UNIVERSITY OF CALIFORNIA, SAN DIEGO

# Classical Limit on Quantum Mechanics for Unbounded Observables

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requirements for the degree
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in

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by

Pun Wai Tong

Committee in charge:

Professor Bruce K. Driver, Chair
Professor Kim Griest
Professor Todd Kemp
Professor Laurence B. Milstein
Professor Jacob Sterbenz

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EPIGRAPH

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—Jay Chou (周杰倫)

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## VITA

| | |
|---|---|
| 2007 | B. S. in Mathematics, The Chinese University of Hong Kong |
| 2009 | M.A. in Mathematics, The Chinese University of Hong Kong |
| 2010-2015 | Graduate Teaching Assistant, University of California, San Diego |
| 2016 | Ph. D. in Mathematics, University of California, San Diego |

## PUBLICATIONS

Tong, P.W., “Conjugate points and Singularity Theorem in Space-Time”, The Chinese University of Hong Kong, 2009.

Driver, B.K.; Tong, P.W., “Powers of Symmetric Differential Operators I,” submitted, *Journal of Functional Analysis*, 2016.

Driver, B.K.; Tong, P.W., “On the Classical Limit of Quantum Mechanics I,” submitted, *Communication in Mathematical Physics*, 2016.

ABSTRACT OF THE DISSERTATION

**Classical Limit on Quantum Mechanics for Unbounded Observables**

by

Pun Wai Tong

Doctor of Philosophy in Mathematics

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Professor Bruce K. Driver, Chair

This dissertation is divided into two parts. In Part I of this dissertation— On the Classical Limit of Quantum Mechanics, we extend a method introduced by Hepp in 1974 for studying the asymptotic behavior of quantum expectations in the limit as Plank's constant ($\hbar$) tends to zero. The goal is to allow for unbounded observables which are (non-commutative) polynomial functions of the position and momentum operators. [This is in contrast to Hepp's original paper where the "observables" were, roughly speaking, required to be bounded functions of the position and momentum operators.] As expected the leading order contributions of the quantum expectations come from evaluating the "symbols" of the observables along the classical trajectories

while the next order contributions (quantum corrections) are computed by evolving the $\hbar = 1$ observables by a linear canonical transformations which is determined by the second order pieces of the quantum mechanical Hamiltonian.

Part II of the dissertation — Powers of Symmetric Differential Operators is devoted to operator theoretic properties of a class of linear symmetric differential operators on the real line. In more detail, let $L$ and $\tilde{L}$ be a linear symmetric differential operator with polynomial coefficients on $L^2(m)$ whose domain is the Schwartz test function space, $\mathcal{S}$. We study conditions on the polynomial coefficients of $L$ and $\tilde{L}$ which implies operator comparison inequalities of the form $\left(\bar{\tilde{L}} + \tilde{C}\right)^r \leq C_r\left(\bar{L} + C\right)^r$ for all $0 \leq r < \infty$. These comparison inequalities (along with their generalizations allowing for the parameter $\hbar > 0$ in the coefficients) are used to supply a large class of Hamiltonian operators which verify the assumptions needed for the results in Part I of this dissertation.

# Chapter 1

# Introduction

The whole dissertation is divided into two main parts— “On the Classical Limit of Quantum Mechanics” and “Powers of Symmetric Differential Operators” which are introduced in Sections 1 and 2 below respectively in this chapter. Definitions, notations and symbols in these two parts are independent. We may redefine some definitions, notations and symbols if necessary.

## 1 On the Classical Limit of Quantum Mechanics

This section is the introduction of Part I below in this dissertation. In the limit where Planck's constant ($\hbar$) tends to zero, quantum mechanics is supposed to reduce to the laws of classical mechanics and their connection was first shown by P. Ehrenfest in [5]. There is in fact a very large literature devoted in one way or another to this theme. Although it is not our intent nor within our ability to review this large literature here, nevertheless the interested reader can find more information by searching for terms like, correspondence principle, WKB approximation, pseudo-differential operators, micro-local analysis, Moyal brackets, star products, deformation quantization, Gaussian wave packet, and stationary phase approximation in the context of Feynmann path integrals to name a few. For more general background pertaining to quantum mechanics and its classical limit the reader may wish to consult (for example) [6, 15, 17, 22, 24, 42]. In Part I we wish to concentrate on a formulation and a method to understand the classical limit of quantum mechanics which was introduced by Hepp [18] in 1974.

Part I is an elaboration on Hepp's method to allow for unbounded observables which was motivated by Rodnianski and Schlein's [33] treatment of the mean field dynamics associated to Bose Einstein condensation. There is large literature related to Hepp's method, see for example [1, 8–14, 23, 33, 40, 41] and more recently [4]. The nice papers by Zucchini, (see Theorem 5.8 of [41] and Theorem 5.10 of [40] ) are closely related to this work. In these papers, Zucchini (using ideas of Ginibre and Velo in [8, 9]) studies the classical limit for unbounded observables which are at most quadratic in the position and momentum observables with Hamiltonian operators which are in standard Shrödinger form. In Part I, we consider observables and Hamiltonians which are non-commutative polynomials (of arbitrary large degree) in the postition and momentum variables. In order to emphasize the main ideas and to not be needlessly encumbered by more complicated notation we will restrict our attention to systems with only one degree of freedom. Before summarizing the main results of Part I, we first need to introduce some notation. [See Chapter 2 below for more details on the basic setup-used in Part I.]

## 1.1 Basic Setup

Let $\alpha_0 = (\xi + i\pi)/\sqrt{2} \in \mathbb{C}$ ($\mathbb{C} \cong T^*\mathbb{R}$ is to be thought of as phase space), $H(\theta, \theta^*)$ be a symmetric [see Notation 2.8] non-commutative polynomial in two indeterminates, $\{\theta, \theta^*\}$, $H^{\mathrm{cl}}(z) := H(z, \bar{z})$ for all $z \in \mathbb{C}$ be the **symbol** of $H$. [By Remark 2.15 below, we know $H^{\mathrm{cl}}$ is real valued.] A differentiable function, $\alpha(t) \in \mathbb{C}$, is said to satisfy Hamilton's equations of motion with an initial condition

$\alpha_0 \in \mathbb{C}$ if

$$i\dot{\alpha}(t) = \left(\frac{\partial}{\partial\bar{\alpha}} H^{\mathrm{cl}}\right)(\alpha(t)) \text{ and } \alpha(0) = \alpha_0. \tag{1.1}$$

[See Section 1 in Chapter 2 where we recall that Eq. (1.1) is equivalent to the standard real form of Hamilton's equations of motion.] Further, let $\Phi(t, \alpha_0) = \alpha(t)$ (where $\alpha(t)$ is the solution to Eq. (1.1) ) be the flow associated to Eq. (1.1) and $\Phi'(t, \alpha_0) : \mathbb{C} \to \mathbb{C}$ be the real-linear differential of this flow relative to its starting point, i.e. for all $z \in \mathbb{C}$ let

$$\Phi'(t, \alpha_0) z := \frac{d}{ds}|_{s=0} \Phi(t, \alpha_0 + sz). \tag{1.2}$$

As $z \to \Phi'(t, \alpha_0) z$ is a real-linear function of $z$, for each $\alpha_0 \in \mathbb{C}$ there exists unique complex valued functions $\gamma(t)$ and $\delta(t)$ such that

$$\Phi'(t, \alpha_0) z = \gamma(t) z + \delta(t) \bar{z}. \tag{1.3}$$

where $\gamma(0) = 1$ and $\delta(0) = 0$.

We now turn to the quantum mechanical setup. Let $L^2(m) := L^2(\mathbb{R}, m)$ be the Hilbert space of square integrable complex valued functions on $\mathbb{R}$ relative to Lebesgue measure, $m$. The inner product on $L^2(m)$ is taken to be

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x) \bar{g}(x) \, dm(x) \ \forall \ f, g \in L^2(m) \tag{1.4}$$

and the corresponding norm is $\|f\| = \|f\|_2 = \sqrt{\langle f, f \rangle}$. [Note that we are using the mathematics convention that $\langle f, g \rangle$ is linear in the first variable and conjugate linear in the second.] We say $A$ is an operator on $L^2(m)$ if $A$ is a linear (possibly unbounded) operator from a dense subspace, $D(A)$, to $L^2(m)$. As usual if $A$ is closable, then its adjoint, $A^*$, also has a dense domain and $A^{**} = \bar{A}$ where $\bar{A}$ is the closure of $A$.

**Notation 1.1.** *As is customary, let* $\mathcal{S} := \mathcal{S}(\mathbb{R}) \subset L^2(m)$ *denote Schwartz space of smooth rapidly decreasing complex valued functions on* $\mathbb{R}$.

**Definition 1.2** (Formal Adjoint)**.** *If* $A$ *is a closable operator on* $L^2(m)$ *such that* $D(A) = \mathcal{S}$ *and* $\mathcal{S} \subset D(A^*)$, *then we define the* ***formal adjoint*** *of* $A$ *to be the operator,* $A^\dagger := A^*|_{\mathcal{S}}$. *Thus* $A^\dagger$ *is the unique operator with* $D\left(A^\dagger\right) = \mathcal{S}$ *such that* $\langle Af, g\rangle = \left\langle f, A^\dagger g\right\rangle$ *for all* $f, g \in \mathcal{S}$.

