UNIVERSITY OF CALIFORNIA, SAN DIEGO

An Approximation of Wiener Measure on Manifolds with Non-positive Sectional Curvature

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Thomas A Laetsch

Committee in charge:

Professor Bruce K. Driver, Chair
Professor Patrick J. Fitzsimmons
Professor George M. Fuller
Professor Laurence B. Milstein
Professor Jason R. Schweinsberg

2012

The dissertation of Thomas A Laetsch is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2012

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

The greatest thanks goes to Professor Bruce Driver. His willingness to improve my understanding and his patience while teaching me were matched by his kindness and consideration while I went through several personal trials during my stay in San Diego. My successes are due in large part to the many hours of conversation inside and outside his office.

I thank my family for their encouragement and keeping perspective on the true importances in life. Mom, Dad, Mike, Victoria, Ted, Mackenzie, Dana, and Anthony - none of this would have been possible without you.

I thank the many friends I've made while here who have shared their good humor and enthusiasm with me. Particularly, Caleb and Leslie, Chad, Dan, Jonny, Maryann, Matt, Matti, Miles, Ravi, Robby, and Tyanna who have truly become a second family to me throughout the last several years.

VITA

2007	B. S. in Mathematics, University of Arizona
2010	M. A. in Mathematics, UC, San Diego
2007-2011	Graduate Teaching Assistant, UC, San Diego
2011-2012	Head Teaching Assistant, UC, San Diego
2012	Ph. D. in Mathematics, UC, San Diego

ABSTRACT OF THE DISSERTATION

An Approximation of Wiener Measure on Manifolds with Non-positive Sectional Curvature

by

Thomas A Laetsch

Doctor of Philosophy in Mathematics

University of California, San Diego, 2012

Professor Bruce K. Driver, Chair

An interpretation is given for the informal path integral expression,

$$\frac{1}{Z}\int_{\sigma\in W(M)} f(\sigma)e^{-\frac{1}{2}E(\sigma)}\mathcal{D}\sigma,$$

where Z is a "normalization" constant, $W(M)$ consists of continuous paths σ on M parametrized on $[0,1]$, $E(\sigma)$ is the energy of the path σ, and $\mathcal{D}\sigma$ is Lebesgue type measure. Approximating the path space by finite dimensional manifolds $H_{\mathcal{P}}(M)$ consisting of the piecewise geodesic paths adapted to a partition $\mathcal{P}$ of $[0,1]$, it is proved that when M has non-positive sectional curvature, then as $\mathrm{mesh}(\mathcal{P})\to 0$,

$$\frac{1}{Z_{\mathcal{P}}}e^{-1/2E(\sigma)}\,\mathrm{dVol}_{G_{\mathcal{P}}}(\sigma)\to\exp\left\{-\frac{2+\sqrt{3}}{20\sqrt{3}}\int_0^1 \mathrm{Scal}(\sigma(s))ds\right\}d\nu(\sigma).$$

Here $Z_{\mathcal{P}}$ is a normalization constant, $G_{\mathcal{P}}$ is an L^2-metric on $H_{\mathcal{P}}(M)$, $\mathrm{Vol}_{G_{\mathcal{P}}}$ is the Riemannian volume measure induced from $G_{\mathcal{P}}$, Scal is the scalar curvature on M, and ν is the Wiener measure on $W(M)$. This allows one to rigorously interpret the heuristic measure given above by $Z^{-1}\exp\{-\frac{1}{2}E(\sigma)\}\mathcal{D}\sigma$ as the measure $\exp\left\{-\frac{2+\sqrt{3}}{20\sqrt{3}}\int_0^1 \mathrm{Scal}(\sigma(s))ds\right\}d\nu(\sigma)$.

Chapter 1

Introduction

The goal of this paper is to give a rigorous interpretation of a Feynman path integral on a Riemannian manifold (M, g), which is heuristically of the form,

$$\frac{1}{Z}\int_{W^o(M)} f(\sigma)e^{-\frac{1}{2}E(\sigma)}\mathcal{D}\sigma. \tag{1.1}$$

Here $W^o(M)$ is the collection of continuous paths $\sigma : [0,1] \to M$ with $\sigma(0) = o$, $E(\sigma) = \int_0^1 |\sigma'(s)|^2 ds$ is the energy of a path σ, Z is a normalization constant, and $\mathcal{D}\sigma$ is a Lebesgue type measure on $W^o(M)$. Truly this expression is not rigorous for several reasons: there is no infinite dimensional Lebesgue measure, Z can be interpreted as 0 or ∞, and "most" paths $\sigma \in W^o(M)$ are nowhere differentiable, leaving E to be well-defined. Nonetheless, expressions as in Eq. (1.1) arise frequently in physics literature and can be understood as the prescription for the path integral quantization of the Hamiltonian operator as well as the path integral formula for the heat kernel of the Schrödinger operator. Theorem 1.4 gives a possible realization of the heuristic measure $Z^{-1}\exp\{-\frac{1}{2}E(\sigma)\}\mathcal{D}\sigma$ on manifolds with non-positive sectional curvature as $\exp\{-\tau_G\int_0^1 \mathrm{Scal}(\sigma(s))ds\}d\nu(\sigma)$, where $\tau_G = (2+\sqrt{3})/(20\sqrt{3})$, Scal is the scalar curvature on M, and ν is the Wiener measure on $W^o(M)$.

Much of the current interest concerning path integrals in physics began with Feynman in [8] and has since grown deeply. The role of path integrals in quantum mechanics is surveyed by Gross in [11] and detailed more by Feynman and Hibbs in [9] as well as Glimm and Jaffe in [10]. Volumes of work have come out to move these stochastic techniques onto the manifold setting of which [5, 13, 16, 6] have been invaluable in completing this paper.

It has been asserted that the correct form of the quantization of the Hamiltonian $\frac{1}{2}g^{ij}p_ip_j + V$ is given by $-\hbar^2(\frac{1}{2}\nabla + \tau\,\mathrm{Scal}) + V$ where $\hbar$ is Planck's constant and $\tau \in \mathbb{R}$ is a constant which depends on the interpretation of the path integral. For example, in [1], $\tau = 0$ or $\tau = \frac{1}{6}$. Our work gives the value $\tau = (2+\sqrt{3})/(20\sqrt{3})$. However, in [17], Lim derives a form that is dissimilar and does not lend itself to the Feynman-Kac formula for interpretation of the quantized Hamiltonian.

1.1 The heat semi-group as a path integral.

Given the Laplace-Beltrami operator Δ on M, the solution to the heat equation with boundary data $f : M \to \mathbb{R}$,

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) & (t,x) \in (0,\infty) \times M \\ u(0,x) = f(x) & x \in M \end{cases}$$

will be suggestively written as

$$u(t,x) =: e^{\frac{t}{2}\Delta} f(x). \tag{1.2}$$

The fundamental solution to the heat equation will be denoted $p_t(x,y)$ and for x near y we have,

$$p_t(x,y) \approx \left(\frac{1}{2\pi t}\right)^{d/2} e^{\frac{\mathrm{dist}(x,y)^2}{2t}}, \tag{1.3}$$

where $\mathrm{dist}(x,y)$ is the geodesic distance between x and y. With this notation we have the following representation of the heat semi-group on M,

$$e^{\frac{t}{2}\Delta} f(o) = \int_M p_t(o,y) f(y) dy. \tag{1.4}$$

Definition 1.1. *The Wiener space $(W(M),\nu)$ is the probability space consisting of continuous paths $\sigma : [0,1] \to M$ such that $\sigma(0) = o$. The Wiener measure ν*

associated with M is the unique probability measure on $W(M)$ such that

$$\int_{W(M)} f(\sigma)d\nu(\sigma)$$
$$= \int_{M^n} F(x_1, ..., x_n) \prod_{i=1}^{n} p_{\Delta_i s}(x_{i-1}, x_i)\, \mathrm{dVol}(x_1) \cdots \mathrm{dVol}(x_n), \tag{1.5}$$

for all functions of the form $f(\sigma) = F(\sigma(s_1), ..., \sigma(s_n))$ with F bounded and continuous on M^n, $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n = 1\}$ is a partition of $[0,1]$, $\Delta_i s = s_i - s_{i-1}$, Vol *is the Riemannian volume form on M, and $x_0 := o$.*

With this description of Wiener measure, setting $t = 1$, we can rewrite Eq. (1.4) as the path integral,

$$e^{\frac{1}{2}\Delta} f(o) = \int_{W(M)} f(\sigma(1))d\nu(\sigma). \tag{1.6}$$

More generally we have the path integral representation for the heat semi-group,

$$e^{\frac{t}{2}\Delta} f(o) = \int_{W_t(M)} f(\sigma(t))d\nu_t(\sigma) \tag{1.7}$$

where $W_t(M)$ and ν_t are defined analogously as above with the parameterization interval $[0,1]$ replaced with $[0,t]$.

1.2 "Derivation" of Eq. (1.1).

As before, let (M, g, o) be a Riemannian manifold with metric g and fixed point $o \in M$. For this endeavor, we will use the following. We now consider the operator $H = -\frac{1}{2}\Delta + M_V$ acting on $L^2(M, d\,\mathrm{Vol})$ where M_V is multiplication by the potential $V \in C(M \to \mathbb{R})$, and Vol is the Riemannian volume form on M. Motivated by Eq. (1.7), we set out to represent $e^{-tH}f$ by a path integral.

Theorem 1.2 (Trotter's product formula)**.** *If A and B are matrices of the same dimensions, then*

$$e^{A+B} = \lim_{n\to\infty} \left[e^{\frac{A}{n}} e^{\frac{B}{n}}\right]^n. \tag{1.8}$$

The necessity of this revolves around the fact that generally $e^{A+B} \neq e^A e^B$ since equality depends on the commutativity properties of A and B. There have been many works to further understand and generalize Trotter's product formula, a short list includes [3, 2, 7, 15, 14, 21, 19, 20, 18]. However, this section is solely for intuition and heuristics, we therefore make no attempt to legitimize our use of Theorem 1.2 any further.

Once again defining $H = -\frac{1}{2}\Delta + M_V$, using Eqs. (1.4) and (1.3), and applying Trotter's product formula with $A = \frac{t}{2}\Delta$ and $B = -tM_V$ gives,

$$\begin{aligned} e^{-tH} f(o) &= \lim_{n\to\infty} \left[e^{\frac{t}{2n}\Delta} e^{\frac{t}{n} M_V} \right]^n f(o) \\ &= \lim_{n\to\infty} \left\{ \int_{M^n} e^{-\frac{t}{n}\sum_{i=1}^n V(x_i)} f(x_n) \prod_{i=1}^n p_{t/n}(x_{i-1}, x_i)\, \mathrm{dVol}(\mathbf{x}) \right\} \\ &= \lim_{n\to\infty} \left\{ \left(2\pi \frac{t}{n}\right)^{-dn/2} \int_{M^n} e^{-\sum_{i=1}^n \left(\frac{\mathrm{dist}(x_i, x_{i-1})^2}{2t/n} + \frac{t}{n} V(x_i) \right)} f(x_n)\, \mathrm{dVol}(\mathbf{x}) \right\} \end{aligned}$$

where we use the convention $\mathrm{dVol}(\mathbf{x}) = \mathrm{dVol}(x_1) \cdots \mathrm{dVol}(x_n)$.

If $\sigma : [0, t] \to M$ is defined by $\sigma(s_i) = x_i$ and is geodesic on all intervals (s_{i-1}, s_i) for $1 \leq i \leq n$ then,

$$\sum_{i=1}^n \left(\frac{\mathrm{dist}(x_i, x_{i-1})^2}{2t/n} + \frac{t}{n} V(x_i) \right) = \int_0^t \left\{ \frac{1}{2} |\sigma'(s)|^2 + V(\sigma(\overline{s})) \right\} ds,$$

where $\overline{s} = s_i$ when $s \in (s_{i-1}, s_i]$. We therefore have,

$$\begin{aligned} e^{-tH} f(o) &= \lim_{n\to\infty} \left\{ \left(2\pi \frac{t}{n}\right)^{-dn/2} \int_{H_{\mathcal{P},t}(M)} e^{-\int_0^t \left\{ \frac{1}{2}|\sigma'(s)|^2 + V(\sigma(\overline{s})) \right\} ds} f(\sigma(t))\, \mathrm{dVol}(\sigma) \right\} \\ &= \frac{1}{Z} \int_{W_t(M)} f(\sigma(t)) e^{-\frac{1}{2}\int_0^1 \left\{ |\sigma'(s)|^2 + V(\sigma(s)) \right\} ds} \mathcal{D}\sigma. \end{aligned} \tag{1.9}$$

Here $H_{\mathcal{P},t}(M)$ is used to represent the collection of paths $\sigma \in W_t(M)$ which are piecewise geodesic with respect to the partition $\mathcal{P}$. Compared with Eq. (1.1), we see now that arguing by Trotter's product formula "derives" the heuristic expression.

1.3 The main theorem

Motivated by the "derivation" of Eq. (1.1) in Section 1.2, we make the following definition.

Definition 1.3. *Given the partition $\mathcal{P} = \{0 = s_0 < s_1 < ... < s_n = 1\}$ of $[0,1]$, the space of continuous piecewise geodesic paths on M with respect to $\mathcal{P}$ will be denoted $H_{\mathcal{P}}(M)$. That is,*

$$H_{\mathcal{P}}(M) = \left\{\sigma \in W(M) : \sigma|_{(s_{i-1},s_i)} \text{ is a geodesic for } 1 \leq i \leq n\right\}.$$

The space $H_{\mathcal{P}}(M)$ is a finite dimensional subspace of $W(M)$. We make $H_{\mathcal{P}}(M)$ into a Riemannian manifold by defining the metric $G_{\mathcal{P}}$, where if $X, Y \in T_\sigma H_{\mathcal{P}}(M)$,

$$G_{\mathcal{P}}(X,Y) := \int_0^1 g(X(s), Y(s))ds. \tag{1.10}$$

Here we are naturally identifying the space $T_\sigma H_{\mathcal{P}}(M)$ with the continuous piecewise Jacobi fields along the path σ. In turn this metric induces a Riemannian volume form $\mathrm{Vol}_{G_{\mathcal{P}}}$ on M and we define the measure $\nu_{G_{\mathcal{P}}}$ on M by,

$$d\nu_{G_{\mathcal{P}}} = \frac{1}{Z_{G_{\mathcal{P}}}} e^{-\frac{1}{2}E}\, \mathrm{dVol}_{G_{\mathcal{P}}}, \tag{1.11}$$

where $E(\sigma) = \int_0^1 |\sigma'(s)|^2 ds$ is the energy of the path and $Z_{G_{\mathcal{P}}}$ is a normalization constant that can be calculated as,

$$Z_{G_{\mathcal{P}}} = \int_{H_{\mathcal{P}}(\mathbb{R}^d)} e^{-\frac{1}{2}E(\omega)}\, \mathrm{dVol}_{G_{\mathcal{P}}}(\omega). \tag{1.12}$$

That is, $Z_{G_{\mathcal{P}}}$ is the normalization constant that for the case $M = \mathbb{R}^d$ makes $\nu_{G_{\mathcal{P}}}$ into a probability measure. We are now are prepared to state the main theorem proved in this paper.

Theorem 1.4. *Let (M, g, o) be a Riemannian manifold with metric g and fixed point $o \in M$. Assume the curvature and its derivative on M are bounded and the sectional curvature on M is non-positive. Then given a continuous and bounded map $f : W(M) \to \mathbb{R}$,*

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma) d\nu_{G_{\mathcal{P}}}(\sigma) = \int_{W(M)} f(\sigma) e^{-\tau_G \int_0^1 \mathrm{Scal}(\sigma(s))ds} d\nu(\sigma). \tag{1.13}$$

Here $\mathcal{P} = \{0, 1/n, 2/n, ..., 1\}$ *is the equally spaced partition of* $[0,1]$, $\tau_G = \frac{2+\sqrt{3}}{20\sqrt{3}}$, Scal *is the scalar curvature of* M, $\nu_{G_{\mathcal{P}}}$ *is defined in Eq. (1.11), and* ν *is the Wiener measure on* $W(M)$.

Notice the similarity of the limit on the left hand side of Eq. (1.13) with that leading to Eq. (1.9). If we take the point of view that Theorem 1.4 is the prescription for the path integral quantization of the Hamiltonian $\hat{H}$, the Feynman-Kac formula gives

$$\hat{H} = -\frac{1}{2}\Delta^2 + V + \tau_G \operatorname{Scal}. \tag{1.14}$$

The measure $\nu_{G_{\mathcal{P}}}$ is considered to be an approximation to the Wiener measure ν on the space of piecewise geodesic paths, which is an approach that has been capitalized on before, see for example [1, 17, 22, 4].

1.4 A theorem in the flat case

This section is dedicated to proving a much simplified theorem resembling that of Theorem 1.4 in the case that $M = \mathbb{R}^d$. In this setting, the curvature term disappears and, up to a multiplicative constant, all translation invariant measures are equal. Much of the notation introduced here will be reintroduced in our more general setting.

Definition 1.5. *The symbol* μ *will be used to denote Wiener measure on* $W(\mathbb{R}^d)$. $\{b_s\}_{s\in[0,t]}$ *is taken to be an* $\mathbb{R}^d$*-valued Brownian motion, and given the partition* $\mathcal{P} = \{0 = s_0 < s_1 < \cdots s_n = t\}$ *of* $[0,t]$, *set*

$$\begin{aligned} b_s^{\mathcal{P}} &:= \sum_{i=1}^{n} 1_{J_i}(s)\left[\frac{\Delta_i b}{\Delta_i s}(s - s_{i-1}) + b_{s_{i-1}}\right] \\ &= \sum_{i=1}^{n} 1_{J_i}(s)\left[\frac{\Delta_i b}{\Delta_i s}(s - s_{i-1}) + \sum_{j=1}^{i-1}\Delta_j b\right]. \end{aligned} \tag{1.15}$$

Here $J_0 = [0, s_1]$, $J_i = (s_{i-1}, s_i]$ *for* $i \geq 1$, $\Delta_i b := b_{s_i} - b_{s_{i-1}}$, $\Delta_i s := s_i - s_{i-1}$ *and* $\sum_{j=1}^{0}\Delta_j b \equiv 0$.

For the following calculations, we are going to define the (linear) parameterization $\varphi : (\mathbb{R}^d)^n \to H_{\mathcal{P}}(\mathbb{R}^d)$ by the very canonical "connect the dots" mapping φ defined by $\varphi(a_1, ..., a_n)(s_i) = a_i$ for each $i = 1, ..., n$. Notice that this completely defines $\varphi(\mathbf{a})$ since it is required to be piecewise linear between the partition points. Explicitly,

$$\varphi(x_1, ..., x_n)(s) = \sum_{i=1}^{n} 1_{J_i}(s) \left[\frac{\Delta_i x}{\Delta_i s}(s - s_{i-1}) + x_{i-1} \right] \tag{1.16}$$

where the notation is similar to above. Because of future calculations, we make a couple notes about the map φ here. The first is, if we let $x_i = (x_i^1, ..., x_i^d) \in \mathbb{R}^d$,

$$g_{i,a}(s) := \partial_{x_i^a} \varphi(\mathbf{x})(s) = \left\{ 1_{J_i}(s) \frac{s - s_{i-1}}{\Delta_i s} + 1_{J_{i+1}}(s) \left[1 - \frac{s - s_i}{\Delta_{i+1} s} \right] \right\} e_a, \tag{1.17}$$

where $\{e_1, ..., e_d\}$ is the standard basis in $\mathbb{R}^d$. The second is that, comparing the similarities between Equations 1.15 and 1.16,

$$\varphi(b_{s_1}, ..., b_{s_n}) = b^{\mathcal{P}}. \tag{1.18}$$

Proposition 1.6. *Define $g_{i,a}(s) := \partial_{x_i^a} \varphi(\mathbf{x})(s)$. Let $\mathcal{G}$ be any inner product on $H_{\mathcal{P}}(\mathbb{R}^d)$ and $\mathrm{Vol}_{\mathcal{G}}$ be the volume measure associated to $\mathcal{G}$. Define*

$$Z_{\mathcal{G}} := \int_{H_{\mathcal{P}}(\mathbb{R}^d)} e^{-\frac{1}{2}E} d\,\mathrm{Vol}_{\mathcal{G}}\,. \tag{1.19}$$

Then,

$$Z_{\mathcal{G}} = \sqrt{\det\left[\mathcal{G}(g_{i,a}, g_{j,c})\right]} \prod_{i=1}^{n} (2\pi \Delta_i s)^{d/2}, \tag{1.20}$$

where $i, j \in \{1, ..., n\}$, $a, c \in \{1, ..., d\}$ and $[\mathcal{G}(g_{i,a}, g_{j,c})]$ represents the $n \times n$ block matrix with $d \times d$ blocks where the $(a, c)^{th}$ element of the $(i, j)^{th}$ $d \times d$ block is $\mathcal{G}(g_{i,a}, g_{j,c})$.

Proof. Considering Eq. (1.17), notice that $g_{i,a}$ is independent of $\mathbf{x}$. We also have

$$\mathrm{dVol}_{\mathcal{G}} = \sqrt{\det\left[\mathcal{G}(g_{i,a}, g_{j,c})\right]} d\mathbf{x} \tag{1.21}$$

so that

$$Z_{\mathcal{G}} = \int_{(\mathbb{R}^d)^n} e^{-\frac{1}{2}E(\varphi(\mathbf{x}))}\sqrt{\det\left[\mathcal{G}(g_{i,a}, g_{j,c})\right]}d\mathbf{x} \tag{1.22}$$

$$= \sqrt{\det\left[\mathcal{G}(g_{i,a}, g_{j,c})\right]}\int_{(\mathbb{R}^d)^n} e^{-\frac{1}{2}E(\varphi(\mathbf{x}))}d\mathbf{x}. \tag{1.23}$$

Moreover,

$$\begin{aligned} E(\varphi(\mathbf{x})) &= \int_0^t \left\|\varphi(\mathbf{x})'(s)\right\|^2 ds \\ &= \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \left\|\frac{\Delta_i x}{\Delta_i s}\right\|^2 ds \\ &= \sum_{i=1}^n \frac{\|\Delta_i x\|^2}{\Delta_i s} \end{aligned}$$

and therefore

$$e^{-\frac{1}{2}E(\varphi(\mathbf{x}))} = e^{-\frac{1}{2}\sum_{i=1}^n \frac{\|\Delta_i x\|^2}{\Delta_i s}}. \tag{1.24}$$

In particular,

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}E(\varphi(\mathbf{x}))}d\mathbf{x} = \int_{\mathbb{R}^n} e^{-\frac{1}{2}\sum_{i=1}^n \frac{\|x_i - x_{i-1}\|^2}{\Delta_i s}} dx_1 dx_2 \cdots dx_n \tag{1.25}$$

$$= \prod_{i=1}^n (2\pi\Delta_i s)^{d/2} \tag{1.26}$$

and hence

$$Z_{\mathcal{G}} = \sqrt{\det\left[\mathcal{G}(g_{i,a}, g_{j,c})\right]}\prod_{i=1}^n (2\pi\Delta_i s)^{d/2}. \tag{1.27}$$

□

Proposition 1.7. *Let $f : W(\mathbb{R}^d) \to \mathbb{R}$ be bounded and μ-measurable. Then*

$$\frac{1}{Z_{\mathcal{G}}}\int_{H_{\mathcal{P}}(\mathbb{R}^d)} f e^{-\frac{1}{2}E} d\,\mathrm{Vol}_{\mathcal{G}} = \mathbb{E}[f(b^{\mathcal{P}})]. \tag{1.28}$$

Proof. Continuing with the notation from Proposition 1.6, we have shown that,

$$Z_{\mathcal{G}} = \sqrt{\det\left[\mathcal{G}(g_{i,a}, g_{j,c})\right]}\prod_{i=1}^n (2\pi\Delta_i s)^{d/2}.$$

Therefore,

$$\frac{1}{Z_{\mathcal{G}}}\int_{H_{\mathcal{P}}(\mathbb{R}^d)} fe^{-\frac{1}{2}E} d\,\mathrm{Vol}_{\mathcal{G}} = \int_{(\mathbb{R}^d)^n} f(\varphi(\mathbf{a}))\left(\prod_{i=1}^{n}(2\pi\Delta_i s)^{-d/2}\right)e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{\|x_i-x_{i-1}\|^2}{\Delta_i s}}d\mathbf{x}$$
$$= \int_{(\mathbb{R}^d)^n} f(\varphi(\mathbf{a}))\prod_{i=1}^{n}\frac{e^{-\frac{1}{2}\frac{\|a_i-a_{i-1}\|^2}{\Delta_i s}}}{(2\pi\Delta_i s)^{d/2}}d\mathbf{a}$$
$$= \int_{(\mathbb{R}^d)^n} f(\varphi(\mathbf{a}))\prod_{i=1}^{n}p_{\Delta_i s}(a_{i-1},a_i)d\mathbf{a}$$
$$= \mathbb{E}\left[f(\varphi(b_{s_1},b_{s_2},...,b_{s_n}))\right]$$
$$= \mathbb{E}\left[f(b^{\mathcal{P}})\right].$$

□

Theorem 1.8. *Using the same notation as Proposition 1.7, given a bounded continuous map $f : W(\mathbb{R}^d) \to \mathbb{R}$,*

$$\lim_{|\mathcal{P}|\to 0}\frac{1}{Z_{\mathcal{G}}}\int_{H_{\mathcal{P}}(\mathbb{R}^d)} fe^{-\frac{1}{2}E} d\,\mathrm{Vol}_{\mathcal{G}} = \int_{W(\mathbb{R}^d)} f d\mu. \tag{1.29}$$

Proof. From the result of Proposition 1.7 along with the knowledge that μ is the law of the Brownian process $b.$, we only need to show that

$$\mathbb{E}[f(b^{\mathcal{P}})] \to \mathbb{E}[f(b.)].$$

This follows from the dominated convergence theorem and the easily seen fact that $b^{\mathcal{P}} \to b$ pointwise. □

Chapter 2

Background Material

2.1 The Wiener Space

In Definition 1.1 we defined the Wiener space $W(M)$ and Wiener measure ν. Intuitively, one might understand the Wiener space as the collection of paths a particle might make in a given interval of time. That is, if at time 0 we know that a particle is at point o, then we collect all the possible paths the particle might travel for the first increment of time. When $M = \mathbb{R}^d$ and $o = 0$ the space $W(\mathbb{R}^d)$ is often called the *Classical Wiener Space*.

It is well known that the Wiener measure on $W(\mathbb{R}^d)$ is the law of a $\mathbb{R}^d$-valued Brownian motion, and conversely, the evaluation maps $b_s(\omega) = \omega(s)$ are an $\mathbb{R}^d$-valued Brownian motion under the Wiener measure. The analogous statements can be said for the Wiener measure on $W(M)$ and an M-valued Brownian motion, although we do not explore this further in what follows. The interested reader is referred to [5, 13, 6] for the definition and treatment of a manifold-valued Brownian motion.

Notation 2.1. *We will be frequently moving between the spaces $W(\mathbb{R}^d)$ and $W(M)$ and therefore it is useful to fix the symbol μ as the Wiener measure on $W(\mathbb{R}^d)$ and keep the symbol ν as the Wiener measure on $W(M)$. Also, we will let $\{b_s\}_{s\in[0,1]}$ represent the evaluation maps on $W(\mathbb{R}^d)$, which, as mentioned above, is an $\mathbb{R}^d$-valued Brownian motion under the measure μ.*

2.2 Important subspaces of $W(M)$

The subspace $H(M) \subset W(M)$ defined by

$$H(M) := \{\sigma \in W(M) : \sigma \text{ is absolutely continuous}, \int_0^1 \|\sigma'(s)\|^2 ds < \infty\} \tag{2.1}$$

is called the *Cameron-Martin space on M*. The quantity $E(\sigma)$ defined for $\sigma \in H(M)$ is given by

$$E(\sigma) := \int_0^1 \|\sigma'(s)\|^2 ds \tag{2.2}$$

and is called the *energy* of σ. Hence, we can describe $H(M)$ as the collection of absolutely continuous paths in $W(M)$ with finite energies.

If $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n = 1\}$ is a partition of $[0, 1]$, then we can define the finite dimensional submanifold of $H(M)$ by collecting the paths which are piecewise geodesic between the partition points. That is,

$$H_{\mathcal{P}}(M) := \{\sigma \in W(M) : \frac{\nabla}{ds}\sigma'(s) = 0 \text{ for } s \notin \mathcal{P}\}, \tag{2.3}$$

where $\frac{\nabla}{ds}$ is the covariant derivative defined in Eq. (2.11). Note that we have the following containment, $H_{\mathcal{P}}(M) \subset H(M) \subset W(M)$. Intuitively, we consider $H_{\mathcal{P}}(M)$ as the piecewise geodesic approximations to the paths in $W(M)$. In the flat case, $H_{\mathcal{P}}(\mathbb{R}^d)$ is the collection of continuous piecewise linear paths parameterized on $[0, 1]$ approximating those paths in $W(\mathbb{R}^d)$, which leads us to create the map $b^{\mathcal{P}} : W(\mathbb{R}^d) \to H_{\mathcal{P}}(\mathbb{R}^d)$ by

$$b_s^{\mathcal{P}} := \sum_{i=1}^{n} 1_{J_i}(s) \left[\frac{\Delta_i b}{\Delta_i s}(s - s_{i-1}) + b_{s_{i-1}}\right] \tag{2.4}$$

where we are using the notation

$$J_i = \begin{cases} [0, s_1] & i = 1 \\ (s_{i-1}, s_i] & 1 < i \le n \end{cases} \tag{2.5}$$

and

$$\Delta_i b := b_{s_i} - b_{s_{i-1}} \tag{2.6}$$

$$\Delta_i s := s_i - s_{i-1}. \tag{2.7}$$

Explicitly, taking $s \in [0,1]$ with $s_{i-1} < s \leq s_i$, we have that

$$\begin{aligned} b_s^{\mathcal{P}}(\omega) &= \frac{b_{s_i}(\omega) - b_{s_{i-1}}(\omega)}{s_i - s_{i-1}}(s - s_{i-1}) + b_{s_{i-1}}(\omega) \\ &= \frac{\omega(s_i) - \omega(s_{i-1})}{s_i - s_{i-1}}(s - s_{i-1}) + \omega(s_{i-1}) \end{aligned} \tag{2.8}$$

which is the piecewise linear approximation to ω with respect to the partition $\mathcal{P}$. Similarly then, we consider $b^{\mathcal{P}}$ as the piecewise linear approximation to Brownian motion b. Something to notice here is that the restriction of $b^{\mathcal{P}}$ to $H_{\mathcal{P}}(\mathbb{R}^d)$ is just evaluation where if $\omega \in H_{\mathcal{P}}(\mathbb{R}^d)$ then $b_s^{\mathcal{P}}(\omega) = \omega(s)$ and therefore, on $H_{\mathcal{P}}(\mathbb{R}^d)$, $b^{\mathcal{P}} = b$.

Notation 2.2. *Since we will frequently use notation such as that in Eq. (2.6) and Eq. (2.7) let us agree that if $f : [0,1] \to \mathbb{R}^d$ is any map, then*

$$\Delta_i f := f(s_i) - f(s_{i-1}). \tag{2.9}$$

2.3 The tangent space $T_\sigma H_{\mathcal{P}}(M)$

Take $\sigma \in H_{\mathcal{P}}(M)$. We can form a tangent vector $X \in T_\sigma H_{\mathcal{P}}(M)$ by taking a smoothly varying one-parameter family $\{\sigma_t\}_{t\in(-\epsilon,\epsilon)} \subset H_{\mathcal{P}}(M)$ such that $\sigma_0 = \sigma$ and setting $X = \frac{d}{dt}\big|_{t=0}\sigma_t$. Hence, fixing $s \in [0,1]$ we can realize X as a vector field along σ by $X(s) = \frac{d}{dt}\big|_{t=0}\sigma_t(s) \in T_{\sigma(s)}(M)$. Moreover, between partition points in $\mathcal{P}$, σ_t is a family of geodesic curves passing through the geodesic curve described by σ, which therefore tells us that X will be a Jacobi field between the partition points. Thusly, between partition points, X must satisfy Jacobi's equation,

$$\frac{\nabla^2}{ds^2}X(s) = R(\sigma'(s), X(s))\sigma'(s), \tag{2.10}$$

where R is the curvature tensor on M. In this way we can (and do!) identify $T_\sigma H_{\mathcal{P}}(M)$ with the continuous piecewise Jacobi fields along σ.

A full statement and proof of this fact can be found in [1, Proposition 4.4] and is restated here for completeness.

Proposition 2.3. *Let $\sigma \in H_{\mathcal{P}}(M)$, then $X \in T_\sigma H_{\mathcal{P}}(M)$ if and only if X satisfies Eq. (2.10) on $[0,1]\backslash\mathcal{P}$.*

2.4 Geometric basics

The pointed d-dimensional Riemannian manifold (M, g, o) with metric g and fixed point $o \in M$ will be endowed with the Levi-Civita covariant derivative ∇. If $\sigma : [0,1] \to M$ is a path with $\sigma(0) = o$, we use the symbol $//_s(\sigma) : T_oM \to T_{\sigma(s)}M$ to represent parallel translation along σ with respect to ∇. For a C^1 vector field X along the path σ, define $\frac{\nabla}{ds}$ by

$$\frac{\nabla}{ds}X(s) := //_s(\sigma)\frac{d}{ds}\left\{//_s^{-1}(\sigma)X(s)\right\}. \tag{2.11}$$

Equivalently, if X is a curve in TM parameterized on $[0,1]$ with $X(0) \in T_oM$, we have

$$\frac{\nabla}{ds}X(s) := //_s(\pi(X))\frac{d}{ds}\left\{//_s^{-1}(\pi(X))X(s)\right\} \tag{2.12}$$

where $\pi : TM \to M$ is the standard projection.

The curvature tensor R is defined by

$$R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{2.13}$$

for all vector fields X, Y and Z on M. The Ricci tensor is then defined as

$$\mathrm{Ric}(X) := \sum_i^d R(X, e_i)e_i \tag{2.14}$$

for the vector field X on M and orthonormal frame $\{e_i\}_{i=1}^d$ and the scalar curvature on M is given by

$$\mathrm{Scal} := \sum_{i=1}^d g(\mathrm{Ric}(e_i), e_i). \tag{2.15}$$

Notice that for a given $p \in M$, $\mathrm{Ric}|_{T_p(M)}$ is a linear map $T_p(M) \to T_p(M)$. Therefore, $\mathrm{Scal}(p) = \mathrm{tr}(\mathrm{Ric}|_{T_p(M)})$.

We fix an isometry $u_0 : \mathbb{R}^d \to T_o(M)$ and from henceforth identify $T_o(M)$ with $\mathbb{R}^d$. Some of the work of this paper will be translating statements between the spaces $W(M)$, $H(M)$, and $H_{\mathcal{P}}(M)$ and the spaces $W(\mathbb{R}^d)$, $H(\mathbb{R}^d)$, and $H_{\mathcal{P}}(\mathbb{R}^d)$. In doing so many proofs become tractable; however, this does lead us to introduce the reader to more notation.

Notation 2.4. *If $\pi : TM \to M$ is the projection, $f : \mathbb{R}^d (= T_o(M)) \to T_pM$ is an isometry, we define*

1. $\Omega_f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ *by* $\Omega_f(a,b)c = f^{-1}R(fa, fb)fc$.

2. $\mathrm{Ric}_f : \mathbb{R}^d \to \mathbb{R}^d$ *is the linear map defined by* $\mathrm{Ric}_f(v) = \sum_{i=1}^d \Omega_f(v, \varepsilon_i)\varepsilon_i$ *where* $\{\varepsilon_i\}_{i=1}^d$ *is an orthonormal basis for* $\mathbb{R}^d$.

3. *Given a curves $h : [0,1] \to \mathbb{R}^d$ and $\sigma : [0,1] \to M$, we define the vector field X_σ^h along σ by $X_\sigma^h(s) = //_s(\sigma)h(s)$.*

Some basic properties of these objects are listed below, the proofs of which are straight forward and are postponed until Appendix C.

Proposition 2.5. *Using the notation above, we have the following properties*

1. *For $v \in T_oM$ we have that* $\mathrm{Ric}_f(v) = f^{-1}\,\mathrm{Ric}(fv)$.

2. *If σ is a curve in $H(M)$ starting at $o \in M$ and $u(s) := //_s(\sigma)$, then* $\mathrm{tr}(\mathrm{Ric}_{u(s)}) = \mathrm{Scal}(\sigma(s))$.

3. $\mathrm{tr}(\Omega_f(v, \cdot)v) = -\langle \mathrm{Ric}_f\, v, v\rangle$.

2.5 Cartan's Development Map

Definition 2.6. *Cartan's Development Map is a diffeomorphism, $\phi : H(\mathbb{R}^d) \to H(M)$ defined by the functional equation,*

$$\sigma'(s) = //_s(\sigma)b'(s), \qquad \sigma(0) = o$$

where $\sigma = \phi(b)$.

By smooth dependence on parameters, ϕ is smooth, and uniqueness of solutions implies that ϕ is injective.

Definition 2.7. *The Anti-Development Map, $\phi^{-1} : H(M) \to H(\mathbb{R}^d)$ is defined by $b = \phi^{-1}(\sigma)$ where*

$$b(s) = \int_0^s //_r^{-1}(\sigma)\sigma'(r)dr.$$

By similar arguments as above, ϕ^{-1} is smooth and injective and shows that $\phi : H(\mathbb{R}^d) \to H(M)$ is a diffeomorphism, as asserted. To understand intuitively what ϕ does, consider N to be a sphere in $\mathbb{R}^3$. The map $\phi : H(\mathbb{R}^2) \to H(N)$ imprints a curve drawn on the flat 2-dimensional surface onto N by rolling N along the curve without sliding or twisting. This is why ϕ is often referred to as the *Rolling Map*.

We also make note of the following important facts,

1. ϕ is a bijection between $H_{\mathcal{P}}(\mathbb{R}^d)$ and $H_{\mathcal{P}}(M)$, $\phi(H_{\mathcal{P}}(\mathbb{R}^d)) = H_{\mathcal{P}}(M)$.

2. This further implies that $T_\sigma H_{\mathcal{P}}(M)$ is an embedded submanifold of $T_\sigma H(M)$, since $T_\sigma H_{\mathcal{P}}(\mathbb{R}^d)$ is an embedded submanifold of $T_\sigma H(\mathbb{R}^d)$.

A more detailed account of the developement can be found in [5, 13].

