sigma(\omega) > 0.$$

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Solution to 4. Suppose that $\{U, V\}$ is a disconnection of X, i.e. $X = U \cup V$, U, V are open non-empty disjoint sets and suppose $x \in U$ and $y \in V$ and there exists $\sigma \in C([0,1], X)$ such that $\sigma(0) = x$ and $\sigma(1) = y$. Then $\sigma([0,1])$ is connected in X being the continuous image of a connected set. But this gives rise to a contradiction, since $\{\sigma([0,1]) \cap U, \sigma([0,1]) \cap V\}$ is a disconnection of $\sigma([0,1])$. ■

Solution to 5. Let $\{V_n\}_{n=1}^\infty$ be a basis for the topology τ and for each n choose $x_n \in V_n$. We will now show $D := \{x_n\}_{n=1}^\infty \subset X$ is a dense set. Indeed if $x \in X$ and $V \in \tau_x$. Then there exists $V_n \subset V$ such that $x \in V_n$ and hence
$$x_n \in V_n \cap D \subset V \cap D$$
which show $D \cap V \neq \emptyset$ and hence $x \in \bar{D}$.

Now let $X := \mathbb{R}$ with $\tau = \{\emptyset\} \cup \{V \subset \mathbb{R} : 0 \in V\}$. It is easily verified that τ is a topology and $\{0\}$ and $\{0, x\}$ is a neighborhood base of 0 and $x \neq 0$ respectively. Therefore τ is first countable. The smallest basis for the topology τ is the collection of sets $\{\{0, x\} : x \in \mathbb{R}\}$ which is uncountable and hence $(\mathbb{R}, \tau)$ is **not** second countable. Finally, let

$D := \{0\}$, then $x \in \bar{D}$ iff $V \cap \{0\} \neq \emptyset$ for all $V \in \tau_x$. But this is clearly true for any $x \in \mathbb{R}$ since all non-empty open sets contain 0. Hence $\bar{D} = \mathbb{R}$ and this space is separable. ■

Solution to 6. Since $\cos(x)$ is monotonic on $[0, \pi/2]$, it follows by the Stone-Weirstrass theorem that polynomials in $\cos(x)$ are dense in

$$I_{\pi/2} := \{g \in C([0, \pi/2]) : g(\pi/2) = 0\}$$

and so by linearity of the integral and the dominated convergence theorem we have

$$\int_0^{\pi/2} f(x)\, g(x)\, dx = 0 \tag{1}$$

for all $g \in I_{\pi/2}$. Take $g(x) = \bar{f}(x) \cdot \cos x$ (so that $g \in I_{\pi/2}$) in Eq. (1) to find,

$$0 = \int_0^{\pi/2} f(x) \cos x \cdot \bar{f}(x)\, dx = \int_0^{\pi/2} |f(x)|^2 \cos x\ dx.$$

Therefore, $|f(x)|^2 = 0$ a.e. and since f is continuous, $f \equiv 0$.

Alternatively: By considering the real and imaginary parts of f separately, it suffices to consider the case where f is real. If f were not identically zero, then there would exist $a \in [0, \pi/2]$ and $\varepsilon > 0$ such that $f(x) > 0$ (or $f(x) < 0$) for $x \in [0, \pi/2] \cap (a - \varepsilon, a + \varepsilon)$ and we could choose $g \in C_c((0, \pi/2), [0, 1])$ such that g is supported in $[0, \pi/2] \cap (a - \varepsilon, a + \varepsilon)$, $fg \geq 0$, and $fg > 0$ somewhere on $[0, \pi/2] \cap (a - \varepsilon, a + \varepsilon)$. But this implies

$$\int_0^{\pi/2} f(x)\, g(x)\, dx > 0$$

which leads to a contradiction with Eq. (1) since $g \in I_{\pi/2}$. ■