**Definition 1.3** (Annihilation and Creation operators)**.** *For* $\hbar > 0$, *let* $a_\hbar$ *be the* ***annihilation operator*** *acting on* $L^2(m)$ *defined so that* $D(a_\hbar) = \mathcal{S}$ *and*

$$(a_\hbar f)(x) := \sqrt{\frac{\hbar}{2}}\left(xf(x) + \partial_x f(x)\right) \text{ for } f \in \mathcal{S}. \tag{1.5}$$

*The corresponding* ***creation operator*** *is* $a_\hbar^\dagger$ *– the formal adjoint of* $a_\hbar$, *i.e.*

$$\left(a_\hbar^\dagger f\right)(x) := \sqrt{\frac{\hbar}{2}}\left(xf(x) - \partial_x f(x)\right) \text{ for } f \in \mathcal{S}. \tag{1.6}$$

*We write* $a$ *and* $a^\dagger$ *for* $a_\hbar$ *and* $a_\hbar^\dagger$ *respectively when* $\hbar = 1$.

Notice that both the creation $\left(a_\hbar^\dagger\right)$ and annihilation $(a_\hbar)$ operators preserve $\mathcal{S}$ and satisfy the canonical commutation relations (CCRs),

$$\left[a_\hbar, a_\hbar^\dagger\right] = \hbar I|_{\mathcal{S}}. \tag{1.7}$$

For each $t \in \mathbb{R}$ and $\alpha_0 \in \mathbb{C}$ we also define two operators, $a(t, \alpha_0)$ and $a^\dagger(t, \alpha_0)$ acting on $\mathcal{S}$ by,

$$a(t, \alpha_0) = \gamma(t)\, a + \delta(t)\, a^\dagger \text{ and} \tag{1.8}$$

$$a^\dagger(t, \alpha_0) = \bar{\gamma}(t)\, a^\dagger + \bar{\delta}(t)\, a, \tag{1.9}$$

where $\gamma(t)$ and $\delta(t)$ are determined as in Eq. (1.3). Because we are going to fix

$\alpha_0 \in \mathbb{C}$ once and for all in Part I we will simply write $a(t)$ and $a^\dagger(t)$ for $a(t,\alpha_0)$ and $a^\dagger(t,\alpha_0)$ respectively. These operators still satisfy the CCRs, indeed making use of Eq. (2.12) below we find,

$$\begin{aligned}\left[a(t), a^\dagger(t)\right] &= \left[\bar{\gamma}(t)\, a^\dagger + \bar{\delta}(t)\, a, \gamma(t)\, a + \delta(t)\, a^\dagger\right] \\ &= \left(|\gamma(t)|^2 - |\delta(t)|^2\right) I = I. \end{aligned} \tag{1.10}$$

This result also may be deduced from Theorem 5.13 below.

**Definition 1.4** (Harmonic Oscillator Hamiltonian)**.** *The **Harmonic Oscillator Hamiltonian** is the positive self-adjoint operator on $L^2(m)$ defined by*

$$\mathcal{N}_\hbar := a_\hbar^* \bar{a}_\hbar = \hbar a^* \bar{a}. \tag{1.11}$$

*As above we write $\mathcal{N}$ for $\mathcal{N}_1$ and refer to $\mathcal{N}$ as the **Number operator**.*

**Remark 1.5.** *The operator, $\mathcal{N}_\hbar$, is self-adjoint by a well know theorem of von Neumann (see for example Theorem 3.24 on p. 275 in [20]). It is also standard and well known (or see Corollary 3.26 below) that*

$$D(a_\hbar^*) = D(\bar{a}_\hbar) = D\left(\mathcal{N}_\hbar^{1/2}\right) = D(\partial_x) \cap D(M_x).$$

**Definition 1.6** (Weyl Operators)**.** *For $\alpha := (\xi + i\pi)/\sqrt{2} \in \mathbb{C}$ as in Eq. (2.1), define the unitary **Weyl Operator** $U(\alpha)$ on $L^2(m)$ by*

$$U(\alpha) = e^{\left(\overline{\alpha \cdot a^\dagger - \bar{\alpha} \cdot a}\right)} = e^{i\left(\overline{\pi M_x - \frac{\xi}{i}\partial_x}\right)}. \tag{1.12}$$

*More generally, if $\hbar > 0$, let*

$$U_\hbar(\alpha) = U\left(\frac{\alpha}{\sqrt{\hbar}}\right) = \exp\left(\frac{1}{\hbar}\left(\overline{\alpha \cdot a_\hbar^\dagger - \bar{\alpha} \cdot a_\hbar}\right)\right). \tag{1.13}$$

The symmetric operator, $i\left(\alpha \cdot a_\hbar^\dagger - \bar{\alpha} \cdot a_\hbar\right)$, can be shown to be essentially self adjoint on $\mathcal{S}$ by the same methods used to show $\frac{1}{i}\partial_x$ is essentially self adjoint on $C_c^\infty(\mathbb{R})$ in Proposition 9.29 of [15]. Hence the Weyl operators, $U_\hbar(\alpha)$, are well defined unitary operators by Stone's theorem. Alternatively, see Proposition 2.4 below for an explicit description of $U_\hbar(\alpha)$.

**Definition 1.7.** *Given an operator $A$ on $L^2(m)$ let*

$$\langle A \rangle_\psi := \langle A\psi, \psi \rangle$$

*denote the* ***expectation*** *of $A$ relative to a normalized state $\psi \in D(A)$. The* ***variance*** *of $A$ relative to a normalized state $\psi \in D(A^2)$ is then defined as*

$$\mathrm{Var}_\psi(A) := \left\langle A^2 \right\rangle_\psi - \langle A \rangle_\psi^2 .$$

From Corollary 3.6 below; if $\psi \in \mathcal{S}$ is a normalized state and $P(\theta, \theta^*)$ is a non-commutative polynomial in two variables $\{\theta, \theta^*\}$, then

$$\begin{aligned} \left\langle P\left(a_\hbar, a_\hbar^\dagger\right)\right\rangle_{U_\hbar(\alpha)\psi} &= P(\alpha, \bar{\alpha}) + O\left(\sqrt{\hbar}\right) \\ \mathrm{Var}_{U_\hbar(\alpha)\psi}\left(P\left(a_\hbar, a_\hbar^*\right)\right) &= O\left(\sqrt{\hbar}\right). \end{aligned}$$

Consequently, $U_\hbar(\alpha)\psi$ is a state which is concentrated in phase space near the $\alpha$ and are therefore reasonable quantum mechanical approximations of the classical state $\alpha$.

**Definition 1.8** (Non-Commutative Laws)**.** *If $A_1, \ldots, A_k$ are operators on $L^2(m)$ having a common dense domain $D$ such that $A_j D \subset D$, $D \subset D\left(A_j^*\right)$, and $A_j^* D \subset D$ for $1 \le j \le k$, then for a unit vector, $\psi \in D$, and a non-commutative polynomial,*

$$\mathbf{P} := P\left(\theta_1, \ldots, \theta_k, \theta_1^*, \ldots, \theta_k^*\right)$$

*in $2k$ indeterminants, we let*

$$\mu(\mathbf{P}) := \langle P(A_1, \ldots, A_k, A_1^*, \ldots, A_k^*) \rangle_\psi = \langle P(A_1, \ldots, A_k, A_1^*, \ldots, A_k^*)\, \psi, \psi \rangle .$$

*The linear functional, $\mu$, on the linear space of non-commutative polynomials in $2k$ – variables is referred to as the **law** of $(A_1, \ldots, A_k)$ relative to $\psi$ and we will in the sequel denote $\mu$ by $\mathrm{Law}_\psi(A_1, \ldots, A_k)$.*

## 1.2 Main results

Theorem 1.17 and Corollaries 1.19 and 1.21 below on the convergence of correlation functions are the main results of Part I. [The proofs of these results will be given Chapter 9.] The results of Part I will be proved under the Assumption 1.11 described below. First we need a little more notation.

**Definition 1.9** (Subspace Symmetry)**.** *Let $S$ be a dense subspace of a Hilbert space $\mathcal{K}$ and $A$ be an operator on $\mathcal{K}$. We say $A$ is **symmetric on** $S$ provided, $S \subseteq D(A)$ and $A|_S \subseteq A|_S^*$, i.e. $\langle Af, g \rangle = \langle f, Ag \rangle$ for all $f, g \in S$.*

We now introduce three different partial ordering on symmetric operators on a Hilbert space.

**Notation 1.10.** *Let $S$ be a dense subspace of a Hilbert space, $\mathcal{K}$, and $A$ and $B$ be two densely defined operators on $\mathcal{K}$.*

1. *We write $A \preceq_S B$ if both $A$ and $B$ are symmetric on $S$ and*

$$\langle A\psi, \psi \rangle_{\mathcal{K}} \le \langle B\psi, \psi \rangle_{\mathcal{K}} \text{ for all } \psi \in S.$$

2. *We write $A \preceq B$ if $A \preceq_{D(B)} B$, i.e. $D(B) \subset D(A)$, $A$ and $B$ are both*

*symmetric on* $D(B)$, *and*

$$\langle A\psi,\psi\rangle_{\mathcal{K}} \leq \langle B\psi,\psi\rangle_{\mathcal{K}} \text{ for all } \psi \in D(B).$$

3. *If* $A$ *and* $B$ *are non-negative (i.e.* $0 \preceq A$ *and* $0 \preceq B$*) self adjoint operators on a Hilbert space* $\mathcal{K}$, *then we say* $A \leq B$ *if and only if* $D\left(\sqrt{B}\right) \subseteq D\left(\sqrt{A}\right)$ *and*

$$\left\|\sqrt{A}\psi\right\| \leq \left\|\sqrt{B}\psi\right\| \text{ for all } \psi \in D\left(\sqrt{B}\right).$$

Interested readers may read Section 10.3 of [34] to learn more properties and relations among these different partial orderings. Let us now record the main assumptions which will be needed for the main theorems in Part I. In this assumption, $\mathbb{R}\langle\theta,\theta^*\rangle$ denotes the subspace of non-commutative polynomials with real coefficients, see Section 4 in Chapter 2.