2.6 An Important Previous Result

In Eq. (1.11) we defined the measure $\nu_{G_{\mathcal{P}}}$ on $H_{\mathcal{P}}(M)$ which resulted from our choice of metric on $H_{\mathcal{P}}(M)$, $G_{\mathcal{P}}$. In 1999, Andersson and Driver in [1] introduced a similar measure $\nu_{S_{\mathcal{P}}}$ on $H_{\mathcal{P}}(M)$ with

$$\nu_{S_{\mathcal{P}}} = \frac{1}{Z_{S_{\mathcal{P}}}} e^{-\frac{1}{2}E} \, \mathrm{dVol}_{S_{\mathcal{P}}}, \tag{2.16}$$

$\mathrm{Vol}_{S_{\mathcal{P}}}$ is the Riemannian volume form on $H_{\mathcal{P}}(M)$ defined by the metric $S_{\mathcal{P}}$ given by,

$$S_{\mathcal{P}}(X,Y) = \sum_{i=1}^{n} g\left(\frac{\nabla}{ds} X(s_{i-1}+), \frac{\nabla}{ds} Y(s_{i-1}+)\right) \Delta_i s, \tag{2.17}$$

and $Z_{S_{\mathcal{P}}}$ is the normalization constant defined by

$$Z_{S_{\mathcal{P}}} := (2\pi)^{dn/2}. \tag{2.18}$$

Remark 2.8. *Similar as to the definition of $Z_{G_{\mathcal{P}}}$ in Eq. (1.12), the constant $Z_{S_{\mathcal{P}}}$ can be calculated by,*

$$Z_{S_{\mathcal{P}}} = \int_{H_{\mathcal{P}}(\mathbb{R}^d)} e^{-\frac{1}{2}E} \, \mathrm{dVol}_{S_{\mathcal{P}}} \tag{2.19}$$

to arrive at Eq. (2.18).

With this measure, Andersson and Driver showed the following.

Theorem 2.9 ([1, Theorem 1.8]). *Suppose that the curvature and its derivative are bounded on M. If $f : W(M) \to \mathbb{R}$ is bounded and continuous then,*

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma) d\nu_{S_{\mathcal{P}}} = \int_{W(M)} f(\sigma) d\nu. \tag{2.20}$$

From here our approach is to compare the measure $\nu_{G_{\mathcal{P}}}$ with $\nu_{S_{\mathcal{P}}}$. Once again we will find ourselves communicating between the spaces $H_{\mathcal{P}}(M)$ and $H_{\mathcal{P}}(\mathbb{R}^d)$ and hence it is useful to introduce some notation.

Notation 2.10. *When $M = \mathbb{R}^d$ we use $\mu_{S_{\mathcal{P}}}$ to represent the measure $\nu_{S_{\mathcal{P}}}$.*

2.7 Important relations between μ, $\mu_{S_{\mathcal{P}}}$, and $\nu_{S_{\mathcal{P}}}$.

The following fact that we state as a Theorem is proved in [1, Theorem 4.10 and Corollary 4.13] and left unproved here.

Theorem 2.11. *Continuing with the notation introduced above, $\mu_{S_{\mathcal{P}}} = \mathrm{Law}_\mu(b^{\mathcal{P}})$ and $\nu_{S_{\mathcal{P}}} = \mathrm{Law}_\mu(\phi(b^{\mathcal{P}}))$. In particular, $\mu_{S_{\mathcal{P}}}$ is the pullback of $\nu_{S_{\mathcal{P}}}$ by ϕ, $\mu_{S_{\mathcal{P}}} = \phi^*\nu_{S_{\mathcal{P}}}$; that is, for any Borel set $A \subset H_{\mathcal{P}}(\mathbb{R}^d)$, $\mu_{S_{\mathcal{P}}}(A) = \nu_{S_{\mathcal{P}}}(\phi(A))$.*

The remainder of this section will be spent exploring a couple of the consequences of this theorem. The first corollary is immediate. What follows helps establish notation that we be used frequently.

Corollary 2.12. *Given a partition $\mathcal{P} = \{0 = s_0 < s_1 < ... < s_n = 1\}$ of $[0, 1]$ and a cylinder function $f : W(\mathbb{R}^d) \to \mathbb{R}$ with $f(\omega) = F(\omega(s_0), \omega(s_1), ..., \omega(s_n))$, where $F : (\mathbb{R}^d)^n \to \mathbb{R}$ is bounded and continuous, then*

$$\int_{W(\mathbb{R}^d)} f d\mu = \int_{H_{\mathcal{P}}(\mathbb{R}^d)} f d\mu_{S_{\mathcal{P}}} = \int_{H_{\mathcal{P}}(M)} f(\phi^{-1}) d\nu_{S_{\mathcal{P}}}. \tag{2.21}$$

In particular, $\nu_{S_{\mathcal{P}}}$ is a probability measure on $H_{\mathcal{P}}(M)$.

Proof. This is a direct consequence of Theorem 2.11 once we realize that $f = f(b^{\mathcal{P}})$. For the second claim, set $f \equiv 1$ and we see that $\nu_{S_{\mathcal{P}}}(H_{\mathcal{P}}(M)) = \mu(W(\mathbb{R}^d)) = 1$. □

Before establishing the next corollary, we need to agree upon some notation. We first define the map $u_s^{\mathcal{P}}$ on $W(\mathbb{R}^d)$ by,

$$u_s^{\mathcal{P}} := //_s(\phi(b^{\mathcal{P}})). \tag{2.22}$$

That is, for $\omega \in W(\mathbb{R}^d)$ and $\sigma^{\mathcal{P}} := \phi(b^{\mathcal{P}}(\omega))$, $u_s^{\mathcal{P}}(\omega) : T_oM \to T_{\sigma^{\mathcal{P}}(s)}M$ is the linear isometry given by,

$$u_s^{\mathcal{P}}(\omega) = //_s(\sigma^{\mathcal{P}}). \tag{2.23}$$

In turn this let us define the random variables $\mathcal{R}_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}} : W(\mathbb{R}^d) \to \mathbb{R}$ by

$$\mathcal{R}_{\mathcal{P}} = \sum_{i=1}^{n} \langle \mathrm{Ric}_{u^{\mathcal{P}}_{s_{i-1}}} \Delta_i b^{\mathcal{P}}, \Delta_i b^{\mathcal{P}} \rangle, \tag{2.24}$$

$$\mathcal{S}_{\mathcal{P}} = \sum_{i=1}^{n} \mathrm{Scal}(\phi(b^{\mathcal{P}})|_{s_{i-1}}) \Delta_i s, . \tag{2.25}$$

Here $\phi(b^{\mathcal{P}})|_{s_{i-1}}(\omega) := \phi(b^{\mathcal{P}}(\omega))(s_{i-1})$. It is important to note that from Proposition 2.5,

$$\mathrm{tr}(\mathrm{Ric}_{u_s^{\mathcal{P}}}) = \mathrm{Scal}(\phi(b^{\mathcal{P}})|_s) : W(\mathbb{R}^d) \to \mathbb{R}. \tag{2.26}$$

Corollary 2.13. *Given $f : \mathbb{R}^2 \to \mathbb{R}$ continuous, then*

$$\int_{H_{\mathcal{P}}(\mathbb{R}^d)} f(\mathcal{R}_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}) d\mu_{S_{\mathcal{P}}} = \int_{W(\mathbb{R}^d)} f(\mathcal{R}_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}) d\mu. \tag{2.27}$$

Proof. Again, what we notice is that $\mathcal{R}_{\mathcal{P}}(b^{\mathcal{P}}) = \mathcal{R}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{P}}(b^{\mathcal{P}}) = \mathcal{S}_{\mathcal{P}}$ so that by Theorem 2.11,

$$\int_{W(\mathbb{R}^d)} f(\mathcal{R}_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}) d\mu = \int_{W(\mathbb{R}^d)} f(\mathcal{R}_{\mathcal{P}}(b^{\mathcal{P}}), \mathcal{S}_{\mathcal{P}}(b^{\mathcal{P}})) d\mu = \int_{H_{\mathcal{P}}(\mathbb{R}^d)} f(\mathcal{R}_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}) d\mu_{S_{\mathcal{P}}}.$$

□

We are now in the position to prove a size estimates that will be used in the proof of the main result. We apply the above Corollary 2.13 by allowing $f(x, y) = e^{p(x-y)}$ where $p \in \mathbb{R}$.

Lemma 2.14. *For $p \in \mathbb{R}$ there is a constant C depending only on* d*, p, and the bound on the curvature of M such that*

$$1 \leq \int_{H_{\mathcal{P}}(\mathbb{R}^d)} e^{p(\mathcal{R}_{\mathcal{P}} - \mathcal{S}_{\mathcal{P}})} d\mu_{S_{\mathcal{P}}} \leq e^{C|\mathcal{P}|}. \tag{2.28}$$

Proof. By Corollary 2.13,

$$\int_{H_{\mathcal{P}}(\mathbb{R}^d)} e^{p(\mathcal{R}_{\mathcal{P}} - \mathcal{S}_{\mathcal{P}})} d\mu_{S_{\mathcal{P}}} = \mathbb{E}\left[e^{p(\mathcal{R}_{\mathcal{P}} - \mathcal{S}_{\mathcal{P}})}\right].$$

The result then follows as a direct application of Corollary F.4 below. □

The above result and proof are also given in [1, Proposition 6.4].

Lemma 2.15. *For $p \in \mathbb{R}$ there is a constant C depending only on* d*, p, and the bound on the curvature of M such that*

$$\int_{H_{\mathcal{P}}(\mathbb{R}^d)} \left| \exp\left\{ p(\mathcal{S}_{\mathcal{P}}(\omega) - \int_0^1 \mathrm{Scal}(\phi(\omega)(s))ds) \right\} - 1 \right| d\mu_{S_{\mathcal{P}}}(\omega) \leq C|\mathcal{P}|^{1/2}. \tag{2.29}$$

Proof. There is a bound $\Lambda = \Lambda(\text{curvature}) < \infty$ such that for $\omega \in H_{\mathcal{P}}(\mathbb{R}^d)$,

$$\begin{aligned}
\left| \mathcal{S}_{\mathcal{P}}(\omega) - \int_0^1 \mathrm{Scal}(\phi(\omega)(s))ds \right| &= \left| \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \left\{ \mathrm{Scal}(\phi(\omega)(s)) - \mathrm{Scal}(\phi(\omega)(s_{i-1})) \right\} ds \right| \\
&\leq \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \left| \mathrm{Scal}(\phi(\omega)(s)) - \mathrm{Scal}(\phi(\omega)(s_{i-1})) \right| ds \\
&\leq \Lambda \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \| \omega(s) - \omega(s_{i-1}) \| \, ds \\
&= \Lambda \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \left\| b_s^{\mathcal{P}}(\omega) - b_{s_{i-1}}^{\mathcal{P}}(\omega) \right\| ds \\
&\leq \Lambda \sum_{i=1}^n \Delta_i s \left\| \Delta_i b^{\mathcal{P}} \right\|,
\end{aligned}$$

where we recall that $b^{\mathcal{P}}$ is the identity on $H_{\mathcal{P}}(\mathbb{R}^d)$. From here,

$$\begin{aligned}
&\int_{H_{\mathcal{P}}(\mathbb{R}^d)} \left| \exp\left\{ p(\mathcal{S}_{\mathcal{P}}(\omega) - \int_0^1 \mathrm{Scal}(\phi(\omega)(s))ds) \right\} - 1 \right| d\mu_{S_{\mathcal{P}}}(\omega) \\
&\leq p\Lambda \sum_{i=1}^n \Delta_i s \int_{H_{\mathcal{P}}(\mathbb{R}^d)} \|\Delta_i b^{\mathcal{P}}\| \exp\left\{ p\Lambda \sum_{j=1}^n \Delta_j s \|\Delta_j b^{\mathcal{P}}\| \right\} d\mu_{S_{\mathcal{P}}} \\
&= p\Lambda \sum_{i=1}^n \Delta_i s \mathbb{E}\left[\|\Delta_i b\| \exp\left\{ p\Lambda \sum_{j=1}^n \Delta_j s \|\Delta_j b\| \right\} \right] \\
&= p\Lambda \sum_{i=1}^n \Delta_i s \mathbb{E}\left[\|\Delta_i b\| e^{p\Lambda \Delta_i s \|\Delta_i b\|} \right] \prod_{j\neq i} \mathbb{E}\left[e^{p\Lambda \Delta_j s \|\Delta_j b\|} \right] \\
&= p\Lambda \sum_{i=1}^n (\Delta_i s)^{3/2} \mathbb{E}\left[\|b_1\| e^{p\Lambda (\Delta_i s)^{3/2} \|b_1\|} \right] \prod_{j\neq i} \mathbb{E}\left[e^{p\Lambda (\Delta_j s)^{3/2} \|b_1\|} \right] \\
&\leq p\Lambda |\mathcal{P}|^{1/2} \mathbb{E}\left[\|b_1\| e^{p\Lambda |\mathcal{P}|^{3/2} \|b_1\|} \right] \prod_{i=1}^n \left(1 + p\Lambda (\Delta_i s)^{3/2} \mathbb{E}\left[\|b_1\| e^{p\Lambda (\Delta_i s)^{3/2} \|b_1\|} \right] \right) \\
&\leq C|\mathcal{P}|^{1/2}.
\end{aligned}$$

Here the first inequality comes from $|e^a - 1| \leq e^{|a|} - 1 \leq |a|e^{|a|}$, the penultimate inequality comes from $e^{ax} \leq 1 + axe^{ax}$ for $a, x \geq 0$, and the final inequality follows from the fact that

$$\prod_{i=1}^n \left(1 + p\Lambda (\Delta_i s)^{3/2} \mathbb{E}\left[\|b_1\| e^{p\Lambda (\Delta_i s)^{3/2} \|b_1\|} \right] \right) \leq \exp\left\{ p\Lambda |\mathcal{P}|^{1/2} \mathbb{E}\left[\|b_1\| e^{p\Lambda |\mathcal{P}|^{3/2} \|b_1\|} \right] \right\}.$$

□

The final result discussed in this section introduces the maps $\rho_{\mathcal{P}}$ and $\tilde{\rho}_{\mathcal{P}}$ which are a major focus throughout the sequel. Given the measure $\nu_{G_{\mathcal{P}}}$ in Eq. (1.11), we let $\rho_{\mathcal{P}} : H_{\mathcal{P}}(M) \to \mathbb{R}$ be the Lebesgue-Radon-Nicodym derivative with respect to $\nu_{S_{\mathcal{P}}}$,

$$d\nu_{G_{\mathcal{P}}} = \rho_{\mathcal{P}} d\nu_{S_{\mathcal{P}}}. \tag{2.30}$$

We then define $\tilde{\rho}_{\mathcal{P}} : W(\mathbb{R}^d) \to \mathbb{R}$ as

$$\tilde{\rho}_{\mathcal{P}} = \rho_{\mathcal{P}}(\phi(b^{\mathcal{P}})). \tag{2.31}$$

Proposition 2.16. *Let $f : W(M) \to \mathbb{R}$ be bounded and continuous. Then,*

$$\int_{H_{\mathcal{P}}(M)} f d\nu_{G_{\mathcal{P}}} = \int_{H_{\mathcal{P}}(\mathbb{R}^d)} f(\phi)\tilde{\rho}_{\mathcal{P}} d\mu_{S_{\mathcal{P}}} = \int_{W(\mathbb{R}^d)} f(\phi(b^{\mathcal{P}}))\tilde{\rho}_{\mathcal{P}} d\mu. \tag{2.32}$$

Proof. By the definition of $\rho_{\mathcal{P}}$,

$$\int_{H_{\mathcal{P}}(M)} f d\nu_{G_{\mathcal{P}}} = \int_{H_{\mathcal{P}}(M)} f \rho_{\mathcal{P}} d\nu_{S_{\mathcal{P}}}.$$

By Theorem 2.11,

$$\int_{H_{\mathcal{P}}(M)} f \rho_{\mathcal{P}} d\nu_{S_{\mathcal{P}}} = \int_{H_{\mathcal{P}}(\mathbb{R}^d)} f(\phi)\rho_{\mathcal{P}}(\phi) d\mu_{S_{\mathcal{P}}},$$

and we finish by another application of Theorem 2.11 and by noticing that $\rho_{\mathcal{P}}(\phi) = \tilde{\rho}_{\mathcal{P}}$ on $H_{\mathcal{P}}(\mathbb{R}^d)$. □

Chapter 3

Setup for the Proof of Theorem 1.4

As previously mentioned, $T_\sigma H_{\mathcal{P}}(M)$ is naturally identified with continuous piecewise Jacobi fields along σ with respect to the partition $\mathcal{P}$. Recall that a Jacobi field X along σ is one which satisfies,

$$\frac{\nabla^2}{ds^2} X(s) = R(\sigma'(s), X(s))\sigma'(s). \tag{3.1}$$

The following proposition is proved in [1, Proposition 4.4].

Proposition 3.1. *For $\sigma \in H_{\mathcal{P}}(M)$ and $h \in H(\mathbb{R}^d)$, define $u(s) := //_s(\sigma)$ and $\omega := \phi^{-1}(\sigma) \in H_{\mathcal{P}}(\mathbb{R}^d)$. Then $X_\sigma^h \in T_\sigma H_{\mathcal{P}}(M)$ if and only if h satisfies*

$$h''(s) = \Omega_{u(s)}(\omega'(s), h(s))\omega'(s) \quad on \quad [0,1]\backslash\mathcal{P}. \tag{3.2}$$

Following the notation of Proposition 3.1, $\omega'(s) = \frac{\Delta_i \omega}{\Delta_i s} = \frac{\Delta_i b^{\mathcal{P}}(\omega)}{\Delta_i s}$ when $s \in (s_{i-1}, s_i)$. Moreover, $u(s) = //_s(\phi(\omega)) = u_s^{\mathcal{P}}(\omega)$, where $u^{\mathcal{P}}$ is defined in Eq. (2.22). Hence, for each $i \in \{1, ..., n\}$ and $s \in (s_{i-1}, s_i)$, we can rewrite Eq. (3.2) as,

$$h''(s) = \Omega_{u_s^{\mathcal{P}}(\omega)} \left(\frac{\Delta_i b^{\mathcal{P}}(\omega)}{\Delta_i s}, h(s) \right) \frac{\Delta_i b^{\mathcal{P}}(\omega)}{\Delta_i s}. \tag{3.3}$$

This motivates the definition of the following operators,

$$A_i^{\mathcal{P}}(s) := \Omega_{u^{\mathcal{P}}_{s+s_{i-1}}} \left(\frac{\Delta_i b^{\mathcal{P}}}{\Delta_i s}, \cdot \right) \frac{\Delta_i b^{\mathcal{P}}}{\Delta_i s}, . \tag{3.4}$$

with $1 \le i \le n$ and $s \in (s_{i-1}, s_i)$. More explicitly, for each $\omega \in W(\mathbb{R}^d)$ and $s \in (s_{i-1}, s_i)$, $A_i^{\mathcal{P}}(s)(\omega)$ is the linear operator $\mathbb{R}^d \to \mathbb{R}^d$ defined by,

$$A_i^{\mathcal{P}}(s)(\omega)\mathbf{x} = \Omega_{u^{\mathcal{P}}_{s+s_{i-1}}(\omega)}\left(\frac{\Delta_i b^{\mathcal{P}}(\omega)}{\Delta_i s}, \mathbf{x}\right)\frac{\Delta_i b^{\mathcal{P}}(\omega)}{\Delta_i s}. \tag{3.5}$$

In what follows we typically suppress ω from the notation. It is important to notice that from Proposition 2.5,

$$\operatorname{tr}(A_i^{\mathcal{P}}(0)) = -\left\langle \operatorname{Ric}_{u^{\mathcal{P}}_{s_{i-1}}} \frac{\Delta_i b^{\mathcal{P}}}{\Delta_i s}, \frac{\Delta_i b^{\mathcal{P}}}{\Delta_i s}\right\rangle. \tag{3.6}$$

Moreover, the assumption that the curvature and its derivative are bounded on M is equivalent to the existence of some $\kappa < \infty$ such that

$$\sup_s \|A_i^{\mathcal{P}}(s)\| \le \kappa \frac{\|\Delta_i b\|^2}{(\Delta_i s)^2} \tag{3.7}$$

$$\sup_s \|\frac{d}{ds} A_i^{\mathcal{P}}(s)\| \le \kappa \frac{\|\Delta_i b\|^3}{(\Delta_i s)^3}. \tag{3.8}$$

3.1 Defining the basis $\{f_{i,a}\}$

From here on, unless mentioned otherwise, we will assume that $\mathcal{P} = \{s_0 = 0 < s_1 = 1/n < \cdots < s_n = 1\}$ is the equally spaced partition on $[0, 1]$ and $\Delta := \frac{1}{n} = |\mathcal{P}|$. We will also assume that the curvature of M and its derivative are both bounded and that M has non-positive sectional curvature.

With this in mind, for each $s \in [0, \Delta]$ and $i \in \{1, ..., n\}$, consider the differential equation,

$$\frac{d^2}{ds^2} Z_i^{\mathcal{P}}(s) = A_i^{\mathcal{P}}(s) Z_i^{\mathcal{P}}(s) \tag{3.9}$$

with $Z_i^{\mathcal{P}}(s) : \mathbb{R}^d \to \mathbb{R}^d$ a linear map. Applying existence and uniqueness of ordinary differential equations, we define the following solutions to Eq. (3.9):

$$S_i^{\mathcal{P}} = Z_i^{\mathcal{P}} \text{ with initial conditions } S_i^{\mathcal{P}}(0) = 0, \frac{d}{ds} S_i^{\mathcal{P}}(0) = I \tag{3.10}$$

$$C_i^{\mathcal{P}} = Z_i^{\mathcal{P}} \text{ with initial conditions } C_i^{\mathcal{P}}(0) = I, \frac{d}{ds} C_i^{\mathcal{P}}(0) = 0 \tag{3.11}$$

$$V_i^{\mathcal{P}} = Z_i^{\mathcal{P}} \text{ with initial conditions } V_i^{\mathcal{P}}(0) = S_{i-1}^{\mathcal{P}}(\Delta), \frac{d}{ds} V_i^{\mathcal{P}}(0) = -F_{i-1}^{\mathcal{P}} \tag{3.12}$$

where

$$F_i^{\mathcal{P}} = S_{i+1}^{\mathcal{P}}(\Delta)^{-1} C_{i+1}^{\mathcal{P}}(\Delta) S_i^{\mathcal{P}}(\Delta). \tag{3.13}$$

With the above definitions,

$$V_{i+1}^{\mathcal{P}}(s) = C_{i+1}^{\mathcal{P}}(s) S_i^{\mathcal{P}}(\Delta) - S_{i+1}^{\mathcal{P}}(s) F_i^{\mathcal{P}}. \tag{3.14}$$

The fact that $S_i^{\mathcal{P}}(\Delta)$ has an inverse is not immediate. However, we are guaranteed the inverse exists when we restrict ourselves to manifolds of non-positive sectional curvature (see Section E). Intuitively, one might understand this fact by realizing that in the case where M has constant negative sectional curvature, $S_i^{\mathcal{P}}$ grows like sinh, which remains away from 0 for positive arguments, whereas if M has constant positive sectional curvature, $S_i^{\mathcal{P}}$ grows like sine, which frequently returns to 0 and therefore fails to be invertible.

We now define the basis $\{f_{i,a}^{\mathcal{P}} : 1 \le i \le n,\ 1 \le a \le d\}$ which satisfy Eq. (3.2) with the boundary conditions

$$f_{i,a}^{\mathcal{P}}(0) = 0 \tag{3.15}$$

$$\frac{d}{ds} f_{i,a}^{\mathcal{P}}(s_j+) = \begin{cases} e_a & j = i-1 \\ -F_i^{\mathcal{P}} e_a & j = i \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

Using Eq. (3.10), Eq. (3.11), and Eq. (3.14),

$$f_{i,a}^{\mathcal{P}}(s) = 1_{J_i}(s) S_i^{\mathcal{P}}(s - s_{i-1}) e_a + 1_{J_{i+1}}(s) V_{i+1}^{\mathcal{P}}(s - s_i) e_a \tag{3.17}$$

where $\{e_a\}_{a=1}^d$ is the standard basis for $\mathbb{R}^d$. This set of maps induces a basis

$$\mathcal{F}_{\mathcal{P}} := \left\{ X_{\phi(b^{\mathcal{P}})}^{f_{i,a}^{\mathcal{P}}} : 1 \le i \le n,\ 1 \le a \le d \right\} \tag{3.18}$$

of $T_{\phi(b^{\mathcal{P}})} H_{\mathcal{P}}(M)$ where the meaning of X_σ^h is established in Notation 2.4.

3.2 The matrix $G^{\mathcal{F}_{\mathcal{P}}}$

Define $G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}$ as the $n \times n$ block diagonal matrix with $d \times d$ blocks given by

$$G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}} := \left[G_{\mathcal{P}} \left(X_{\phi(b^{\mathcal{P}})}^{f_{i,a}^{\mathcal{P}}}, X_{\phi(b^{\mathcal{P}})}^{f_{j,c}^{\mathcal{P}}} \right) : 1 \le i, j \le n,\ 1 \le a, c \le d \right]. \tag{3.19}$$

That is, $G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}$ is the matrix representation of the metric $G_{\mathcal{P}}$ under the basis $\mathcal{F}_{\mathcal{P}}$ in Eq. (3.18). We write the $(i,j)^{th}$ block of $G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}$ as

$$\left[G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}\right]_{i,j} := \left[G_{\mathcal{P}}\left(X_{\phi(b^{\mathcal{P}})}^{f_{i,a}^{\mathcal{P}}}, X_{\phi(b^{\mathcal{P}})}^{f_{j,c}^{\mathcal{P}}}\right) : 1 \leq a, c \leq d\right] \tag{3.20}$$

and make note that as the mesh of the partition decreases to zero, n goes off to infinity while d remains constant.

Let $i, j \in \{1, ..., n\}$, $a, c \in \{1, ..., d\}$, and define $S_{n+1}^{\mathcal{P}} = V_{n+1}^{\mathcal{P}} \equiv 0$,

$$G_{\mathcal{P}}\left(X_{\phi(b^{\mathcal{P}})}^{f_{i,a}^{\mathcal{P}}}, X_{\phi(b^{\mathcal{P}})}^{f_{j,c}^{\mathcal{P}}}\right) = \int_0^1 g\left(X_{\phi(b^{\mathcal{P}})}^{f_{i,a}^{\mathcal{P}}}(s), X_{\phi(b^{\mathcal{P}})}^{f_{j,c}^{\mathcal{P}}}(s)\right) ds \tag{3.21}$$

$$= \int_0^1 \left\langle f_{i,a}^{\mathcal{P}}(s), f_{j,c}^{\mathcal{P}}(s)\right\rangle ds \tag{3.22}$$

where $\langle\cdot,\cdot\rangle$ indicates the inner-product on $T_o(M) = \mathbb{R}^d$ defined by g. Using Eq. (3.17),

$$\begin{aligned}\int_0^1 \left\langle f_{i,a}^{\mathcal{P}}(s), f_{j,c}^{\mathcal{P}}(s)\right\rangle ds = \delta_{i,j}&\left\langle e_a, \left[\int_0^\Delta (S_i^{\mathcal{P}}(s)^{tr} S_i^{\mathcal{P}}(s) + V_{i+1}^{\mathcal{P}}(s)^{tr} V_{i+1}^{\mathcal{P}}(s))ds\right] e_c\right\rangle \\ &+ \delta_{i,j+1}\left\langle e_a, \left[\int_0^\Delta S_{j+1}^{\mathcal{P}}(s)^{tr} V_{j+1}^{\mathcal{P}}(s)ds\right] e_c\right\rangle \\ &+ \delta_{i+1,j}\left\langle e_a, \left[\int_0^\Delta V_{i+1}^{\mathcal{P}}(s)^{tr} S_{i+1}^{\mathcal{P}}(s)ds\right] e_c\right\rangle,\end{aligned} \tag{3.23}$$

which implies that,

$$\left[G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}\right]_{i,j} = \begin{cases} \int_0^\Delta (S_i^{\mathcal{P}}(s)^{tr} S_i^{\mathcal{P}}(s) + V_{i+1}^{\mathcal{P}}(s)^{tr} V_{i+1}^{\mathcal{P}}(s))ds & i = j \\ \int_0^\Delta V_{i+1}^{\mathcal{P}}(s)^{tr} S_{i+1}^{\mathcal{P}}(s)ds & i+1 = j \\ \int_0^\Delta S_{j+1}^{\mathcal{P}}(s)^{tr} V_{j+1}^{\mathcal{P}}(s)ds & i = j+1 \\ 0 & \text{otherwise} \end{cases} \tag{3.24}$$

Equivalently,

$$G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}} = \begin{pmatrix} D_1 & M_2 & 0 & 0 \\ M_2^{tr} & D_2 & M_3 & 0 \\ 0 & \ddots & \ddots & M_n \\ 0 & 0 & M_n^{tr} & D_n \end{pmatrix} \tag{3.25}$$

where

$$D_i = \begin{cases} \int_0^\Delta \left(S_i^{\mathcal{P}}(s)^{tr} S_i^{\mathcal{P}}(s) + V_{i+1}^{\mathcal{P}}(s)^{tr} V_{i+1}^{\mathcal{P}}(s) \right) ds & 1 \le i < n \\ \int_0^\Delta S_n^{\mathcal{P}}(s)^{tr} S_n^{\mathcal{P}}(s) ds & i = n \end{cases} \tag{3.26}$$

and

$$M_i = \int_0^\Delta V_i^{\mathcal{P}}(s)^{tr} S_i^{\mathcal{P}}(s) ds \quad 2 \le i \le n. \tag{3.27}$$

3.3 $G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}$ in the flat case

In the case that $M = \mathbb{R}^d$, Eq. (3.9) becomes $\frac{d}{ds} Z^{\mathcal{P}}(s) = 0$ yielding that $Z^{\mathcal{P}}(s) = Z^{\mathcal{P}}(0) + s\left(\frac{d}{ds}\big|_0 Z^{\mathcal{P}}(s) \right)$. Therefore $S_i^{\mathcal{P}}(s) = sI$, $C_i^{\mathcal{P}}(s) = I$, and $V_{i+1}^{\mathcal{P}}(s) = (\Delta - s)I$. Particularly,

$$f_{i,a} = \left\{ (s - s_{i-1}) 1_{J_i}(s) + (s_{i+1} - s) 1_{J_{i+1}}(s) \right\} e_a. \tag{3.28}$$

Using Eqs. (3.25), (3.26), and (3.27) we calculate

$$G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}(\mathbb{R}^d) = \frac{\Delta^3}{6} \begin{pmatrix} 4I & I & 0 & 0 \\ I & 4I & I & 0 \\ 0 & \ddots & \ddots & I \\ 0 & 0 & I & 2I \end{pmatrix}. \tag{3.29}$$

Since this matrix will prove important to understand throughout the remainder, we use the notation $\mathcal{L}_{\mathcal{P}} := G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}(\mathbb{R}^d)$. That is,

$$\mathcal{L}_{\mathcal{P}} := \frac{\Delta^3}{6} \begin{pmatrix} 4I & I & 0 & 0 \\ I & 4I & I & 0 \\ 0 & \ddots & \ddots & I \\ 0 & 0 & I & 2I \end{pmatrix}. \tag{3.30}$$

Proposition 3.2. *The normalization constant $Z_{G_{\mathcal{P}}}$ defined in Eq. (1.12) is given by,*

$$Z_{G_{\mathcal{P}}} = \sqrt{(2\pi)^{nd} \frac{\det(\mathcal{L}_{\mathcal{P}})}{\Delta^{nd}}}. \tag{3.31}$$

Proof. In Proposition 1.6, we found that,

$$Z_{G_{\mathcal{P}}} = \sqrt{\det(G_{\mathcal{P}}(g_{i,a}, g_{j,c}))} \prod_{i=1}^{n} (2\pi\Delta_i s)^{d/2}$$
$$= \sqrt{(2\pi\Delta)^{nd} \det(G_{\mathcal{P}}(g_{i,a}, g_{j,c}))},$$

where $g_{i,a}$ were defined in Eq. (1.17). However, comparing Eqs. (1.17) and (3.28), we notice that, $\frac{1}{\Delta} f_{i,a} = g_{i,a}$ (where we recall that $\Delta_i s = \Delta$ for each i), implying that,

$$\det(G_{\mathcal{P}}(g_{i,a}, g_{j,c})) = \frac{1}{\Delta^{2nd}} \det(G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}(\mathbb{R}^d)) = \frac{1}{\Delta^{2nd}} \det(\mathcal{L}_{\mathcal{P}}).$$

Therefore,

$$Z_{G_{\mathcal{P}}} = \sqrt{\frac{(2\pi)^{nd} \det(\mathcal{L}_{\mathcal{P}})}{\Delta^{nd}}}.$$

□

3.4 A simplified expression for $\tilde{\rho}_{\mathcal{P}}$

We now return to the more general manifold M. In Eq. (2.30) we defined the Lebesgue-Radon-Nikodym derivative $\rho_{\mathcal{P}}$, which by the definition of the Riemannian volume form,

$$\rho_{\mathcal{P}}(\sigma) = \frac{Z_{S_{\mathcal{P}}}}{Z_{G_{\mathcal{P}}}} \sqrt{\frac{\det\left\{G_{\mathcal{P}}(X_\sigma^{h_{i,a}}, X_\sigma^{h_{j,c}})\right\}}{\det\left\{S_{\mathcal{P}}(X_\sigma^{h_{i,a}}, X_\sigma^{h_{j,c}})\right\}}} \tag{3.32}$$

where the collection $\{h_{i,a} : 1 \le i \le n,\ 1 \le a \le d\}$ forms a basis of the the space consisting of continuous maps satisfying Eq. (3.2) with $h_{i,a}(0) = 0$. If we define $S_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}$ analogously to $G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}$, where $S_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}$ is the matrix representation of the metric $S_{\mathcal{P}}$ using the basis $\mathcal{F}_{\mathcal{P}}$, then we have,

$$\tilde{\rho}_{\mathcal{P}} = \rho_{\mathcal{P}}(\phi(b^{\mathcal{P}})) = \frac{Z_{S_{\mathcal{P}}}}{Z_{G_{\mathcal{P}}}} \sqrt{\frac{\det\left(G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}\right)}{\det\left(S_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}\right)}}. \tag{3.33}$$

Define the remainder of the matrix $G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}$ by

$$\mathcal{R}_{\mathcal{P}} := G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}} - \mathcal{L}_{\mathcal{P}}. \tag{3.34}$$

From Eqs. (2.18) and (3.33) along with Proposition 3.2,

$$\begin{aligned}\tilde{\rho}_{\mathcal{P}} &= \sqrt{\frac{\Delta^{nd}\det\left(G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}\right)}{\det(\mathcal{L}_{\mathcal{P}})\det\left(S_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}\right)}} \\ &= \sqrt{\frac{\Delta^{nd}\det\left(\mathcal{L}_{\mathcal{P}}+\mathcal{R}_{\mathcal{P}}\right)}{\det\left(\mathcal{L}_{\mathcal{P}}\right)\det\left(S_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}\right)}}. \end{aligned} \tag{3.35}$$

One final step in simplifying the form of $\tilde{\rho}_{\mathcal{P}}$ is aided by the following Proposition, proved in [17, Theorem 4.7],

Proposition 3.3. *Using the notation defined above,*

$$\det\left(S_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}\right) = \Delta^{nd}. \tag{3.36}$$

As an immediate corollary of Proposition 3.3 in combination with Eq. (3.35) we have,

$$\tilde{\rho}_{\mathcal{P}} = \sqrt{\frac{\det\left(\mathcal{L}_{\mathcal{P}}+\mathcal{R}_{\mathcal{P}}\right)}{\det\left(\mathcal{L}_{\mathcal{P}}\right)}} \tag{3.37}$$

3.5 Properties of $\mathcal{L}_{\mathcal{P}}$

This section is devoted to proving certain properties of $\mathcal{L}_{\mathcal{P}}$ summarized below in Theorem 3.4. The proof of the theorem can safely be skipped if the reader is willing to accept the statement. We use $\{e_a : 1 \le a \le d\}$ to denote the standard basis in $\mathbb{R}^d$.