**Assumption 1.11.** *We say* $H(\theta,\theta^*) \in \mathbb{R}\langle\theta,\theta^*\rangle$ *satisfies Assumption 1. if,* $H$ *is symmetric (see Definition 2.10),* $d = \deg_\theta H \geq 2$ *(see Notation 2.8) is even and* $H_\hbar := \overline{H\left(a_\hbar, a_\hbar^\dagger\right)}$ *satisfies; there exists constants* $C > 0$, $C_\beta > 0$ *for* $\beta \geq 0$, *and* $1 \geq \eta > 0$ *such that for all* $\hbar \in (0,\eta)$,

1. $H_\hbar$ *is self-adjoint and* $H_\hbar + C \succeq I$, *and*

2. *for all* $\beta \geq 0$,

$$\mathcal{N}_\hbar^\beta \preceq C_\beta (H_\hbar + C)^\beta. \tag{1.14}$$

The next Proposition provides a simple class of example $H \in \mathbb{R}\langle\theta,\theta^*\rangle$ satisfying Assumption 1.11 whose infinite dimensional analogues feature in some of the papers involving Bose-Einstein condensation, see for example, [1,33].

**Proposition 1.12** ($p(\theta^*\theta)$ – examples)**.** *Let* $p(x) \in \mathbb{R}[x]$ *(the polynomials in* $x$ *with real coefficients) and suppose* $\deg(p) \geq 1$ *and the leading order coefficient is positive. Then* $H(\theta,\theta^*) = p(\theta^*\theta) \in \mathbb{R}\langle\theta,\theta^*\rangle$ *will satisfy the hypothesis of Assumption 1.11.*

**Proof.** First we will show

$$H_\hbar = \overline{p\left(a_\hbar^\dagger a_\hbar\right)} = p\left(\mathcal{N}_\hbar\right).$$

We know that $p\left(\mathcal{N}_\hbar\right)$ is self-adjoint and by Corollaries 3.17 and 3.30 below we have

$$p\left(\mathcal{N}_\hbar\right) = p\left(a_\hbar^* \bar{a}_\hbar\right) = p\left(\overline{a_\hbar^\dagger}\bar{a}_\hbar\right) \subset \overline{p\left(a_\hbar^\dagger a_\hbar\right)}.$$

Taking adjoint of this inclusion implies

$$p\left(a_\hbar^\dagger a_\hbar\right)^* = \overline{p\left(a_\hbar^\dagger a_\hbar\right)}^* \subset p\left(\mathcal{N}_\hbar\right)^* = p\left(\mathcal{N}_\hbar\right).$$

However, since $p\left(a_\hbar^\dagger a_\hbar\right)$ is symmetric we also have

$$p\left(a_\hbar^\dagger a_\hbar\right) \subset p\left(a_\hbar^\dagger a_\hbar\right)^* = \overline{p\left(a_\hbar^\dagger a_\hbar\right)}^* \subset p\left(\mathcal{N}_\hbar\right)$$

which implies

$$\overline{p\left(a_\hbar^\dagger a_\hbar\right)} \subset p\left(\mathcal{N}_\hbar\right).$$

Since there exists $C > 0$ and $C_\beta$ for any $\beta \geq 0$ such that $x \leq C_\beta\left(p\left(x\right) + C\right)$ for $x \geq 0$, it follows by the spectral theorem that $H_\hbar$ satisfies Eq. (1.14). ■

The next example provides a much broader class of $H \in \mathbb{R}\langle\theta, \theta^*\rangle$ satisfying Assumption 1.11 while the corresponding operators, $H_\hbar$, no longer typically commute with the number operator.

**Example 1.13** (Example Hamiltonians)**.** *Let* $m \geq 1$, $b_k \in \mathbb{R}\left[x\right]$ *for* $0 \leq k \leq m$, *and*

$$H\left(\theta, \theta^*\right) := \sum_{k=0}^{m} \frac{\left(-1\right)^k}{2^k}\left(\theta - \theta^*\right)^k b_k\left(\frac{1}{\sqrt{2}}\left(\theta + \theta^*\right)\right)\left(\theta - \theta^*\right)^k. \tag{1.15}$$

*With the use of Eqs. (1.5) and (1.6), it follows*

$$H_{\hbar} = \sum_{k=0}^{m} \hbar^k \partial_x^k M_{b_k\left(\sqrt{\hbar}x\right)} \partial_x^k \text{ on } \mathcal{S} \tag{1.16}$$

*If*

1. *each* $b_k(x)$ *is an even polynomial in* $x$ *with positive leading order coefficient, and* $b_m > 0$, *and*
2. $\deg_x(b_0) \geq 2$ *and* $\deg_x(b_k) \leq \deg_x(b_{k-1})$ *for* $1 \leq k \leq m$,

*then by Corollary 1.41 below,* $H(\theta, \theta^*)$ *satisfies Assumption 1.11. In particular, if* $m > 0$ *and* $V \in \mathbb{R}[x]$ *such that* $\deg_x V \in 2\mathbb{N}$ *such that* $\lim_{x\to\infty} V(x) = \infty$, *then*

$$H(\theta, \theta^*) = -\frac{m}{2}\left(\frac{\theta - \theta^*}{\sqrt{2}}\right)^2 + V\left(\frac{1}{\sqrt{2}}(\theta + \theta^*)\right) \text{ and} \tag{1.17}$$

$$H\left(a_{\hbar}, a_{\hbar}^{\dagger}\right) = -\frac{1}{2}\hbar m \partial_x^2 + V\left(\sqrt{\hbar}x\right) \tag{1.18}$$

*satisfies Assumption 1.11.*

**Remark 1.14.** *The essential self-adjointness of* $H\left(a_{\hbar}, a_{\hbar}^{\dagger}\right)$ *in Eq. (1.18) and all of its non-negative integer powers on* $\mathcal{S}$ *may be deduced using results in Chernoff [3] and Kato [21]. This fact along with Eq. (1.14) restricted to hold on* $\mathcal{S}$ *and for* $\beta \in \mathbb{N}$ *could be combined together to prove Eq. (1.14) for all* $\beta \geq 0$ *as is explained in Lemma 14.13 below.*

*Using Theorem B.2, for any symmetric noncommutative polynomial,* $H(\theta, \theta^*) \in \mathbb{R}\langle\theta, \theta^*\rangle$, *there exists polynomials,* $b_l\left(\sqrt{\hbar}, x\right) \in \mathbb{R}\left[\sqrt{\hbar}, x\right]$, *(polynomials in* $\sqrt{\hbar}$ *and* $x$ *with real coefficients), such that*

$$H\left(a_{\hbar}, a_{\hbar}^{\dagger}\right) = \sum_{k=0}^{m} \hbar^k \partial_x^k M_{b_k\left(\sqrt{\hbar}, \sqrt{\hbar}x\right)} \partial_x^k \text{ on } \mathcal{S}.$$

*If it so happens that these* $b_k\left(\sqrt{\hbar},\sqrt{\hbar}x\right)$ *satisfy the assumptions of Corollary 1.41 below, then Assumption 1.11 will hold for this* $H$.

**Example 1.15.** *Let*

$$H\left(\theta,\theta^*\right)=\theta^4+\theta^{*4}-\frac{7}{8}\left(\theta-\theta^*\right)\left(\theta+\theta^*\right)^2\left(\theta-\theta^*\right)\in\mathbb{R}\left\langle\theta,\theta^*\right\rangle. \tag{1.19}$$

*By using product rule repeatedly with Eqs. (1.5) and (1.6), it follows that*

$$H\left(a_h,a_h^\dagger\right)=\hbar^2\partial_x^2 b_2\left(\sqrt{\hbar},\sqrt{\hbar}x\right)\partial_x^2-\hbar\partial_x b_1\left(\sqrt{\hbar},\sqrt{\hbar}x\right)\partial_x+b_0\left(\sqrt{\hbar},\sqrt{\hbar}x\right)$$

*where*

$$b_0\left(\sqrt{\hbar},x\right)=\frac{1}{2}x^4+\frac{3h^2}{2},\ \ b_1\left(\sqrt{\hbar},x\right)=\frac{1}{2}x^2,\ \ \text{and}\ b_2\left(\sqrt{\hbar},x\right)=\frac{1}{2}.$$

*These polynomials satisfy the assumptions of Corollary 1.41 and therefore* $H\left(\theta,\theta^*\right)$ *in Eq. (1.19) satisfies Assumption 1.11.*

**Notation 1.16.** *Given a non-commutative polynomial*

$$P\left(\{\theta_i,\theta_i^*\}_{i=1}^n\right):=P\left(\theta_1,\dots,\theta_n,\theta_1^*,\dots,\theta_n^*\right)\in\mathbb{C}\left\langle\theta_1,\dots,\theta_n,\theta_1^*,\dots,\theta_n^*\right\rangle, \tag{1.20}$$

*in* $2n$ *– indeterminants,*

$$\Lambda_n:=\left\{\theta_1,\dots,\theta_n,\theta_1^*,\dots,\theta_n^*\right\}, \tag{1.21}$$

*let* $p_{\min}$ *denote the minimum degree among all non-constant monomials terms appearing in* $P\left(\{\theta_i,\theta_i^*\}_{i=1}^n\right)$. *In more detail there is a constant,* $P_0\in\mathbb{C}$, *such that* $P\left(\theta_1,\dots,\theta_n,\theta_1^*,\dots,\theta_n^*\right)-P_0$ *may be written as a linear combination in words in the alphabet,* $\Lambda_n$, *which have length no smaller than* $p_{\min}$.

**Theorem 1.17.** *Suppose* $H(\theta,\theta^*) \in \mathbb{R}\langle\theta,\theta^*\rangle$, $d = \deg_\theta H > 0$ *and* $1 \geq \eta > 0$ *satisfy Assumptions 1.11,* $\alpha_0 \in \mathbb{C}$, $\psi \in \mathcal{S}$ *is an* $L^2(m)$ *– normalized state and then let;*

1. $\alpha(t) \in \mathbb{C}$ *be the solution (which exists for all time by Proposition 3.8) to Hamilton's (classical) equations of motion (1.1),*

2. $a(t) = a(t,\alpha_0)$ *be the annihilation operator on* $L^2(m)$ *as in Eq. (1.8), and*

3. $A_\hbar(t)$ *denote* $a_\hbar$ *in the Heisenberg picture, i.e.*

$$A_\hbar(t) := e^{iH_\hbar t/\hbar} a_\hbar e^{-iH_\hbar t/\hbar}. \tag{1.22}$$

*If* $\{t_i\}_{i=1}^n \subset \mathbb{R}$ *and* $P(\{\theta_i,\theta_i^*\}_{i=1}^n) \in \mathbb{C}\langle\theta_1,\ldots,\theta_n,\theta_1^*,\ldots,\theta_n^*\rangle$ *is a non-commutative polynomial in* $2n$ *– indeterminants, then for* $0 < \hbar < \eta$*, we have*

$$\begin{aligned} &\left\langle P\left(\left\{A_\hbar(t_i) - \alpha(t_i), A_\hbar^\dagger(t_i) - \overline{\alpha}(t_i)\right\}_{i=1}^n\right)\right\rangle_{U_\hbar(\alpha_0)\psi} \\ &\quad = \left\langle P\left(\left\{\sqrt{\hbar}a(t_i), \sqrt{\hbar}a^\dagger(t_i)\right\}_{i=1}^n\right)\right\rangle_\psi + O\left(\hbar^{\frac{p_{\min}+1}{2}}\right). \end{aligned} \tag{1.23}$$

**Remark 1.18.** *The left member of Eq. (1.23) is well defined because; 1)* $U_\hbar(\alpha_0)\mathcal{S} = \mathcal{S}$ *(see Proposition 2.4) and 2)* $e^{itH_\hbar/\hbar}\mathcal{S} = \mathcal{S}$ *(see Proposition 6.3) from which it follows that* $A_\hbar(t)$ *and* $A_\hbar(t)^\dagger = e^{iH_\hbar t/\hbar}a_\hbar^\dagger e^{-iH_\hbar t/\hbar}$ *both preserve* $\mathcal{S}$ *for all* $t \in \mathbb{R}$.

This theorem is a variant of the results in Hepp [18] which now allows for unbounded observables. It should be emphasized that the operators, $a(t)$, are constructed using only knowledge of solutions to the classical ordinary differential equations of motions while the construction of $A_\hbar(t)$ requires knowledge of the quantum mechanical evolution. As an easy consequence of Theorem 1.17 we may conclude that

$$\mathrm{Law}_{U_\hbar(\alpha_0)\psi}\left(\{A_\hbar(t_i)\}_{i=1}^n\right) \cong \mathrm{Law}_\psi\left(\left\{\alpha(t_i) + \sqrt{\hbar}a(t_i)\right\}_{i=1}^n\right) \text{ for } 0 < \hbar \ll 1. \tag{1.24}$$

The precise meaning of Eq. (1.24) is given in the following corollary.

**Corollary 1.19.** *If we assume the same conditions and notations as in Theorem 1.17, then (for* $0<\hbar<\eta$*)*

$$\begin{aligned}&\left\langle P\left(\left\{A_{\hbar}\left(t_{i}\right),A_{\hbar}^{\dagger}\left(t_{i}\right)\right\}_{i=1}^{n}\right)\right\rangle_{U_{\hbar}\left(\alpha_{0}\right)\psi}\\&\quad=\left\langle P\left(\left\{\alpha\left(t_{i}\right)+\sqrt{\hbar}a\left(t_{i}\right),\overline{\alpha}\left(t_{i}\right)+\sqrt{\hbar}a^{\dagger}\left(t_{i}\right)\right\}_{i=1}^{n}\right)\right\rangle_{\psi}+O\left(\hbar\right).\end{aligned}\tag{1.25}$$

By expanding out the right side of Eq.(1.25), it follows that

$$\begin{aligned}&\left\langle P\left(\left\{A_{\hbar}\left(t_{i}\right),A_{\hbar}^{\dagger}\left(t_{i}\right)\right\}_{i=1}^{n}\right)\right\rangle_{U_{\hbar}\left(\alpha_{0}\right)\psi}\\&\quad=P\left(\left\{\alpha\left(t_{i}\right),\bar{\alpha}\left(t_{i}\right)\right\}_{i=1}^{n}\right)+\sqrt{\hbar}\left\langle P_{1}\left(\left\{\alpha\left(t_{i}\right):a\left(t_{i}\right),a^{\dagger}\left(t_{i}\right)\right\}_{i=1}^{n}\right)\right\rangle_{\psi}+O\left(\hbar\right)\end{aligned}\tag{1.26}$$

where $P_1\left(\left\{\alpha\left(t_i\right):\theta_i,\theta_i^*\right\}_{i=1}^n\right)$ is a degree one homogeneous polynomial of $\{\theta_i,\theta_i^*\}_{i=1}^n$ with coefficients depending smoothly on $\{\alpha\left(t_i\right)\}_{i=1}^n$ . Equation (1.26) states that the quantum expectation values,

$$\left\langle P\left(\left\{A_{\hbar}\left(t_{i}\right),A_{\hbar}^{\dagger}\left(t_{i}\right)\right\}_{i=1}^{n}\right)\right\rangle_{U_{\hbar}\left(\alpha_{0}\right)\psi},\tag{1.27}$$

closely track the corresponding classical values $P\left(\left\{\alpha\left(t_i\right),\bar{\alpha}\left(t_i\right)\right\}_{i=1}^n\right)$ . The $\sqrt{\hbar}$ term in Eq. (1.26) represent the first quantum corrections (or fluctuations ) beyond the leading order classical behavior.

**Remark 1.20.** *If both* $H\left(\theta,\theta^*\right),\ \widetilde{H}\left(\theta,\theta^*\right)\in\mathbb{R}\left\langle\theta,\theta^*\right\rangle$ *both satisfy Assumption 1.11 and are such that* $H^{\mathrm{cl}}\left(\alpha\right):=H\left(\alpha,\overline{\alpha}\right)$ *and* $\widetilde{H}^{\mathrm{cl}}\left(\alpha\right):=\widetilde{H}\left(\alpha,\overline{\alpha}\right)$ *are equal modulo a constant, then Eq. (1.26) also holds with the* $A_{\hbar}\left(t_i\right)$ *and* $A_{\hbar}^{\dagger}\left(t_i\right)$ *appearing on the left side of this equation being replaced by*

$$e^{i\tilde{H}_{\hbar}t_i/\hbar}a_{\hbar}e^{-i\tilde{H}_{\hbar}t_i/\hbar}\ \text{and}\ e^{i\tilde{H}_{\hbar}t_i/\hbar}a_{\hbar}^{\dagger}e^{-i\tilde{H}_{\hbar}t_i/\hbar}$$

*where* $\tilde{H}_\hbar := \overline{\tilde{H}\left(a_\hbar, a_\hbar^\dagger\right)}$. *In other words, if we view* $H$ *and* $\tilde{H}$ *as two "quantizations"of* $H^{\mathrm{cl}}$, *then the quantum expectations relative to* $H$ *and* $\tilde{H}$ *agree up to order* $\sqrt{\hbar}$.