Theorem 3.4. *There exists an orthonormal basis of eigenvectors of $\mathcal{L}_{\mathcal{P}}$, $\{u_{k,a} : 1 \le k \le n, 1 \le a \le d\}$ with,*

$$u_{k,a}^{\mathcal{P}} := \beta_k^{\mathcal{P}} \begin{pmatrix} \alpha_k^1 e_a \\ \alpha_k^2 e_a \\ \vdots \\ \alpha_k^n e_a \end{pmatrix}. \tag{3.38}$$

Here, for $1 \le k < n$, $\alpha_k^m = \sin(m\theta_k^{\mathcal{P}})$ with $\{\theta_k^{\mathcal{P}} : 1 \le k < n\} \subset (0, \pi)$ a monotonically increasing sequence given by,

$$\theta_k^{\mathcal{P}} = \frac{\pi(k + r_k)}{n+1} \tag{3.39}$$

where

$$\frac{k}{2(n+1)} \leq r_k \leq \left(\frac{4k}{n+1} \wedge 1\right). \tag{3.40}$$

For $k = n$, $\alpha_n^m = \gamma_n^m - \gamma_n^{-m}$ *with* $\gamma_n \in (-2, -3/2)$*. If* $1 \leq k < n$*, the normalization constants* β_k *are given by*

$$(\beta_k^{\mathcal{P}})^2 = \frac{2}{n}\left(\frac{1}{1-\epsilon_k}\right) \tag{3.41}$$

where $|\epsilon_k| = O(1/n)$ *and thusly* $\beta_k^{\mathcal{P}} = O(1/\sqrt{n})$*. For* $k = n$*,*

$$(\beta_n^{\mathcal{P}})^2 = \left(\frac{\gamma_n^{2(n+1/2)} - \gamma_n^{-2(n+1/2)}}{\gamma_n - \gamma_n^{-1}} - 2n - 1\right)^{-1}. \tag{3.42}$$

In particular, for large enough n, $\beta_n^{\mathcal{P}} < (3/2)^{-n}$*. Further, if* $1 \leq k < n-1$*, then* $\left|(\beta_{k+1}^{\mathcal{P}})^2 - (\beta_k^{\mathcal{P}})^2\right| = O(1/n^2)$*.*

The eigenvalues $\lambda_{k,a}^{\mathcal{P}}$*, defined so that* $\mathcal{L}_{\mathcal{P}} u_{k,a} = \lambda_{k,a}^{\mathcal{P}} u_{k,a}$*, are given by* $\lambda_{k,a}^{\mathcal{P}} = \frac{\Delta^3}{3}\left(2 + \cos(\theta_k^{\mathcal{P}})\right) =: \lambda_k^{\mathcal{P}}$ *for* $1 \leq k < n$*, and* $\lambda_{n,a}^{\mathcal{P}} = \frac{\Delta^3}{6}(4 + \gamma_n + \gamma_n^{-1}) =: \lambda_n^{\mathcal{P}}$ *when* $k = n$*, which further implies that*

$$\frac{\Delta^3}{4} \leq \|\mathcal{L}_{\mathcal{P}}\| \leq \Delta^3. \tag{3.43}$$

Finally, there exists an upper triangular matrix $\mathcal{A}_{\mathcal{P}}$ *such that*

$$\mathcal{L}_{\mathcal{P}} = \mathcal{A}_{\mathcal{P}} \mathcal{A}_{\mathcal{P}}^{tr}. \tag{3.44}$$

Here $\mathcal{A}_{\mathcal{P}}$ *is invertible and*

$$\|[\mathcal{A}_{\mathcal{P}}^{-1}]_{i,j}\|^2 \leq \begin{cases} \frac{3}{\Delta^3}\left(\frac{1}{2}\right)^{j-i} & j \geq i \\ 0 & j < i \end{cases}. \tag{3.45}$$

3.5.1 The Matrix l_n

Let l_n be the $n \times n$ matrix given by

$$l_n = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & \ddots & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}. \tag{3.46}$$

Our first goal is to find the eigenvalues and eigenvectors of l_n. To do so, we write l_n as

$$l_n = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & \ddots & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -4 \end{pmatrix} + \begin{pmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$
$$=: \mathcal{D}_n + 6I_n.$$

The usefulness of this is that now $\mathcal{D}_n$ looks like the discreetization of the Laplace operator acting on functions with the boundary conditions $f(0) = 0$ and $f(n+1) = -2f(n)$. We know that the eigenvectors of the Laplace operator have the form $f(j) = az^j + bz^{-j}$. Enforcing the boundary conditions we have that $a = -b$ from $f(0) = 0$ and that $z^{n+1} - z^{-(n+1)} = -2(z^n - z^{-n})$ from $f(n+1) = -2f(n)$. Rewriting this last equation yields,

$$z^{2(n+1)} + 2z^{2n+1} - 2z - 1 = 0. \tag{3.47}$$

Before we discuss solutions to the above equation, we first will find the eigenvalues. With $f(j) = a(z^j - z^{-j})$ for $1 \leq j \leq n$,

$$\begin{aligned} f(j+1) - 2f(j) + f(j-1) &= a(z^{j+1} - z^{-j-1} - 2z^j + 2z^{-j} + z^{j-1} - z^{-j+1}) \\ &= a(z^j - z^{-j})(z + z^{-1} - 2) \\ &= (z + z^{-1} - 2)f(j). \end{aligned}$$

Therefore $\mathcal{D}_n f(j) = (z + z^{-1} - 2)f(j)$ and hence $l_n f(j) = (z + z^{-1} + 4)f(j)$. To summarize these statements,

Proposition 3.5. *For any $a, z \in \mathbb{C}$ with z satisfying equation 3.47, the map $f : \{1, 2, ..., n\} \to \mathbb{C}$ given by $f(j) = a(z^j - z^{-j})$ is an eigenvector of l_n with eigenvalue $z + z^{-1} + 4$.*

3.5.2 Solving Equation 3.47

Lemma 3.6. *There exists $\gamma_n \in \mathbb{R}$ with $-2 < \gamma_n < -\frac{3}{2}$ such that if $n \geq 2$ then γ_n solves equation 3.47. Moreover as $n \to \infty$, $\gamma_n \to -2$.*

Proof. If we set $g_n(z) := z^{2(n+1)}(1+2z^{-1})-(1+2z)$, it is easy to see that $g_n(-2) = 3$ and that $g_n(-3/2) < 0$ when $n \geq 2$. Moreover, given any $x \in (-2, -3/2)$, $\lim_{n\to\infty} g_n(x) \to -\infty$. Applying the intermediate value theorem to these facts finishes the proof. □

Proposition 3.7. *There exist numbers* $\{\theta_k\}_{k=1}^{n-1} \subset (0,\pi)$ *with* $\theta_k < \frac{\pi(k+1)}{n+1} < \theta_{k+1}$ *such that* $e^{i\theta_k}$ *solves equation 3.47. Moreover, there is a strictly increasing function* $\varphi \in C^1([0,\pi] \to [0,\pi])$*, independent of* n*, where* $\varphi(0) = 0$*,* $\varphi(\pi) = 2\pi$ *and* $1 \leq \varphi' \leq 4$ *where* θ_k *solves*

$$\theta_k - \frac{1}{2(n+1)}\varphi(\theta_k) = \frac{\pi k}{n+1}.$$

Proof. Suppose that $\theta \in (0,\pi)$ and that $z = e^{i\theta}$ solve 3.47. By factoring we have

$$e^{2(n+1)i\theta}(1+2e^{-i\theta}) = (1+2e^{i\theta}).$$

Setting $\zeta = 1 + 2e^{i\theta}$, we then have

$$e^{2(n+1)i\theta} = \frac{\zeta}{\overline{\zeta}}.$$

Now, since $|\zeta/\overline{\zeta}| = 1$, we can find some $\mathbb{R}$-valued map, $\varphi(\theta)$, such that

$$e^{i\varphi(\theta)} = \frac{\zeta}{\overline{\zeta}},$$

which then implies that

$$\exp\left\{2(n+1)i\left(\theta - \frac{1}{2(n+1)}\varphi(\theta)\right)\right\} = 1$$

which in turn implies that $\theta - \frac{1}{2(n+1)}\varphi(\theta) = \frac{\pi k}{n+1}$ for some $k \in \mathbb{N}$.

With $\zeta = 1+2e^{i\theta}$, we have $\zeta/\overline{\zeta} = a\,\{(1 + 4(\cos(\theta) + \cos(2\theta)) + i4(\sin(\theta) + \sin(2\theta))\}$ where a is a normalization constant. Hence, if r_1 and r_2 are the two roots of $1 + 4(\cos(x) + \cos(2x))$ on the interval $(0,\pi)$ we can define φ as

$$\varphi(\theta) := \begin{cases} \tan^{-1}\left(\frac{4(\sin(\theta)+\sin(2\theta))}{1+4(\cos(\theta)+\cos(2\theta))}\right) & \theta \in [0, r_1] \\ \pi + \tan^{-1}\left(\frac{4(\sin(\theta)+\sin(2\theta))}{1+4(\cos(\theta)+\cos(2\theta))}\right) & \theta \in (r_1, r_2] \\ 2\pi + \tan^{-1}\left(\frac{4(\sin(\theta)+\sin(2\theta))}{1+4(\cos(\theta)+\cos(2\theta))}\right) & \theta \in (r_2, \pi] \end{cases} \tag{3.48}$$

in which case φ is strictly increasing and C^1 with $\varphi(0) = 0$ and $\varphi(\pi) = 2\pi$. Moreover, $\varphi'(x) = \frac{4(\cos(x)+2)}{5+4\cos(x)}$ and therefore $1 \leq |\varphi'(x)| \leq 4$.

With this definition of φ, for $1 \leq k \leq n-1$, define θ_k by

$$\theta_k - \frac{1}{2(n+1)}\varphi(\theta_k) = \frac{\pi k}{n+1}. \tag{3.49}$$

We have left to show that such a θ_k exists and that $\theta_k < \frac{\pi(k+1)}{n+1} < \theta_{k+1}$ for each $1 \leq k \leq n-2$. Notice that for any $x \in (0, \pi)$, $\pi k < \pi k + \frac{1}{2}\varphi(x) < \pi(k+1)$, so therefore if we define

$$g(x) := x - \frac{1}{2(n+1)}\varphi(x) - \frac{\pi k}{n+1}$$

then $g(\pi k/(n+1)) < 0$ and $g(\pi(k+1)/(n+1)) > 0$ and an application of the intermediate value theorem finishes the proof. □

Corollary 3.8. *Defining* $\{\theta_k\}$ *as above,*

$$\frac{\pi}{2(n+1)^2} \leq \left|\theta_{k+1} - (\theta_k + \frac{\pi}{n+1})\right| \leq \frac{4\pi}{(n+1)^2}. \tag{3.50}$$

Proof. Using proposition 3.7,

$$\left|\theta_{k+1} - (\theta_k + \frac{\pi}{n+1})\right| = \frac{1}{2(n+1)}(\varphi(\theta_{k+1}) - \varphi(\theta_k)).$$

Now it's just a matter of applying the mean value theorem and the bounds on φ',

$$\begin{aligned}
\varphi(\theta_{k+1}) - \varphi(\theta_k) &\leq 4(\theta_{k+1} - \theta_k) \\
&\leq 4\left(\frac{\pi(k+2)}{n+1} - \frac{\pi k}{n+1}\right) \\
&= \frac{8\pi}{n+1}
\end{aligned}$$

and

$$\begin{aligned}
\varphi(\theta_{k+1}) - \varphi(\theta_k) &\geq \theta_{k+1} - \theta_k \\
&\geq \frac{\pi}{n+1}
\end{aligned}$$

where the last inequality follows since φ is an increasing function and $\theta_{k+1} - \theta_k = \frac{\pi}{n+1} + \varphi(\theta_{k+1}) - \varphi(\theta_k)$. □

Because of the bounds on θ_k given in proposition 3.7, we can write

$$\theta_k = \frac{\pi(k + r_k)}{n+1} \tag{3.51}$$

for some $r_k \in (0, 1)$. Using the previous corollary we can give better bounds on r_k, which will be used in later estimates.

Lemma 3.9. *Using the notation in Eq. 3.51,*

$$\frac{k}{2(n+1)} \leq r_k \leq \left(\frac{4k}{n+1} \wedge 1\right) \tag{3.52}$$

Proof. $r_k < 1$ by Proposition 3.7. Defining $\theta_0 := 0$ and using the recursive relationship from corollary 3.8

$$\theta_k + \frac{\pi}{n+1} + \frac{\pi}{2(n+1)^2} \leq \theta_{k+1} \leq \theta_k + \frac{\pi}{n+1} + \frac{4\pi}{(n+1)^2},$$

we get

$$k\left(\frac{\pi}{n+1} + \frac{\pi}{2(n+1)^2}\right) \leq \theta_k \leq k\left(\frac{\pi}{n+1} + \frac{4\pi}{(n+1)^2}\right) \tag{3.53}$$

which is a quick algebraic rearrangement away from what needed to be shown. □

3.5.3 Main Properties of l_n

Proposition 3.10. *There exists an orthonormal set of eigenvectors of matrix l_n from Equation 3.46. For $1 \leq k < n$, the eigenvectors can be written as*

$$v_k = \beta_k \begin{pmatrix} \sin(\theta_k) \\ \sin(2\theta_k) \\ \vdots \\ \sin(n\theta_k) \end{pmatrix} \tag{3.54}$$

and for $k = n$,

$$v_n = \beta_n \begin{pmatrix} \gamma_n + \gamma_n^{-1} \\ \gamma_n^2 + \gamma_n^{-2} \\ \vdots \\ \gamma_n^n + \gamma_n^{-n} \end{pmatrix}. \tag{3.55}$$

Here θ_k are described in Proposition 3.7 and $\gamma_n \in (-2, -3/2)$ is described in Lemma 3.6. For $1 \leq k < n$, the normalization constants β_k are given by

$$\beta_k^2 = \frac{2}{n}\left(\frac{1}{1-\epsilon_k}\right) \tag{3.56}$$

where

$$\epsilon_k = \frac{1}{2n}\left(\frac{\sin((2n+1)\theta_k)}{\sin(\theta_k)} - 1\right) \tag{3.57}$$

from which $|\epsilon_k| \leq \frac{4}{n}$. For $k = n$,

$$\beta_n^2 = \left(\frac{\gamma_n^{2(n+1/2)} - \gamma_n^{-2(n+1/2)}}{\gamma_n - \gamma_n^{-1}} - 2n - 1\right)^{-1}. \tag{3.58}$$

Moreover, $l_n v_k = \lambda_k v_k$ where for $1 \leq k < n$, $\lambda_k = 2(2+\cos(\theta_k))$ and for $k = n$, $\lambda_n = 4 + \gamma_n + \gamma_n^{-1}$.

Proof. From Proposition 3.5 and Proposition 3.7, for $1 \leq k < n$ we have that v_k is an eigenvector of l_n with eigenvalue λ_k. Also taking into consideration Lemma 3.6, v_n is an eigenvector of l_n with eigenvalue λ_n. Since l_n is symmetric, and the eigenvalues are distinct, it follows that the $\{v_k\}$ are indeed orthogonal.

To prove the properties of the normalization constants β_k, let $a, z \in \mathbb{C}$ and set $f \in \mathbb{R}^n$ with $f(j) = a(z^j - z^{-j})$. Then,

$$\begin{aligned}
\sum_{j=1}^{n} f(j)^2 &= a^2 \sum_{j=1}^{n} \left(z^j - z^{-j}\right)^2 \\
&= a^2 \sum_{j=1}^{n} \left(z^{2j} + z^{-2j} - 2\right) \\
&= a^2 \left(-2n + \sum_{j=-n}^{j=n} z^{2j} - 1\right) \\
&= a^2 \left(-2n - 1 + \frac{z^{2(n+1/2)} - z^{-2(n+1/2)}}{z - z^{-1}}\right).
\end{aligned}$$

Hence for $a = \frac{1}{2i}$ and $z = e^{i\theta}$ then $f(j) = \sin(j\theta_k)$ and,

$$\|f\|^2 = \sum_{j=1}^{n} f(j)^2 = \frac{n}{2} + \frac{1}{4}\left(1 - \frac{\sin(2(n+1/2)\theta)}{\sin(\theta)}\right) \tag{3.59}$$

which proves equations 3.56 and 3.57. Alternatively, if $z = \gamma_n$ and $a = 1$,

$$\|f\|^2 = \sum_{j=1}^{n} f(j)^2 = -2n - 1 + \frac{\gamma_n^{2(n+1/2)} - \gamma_n^{-2(n+1/2)}}{\gamma_n - \gamma_n^{-1}} \tag{3.60}$$

which proves equation 3.58. For the bound on ϵ_k, note that $\sin(\theta_k) \geq \sin\left(\frac{\pi k}{n+1}\right) \wedge \sin\left(\frac{\pi(k+1)}{n+1}\right)$. Also,

$$\begin{aligned}
\sin\left((2n+1)\theta_k\right) &= \sin\left([2(n+1)-1]\frac{\pi(k+r_k)}{n+1}\right) \\
&= \sin\left(2\pi(k+r_k) - \theta_k\right) \\
&= \sin\left(2\pi(k+r_k)\right)\cos\left(\theta_k\right) - \cos\left(2\pi(k+r_k)\right)\sin\left(\theta_k\right) \\
&= \sin(2\pi r_k)\cos(\theta_k) - \cos(2\pi(k+r_k))\sin(\theta_k).
\end{aligned}$$

Hence

$$\begin{aligned}
\left|\frac{\sin((2n+1)\theta_k)}{\sin(\theta_k)}\right| &= \left|\frac{\sin(2\pi r_k)}{\sin(\theta_k)}\cos(\theta_k) - \cos(2\pi(k+r_k))\right| \\
&\leq 1 + \left|\frac{\sin(2\pi r_k)}{\sin(\theta_k)}\right| \\
&\leq 1 + \frac{|\sin(2\pi r_k)|}{\sin\left(\frac{\pi k}{n+1}\right) \wedge \sin\left(\frac{\pi(k+1)}{n+1}\right)} \\
&\leq 1 + \frac{\left|\sin\left(S_k \frac{\pi k}{n+1}\right)\right|}{\sin\left(\frac{\pi k}{n+1}\right) \wedge \sin\left(\frac{\pi(k+1)}{n+1}\right)} \\
&\leq 1 + S_k \\
&\leq 9.
\end{aligned}$$

where $S_k \in [1, 8]$ is chosen (using Lemma 3.9) so that

$$\sup_{s \in [\frac{1}{2}, 4]} \left|\sin\left(2\pi \frac{sk}{n+1}\right)\right| = \left|\sin\left(S_k \frac{\pi k}{n+1}\right)\right|.$$

Therefore $|\epsilon_k| \leq \frac{4}{n}$. □

Corollary 3.11. *For $n \geq 5$,*

$$\left|\beta_k^2\right| \leq \frac{10}{n} \tag{3.61}$$

and

$$|\beta_{k+1}^2 - \beta_k^2| \leq \frac{4}{n^2\left(1 - \frac{4}{n}\right)^2} \tag{3.62}$$

$$\leq \frac{100}{n^2} \tag{3.63}$$

Proof. The first claim is immediate since from Proposition 3.10, $\beta_k^2 = \frac{2}{n}(1 - \epsilon_k)^{-1}$ with $|\epsilon_k| \leq 4/n$. For the second claim,

$$\begin{aligned}
|\beta_{k+1}^2 - \beta_k^2| &= \frac{1}{2n}\left(\frac{|\epsilon_k - \epsilon_{k+1}|}{(1-\epsilon_k)(1-\epsilon_{k+1})}\right) \\
&\leq \frac{1}{2n}\left(\frac{\frac{8}{n}}{\left(1 - \frac{4}{n}\right)^2}\right) \\
&= \frac{4}{n^2}\frac{1}{\left(1 - \frac{4}{n}\right)^2}
\end{aligned}$$

and again, the result follows. □

We have proved all we need to present the proof of Theorem 3.4. However, we end this section with a lemma which will not be needed for the main result of this paper, but is interesting in that it gives an explicit formula for the determinant of l_n.

Lemma 3.12. *We can calculate the determinant of l_n as*

$$\det(l_n) = \frac{1}{2}\left\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\right\} \tag{3.64}$$

$$= \cosh(n \log(2+\sqrt{3})). \tag{3.65}$$

Proof. Set $d_n = \det(l_n)$. Then,

$$d_n = 4\begin{vmatrix} 4 & 1 & 0 & 0 \\ 1 & \ddots & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & \ddots & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix}$$

$$= 4\begin{vmatrix} 4 & 1 & 0 & 0 \\ 1 & \ddots & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 4 & 1 & 0 \\ 1 & \ddots & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= 4d_{n-1} - d_{n-2}.$$

Therefore, we can write

$$\begin{pmatrix} d_n \\ d_{n-1} \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d_{n-1} \\ d_{n-2} \end{pmatrix}$$

$$\begin{pmatrix} d_n \\ d_{n-1} \end{pmatrix} =: A \begin{pmatrix} d_{n-1} \\ d_{n-2} \end{pmatrix}$$

Now we use elementary linear algebra and find that the eigenvalues of A are $\lambda_+ := 2 + \sqrt{3}$ and $\lambda_- := 2 - \sqrt{3}$ with eigenvectors $v_+ := \begin{pmatrix} \lambda_+ \\ 1 \end{pmatrix}$ and $v_- := \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix}$. With $d_1 = 2$ and $d_2 = 7$, from here we use this information to diagonalize A and solve for d_n with

$$\begin{pmatrix} d_n \\ d_{n-1} \end{pmatrix} = A^{n-2} \begin{pmatrix} d_2 \\ d_1 \end{pmatrix}.$$

The second equality follows since $(2+\sqrt{3})^{-1} = 2 - \sqrt{3}$. Therefore,

$$\begin{aligned} \frac{1}{2}\left\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\right\} &= \frac{1}{2}\left\{(2+\sqrt{3})^n + (2+\sqrt{3})^{-n}\right\} \\ &= \frac{1}{2}\left\{e^{n\log(2+\sqrt{3})} + e^{-n\log(2+\sqrt{3})}\right\} \\ &= \cosh(n\log(2+\sqrt{3})). \end{aligned}$$

$\square$

Proposition 3.13. *As above, let* $d_n = \det(l_n)$ *with* $d_1 = 2$. *Define the upper triangular* $n \times n$ *matrix* a_n *by,*

$$a_n = \begin{pmatrix} \sqrt{\frac{d_n}{d_{n-1}}} & \sqrt{\frac{d_{n-2}}{d_{n-1}}} & 0 & 0 \\ 0 & \sqrt{\frac{d_{n-1}}{d_{n-2}}} & \sqrt{\frac{d_{n-3}}{d_{n-2}}} & 0 \\ 0 & 0 & \ddots & \sqrt{\frac{1}{d_1}} \\ 0 & 0 & 0 & \sqrt{d_1} \end{pmatrix}. \tag{3.66}$$

Then

$$l_n = a_n a_n^{tr} \tag{3.67}$$

and

$$\left| \left(a_n^{-1}\right)_{i,j} \right|^2 \leq \begin{cases} \frac{1}{2}\left(\frac{1}{2}\right)^{j-i} & j \geq i \\ 0 & j < i \end{cases}. \tag{3.68}$$

Proof. Define $d_0 := 1$ in which case the diagonal elements can all be written as $(a_n)_{i,i} = \sqrt{d_i/d_{i-1}}$ for $1 \leq i \leq n$ and the super diagonal elements are given by $(a_n)_{i,i+1} = \sqrt{d_{i-2}/d_{i-1}}$ for $2 \leq i \leq n$. From Lemma 3.12, $d_n = k_1\lambda_+^{n-1} + k_1\lambda_-^{n-1}$ where $\lambda_+ = 2 + \sqrt{3}$ and $\lambda_- = 2 - \sqrt{3}$. Note that $\lambda_+^{-1} = \lambda_-$ and $\lambda_+ + \lambda_- = 4$. In particular, for $i \geq 2$,

$$\begin{aligned} d_i + d_{i-2} &= k_1(\lambda_+^{i-1} + \lambda_+^{i-3}) + k_2(\lambda_-^{i-1} + \lambda_-^{i-3}) \\ &= k_1\lambda_+^{i-2}(\lambda_+ + \lambda_+^{-1}) + k_2\lambda_-^{i-2}(\lambda_- + \lambda_-^{-1}) \\ &= k_1\lambda_+^{i-2}(\lambda_+ + \lambda_-) + k_2\lambda_-^{i-2}(\lambda_- + \lambda_+) \\ &= 4d_{i-1}. \end{aligned}$$

Simple matrix multiplication leads to

$$(a_n a_n^{tr})_{i,i} = \begin{cases} d_1 = 2 & i = 1 \\ \frac{d_i + d_{i-2}}{d_{i-1}} = \frac{4d_{i-1}}{d_{i-1}} = 4 & 2 \leq i \leq n \end{cases}$$

and for the super/sub diagonal elements

$$(a_n a_n^{tr})_{i,i+1} = (a_n a_n^{tr})_{i+1,i} = \sqrt{\frac{d_{i-2}}{d_{i-1}}}\sqrt{\frac{d_{i-1}}{d_{i-2}}} = 1.$$

With all other elements equal to 0 we conclude that $l_n = a_n a_n^{tr}$. To establish the inequality in Eq. (3.68), for a general matrix of the form

$$A = \begin{pmatrix} \alpha_n & \beta_{n-1} & 0 & 0 \\ 0 & \alpha_{n-1} & \beta_{n-2} & 0 \\ 0 & 0 & \ddots & \beta_1 \\ 0 & 0 & 0 & \alpha_1 \end{pmatrix}$$

the explicit formula for the inverse is,

$$A^{-1} = \begin{pmatrix} \frac{1}{\alpha_n} & -\frac{\beta_{n-1}}{\alpha_n\alpha_{n-1}} & \frac{\beta_{n-1}\beta_{n-2}}{\alpha_n\alpha_{n-1}\alpha_{n-2}} & \cdots & \frac{\prod_{i=1}^{n-1}(-\beta_i)}{\prod_{i=1}^{n}\alpha_i} \\ 0 & \frac{1}{\alpha_{n-1}} & -\frac{\beta_{n-2}}{\alpha_{n-1}\alpha_{n-2}} & \cdots & \frac{\prod_{i=1}^{n-2}(-\beta_i)}{\prod_{i=1}^{n-1}\alpha_i} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \frac{1}{\alpha_2} & -\frac{\beta_1}{\alpha_2\alpha_1} \\ 0 & 0 & 0 & 0 & \frac{1}{\alpha_1} \end{pmatrix}.$$

Therefore, for $1 \le i \le n$ and $i < j \le n$,

$$\begin{aligned} |(a_n)_{i,j}|^2 &= \frac{(d_{n-i-1}/d_{n-i})(d_{n-i-2}/d_{n-i-1})\cdots(d_{n-j}/d_{n-j+1})}{(d_{n-i+1}/d_{n-i})(d_{n-i}/d_{n-i-1})\cdots(d_{n-j+1}/d_{n-j})} \\ &= \frac{(d_{n-j})^2}{d_{n-i}d_{n-i+1}} \\ &< \left(\frac{d_{n-j}}{d_{n-i}}\right)^2 \\ &= \left(\frac{1}{2+\sqrt{3}}\right)^{2(j-i)} \left(\frac{1+(2+\sqrt{3})^{2(j-n)}}{1+(2+\sqrt{3})^{2(i-n)}}\right)^2 \\ &\le 2\left(\frac{1}{2+\sqrt{3}}\right)^{2(j-i)} \\ &= 2\left(\frac{1}{2}\right)^{j-i}\left(\frac{2}{2+\sqrt{3}}\right)^{j-i}\left(\frac{1}{2+\sqrt{3}}\right)^{j-i} \\ &\le \left(\frac{1}{2}\right)^{j-i}\left(\frac{2}{2+\sqrt{3}}\right)^{2} \\ &< \frac{1}{2}\left(\frac{1}{2}\right)^{j-i}. \end{aligned}$$

The diagonal elements of a_n^{-1} are all $\frac{1}{4}$ with exception of the last diagonal entry, which is $\frac{1}{2}$. This finishes the proof of Eq. (3.68). □

We now finish this section with the (nearly immediate) proof of Theorem 3.4.

Proof of Theorem 3.4. First note that $\mathcal{L}_{\mathcal{P}} = \frac{\Delta^3}{6} l_n \otimes I_d$ where I_d is the $d \times d$ identity. For the assertions concerning the eigenvectors $u^{\mathcal{P}}_{k,a}$, the eigenvalues $\lambda^{\mathcal{P}}_{k,a}$, and the normalization constants $\beta^{\mathcal{P}}_k$, use Proposition B.9, Proposition 3.10, and Corollary 3.11. For those involving $\theta^{\mathcal{P}}_k$, use Proposition 3.7, Corollary 3.8, and Lemma 3.9. Finally, for the proofs involving $\mathcal{A}_{\mathcal{P}}$, Using Proposition 3.13, $\mathcal{L}_{\mathcal{P}} = \mathcal{A}_{\mathcal{P}} \mathcal{A}^{tr}_{\mathcal{P}}$ with $\mathcal{A}_{\mathcal{P}} = \sqrt{\frac{\Delta^3}{6}} a_n \otimes I_d$ and hence $\mathcal{A}^{-1}_{\mathcal{P}} = \sqrt{\frac{6}{\Delta^3}} a^{-1}_n \otimes I_d$. $\square$

Chapter 4

Size Estimates and Uniform Integrability

In Theorem 3.4 we show that there exists an upper triangular block matrix $\mathcal{A}_\mathcal{P}$ such that

$$\mathcal{L}_\mathcal{P} = \mathcal{A}_\mathcal{P}\mathcal{A}_\mathcal{P}^{tr}. \tag{4.1}$$

From Eq. (3.37),

$$\begin{aligned} \tilde{\rho}_\mathcal{P} &= \sqrt{\frac{\det(\mathcal{L}_\mathcal{P} + \mathcal{R}_\mathcal{P})}{\det \mathcal{L}_\mathcal{P}}} \\ &= \sqrt{\det\left(I + \mathcal{A}_\mathcal{P}^{-1}\mathcal{R}_\mathcal{P}(\mathcal{A}_\mathcal{P}^{-1})^{tr}\right)}. \end{aligned} \tag{4.2}$$

Moreover, each of the $d \times d$ blocks of $\mathcal{A}_\mathcal{P}^{-1}$ is bounded,

$$\|[\mathcal{A}_\mathcal{P}^{-1}]_{i,j}\|^2 \leq \begin{cases} \frac{3}{\Delta^3}\left(\frac{1}{2}\right)^{j-i} & j \geq i \\ 0 & j < i \end{cases} \tag{4.3}$$

To ease notation for the remainder of this chapter, we also introduce

$$K_i^\mathcal{P} := \sup_{0 \leq s \leq \Delta} \|A_i^\mathcal{P}(s)\| \tag{4.4}$$

where $A_i^\mathcal{P}$ is defined in Eq. (3.4). Using Eq. (3.7),

$$K_i^\mathcal{P} \leq \kappa \frac{\|\Delta_i b\|^2}{\Delta^2}. \tag{4.5}$$

4.1 Estimates on the remainder $\mathcal{R}_\mathcal{P}$

This section is devoted to giving estimates on the non-zero $d \times d$ blocks of the remainder $\mathcal{R}_\mathcal{P}$, which will in turn be used to estimate the size of $\tilde{\rho}_\mathcal{P}$ sufficient for proving uniform integrability. From Eq. (3.34) along with the definition of $\mathcal{L}_\mathcal{P}$ in Eq. (3.30) and the form of $G_\mathcal{P}^{\mathcal{F}_\mathcal{P}}$ in Eq. (3.26), the non-zero $d \times d$ blocks of $\mathcal{R}_\mathcal{P}$ are of the form

$$[\mathcal{R}_\mathcal{P}]_{i,i} = \begin{cases} \int_0^\Delta \left(V_{i+1}^\mathcal{P}(s)^{\text{tr}} V_{i+1}^\mathcal{P}(s) + S_i^\mathcal{P}(s)^{\text{tr}} S_i^\mathcal{P}(s)\right) ds - \frac{2\Delta^3}{3} I & i < n \\ \int_0^\Delta \left(S_i^\mathcal{P}(s)^{\text{tr}} S_i^\mathcal{P}(s)\right) ds - \frac{\Delta^3}{3} I & i = n \end{cases} \tag{4.6}$$

$$= \begin{cases} \int_0^\Delta \left\{\left(V_{i+1}^\mathcal{P}(s)^{\text{tr}} V_{i+1}^\mathcal{P}(s) - (\Delta - s)^2 I\right) + \left(S_i^\mathcal{P}(s)^{\text{tr}} S_i^\mathcal{P}(s) - s^2 I\right)\right\} ds & i < n \\ \int_0^\Delta \left(S_n^\mathcal{P}(s)^{\text{tr}} S_n^\mathcal{P}(s) - s^2 I\right) ds & i = n \end{cases}. \tag{4.7}$$

and for $1 \le i < n$,

$$[\mathcal{R}_\mathcal{P}]_{i,i+1} = [\mathcal{R}_\mathcal{P}]_{i+1,i}^{tr} = \int_0^\Delta V_{i+1}^\mathcal{P}(s)^{tr} S_{i+1}^\mathcal{P}(s) ds - \frac{\Delta^3}{6} I \tag{4.8}$$

$$= \int_0^\Delta (V_{i+1}^\mathcal{P}(s)^{tr} S_{i+1}^\mathcal{P}(s) - (\Delta - s) s I) ds \tag{4.9}$$

Written suggestively in Eqs. (4.7) and (4.9), we set out to estimate $\|V_{i+1}^\mathcal{P}(s)^{\text{tr}} V_{i+1}^\mathcal{P}(s) - (\Delta - s)^2 I\|$, $\|S_i^\mathcal{P}(s)^{\text{tr}} S_i^\mathcal{P}(s) - s^2 I\|$, and $\|V_{i+1}^\mathcal{P}(s)^{tr} S_{i+1}^\mathcal{P}(s) - (\Delta - s) s I\|$.

Lemma 4.1. *Let $s \in [0, \Delta]$. The following inequalities hold for $S_i^\mathcal{P}$, $C_i^\mathcal{P}$, and $F_i^\mathcal{P}$ defined in Eqs. (3.10), (3.11), and (3.13).*

$$\|S_i^\mathcal{P}(s)\| \le \frac{\sinh(\sqrt{K_i^\mathcal{P}} s)}{\sqrt{K_i^\mathcal{P}}} \le s \cosh(\sqrt{K_i^\mathcal{P}} s) \tag{4.10}$$

$$\|C_i^\mathcal{P}(s)\| \le \cosh(\sqrt{K_i^\mathcal{P}} s) \tag{4.11}$$

$$\|F_i^\mathcal{P}\| \le \cosh(\sqrt{K_i^\mathcal{P}} \Delta) \cosh(\sqrt{K_{i+1}^\mathcal{P}} \Delta) \tag{4.12}$$

Proof. Eqs. (4.10) and (4.11) are direct consequences of Proposition D.1 in combination with the inequality $x^{-1}\sinh(x) \leq \cosh(x)$. For Eq. (4.12),

$$\begin{aligned}
\|F_i^{\mathcal{P}}\| &= \|S_{i+1}^{\mathcal{P}}(\Delta)^{-1}C_{i+1}^{\mathcal{P}}(\Delta)S_i^{\mathcal{P}}(\Delta)\| \\
&\leq \left\| C_{i+1}^{\mathcal{P}}(\Delta)\frac{S_i^{\mathcal{P}}(\Delta)}{\Delta} \right\| && \text{from Proposition E.3} \\
&\leq \cosh(\sqrt{K_{i+1}^{\mathcal{P}}}\Delta)\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta) && \text{from Eqs. (4.10) \& (4.11).}
\end{aligned}$$

□

Lemma 4.2. *For* $s \in [0, \Delta]$,

$$\left\|S_i^{\mathcal{P}}(s) - sI\right\| \leq s\left(\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta) - 1\right), \tag{4.13}$$

$$\left\|V_{i+1}^{\mathcal{P}}(s) - \frac{S_i^{\mathcal{P}}(\Delta)}{\Delta}(\Delta - s)\right\| \leq \frac{s}{\Delta}(\Delta - s)\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta)\left(\cosh(4\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1\right), \tag{4.14}$$

and

$$\left\|V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)I\right\| \quad \leq \frac{s}{\Delta}(\Delta - s)\left(\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta)\cosh(4\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1\right). \tag{4.15}$$

Proof. Eq. (4.13) is a direct consequence of Proposition D.1 along with the fact that cosh is monotonically increasing on $[0, \infty)$. For Eq. (4.14) we first apply Proposition D.2 along with the definition of $V_{i+1}^{\mathcal{P}}$ in Eq. (3.14) to yield,

$$\begin{aligned}
&\left\|V_{i+1}^{\mathcal{P}}(s) - \frac{S_i^{\mathcal{P}}(\Delta)}{\Delta}(\Delta - s)\right\| \\
&\leq s\left(1 - \frac{s}{\Delta}\right)\left[\|S_i^{\mathcal{P}}(\Delta)\|K_{i+1}^{\mathcal{P}}\Delta\cosh(\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) + \|F_i^{\mathcal{P}}\|\left(\cosh(\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1\right)\right].
\end{aligned}$$

Next, use Eqs. (4.10) and (4.12) for $\|S_i^{\mathcal{P}}(\Delta)\|$ and $\|F_i^{\mathcal{P}}\|$,

$$\begin{aligned}
&\leq s\left(1 - \frac{s}{\Delta}\right)\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta)\cosh(\sqrt{K_{i+1}^{\mathcal{P}}}\Delta)\left(K_{i+1}^{\mathcal{P}}\Delta^2 + \cosh(\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1\right) \\
&\leq s\left(1 - \frac{s}{\Delta}\right)\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta)\cosh(\sqrt{K_{i+1}^{\mathcal{P}}}\Delta)\left(\left(K_{i+1}^{\mathcal{P}}\Delta^2 + 1\right)\cosh(\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1\right) \\
&\leq s\left(1 - \frac{s}{\Delta}\right)\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta)\cosh(\sqrt{K_{i+1}^{\mathcal{P}}}\Delta)\left(\cosh^2(\sqrt{2K_{i+1}^{\mathcal{P}}}\Delta) - 1\right) \\
&\leq s\left(1 - \frac{s}{\Delta}\right)\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta)\left(\cosh(4\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1\right).
\end{aligned}$$

Here, the final inequality follows from the calculus fact that for $a, b \geq 0$, $\cosh(a)\cosh(b) \leq \cosh(a+b)$ and $\cosh(a)(\cosh(b)-1) \leq \cosh(a)\cosh(b) - 1$.

Finally, for Eq. (4.15) we rewrite $V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)I$ as

$$
\begin{aligned}
&V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)I \\
&= V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)I + \frac{S_i^{\mathcal{P}}(\Delta)}{\Delta}(\Delta - s) - \frac{S_i^{\mathcal{P}}(\Delta)}{\Delta}(\Delta - s) \\
&= \left(V_{i+1}^{\mathcal{P}}(s) - \frac{S_i^{\mathcal{P}}(\Delta)}{\Delta}(\Delta - s)\right) + (1 - \frac{s}{\Delta})\left(S_i^{\mathcal{P}}(\Delta) - \Delta I\right).
\end{aligned}
$$

Applying Eqs. (4.13) and (4.14) to the appropriate terms of this sum,

$$
\begin{aligned}
&\left\|V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)I\right\| \\
&\leq \left\|V_{i+1}^{\mathcal{P}}(s) - \frac{S_i^{\mathcal{P}}(\Delta)}{\Delta}(\Delta - s)\right\| + (1 - \frac{s}{\Delta})\left\|S_i^{\mathcal{P}}(\Delta) - \Delta I\right\| \\
&\leq \frac{s}{\Delta}(\Delta - s)\left[\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta)\left(\cosh(4\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1\right) + \left(\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta) - 1\right)\right] \\
&= \frac{s}{\Delta}(\Delta - s)\left(\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta)\cosh(4\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1\right)
\end{aligned}
$$

□

We have finally arrived at the point to give estimates for $\|V_{i+1}^{\mathcal{P}}(s)^{tr}V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)^2 I\|$, $\|S_i^{\mathcal{P}}(s)^{\mathrm{tr}}S_i^{\mathcal{P}} - s^2 I\|$, and $\|V_{i+1}^{\mathcal{P}}(s)^{tr}S_{i+1}^{\mathcal{P}}(s) - (\Delta - s)sI\|$. To understand how the previous lemma will be used, we write

$$V_{i+1}^{\mathcal{P}}(s) = \left(V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)I\right) + (\Delta - s)I$$

so that,

$$
\begin{aligned}
V_{i+1}^{\mathcal{P}}(s)^{\mathrm{tr}}V_{i+1}^{\mathcal{P}}(s) = &\left(V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)I\right)^{\mathrm{tr}}\left(V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)I\right) \\
&+ (\Delta - s)\left[\left(V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)I\right)^{\mathrm{tr}} + \left(V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)I\right)\right] \\
&+ (\Delta - s)^2 I
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\left\|V_{i+1}^{\mathcal{P}}(s)^{\mathrm{tr}}V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)^2 I\right\| \leq &\left\|V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)I\right\|^2 \\
&+ 2(\Delta - s)\left\|V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)I\right\|. \qquad (4.16)
\end{aligned}
$$

Conducting the analogous manipulation for $S_i^{\mathcal{P}}(s)^{\mathrm{tr}} S_i^{\mathcal{P}}(s) - s^2 I$,

$$\left\| S_i^{\mathcal{P}}(s)^{\mathrm{tr}} S_i^{\mathcal{P}}(s) - s^2 I \right\| \leq \left\| S_i^{\mathcal{P}}(s) - sI \right\|^2 + 2s \left\| S_i^{\mathcal{P}}(s) - sI \right\|. \tag{4.17}$$

For the final bound,

$$\begin{aligned} &V_{i+1}^{\mathcal{P}}(s)^{tr} S_{i+1}^{\mathcal{P}}(s) \\ &= (V_{i+1}^{\mathcal{P}} - (\Delta - s)I + (\Delta - s)I)^{tr} (S_{i+1}^{\mathcal{P}}(s) - sI + sI) \\ &= (V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)I) S_{i+1}^{\mathcal{P}}(s) + (\Delta - s)(S_{i+1}^{\mathcal{P}}(s) - sI) + (\Delta - s)sI. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|V_{i+1}^{\mathcal{P}}(s)^{tr} S_{i+1}^{\mathcal{P}}(s) - (\Delta - s)sI\| \\ &\leq \|V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)I\| \left(\|S_{i+1}^{\mathcal{P}}(s) - sI\| + s\right) + (\Delta - s)\|S_{i+1}^{\mathcal{P}}(s) - sI\|. \end{aligned} \tag{4.18}$$

Proposition 4.3. *For $s \in [0, \Delta]$,*

$$\begin{aligned} &\left\| V_{i+1}^{\mathcal{P}}(s)^{\mathrm{tr}} V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)^2 I \right\| \\ &\quad \leq \left(\frac{s}{\Delta}\right)^2 (\Delta - s)^2 \left(\cosh(2\sqrt{K_i^{\mathcal{P}}}\Delta) \cosh(8\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1 \right) \\ &\quad + 2\frac{s}{\Delta}(\Delta - s)^2 \left(\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta) \cosh(4\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1 \right) &(4.19) \\ &\quad \leq 3(\Delta - s)^2 \left(\cosh(2\sqrt{K_i^{\mathcal{P}}}\Delta) \cosh(8\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1 \right), &(4.20) \end{aligned}$$

$$\left\| S_i^{\mathcal{P}}(s)^{\mathrm{tr}} S_i^{\mathcal{P}}(s) - s^2 I \right\| \leq 3s^2 \left(\cosh(2\sqrt{K_i^{\mathcal{P}}}\Delta) - 1 \right) \tag{4.21}$$

and

$$\begin{aligned} &\|V_{i+1}^{\mathcal{P}}(s)^{tr} S_{i+1}^{\mathcal{P}}(s) - (\Delta - s)sI\| \\ &\quad \leq 2\frac{s^2}{\Delta}(\Delta - s) \left(\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta) \cosh(5\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1 \right) \\ &\quad + (\Delta - s)s \left(\cosh(\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1 \right) &(4.22) \\ &\quad \leq 3s(\Delta - s) \left(\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta) \cosh(5\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1 \right). &(4.23) \end{aligned}$$

Proof. We start with the calculus facts that $(\cosh(a) - 1)(\cosh(c) - 1) \leq \cosh(a)\cosh(c) - 1$, $\cosh(a)\cosh(c) \leq \cosh(a + c)$, and $\cosh(a) \leq \cosh(ra)$ for any $a, c \in \mathbb{R}$ and $r \geq 1$. The inequalities then follow by combining Eqs. (4.16), (4.17), and (4.18) with Lemma 4.2 along with the fact that $s/\Delta \leq 1$. □

The following theorem is the main result used from this section for the remainder of this chapter.