**Corollary 1.21.** *Under the same conditions in Theorem 1.17, we let* $\psi_\hbar = U_\hbar(\alpha_0)\psi$. *As* $\hbar \to 0^+$, *we have*

$$\left\langle P\left(\left\{A_\hbar(t_i), A_\hbar^\dagger(t_i)\right\}_{i=1}^n\right)\right\rangle_{\psi_\hbar} \to P\left(\{\alpha_i(t), \bar{\alpha}_i(t)\}_{i=1}^n\right), \tag{1.28}$$

*and*

$$\left\langle P\left(\left\{\frac{A_\hbar(t_i) - \alpha(t_i)}{\sqrt{\hbar}}, \frac{A_\hbar^\dagger(t_i) - \bar{\alpha}(t_i)}{\sqrt{\hbar}}\right\}_{i=1}^n\right)\right\rangle_{\psi_\hbar} \to \left\langle P\left(\left\{a(t_i), a^\dagger(t_i)\right\}_{i=1}^n\right)\right\rangle_\psi. \tag{1.29}$$

*We abbreviate this convergence by saying*

$$\mathrm{Law}_{\psi_\hbar}\left(\left\{\frac{A_\hbar(t_i) - \alpha(t_i)}{\sqrt{\hbar}}, \frac{A_\hbar^\dagger(t_i) - \bar{\alpha}(t_i)}{\sqrt{\hbar}}\right\}_{i=1}^n\right) \to \mathrm{Law}_\psi\left(\left\{a(t_i), a^\dagger(t_i)\right\}_{i=1}^n\right).$$

## 1.3 Comparison with Hepp

The primary difference between our results and Hepp's results in [18] is that we allow for non-bounded (polynomial in $a_\hbar$ and $a_\hbar^\dagger$) observables where as Hepp's "observables" are unitary operators of the form

$$U_\hbar(z) = \exp\left(\overline{za_\hbar - \bar{z}a_\hbar^\dagger}\right) \text{ for } z \in \mathbb{C}.$$

As these observables are bounded operators, Hepp is able to prove his results under weaker growth and regularity conditions of the potential function $V$. [Compared to our Assumption 1.11 of Part I, Hepp's Hamiltonian operators, however, are not in form of any arbitrary order of differential operators.] For the most part Hepp primarily works with Hamiltonian operators in the Schrödinger form of Eq. (1.17) where the potential function, $V$, is not necessarily restricted to be a polynomial function. The analogue of Corollary 1.21 (for $n = 1$) in Hepp [18], is his Theorem 2.1 which states; if $z \in \mathbb{C}$ and $\psi \in L^2(\mathbb{R})$, then

$$\lim_{\hbar \downarrow 0} \left\langle \exp\left( \overline{z \frac{a_\hbar - \alpha(t)}{\sqrt{\hbar}} - \bar{z} \frac{a_\hbar^\dagger - \bar{\alpha}(t)}{\sqrt{\hbar}}} \right) \right\rangle_{\psi_{\hbar(t)}} = \left\langle \exp\left( \overline{za(t) - \bar{z}a^\dagger(t)} \right) \right\rangle_\psi ,$$

where $\psi_\hbar(t) := e^{-iH_\hbar t/\hbar} U_\hbar(\alpha_0)\psi$.

# 2 Powers of Symmetric Differential Operators

This section is the introduction of Part II below in this dissertation.

Let $L^2(m)$ be the same Hilbert space as above equipped with the inner product defined in Eq. (1.4) and $\|f\| := \sqrt{\langle f, f \rangle}$. For simplicity, we will denote $dm(x)$ in Eq. (1.4) as $dx$.

**Notation 1.22.** *Let $C^\infty(\mathbb{R}) = C^\infty(\mathbb{R}, \mathbb{C})$ denote smooth functions from $\mathbb{R}$ to $\mathbb{C}$, $C_c^\infty(\mathbb{R})$ denote those $f \in C^\infty(\mathbb{R})$ which have compact support, and $\mathcal{S} := \mathcal{S}(\mathbb{R}) \subset C^\infty(\mathbb{R})$ be the subspace of Schwartz test functions, i.e. those $f \in C^\infty(\mathbb{R})$ such that $f$ and its derivatives vanish at infinity faster than $|x|^{-n}$ for all $n \in \mathbb{N}$.*

**Notation 1.23.** *Let $C^\infty(\mathbb{R}) = C^\infty(\mathbb{R}, \mathbb{C})$. Also, let $\partial : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ denote the differentiation operator, i.e. $\partial f(x) = f'(x) = \frac{d}{dx} f(x)$.*

**Notation 1.24.** *Given a function $f : \mathbb{R} \to \mathbb{C}$, we let $M_f g := fg$ for all functions $g : \mathbb{R} \to \mathbb{C}$, i.e. $M_f$ denotes the linear operator given by multiplication by $f$. Notice that if $f \in C^\infty(\mathbb{R})$ then we may view $M_f$ as a linear operator from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$.*

For the purposes of this part, a $d^{\text{th}}$–**order linear differential operator** on $C^\infty(\mathbb{R})$ with $d \in \mathbb{N}$ is an operator $L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ which may be expressed as

$$L = \sum_{k=0}^{d} M_{a_k} \partial^k = \sum_{k=0}^{d} a_k \partial^k \tag{1.30}$$

for some $\{a_k\}_{k=0}^d \subset C^\infty(\mathbb{R}, \mathbb{C})$. The **symbol** of $L$, $\sigma = \sigma_L$, is the function on $\mathbb{R} \times \mathbb{R}$ defined by

$$\sigma_L(x, \xi) := \sum_{k=0}^{d} a_k(x)(i\xi)^k. \tag{1.31}$$

**Remark 1.25.** *The action of $L$ on $C_c^\infty(\mathbb{R})$ completely determines the coefficients, $\{a_k\}_{k=0}^d$. Indeed, suppose that $x_0 \in \mathbb{R}$ and $0 \le k \le d$ and let $\varphi(x) := (x - x_0)^k \chi(x)$ where $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi = 1$ in a neighborhood of $x_0$. Then an elementary computation shows $k! \cdot a_k(x_0) = (L\varphi)(x_0)$. In particular if $L\varphi \equiv 0$ for all $\varphi \in C_c^\infty(\mathbb{R})$ then $a_k \equiv 0$ for $0 \le k \le d$ and hence $L\varphi \equiv 0$ for all $\varphi \in C^\infty(\mathbb{R})$.*

**Definition 1.26** (Formal adjoint and symmetry)**.** *Suppose $L$ is a linear differential operator on $C^\infty(\mathbb{R})$ as in Eq. (1.30). Then $L^\dagger : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ denote the* ***formal adjoint*** *of $L$ given by the differential operator,*

$$L^\dagger = \sum_{k=0}^{d} (-1)^k \partial^k M_{\bar{a}_k} \text{ on } C^\infty(\mathbb{R}). \tag{1.32}$$

*Moreover $L$ is said to be* ***symmetric*** *if $L^\dagger = L$ on $C^\infty(\mathbb{R})$.*

**Remark 1.27.** *Using Remark 1.25, one easily shows $L^\dagger$ may alternatively be characterized as that unique $d^{th}$–order differential operator on $C^\infty(\mathbb{R})$ such that*

$$\langle Lf, g\rangle = \left\langle f, L^\dagger g\right\rangle \text{ for all } f, g \in C_c^\infty(\mathbb{R}). \tag{1.33}$$

*From this characterization it is then easily verified that;*

1. *The dagger operation is an involution, in particular $L^{\dagger\dagger} = L$ and if $S$ is another linear differential operator on $C^\infty(\mathbb{R})$, then $(LS)^\dagger = S^\dagger L^\dagger$.*

2. *$L$ is symmetric iff $\langle Lf, g\rangle = \langle f, Lg\rangle$ for all $f, g \in C_c^\infty(\mathbb{R})$.*

Proposition 10.2 below shows if $\{a_k\}_{k=0}^d \subset C^\infty(\mathbb{R},\mathbb{R})$, then $L = L^\dagger$ iff $d = 2m$ is even and there exists $\{b_l\}_{l=0}^m \subset C^\infty(\mathbb{R},\mathbb{R})$ such that

$$L = L(\{b_l\}_{l=0}^m) := \sum_{l=0}^m (-1)^l \partial^l b_l(x)\partial^l. \tag{1.34}$$

The factor of $(-1)^l$ is added for later convenience. The coefficients $\{b_l\}_{l=0}^m$ are uniquely determined by $\{a_{2l}\}_{l=0}^m$ (the even coefficients in Eq. (1.30) and in turn the coefficients $\{a_k\}_{k=0}^{2m}$ are determined by the $\{b_l\}_{l=0}^m$, see Lemma 10.4 and Theorem 10.7 respectively. We say that $L$ is written in **divergence form** when $L$ is expressed as in Eq. (1.34).

From now on let us assume that $\{a_k\}_{k=0}^d \subset C^\infty(\mathbb{R},\mathbb{R})$ and $L$ is given as in Eq. (1.30). For each $n \in \mathbb{N}$, $L^n$ is a $dn$ order differential operator on $C^\infty(\mathbb{R})$ and hence there exists $\{A_k\}_{k=0}^{2mn} \subset C^\infty(\mathbb{R},\mathbb{C})$ such that

$$L^n = \sum_{k=0}^{dn} A_k \partial^k. \tag{1.35}$$

If we further assume that $L$ is symmetric (so $d = 2m$ for some $m \in \mathbb{N}_0$), then by Remark 1.27 $L^n$ is a symmetric $2mn$ - order differential operator. Therefore by Proposition 10.2, there exists $\{B_\ell\}_{\ell=0}^{mn} \subset C^\infty(\mathbb{R},\mathbb{R})$ so that $L^n$ may be written in divergence form as

$$L^n = \sum_{\ell=0}^{mn} (-1)^\ell \partial^\ell B_\ell \partial^\ell. \tag{1.36}$$

Information about the coefficients $\{A_k\}_{k=0}^{2mn}$ and $\{B_\ell\}_{\ell=0}^{mn}$ in terms of the divergence form coefficients $\{b_l\}_{l=0}^m$ of $L$ may be found in Propositions 11.7 and Proposition 11.8 respectively.

Let $\mathbb{R}[x]$ be the space of polynomial functions in one variable, $x$, with real coefficients.

**Remark 1.28.** *If the coefficients,* $\{a_k\}_{k=0}^{d=2m}$ *, of* $L$ *in the Eq. (1.30) are in* $\mathbb{R}[x]$ *, then* $L$ *and* $L^{\dagger}$ *are both linear differential operator on* $C^{\infty}(\mathbb{R})$ *which leave* $\mathcal{S}$ *invariant. Moreover by simple integration by parts Eq. (1.33) holds with* $C_c^{\infty}(\mathbb{R})$ *replaced by* $\mathcal{S}$*, i.e.* $\langle Lf, g\rangle = \left\langle f, L^{\dagger}g\right\rangle$ *for all* $f, g \in \mathcal{S}$*.*

**Notation 1.29.** *For the remainder of this introduction we are going to assume* $L$ *is symmetric* ($L = L^{\dagger}$)*,* $L$ *is given in divergence form as in Eq. (1.34) with* $\{b_l\}_{l=0}^{m} \subset \mathbb{R}[x]$ *, and we now view* $L$ *as an operator on* $L^2(\mathbb{R}, m)$ *with* $\mathcal{D}(L) = \mathcal{S} \subset L^2(m)$ *. In other words, we are going to replace* $L$ *by* $L|_{\mathcal{S}}$*.*

The main results of Part II will now be summarized in the next two sections.

## 2.1 Essential self-adjointness results

**Theorem 1.30.** *Let* $m \in \mathbb{N}$*,* $\{b_l\}_{l=0}^{m} \subset \mathbb{R}[x]$ *with* $b_m(x) \neq 0$ *and assume;*

1. *either* $\inf_x b_l(x) > 0$ *or* $b_l \equiv 0$ *and*
2. $\deg(b_l) \leq \max\{\deg(b_0), 0\}$ *whenever* $1 \leq l \leq m$*. [The zero polynomial is defined to be of degree* $-\infty$*.]*

*If* $L$ *is the unbounded operator on* $L^2(m)$ *as in Notation 1.29, then* $L^n$ *(for which* $\mathcal{D}(L^n)$ *is still* $\mathcal{S}$*) is essentially self-adjoint for all* $n \in \mathbb{N}$*.*

**Remark 1.31.** *Notice that assumption 1 of Theorem 1.30 implies* $\deg(b_l)$ *is even and the leading order coefficient of* $b_l$ *is positive unless* $b_l \equiv 0$*.*

Let us recall [Subspace Symmetry] in Definition 1.9 Let $S$ be a dense subspace of a Hilbert space $\mathcal{K}$ and $A$ be a linear operator on $\mathcal{K}$. Then $A$ is said to be symmetric on $S$ if $S \subseteq \mathcal{D}(A)$ and

$$\langle A\psi, \psi\rangle_{\mathcal{K}} = \langle \psi, A\psi\rangle_{\mathcal{K}} \text{ for all } \psi \in S.$$

The equality is equivalent to say $A|_S \subseteq (A|_S)^*$ or $A \subseteq A^*$ if $\mathcal{D}(A) = S$.

**Remark 1.32.** *Using Remark 1.27, it is easy to see that $L$ with polynomial coefficients is symmetric on $C^{\infty}(\mathbb{R})$ as in Definition 1.26 if and only if $L$ is symmetric on $\mathcal{S}$ as in Definition 1.9.*

Therefore, there are three different partial ordering $\preceq_S$, $\preceq$ and $\leq$ (see Notation 1.10) on symmetric operators on a Hilbert space.

There is a sizable literature dealing with similar essential self-adjointness in Theorem 1.30, see for example [3, 21, 31]. Suppose that $b_2$, $b_1$, and $b_0$ are smooth real-valued functions of $x \in \mathbb{R}$ and $T$ is a differential operator on $C_c^{\infty}(\mathbb{R}) \subseteq L^2(m)$ defined by,

$$T = -\partial b_2(x)\,\partial + b_0(x) + i\left(b_1(x)\,\partial + \partial b_1(x)\right).$$

Kato [21] shows $T^n$ is essentially self-adjoint for all $n \in \mathbb{N}$ when $b_2 = 1$, $b_1 = 0$ and $-a - b|x|^2 \preceq_{C_c^{\infty}(\mathbb{R})} T$ for some constants $a$ and $b$. Chernoff [3] gives the same conclusion under certain assumptions on $b_2$ and $T$. For example, Chernoff's assumptions would hold if $b_2$, $b_1$ and $b_0$ are real valued polynomial functions such that $\deg(b_2) \leq 2$ and $b_2$ is positive and $T$ is semi-bounded on $C_c^{\infty}(\mathbb{R})$. In contrast, Theorem 1.30 allows for higher order differential operators but does not allow for non-polynomial coefficients. [However, the methods in Part II can be pushed further in order to allow for certain non-polynomial coefficients.]

There are also a number of results regarding essential self-adjointness in the pseudo-differential operator literature, the reader may be referred to, for example, [6, 26, 27, 36, 39, 42]. In fact, our proof of Theorem 1.30 will be an adaptation of an approach found in Theorem 3.1 in [26].

## 2.2 Operator Comparison Theorems

Motivated by $\hbar$ scaled quantization we picked in Definition 1.3 and the important paper by [18], we will define a scaled version of $L$ (see Notation 1.33) where for any $\hbar > 0$ we make the following replacements in Eq. (1.34),

$$x \to \sqrt{\hbar} M_x \text{ and } \partial \to \sqrt{\hbar}\partial. \tag{1.37}$$

For reasons explained in Theorem B.2 of the appendix, we are lead to consider a more general class of operators parametrized by $\hbar > 0$.

**Notation 1.33.** *Let*

$$\{b_{l,\hbar}(\cdot) : 0 \le l \le m \textit{ and } \hbar > 0\} \subset \mathbb{R}[x], \tag{1.38}$$

*and then define*

$$L_\hbar = L\left(\left\{\hbar^l b_{l,\hbar}(\sqrt{\hbar}\,(\cdot))\right\}_{l=0}^m\right) = \sum_{l=0}^m (-\hbar)^l \partial^l b_{l,\hbar}(\sqrt{\hbar}\,(\cdot))\partial^l \textit{ on } \mathcal{S}. \tag{1.39}$$

We now record an assumption which is needed in a number of the results stated below.

**Assumption 1.34.** *Let $m \in \mathbb{N}_0$. We say $\{b_{l,\hbar}\}_{l=0}^m \subset \mathbb{R}[x]$ and $\eta > 0$ satisfies Assumption 1.34 if the following conditions hold.*

1. *For $0 \le l \le m$, $b_{l,\hbar}(x) = \sum_{j=0}^{2m_l} \alpha_{l,j}(\hbar)\, x^j$ is a real polynomial of $x$ where $\alpha_{l,j}$ is a real continuous function on $[0,\eta]$.*

2. *For all $0 < \hbar < \eta$,*

$$2m_l = \deg(b_{l,\hbar}) \le \deg(b_{l-1,\hbar}) = 2m_{l-1} \textit{ for } 1 \le l \le m. \tag{1.40}$$

*3. We have,*

$$c_{b_m} := \inf_{x\in\mathbb{R},0<\hbar<\eta} b_{m,\hbar}(x) > 0 \text{ and} \tag{1.41}$$

$$c_\alpha := \min_{0\le l\le m} \inf_{0<\hbar<\eta} \alpha_{l,2m_l}(\hbar) > 0, \tag{1.42}$$

*i.e.* $b_{m,\hbar}(x)$ *is positive uniformly in* $x \in \mathbb{R}$ *and* $0 < \hbar < \eta$ *and leading orders,* $\alpha_{l,2m_l}(\hbar)$, *of all* $b_{l,\hbar} \in \mathbb{R}[x]$ *are uniformly strictly positive.*

**Remark 1.35.** *Conditions* (1) *and* (3) *of Assumption 1.34 implies there exists* $A \in (0,\infty)$ *so that*

$$\min_{0\le l\le m} \inf_{0<\hbar<\eta} \inf_{|x|\ge A} b_{l,\hbar}(x) > 0.$$

*Furthermore, if* $k \ge 1$, $0 \le l_1, \dots, l_k \le m$, *and* $q_\hbar(x) = b_{l_1,\hbar}(x)\dots b_{l_k,\hbar}(x) \in \mathbb{R}[x]$, *then*

$$q_\hbar(x) = \sum_{i=0}^{2M} Q_i(\hbar)\cdot x^i$$

*where* $M = m_{l_1} + \ldots + m_{l_k}$, *each of the coefficients,* $Q_i(\hbar)$ *is uniformly bounded for* $0 < \hbar < \eta$, *and*

$$\inf_{0<\hbar<\eta} Q_{2M}(\hbar) = \inf_{0<\hbar<\eta} \alpha_{l_1,m_{l_1}}\dots\alpha_{l_k,m_{l_k}}(\hbar) \ge c_\alpha^k > 0.$$

*From these remarks one easily shows* $\inf_{0<\hbar<\eta}\inf_{x\in\mathbb{R}} q_\hbar(x) > -\infty$.

The second main goal of Part II is to find criteria on two symmetric differential operators $L_\hbar$ and $\tilde{L}_\hbar$ so that for each $n \in \mathbb{N}$, there exists $K_n < \infty$ such that

$$L_\hbar^n \preceq_{\mathcal{S}} K_n\left(\tilde{L}_\hbar^n + I\right). \tag{1.43}$$

(As usual $I$ denotes the identity operator here and $\preceq_{\mathcal{S}}$ is as in Notation 1.10.) For some perspective let us recall the Löwner-Heinz inequality.