Theorem 4.4. *For* $1 \le i \le n$,

$$\left\| [\mathcal{R}_{\mathcal{P}}]_{i,i} \right\| \le 2\Delta^3 \left(\cosh(2\sqrt{K_i^{\mathcal{P}}}\Delta) \cosh(8\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1 \right) \tag{4.24}$$

and for $1 \le i < n$,

$$\left\| [\mathcal{R}_{\mathcal{P}}]_{i,i+1} \right\| = \left\| [\mathcal{R}_{\mathcal{P}}]_{i+1,i} \right\| \le \frac{\Delta^3}{2} \left(\cosh(\sqrt{K_i^{\mathcal{P}}}\Delta) \cosh(5\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1 \right). \tag{4.25}$$

Moreover, this implies that,

$$\left\| \left[\mathcal{A}_{\mathcal{P}}^{-1} \mathcal{R}_{\mathcal{P}} (\mathcal{A}_{\mathcal{P}}^{-1})^{tr} \right]_{i,i} \right\| \le \sum_{j=i}^{n} \lambda_{i,j} \left(\cosh(30\sqrt{K_j^{\mathcal{P}}}\Delta) \cosh(120\sqrt{K_{j+1}^{\mathcal{P}}}\Delta) - 1 \right) \tag{4.26}$$

where $\lambda_{i,j} := \left(\frac{1}{2}\right)^{j-i} \left(\sum_{j=i}^{n} \left(\frac{1}{2}\right)^{j-i} \right)^{-1} = \left(\frac{1}{2}\right)^{j-i} \left(2 - \left(\frac{1}{2}\right)^{n-i} \right)^{-1}$ *are chosen so that* $\sum_{j=i}^{n} \lambda_{i,j} = 1$.

Proof. In consideration of Eqs. (4.7) and (4.9), Eqs. (4.24) and (4.25) come from integrating the bounds given in Proposition 4.3 with respect to s for $s \in [0, \Delta]$.

For Eq. (4.26), recall that $\mathcal{A}_{\mathcal{P}}$ is upper diagonal, and hence so is $\mathcal{A}_{\mathcal{P}}^{-1}$, and that $\mathcal{R}_{\mathcal{P}}$ is tri-diagonal, yielding

$$\begin{aligned}
[\mathcal{A}_{\mathcal{P}}^{-1} \mathcal{R}_{\mathcal{P}} (\mathcal{A}_{\mathcal{P}}^{-1})^{tr}]_{i,i} &= \sum_{j,k=1}^{n} [\mathcal{A}_{\mathcal{P}}^{-1}]_{i,j} [\mathcal{R}_{\mathcal{P}}]_{j,k} [(\mathcal{A}_{\mathcal{P}}^{-1})^{tr}]_{k,i} \\
&= \sum_{j=i}^{n} \big\{ [\mathcal{A}_{\mathcal{P}}^{-1}]_{i,j} [\mathcal{R}_{\mathcal{P}}]_{j,j} [(\mathcal{A}_{\mathcal{P}}^{-1})^{tr}]_{j,i} + [\mathcal{A}_{\mathcal{P}}^{-1}]_{i,j+1} [\mathcal{R}_{\mathcal{P}}]_{j+1,j} [(\mathcal{A}_{\mathcal{P}}^{-1})^{tr}]_{j,i} \\
&\qquad + [\mathcal{A}_{\mathcal{P}}^{-1}]_{i,j} [\mathcal{R}_{\mathcal{P}}]_{j,j+1} [(\mathcal{A}_{\mathcal{P}}^{-1})^{tr}]_{j+1,i} \big\}
\end{aligned}$$

where we keep the convention that for $j = n$, $[\cdot]_{n,n+1} = [\cdot]_{n+1,n} = 0$. Therefore,

$$
\begin{aligned}
&\|[\mathcal{A}_{\mathcal{P}}^{-1}\mathcal{R}_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}}^{-1})^{tr}]_{i,i}\| \\
&\quad\le \sum_{j=i}^{n}\left\{\|[\mathcal{A}_{\mathcal{P}}^{-1}]_{i,j}\|^2\|[\mathcal{R}_{\mathcal{P}}]_{j,j}\| + 2\|[\mathcal{A}_{\mathcal{P}}^{-1}]_{i,j+1}\|\|[\mathcal{A}_{\mathcal{P}}^{-1}]_{i,j}\|\|[\mathcal{R}_{\mathcal{P}}]_{j+1,j}\|\right\} \\
&\quad\le \sum_{j=i}^{n}\left\{3\left(\frac{1}{2}\right)^{j-i}\left[2\left(\cosh(2\sqrt{K_j^{\mathcal{P}}}\Delta)\cosh(8\sqrt{K_{j+1}^{\mathcal{P}}}\Delta)-1\right)\right.\right. \\
&\qquad\qquad\left.\left.+\frac{1}{2}\left(\cosh(\sqrt{K_j^{\mathcal{P}}}\Delta)\cosh(5\sqrt{K_{j+1}^{\mathcal{P}}}\Delta)-1\right)\right]\right\} \\
&\quad\le \sum_{j=i}^{n}\frac{15}{2}\left(\frac{1}{2}\right)^{j-i}\left(\cosh(2\sqrt{K_j^{\mathcal{P}}}\Delta)\cosh(8\sqrt{K_{j+1}^{\mathcal{P}}}\Delta)-1\right) \\
&\quad\le \sum_{j=i}^{n}15\lambda_{i,j}\left(\cosh(2\sqrt{K_j^{\mathcal{P}}}\Delta)\cosh(8\sqrt{K_{j+1}^{\mathcal{P}}}\Delta)-1\right) \\
&\quad\le \sum_{j=i}^{n}\lambda_{i,j}\left(\cosh(30\sqrt{K_j^{\mathcal{P}}}\Delta)\cosh(120\sqrt{K_{j+1}^{\mathcal{P}}}\Delta)-1\right)
\end{aligned}
$$

wherein the last inequality we used Proposition A.5 where for $\alpha \ge 1$, $\alpha(\cosh(a)\cosh(b) - 1) \le \cosh(\alpha a)\cosh(\alpha b) - 1$. □

4.2 An Important Lemma

Lemma 4.5. *Let $p \in \mathbb{N}$ and $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha < \frac{1}{4p}, 0 \le \beta$, and $-1 \le \gamma$. Set $\mathbf{B}_n := (\Delta_1 b, \cdots, \Delta_n b)$. Then*

$$\limsup_{n\to\infty}\mathbb{E}\left[\left(1+\frac{\beta}{n}\left(e^{\alpha\|\mathbf{B}_n\|^2}+\gamma\right)\right)^{np}\right] < \infty. \tag{4.27}$$

Proof. We define the deterministic functions $g(x) := 1 + \frac{\beta}{n}(e^{\alpha x} + \gamma)$ and $f(x) := g(x)^{np} = (1+\frac{\beta}{n}(e^{\alpha x} + \gamma))^{np}$. We also define the stochastic process $Q_t^n := \sum_{i=1}^n \|b_{t\wedge s_i} - b_{t\wedge s_{i-1}}\|^2$. With this notation, Eq. (4.27) becomes

$$\limsup_{n\to\infty}\mathbb{E}[f(Q_1^n)] < \infty. \tag{4.28}$$

We now use Itô's Lemma to get an estimate on $\mathbb{E}[f(Q_t^n)]$. To start, $dQ_s^n = 2\|b_s - b_{\underline{s}}\|db_s + \mathrm{d}\,ds$ and $d[Q^n]_s = 4\|b_s - b_{\underline{s}}\|^2 ds$ where $\underline{s} = s_{i-1}$ whenever $s \in (s_{i-1}, s_i]$.

Therefore,

$$\begin{aligned}\mathbb{E}[f(Q^n_t) - f(Q^n_0)] &= \mathbb{E}\left[\int_0^t f'(Q^n_s)dQ^n_s + \frac{1}{2}\int_0^t f''(Q^n_s)d[Q^n]_s\right] \\ &= \mathbb{E}\left[\mathrm{d}\int_0^t f'(Q^n_s)ds + 4\int_0^t f''(Q^n_s)\|b_s - b_{\underline{s}}\|^2 ds\right] \\ &= \mathrm{d}\int_0^t \mathbb{E}[f'(Q^n_s)]ds + 4\int_0^t \mathbb{E}[f''(Q^n_s)\|b_s - b_{\underline{s}}\|^2]ds\end{aligned}$$

where in the second equality we dropped the martingale term. Calculating the derivatives of f,

$$\begin{aligned}f'(x) &= \beta\alpha p e^{\alpha x} g(x)^{np-1} \\ f''(x) &= \beta^2\alpha^2 p(p - \frac{1}{n})e^{2\alpha x} g(x)^{np-2} + \beta\alpha^2 p e^{\alpha x} g(x)^{np-1}.\end{aligned}$$

This in combination with the fact that $g(x) \geq 1$ by our choices of β and γ implies that there exists a constant $C < \infty$ independent of n such that

$$\mathbb{E}[f(Q^n_t) - f(Q^n_0)] \leq C\int_0^t \mathbb{E}[e^{2\alpha Q^n_s}(1 + \|b_s - b_{\underline{s}}\|^2)g(Q^n_s)^{np-1}]ds. \tag{4.29}$$

From here we want to show that there exists another constant $\tilde{C} < \infty$ independent of n such that,

$$\begin{aligned}\mathbb{E}[f(Q^n_t) - f(Q^n_0)] &\leq \tilde{C}\int_0^t \mathbb{E}[g(Q^n_s)^{np}]ds \\ &= \tilde{C}\int_0^t \mathbb{E}[f(Q^n_s)]ds.\end{aligned}$$

Before we do this, let us first understand why this will be enough to finish the proof. If such a $\tilde{C}$ exists, then we will have

$$\begin{aligned}\mathbb{E}[f(Q^n_t)] &\leq \mathbb{E}[f(Q^n_0)] + \tilde{C}\int_0^t \mathbb{E}[f(Q^n_s)]ds \\ &= \left(1 + \frac{\beta}{n}(1+\gamma)\right)^{np} + \tilde{C}\int_0^t \mathbb{E}[f(Q^n_s)]ds \\ &\leq e^{\beta p(1+\gamma)} + \tilde{C}\int_0^t \mathbb{E}[f(Q^n_s)]ds.\end{aligned}$$

Applying Gronwall's inequality to the function $t \mapsto \mathbb{E}[f(Q^n_t)]$,

$$\mathbb{E}[f(Q^n_t)] \leq e^{\beta p(1+\gamma) + \tilde{C}t},$$

noting the the right hand side is independent of n. In particular, for any $t \in [0,1]$,

$$\limsup_{n\to\infty} \mathbb{E}[f(Q_t^n)] \leq e^{\beta p(1+\gamma)+\tilde{C}t} < \infty, \tag{4.30}$$

which concludes the proof as soon as the existence of $\tilde{C}$ is established.

From Eq. (4.29), to prove the existence of $\tilde{C}$ it will suffice to show that there exists a constant Λ independent of n such that $\mathbb{E}[e^{2\alpha Q_s^n}(1+\|b_s - b_{\underline{s}}\|^2)g(Q_s^n)^{np-1}] \leq \Lambda\mathbb{E}[g(Q_s^n)^{np}]$. Using Holder's inequality,

$$\begin{aligned}\mathbb{E}[e^{2\alpha Q_s^n}(1+\|b_s - b_{\underline{s}}\|^2)g(Q_s^n)^{np-1}] &\leq \mathbb{E}[e^{2\alpha np Q_s^n}(1+\|b_s - b_{\underline{s}}\|^2)^{np}]^{\frac{1}{np}}\mathbb{E}[g(Q_s^n)^{np}]^{\frac{np-1}{np}}\\ &\leq \mathbb{E}[e^{2\alpha np Q_s^n}(1+\|b_s - b_{\underline{s}}\|^2)^{np}]^{\frac{1}{np}}\mathbb{E}[g(Q_s^n)^{np}],\end{aligned}$$

where we once again used that $g \geq 1$ for the second inequality. Therefore it is sufficient to find such a Λ with $\mathbb{E}[e^{2\alpha np Q_s^n}(1+\|b_s - b_{\underline{s}}\|^2)^{np}]^{\frac{1}{np}} \leq \Lambda$. If $\{Z_i\}_{i=1}^\infty$ are i.i.d. $N^{\mathrm{d}}(0,1)$ random variables and $s \in (s_{j-1}, s_j]$, then

$$\begin{aligned} Q_s^n &= \sum_{i=1}^{j-1}\|\Delta_i b\|^2 + \|b_s - b_{\underline{s}}\|^2\\ &\overset{d}{=} \sum_{i=1}^{j-1}\frac{1}{n}\|Z_i\|^2 + (s-\underline{s})\|Z_j\|^2\end{aligned}$$

and

$$1+\|b_s - b_{\underline{s}}\|^2 \overset{d}{=} 1+(s-\underline{s})\|Z_j\|^2.$$

Therefore,

$$\begin{aligned}\mathbb{E}[e^{2\alpha np Q_s^n}(1+(b_s-b_{\underline{s}})^2)^{np}]^{\frac{1}{np}} &= \mathbb{E}[\left(\prod_{i=1}^{j-1} e^{2\alpha p\|Z_i\|^2}\right) e^{2\alpha p\|Z_j\|^2}(1+(s-\underline{s})\|Z_j\|^2)^{np}]^{\frac{1}{np}}\\ &= \left(\prod_{i=1}^{j-1}\mathbb{E}[e^{2\alpha p\|Z_i\|^2}]\right)^{\frac{1}{np}}\mathbb{E}[e^{2\alpha p\|Z_j\|^2}(1+(s-\underline{s})\|Z_j\|^2)^{np}]^{\frac{1}{np}}\\ &= \mathbb{E}[e^{2\alpha p\|Z_1\|^2}]^{\frac{j-1}{np}}\mathbb{E}[e^{2\alpha p\|Z_j\|^2}(1+(s-\underline{s})\|Z_j\|^2)^{np}]^{\frac{1}{np}}\\ &\leq \mathbb{E}[e^{2\alpha p\|Z_1\|^2}]^{\frac{1}{p}}\mathbb{E}[e^{2\alpha p\|Z_j\|^2}(1+\frac{1}{n}\|Z_j\|^2)^{np}]^{\frac{1}{np}}.\end{aligned}$$

With $\alpha < \frac{1}{4p}$, $\mathbb{E}[e^{2\alpha p\|Z_1\|^2}]^{\frac{1}{p}} = (1-4\alpha p)^{-\frac{1}{p}}$. For the second term, find some δ with $\alpha < \delta < \frac{1}{4p}$ and set $m = \inf\{l \in \mathbb{N} : l \geq \frac{\delta}{\delta-\alpha}\}$, the ceiling of $\frac{\delta}{\delta-\alpha}$. Again using

Holder's inequality,

$$\begin{aligned}\mathbb{E}[e^{2\alpha p\|Z_j\|^2}(1+\frac{1}{n}\|Z_j\|^2)^{np}] &\le \mathbb{E}[e^{2\delta p\|Z_j\|^2}]^{\frac{\alpha}{\delta}}\mathbb{E}[(1+\frac{1}{n}\|Z_j\|^2)^{np\frac{\delta}{\delta-\alpha}}]^{\frac{\delta-\alpha}{\delta}} \\ &\le (1-4\delta p)^{-\frac{\alpha}{\delta}}\mathbb{E}[(1+\frac{1}{n}\|Z_j\|^2)^{npm}]^{\frac{\delta-\alpha}{\delta}}\end{aligned}$$

Using the binomial formula,

$$\begin{aligned}\mathbb{E}[(1+\frac{1}{n}\|Z_j\|^2)^{npm}] &= \sum_{k=0}^{npm}\begin{pmatrix} npm \\ k \end{pmatrix}\left(\frac{1}{n}\right)^k \mathbb{E}[\|Z_j\|^{2k}] \\ &= \sum_{k=0}^{npm}\begin{pmatrix} npm \\ k \end{pmatrix}\left(\frac{1}{n}\right)^k \frac{(2k)!}{2^k k!} \\ &\le \sum_{k=0}^{npm}\begin{pmatrix} npm \\ k \end{pmatrix}\left(\frac{1}{n}\right)^k \frac{1}{2^k}\frac{e}{\sqrt{\pi}}\left(\frac{4}{e}\right)^k k^k \\ &\le \sum_{k=0}^{npm}\begin{pmatrix} npm \\ k \end{pmatrix}\frac{e}{\sqrt{\pi}}\left(\frac{2pm}{e}\right)^k \\ &= \frac{e}{\sqrt{\pi}}\left(1+\frac{2pm}{e}\right)^{npm}.\end{aligned}$$

Here the third line comes from Stirling's approximation, $\sqrt{2\pi}N^{N+\frac{1}{2}}e^{-N} \le N! \le eN^{N+\frac{1}{2}}e^{-N}$. Putting these pieces together,

$$\begin{aligned}&\mathbb{E}[e^{2\alpha npQ_s^n}(1+\|b_s-b_{\underline{s}}\|^2)^{np}]^{\frac{1}{np}} \\ &\le \mathbb{E}[e^{2\alpha p\|Z_1\|^2}]^{\frac{1}{p}}\mathbb{E}[e^{2\alpha p\|Z_j\|^2}(1+\frac{1}{n}\|Z_j\|^2)^{np}]^{\frac{1}{np}} \\ &\le (1-4\alpha p)^{-\frac{1}{p}}\left[(1-4\delta p)^{-\frac{\alpha}{\delta}}\left(\frac{e}{\sqrt{\pi}}\left(1+\frac{2pm}{e}\right)^{npm}\right)^{\frac{\delta-\alpha}{\delta}}\right]^{\frac{1}{np}} \\ &= (1-4\alpha p)^{-\frac{1}{p}}\left[(1-4\delta p)^{-\frac{\alpha}{\delta}}\left(\frac{e}{\sqrt{\pi}}\right)^{\frac{\delta-\alpha}{\delta}}\right]^{\frac{1}{np}}\left(1+\frac{2pm}{e}\right)^{m\frac{\delta-\alpha}{\delta}} \\ &\le (1-4\alpha p)^{-\frac{1}{p}}(1-4\delta p)^{-\frac{\alpha}{\delta}}\left(\frac{e}{\sqrt{\pi}}\right)^{\frac{\delta-\alpha}{\delta}}\left(1+\frac{2pm}{e}\right)^{m\frac{\delta-\alpha}{\delta}} \\ &=: \Lambda < \infty\end{aligned}$$

where as desired Λ is independent of n. □

4.3 Bounds on $\tilde{\rho}_{\mathcal{P}}$ and Uniform Integrability

Lemma 4.6. *Let* $\{\lambda_{i,j} : 1 \le i \le n, i \le j \le n\}$ *be defined as in Theorem 4.4. Define* $\{p_{i,j} : 1 \le i \le n, i \le j \le n\}$ *by*

$$p_{i,j} := \begin{cases} \frac{1}{2}(\lambda_{i,j} + \lambda_{i,j-1}) & j > i \\ \frac{1}{2}(\lambda_{i,i} + \lambda_{i,n}) & j = i \end{cases}. \tag{4.31}$$

Then,

$$\det\left(I + \mathcal{A}_{\mathcal{P}}^{-1}\mathcal{R}_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}}^{tr})^{-1}\right) \le \prod_{i=1}^{n}\left(\sum_{j=i}^{n} p_{i,j}\cosh(240\sqrt{K_j^{\mathcal{P}}}\Delta)\right)^d. \tag{4.32}$$

Moreover, $\sum_{j=i}^{n} p_{i,j} = 1$ *and* $\sum_{i=1}^{j} p_{i,j} < 3$

Proof. By Lemma B.8, the matrix $I+\mathcal{A}_{\mathcal{P}}^{-1}\mathcal{R}_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}}^{tr})^{-1}$ is symmetric positive definite, so we can apply Fischer's inequalty (see [12, Theorem 7.8.3]),

$$\begin{aligned}\det(I + \mathcal{A}_{\mathcal{P}}^{-1}\mathcal{R}_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}}^{tr})^{-1}) &\le \prod_{i=1}^{n}\det([I + \mathcal{A}_{\mathcal{P}}^{-1}\mathcal{R}_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}}^{tr})^{-1}]_{i,i}) \\ &\le \prod_{i=1}^{n}\left(1 + \|[\mathcal{A}_{\mathcal{P}}^{-1}\mathcal{R}_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}}^{tr})^{-1}]_{i,i}\|\right)^d.\end{aligned}$$

From Theorem 4.4,

$$\begin{aligned}&\le \prod_{i=1}^{n}\left(1 + \sum_{j=i}^{n}\lambda_{i,j}\left(\cosh(30\sqrt{K_j^{\mathcal{P}}}\Delta)\cosh(120\sqrt{K_{j+1}^{\mathcal{P}}}\Delta) - 1\right)\right)^d \\ &= \prod_{i=1}^{n}\left(\sum_{j=i}^{n}\lambda_{i,j}\cosh(30\sqrt{K_j^{\mathcal{P}}}\Delta)\cosh(120\sqrt{K_{j+1}^{\mathcal{P}}}\Delta)\right)^d\end{aligned}$$

Using that $xy \le \frac{1}{2}(x^2 + y^2)$,

$$\begin{aligned}&\le \prod_{i=1}^{n}\left(\sum_{j=i}^{n}\frac{1}{2}(\lambda_{i,j} + \lambda_{i,j-1})\cosh^2(120\sqrt{K_j^{\mathcal{P}}}\Delta)\right)^d \\ &\le \prod_{i=1}^{n}\left(\sum_{j=i}^{n} p_{i,j}\cosh(240\sqrt{K_j^{\mathcal{P}}}\Delta)\right)^d\end{aligned}$$

where we define $\lambda_{i,i-1} := 0$ and use the inequality $\cosh^2(x) \le \cosh(2x)$.

We now calculate,

$$\begin{aligned}
\sum_{j=i}^{n} p_{i,j} &= \frac{1}{2}\left(\lambda_{i,i} + \lambda_{i,n} + \sum_{j=i+1}^{n}(\lambda_{i,j} + \lambda_{i,j-1})\right) \\
&= \frac{1}{2}\left(2\sum_{j=i}^{n}\lambda_{i,j}\right) \\
&= 1.
\end{aligned}$$

From the definition of $\lambda_{i,j} = \left(\frac{1}{2}\right)^{j-i}\left(2 - \left(\frac{1}{2}\right)^{n-i}\right)^{-1} \leq \left(\frac{1}{2}\right)^{j-i}$. From here,

$$\begin{aligned}
p_{i,j} &= \begin{cases} \frac{1}{2}(\lambda_{i,j} + \lambda_{i,j-1}) & j > i \\ \frac{1}{2}(\lambda_{i,i} + \lambda_{i,n}) & j = i \end{cases} \\
&\leq \begin{cases} \frac{3}{2}\left(\frac{1}{2}\right)^{j-i} & j > i \\ \frac{3}{4} & j = i \end{cases}
\end{aligned}$$

and therefore,

$$\begin{aligned}
\sum_{i=1}^{j} p_{i,j} &\leq \sum_{i=1}^{j} \frac{3}{2}\left(\frac{1}{2}\right)^{j-i} \\
&= \frac{3}{2}\left(\frac{1-\left(\frac{1}{2}\right)^{j}}{1-\frac{1}{2}}\right) \\
&< 3.
\end{aligned}$$

□

Proposition 4.7. *Let $\zeta > 0$, $p \in \mathbb{N}$, and $\{p_{i,j} : 1 \leq i \leq n, i \leq j \leq n\}$ be defined as in Lemma 4.6. Then,*

$$\limsup_{n\to\infty} \mathbb{E}\left[\prod_{i=1}^{n}\left(\sum_{j=i}^{n} p_{i,j}\cosh(\zeta\|\Delta_j b\|\right)^{p}\right] < \infty. \tag{4.33}$$

Proof. For convenience define $x_j := \zeta\|\Delta_j b\|$. Using the geometric-arithmetic mean inequality,

$$\begin{aligned}
\prod_{i=1}^{n}\left(\sum_{j=i}^{n} p_{i,j}\cosh(x_j)\right)^{p} &\leq \left(\sum_{i=1}^{n}\frac{1}{n}\left(\sum_{j=i}^{n} p_{i,j}\cosh(x_j)\right)\right)^{np}\\
&= \left(\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{p_{i,j}}{n}\cosh(x_j)\right)^{np}\\
&= \left(\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{p_{i,j}}{n} + \sum_{i=1}^{n}\sum_{j=i}^{n}\frac{p_{i,j}}{n}(\cosh(x_j)-1)\right)^{np}\\
&= \left(1+\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{p_{i,j}}{n}(\cosh(x_j)-1)\right)^{np}\\
&= \left(1+\sum_{j=1}^{n}\sum_{i=1}^{j}\frac{p_{i,j}}{n}(\cosh(x_j)-1)\right)^{np}\\
&< \left(1+\frac{3}{n}\sum_{j=1}^{n}(\cosh(x_j)-1)\right)^{np}
\end{aligned}$$

Where we used Lemma 4.6 to realize $\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{p_{i,j}}{n} = \sum_{i=1}^{n}\frac{1}{n} = 1$ and $\sum_{i=1}^{j} p_{i,j} < 3$. Estimating the sum on the right hand side of the inequality,

$$\begin{aligned}
\sum_{j=1}^{n}(\cosh(x_j)-1) &= \sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{x_j^{2k}}{(2k)!}\\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{n}\frac{x_j^{2k}}{(2k)!}\\
&\leq \sum_{k=1}^{\infty}\frac{\left(\sum_{j=1}^{n} x_j^2\right)^k}{(2k)!}\\
&\leq \sum_{k=1}^{\infty}\frac{\|\mathbf{x}\|^{2k}}{(2k)!}\\
&= \cosh(\|\mathbf{x}\|)-1
\end{aligned}$$

where $\mathbf{x} := (x_1, ..., x_n)$. Fix some $\alpha \in (0, \frac{1}{4\zeta^2 p})$ and use Lemma A.3 to find some $C_\alpha < \infty$ such that

$$\begin{aligned}
\cosh(\|\mathbf{x}\|)-1 &\leq C_\alpha(e^{\alpha\|\mathbf{x}\|^2}-1)\\
&= C_\alpha(e^{\tilde{\alpha}\|\mathbf{B}_n\|^2}-1)
\end{aligned}$$

where $\mathbf{B}_n := (\Delta_1 b, \cdots, \Delta_n b)$ and $\tilde{\alpha} = \zeta^2\alpha \in (0, \frac{1}{4p})$. Therefore, using the above inequalities and Lemma 4.5,

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{E}\left[\prod_{i=1}^{n}\left(\sum_{j=i}^{n} p_{i,j}\cosh(\zeta\|\Delta_j b\|\right)^p\right] \\
&\leq \limsup_{n\to\infty} \mathbb{E}\left[\left(1 + \frac{3}{n}\sum_{j=1}^{n}(\cosh(\zeta\|\Delta_j b\|) - 1)\right)^{np}\right] \\
&\leq \limsup_{n\to\infty} \mathbb{E}\left[\left(1 + \frac{3C_\alpha}{n}(e^{\tilde{\alpha}\|\mathbf{B}_n\|^2} - 1)\right)^{np}\right] \\
&< \infty.
\end{aligned}$$

□

The following Theorem is in fact just a corollary of what we've shown thus far.

Theorem 4.8. *Let $\mathcal{P} = \{0, 1/n, 2/n, ..., 1\}$ be an equally spaced partition of $[0,1]$. Then for any $p \in \mathbb{N}$,*

$$\limsup_{|\mathcal{P}|\to 0} \mathbb{E}\left[\left(\det(I + \mathcal{A}_{\mathcal{P}}^{-1}\mathcal{R}_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}}^{tr})^{-1})\right)^p\right] < \infty. \tag{4.34}$$

In particular, given some $p \in \mathbb{N}$, there exists $N \in \mathbb{N}$ and $C < \infty$ such that

$$\sup_{n\geq N}\{\mathbb{E}[(\tilde{\rho}_{\mathcal{P}})^p] : \#(\mathcal{P}) = n\} < C, \tag{4.35}$$

where $\{\tilde{\rho}_{\mathcal{P}} : \#(\mathcal{P}) = n, n \in \mathbb{N}\}$ is collection of functions defined in Eq. (3.37). This further implies that $\{\tilde{\rho}_{\mathcal{P}} : \#(\mathcal{P}) = n, n \in \mathbb{N}\}$ are uniformly integrable.

Proof. From Lemma 4.6 and Eq. (4.5),

$$\begin{aligned}
\left(\det(I + \mathcal{A}_{\mathcal{P}}^{-1}\mathcal{R}_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}}^{tr})^{-1})\right)^p &\leq \prod_{i=1}^{n}\left(\sum_{j=i}^{n} p_{i,j}\cosh(240\sqrt{K_j^{\mathcal{P}}}\Delta)\right)^{dp} \\
&\leq \prod_{i=1}^{n}\left(\sum_{j=i}^{n} p_{i,j}\cosh(240\sqrt{\kappa}\|\Delta_j b\|)\right)^{dp}.
\end{aligned}$$

Applying Proposition 4.7,

$$\limsup_{|\mathcal{P}|\to 0} \mathbb{E}\left[\prod_{i=1}^{n}\left(\sum_{j=i}^{n} p_{i,j}\cosh(240\sqrt{\kappa}\|\Delta_j b\|)\right)^{dp}\right] < \infty,$$

concluding the proof of Eq. (4.34). From here, Eq. (4.35) is simply a matter of combining the definitions of $\tilde{\rho}_{\mathcal{P}}$ and $\limsup$. Finally, the claim of uniform integrability follows by applying Proposition A.6 below. $\square$

Chapter 5

The L^1 Limit

In Section 3.5 we find that the matrix $\mathcal{L}_{\mathcal{P}}$ is positive definite. This implies that $\mathcal{L}_{\mathcal{P}}^{1/2}$ exists and is invertible, allowing us to write

$$\mathcal{L}_{\mathcal{P}} + \mathcal{R}_{\mathcal{P}} = \mathcal{L}_{\mathcal{P}}^{1/2} \left(I + \mathcal{L}_{\mathcal{P}}^{-1/2} \mathcal{R}_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}^{-1/2} \right) \mathcal{L}_{\mathcal{P}}^{1/2}. \tag{5.1}$$

In turn, this lets us represent Eq. (3.37) as

$$\tilde{\rho}_{\mathcal{P}} = \sqrt{\det \left(I + \mathcal{L}_{\mathcal{P}}^{-1/2} \mathcal{R}_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}^{-1/2} \right)}. \tag{5.2}$$

5.1 The space $H_{\mathcal{P}}^{\varepsilon}$

Let $\varepsilon > 0$. We are going to restrict the space that we're working on more by defining the subspace $H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d) \subset H_{\mathcal{P}}(\mathbb{R}^d)$,

$$H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d) := \left\{ \vee_{i=1}^{n} \|\Delta_i b^{\mathcal{P}}\| < \varepsilon \right\}. \tag{5.3}$$

We can alternatively define it in the following equivalent ways

$$\begin{aligned} H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d) &= \cap_{i=1}^{n} \left\{ \|\Delta_i b^{\mathcal{P}}\| < \varepsilon \right\} & (5.4) \\ &= \left\{ \omega \in H_{\mathcal{P}}(\mathbb{R}^d) : \vee_{i=1}^{n} \|\Delta_i \omega\| < \varepsilon \right\} & (5.5) \\ &= \left\{ \omega \in H_{\mathcal{P}}(\mathbb{R}^d) : \vee_{i=1}^{n} \int_{s_{i-1}}^{s_i} \|\omega'(s)\| ds < \varepsilon \right\}. & (5.6) \end{aligned}$$

From here we can define $H_{\mathcal{P}}^{\varepsilon}(M) \subset H_{\mathcal{P}}(M)$ by

$$H_{\mathcal{P}}^{\varepsilon}(M) = \phi(H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)) \tag{5.7}$$

where ϕ is Cartan's Development discussed in Section 2.5. It is worth noting here that if $\sigma = \phi(\omega)$, then

$$\begin{aligned}\|\omega'(s)\|^2 &= \langle \omega'(s), \omega'(s)\rangle \\ &= g(//_s(\sigma)\omega'(s), //_s(\sigma)\omega'(s)) \\ &= g(\sigma'(s), \sigma'(s)).\end{aligned}$$

So in consideration of Eq (5.6), we might have also chosen to define $H^{\varepsilon}_{\mathcal{P}}(M)$ by

$$H^{\varepsilon}_{\mathcal{P}}(M) = \left\{\sigma \in H_{\mathcal{P}}(M) : \vee_{i=1}^{n} \int_{s_{i-1}}^{s_i} \|\sigma'(s)\| ds < \varepsilon\right\}. \tag{5.8}$$

One motivation for defining $H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)$ is that in calculating the limit of $\tilde{\rho}_{\mathcal{P}}$ as $|\mathcal{P}| \to 0$, we conduct an expansion of the $\tilde{\rho}_{\mathcal{P}}$ in powers of the $\Delta_i b^{\mathcal{P}}$, which makes it advantageous to have a size estimate for $\|\Delta_i b^{\mathcal{P}}\|$. Moreover, $H_{\mathcal{P}}(M)\backslash H^{\varepsilon}_{\mathcal{P}}(M)$ is quite small in a manner that is explored below in the following two lemmas. Lemma 5.1 is proved in [1, Proposition 5.13] and left unproved here.

Lemma 5.1. *For any $\varepsilon > 0$, there is a constant $C = C(\mathrm{d}) < \infty$ such that*

$$\nu_{S_{\mathcal{P}}}(H_{\mathcal{P}}(M)\backslash H^{\varepsilon}_{\mathcal{P}}(M)) \le \frac{C}{\varepsilon^2} \exp\left\{-\frac{\varepsilon^2}{4|\mathcal{P}|}\right\}. \tag{5.9}$$

Lemma 5.2. *Given an equally spaced partition $\mathcal{P}$ with $|\mathcal{P}| = 1/n$, for $\varepsilon > 0$ and sufficiently small $|\mathcal{P}|$, there exists a $C = C(\mathrm{d}) < \infty$ such that*

$$\nu_{G_{\mathcal{P}}}(H_{\mathcal{P}}(M)\backslash H^{\varepsilon}_{\mathcal{P}}(M)) \le \frac{C}{\sqrt{|\mathcal{P}|}\varepsilon} \exp\{-\frac{\varepsilon^2}{8|\mathcal{P}|}\}. \tag{5.10}$$

Proof. Set $\Gamma = H_{\mathcal{P}}(M)\backslash H^{\varepsilon}_{\mathcal{P}}(M)$. Let $B_i := \{\|\Delta_i b\| > \varepsilon\} = \{\|\Delta_i b^{\mathcal{P}}\| > \varepsilon\}$, $B := \cup_i B_i$. We have, $\phi^{-1}(\Gamma) = H_{\mathcal{P}}(\mathbb{R}^d)\backslash H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d) = B^{\mathcal{P}} \cap H_{\mathcal{P}}(\mathbb{R}^d)$, and $(b^{\mathcal{P}})^{-1}(B^{\mathcal{P}} \cap$

$H_{\mathcal{P}}(\mathbb{R}^d)) = \cup_i\{\|\Delta_i b^{\mathcal{P}}\| > \varepsilon\} = B$. Therefore, $\phi(b^{\mathcal{P}})^{-1}(\Gamma) = B$ and

$$\begin{aligned}
\nu_{G_{\mathcal{P}}}(\Gamma) &= \int_\Gamma d\nu_{G_{\mathcal{P}}} \\
&= \int_\Gamma \rho_{\mathcal{P}} d\nu_{S_{\mathcal{P}}} \\
&= \int_B \tilde{\rho}_{\mathcal{P}} d\mu \\
&\leq \sum_{i=1}^n \int_{B_i} \tilde{\rho}_{\mathcal{P}} d\mu \\
&\leq \sum_{i=1}^n \left(\mathbb{E}[1_{B_i}]\right)^{1/2} \left(\mathbb{E}[(\tilde{\rho}_{\mathcal{P}})^2]\right)^{1/2}.
\end{aligned}$$

Here the second equality comes from $\nu_{G_{\mathcal{P}}} = \rho_{\mathcal{P}}\nu_{S_{\mathcal{P}}}$, the third equality comes from the fact that $\nu_{S_{\mathcal{P}}}$ is the law of $\phi(b^{\mathcal{P}})$ under μ (see Theorem 2.11), and the final inequality is a simple application of Hölder's inequality. Applying Theorem 4.8 with n sufficiently large, find a constant $C_1 < \infty$ such that

$$\sup\{\left(\mathbb{E}[(\tilde{\rho}_{\mathcal{P}})^2]\right)^{1/2} : \#(\mathcal{P}) = n, n \text{ sufficiently large } \} \leq C_1.$$

Further, from Lemma A.1 with $k = 0$ and $a = \varepsilon/\sqrt{|\mathcal{P}|}$, there is a $C_2 < \infty$ such that

$$\mathbb{E}[1_{B_i}] \leq \frac{C_2|\mathcal{P}|}{\varepsilon^2} \exp\{-\frac{\varepsilon^2}{4|\mathcal{P}|}\}.$$

Therefore,

$$\begin{aligned}
\nu_{G_{\mathcal{P}}}(\Gamma) &\leq \sum_{i=1}^n \left(\mathbb{E}[1_{B_i}]\right)^{1/2} \left(\mathbb{E}[(\tilde{\rho}_{\mathcal{P}})^2]\right)^{1/2} \\
&\leq C \sum_{i=1}^n \frac{\sqrt{|\mathcal{P}|}}{\varepsilon} \exp\{-\frac{\varepsilon^2}{8|\mathcal{P}|}\} \\
&= C \frac{\sqrt{|\mathcal{P}|}}{|\mathcal{P}|\varepsilon} \exp\{-\frac{\varepsilon^2}{8|\mathcal{P}|}\} \\
&= \frac{C}{\sqrt{|\mathcal{P}|}\varepsilon} \exp\{-\frac{\varepsilon^2}{8|\mathcal{P}|}\},
\end{aligned}$$

where we are certainly using the fact that $\mathcal{P}$ is equally spaced and $|\mathcal{P}| = 1/n$. □

The way in which we apply Lemmas 5.1 and 5.2 is in the following immediate corollary.