**Theorem 1.36** (Löwner-Heinz inequality)**.** *If $A$ and $B$ are two non-negative self-adjoint operators on a Hilbert space, $\mathcal{K}$, such that $A \leq B$, then $A^r \leq B^r$ for $0 \leq r \leq 1$.*

Löwner proved this result for finite dimensional matrices in [25] and Heinz extended it to bounded operators in a Hilbert space in [16]. Later, both Heinz in [16] and Kato in Theorem 2 of [19] extended the result for unbounded operators, also see Proposition 10.14 of [35]. There is a large literature on so called "operator monotone functions," e.g. [2, 7], Theorem 18 of [29], [30] and [38]. It is well known (see Section 10.3 of [35] for more background) that $f(x) = x^r$ is not an operator monotone for $r > 1$, see [35, Example 10.3] for example. This indicates that proving operator inequalities of the form in Eq. (1.43) is somewhat delicate. Our main result in this direction is the subject of the next theorem.

**Theorem 1.37** (Operator Comparison Theorem)**.** *Suppose that $\tilde{L}_\hbar$ and $L_\hbar$ are two linear differential operators on $\mathcal{S}$ given by*

$$\tilde{L}_\hbar = \sum_{l=0}^{m_{\tilde{L}}} (-\hbar)^l \partial^l \tilde{b}_{l,\hbar}(\sqrt{\hbar}x)\partial^l \text{ and } L_\hbar = \sum_{l=0}^{m_L} (-\hbar)^l \partial^l b_{l,\hbar}(\sqrt{\hbar}x)\partial^l,$$

*with polynomial coefficients, $\left\{\tilde{b}_{l,\hbar}(x)\right\}_{l=0}^{m_{\tilde{L}}}$ and $\{b_{l,\hbar}(x)\}_{l=0}^{m_L}$ satisfying Assumption 1.34 with constants $\eta_{\tilde{L}}$ and $\eta_L$ respectively. Let $\eta = \min\{\eta_{\tilde{L}}, \eta_L\}$. If we further assume that $m_{\tilde{L}} \leq m_L$ and there exists $c_1$ and $c_2$ such that*

$$\left|\tilde{b}_{l,\hbar}(x)\right| \leq c_1 \left(b_{l,\hbar}(x) + c_2\right) \;\; \forall \; 0 \leq l \leq m_{\tilde{L}} \text{ and } 0 < \hbar < \eta, \tag{1.44}$$

*then for any $n \in \mathbb{N}$ there exists $C_1$ and $C_2$ such that*

$$\tilde{L}_\hbar^n \preceq_{\mathcal{S}} C_1 \left(L_\hbar^n + C_2\right) \text{ for all } 0 < \hbar < \eta. \tag{1.45}$$

**Corollary 1.38.** *If $\{b_{l,\hbar}(x)\}_{l=0}^{m_L}$ and $\eta > 0$ satisfy Assumption 1.34, then there* *exists $C \in \mathbb{R}$ such that $CI \preceq_{\mathcal{S}} L_\hbar$ for all $0 < \hbar < \eta$.*

**Proof.** Define $\tilde{L}_\hbar = I$, i.e. we are taking $m_{\tilde{L}} = 0$ and $\tilde{b}_{0,\hbar}(x) = 1$. It then follows from Theorem 1.37 with $n = 1$ that there exists $C_1, C_2 \in (0, \infty)$ such that $I = \tilde{L}_\hbar \preceq_{\mathcal{S}} C_1 L_\hbar + C_1 C_2$ and hence $L_\hbar + C_2 \succeq_{\mathcal{S}} C_1^{-1} I$. ■

A similar result to Theorem 1.37 may be found in the Theorem 1.1 of [38]. The paper [38] compares the standard Laplacian $-\triangle$ with an operator of $H_0$ in the form of $-\sum_{i,j}^{d} \partial_i c_{ij}(x) \partial_j$ with coefficients $\{c_{ij}\}_{i,j=1}^{d}$ lying in a Sobolev spaces $W^{m+1,\infty}\left(\mathbb{R}^d\right)$ for some $m \in \mathbb{N}$ and $\mathcal{D}(H_0) = W^{\infty,2}\left(\mathbb{R}^d\right)$. The theorem shows that if $H_0$ is a symmetric, positive and subelliptic of order $\gamma \in (0,1]$ then $\overline{H_0}$ is positive self-adjoint and for all $\alpha \in \left[0, \frac{m+1+\gamma^{-1}}{2}\right]$, there exists $C_\alpha$ such that

$$(-\Delta)^{2\alpha\gamma} \leq C_\alpha \left(I + \overline{H_0}\right)^{2\alpha}.$$

Theorem 1.37 also has a similar flavor to results in Nelson [28]. However, we have not seen how to use Nelson's result in our context.

As a corollary of Theorem 1.30 and aspects of the proof of Theorem 1.37 given in Chapter 14 below, we have the following corollaries which are proved in Section 3 in Chapter 14 below.

**Corollary 1.39.** *Supposed $\{b_{l,\hbar}(x)\}_{l=0}^{m} \subset \mathbb{R}[x]$ and $\eta > 0$ satisfies Assumption 1.34, $L_\hbar$ is the operator in the Eq. (1.39), and suppose that $C \geq 0$ has been chosen so that $0 \preceq_{\mathcal{S}} L_\hbar + CI$ for all $0 < \hbar < \eta$. (The existence of $C$ is guaranteed by Corollary 1.38.) Then for any $0 < \hbar < \eta$, $\bar{L}_\hbar + CI$ is a non-negative self-adjoint operator on $L^2(m)$ and $\mathcal{S}$ is a core for $\left(\bar{L}_\hbar + C\right)^r$ for all $r \geq 0$.*

**Corollary 1.40.** *Suppose that $\tilde{L}_\hbar$ and $L_\hbar$ are two linear differential operators and $\eta > 0$ as in Theorem 1.37. If $C \geq 0$ and $\tilde{C} \geq 0$ are chosen so that $L_\hbar + C \succeq_{\mathcal{S}} I$ and $\tilde{L}_\hbar + \tilde{C} \succeq_{\mathcal{S}} 0$ (as is possible by Corollary 1.38), then $\overline{\tilde{L}_\hbar} + \tilde{C}$ and $\bar{L}_\hbar + C$ are* *non-negative self adjoint operators and for each $r \geq 0$ there exists $C_r$ such that*

$$\left(\overline{\tilde{L}_\hbar} + \tilde{C}\right)^r \preceq C_r \left(\bar{L}_\hbar + C\right)^r \ \forall\ 0 < \hbar < \eta. \tag{1.46}$$

From Definitions 1.3 and 1.4 and Corollary 3.17, the positive self-adjoint **number operator**, $\mathcal{N}$, on $L^2(m)$ is defined as the closure of

$$-\frac{1}{2}\partial^2 + \frac{1}{2}x^2 - \frac{1}{2} \text{ on } \mathcal{S}. \tag{1.47}$$

The next corollary is a direct consequence from Corollaries 1.39 and 1.40 where $\overline{\tilde{L}_\hbar} = \mathcal{N}_\hbar$ in Eq. (1.46).

**Corollary 1.41.** *Suppose $m \geq 1$, $\{b_{l,\hbar}(\cdot)\}_{l=0}^m \subset \mathbb{R}[x]$ and $\eta > 0$ satisfy Assumption 1.34, and $L_\hbar$ is the operator on $\mathcal{S}$ defined in Eq. (1.39). If $C \geq 0$ is chosen so that $I \preceq_{\mathcal{S}} L_\hbar + C$ (see Corollary 1.38), then;*

1. *$\bar{L}_\hbar + C$ is a non-negative self-adjoint operator on $L^2(m)$ for all $0 < \hbar < \eta$.*

2. *$\mathcal{S}$ is a core for $\left(\bar{L}_\hbar + C\right)^r$ for all $r \geq 0$ and $0 < \hbar < \eta$.*

3. *If we further supposed $\deg(b_{0,\hbar}) \geq 2$, then there exists $C_r > 0$ such that*

$$\mathcal{N}_\hbar^r \preceq C_r \left(\bar{L}_\hbar + C\right)^r \tag{1.48}$$

*for all $0 < \hbar < \eta$ and $r \geq 0$.*[1]

[1]By the spectral theorem one shows $D\left(\left|\bar{L}_\hbar\right|^r\right) = D\left(\left(\bar{L}_\hbar + C\right)^r\right)$.