Corollary 5.3. *Let $f : W(M) \to \mathbb{R}$ be bounded and continuous with $\sup_x |f(x)| \leq \Lambda < \infty$. Then,*

$$\left| \int_{H_{\mathcal{P}}(M)} f d\nu_{S_{\mathcal{P}}} - \int_{H_{\mathcal{P}}^{\varepsilon}(M)} f d\nu_{S_{\mathcal{P}}} \right| \leq \frac{\tilde{\Lambda}}{\varepsilon^2} e^{-\frac{\varepsilon^2}{4|\mathcal{P}|}} \tag{5.11}$$

and

$$\left| \int_{H_{\mathcal{P}}(M)} f d\nu_{G_{\mathcal{P}}} - \int_{H_{\mathcal{P}}^{\varepsilon}(M)} f d\nu_{G_{\mathcal{P}}} \right| \leq \frac{\tilde{\Lambda}}{\sqrt{|\mathcal{P}|}\varepsilon} e^{-\frac{\varepsilon^2}{8|\mathcal{P}|}} \tag{5.12}$$

where $\tilde{\Lambda} = \tilde{\Lambda}(\Lambda, \mathrm{d}) < \infty$.

We conclude this section with a few estimates which motivate several of the bounds that we make in the sequel.

Lemma 5.4. *Take $c > 0$ and $p \in \mathbb{N}$ with $p \geq 2$. For sufficiently small $|\mathcal{P}|$, there exists a $C = C(\mathrm{d}, c, p) < \infty$ such that,*

$$\int_{W(\mathbb{R}^d)} \|\Delta_i b\|^p \exp\left\{ c \sum_{i=1}^{n} \|\Delta_i b\|^2 \right\} d\mu \leq C \Delta_i s |\mathcal{P}|^{\frac{p-2}{2}} \leq C|\mathcal{P}|^{\frac{p}{2}}. \tag{5.13}$$

In particular, if $\Gamma \subset \{1, ..., n\}$ with $\#(\Gamma) = m$,

$$\sum_{i \in \Gamma} \int_{W(\mathbb{R}^d)} \|\Delta_i b\|^p \exp\left\{ c \sum_{j=1}^{n} \|\Delta_j b\|^2 \right\} d\mu \leq C \left(\sum_{i \in \Gamma} \Delta_i s \right) |\mathcal{P}|^{\frac{p-2}{2}} \leq Cm|\mathcal{P}|^{\frac{p}{2}}, \tag{5.14}$$

implying,

$$\sum_{i=1}^{n} \int_{W(\mathbb{R}^d)} \|\Delta_i b\|^p \exp\left\{ c \sum_{j=1}^{n} \|\Delta_j b\|^2 \right\} d\mu \leq C|\mathcal{P}|^{\frac{p-2}{2}}. \tag{5.15}$$

Proof. Notice first,

$$\begin{aligned} &\int_{W(\mathbb{R}^d)} \|\Delta_i b\|^p \exp\left\{ c \sum_{i=1}^{n} \|\Delta_i b\|^2 \right\} d\mu \\ &\quad = \mathbb{E}\left[\|\Delta_i b\|^p \exp\left\{ c\|\Delta_i b\|^2 \right\} \right] \mathbb{E}\left[\exp\left\{ c \sum_{j \neq i} \|\Delta_j b\|^2 \right\} \right]. \end{aligned}$$

By Lemma A.2, $\limsup_{|\mathcal{P}|\to 0} \mathbb{E}\left[\exp\left\{c\sum_{j\neq i}\|\Delta_j b\|^2\right\}\right] = e^{\mathrm{d}\,c}$ and thus

$$\begin{aligned}
&\mathbb{E}\left[\|\Delta_i b\|^p \exp\left\{c\|\Delta_i b\|^2\right\}\right]\mathbb{E}\left[\exp\left\{c\sum_{j\neq i}\|\Delta_j b\|^2\right\}\right] \\
&\quad\le 2e^{\mathrm{d}\,c}\mathbb{E}\left[\|\Delta_i b\|^p \exp\left\{c\|\Delta_i b\|^2\right\}\right] \\
&\quad= 2ce^{\mathrm{d}\,c}(\Delta_i s)^{p/2}\mathbb{E}\left[\|b_1\|^3 \exp\left\{c\Delta_i s\|b_1\|^2\right\}\right] \\
&\quad\le 2c\Delta_i s e^{\mathrm{d}\,c}|\mathcal{P}|^{\frac{p-2}{2}}\mathbb{E}\left[\|b_1\|^p \exp\left\{c|\mathcal{P}|\|b_1\|^2\right\}\right] \\
&\quad\le C\Delta_i s|\mathcal{P}|^{\frac{p-2}{2}}
\end{aligned}$$

where C is as desired. □

It's important to note that the constant C used in Lemma 5.4 only gets better as $c \to 0$, which is a fact that will be used in the following corollary.

Corollary 5.5. *Take $c > 0$ and suppose that Y is a random variable on $H_{\mathcal{P}}(\mathbb{R}^d)$ such that $|Y| \le c\sum_{i=1}^n \|\Delta_i b^{\mathcal{P}}\|^3$. For sufficiently small $|\mathcal{P}|$ and ε, there exists a $C = C(\mathrm{d}, \mathit{curvature}, c) < \infty$ such that,*

$$\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} \left(e^Y - 1\right) d\mu_{S_{\mathcal{P}}} \le C\sqrt{|\mathcal{P}|}. \tag{5.16}$$

Moreover, this implies that for any $p \in \mathbb{N}$,

$$\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} \left(e^Y - 1\right)^p d\mu_{S_{\mathcal{P}}} \le C\sqrt{|\mathcal{P}|} \tag{5.17}$$

where here C also depends on p.

Proof. From the inequality $|e^a - 1| \leq e^{|a|} - 1 \leq |a|e^{|a|}$ for any $a \in \mathbb{R}$,

$$\begin{aligned}
&\int_{H^{\varepsilon}_{\mathcal{P}}(M)} \left|e^{Y} - 1\right| d\mu_{S_{\mathcal{P}}} \\
&\quad \leq \int_{H^{\varepsilon}_{\mathcal{P}}(M)} \left(\exp\left\{c\sum_{i=1}^{n} \|\Delta_i b^{\mathcal{P}}\|^3\right\} - 1\right) d\mu_{S_{\mathcal{P}}} \\
&\quad \leq c\sum_{i=1}^{n} \int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} \|\Delta_i b^{\mathcal{P}}\|^3 \exp\left\{c\varepsilon\sum_{i=1}^{n} \|\Delta_i b^{\mathcal{P}}\|^2\right\} d\mu_{S_{\mathcal{P}}} \\
&\quad \leq c\sum_{i=1}^{n} \int_{H_{\mathcal{P}}(\mathbb{R}^d)} \|\Delta_i b^{\mathcal{P}}\|^3 \exp\left\{c\varepsilon\sum_{i=1}^{n} \|\Delta_i b^{\mathcal{P}}\|^2\right\} d\mu_{S_{\mathcal{P}}} \\
&\quad \leq c\sum_{i=1}^{n} \int_{W(\mathbb{R}^d)} \|\Delta_i b\|^3 \exp\left\{c\varepsilon\sum_{i=1}^{n} \|\Delta_i b\|^2\right\} d\mu \\
&\quad \leq C\sqrt{|\mathcal{P}|}
\end{aligned}$$

where the last inequality follows from Lemma 5.4 and C is as desired since our bound only becomes better as $\varepsilon \to 0$, as noted in the remark following Lemma 5.4. The last claim is a corollary to the first using Lemma A.4, which tells us $(e^Y - 1)^p \leq e^{p|Y|} - 1$. □

Corollary 5.6. *Take $c > 0$ and suppose that $\Gamma \subset \{1, ..., n\}$ with $\#(\Gamma) = m$. Suppose that Y is a random variable on $H_{\mathcal{P}}(\mathbb{R}^d)$ such that $|Y| \leq c\sum_{i\in\Gamma} \|\Delta_i b\|^2$. For sufficiently small $|\mathcal{P}|$ and ε, there exists a $C = C(\mathrm{d}, curvature, c) < \infty$ such that,*

$$\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} (e^Y - 1) d\mu_{S_{\mathcal{P}}} \leq Cm|\mathcal{P}|. \tag{5.18}$$

Moreover, this implies that for $p \in \mathbb{N}$,

$$\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} (e^Y - 1)^p d\mu_{S_{\mathcal{P}}} \leq Cm|\mathcal{P}|, \tag{5.19}$$

where here C also depends on p.

Proof. This proof is nearly identical as that for Corollary 5.5 with

$$\begin{aligned}
|e^Y - 1| &\leq c\sum_{i\in\Gamma} \|\Delta_i b\|^2 e^{c\sum_{j\in\Gamma}\|\Delta_j b\|^2} \\
&\leq c\sum_{i\in\Gamma} \|\Delta_i b\|^2 e^{c\sum_{j=1}^{n}\|\Delta_j b\|^2}.
\end{aligned}$$

□

The following estimate has a similar form also proved in [1, Proposition 6.4]; however, for our purposes we would like to keep track of ε, forcing us to be slightly more careful than the aforementioned result.

Proposition 5.7. *Define* $\mathcal{R}_{\mathcal{P}}$ *and* $\mathcal{S}_{\mathcal{P}}$ *as in Eqs. (2.24) and (2.25). With* $|\mathcal{P}|$ *sufficiently small, for any* $p \in \mathbb{R}$ *and* $\varepsilon > 0$ *there exists constants* $C_1 = C_1(p, \mathrm{d}, \mathit{curvature})$ *and* $C_2 = C_2(p, \mathrm{d}, \mathit{curvature})$ *such that,*

$$1 - \frac{C_1}{\varepsilon^2} e^{-C_2 \frac{\varepsilon^2}{|\mathcal{P}|}} \leq \int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} e^{p(\mathcal{R}_{\mathcal{P}} - \mathcal{S}_{\mathcal{P}})} d\mu_{S_{\mathcal{P}}} \leq e^{C_1 |\mathcal{P}|} + \frac{C_1}{\varepsilon^2} e^{-C_2 \frac{\varepsilon^2}{|\mathcal{P}|}}. \tag{5.20}$$

In particular, this implies that

$$\left| \int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} e^{p(\mathcal{R}_{\mathcal{P}} - \mathcal{S}_{\mathcal{P}})} d\mu_{S_{\mathcal{P}}} - 1 \right| \leq e^{C_1 |\mathcal{P}|} - 1 + \frac{C_1}{\varepsilon^2} e^{-C_2 \frac{\varepsilon^2}{|\mathcal{P}|}}, \tag{5.21}$$

and hence

$$\left| \int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} \left(e^{p(\mathcal{R}_{\mathcal{P}} - \mathcal{S}_{\mathcal{P}})} - 1 \right) d\mu_{S_{\mathcal{P}}} \right| \leq e^{C_1 |\mathcal{P}|} - 1 + \frac{C_1}{\varepsilon^2} e^{-C_2 \frac{\varepsilon^2}{|\mathcal{P}|}} + \frac{C_1}{\sqrt{|\mathcal{P}|}\varepsilon} e^{-\frac{\varepsilon^2}{8|\mathcal{P}|}}. \tag{5.22}$$

Proof. Following the notation in Lemma 5.2 by setting $B_i = \{\|\Delta_i b\| > \varepsilon\}$, by Theorem 2.11,

$$\begin{aligned}
\int_{H_{\mathcal{P}}(\mathbb{R}^d) \setminus H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} e^{p(\mathcal{R}_{\mathcal{P}} - \mathcal{S}_{\mathcal{P}})} d\mu_{S_{\mathcal{P}}} &= \int_{\cup_{i=1}^n \{\Delta_i b \geq \varepsilon\}} e^{p(\mathcal{R} - \mathcal{S})} d\mu \\
&\leq \sum_{i=1}^n \int_{B_i} e^{p(\mathcal{R} - \mathcal{S})} d\mu \\
&\leq \sum_{i=1}^n \mathbb{E}\left[1_{B_i} e^{|p|\Lambda(\sum_{j=1}^n \|\Delta_j b\|^2 + \mathrm{d})} \right] \\
&= \sum_{i=1}^n e^{|p|\Lambda \mathrm{d}} \mathbb{E}\left[e^{|p|\Lambda \sum_{j \neq i} \|\Delta_j b\|^2} \right] \mathbb{E}\left[1_{B_i} e^{|p|\Lambda \|\Delta_i b\|^2} \right]
\end{aligned}$$

where $\Lambda < \infty$ depends solely on the curvature of the manifold. From Lemma A.1, for sufficiently small $|\mathcal{P}|$, we can take $\Lambda_2 = \Lambda_2(p, \mathrm{d}, \mathrm{curvature})$ with $0 < \Lambda_2 < \infty$ such that,

$$\begin{aligned}
\mathbb{E}\left[1_{B_i} e^{|p|\Lambda \|\Delta_i b\|^2} \right] &\leq \frac{\Lambda_2}{(\varepsilon / \sqrt{|\mathcal{P}|})^2} e^{-\Lambda_2 (\varepsilon / \sqrt{|\mathcal{P}|})^2} \\
&= \frac{|\mathcal{P}| \Lambda_2}{\varepsilon^2} e^{-\Lambda_2 \frac{\varepsilon^2}{|\mathcal{P}|}}.
\end{aligned}$$

Lemma A.2 ensures that for sufficiently small $|\mathcal{P}|$ the quantity $\mathbb{E}\left[e^{|p|\Lambda\sum_{j\neq i}\|\Delta_j b\|^2}\right]$ remains bounded (depending only on d, p, and curvature), which at last yields,

$$\begin{aligned}\sum_{i=1}^{n} e^{|p|\Lambda\,\mathrm{d}}\mathbb{E}\left[e^{|p|\Lambda\sum_{j\neq i}\|\Delta_j b\|^2}\right]\mathbb{E}\left[1_{B_i}e^{|p|\Lambda\|\Delta_i b\|^2}\right] &\leq n\tilde{C}_1\frac{|\mathcal{P}|\Lambda_2}{\varepsilon^2}e^{-\Lambda_2\frac{\varepsilon^2}{|\mathcal{P}|}}\\ &= \tilde{C}_1\frac{\Lambda_2}{\varepsilon^2}e^{-\Lambda_2\frac{\varepsilon^2}{|\mathcal{P}|}}\\ &\leq \frac{C_1}{\varepsilon^2}e^{-C_2\frac{\varepsilon^2}{|\mathcal{P}|}}.\end{aligned}$$

The first claim follows by applying Lemma 2.14. The second claim is a triviality, and the third is as well upon realizing that $\mu_{S_{\mathcal{P}}}(H_{\mathcal{P}}(\mathbb{R}^d)) = 1$ and by Lemma 5.1, $\mu_{S_{\mathcal{P}}}(H_{\mathcal{P}}(\mathbb{R}^d)\backslash H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)) \leq \frac{C_1}{\sqrt{|\mathcal{P}|}\varepsilon}e^{-\frac{\varepsilon^2}{8|\mathcal{P}|}}$. □

5.2 Taylor Expansions

From Eqs. (3.25), (3.26), and (3.27) it is clear that understanding the behavior of $\tilde{\rho}_{\mathcal{P}}$, we need to understand $\int_0^{\Delta}\{S_i^{\mathcal{P}}(s)^{\mathrm{tr}}S_i^{\mathcal{P}}(s)+V_{i+1}^{\mathcal{P}}(s)^{\mathrm{tr}}V_{i+1}^{\mathcal{P}}(s)\}ds$ and $\int_0^{\Delta}V_{i+1}^{\mathcal{P}}(s)^{\mathrm{tr}}S_{i+1}^{\mathcal{P}}(s)ds$ for each $i \in \{1,...,n\}$ (where, as always, we define $V_{n+1}^{\mathcal{P}} = S_{n+1}^{\mathcal{P}} \equiv 0$). To do this, we Taylor expand each of the $S_i^{\mathcal{P}}, C_i^{\mathcal{P}}$, and $V_{i+1}^{\mathcal{P}}$ in powers of $\Delta_i b^{\mathcal{P}}$, where it will be seen that an expansion to degree 3 is sufficient to understand the behavior of $\tilde{\rho}_{\mathcal{P}}$ in the limit $|\mathcal{P}| \to 0$.

Notation 5.8. *If $s \mapsto A(s) \in \mathrm{Hom}(\mathbb{R}^d)$ is a curve parameterized on an interval J and $r > 0$, then we will write $A = O(r)$ to mean that there exists some constant $c > 0$ such that $\sup_{s\in J}\|A(s)\| \leq cr$. Alternatively, we will write $A(s) = O(r)$ when there exists some constant $c > 0$ such that $\|A(s)\| \leq cr$. In both cases, c will be referred to as the bounding constant.*

Using the above notation, Eqs. (3.7) and (3.8) can be restated as $A_i^{\mathcal{P}}\Delta^2 = O(\|\Delta_i b\|^2)$ and $(\frac{d}{ds}A_i^{\mathcal{P}})\Delta^3 = O(\|\Delta_i b\|^3)$ with bounding constant κ.

Lemma 5.9. *For sufficiently small $\varepsilon > 0$ and $s \in (0, \Delta]$,*

$$S_i^{\mathcal{P}}(s) = sI + \frac{s^3}{6} A_i^{\mathcal{P}}(0) + O(s\|\Delta_i b^{\mathcal{P}}\|^3), \tag{5.23}$$

$$C_i^{\mathcal{P}}(s) = I + \frac{s^2}{2} A_i^{\mathcal{P}}(0) + O(\|\Delta_i b^{\mathcal{P}}\|^3), \tag{5.24}$$

$$\left[\frac{S_i^{\mathcal{P}}(s)}{s}\right]^{-1} = I - \frac{s^2}{6} A_i^{\mathcal{P}}(0) + O(\|\Delta_i b^{\mathcal{P}}\|^3), \tag{5.25}$$

$$F_i^{\mathcal{P}} = I + \frac{\Delta^2}{6} A_i^{\mathcal{P}}(0) + \frac{\Delta^2}{3} A_{i+1}^{\mathcal{P}}(0) + O(\|\Delta_i b^{\mathcal{P}}\|^3 + \|\Delta_{i+1} b^{\mathcal{P}}\|^3), \tag{5.26}$$

and

$$\begin{aligned} V_{i+1}^{\mathcal{P}}(s) = (\Delta - s)I + \left(\frac{\Delta^3 - s\Delta^2}{6}\right) A_i^{\mathcal{P}}(0) + \left(\frac{3\Delta s^2 - 2\Delta^2 s - s^3}{6}\right) A_{i+1}^{\mathcal{P}}(0) \\ + O(\Delta\{\|\Delta_i b^{\mathcal{P}}\|^3 + \|\Delta_{i+1} b^{\mathcal{P}}\|^3\}) \end{aligned} \tag{5.27}$$

on $H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)$. Moreover, the bounding constant can be taken independent of i and further depends only on ε and the curvature of the manifold, and remains bounded as $\varepsilon \to 0$.

Proof. By Proposition D.1 and Eqs. (3.7) and (3.8),

$$\begin{aligned} \|S_i^{\mathcal{P}}(s) - sI - \frac{s^3}{6} A_i^{\mathcal{P}}(0)\| &\le \frac{\kappa s}{12}\|\Delta_i b^{\mathcal{P}}\|^3 + \frac{\kappa s}{6}\left(\sum_{j=2}^{\infty} \frac{\kappa^j \|\Delta_i b\|^{2j}}{(2j+1)!}\right) && (5.28) \\ &\le \frac{\kappa s}{12}\|\Delta_i b^{\mathcal{P}}\|^3 + \frac{\kappa s}{6}\|\Delta_i b^{\mathcal{P}}\|^3 \left(\sum_{j=2}^{\infty} \frac{\kappa^j \varepsilon^{2j-3}}{(2j+1)!}\right) && (5.29) \\ &\le c(\varepsilon, \kappa) s \|\Delta_i b^{\mathcal{P}}\|^3 && (5.30) \end{aligned}$$

and

$$\begin{aligned} \|C_i^{\mathcal{P}}(s) - I - \frac{s^2}{2} A_i^{\mathcal{P}}(0)\| &\le \frac{1}{6}\|\Delta_i b^{\mathcal{P}}\|^3 + \frac{\kappa}{2}\left(\sum_{j=2}^{\infty} \frac{\kappa^j \|\Delta_i b\|^{2j}}{(2j)!}\right) && (5.31) \\ &\le \frac{\kappa}{6}\|\Delta_i b^{\mathcal{P}}\|^3 + \frac{\kappa}{2}\|\Delta_i b^{\mathcal{P}}\|^3 \left(\sum_{j=2}^{\infty} \frac{\kappa^j \varepsilon^{2j-3}}{(2j)!}\right) && (5.32) \\ &\le c(\varepsilon, \kappa) \|\Delta_i b^{\mathcal{P}}\|^3. && (5.33) \end{aligned}$$

which concludes the proof of Eqs. (5.23) and (5.24). Notice that we can take $c(\varepsilon,\kappa)$ to remain bounded as $\varepsilon \to 0$, which implies that for ε small enough,

$$\begin{aligned}\left\|\frac{S_i^{\mathcal{P}}(s)}{s} - I\right\| &\le \frac{1}{6}\|\Delta_i b^{\mathcal{P}}\|^2 + c(\varepsilon,\kappa)\|\Delta_i b^{\mathcal{P}}\|^3 \\ &\le \varepsilon^2 + c(\varepsilon,\kappa)\varepsilon^3 < 1.\end{aligned}$$

Therefore,

$$\begin{aligned}\left[\frac{S_i^{\mathcal{P}}(s)}{s}\right]^{-1} &= \left[I + \frac{s^2}{6}A_i^{\mathcal{P}}(0) + \left(\frac{S_i^{\mathcal{P}}(s)}{s} - I - \frac{s^2}{6}A_i^{\mathcal{P}}(0)\right)\right]^{-1} \\ &= I - \frac{s^2}{6}A_i^{\mathcal{P}}(0) - \left[\frac{S_i^{\mathcal{P}}(s)}{s} - I - \frac{s^2}{6}A_i^{\mathcal{P}}(0)\right] + \sum_{j=2}^{\infty}(-1)^j\left(\frac{S_i^{\mathcal{P}}(s)}{s} - I\right)^j\end{aligned}$$

with

$$\begin{aligned}\left\|\sum_{j=2}^{\infty}(-1)^j\left(\frac{S_i^{\mathcal{P}}(s)}{s} - I\right)^j\right\| &\le \sum_{j=2}^{\infty}\left(\frac{\kappa}{6}\|\Delta_i b^{\mathcal{P}}\|^2 + c(\varepsilon,\kappa)\|\Delta_i b^{\mathcal{P}}\|^3\right)^j \\ &\le \sum_{j=2}^{\infty}\left\{\left(\frac{\kappa}{6} + \varepsilon c(\varepsilon,\kappa)\right)^j \|\Delta_i b^{\mathcal{P}}\|^{2j}\right\} \\ &\le \|\Delta_i b^{\mathcal{P}}\|^3 \sum_{j=2}^{\infty}\left\{\left(\frac{\kappa}{6} + \varepsilon c(\varepsilon,\kappa)\right)^j \varepsilon^{2j-3}\right\} \\ &\le \tilde{c}(\varepsilon,\kappa)\|\Delta_i b^{\mathcal{P}}\|^3.\end{aligned}$$

Combining this with Eq. (5.23) proves Eq. (5.25). For Eq. (5.26),

$$\begin{aligned}F_i^{\mathcal{P}} &= \left(\frac{S_{i+1}^{\mathcal{P}}(\Delta)}{\Delta}\right)^{-1} C_{i+1}^{\mathcal{P}}(\Delta)\left(\frac{S_i^{\mathcal{P}}(\Delta)}{\Delta}\right) \\ &= \left(I - \frac{\Delta^2}{6}A_{i+1}^{\mathcal{P}}(0) + O(\|\Delta_{i+1}b^{\mathcal{P}}\|^3\right)\left(I + \frac{\Delta^2}{2}A_{i+1}^{\mathcal{P}}(0) + O(\|\Delta_{i+1}b^{\mathcal{P}}\|^3)\right) \\ &\quad \times \left(I + \frac{\Delta^2}{6}A_i^{\mathcal{P}}(0) + O(\|\Delta_i b^{\mathcal{P}}\|^3)\right) \\ &= I + \frac{\Delta^2}{6}A_i^{\mathcal{P}}(0) + \frac{\Delta^2}{3}A_{i+1}^{\mathcal{P}}(0) + O(\|\Delta_i b^{\mathcal{P}}\|^3 + \|\Delta_{i+1}b^{\mathcal{P}}\|^3).\end{aligned}$$

Finally,

$$\begin{aligned}
V_{i+1}^{\mathcal{P}}(s) &= C_{i+1}^{\mathcal{P}}(s)S_i^{\mathcal{P}}(\Delta) - S_{i+1}^{\mathcal{P}}(s)F_i^{\mathcal{P}} \\
&= \left(I + \frac{s^2}{2}A_{i+1}^{\mathcal{P}}(0) + O(\|\Delta_{i+1}b^{\mathcal{P}}\|^3)\right)\left(\Delta I + \frac{\Delta^3}{6}A_i^{\mathcal{P}}(0) + O(\Delta\|\Delta_i b^{\mathcal{P}}\|^3)\right) \\
&\quad - \left(sI + \frac{s^3}{6}A_{i+1}^{\mathcal{P}}(0) + O(s\|\Delta_{i+1}b^{\mathcal{P}}\|^3\right) \\
&\quad \times \left(I + \frac{\Delta^2}{6}A_i^{\mathcal{P}}(0) + \frac{\Delta^2}{3}A_{i+1}^{\mathcal{P}}(0) + O(\|\Delta_i b^{\mathcal{P}}\|^3 + \|\Delta_{i+1}b^{\mathcal{P}}\|^3)\right) \\
&= (\Delta - s)I + \left(\frac{\Delta^3 - s\Delta^2}{6}\right)A_i^{\mathcal{P}}(0) + \left(\frac{3\Delta s^2 - 2\Delta^2 s - s^3}{6}\right)A_{i+1}^{\mathcal{P}}(0) \\
&\quad + O(\Delta\{\|\Delta_i b^{\mathcal{P}}\|^3 + \|\Delta_{i+1}b^{\mathcal{P}}\|^3\}).
\end{aligned}$$

Since the bounding constants for each of Eqs. (5.23), (5.24), and (5.25) is as claimed, it follows from the above calculations that the same is true for Eqs. (5.26) and (5.27). □

Proposition 5.10. *For sufficiently small $\varepsilon > 0$,*

$$\int_0^\Delta S_i^{\mathcal{P}}(s)^{\mathrm{tr}} S_i^{\mathcal{P}}(s)ds = \frac{\Delta^3}{3}I + \frac{\Delta^5}{15}A_i^{\mathcal{P}}(0) + O(\Delta^3\|\Delta_i b^{\mathcal{P}}\|^3), \tag{5.34}$$

$$\begin{aligned}\int_0^\Delta V_{i+1}^{\mathcal{P}}(s)^{\mathrm{tr}} V_{i+1}^{\mathcal{P}}(s)ds &= \frac{\Delta^3}{3}I + \frac{\Delta^5}{9}A_i^{\mathcal{P}}(0) - \frac{2\Delta^5}{45}A_{i+1}^{\mathcal{P}}(0) \\ &\quad + O(\Delta^3\{\|\Delta_i b^{\mathcal{P}}\|^3 + \|\Delta_{i+1}b^{\mathcal{P}}\|^3\})\end{aligned} \tag{5.35}$$

$$\begin{aligned}\int_0^\Delta V_{i+1}^{\mathcal{P}}(s)^{\mathrm{tr}} S_{i+1}^{\mathcal{P}}(s)ds &= \frac{\Delta^3}{6}I + \frac{13\Delta^5}{360}A_i^{\mathcal{P}}(0) - \frac{7\Delta^5}{360}A_{i+1}^{\mathcal{P}}(0) \\ &\quad + O(\Delta^3\{\|\Delta_i b^{\mathcal{P}}\|^3 + \|\Delta_{i+1}b^{\mathcal{P}}\|^3\})\end{aligned} \tag{5.36}$$

on $H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)$. Moreover, the bounding constant can be taken independent of i and further depends only on ε and the curvature of the manifold, and remains bounded as $\varepsilon \to 0$.

Proof. This follows from multiplying together the appropriate operators using the estimates from Lemma 5.9 and integrating over $[0, \Delta]$ keeping in mind that $(A_i^{\mathcal{P}})^{\mathrm{tr}} = A_i^{\mathcal{P}}$. □

In light of Proposition 5.10, we decompose $\mathcal{R}_{\mathcal{P}}$ (on $H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)$) in the following way,

$$\mathcal{R}_{\mathcal{P}} = \mathcal{U}_{\mathcal{P}} + \mathcal{E}_{\mathcal{P}} \tag{5.37}$$

where $\mathcal{U}_{\mathcal{P}}$ is defined by

$$[\mathcal{U}_{\mathcal{P}}]_{i,j} = \frac{\Delta^5}{360}\begin{cases} 16(4A_i^{\mathcal{P}}(0) - A_{i+1}^{\mathcal{P}}(0)) & i = j \\ 13A_i^{\mathcal{P}}(0) - 7A_{i+1}^{\mathcal{P}}(0) & i = j+1, i+1 = j \\ 0 & \text{otherwise} \end{cases} \tag{5.38}$$

with $A_{n+1}^{\mathcal{P}} \equiv 0$, and the blocks of $\mathcal{E}_{\mathcal{P}}$ have size estimates,

$$[\mathcal{E}_{\mathcal{P}}]_{i,j} = \begin{cases} O(\Delta^3\{\|\Delta_i b^{\mathcal{P}}\|^3 + \|\Delta_{i+1} b^{\mathcal{P}}\|^3\}) & i = j, i = j+1, i+1 = j \\ 0 & \text{otherwise} \end{cases} \tag{5.39}$$

where the bounding constant can be taken independent of i and further depends only on ε and the curvature of the manifold, and remains bounded as $\varepsilon \to 0$.

5.3 Proof of Theorem 1.4

Here we restate the main result of this paper and dedicate the remainder of this section to the proof.

Theorem 5.11. *Let (M, g, o) be a Riemannian manifold with metric g and fixed pointed $o \in M$. Assume the curvature and its derivative on M are bounded and the sectional curvature on M is non-positive. Then given a continuous and bounded map $f : W(M) \to \mathbb{R}$,*

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma) d\nu_{G_{\mathcal{P}}}(\sigma) = \int_{W(M)} f(\sigma) e^{-\tau_G \int_0^1 \mathrm{Scal}(\sigma(s))ds} d\nu(\sigma). \tag{5.40}$$

Here $\mathcal{P} = \{0, 1/n, 2/n, ..., 1\}$ is the equally spaced partition of $[0, 1]$, τ_G is defined in Eq. (5.67), Scal *is the scalar curvature of M, $\nu_{G_{\mathcal{P}}}$ is defined in Eq. (1.11), and ν is the Wiener measure on $W(M)$.*

Our first step is to give another representation of $\tilde{\rho}_{\mathcal{P}}$. Eq (5.2) implies that

$$\tilde{\rho}_{\mathcal{P}} = \sqrt{\det\left(I + \mathcal{L}_{\mathcal{P}}^{-1/2} \mathcal{R}_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}^{-1/2}\right)} \tag{5.41}$$

$$= \sqrt{\det\left(I + \mathcal{L}_{\mathcal{P}}^{-1/2} (\mathcal{U}_{\mathcal{P}} + \mathcal{E}_{\mathcal{P}}) \mathcal{L}_{\mathcal{P}}^{-1/2}\right)}. \tag{5.42}$$

On the event $\mathcal{A} \cap H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)$ where $\mathcal{A}$ is defined in Proposition 5.14, for small enough ε and $|\mathcal{P}|$, $\|\mathcal{L}_{\mathcal{P}}^{-1/2}(\mathcal{U}_{\mathcal{P}} + \mathcal{E}_{\mathcal{P}})\mathcal{L}_{\mathcal{P}}^{-1/2}\| < 1$ and we can apply Lemma B.7, yielding

$$\tilde{\rho}_{\mathcal{P}} = \exp\frac{1}{2}\left\{\mathrm{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{U}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}) + \mathrm{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{E}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}) + \Psi_2(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})\right\} \tag{5.43}$$

with

$$\left|\Psi_2(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})\right| \leq \frac{\|\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}\|_2^2}{1 - \|\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}\|}. \tag{5.44}$$

Proposition 5.16 below shows that $\exp\frac{1}{2}\{\mathrm{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{U}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})\}$ is what contributes in the limit as $|\mathcal{P}| \to 0$, while the other factors vanish. This will then turns our focus to understanding the behavior of $\mathrm{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{U}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})$.

Lemma 5.12. *Let U be an $n \times n$ symmetric tri-diagonal block matrix with $d \times d$ blocks. Then,*

$$\mathrm{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}U\mathcal{L}_{\mathcal{P}}^{-1/2}) = \sum_{m,k=1}^{n}\frac{(\beta_k^{\mathcal{P}})^2}{\lambda_k^{\mathcal{P}}}\left[(\alpha_k^m)^2\,\mathrm{tr}([U]_{m,m}) + 2\alpha_k^m\alpha_k^{m+1}\,\mathrm{tr}([U]_{m,m+1})\right], \tag{5.45}$$

where $\beta_k^{\mathcal{P}}, \lambda_k^{\mathcal{P}}$, and α_k^m are as in Theorem 3.4. Moreover, this implies that there exists constants $C = C(\mathrm{d}) < \infty$ and $\Lambda < \infty$ such that

$$|\,\mathrm{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}U\mathcal{L}_{\mathcal{P}}^{-1/2})| \leq \frac{\Lambda}{\Delta^3}\sum_{m=1}^{n}\left(|tr([U]_{m,m})| + |\,\mathrm{tr}([U]_{m,m+1})|\right) \tag{5.46}$$

$$\leq \frac{C}{\Delta^3}\sum_{m=1}^{n}\left(\|[U]_{m,m}\| + \|[U]_{m,m+1}\|\right), \tag{5.47}$$

where we define $[U]_{n,n+1} := 0$.

Proof. The orthonormal basis of eigenvectors $u_{k,a}$ of $\mathcal{L}_{\mathcal{P}}$ from Theorem 3.4 are also eigenvectors for $\mathcal{L}_{\mathcal{P}}^{-1}$ with respective eigenvalues $1/\lambda_k^{\mathcal{P}}$. Since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$,

$$\begin{aligned}\mathrm{tr}\left(\mathcal{L}_{\mathcal{P}}^{-1/2}U\mathcal{L}_{\mathcal{P}}^{-1/2}\right) &= \mathrm{tr}\left(U\mathcal{L}_{\mathcal{P}}^{-1}\right)\\ &= \sum_{k=1}^{n}\sum_{a=1}^{d} U\mathcal{L}_{\mathcal{P}}^{-1}u_{k,a}\cdot u_{k,a}\\ &= \sum_{k=1}^{n}\sum_{a=1}^{d}\frac{1}{\lambda_k^{\mathcal{P}}}Uu_{k,a}\cdot u_{k,a}.\end{aligned}$$

Eq. (5.45) now follows by applying Lemma B.11.

For the size estimate, use Corollary B.5 below to see that $|\operatorname{tr}([U]_{i,j})| \leq d\|[U]_{i,j}\|$. The estimates for $\beta_k^{\mathcal{P}}, \lambda_k^{\mathcal{P}}$, and α_k^m in Theorem 3.4 implies the existence of $\Lambda = \Lambda(\mathrm{d}, \mathrm{curvature}) < \infty$ such that

$$\begin{aligned}\sum_{m,k=1}^{n} \frac{(\beta_k^{\mathcal{P}})^2}{\lambda_k^{\mathcal{P}}} &\left[(\alpha_k^m)^2 \operatorname{tr}([U]_{m,m}) + 2\alpha_k^m \alpha_k^{m+1} \operatorname{tr}([U]_{m,m+1})\right] \\ &\leq \frac{\Lambda}{\Delta^3} \sum_{k,m=1}^{n} \frac{1}{n} \left(|\operatorname{tr}([U]_{m,m})| + |\operatorname{tr}([U]_{m,m+1})|\right) \\ &= \frac{\Lambda}{\Delta^3} \sum_{m=1}^{n} \left(|\operatorname{tr}([U]_{m,m})| + |\operatorname{tr}([U]_{m,m+1})|\right).\end{aligned}$$

Combining these facts gives the necessary size estimate. □

Corollary 5.13. *Let $\mathcal{U}_{\mathcal{P}}$ and $\mathcal{E}_{\mathcal{P}}$ be as in Eqs. (5.38) and (5.39) respectively. Then there exists some $C = C(\mathrm{d}, curvature) < \infty$ such that,*

$$|\operatorname{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2} \mathcal{U}_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}^{-1/2})| \leq C \sum_{i=1}^{n} \|\Delta_i b\|^2 \tag{5.48}$$

and

$$|\operatorname{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2} \mathcal{E}_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}^{-1/2})| \leq C \sum_{i=1}^{n} \|\Delta_i b\|^3. \tag{5.49}$$

Proof. Both $\mathcal{U}_{\mathcal{P}}$ and $\mathcal{E}_{\mathcal{P}}$ are symmetric, so we can therefore apply Lemma 5.12 to find some $\Lambda = \Lambda(\mathrm{d}) < \infty$ with

$$|\operatorname{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2} \mathcal{U}_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}^{-1/2}| \leq \frac{\Lambda}{\Delta^3} \sum_{i=1}^{n} \left(|\operatorname{tr}([\mathcal{U}_{\mathcal{P}}]_{i,i})| + |\operatorname{tr}([\mathcal{U}_{\mathcal{P}}]_{i,i+1})|\right)$$

and

$$|\operatorname{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2} \mathcal{E}_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}^{-1/2}| \leq \frac{\Lambda}{\Delta^3} \sum_{i=1}^{n} \left(\|[\mathcal{E}_{\mathcal{P}}]_{i,i}\| + \|[\mathcal{E}_{\mathcal{P}}]_{i,i+1}\|\right).$$

Eq. (5.39) along with the above estimate is enough to imply Eq. (5.49). To finish the proof of Eq. (5.48), it is sufficient to show that there is some $\tilde{C} = \tilde{C}(\mathrm{d}, \mathrm{curvature}) < \infty$ such that $|\operatorname{tr}([\mathcal{U}_{\mathcal{P}}]_{m,m})|$ and $|\operatorname{tr}([\mathcal{U}_{\mathcal{P}}]_{m,m+1})|$ are bounded by $\tilde{C}(\|\Delta_i b\|^2 + \|\Delta_{m+1} b\|^2)$. However, the bound on curvature along with Eqs. (3.7) and (5.38) imply just such a $\tilde{C}$. □

Proposition 5.14. *Define the event* $\mathcal{A}_{\mathcal{P}} \subset W(\mathbb{R}^d)$ *by*

$$\mathcal{A}_{\mathcal{P}} = \left\{ \sum_{i=1}^{n} \|\Delta_i b^{\mathcal{P}}\|^4 \leq \frac{1}{4} \right\}.$$

For sufficiently small ε *and* $|\mathcal{P}|$*, there is a constant* $C = C(\mathrm{d}, \textit{curvature}) < \infty$ *such that*

$$\left|\mathrm{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{E}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})\right| + \left|\Psi_2(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})\right| \leq C \sum_{i=1}^{d} \|\Delta_i b\|^3 \tag{5.50}$$

on $H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d) \cap \mathcal{A}_{\mathcal{P}}$*.*

Proof. Focusing on the second term, from Corollary B.4 and Theorem 3.4,

$$\begin{aligned} \|\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}\|_2^2 &\leq \|\mathcal{L}_{\mathcal{P}}^{-1}\|^2\|\mathcal{R}_{\mathcal{P}}\|_2^2 \\ &\leq 16\Delta^{-6}\|\mathcal{R}_{\mathcal{P}}\|_2^2. \end{aligned}$$

From Theorem 4.4,

$$\begin{aligned} \|[\mathcal{R}_{\mathcal{P}}]_{i,i}\|\,, \|[\mathcal{R}_{\mathcal{P}}]_{i,i+1}\|\,, \|[\mathcal{R}_{\mathcal{P}}]_{i+1,i}\| &\leq 2\Delta^3 \left(\cosh(2\sqrt{K_i^{\mathcal{P}}}\Delta)\cosh(6\sqrt{K_{i+1}^{\mathcal{P}}}\Delta) - 1\right) \\ &\leq 2\Delta^3 \left(\cosh(2\kappa\|\Delta_i b\|)\cosh(6\kappa\|\Delta_{i+1} b\|) - 1\right) \end{aligned}$$

where the second inequality follows from Eq. (4.5). Further, with $\|\Delta_i b\| \leq \varepsilon$ for each i,

$$\begin{aligned} &(\cosh(2\kappa\|\Delta_i b\|)\cosh(6\kappa\|\Delta_{i+1} b\|) - 1) \\ &\quad= (\cosh(2\kappa\|\Delta_i b\|) - 1)(\cosh(6\kappa\|\Delta_{i+1} b\|) - 1) + (\cosh(2\kappa\|\Delta_i b\|) - 1) \\ &\qquad + (\cosh(6\kappa\|\Delta_{i+1} b\|) - 1) \\ &\quad\leq \tilde{C}(\varepsilon,\kappa)(\|\Delta_i b\|^2 + \|\Delta_{i+1} b\|^2). \end{aligned}$$

For sufficiently small ε we can take $\tilde{C}^2 \le (384d)^{-1}$. From Lemma B.6,

$$\begin{aligned}
\|\mathcal{R}_{\mathcal{P}}\|_2^2 &\le \mathrm{d}\sum_{i=1}^{n}\left(\|[\mathcal{R}_{\mathcal{P}}]_{i,i}\|^2 + \|[\mathcal{R}_{\mathcal{P}}]_{i,i+1}\|^2 + \|[\mathcal{R}_{\mathcal{P}}]_{i+1,i}\|^2\right)\\
&\le 12\,\mathrm{d}\,\tilde{C}^2\Delta^6\sum_{i=1}^{n}\left(\|\Delta_i b\|^2 + \|\Delta_{i+1} b\|^2\right)^2\\
&\le 24\,\mathrm{d}\,\tilde{C}^2\Delta^6\sum_{i=1}^{n}\|\Delta_i b\|^4\\
&\le \frac{1}{16}\Delta^6\sum_{i=1}^{n}\|\Delta_i b\|^4.
\end{aligned}$$

Therefore $\|\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}\|_2^2 \le \sum_{i=1}^{n}\|\Delta_i b\|^4$. This further implies,

$$\begin{aligned}
\|\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}\| &\le \|\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}\|_2\\
&\le \sqrt{\sum_{i=1}^{n}\|\Delta_i b\|^4}\\
&\le \frac{1}{2}
\end{aligned}$$

on $\mathcal{A}_{\mathcal{P}}$. Hence, $1 - \|\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}\| \ge \frac{1}{2}$ and we have

$$\begin{aligned}
\left|\Psi_2(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})\right| &\le \frac{\|\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}\|_2^2}{1 - \|\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}\|}\\
&\le 2\sum_{i=1}^{n}\|\Delta_i b\|^4.
\end{aligned}$$

Eq. (5.49) in Corollary 5.13 above gives the existence of some $\Lambda < \infty$ depending only on curvature and d such that,

$$|\operatorname{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{E}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})| \le \Lambda\sum_{i=1}^{n}\|\Delta_i b\|^3.$$

Therefore on $\mathcal{A}_{\mathcal{P}}$,

$$\begin{aligned}
\left|\operatorname{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{E}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})\right| + \left|\Psi_2(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})\right| &\le \Lambda\sum_{i=1}^{n}\|\Delta_i b^{\mathcal{P}}\|^3 + 2\sum_{i=1}^{n}\|\Delta_i b\|^4\\
&\le (\Lambda + 2\varepsilon)\sum_{i=1}^{n}\|\Delta_i b^{\mathcal{P}}\|^3\\
&\le C\sum_{i=1}^{n}\|\Delta_i b^{\mathcal{P}}\|^3
\end{aligned}$$

which conclude the proof. $\square$

Lemma 5.15. *For $\beta_k^{\mathcal{P}}, \alpha_k^m$ and $\lambda_k^{\mathcal{P}}$ as in Theorem 3.4,*

$$\operatorname{tr}\left(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{U}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}\right) = \sum_{m,k=1}^{n} \frac{(\beta_k^{\mathcal{P}})^2}{\lambda_k^{\mathcal{P}}}\left[(\alpha_k^m)^2 \operatorname{tr}([\mathcal{U}_{\mathcal{P}}]_{m,m}) + 2\alpha_k^m\alpha_k^{m+1}\operatorname{tr}([\mathcal{U}_{\mathcal{P}}]_{m,m+1})\right] \tag{5.51}$$

$$= -\sum_{m,k=1}^{n} \frac{(\beta_k^{\mathcal{P}})^2}{\lambda_k^{\mathcal{P}}}\Delta^3\langle \operatorname{Ric}_{u(s_{m-1})}\Delta_m b, \Delta_m b\rangle \xi_{k,m} \tag{5.52}$$

where for $1 < m \le n$ (with $\alpha_k^{n+1} := 0$),

$$\xi_{k,m} = \frac{2}{45}\left[4(\alpha_k^m)^2 - (\alpha_k^{m-1})^2\right] + \frac{1}{180}\alpha_k^m\left[13\alpha_k^{m+1} - 7\alpha_k^{m-1}\right] \tag{5.53}$$

and for $m = 1$,

$$\xi_{k,m} = \frac{8}{45}(\alpha_k^m)^2 + \frac{13}{180}\alpha_k^m\alpha_k^{m+1}. \tag{5.54}$$

This further implies that there is some constant $C = C(\mathrm{d}, \textit{curvature}) < \infty$ such that,

$$\left|\operatorname{tr}\left(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{U}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}\right)\right| \le C\sum_{m=1}^{n}\|\Delta_m b\|^2. \tag{5.55}$$

Proof. Eq. (5.51) follows by applying Lemma 5.12. Eq. (5.55) is just a restatement of Eq. (5.48) in Corollary 5.13 above. By the definition of $\mathcal{U}_{\mathcal{P}}$ in Eq 5.38,

$$\begin{aligned}\operatorname{tr}([\mathcal{U}_{\mathcal{P}}]_{m,m}) &= \frac{2\Delta^5}{45}\operatorname{tr}(4A_m^{\mathcal{P}}(0) - A_{m+1}^{\mathcal{P}}(0)) \quad \text{and,}\\ \operatorname{tr}([\mathcal{U}_{\mathcal{P}}]_{m,m+1}) &= \frac{\Delta^5}{360}\operatorname{tr}(13A_m^{\mathcal{P}}(0) - 7A_{m+1}^{\mathcal{P}}(0)).\end{aligned}$$

This implies,

$$\begin{aligned}&(\alpha_k^m)^2\operatorname{tr}([\mathcal{U}_{\mathcal{P}}]_{m,m}) + 2\alpha_k^m\alpha_k^{m+1}\operatorname{tr}([\mathcal{U}_{\mathcal{P}}]_{m,m+1})\\ &\quad = \Delta^5\left[\operatorname{tr}(A_m^{\mathcal{P}}(0))\left(\frac{8}{45}(\alpha_k^m)^2 + \frac{13}{180}\alpha_k^m\alpha_k^{m+1}\right)\right.\\ &\qquad\left. - \operatorname{tr}(A_{m+1}^{\mathcal{P}}(0))\left(\frac{2}{45}(\alpha_k^m)^2 + \frac{7}{180}\alpha_k^m\alpha_k^{m+1}\right)\right].\end{aligned}$$

Hence,

$$\sum_{m=1}^{n} (\alpha_k^m)^2 \operatorname{tr}([\mathcal{U}_{\mathcal{P}}]_{m,m}) + 2\alpha_k^m \alpha_k^{m+1} \operatorname{tr}([\mathcal{U}_{\mathcal{P}}]_{m,m+1}) = \sum_{m=1}^{n} \Delta^5 \operatorname{tr}(A_m^{\mathcal{P}}(0))\xi_{k,m}.$$

This leads to Eq. (5.52) by Eqs. (3.4) and (3.6). □

We now have the necessary bounds to show that in the limit as $|\mathcal{P}| \to 0$, we need only concern ourselves with the $\operatorname{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{U}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})$ term.

Proposition 5.16. *Let $\mathcal{A}_{\mathcal{P}}$ be as in Lemma 5.15. Let $X_{\mathcal{P}} := \frac{1}{2}\operatorname{tr}\left(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{U}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}\right)$. Then for sufficiently small $|\mathcal{P}|$ and ε, there exists a $C < \infty$ depending only on* d *and the bound on the curvature of M such that,*

$$\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)} \left|\tilde{\rho}_{\mathcal{P}} - e^{X_{\mathcal{P}}}\right| d\mu_{S_{\mathcal{P}}} \le C|\mathcal{P}|^{1/4}. \tag{5.56}$$

Proof. Let $Y_{\mathcal{P}} := \frac{1}{2}\left(\operatorname{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{E}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2}) + \Psi_2(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})\right)$, so that from Eq. (5.43),

$$\left|\tilde{\rho}_{\mathcal{P}} - e^{X_{\mathcal{P}}}\right| = \left|e^{X_{\mathcal{P}}}(e^{Y_{\mathcal{P}}} - 1)\right|.$$

Given any $a > 0$, using Eq. (5.55) in Lemma 5.15,

$$\int_{H_{\mathcal{P}}(\mathbb{R}^d)} e^{aX_{\mathcal{P}}} d\mu_{S_{\mathcal{P}}} \le \mathbb{E}\left[e^{aC\sum_{i=1}^{n}\|\Delta_i b\|^2} d\mu\right] \to e^{daC} < \infty, \tag{5.57}$$

where the right hand side follows from Lemma A.2. By Hölder's inequality,

$$\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)\cap\mathcal{A}_{\mathcal{P}}} \left|\tilde{\rho}_{\mathcal{P}} - e^{X_{\mathcal{P}}}\right| d\mu_{S_{\mathcal{P}}} \le \left(\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)\cap\mathcal{A}_{\mathcal{P}}} e^{2X_p} d\mu_{S_{\mathcal{P}}}\right)^{1/2} \left(\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)\cap\mathcal{A}_{\mathcal{P}}} (e^{Y_p} - 1)^2 d\mu_{S_{\mathcal{P}}}\right)^{1/2}.$$

From Eq. (5.55) in Lemma 5.15 and Eq. (5.50) in Proposition 5.14, there is some

$\Lambda = \Lambda(\mathrm{d}, \mathrm{curvature}) < \infty$ such that,

$$\left(\int_{H^\varepsilon_{\mathcal{P}}(\mathbb{R}^d)\cap\mathcal{A}_{\mathcal{P}}} e^{2X_p} d\mu_{S_{\mathcal{P}}}\right)^{1/2} \left(\int_{H^\varepsilon_{\mathcal{P}}(\mathbb{R}^d)\cap\mathcal{A}_{\mathcal{P}}} (e^{Y_p}-1)^2 d\mu_{S_{\mathcal{P}}}\right)^{1/2}$$
$$\le \left(\int_{H^\varepsilon_{\mathcal{P}}(\mathbb{R}^d)\cap\mathcal{A}_{\mathcal{P}}} e^{\Lambda\sum_{i=1}^n \|\Delta_i b^{\mathcal{P}}\|^2} d\mu_{S_{\mathcal{P}}}\right)^{1/2} \left(\int_{H^\varepsilon_{\mathcal{P}}(\mathbb{R}^d)\cap\mathcal{A}_{\mathcal{P}}} (e^{\Lambda\sum_{i=1}^n \|\Delta_i b^{\mathcal{P}}\|^3}-1)^2 d\mu_{S_{\mathcal{P}}}\right)^{1/2}$$
$$\le \left(\int_{H_{\mathcal{P}}(\mathbb{R}^d)} e^{\Lambda\sum_{i=1}^n \|\Delta_i b^{\mathcal{P}}\|^2} d\mu_{S_{\mathcal{P}}}\right)^{1/2} \left(\int_{H^\varepsilon_{\mathcal{P}}(\mathbb{R}^d)} (e^{\Lambda\sum_{i=1}^n \|\Delta_i b^{\mathcal{P}}\|^3}-1)^2 d\mu_{S_{\mathcal{P}}}\right)^{1/2}$$
$$= \mathbb{E}\left[e^{\Lambda\sum_{i=1}^n \|\Delta_i b\|^2} d\mu\right]^{1/2} \left(\int_{H^\varepsilon_{\mathcal{P}}(\mathbb{R}^d)} (e^{\Lambda\sum_{i=1}^n \|\Delta_i b^{\mathcal{P}}\|^3}-1)^2 d\mu_{S_{\mathcal{P}}}\right)^{1/2}.$$
$$\le C|\mathcal{P}|^{1/4}$$

Here the last inequality follows from Corollary 5.5. On the compliment of $\mathcal{A}_{\mathcal{P}}$,

$$\int_{\mathcal{A}^c_{\mathcal{P}}} \left|\tilde{\rho}_{\mathcal{P}} - e^{X_{\mathcal{P}}}\right| d\mu_{S_{\mathcal{P}}} \le \int_{\mathcal{A}^c_{\mathcal{P}}} \left(|\tilde{\rho}_{\mathcal{P}}| + \left|e^{X_{\mathcal{P}}}\right|\right) d\mu_{S_{\mathcal{P}}}$$
$$\le \left(\mathbb{E}\left[|\tilde{\rho}_{\mathcal{P}}|^2\right]^{1/2} + \mathbb{E}\left[e^{2X_{\mathcal{P}}}\right]^{1/2}\right) \mathbb{E}\left[1_{\left\{\sum_{i=1}^n \|\Delta_i b\|^4 > 1/4\right\}}\right]^{1/2}.$$

Arguing as above and using Theorem 4.8 ensures that $\mathbb{E}\left[|\tilde{\rho}_{\mathcal{P}}|^2\right]^{1/2} + \mathbb{E}\left[e^{2X_{\mathcal{P}}}\right]^{1/2}$ stays bounded for sufficiently small $|\mathcal{P}|$. Noticing that

$$\left\{\sum_{i=1}^n \|\Delta_i b\|^4 > \frac{1}{4}\right\} \subset \bigcup_{i=1}^n \left\{\|\Delta_i b\|^4 > \frac{1}{4n}\right\}$$
$$\stackrel{d}{=} \cup_{i=1}^n \left\{\|Z_i\| > \left(\frac{n}{4}\right)^{1/4}\right\},$$

where $\{Z_i\}$ are i.i.d. with $Z_i \stackrel{d}{=} b_1$. Lemma A.1 gives for sufficiently small $|\mathcal{P}|$,

$$\mathbb{E}\left[1_{\left\{\sum_{i=1}^n \|\Delta_i b\|^4 > 1/4\right\}}\right] \le \sum_{i=1}^n \frac{\tilde{C}}{\sqrt{n}} e^{-\frac{\sqrt{n}}{16}}$$
$$= \sqrt{n}\tilde{C} e^{-\frac{\sqrt{n}}{16}}$$
$$= \frac{\tilde{C}}{\sqrt{|\mathcal{P}|}} e^{-\frac{1}{16\sqrt{|\mathcal{P}|}}}$$
$$\le C\sqrt{|\mathcal{P}|}.$$

Combining these facts imply the claim. □

An important fact that will be used is that from the definition of α_k^m in Theorem 3.4, we have for $1 \leq m \leq n-1$

$$\begin{aligned}(\alpha_k^m)^2 &= \sin^2(m\theta_k^{\mathcal{P}}) \\ &= \frac{1}{2}\left(1 - \cos(2m\theta_k^{\mathcal{P}})\right)\end{aligned}$$

and if also $m+1 < n$,

$$\begin{aligned}\alpha_k^m \alpha_k^{m+1} &= -\frac{1}{4}(e^{im\theta_k^{\mathcal{P}}} - e^{-im\theta_k^{\mathcal{P}}})(e^{i(m+1)\theta_k^{\mathcal{P}}} - e^{-im\theta_k^{\mathcal{P}}}) \\ &= \frac{1}{4}\left(e^{i\theta_k^{\mathcal{P}}} + e^{-i\theta_k^{\mathcal{P}}} - (e^{i(2m+1)\theta_k^{\mathcal{P}}} + e^{-i(2m+1)\theta_k^{\mathcal{P}}})\right) \\ &= \frac{1}{2}\left(\cos(\theta_k^{\mathcal{P}}) - \cos((2m+1)\theta_k^{\mathcal{P}})\right).\end{aligned}$$

Therefore, for $1 < m < n-1$ and $\xi_{k,m}$ defined in Lemma 5.15,

$$\begin{aligned}\xi_{k,m} &= \frac{2}{45}\left[4(\alpha_k^m)^2 - (\alpha_k^{m-1})^2\right] + \frac{1}{180}\alpha_k^m\left[13\alpha_k^{m+1} - 7\alpha_k^{m-1}\right] \\ &= \frac{2}{45}\left[2\{1 - \cos(2m\theta_k^{\mathcal{P}})\} - \frac{1}{2}\{1 - \cos(2(m-1)\theta_k^{\mathcal{P}})\}\right] \\ &\quad + \frac{1}{180}\left[\frac{13}{2}\{\cos(\theta_k^{\mathcal{P}}) - \cos((2m+1)\theta_k^{\mathcal{P}})\} - \frac{7}{2}\{\cos(\theta_k^{\mathcal{P}}) - \cos((2m-1)\theta_k^{\mathcal{P}})\}\right] \\ &= \frac{1}{15} + \frac{1}{60}\cos(\theta_k^{\mathcal{P}}) + \mathrm{rem}_{k,m} \qquad (5.58)\end{aligned}$$

where the remainder term is given by,

$$\begin{aligned}\mathrm{rem}_{k,m} &= -\frac{13}{360}\cos((2m+1)\theta_k^{\mathcal{P}}) - \frac{4}{45}\cos(2m\theta_k^{\mathcal{P}}) + \frac{7}{360}\cos((2m-1)\theta_k^{\mathcal{P}}) \\ &\quad + \frac{1}{45}\cos(2(m-1)\theta_k^{\mathcal{P}}) \qquad (5.59) \\ &= \cos(2m\theta_k^{\mathcal{P}})\left\{\frac{1}{45}(\cos(2\theta_k^{\mathcal{P}}) - 4) - \frac{1}{60}\cos(\theta_k^{\mathcal{P}})\right\} \\ &\quad + \sin(2m\theta_k^{\mathcal{P}})\left\{\frac{1}{45}\sin(2\theta_k^{\mathcal{P}}) - \frac{1}{60}\sin(\theta_k^{\mathcal{P}})\right\}. \qquad (5.60)\end{aligned}$$

It's useful to write $\xi_{k,m}$ in this fashion since what follows shows that as $|\mathcal{P}| \to 0$, the terms involving $\mathrm{rem}_{k,m}$ vanish. Intuitively, one might hope for this to be the case since as $n \to \infty$ and we sum over something of the form $f(\theta_k^{\mathcal{P}})e^{2im\theta_k^{\mathcal{P}}}$, it begins to look like an application of the Riemann-Lebesgue Lemma. We formalize this presently.

Proposition 5.17. *Take $\delta > 0$ and define $\partial_\delta = \partial_\delta(n) := \{k \in \mathbb{N} : \frac{\pi k}{n+1} \leq \delta$ or $\frac{\pi k}{n+1} \geq \pi - \delta\}$ and $\Omega_\delta = \Omega_\delta(n) := \{k \in \mathbb{N} : \delta < \frac{\pi k}{n+1} < \pi - \delta\}$. Let $f : [0, \pi] \to \mathbb{R}$ be Lipshitz and bounded where we set $\Lambda < \infty$ with $\sup_x |f(x)| \leq \Lambda$ and $|f(x) - f(y)| \leq \Lambda|x - y|$ for each $x, y \in [0, \pi]$. Then there exists a constant $C = C(\Lambda) < \infty$ such that if $j \in \Omega_\delta$,*

$$\left| \sum_{k=1}^{n} (\beta_k^{\mathcal{P}})^2 f(\theta_k^{\mathcal{P}}) e^{i2j\theta_k^{\mathcal{P}}} \right| \leq \frac{C}{n \sin(\delta)}. \tag{5.61}$$

And for any $j = 1, 2, ..., n$,

$$\left| \sum_{k=1}^{n} (\beta_k^{\mathcal{P}})^2 f(\theta_k^{\mathcal{P}}) e^{i2j\theta_k^{\mathcal{P}}} \right| \leq C. \tag{5.62}$$

Proof. For notational convenience, define $\zeta_k := \theta_k - \frac{\pi k}{n+1}$. Using the notation and results from Theorem 3.4, we start with the following estimates.

$$|(\beta_{k+1}^{\mathcal{P}})^2 - (\beta_k^{\mathcal{P}})^2| = O(\frac{1}{n^2}),$$

$$|\theta_{k+1}^{\mathcal{P}} - \theta_k^{\mathcal{P}}| = \frac{\pi}{n+1} |1 + r_{k+1} - r_k| = O(\frac{1}{n}),$$

and

$$|\zeta_{k+1} - \zeta_k| = \frac{\pi}{n+1} |r_{k+1} - r_k| = O(\frac{1}{n}).$$

Therefore, there exists a $c = c(\Lambda) < \infty$ such that

$$\begin{aligned}
&\left|(\beta_{k+1}^{\mathcal{P}})^2 f(\theta_{k+1}^{\mathcal{P}}) e^{i2j\zeta_{k+1}} - (\beta_k^{\mathcal{P}})^2 f(\theta_k^{\mathcal{P}}) e^{i2j\zeta_k}\right| \\
&\quad \leq \Lambda \left|(\beta_{k+1}^{\mathcal{P}})^2 - (\beta_k^{\mathcal{P}})^2\right| + O(\frac{1}{n}) \left[\left|f(\theta_{k+1}^{\mathcal{P}}) - f(\theta_k^{\mathcal{P}})\right| + \Lambda \left|e^{i2j\zeta_{k+1}} - e^{i2j\zeta_k}\right|\right] \\
&\quad \leq \frac{c}{n^2}.
\end{aligned}$$

Now, define the partial sum $S_m := \sum_{k=1}^{m} e^{i2j\frac{\pi k}{n+1}}$ and $S_0 := 0$. Using the fact that S_m is a geometric series,

$$\begin{aligned}
S_m &= \frac{1 - e^{i2j\frac{\pi(m+1)}{n+1}}}{1 - e^{i2j\frac{\pi}{n+1}}} - 1 \\
&= \frac{1}{2i} \frac{e^{i2j\frac{\pi(m+1/2)}{n+1}} - e^{ij\frac{\pi}{n+1}}}{\sin\left(\frac{\pi j}{n+1}\right)} - 1
\end{aligned}$$

so that

$$|S_m| \le \frac{1}{\sin(\delta)} + 1.$$

Applying summation by parts,

$$\begin{aligned}
\sum_{k=1}^{n} (\beta_k^{\mathcal{P}})^2 f(\theta_k^{\mathcal{P}}) e^{i2j\theta_k^{\mathcal{P}}} &= (\beta_n^{\mathcal{P}})^2 f(\theta_n^{\mathcal{P}}) e^{i2j\theta_n^{\mathcal{P}}} + \sum_{k=1}^{n-1} (\beta_k^{\mathcal{P}})^2 f(\theta_k^{\mathcal{P}}) e^{i2j\zeta_k} e^{i2j\frac{\pi k}{n+1}} \\
&= (\beta_n^{\mathcal{P}})^2 f(\theta_n^{\mathcal{P}}) e^{i2j\theta_n^{\mathcal{P}}} + \sum_{k=1}^{n-1} (\beta_k^{\mathcal{P}})^2 f(\theta_k^{\mathcal{P}}) e^{i2j\zeta_k} (S_k - S_{k-1}) \\
&= (\beta_n^{\mathcal{P}})^2 f(\theta_n^{\mathcal{P}}) e^{i2j\theta_n^{\mathcal{P}}} (1 + e^{i2j\frac{\pi n}{n+1}}) - (\beta_1^{\mathcal{P}})^2 f(\theta_1^{\mathcal{P}}) e^{i2j\theta_1^{\mathcal{P}}} \\
&\quad + \sum_{k=1}^{n-1} \left((\beta_{k+1}^{\mathcal{P}})^2 f(\theta_{k+1}^{\mathcal{P}}) e^{i2j\zeta_{k+1}} - (\beta_k^{\mathcal{P}})^2 f(\theta_k^{\mathcal{P}}) e^{i2j\zeta_k} \right) S_k
\end{aligned}$$

and now using the above estimates and noting from Theorem 3.4, $(\beta_n^{\mathcal{P}})^2 < O(1/n)$,

$$\begin{aligned}
\left| \sum_{k=1}^{n} \beta_k^2 f(\theta_k) e^{i2j\theta_k} \right| &\le O(\frac{\Lambda}{n}) + \left(1 + \frac{1}{\sin(\delta)}\right) \sum_{k=1}^{n-1} \frac{c}{n^2} \\
&\le O(\frac{\Lambda}{n}) + \left(1 + \frac{1}{\sin \delta}\right) \frac{c}{n} \\
&\le \frac{C}{n \sin(\delta)}.
\end{aligned}$$

The second claim is nearly immediate since from Theorem 3.4, $(\beta_k^{\mathcal{P}})^2 = O(1/n)$ and hence

$$\begin{aligned}
\left| \sum_{k=1}^{n} \beta_k^2 f(\theta_k) e^{i2j\theta_k} \right| &\le \sum_{k=1}^{n} \Lambda O(\frac{1}{n}) \\
&\le C.
\end{aligned}$$

□

Corollary 5.18. *Let $\delta > 0$. Using the notation from Proposition 5.17, there exists a $C = C(curvature) < \infty$ such that,*

$$\left| \sum_{m,k=1}^{n} \frac{(\beta_k^{\mathcal{P}})^2}{\lambda_k^{\mathcal{P}}} \Delta^3 \langle \mathrm{Ric}_{u(s_{m-1})} \Delta_m b, \Delta_m b \rangle \operatorname{rem}_{k,m} \right| \le \frac{C}{n \sin(\delta)} \sum_{m=1}^{n} \|\Delta_m b\|^2 + C \sum_{m \in \partial_\delta} \|\Delta_m b\|^2. \tag{5.63}$$

Proof. Using Proposition 5.17 and Eq. (5.60),

$$\begin{aligned}
&\left|\sum_{m=1}^{n}\sum_{k=1}^{n}\frac{(\beta_k^{\mathcal{P}})^2}{\lambda_k^{\mathcal{P}}}\Delta^3\langle \mathrm{Ric}_{u(s_{m-1})}\,\Delta_m b,\Delta_m b\rangle\,\mathrm{rem}_{k,m}\right| \\
&\quad\le \sum_{m=1}^{n}\left|\langle \mathrm{Ric}_{u(s_{m-1})}\,\Delta_m b,\Delta_m b\rangle\sum_{k=1}^{n}\frac{(\beta_k^{\mathcal{P}})^2}{\lambda_k^{\mathcal{P}}}\Delta^3\,\mathrm{rem}_{k,m}\right| \\
&\quad\le \frac{C}{n\sin(\delta)}\sum_{m\in\Omega_\delta}\|\Delta_m b\|^2 + C\sum_{m\in\partial_\delta}\|\Delta_m b\|^2 \\
&\quad\le \frac{C}{n\sin(\delta)}\sum_{m=1}^{n}\|\Delta_m b\|^2 + C\sum_{m\in\partial_\delta}\|\Delta_m b\|^2.
\end{aligned}$$

□

From here we are able to show that what actually contributes in the limit allows us to ignore the $\mathrm{rem}_{k,m}$ term. Moreover, we will see that those boundary cases of $\xi_{k,m}$ for $m = 1, n-1, n$ are also negligible and allow us to further simplify our expression when passing to the limit.

Proposition 5.19. *Following the notation in Proposition 5.17, for sufficiently small* $|\mathcal{P}|$,

$$\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)} \left|e^{X_{\mathcal{P}}} - e^{y_{\mathcal{P}}}\right| d\mu_{S_{\mathcal{P}}} \le C n^{-1/4} = C|\mathcal{P}|^{1/4} \tag{5.64}$$

where $C = C(\mathrm{d}, \textit{curvature}) < \infty$,

$$y_{\mathcal{P}} := -\frac{1}{2}\sum_{m=2}^{n-2}\sum_{k=1}^{n}\frac{(\beta_k^{\mathcal{P}})^2}{\lambda_k^{\mathcal{P}}}\Delta^3\left(\frac{1}{15}+\frac{1}{60}\cos(\theta_k^{\mathcal{P}})\right)\langle \mathrm{Ric}_{u(s_{m-1})}\,\Delta_m b,\Delta_m b\rangle$$

and as before $X_{\mathcal{P}} := \frac{1}{2}\operatorname{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{U}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})$.

Proof. We first write,

$$\begin{aligned}
|e^{X_{\mathcal{P}}} - e^{-y_{\mathcal{P}}}| &= e^{y_{\mathcal{P}}}|e^{X_{\mathcal{P}}-y_{\mathcal{P}}} - 1| \\
&\le e^{y_{\mathcal{P}}}\left(e^{\partial X_{\mathcal{P}}}|e^{Z_{\mathcal{P}}} - 1| + |e^{\partial X_{\mathcal{P}}} - 1|\right) \\
&= e^{\tilde{y}_{\mathcal{P}}}|e^{Z_{\mathcal{P}}} - 1| + e^{y_{\mathcal{P}}}|e^{\partial X_{\mathcal{P}}} - 1|
\end{aligned}$$

where,

$$\partial X_{\mathcal{P}} := \sum_{m\in\{1,n-1,n\}} \sum_{k=1}^{n} \frac{(\beta_k^{\mathcal{P}})^2}{\lambda_k^{\mathcal{P}}} \Delta^3 \langle \mathrm{Ric}_{u(s_{m-1})}\, \Delta_m b^{\mathcal{P}}, \Delta_m b^{\mathcal{P}} \rangle \xi_{k,m}$$

$$\tilde{y}_{\mathcal{P}} := y_{\mathcal{P}} + \partial X_{\mathcal{P}},$$

and

$$\begin{aligned} Z_{\mathcal{P}} &:= (X_{\mathcal{P}} - \partial X_{\mathcal{P}}) - y_{\mathcal{P}} \\ &= -\sum_{m=2}^{n-2} \sum_{k=1}^{n} \frac{(\beta_k^{\mathcal{P}})^2}{\lambda_k^{\mathcal{P}}} \Delta^3 \langle \mathrm{Ric}_{u(s_{m-1})}\, \Delta_m b^{\mathcal{P}}, \Delta_m b^{\mathcal{P}} \rangle \,\mathrm{rem}_{k,m} \,. \end{aligned}$$

Note that there is some $\Lambda = \Lambda(\text{curvature})$ so that we have the following size estimates,

$$\begin{aligned} |\partial X_{\mathcal{P}}| &\le \sum_{k=1}^{n} \frac{\Lambda}{n} \left(\|\Delta_1 b^{\mathcal{P}}\|^2 + \|\Delta_{n-1} b^{\mathcal{P}}\|^2 + \|\Delta_n b^{\mathcal{P}}\|^2 \right) \\ &\le \Lambda \left(\|\Delta_1 b^{\mathcal{P}}\|^2 + \|\Delta_{n-1} b^{\mathcal{P}}\|^2 + \|\Delta_n b^{\mathcal{P}}\|^2 \right), \\ |y_{\mathcal{P}}|, |\tilde{y}_{\mathcal{P}}| &\le \Lambda \sum_{m=1}^{n} \|\Delta_m b^{\mathcal{P}}\|^2, \end{aligned}$$

and from Corollary 5.18, with $\delta = 1/\sqrt{n}$,

$$|Z_{\mathcal{P}}| \le \frac{\Lambda}{\sqrt{n}} \sum_{m=1}^{n} \|\Delta_m b^{\mathcal{P}}\|^2 + \Lambda \sum_{m\in\partial_\delta} \|\Delta_m b^{\mathcal{P}}\|^2.$$

Using Hölder's inequality with the above estimates and once again making use of the fact that $(e^a - 1)^2 \le 2|a|e^{2|a|}$,

$$\begin{aligned} \int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)} |e^{X_{\mathcal{P}}} - e^{y_{\mathcal{P}}}| d\mu_{S_{\mathcal{P}}} \le& \left[\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)} e^{2\tilde{y}} d\mu_{S_{\mathcal{P}}} \right]^{1/2} \left[\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)} (e^{Z} - 1)^2 d\mu_{S_{\mathcal{P}}} \right]^{1/2} \\ &+ \left[\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)} e^{2y} d\mu_{S_{\mathcal{P}}} \right]^{1/2} \left[\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)} (e^{|\partial X|} - 1)^2 d\mu_{S_{\mathcal{P}}} \right]^{1/2} \end{aligned}$$

where we have simplified notation by dropping the subscript $\mathcal{P}$ from $y, \tilde{y}, X$, and Z. From Corollary 5.6, for sufficiently small ε and $|\mathcal{P}|$ here there is some $\tilde{\Lambda} =$

$\tilde{\Lambda}(\mathrm{d}, \text{curvature})$ such that,

$$\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} e^{2y} d\mu_{S_{\mathcal{P}}} \leq \tilde{\Lambda},$$
$$\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} e^{2\tilde{y}} d\mu_{S_{\mathcal{P}}} \leq \tilde{\Lambda},$$

and

$$\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} (e^{|\partial X|} - 1)^2 d\mu_{S_{\mathcal{P}}} \leq \tilde{\Lambda}|\mathcal{P}|.$$

For the term involving Z,

$$\begin{aligned}\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} (e^{Z} - 1)^2 d\mu_{S_{\mathcal{P}}} &\leq 2\mathbb{E}\left[|Z|e^{2|Z|}\right] \\ &\leq \mathbb{E}\left[\left(\frac{\Lambda}{\sqrt{n}}\sum_{m=1}^{n}\|\Delta_m b\|^2 + \Lambda\sum_{m\in\partial_\delta}\|\Delta_m b\|^2\right)\exp\left\{\Lambda\sum_{m=1}^{n}\|\Delta_m b\|^2\right\}\right]^{1/2}\end{aligned}$$

Using Lemma 5.4,

$$\mathbb{E}\left[\frac{\Lambda}{\sqrt{n}}\sum_{m=1}^{n}\|\Delta_m b\|^2 e^{2\Lambda\sum_{m=1}^{n}\|\Delta_m b\|^2}\right] \leq \frac{\tilde{\Lambda}}{\sqrt{n}} = \tilde{\Lambda}|\mathcal{P}|^{1/2},$$

and

$$\mathbb{E}\left[\tilde{C}\sum_{m\in\partial_\delta}\|\Delta_m b\|^2 e^{2\tilde{C}\sum_{m=1}^{n}\|\Delta_m b\|^2}\right] \leq \tilde{\Lambda}\delta = \frac{\tilde{\Lambda}}{\sqrt{n}} = \tilde{\Lambda}|\mathcal{P}|^{1/2}$$

Here we used $\mathbb{E}$ as integration on $W(\mathbb{R}^d)$ against Wiener measure. Hence $2\mathbb{E}\left[|Z|e^{2|Z|}\right] \leq 2\tilde{\Lambda}|\mathcal{P}|^{-1/2} = 2\tilde{\Lambda}|\mathcal{P}|^{1/2}$. Therefore,

$$\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} |e^{X_{\mathcal{P}}} - e^{y_{\mathcal{P}}}| d\mu_{S_{\mathcal{P}}} \leq Cn^{-1/4} = C|\mathcal{P}|^{1/4}.$$

which is what we wanted to show. □

In Proposition 5.19 we defined the process $y_{\mathcal{P}}$. Rearranging the expression and using the definition of $\lambda_k^{\mathcal{P}}$ from Theorem 3.4,

$$y_{\mathcal{P}} = -\frac{1}{40}\left\{\sum_{k=1}^{n}(\beta_k^{\mathcal{P}})^2\frac{4+\cos(\theta_k^{\mathcal{P}})}{2+\cos(\theta_k^{\mathcal{P}})}\right\}\left\{\sum_{m=2}^{n-2}\langle \mathrm{Ric}_{u(s_{m-1})}\,\Delta_m b^{\mathcal{P}}, \Delta_m b^{\mathcal{P}}\rangle\right\}. \tag{5.65}$$

$$=: -\tau_{\mathcal{P}}\sum_{m=2}^{n-2}\langle \mathrm{Ric}_{u(s_{m-1})}\,\Delta_m b^{\mathcal{P}}, \Delta_m b^{\mathcal{P}}\rangle. \tag{5.66}$$

Proposition 5.20 below shows that

$$\tau_{\mathcal{P}} \to \frac{1}{20}\int_0^1 \frac{4+\cos(\pi x)}{2+\cos(\pi x)}dx = \frac{2+\sqrt{3}}{20\sqrt{3}} = \tau_G. \tag{5.67}$$

Proposition 5.20. *Suppose that* $0 \le r < s \le 1$ *and* $f \in C^1([0,\pi] \to \mathbb{R})$. *Then,*

$$\lim_{n\to\infty} \sum_{\{k:r<k/n<s\}} (\beta_k^{\mathcal{P}})^2 f(\theta_k^{\mathcal{P}}) = 2\int_r^s f(\pi t)dt. \tag{5.68}$$

Proof. Notice that $\sum \frac{2}{n} f\left(\frac{\pi k}{n}\right)$ is the Riemann sum approximation to $2\int f(\pi t)dt$. Hence it suffices to show that $|\sum \left\{(\beta_k^{\mathcal{P}})^2 f(\theta_k^{\mathcal{P}}) - \frac{2}{n} f\left(\frac{\pi k}{n}\right)\right\}| \to 0$, which will be done by showing that the summand $(\beta_k^{\mathcal{P}})^2 f(\theta_k^{\mathcal{P}}) - \frac{2}{n} f\left(\frac{\pi k}{n}\right)$ is $O(1/n^2)$. To this end we write,

$$\left|(\beta_k^{\mathcal{P}})^2 f(\theta_k^{\mathcal{P}}) - \frac{2}{n} f\left(\frac{\pi k}{n}\right)\right| \le |\beta_k^{\mathcal{P}}|^2 \left|f(\theta_k^{\mathcal{P}}) - f\left(\frac{\pi k}{n}\right)\right| + \left|f\left(\frac{\pi k}{n}\right)\right| \left|(\beta_k^{\mathcal{P}})^2 - \frac{2}{n}\right|.$$

From Theorem 3.4,

$$\begin{aligned} |\beta_k^{\mathcal{P}}|^2 &= O\left(\frac{1}{n}\right), \\ \left|\theta_k^{\mathcal{P}} - \frac{\pi k}{n}\right| &= \left|\frac{\pi r_k n - \pi k}{n(n+1)}\right| = O\left(\frac{1}{n}\right), \\ \left|(\beta_k^{\mathcal{P}})^2 - \frac{2}{n}\right| &= \frac{2}{n}\left(\frac{\epsilon_k}{1-\epsilon_k}\right) = O\left(\frac{1}{n^2}\right). \end{aligned}$$

Here $f \in C^1([0,\pi] \to \mathbb{R})$, $|f(\theta_k^{\mathcal{P}}) - f(\pi k/n)| = O(|\theta_k^{\mathcal{P}} - \pi k/n|) = O(1/n)$ and $|f(\pi k/n)| = O(1)$. Therefore, $(\beta_k^{\mathcal{P}})^2 f(\theta_k^{\mathcal{P}}) - \frac{2}{n} f\left(\frac{\pi k}{n}\right) = O(1/n^2)$. □

Proposition 5.21. *Defining* $\mathcal{R}_{\mathcal{P}}$ *as in Eq. (2.24),* $y_{\mathcal{P}}$ *and* $\tau_{\mathcal{P}}$ *as in Eqs. (5.65) and (5.66), and* τ_G *as in Eq. (5.67), there is a constant* $C = C(\mathrm{d}, \mathit{curvature}) < \infty$ *such that,*

$$\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)} |e^{y_{\mathcal{P}}} - e^{-\tau_G \mathcal{R}_{\mathcal{P}}}| d\mu_{S_{\mathcal{P}}} \le C(\sqrt{|\tau_{\mathcal{P}} - \tau_G|} + \sqrt{|\mathcal{P}|}). \tag{5.69}$$

Proof. Start with

$$\tau_G \mathcal{R}_{\mathcal{P}} + y_{\mathcal{P}} = (\tau_G - \tau_{\mathcal{P}})\mathcal{R}_{\mathcal{P}} + \tau_{\mathcal{P}} \partial \mathcal{R}_{\mathcal{P}}$$

where

$$\partial\mathcal{R}_{\mathcal{P}} := \langle \mathrm{Ric}_{u(s_0)}\, \Delta_1 b^{\mathcal{P}}, \Delta_1 b^{\mathcal{P}}\rangle + \langle \mathrm{Ric}_{u(s_{n-2})}\, \Delta_{n-1} b^{\mathcal{P}}, \Delta_{n-1} b^{\mathcal{P}}\rangle \\ + \langle \mathrm{Ric}_{u(s_{n-1})}\, \Delta_n b^{\mathcal{P}}, \Delta_n b^{\mathcal{P}}\rangle.$$

Hence,

$$\begin{aligned} |e^{y_{\mathcal{P}}} - e^{-\tau_G \mathcal{R}_{\mathcal{P}}}| &= e^{y_{\mathcal{P}}} |e^{-(\tau_G \mathcal{R}_{\mathcal{P}} + y_{\mathcal{P}})} - 1| \\ &= e^{y_{\mathcal{P}}} |e^{-(\tau_G - \tau_{\mathcal{P}})\mathcal{R}_{\mathcal{P}}} e^{-\tau_{\mathcal{P}} \partial \mathcal{R}_{\mathcal{P}}} - 1| \\ &\leq e^{y_{\mathcal{P}} - \tau_{\mathcal{P}} \partial\mathcal{R}_{\mathcal{P}}} |e^{-(\tau_G - \tau_{\mathcal{P}})\mathcal{R}_{\mathcal{P}}} - 1| + e^{y_{\mathcal{P}}} |e^{-\tau_{\mathcal{P}} \partial\mathcal{R}_{\mathcal{P}}} - 1|. \end{aligned}$$

As usual, we find some constant $\Lambda = \Lambda(\mathrm{d}, \mathrm{curvature}) < \infty$ such that,

$$|y_{\mathcal{P}}|, |y_{\mathcal{P}} - \tau_{\mathcal{P}} \mathcal{R}_{\mathcal{P}}| \leq \Lambda \sum_{i=1}^{n} \|\Delta_i b\|^2,$$

and

$$|\partial\mathcal{R}_{\mathcal{P}}| \leq \Lambda(\|\Delta_1 b^{\mathcal{P}}\|^2 + \|\Delta_{n-1} b^{\mathcal{P}}\|^2 + \|\Delta_n b^{\mathcal{P}}\|^2).$$

Using Hölder's inequality, Lemma 5.4, Corollary 5.6, and the fact that Proposition 5.20 implies that $\tau_{\mathcal{P}}$ remains bounded as $|\mathcal{P}| \to 0$,

$$\begin{aligned} &\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} |e^{y_{\mathcal{P}}} - e^{-\tau_G \mathcal{R}_{\mathcal{P}}}| d\mu_{S_{\mathcal{P}}} \\ &\leq \left[\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} e^{2y_{\mathcal{P}} - 2\tau_{\mathcal{P}}\partial\mathcal{R}_{\mathcal{P}}} d\mu_{S_{\mathcal{P}}}\right]^{1/2} \left[2|\tau_G - \tau_{\mathcal{P}}| \int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} |\mathcal{R}_{\mathcal{P}}|\, |e^{2|\tau_G - \tau_{\mathcal{P}}||\mathcal{R}_{\mathcal{P}}|} d\mu_{S_{\mathcal{P}}}\right]^{1/2} \\ &\quad + \left[\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} e^{2y_{\mathcal{P}}} d\mu_{S_{\mathcal{P}}}\right]^{1/2} \left[\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} (e^{2|\tau_{\mathcal{P}}||\partial\mathcal{R}_{\mathcal{P}}|} - 1) d\mu_{S_{\mathcal{P}}}\right]^{1/2} \\ &\leq C\left(\sqrt{|\tau_{\mathcal{P}} - \tau_G|} + \sqrt{|\mathcal{P}|}\right). \end{aligned}$$

Proposition 5.20 also implies that $|\tau_{\mathcal{P}} - \tau_G| \to 0$ as $\mathcal{P} \to 0$, which is why we can choose C independent of $\tau_{\mathcal{P}}$ for sufficiently small $\mathcal{P}$ (as noted in the remark following Lemma 5.4). □

We are now ready to piece together the proof of the main theorem.

Proof of Theorem 1.4. Let Λ be the bound on f. In what follows we will liberally use the triangle inequality and Hölder's inequality without explicit mention. Defining $\mathcal{R}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{P}}$ as in Eqs. (2.24) and (2.25) resp.,

$$\begin{aligned}
&\left|\int_{H_{\mathcal{P}}(M)} f(\sigma)d\nu_{G_{\mathcal{P}}}(\sigma) - \int_{W(M)} f(\sigma)e^{-\tau_G \int_0^1 \mathrm{Scal}(\sigma(s))ds}d\nu(\sigma)\right| \\
&\leq \left|\int_{H_{\mathcal{P}}(M)} f(\sigma)d\nu_{G_{\mathcal{P}}}(\sigma) - \int_{H^{\varepsilon}_{\mathcal{P}}(M)} f(\sigma)d\nu_{G_{\mathcal{P}}}(\sigma)\right| \\
&\quad + \left|\int_{H^{\varepsilon}_{\mathcal{P}}(M)} f(\sigma)d\nu_{G_{\mathcal{P}}}(\sigma) - \int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} f(\omega)e^{-\tau_G \mathcal{R}_{\mathcal{P}}(\omega)}d\mu_{S_{\mathcal{P}}}(\omega)\right| \\
&\quad + \left|\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} f(\omega)e^{-\tau_G \mathcal{R}_{\mathcal{P}}(\omega)}d\mu_{S_{\mathcal{P}}}(\omega) - \int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} f(\omega)e^{-\tau_G \mathcal{S}_{\mathcal{P}}(\omega)}d\mu_{S_{\mathcal{P}}}(\omega)\right| \\
&\quad + \left|\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} f(\omega)e^{-\tau_G \mathcal{S}_{\mathcal{P}}(\omega)}d\mu_{S_{\mathcal{P}}}(\omega) - \int_{H_{\mathcal{P}}(M)} f(\sigma)e^{-\tau_G \int_0^1 \mathrm{Scal}(\sigma(s))ds}d\nu_{S_{\mathcal{P}}}(\sigma)\right| \\
&\quad + \left|\int_{H_{\mathcal{P}}(M)} f(\sigma)e^{-\tau_G \int_0^1 \mathrm{Scal}(\sigma(s))ds}d\nu_{S_{\mathcal{P}}}(\sigma) - \int_{W(M)} f(\sigma)e^{-\tau_G \int_0^1 \mathrm{Scal}(\sigma(s))ds}d\nu(\sigma)\right|.
\end{aligned}$$

We have completed most the work to show that each of these differences approaches 0 as $|\mathcal{P}| \to 0$. Indeed, fix $\varepsilon > 0$. Corollary 5.3 gives,

$$\left|\int_{H_{\mathcal{P}}(M)} f(\sigma)d\nu_{G_{\mathcal{P}}}(\sigma) - \int_{H^{\varepsilon}_{\mathcal{P}}(M)} f(\sigma)d\nu_{G_{\mathcal{P}}}(\sigma)\right| \leq \frac{C}{\varepsilon^2}e^{-\frac{\varepsilon^2}{4|\mathcal{P}|}}.$$

The definition of $\tilde{\rho}_{\mathcal{P}}$ along with Propositions 5.16, 5.19, and 5.21 implies,

$$\begin{aligned}
&\left|\int_{H^{\varepsilon}_{\mathcal{P}}(M)} f(\sigma)d\nu_{G_{\mathcal{P}}}(\sigma) - \int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} f(\omega)e^{-\tau_G \mathcal{R}_{\mathcal{P}}(\omega)}d\mu_{S_{\mathcal{P}}}(\omega)\right| \\
&= \left|\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} f(\omega)\left(\tilde{\rho}_{\mathcal{P}} - e^{-\tau_G \mathcal{R}_{\mathcal{P}}(\omega)}\right)d\mu_{S_{\mathcal{P}}}(\omega)\right| \\
&\leq \Lambda \int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} \left|\tilde{\rho}_{\mathcal{P}}(\omega) - e^{-\tau_G \mathcal{R}_{\mathcal{P}}(\omega)}\right| d\mu_{S_{\mathcal{P}}}(\omega) \\
&= \Lambda \int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} \left|(\tilde{\rho}_{\mathcal{P}}(\omega) - e^{X_{\mathcal{P}}}) + (e^{X_{\mathcal{P}}} - e^{y_{\mathcal{P}}}) + (e^{y_{\mathcal{P}}} - e^{-\tau_G \mathcal{R}_{\mathcal{P}}(\omega)})\right| d\mu_{S_{\mathcal{P}}}(\omega) \\
&\leq C\left(\sqrt{|\tau_{\mathcal{P}} - \tau_G|} + |\mathcal{P}|^{1/4}\right),
\end{aligned}$$

where the notation is consistent with that in the aforementioned propositions.

Using Proposition 5.7,

$$\left|\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} f(\omega)e^{-\tau_G \mathcal{R}_{\mathcal{P}}(\omega)}d\mu_{S_{\mathcal{P}}}(\omega) - \int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} f(\omega)e^{-\tau_G \mathcal{S}_{\mathcal{P}}(\omega)}d\mu_{S_{\mathcal{P}}}(\omega)\right|$$
$$\leq \Lambda \left[\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} e^{-2\mathcal{S}_{\mathcal{P}}}d\mu_{S_{\mathcal{P}}}\right]^{1/2}\left[\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} \left(e^{-(\mathcal{R}_{\mathcal{P}}-\mathcal{S}_{\mathcal{P}})}-1\right)^2 d\mu_{S_{\mathcal{P}}}\right]^{1/2}$$
$$= \Lambda \left[\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} e^{-2\mathcal{S}_{\mathcal{P}}}d\mu_{S_{\mathcal{P}}}\right]^{1/2}\left[\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} \left\{\left(e^{-2(\mathcal{R}_{\mathcal{P}}-\mathcal{S}_{\mathcal{P}})}-1\right) - 2\left(e^{-(\mathcal{R}_{\mathcal{P}}-\mathcal{S}_{\mathcal{P}})}-1\right)\right\} d\mu_{S_{\mathcal{P}}}\right]^{1/2}$$
$$\leq C\sqrt{e^{C|\mathcal{P}|}-1+\frac{C}{\varepsilon^2}e^{-\tilde{C}\frac{\varepsilon^2}{|\mathcal{P}|}}+\frac{C}{\sqrt{|\mathcal{P}|}\varepsilon}e^{-\frac{\varepsilon^2}{8|\mathcal{P}|}}}.$$

Let $Z_{\mathcal{P}} := \tau_G\left(\mathcal{S}_{\mathcal{P}}(\omega) - \int_0^1 \mathrm{Scal}(\phi(\omega)(s))ds\right)$. From Theorem 2.11 and Lemma 2.15,

$$\left|\int_{H^{\varepsilon}_{\mathcal{P}}(\mathbb{R}^d)} f(\omega)e^{-\tau_G \mathcal{S}_{\mathcal{P}}(\omega)}d\mu_{S_{\mathcal{P}}}(\omega) - \int_{H_{\mathcal{P}}(M)} f(\sigma)e^{-\tau_G \int_0^1 \mathrm{Scal}(\sigma(s))ds}d\nu_{S_{\mathcal{P}}}(\sigma)\right|$$
$$\leq \Lambda \int_{H_{\mathcal{P}}(\mathbb{R}^d)} e^{\tau_G \int_0^1 |\mathrm{Scal}(\phi(\omega)(s))|ds}\left|e^{-Z_{\mathcal{P}}}-1\right| d\mu_{S_{\mathcal{P}}}(\omega)$$
$$\leq \Lambda \mathbb{E}\left[e^{3\tau_G|b_1|}\right]^{1/2}\left[\int_{H_{\mathcal{P}}(\mathbb{R}^d)} \left|e^{-Z_{\mathcal{P}}}-1\right|^2 d\mu_{S_{\mathcal{P}}}(\omega)\right]^{1/2}$$
$$= \Lambda \mathbb{E}\left[e^{3\tau_G|b_1|}\right]^{1/2}\left[\int_{H_{\mathcal{P}}(\mathbb{R}^d)} \left\{(e^{-2Z_{\mathcal{P}}}-1) - 2(e^{-Z_{\mathcal{P}}}-1)\right\} d\mu_{S_{\mathcal{P}}}(\omega)\right]^{1/2}$$
$$\leq C|\mathcal{P}|^{1/4}.$$

Therefore all that remains is to show that

$$\left|\int_{H_{\mathcal{P}}(M)} f(\sigma)e^{-\tau_G \int_0^1 \mathrm{Scal}(\sigma(s))ds}d\nu_{S_{\mathcal{P}}}(\sigma) - \int_{W(M)} f(\sigma)e^{-\tau_G \int_0^1 \mathrm{Scal}(\sigma(s))ds}d\nu(\sigma)\right| \to 0$$

as $|\mathcal{P}| \to 0$; however, this is the result of Theorem 2.9 and why we chose to compare $\nu_{G_{\mathcal{P}}}$ with $\nu_{S_{\mathcal{P}}}$. □

Appendix A

Calculus and Probabilistic Inequalities

The proof of Eq. (A.1) can be found in [1, Lemma 8.6], but a full proof is included here.

Lemma A.1. *Let* $Z \stackrel{d}{=} N^d(0,1)$*,* $k \geq 0$*, and* $a > 0$*. Then there exists a* $C < \infty$ *depending only on* k *and* d *such that*

$$\mathbb{E}[e^{k\|Z\|} : \|Z\| \geq a] \leq \frac{C}{a^2} e^{-\frac{1}{4}a^2}. \tag{A.1}$$

If we further restrict $k < 1/2$*, then we can take* C *such that*

$$\mathbb{E}[e^{k\|Z\|^2} : \|Z\| \geq a] \leq \frac{C}{a^2} e^{-\frac{1-2k}{4}a^2}. \tag{A.2}$$

Proof. We can find a $\Lambda = \Lambda(k, \mathrm{d}) < \infty$ such that $r^{\mathrm{d}-1} e^{kr} e^{-r^2/2} \leq \Lambda e^{-3r^2/8}$ and $r^{d-1} e^{-(1-2k)r^2/2} \leq \Lambda e^{-3(1-2k)r^2/8}$. Now, let ω_{d-1} be the volume of the unit sphere in $\mathbb{R}^d$, and for $b > 0$ we have,

$$\begin{aligned}
\Lambda\omega_{d-1}(2\pi)^{d/2} \int_a^\infty e^{-\frac{3b}{8}r^2} dr &\leq \Lambda\omega_{d-1}(2\pi)^{d/2} \int_a^\infty \frac{r}{a} e^{-\frac{3b}{8}r^2} dr \\
&= \frac{8\Lambda\omega_{d-1}(2\pi)^{d/2}}{6ba} e^{-\frac{3b}{8}a^2} \\
&\leq \frac{C}{a^2} e^{-\frac{b}{4}a^2}.
\end{aligned}$$

Here, for example, we can take $C = \sqrt{\frac{2}{e}} \times \frac{8\Lambda\omega_{d-1}(2\pi)^{d/2}}{6b}$. Realizing that

$$\begin{aligned}\mathbb{E}[e^{k\|Z\|} : \|Z\| \geq a] &= \omega_{d-1}(2\pi)^{d/2} \int_a^\infty r^{d-1} e^{kr} e^{-\frac{1}{2}r^2} dr \\ &\leq \Lambda\omega_{d-1}(2\pi)^{d/2} \int_a^\infty e^{-\frac{3}{8}r^2} dr\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[e^{k\|Z\|^2} : \|Z\| \geq a] &= \omega_{d-1}(2\pi)^{d/2} \int_a^\infty r^{d-1} e^{-\frac{1-2k}{2}r^2} dr \\ &\leq \Lambda\omega_{d-1}(2\pi)^{d/2} \int_a^\infty e^{-\frac{3(1-2k)}{8}r^2} dr\end{aligned}$$

implies the result. □

Lemma A.2. *Given any $C \geq 0$ and $p \in [1, \infty)$, if $pC\Delta_i s < 1$ for each i, then*

$$\mathbb{E}\left[e^{\frac{p}{2}C\sum_{i=1}^n \|\Delta_i b\|^2}\right] = \prod_{i=1}^n (1 - pC\Delta_i s)^{-d/2} \tag{A.3}$$

$$\to e^{dpC/2} \text{ as } |\mathcal{P}| \to 0. \tag{A.4}$$

Proof. If $Z \overset{d}{=} N^d(0,1)$, the $\mathbb{E}[\exp\{pC\Delta_i s\|Z\|^2/2\}] = (1 - pC\Delta_i s)^{-d/2}$. Therefore the above equalities are elementary using independent increments and scaling of Brownian motion. □

Lemma A.3. *For any $\alpha > 0$, there exists a $C_\alpha < \infty$ such that*

$$\cosh(x) - 1 \leq C_\alpha(e^{\alpha x^2} - 1). \tag{A.5}$$

Proof. If $\alpha \geq 1$, this is obvious by taking $C_\alpha = 1$. For $\alpha < 1$, expand the exponential term,

$$\begin{aligned}e^{\alpha x^2} - 1 &= \sum_{k=1}^\infty \frac{(\alpha x^2)^k}{k!} \\ &= \sum_{k=1}^\infty \alpha^k \frac{(2k)!}{k!} \frac{x^{2k}}{(2k)!}.\end{aligned}$$

Define $\tilde{k} = \inf\{k \in \mathbb{N} : \alpha k \geq 1\}$. Stirling's approximation ensures that $(2k)!/k! > k^k$, so for any $k \geq \tilde{k}$ and any $l \leq k$, $\alpha^l \frac{(2k)!}{k!} \geq 1$. Therefore, setting $C_\alpha := \alpha^{-\tilde{k}}$,

$$\begin{aligned} C_\alpha(e^{\alpha x^2} - 1) &= \sum_{k=1}^{\infty} \alpha^{k-\tilde{k}} \frac{(2k)!}{k!} \frac{x^{2k}}{(2k)!} \\ &\geq \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)!} \\ &= \cosh(x) - 1. \end{aligned}$$

□

Lemma A.4. *For any $x \in \mathbb{R}$ and $p \in \mathbb{N}$,*

$$|e^x - 1|^p \leq e^{p|x|} - 1. \tag{A.6}$$

Proof. A quick look at the Taylor sum of e^x assures that $|e^x - 1| \leq e^{|x|} - 1$, so it suffices to show the above inequality for $x > 0$. In that case, when $p = 1$ this result is trivial. Suppose that we have that this inequality holds for each k for $1 \leq k \leq p$, then

$$\begin{aligned} (e^x - 1)^{p+1} &\leq (e^x - 1)(e^{px} - 1) \\ &= e^{(p+1)x} - e^x - e^{px} + 1 \\ &\leq e^{(p+1)x} - 1. \end{aligned}$$

By induction we arrive at the desired result. □

Proposition A.5. *We have the following estimates on the hyperbolic functions for $x, a, b \geq 0$*

1) $\cosh(x) - 1 \leq \min(x\sinh(x), \sinh(x))$

2) $\sinh(x) \leq \min(x\cosh(x), \cosh(x))$

3) $\cosh(a)\cosh(b) \leq \cosh(a+b)$ *and in particular,* $\cosh^2(a) \leq \cosh(2a)$

4) $\cosh(a)(\cosh(b) - 1) \leq \cosh(a)\cosh(b) - 1$

5) $(\cosh(a) - 1)(\cosh(b) - 1) \leq \cosh(a)\cosh(b) - 1$

6) If $\alpha \geq 1$ then $\alpha(\cosh(a)\cosh(b) - 1) \leq (\cosh(\alpha a)\cosh(\alpha b) - 1)$.

Proof. 1) By the fundamental theorem of calculus,

$$\begin{aligned}\cosh(x) - 1 &= \int_0^x \sinh(t)dt \\ &\leq \sinh(x)\int_0^x dt \\ &= x\sinh(x)\end{aligned}$$

and also that $\sinh(x) - (\cosh(x) - 1) = 1 - e^{-x} \geq 0$.

2) Similar to the previous inequalities, $\sinh(x) = \int_0^x \cosh(t)dt \leq x\cosh(x)$ and $\cosh(x) - \sinh(x) = e^{-x} > 0$.

3) This is a simple identity expanding cosh in terms of exponentials and noting that $|a - b| \leq a + b$

$$\begin{aligned}\cosh(a)\cosh(b) &= \frac{1}{4}\left(e^a + e^{-a}\right)\left(e^b + e^{-b}\right) \\ &= \frac{1}{4}\left(e^{a+b} + e^{-(a+b)} + e^{a-b} + e^{-(a-b)}\right) \\ &= \frac{1}{2}(\cosh(a+b) + \cosh(a-b)) \\ &\leq \frac{1}{2}2\cosh(a+b) \\ &= \cosh(a+b)\end{aligned}$$

4) With $\cosh(a) \geq 1$, $\cosh(a)\cosh(b) - \cosh(a) \leq \cosh(a)\cosh(b) - 1$.

5) $(\cosh(a) - 1)^2 = \cosh^2(a) - 2\cosh(a) + 1 \leq \cosh(2a) - 1$ since $\cosh^2(a) \leq \cosh(2a)$ and $1 - 2\cosh(a) \leq -1$.

6) We first establish that $f(x) := \alpha(\cosh(x) - 1) \leq \tilde{f}(x) := \cosh(\alpha x) - 1$. With $f(0) = 0 = \tilde{f}(0)$ and $f'(x) = \alpha\sinh(x) \leq \alpha\sinh(\alpha x) = \tilde{f}'(x)$ for $\alpha \geq 1$ and $x > 0$ the first claim follows. Now, let $g(x) = \alpha(\cosh(x)\cosh(b) - 1)$ and $\tilde{g}(x) = \cosh(\alpha x)\cosh(\alpha b) - 1$. Then, by what we just established, $g(0) \leq \tilde{g}(0)$. Further $g'(x) = \alpha\sinh(x)\cosh(b) \leq \alpha\sinh(\alpha x)\cosh(\alpha b) = \tilde{g}'(x)$. Hence the proof.

□

Proposition A.6. *Let* $\{X_n : n \in \mathbb{N}\}$ *be a sequence of random variables on the probability space* $(\Omega, \mathbb{P})$*. If there exists a* $p > 1$ *such that*

$$\limsup_{n\to\infty} \mathbb{E}\left[|X_n|^p\right] < \infty, \tag{A.7}$$

then the collection $\{X_n\}$ *is uniformly integrable.*

Proof. Since any finite collection of random variables is uniformly integrable, we may as well assume that $\sup_n \mathbb{E}[|X_n|] =: K < \infty$. Let $a > 0$ and we have

$$\begin{aligned}
\mathbb{E}\left[1_{\{|X_n|\geq a\}}|X_n|\right] &\leq \mathbb{E}\left[1_{\{|X_n|\geq a\}}\left(\frac{|X_n|}{a}\right)^{p-1}|X_n|\right] \\
&\leq \frac{1}{a^{p-1}}\mathbb{E}\left[|X_n|^p\right] \\
&\leq \frac{K}{a^{p-1}} \to 0 \text{ as } a \to \infty.
\end{aligned}$$

□

Appendix B

Some Linear Algebra

This chapter is dedicated to establishing some equalities and inequalities that fall broadly into the category of Linear Algebra. Many have extensions to bounded, compact, or trace class operators on Hilbert Spaces although for our purposes we need only consider finite dimensional matrices. The interested reader is directed to [25, 23] for further investigation.

Notation B.1. *We will assume that V is an N-dimensional complex inner-product space, $\mathcal{L}(V)$ will be the collection of linear operators $V \to V$, and $\{e_i\}_{i=1}^N$ is an orthonormal basis. If $A \in \mathcal{L}(V)$, then the operator $A^* \in \mathcal{L}(V)$ will denote the adjoint of A, $|A| = \sqrt{A^*A}$ will be the absolute value of A, $\|A\| = \sup\{\|Av\| : \|v\| = 1\}$ will denote the operator norm on A, and $\|A\|_2 := \sqrt{\mathrm{tr}(A^*A)} = \sqrt{\mathrm{tr}(|A|^2)}$ will denote the Hilbert-Schmidt norm of A.*

For the Hilbert-Schmidt norm, we can also define it so that it appears more like l_2 norm since,

$$\|A\|_2^2 = \mathrm{tr}(A^*A) = \sum_{i=1}^N \langle A^*Ae_i, e_i\rangle = \sum_{i=1}^N \|Ae_i\|^2 = \sum_{i,j=1}^N |\langle Ae_i, e_j\rangle|^2$$

where we recognize that $\langle Ae_i, e_j\rangle$ is the $(i,j)^{th}$ entry of the matrix representation of A under the basis $\{e_i\}$.

The following theorem generalizes the idea that if $z \in \mathbb{C}$, then we can find some $e^{i\theta}$ in the unit circle such that $z = |z|e^{i\theta}$. The proof is given in [25, Theorem VI.10].

Theorem B.2 (Polar Decomposition)**.** *Given $A \in \mathcal{L}(V)$, there exists a unique operator $U \in \mathcal{L}(V)$ such that U is an isometry when restricted to $(\operatorname{Ker} U)^{\perp}$, $\operatorname{Ran} U = \overline{\operatorname{Ran} A}$, and $U|A| = A$.*

As a consequence of Theorem B.2 we have,

Proposition B.3. *Let $A, B \in \mathcal{L}(V)$. Then,*

$$|\operatorname{tr}(AB)| \leq \|A\| \operatorname{tr}(|B|). \tag{B.1}$$

Proof. Since $|B|$ is self-adjoint, there is an orthonormal basis of eigenvectors of $\{v_i\}_{i=1}^{N}$ of $|B|$. Let $\{\lambda_i\}_{i=1}^{N}$ be the respective eigenvalues with $|B|v_i = \lambda_i v_i$. Let $U \in \mathcal{L}(V)$ be chosen as in Theorem B.2 such that $U|B| = B$. Then,

$$\begin{aligned}
|\operatorname{tr}(AB)| &= \left| \sum_{i=1}^{n} \langle ABv_i, v_i \rangle \right| \\
&= \left| \sum_{i=1}^{n} \langle AU|B|v_i, v_i \rangle \right| \\
&= \left| \sum_{i=1}^{n} \lambda_i \langle AUv_i, v_i \rangle \right| \\
&\leq \sum_{i=1}^{n} \lambda_i \left| \langle AUv_i, v_i \rangle \right| \\
&\leq \|U\| \|A\| \sum_{i=1}^{n} \lambda_i \|v_i\|^2 \\
&= \|A\| \operatorname{tr}(|B|).
\end{aligned}$$

Here the second inequality comes from applying the Cauchy-Schwarz inequality to $|\langle AUv_i, v_i \rangle|$. The last equality comes from the fact that $\|v_i\| = 1$. □

Corollary B.4. *Let A and B be defined as in Proposition B.3 above. Then*

$$\|AB\|_2 \leq \|A\| \|B\|_2. \tag{B.2}$$

In particular,

$$\|A\|_2 \leq \sqrt{N} \|A\|. \tag{B.3}$$

Proof. Recall that for $C_1, C_2 \in \mathcal{L}(V)$, $\operatorname{tr}(C_1C_2) = \operatorname{tr}(C_2C_1)$. Therefore,

$$\begin{aligned}\|AB\|_2^2 &= \operatorname{tr}(B^*A^*AB)\\ &= \operatorname{tr}(BB^*A^*A)\\ &\leq \|A^*A\| \operatorname{tr}(BB^*) \quad \text{by Proposition B.3}\\ &= \|A\|^2 \operatorname{tr}(|B|^2)\\ &= \|A\|^2\|B\|_2^2.\end{aligned}$$

For the second claim, set $B = I$. □

Corollary B.5. *Let $A \in \mathcal{L}(V)$. Then*

$$|\operatorname{tr}(A)| \leq \sqrt{N}\|A\|_2 \leq N\|A\|. \tag{B.4}$$

Proof. Let $\{\lambda_i\}_{i=1}^N$ be the eigenvalues of $|A|$. Since $|\operatorname{tr}(A)| \leq \operatorname{tr}(|A|)$ by Proposition B.3 and using the Cauchy-Schwarz inequality,

$$\begin{aligned}\operatorname{tr}(|A|) &= \sum_{i=1}^N \lambda_i\\ &\leq \left(\sum_{i=1}^N \lambda_i^2\right)^{1/2} \sqrt{N}\\ &= \sqrt{N}\sqrt{\operatorname{tr}(|A|^2)}\\ &= \sqrt{N}\|A\|_2,\end{aligned}$$

the result follows when combing this with the fact that $\|A\|_2 \leq \sqrt{N}\|A\|$ by Corollary B.4. □

Lemma B.6. *Suppose that $n, d \in \mathbb{N}$ and $N = nd$. Let $A \in \mathcal{L}(V)$, where we consider A to be an $n \times n$ block matrix with $d \times d$ blocks. If A is block-tridiagonal then,*

$$\|A\|_2^2 \leq d \sum_{i=1}^n \left(\|[A]_{i,i}\|^2 + \|[A]_{i+1,i}\|^2 + \|[A]_{i,i+1}\|^2\right), \tag{B.5}$$

where $[A]_{i,j}$ represents the $(i,j)^{th}$ $d \times d$ block of A and we define $[A]_{n,n+1} = [A]_{n+1,n} = 0$.

Proof. Since we can calculate $\|A\|_2^2$ as the sum of squares of each entry in A,

$$\begin{aligned}\|A\|_2^2 &= \sum_{i,j=1}^{n} \|[A]_{i,j}\|_2^2 \\ &= \sum_{i=1}^{n} \left(\|[A]_{i+1,i}\|_2^2 + \|[A]_{i,i}\|_2^2 + \|[A]_{i,i+1}\|_2^2\right) \\ &\leq d \sum_{i=1}^{n} \left(\|[A]_{i+1,i}\|^2 + \|[A]_{i,i}\|^2 + \|[A]_{i,i+1}\|^2\right).\end{aligned}$$

Here the final inequality comes from Corollary B.4. □

Lemma B.7. *Let* $U \in \mathcal{L}(V)$ *with* $\|U\| < 1$*. Define* $\mathcal{T}_k(U) = \sum_{m=1}^{k} \frac{(-1)^{m+1}}{m} \operatorname{tr}(U^m)$ *and* $\Psi_{k+1}(U) = \sum_{m=k+1}^{\infty} \frac{(-1)^{m+1}}{m} \operatorname{tr}(U^m)$*. Then,*

$$\det(I + U) = \exp\{\mathcal{T}_k(U) + \Psi_{k+1}(U)\} \tag{B.6}$$

with

$$|\Psi_{k+1}(U)| \leq \frac{N\|U\|^{k+1}}{1 - \|U\|}. \tag{B.7}$$

If we further assume that U *is normal then,*

$$|\Psi_{k+1}(U)| \leq \|U\|_2^2 \frac{\|U\|^{k-1}}{1 - \|U\|}. \tag{B.8}$$

Proof. With $\|U\| < 1$, $\Re(\det(I + U)) > 0$, and hence we have the familiar formula

$$\log(\det(I + U)) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \operatorname{tr}(U^m). \tag{B.9}$$

Eq. (B.6) now results. Eq. (B.7) then comes from estimating the trace as $|\operatorname{tr}(U^m)| \leq N\|U\|^m$ which yields,

$$|\Psi_{k+1}(U)| \leq N \sum_{m=k+1}^{\infty} \frac{\|U\|^m}{m} \leq N \frac{\|U\|^{k+1}}{1 - \|U\|}.$$

If U is normal then, $|U^2| = |U|^2$. Using Proposition B.3, for $m \geq 2$,

$$|\operatorname{tr}(U^m)| \leq \|U\|^{m-2} \operatorname{tr}(|U^2|) = \|U\|^{m-2} \operatorname{tr}(|U|^2) = \|U\|^{m-2}\|U\|_2^2.$$

Thus for $k \geq 1$,

$$|\Psi_{k+1}(U)| \leq \sum_{m=k+1}^{\infty} \frac{\|U\|^{m-2}}{m} \|U\|_2^2 \leq \|U\|_2^2 \frac{\|U\|^{k-1}}{1 - \|U\|}.$$

□

Lemma B.8. *Suppose that $A, B \in \mathcal{L}(V)$ with A invertible. Then ABA^* is positive (semi-)definite if and only if B is.*

Proof. For any $v \in V$, $\langle ABA^*v, v\rangle = \langle BA^*v, A^*v\rangle$. This immediately implies that if B is positive (semi-)definite, then so is ABA^*. However, if we assume that B is not positive definite, then find $w \in V$ such that $\langle Bw, w\rangle \leq 0$. Set $v = (A^*)^{-1}w$ to see that ABA^* cannot be positive definite. A similar calculation holds for the semi-definite case. □

B.1 Kronecker Product Results

Take A and B to be $k \times m$ and $p \times q$ matrices respectively. Define the *Kronecker product* of A with B, denoted $A \otimes B$, by the $k \times m$ block matrix with $p \times q$ blocks,

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,m}B \\ a_{2,1}B & a_{2,2}U_2 & \cdots & a_{2,m}B \\ \vdots & \ddots & \ddots & \vdots \\ a_{k,1}B & a_{k,2}B & \cdots & a_{k,m}B \end{pmatrix}, \tag{B.10}$$

where,

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \ddots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,m} \end{pmatrix}. \tag{B.11}$$

As notation would suggest, the Kronecker product is related to the tensor product. Indeed, if $A : \mathbb{R}^m \to \mathbb{R}^k$ is the matrix representation of the linear operator $\tilde{A} : V_1 \to V_2$ under the bases $\beta_1 \subset V_1$ and $\beta_2 \subset V_2$, and similarly $B : \mathbb{R}^q \to \mathbb{R}^p$ is the matrix representation of the linear operator $\tilde{B} : W_1 \to W_2$ under the bases $\Gamma_1 \subset W_1$ and $\Gamma_2 \subset W_2$, then $A \otimes B$ is the matrix representation of the linear operator $\tilde{A} \otimes \tilde{B} : V_1 \otimes W_1 \to V_2 \otimes W_2$ under the bases $\beta_1 \otimes \Gamma_1$ and $\beta_2 \otimes \Gamma_2$. Under this identification of the tensor product, the vector $v \otimes w$, where $v \in \mathbb{R}^m$ and

$w \in \mathbb{R}^q$ is the given by,

$$v \otimes w = \begin{pmatrix} v_1 w \\ v_2 w \\ \vdots \\ v_m w \end{pmatrix} \tag{B.12}$$

where $v = (v_1, ..., v_m)^{\text{tr}}$. This differs from the more common identification of $v \otimes w$ with the matrix vw^{tr}; however, this ambiguity does not alter the results needed. In particular, it is easy to check that $A \otimes B(v \otimes w) = (Av) \otimes (Bw)$, from which it also follows that if A_1, A_2, B_1, and B_2 are matrices such that $A_1 A_2$ and $B_1 B_2$ exist, then $(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2)$. This implies that for invertible A and B, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, and for vectors v_1, v_2, w_1, and w_2, $\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle$ where $\langle v, u \rangle := v^{\text{tr}} u$.

Proposition B.9. *Suppose A is an $n \times n$ square matrix with eigenvalues $\{\lambda_i^A\}_{i=1}^n$ and B is a $d \times d$ square matrix with eigenvalues $\{\lambda_j^B\}_{j=1}^d$. Then $A \otimes B$ has eigenvalues $\{\lambda_i^A \lambda_j^B : 1 \le i \le n, 1 \le j \le d\}$. Moreover, if u is an eigenvector of A with eigenvector λ^A and v is an eigenvector of B with eigenvalue λ^B, then $u \otimes v$ is an eigenvector of $A \otimes B$ with eigenvalue $\lambda^A \lambda^B$.*

Proof. This result is immediate in the case that A and B are upper triangular matrices. Now if P and Q are such that PAP^{-1} is upper triangular and QBQ^{-1} is upper triangular, then by the comments above, $(P \otimes Q)(A \otimes B)(P \otimes Q)^{-1} = (PAP^{-1}) \otimes (QBQ^{-1})$ is upper triangular and similar to $A \otimes B$. Since similar matrices share eigenvalues, this is enough to conclude the first claim. For the second, we calculate $A \otimes B(u \otimes v) = (Au) \otimes (Bv) = \lambda^A \lambda^B (u \otimes v)$. □

The following corollary is immediate as a special case of the above comments and proposition. It is given special attention here because it is the case that manifests itself within this paper.

Corollary B.10. *If $\{u_i\}_{i=1}^n$ is a basis of eigenvectors of A with respective eigenvalues $\{\lambda_i\}_{i=1}^n$, then $\{u_i \otimes e_a : 1 \le i \le n, 1 \le a \le d\}$ is a basis of eigenvectors of $A \otimes I_d$ with $A \otimes I_d(u_i \otimes e_j) = \lambda_i (u_i \otimes e_j)$. Here I_d is the $d \times d$ identity matrix. If the* *collection* $\{u_i\}$ *is orthonormal, then so are* $\{u_i \otimes e_j\}$*. Moreover, the eigenvalues of* $A \otimes I_d$ *are again given by* $\{\lambda_i\}$ *with each* λ_i *repeated* d *times.*

Lemma B.11. *Let* U *be an* $n \times n$ *tri-diagonal block matrix with* $d \times d$ *blocks given by*

$$U = \begin{pmatrix} U_{1,1} & U_{1,2} & 0 & 0 & 0 \\ U_{2,1} & U_{2,2} & U_{2,3} & 0 & 0 \\ 0 & U_{3,2} & U_{3,3} & U_{3,4} & 0 \\ 0 & 0 & \ddots & \ddots & U_{n-1,n} \\ 0 & 0 & 0 & U_{n,n-1} & U_{n,n} \end{pmatrix}.$$

Let $\{e_a\}_{a=1}^d$ *be the standard basis for* $\mathbb{R}^d$ *and for* $1 \le k \le n$, $u_k = (c_k^1, ..., c_k^n)^{\mathrm{tr}} \in \mathbb{R}^n$*. Set* $\{w_{k,a} : 1 \le a \le d, 1 \le k \le n\} \subset \mathbb{R}^{nd}$ *with*

$$w_{k,j} := \begin{pmatrix} c_k^1 e_j \\ c_k^2 e_j \\ \vdots \\ c_k^n e_j \end{pmatrix} = u_k \otimes e_j.$$

Then,

$$\sum_{a=1}^{d} U w_{k,a} \cdot w_{k,a} = \sum_{i=1}^{n} \left[(c_k^i)^2 \operatorname{tr}(U_{i,i}) + c_k^i c_k^{i+1} (\operatorname{tr}(U_{i,i+1}) + \operatorname{tr}(U_{i+1,i})) \right] \qquad \text{(B.13)}$$

where we set $c_k^{n+1} = \operatorname{tr}(U_{n+1,n}) = \operatorname{tr}(U_{n,n+1}) = 0$*. In particular, if* U *is symmetric so that* $U_{i,i+1} = U_{i+1,i}^{tr}$ *then*

$$\sum_{a=1}^{d} U w_{k,a} \cdot w_{k,a} = \sum_{i=1}^{n} \left[(c_k^i)^2 \operatorname{tr}(U_{i,i}) + 2 c_k^i c_k^{i+1} \operatorname{tr}(U_{i,i+1}) \right]. \qquad \text{(B.14)}$$

Proof. For simplicity, let us agree that $U_{1,0} = 0$. We then have,

$$\begin{aligned}
\sum_{a=1}^{d} Uw_{k,a} \cdot w_{k,a} &= \sum_{a=1}^{d}\sum_{i=1}^{n}\sum_{j=1}^{n} U_{i,j} w_{k,a}^{j} \cdot w_{k,a}^{i} \\
&= \sum_{a=1}^{d}\sum_{i=1}^{n} \left(U_{i,i-1} w_{k,a}^{i-1} + U_{i,i} w_{k,a}^{i} + U_{i,i+1} w_{k,a}^{i+1}\right) \cdot w_{k,a}^{i} \\
&= \sum_{a=1}^{d}\sum_{i=1}^{n} \left(c_k^{i-1} c_k^{i} U_{i,i-1} e_a + (c_k^{i})^2 U_{i,i} e_a + c_k^{i+1} c_k^{i} U_{i,i+1} e_a\right) \cdot e_a \\
&= \sum_{i=1}^{n} \left(c_k^{i-1} c_k^{i} \operatorname{tr}(U_{i,i-1}) + (c_k^{i})^2 \operatorname{tr}(U_{i,i} e_a) + c_k^{i+1} c_k^{i} \operatorname{tr}(U_{i,i+1})\right) \\
&= \sum_{i=1}^{n} \left[(c_k^{i})^2 \operatorname{tr}(U_{i,i}) + c_k^{i} c_k^{i+1} \left(\operatorname{tr}(U_{i,i+1}) + \operatorname{tr}(U_{i+1,i})\right)\right].
\end{aligned}$$

which proves the first claim. The second follows since $\operatorname{tr}(A) = \operatorname{tr}(A^{tr})$. □

Appendix C

Some Geometry

Notation C.1. *If $\pi : TM \to M$ is the projection, $f : T_oM \to T_pM$ is an isometry, recall the following*

1. $\Omega_f : T_oM \times T_oM \times T_oM \to T_oM$ *is defined by* $\Omega_f(a,b)c = f^{-1}R(fa, fb)fc$.

2. $\mathrm{Ric} : T_\pi M \to T_\pi M$ *is the linear map defined by* $\mathrm{Ric}(v) = \sum_{i=1}^d R(v, e_i)e_i$ *where* $\{e_i\}_{i=1}^d$ *is an orthonormal basis for* $T_{\pi(v)}M$.

3. $\mathrm{Ric}_f : T_oM \to T_oM$ *is the linear map defined by* $\mathrm{Ric}_f(v) = \sum_{i=1}^d \Omega_f(v, e_i)e_i$ *where here* $\{e_i\}_{i=1}^d$ *is an orthonormal basis for* T_oM.

4. $\mathrm{Scal} : M \to \mathbb{R}$ *is the defined by* $\mathrm{Scal}(p) = \mathrm{tr}(\mathrm{Ric}|_{T_pM})$. *That is* $\mathrm{Scal}(p)$ *gives the trace of the linear operator* Ric *on the tangent space* T_pM.

Proposition C.2. *Using the notation above, we have the following properties*

1. *For* $v \in T_oM$ *we have that* $\mathrm{Ric}_f(v) = f^{-1}\,\mathrm{Ric}(fv)$.

2. *If* σ *is a curve in* M *starting at* $o \in M$ *and* $u(s) := //_s(\sigma)$, *then* $\mathrm{tr}(\mathrm{Ric}_{u(s)}) = \mathrm{Scal}(\sigma(s))$.

3. $\mathrm{tr}(\Omega_f(v, \cdot)v) = -\langle \mathrm{Ric}_f\, v, v\rangle$.

Proof. Let $\{e_i\}_{i=1}^d$ be an orthonormal basis for T_oM

1. Note that if $\mathcal{E}_i := fe_i$, the $\{\mathcal{E}_i\}$ forms an orthonormal basis for T_pM since f is an isometry. Hence, following our nose from the definition

$$\begin{aligned}\mathrm{Ric}_f(v) &= \sum_{i=1}^d \Omega_f(v, e_i)e_i \\ &= \sum_{i=1}^d f^{-1}R(fv, fe_i)fe_i \\ &= f^{-1}\left(\sum_{i=1}^d R(fv, \mathcal{E}_i)\mathcal{E}_i\right) \\ &= f^{-1}\,\mathrm{Ric}(fv)\end{aligned}$$

2. Using that $\{//_s(\sigma)e_i\}$ forms an orthonormal basis for $T_{\sigma(s)}M$ and that $//_s(\sigma)^{-1}$ is an isometry,

$$\begin{aligned}\mathrm{tr}(\mathrm{Ric}_{u(s)}) &= \sum_{i=1}^d \langle \mathrm{Ric}_{u(s)}\, e_i, e_i\rangle \\ &= \sum_{i=1}^d \langle //_s(\sigma)^{-1}\,\mathrm{Ric}(//_s(\sigma)e_i), e_i\rangle \quad \text{from (1.)} \\ &= \sum_{i=1}^d \langle //_s(\sigma)^{-1}\,\mathrm{Ric}(//_s(\sigma)e_i), //_s(\sigma)^{-1}(//_s(\sigma)e_i)\rangle \\ &= \sum_{i=1}^d \langle \mathrm{Ric}(//_s(\sigma)e_i), //_s(\sigma)e_i\rangle \\ &= \mathrm{Scal}(\sigma(s)).\end{aligned}$$

3. Recall the Bianchi identity $\langle R(a,b)c, d\rangle = -\langle R(a,b)d, c\rangle$. Hence,

$$\begin{aligned}\langle \Omega_f(v, e_i)v, e_i\rangle &= \langle f^{-1}R(fv, fe_i)fv, e_i\rangle \\ &= \langle f^{-1}R(fv, fe_i)fv, f^{-1}fe_i\rangle \\ &= \langle R(fv, fe_i)fv, fe_i\rangle \\ &= -\langle R(fv, fe_i)fe_i, fv\rangle \\ &= \langle \Omega_f(v, e_i)e_i, v\rangle.\end{aligned}$$

Therefore, by summing over i and using the definition of Ric_f, the result follows.

$\square$

Appendix D

ODE Estimates

Proposition D.1. *Let $s > 0$ and J be an interval of $\mathbb{R}$ containing $[0, s]$. Suppose that $z : J \to \mathrm{Hom}(\mathbb{R}^N \to \mathbb{R}^N)$ satisfies $z''(r) = A(r)z(r)$ where $A \in C^1(J \to \mathrm{Hom}(\mathbb{R}^N \to \mathbb{R}^N))$. We also suppose that there exist $K_0, K_1 > 0$ such that $\sup_{r\in[0,s]} \|A(r)\| \le K_0$ and $\sup_{r\in[0,s]} \|A'(r)\| \le K_1$. Then,*

$$\|z(s) - z(0) - sz'(0)\| \le \|z(0)\|(\cosh(\sqrt{K_0}s) - 1) + \|z'(0)\|s\left(\frac{\sinh(\sqrt{K_0}s)}{\sqrt{K_0}s} - 1\right). \tag{D.1}$$

If we assume that $z(0) = 0$ and $z'(0) = I$, then

$$\|z(s) - sI - \frac{s^3}{6}A(0)\| \le \frac{s^4}{12}K_1 + \frac{s}{6}\left(\frac{\sinh(\sqrt{K_0}s)}{\sqrt{K_0}s} - 1 - \frac{1}{6}K_0 s^2\right). \tag{D.2}$$

If instead we assume that $z(0) = I$ and $z'(0) = 0$, then

$$\|z(s) - I - \frac{s^2}{2}A(0)\| \le \frac{s^3}{6}K_1 + \frac{1}{2}\left(\cosh(\sqrt{K_0}s) - 1 - \frac{1}{2}K_0 s^2\right). \tag{D.3}$$

Proof. Start with the following calculus facts,

$$\int_0^s \int_0^{s_{j-1}} \cdots \int_0^{s_2} (s - s_j)(s_j - s_{j-1}) \cdots (s_2 - s_1) ds_1 \cdots ds_j = \frac{s^{2j}}{(2j)!} \tag{D.4}$$

and

$$\int_0^s \int_0^{s_{j-1}} \cdots \int_0^{s_2} (s - s_j)(s_j - s_{j-1}) \cdots (s_2 - s_1) s_1 ds_1 \cdots ds_j = \frac{s^{2j+1}}{(2j+1)!}. \tag{D.5}$$

By Taylor's theorem with integral remainder,

$$\begin{aligned} z(s) &= z(0) + sz'(0) + \int_0^s (s-r)z''(r)dr \\ &= z(0) + sz'(0) + \int_0^s (s-r)A(r)z(r)dr. \end{aligned}$$

From here, iterating Talyor's theorem,

$$\begin{aligned} & z(s) - z(0) - sz'(0) \\ &= \int_0^s (s-r)A(r)z(r)dr \\ &= \int_0^s (s-r)A(r)\left\{z(0) + rz'(0) + \int_0^r (r-t)A(t)z(t)dt\right\} dr \\ & \cdots \\ &= \left(\overbrace{\sum_{j=1}^m \int_{0<s_1<\cdots<s_j<s} (s-s_j)\cdots(s_2-s_1)A(s_j)\cdots A(s_1)ds_1\cdots ds_j}^{I}\right) z(0) \\ &+ \left(\overbrace{\sum_{j=1}^m \int_{0<s_1<\cdots<s_j<s} (s-s_j)\cdots(s_2-s_1)s_1A(s_j)\cdots A(s_1)ds_1\cdots ds_j}^{II}\right) z'(0) \\ &+ \overbrace{\int_{0<s_1<\cdots<s_{m+1}<s} (s-s_{m+1})\cdots(s_2-s_1)A(s_{m+1})\cdots A(s_1)z(s_1)ds_1\cdots ds_{m+1}}^{III}. \end{aligned}$$

Therefore by using Eqs. (D.4) and (D.5) and the bound on A,

$$\|I\| \leq \sum_{j=1}^m \frac{s^{2j}K_0^j}{(2j)!} \leq \cosh(\sqrt{K_0}s) - 1 \tag{D.6}$$

$$\|II\| \leq \sum_{j=1}^m \frac{s^{2j+1}K_0^j}{(2j+1)!} \leq \frac{\sinh(\sqrt{K_0}s)}{\sqrt{K_0}} - s \tag{D.7}$$

and

$$\|III\| \leq \sup_{r\in[0,s]} \|z(r)\| \frac{s^{2(m+1)}}{[2(m+1)]!}. \tag{D.8}$$

Taking $m \to \infty$ completes the proof of Eq. (D.1).

If $z(0) = 0$ and $z'(0) = I$, then

$$\|z(r)\| \leq \frac{\sinh(\sqrt{K_0}r)}{\sqrt{K_0}}.$$

Again iterating Taylor's theorem,

$$\begin{aligned}
z(s) &= sI + \int_0^s (s-r)A(r)\left\{rI + \int_0^r (r-t)A(t)z(t)dt\right\} dr \\
&= sI + \int_0^s (s-r)rA(0)dr + \int_0^s \int_0^r (s-r)rA'(t)dtdr \\
&\quad + \int_0^s \int_0^r (s-r)(r-t)A(r)A(t)z(t)dtdr \\
&= sI + \frac{s^3}{6}A(0) + \int_0^s \int_0^r (s-r)rA'(t)dtdr \\
&\quad + \int_0^s \int_0^r (s-r)(r-t)A(r)A(t)z(t)dtdr
\end{aligned}$$

where the second equality came from $A(r) = A(0) + \int_0^r A'(t)dt$. Hence

$$\begin{aligned}
\|z(s) - sI - \frac{s^3}{6}A(0)\| &\leq \frac{s^4}{12}K_1 + K_0^2 \int_0^s \int_0^r (s-r)(r-t)\frac{\sinh(\sqrt{K_0}t)}{\sqrt{K_0}}dtdr \\
&= \frac{s^4}{12}K_1 + \frac{s}{6}\left(\frac{\sinh(\sqrt{K_0}s)}{\sqrt{K_0}s} - 1 - \frac{1}{6}K_0 s^2\right).
\end{aligned}$$

If $z(0) = I$ and $z'(0) = 0$, then $\|z(r)\| \leq \cosh(\sqrt{K_0}r)$, a similar expansion as above shows

$$\begin{aligned}
z(s) &= I + \int_0^s (s-r)A(0)dr + \int_0^s \int_0^r (s-r)A'(t)dtdr \\
&\quad + \int_0^s \int_0^r (s-r)(r-t)A(r)A(t)z(t)dtdr \\
&= I + \frac{s^2}{2}A(0) + \int_0^s \int_0^r (s-r)A'(t)dtdr \\
&\quad + \int_0^s \int_0^r (s-r)(r-t)A(r)A(t)z(t)dtdr.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|z(s) - I - \frac{s^2}{2}A(0)\| &\leq \frac{s^3}{6}K_1 + K_0^2 \int_0^s \int_0^r (s-r)(r-t)\cosh(\sqrt{K_0}t)dtdr \\
&= \frac{s^3}{6}K_1 + \frac{1}{2}\left(\cosh(\sqrt{K_0}s) - 1 - \frac{1}{2}K_0 s^2\right).
\end{aligned}$$

□

Proposition D.2. *Let $z(s)$ be as in Proposition D.1 and $f(s) := \frac{z(\Delta)-z(0)}{\Delta}s + z(0)$. Then,*

$$||z(s) - f(s)|| \leq s\left(1 - \frac{s}{\Delta}\right)\Big(||z(0)||K\Delta\cosh(\sqrt{K}\Delta) + ||z'(0)||\left(\cosh(\sqrt{K}\Delta) - 1\right)\Big) \tag{D.9}$$

Proof. Let $G(s,t)$ be the Green's function for $s, t \in [0, \Delta]$,

$$G(s,t) := t\left(1 - \frac{s}{\Delta}\right)1_{[0,s)} + s\left(1 - \frac{t}{\Delta}\right)1_{[s,\Delta]}$$

so that we have

$$\begin{aligned} z(s) - f(s) &= \int_0^\Delta G(s,t)(z''(t) + f''(t))dt \\ &= \int_0^\Delta G(s,t)z''(t)dt \\ &= \int_0^\Delta G(s,t)A(t)z(t)dt. \end{aligned}$$

Therefore, using the fact that $0 \leq G(s,t) \leq s\left(1 - \frac{s}{\Delta}\right)$ and the estimate from proposition D.1 , we have

$$\begin{aligned} ||z(s) - f(s)|| &\leq s\left(1 - \frac{s}{\Delta}\right)K\int_0^\Delta \left(||z(0)||\cosh(\sqrt{K}t) + ||z'(0)||\frac{\sinh(\sqrt{K}t)}{\sqrt{K}}\right)dt \\ &= s\left(1 - \frac{s}{\Delta}\right)K\left(||z(0)||\frac{\sinh(\sqrt{K}\Delta)}{\sqrt{K}} + ||z'(0)||\frac{\cosh(\sqrt{K}\Delta) - 1}{K}\right) \\ &\leq s\left(1 - \frac{s}{\Delta}\right)\left(||z(0)||K\Delta\cosh(\sqrt{K}\Delta) + ||z'(0)||\left(\cosh(\sqrt{K}\Delta) - 1\right)\right). \end{aligned}$$

□

Appendix E

Existence and Bound of $S_i(\Delta)^{-1}$ on Manifolds with Non-Positive Sectional Curvature

Lemma E.1. *Let* $t \mapsto \xi(t) \in \mathbb{R}^N$ *for* $t \geq 0$ *be a smooth map with* $\dot{\xi(0)} \neq 0$ *and set* $x(t) := \|\xi(t)\|$*. If* $\xi(0) = 0$ *then*

$$\lim_{t \to 0} \dot{x}(t) = \|\dot{\xi}(0)\|. \tag{E.1}$$

In particular, by defining $\dot{x}(0) = \|\dot{\xi}(0)\|$*,* $\dot{x}$ *is smooth for* $t > 0$ *and continuous for* $t \geq 0$*.*

Proof. By Taylor's theorem we can write $\xi(t) = t\dot{\xi}(0) + O(t^2)$ and $\dot{\xi}(t) = \dot{\xi}(0) + O(t)$. Hence,

$$\langle \xi(t), \dot{\xi}(t) \rangle = t\|\dot{\xi}(0)\|^2 + O(t^2)$$

and

$$\begin{aligned} \|\xi(t)\| &= \left(t^2\|\dot{\xi}(0)\|^2 + O(t^3)\right)^{1/2} \\ &= t\|\dot{\xi}(0)\| \left(1 + O(t^3)\right)^{1/2}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{t\to 0} \dot{x}(t) &= \lim_{t\to 0} \frac{\langle \xi(t), \dot{\xi}(t)\rangle}{\|\xi(t)\|} \\
&= \lim_{t\to 0} \frac{t\|\dot{\xi}(0)\|^2 + O(t^2)}{t\|\dot{\xi}(0)\| \left(1 + O(t^3)\right)^{1/2}} \\
&= \|\dot{\xi}(0)\|.
\end{aligned}$$

□

Proposition E.2. *Suppose that $t \mapsto A(t)$ is a smooth map where $A(t)$ is an $N \times N$ positive semi-definite matrix. Let $t \mapsto \xi(t) \in \mathbb{R}^N$ for $t \geq 0$ be a smooth map with $\ddot{\xi}(t) = A(t)\xi(t), \xi(0) = 0,$ and $\dot{\xi}(0) \neq 0$. Then,*

$$\|\xi(t)\| \geq t\|\dot{\xi}(0)\|. \tag{E.2}$$

Proof. If $x(t) := \|\xi(t)\|$, what we want to show is that $x(t) \geq t\|\dot{\xi}(0)\|$ which, in consideration of the previous Lemma E.1, is equivalent to showing $x(t) \geq t\dot{x}(0)$. Therefore, it suffices to show that $\ddot{x}(t) \geq 0$ for all t so that $\dot{x}$ is increasing and hence $x(t) = \int_0^t \dot{x}(s)ds \geq t\dot{x}(0)$. By assumption $\dot{x}(0) > 0$ and since $x(0) = 0$, there exists some $\varepsilon > 0$ with $x(t) > 0$ for all $t \in (0, \varepsilon)$. For $t \in (0, \varepsilon)$, we calculate,

$$\begin{aligned}
\ddot{x}(t) &= \frac{\langle \ddot{\xi}(t), \xi(t)\rangle + \|\dot{\xi}(t)\|^2}{\|\xi(t)\|} - \frac{\langle \xi(t), \dot{\xi}(t)\rangle^2}{\|\xi(t)\|^3} \\
&= \frac{\langle A(t)\xi(t), \xi(t)\rangle}{\|\xi(t)\|} + \frac{\|\dot{\xi}(t)\|^2\|\xi(t)\|^2 - \langle \xi(t), \dot{\xi}(t)\rangle^2}{\|\xi(t)\|^3} \\
&\geq 0
\end{aligned}$$

where the last inequality comes from the positivity of $A(t)$ and the Cauchy-Schwarz inequality. Now, suppose we set $\tau = \sup\{\varepsilon > 0 : x(t) > 0 \text{ for all } t \in (0, \varepsilon)\}$. Then a continuity argument reveals that if $\tau < \infty$, $x(\tau) \geq \tau\dot{x}(0) > 0$, but by the definition of τ, $x(\tau) = 0$, a contradiction. Therefore $\tau = \infty$ and the above argument shows that $\ddot{x}(t) \geq 0$ for all $t \in (0, \infty)$. □

Proposition E.3. *Let M be a manifold with non-positive sectional curvature. For $1 \leq i \leq n$, let $A_i^{\mathcal{P}}$ be as defined in Eq. (3.4) and $S_i^{\mathcal{P}}$ be as in Eq. (3.10). Then, with $\Delta \in (0, 1]$, $S_i^{\mathcal{P}}(\Delta)^{-1}$ exists and further $\|S_i^{\mathcal{P}}(\Delta)^{-1}\| \leq \frac{1}{\Delta}$.*

Proof. Since we are assuming that M has non-positive sectional curvature, this therefore implies that $A_i^{\mathcal{P}}$ is positive semi-definite. Let $v \in \mathbb{R}^d$ and define $\xi(t) = S_i^{\mathcal{P}}(t)v$. Then, $\xi(0) = 0$ and $\ddot{\xi}(t) = A_i^{\mathcal{P}}(t)\xi(t)$. Using Proposition E.2 along with the fact that $\dot{S}_i^{\mathcal{P}}(0) = I$, we have

$$\begin{aligned}
\|\xi(\Delta)\| &\geq \Delta\|\dot{\xi}(0)\| \\
&= \Delta\|\dot{S}_i^{\mathcal{P}}(0)v\| \\
&= \Delta\|v\|.
\end{aligned}$$

That is

$$\|S_i^{\mathcal{P}}(\Delta)v\| \geq \Delta\|v\|$$

which shows that $S_i^{\mathcal{P}}(\Delta) : \mathbb{R}^d \to \mathbb{R}^d$ is injective and hence invertible and also that the spectrum of $S_i^{\mathcal{P}}(\Delta)$ is bounded below by Δ in magnitude, which implies that $\|S_i^{\mathcal{P}}(\Delta)^{-1}\| \leq \frac{1}{\Delta}$. □

Appendix F

Stochastic Notions

Throughout this section, we let $\{b_s : s \in [0,1]\}$ be an $\mathbb{R}^{\mathrm{d}}$-valued Brownian motion and $\mathcal{F}_t := \sigma\{b_s : 0 \leq s \leq t\}$.

Lemma F.1. *Let R be a $\mathrm{d} \times \mathrm{d}$ random symmetric matrix which is $\mathcal{F}_{s_{i-1}}$-measurable. Then,*

$$\langle R(\Delta_i b), \Delta_i b\rangle = 2\int_{s_{i-1}}^{s_i} \langle R(b_s - b_{s_{i-1}}), db_s\rangle + \mathrm{tr}(R)\Delta_i s. \tag{F.1}$$

Therefore,

$$\langle R(\Delta_i b), \Delta_i b\rangle - \mathrm{tr}(R)\Delta_i s = M_i(\Delta_i s) \tag{F.2}$$

where M_i is the square-integrable martingale given by

$$M_i(t) = 2\int_{s_{i-1}}^{s_{i-1}+t} \langle R(b_s - b_{s_{i-1}}), db_s\rangle \tag{F.3}$$

and, in particular,

$$\mathbb{E}\left[\langle R(\Delta_i b), \Delta_i b\rangle\right] = \mathrm{tr}(R)\Delta_i s. \tag{F.4}$$

Proof. It suffices to assume that R is a constant matrix. In this case, let $f : \mathbb{R}^d \to \mathbb{R}$ be defined by $f(x) = \langle Rx, x\rangle$. Then,

$$\begin{aligned}\frac{\partial}{\partial x^i} f(x) &= \frac{d}{dt}\bigg|_0 \langle R(x + te_i), x + te_i\rangle \\ &= 2\langle Rx, e_i\rangle\end{aligned}$$

and

$$\left(\frac{\partial}{\partial x^i}\right)^2 f(x) = 2\left.\frac{d}{dt}\right|_0 \langle R(x+te_i), e_i\rangle$$
$$= 2\langle Re_i, e_i\rangle.$$

In particular, $\nabla f(x) = Rx$ and $\Delta f(x) = 2\,\mathrm{tr}(R)$. Therefore, by Ito's lemma,

$$f(b_{s_i}) - f(b_{s_{i-1}}) = \int_{s_{i-1}}^{s_i} \langle \nabla f(b_s), db_s\rangle + \frac{1}{2}\int_{s_{i-1}}^{s_i} \Delta f(b_s) ds$$
$$= 2\int_{s_{i-1}}^{s_i} \langle Rb_s, db_s\rangle + \mathrm{tr}(R)\Delta_i s,$$

and hence,

$$\begin{aligned}\langle R(\Delta_i b), \Delta_i b\rangle &= \langle Rb_{s_i}, b_{s_i}\rangle + \langle Rb_{s_{i-1}}, b_{s_{i-1}}\rangle - 2\langle Rb_{s_{i-1}}, b_{s_i}\rangle \\ &= f(b_{s_i}) - f(b_{s_{i-1}}) + 2\left\{\langle Rb_{s_{i-1}}, b_{s_{i-1}}\rangle - \langle Rb_{s_{i-1}}, b_{s_i}\rangle\right\} \\ &= f(b_{s_i}) - f(b_{s_{i-1}}) - 2\int_{s_{i-1}}^{s_i} \langle Rb_{s_{i-1}}, db_s\rangle \\ &= 2\int_{s_{i-1}}^{s_i} \langle R(b_s - b_{s_{i-1}}), db_s\rangle + \mathrm{tr}(R)\Delta_i s.\end{aligned}$$

□

Corollary F.2. *If $\{R_i\}_{i=1}^n$ is a collection of random $\mathrm{d}\times\mathrm{d}$ symmetric matrices such that R_i is $\mathcal{F}_{s_{i-1}}$-measurable then,*

$$\sum_{i=1}^n \left(\langle R_i(\Delta_i b), \Delta_i b\rangle - \mathrm{tr}(R_i)\Delta_i s\right) = M_1 \tag{F.5}$$

where M_1 is the square integrable martingale given by

$$M_t = 2\int_0^t \langle R_s(b_s - b_{\underline{s}}), db_s\rangle \tag{F.6}$$

with $R_s = R_i$ when $s \in (s_{i-1}, s_i]$ and, as usual, $\underline{s} = s_{i-1}$ whenever $s \in (s_{i-1}, s_i]$.

Proof. Apply Lemma F.1 for each i and sum. □

Lemma F.3. *Let $\{X_s : s \in [0,1]\}$ be an $\mathbb{R}^{\mathrm{d}}$-valued process adapted to $\{\mathcal{F}_s : s \in [0,1]\}$. Define the square-integrable martingale Y_t by*

$$Y_t = \int_0^t \langle X_s, db_s\rangle.$$

Suppose that for each $p \in \mathbb{R}$, $\mathbb{E}\left[e^{\frac{p^2}{2}\langle Y\rangle_1}\right] = S_p < \infty$. *Then,*

$$1 \le \mathbb{E}\left[e^{Y_1}\right] \le \sqrt{S_{2p}}. \tag{F.7}$$

Proof. From [24, Chapter VIII, Proposition 1.15], $Z_t^{(p)} := e^{pY_t - \frac{p^2}{2}\langle Y\rangle_t}$ is a (uniformly integrable) martingale and particularly $\mathbb{E}[Z_t] = 1$ for all $t \in [0,1]$. Therefore,

$$\begin{aligned}
1 = \mathbb{E}\left[e^{Y_t - \frac{1}{2}\langle Y\rangle_t}\right] &\le \mathbb{E}\left[e^{Y_t - \frac{1}{2}\langle Y\rangle_t} e^{\frac{1}{2}\langle Y\rangle_t}\right] = \mathbb{E}\left[e^{Y_1}\right] \\
&= \mathbb{E}\left[e^{Y_t - \langle Y\rangle_t} e^{\langle Y\rangle_t}\right] \\
&\le \sqrt{\mathbb{E}\left[e^{2Y_t - 2\langle Y\rangle_t}\right]\mathbb{E}\left[e^{2\langle Y\rangle_t}\right]} \\
&= \sqrt{\mathbb{E}\left[Z_1^{(2)}\right]\mathbb{E}\left[e^{2\langle Y\rangle_t}\right]} \\
&= \sqrt{S_{2p}}.
\end{aligned}$$

□

The proof of the following Corollary follows that of [1, Proposition 8.8].

Corollary F.4. *Suppose that there is some* $K < \infty$ *with* $\|R_i\| \le K$ *for all* i. *Given any* $p \in \mathbb{R}$,

$$1 \le \mathbb{E}\left[\exp\{p\sum_{i=1}^{n}\left(\langle R_i\Delta_i b, \Delta_i b\rangle - \mathrm{tr}(R_i)\right)\}\right] \le e^{2\,\mathrm{d}\,p^2K^2|\mathcal{P}|}. \tag{F.8}$$

Proof. Defining M_t as in Corollary F.2, the quadratic variation is given by,

$$\langle M\rangle_t = 4\int_0^t |R_s(b_s - b_{\underline{s}})|^2 ds \le 4K^2\int_0^t |b_s - b_{\underline{s}}|^2 ds.$$

Therefore,

$$\begin{aligned}
\mathbb{E}[e^{\frac{p^2}{2}\langle M\rangle_1}] &\le \mathbb{E}\left[\exp\{2p^2K^2\int_0^1 |b_s - b_{\underline{s}}|^2 ds\}\right] \\
&= \prod_{i=1}^{n}\mathbb{E}\left[\exp\{2p^2K^2(\Delta_i s)^2\int_0^1 |b_s|^2 ds\}\right].
\end{aligned}$$

An application of Fernique's Theorem tells us that

$$\mathbb{E}\left[\exp\{2p^2K^2(\Delta_i s)^2\int_0^1 |b_s|^2 ds\}\right] \le \exp\left\{2dp^2K^2(\Delta_i s)^2\right\},$$

which together with the above inequalities implies that

$$\mathbb{E}[e^{\frac{p^2}{2}\langle M\rangle_1}] \le \exp\left\{2dp^2K^2|\mathcal{P}|\right\}.$$

Therefore, we can apply Lemma F.3 yielding,

$$1 \le \mathbb{E}\left[\exp\{p\sum_{i=1}^{n}\left(\langle R_i\Delta_i b, \Delta_i b\rangle - \operatorname{tr}(R_i)\right)\}\right] \le e^{2\,\mathrm{d}\,p^2K^2|\mathcal{P}|}.$$

□

Bibliography

[1] L. Andersson and B. K. Driver. Finite dimensional approximations to wiener measure and path integral formulas on manifolds. *Journal of Functional Analysis*, 165:430–498, 1999.

[2] Vincent Cachia, Hagen Neidhardt, and Valentin A. Zagrebnov. Accretive perturbations and error estimates for the Trotter product formula. *Integral Equations Operator Theory*, 39(4):396–412, 2001.

[3] Vincent Cachia, Hagen Neidhardt, and Valentin A. Zagrebnov. Comments on the Trotter product formula error-bound estimates for nonself-adjoint semigroups. *Integral Equations Operator Theory*, 42(4):425–448, 2002.

[4] R. W. R. Darling. On the convergence of Gangolli processes to Brownian motion on a manifold. *Stochastics*, 12(3-4):277–301, 1984.

[5] Bruce K. Driver. Curved Wiener space analysis. In *Real and stochastic analysis*, Trends Math., pages 43–198. Birkhäuser Boston, Boston, MA, 2004.

[6] K. D. Elworthy. *Stochastic differential equations on manifolds*, volume 70 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1982.

[7] Pavel Exner, Hagen Neidhardt, and Valentin A. Zagrebnov. Remarks on the Trotter-Kato product formula for unitary groups. *Integral Equations Operator Theory*, 69(4):451–478, 2011.

[8] R. P. Feynman. Space-time approach to non-relativistic quantum mechanics. *Rev. Modern Physics*, 20:367–387, 1948.

[9] Richard P. Feynman and Albert R. Hibbs. *Quantum mechanics and path integrals*. Dover Publications Inc., Mineola, NY, emended edition, 2010. Emended and with a preface by Daniel F. Styer.

[10] James Glimm and Arthur Jaffe. *Quantum physics*. Springer-Verlag, New York, 1981. A functional integral point of view.

[11] Leonard Gross. Lattice gauge theory; heuristics and convergence. In *Stochastic processes—mathematics and physics (Bielefeld, 1984)*, volume 1158 of *Lecture Notes in Math.*, pages 130–140. Springer, Berlin, 1986.

[12] Roger A. Horn and Charles R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1985.

[13] Elton P. Hsu. *Stochastic analysis on manifolds*, volume 38 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[14] T. Ichinose, H. Neidhardt, and V. A. Zagrebnov. Operator norm convergence of Trotter-Kato product formula. In *Ukrainian Mathematics Congress—2001 (Ukrainian)*, pages 100–106. Natsīonal. Akad. Nauk Ukraïni Īnst. Mat., Kiev, 2002.

[15] Takashi Ichinose, Hagen Neidhardt, and Valentin A. Zagrebnov. Trotter-Kato product formula and fractional powers of self-adjoint generators. *J. Funct. Anal.*, 207(1):33–57, 2004.

[16] Hiroshi Kunita. *Stochastic flows and stochastic differential equations*, volume 24 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[17] Adrian P. C. Lim. Path integrals on a compact manifold with non-negative curvature. *Reviews in Mathematical Physics*, 19:967–1044, 2007.

[18] H. Neidhardt and V. A. Zagrebnov. On error estimates for the Trotter-Kato product formula. *Lett. Math. Phys.*, 44(3):169–186, 1998.

[19] H. Neidhardt and V. A. Zagrebnov. On the operator-norm convergence of the Trotter-Kato product formula. In *Mathematical results in quantum mechanics (Prague, 1998)*, volume 108 of *Oper. Theory Adv. Appl.*, pages 323–334. Birkhäuser, Basel, 1999.

[20] H. Neidhardt and V. A. Zagrebnov. Trotter-Kato product formula and operator-norm convergence. *Comm. Math. Phys.*, 205(1):129–159, 1999.

[21] H. Neidhardt and V. A. Zagrebnov. Trotter-Kato product formula and symmetrically normed ideals. *J. Funct. Anal.*, 167(1):113–147, 1999.

[22] Mark A. Pinsky. Stochastic Riemannian geometry. In *Probabilistic analysis and related topics, Vol. 1*, pages 199–236. Academic Press, New York, 1978.

[23] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York, 1972.

[24] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1994.

[25] Barry Simon. *Trace ideals and their applications*, volume 120 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2005.