# Part I

# On the classical limit of quantum mechanics

# Chapter 2

# Background and Setup

In this chapter we will expand on the basic setup described above and recall some basic facts that will be needed throughout Part I

## 1 Classical Setup

In Part I, we take configuration space to be $\mathbb{R}$ so that our classical state space is $T^*\mathbb{R} \cong \mathbb{R}^2$. [Extensions to higher and to infinite dimensions will be considered elsewhere.] Following Hepp [18], we identify $T^*\mathbb{R}$ with $\mathbb{C}$ via

$$T^*\mathbb{R} \ni (\xi, \pi) \to \alpha := \frac{1}{\sqrt{2}} (\xi + i\pi) . \tag{2.1}$$

Taking in account the "$\sqrt{2}$"above, we set

$$\frac{\partial}{\partial \alpha} := \frac{1}{\sqrt{2}} (\partial_\xi - i\partial_\pi) \text{ and } \frac{\partial}{\partial \bar{\alpha}} := \frac{1}{\sqrt{2}} (\partial_\xi + i\partial_\pi)$$

so that $\frac{\partial}{\partial \alpha}\alpha = 1 = \frac{\partial}{\partial \bar{\alpha}}\bar{\alpha}$ and $\frac{\partial}{\partial \alpha}\bar{\alpha} = 0 = \frac{\partial}{\partial \bar{\alpha}}\alpha$. As usual given a smooth real valued function,[1] $H^{\mathrm{cl}}(\xi, \pi)$, on $T^*\mathbb{R}$ we say $(\xi(t), \pi(t))$ solves Hamilton's equations of

[1]Later $H^{cl}$ will be the symbol of a symmetric element of $H \in \mathbb{C}\langle\theta, \theta^*\rangle$ as described in section 4.

motion provided,

$$\dot{\xi}(t) = H^{\mathrm{cl}}_{\pi}(\xi(t), \pi(t)) \text{ and } \dot{\pi}(t) = -H^{\mathrm{cl}}_{\xi}(\xi(t), \pi(t)) \tag{2.2}$$

where $H^{\mathrm{cl}}_{\pi} := \partial H^{\mathrm{cl}}/\partial\pi$ and $H^{\mathrm{cl}}_{\xi} := \partial H^{\mathrm{cl}}/\partial\xi$ . A simple verifications shows; if

$$\alpha(t) := \frac{1}{\sqrt{2}}(\xi(t) + i\pi(t)),$$

then $(\xi(t), \pi(t))$ solves Hamilton's Eqs. (2.2) iff $\alpha(t)$ satisfies

$$i\dot{\alpha}(t) = \left(\frac{\partial}{\partial\bar{\alpha}}\tilde{H}^{\mathrm{cl}}\right)(\alpha(t)) \tag{2.3}$$

where

$$\tilde{H}^{\mathrm{cl}}(\alpha) := H^{\mathrm{cl}}(\xi, \pi) \text{ where } \alpha = \frac{1}{\sqrt{2}}(\xi + i\pi) \in \mathbb{C}.$$

In the future we will identify $\tilde{H}^{\mathrm{cl}}$ with $H^{\mathrm{cl}}$ and drop the tilde from our notation.

**Example 2.1.** *If $H(\alpha) = |\alpha|^2 + \frac{1}{2}|\alpha|^4$, then the associated Hamiltonian equations of motion are given by*

$$i\dot{\alpha} = \frac{\partial}{\partial\bar{\alpha}}\left(\alpha\bar{\alpha} + \frac{1}{2}\alpha^2\bar{\alpha}^2\right) = \alpha + \alpha^2\bar{\alpha} = \alpha + |\alpha|^2\alpha.$$

**Proposition 2.2.** *Let $z(t) := \Phi'(t, \alpha_0) z$ be the **real** differential of the flow associated to Eq. (1.1) as in Eq. (1.2). Then $z(t)$ satisfies $z(0) = z$ and*

$$i\dot{z}(t) = u(t)\bar{z}(t) + v(t)z(t), \tag{2.4}$$

*where*

$$u(t) := \left(\frac{\partial^2}{\partial\bar{\alpha}^2}H^{\mathrm{cl}}\right)(\alpha(t)) \in \mathbb{C} \text{ and } v(t) = \left(\frac{\partial^2}{\partial\alpha\partial\bar{\alpha}}H^{\mathrm{cl}}\right)(\alpha(t)) \in \mathbb{R}. \tag{2.5}$$

*Moreover, if we express* $z(t) = \gamma(t) z + \delta(t) \bar{z}$ *as in Eq. (1.3) and let*

$$\Lambda(t) := \begin{bmatrix} \gamma(t) & \delta(t) \\ \bar{\delta}(t) & \bar{\gamma}(t) \end{bmatrix}, \tag{2.6}$$

*then*

$$\det \Lambda(t) = |\gamma(t)|^2 - |\delta(t)|^2 = 1 \ \forall \ t \in \mathbb{R}$$

*and*

$$i\dot{\Lambda}(t) = \begin{bmatrix} v(t) & u(t) \\ -\bar{u}(t) & -\bar{v}(t) \end{bmatrix} \Lambda(t) \ \ \textit{and} \ \Lambda(0) = I. \tag{2.7}$$

**Proof.** First recall if $f : \mathbb{C} \to \mathbb{C}$ is a smooth function (not analytic in general), then the **real** differential, $z \to f'(\alpha) z := \frac{d}{ds}|_0 f(\alpha + sz)$, of $f$ at $\alpha$ satisfies

$$f'(\alpha) z = \left( \frac{\partial}{\partial \alpha} f \right)(\alpha) z + \left( \frac{\partial}{\partial \bar{\alpha}} f \right)(\alpha) \bar{z}. \tag{2.8}$$

By definition $\Phi(t, \alpha_0)$ satisfies the differential equation,

$$i\dot{\Phi}(t, \alpha_0) = \left( \frac{\partial}{\partial \bar{\alpha}} H^{\mathrm{cl}} \right)(\Phi(t, \alpha_0)) \ \text{ and } \Phi(0, \alpha_0) = \alpha_0.$$

Differentiating this equation relative to $\alpha_0$ using the chain rule along with Eq. (2.8) shows $z(t) := \Phi'(t, \alpha_0) z$ satisfies Eq. (2.4). The fact that $v(t)$ is real valued follows from its definition in Eq. (2.5) and the fact that $H^{\mathrm{cl}}$ is a real valued function.

Inserting the expression, $z(t) = \gamma(t) z + \delta(t) \bar{z}$, into Eq. (2.4) one shows after a little algebra that,

$$i\dot{\gamma}(t) z + i\dot{\delta}(t) \bar{z} = \big(u(t) \bar{\delta}(t) + v(t) \gamma(t)\big) z + (u(t) \bar{\gamma}(t) + v(t) \delta(t)) \bar{z}$$

from which we conclude that $(\gamma(t), \delta(t)) \in \mathbb{C}^2$ satisfy the equations

$$i\dot{\gamma}(t) = u(t)\,\bar{\delta}(t) + v(t)\,\gamma(t) \text{ and} \tag{2.9}$$

$$i\dot{\delta}(t) = u(t)\,\bar{\gamma}(t) + v(t)\,\delta(t). \tag{2.10}$$

Using these equations we then find;

$$\begin{aligned}\frac{d}{dt}\left(|\gamma|^2 - |\delta|^2\right) &= 2\,\mathrm{Re}\left(\dot{\gamma}\bar{\gamma} - \dot{\delta}\bar{\delta}\right) \\ &= 2\,\mathrm{Re}\left(-i\left(u\bar{\delta} + v\gamma\right)\bar{\gamma} + i\left(u\bar{\gamma} + v\delta\right)\bar{\delta}\right) \\ &= 2\,\mathrm{Re}\left(-iv\,|\gamma|^2 + iv\,|\delta|^2\right) = 0. \end{aligned} \tag{2.11}$$

Since $z(0) = z$, $\gamma(0) = 1$ and $\delta(0) = 1$ and so from Eq. (2.11) we learn

$$\left(|\gamma|^2 - |\delta|^2\right)(t) = \left(|\gamma|^2 - |\delta|^2\right)(0) = 1^2 - 0^2 = 1. \tag{2.12}$$

Finally, Eq. (2.7) is simply the vector form of Eqs. (2.9) and (2.10). ■

**Remark 2.3.** *Equation (2.4) may be thought of as the time dependent Hamiltonian flow,*

$$i\dot{z}(t) = \frac{\partial q(t,\cdot)}{\partial \bar{z}}(z(t))$$

*where $q(t,z) \in \mathbb{R}$ is the quadratic time dependent Hamiltonian defined by*

$$\begin{aligned} q(t:z) =& \frac{1}{2}u(t)\,z^2 + \frac{1}{2}\bar{u}(t)\,\bar{z}^2 + v(t)\,\bar{z}z \\ =& \frac{1}{2}\left(\frac{\partial^2}{\partial\alpha^2}H^{\mathrm{cl}}\right)(\alpha(t))\,z^2 + \frac{1}{2}\left(\frac{\partial^2}{\partial\bar{\alpha}^2}H^{\mathrm{cl}}\right)(\alpha(t))\,\bar{z}^2 \\ &+ \left(\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\bar{\alpha}}H^{\mathrm{cl}}\right)(\alpha(t))\,|z|^2. \end{aligned}$$

# 2 Quantum Mechanical Setup

Recall that our quantum mechanical Hilbert space is taken to be the space of Lebesgue square integrable complex valued functions on $\mathbb{R}$ ($L^2(m) := L^2(\mathbb{R}, m)$) equipped with the usual $L^2(m)$-inner product as in Eq. (1.4). To each $\hbar > 0$ ($\hbar$ is to be thought of as Planck's constant), let

$$q_\hbar := \sqrt{\hbar} M_x \text{ and } p_\hbar := \sqrt{\hbar}\frac{1}{i}\frac{d}{dx} \tag{2.13}$$

interpreted as self-adjoint operators on $L^2(m) := L^2(\mathbb{R}, m)$ with domains

$$D(q_\hbar) = \left\{ f \in L^2(m) : x \to xf(x) \in L^2(m) \right\} \text{ and}$$
$$D(p_\hbar) = D\left(\frac{d}{dx}\right) = \left\{ f \in L^2(m) : x \to f(x) \text{ is A.C. and } f' \in L^2(m) \right\}$$

where A.C. is an abbreviation of absolutely continuous. Using Corollary 3.26 below, the annihilation and creation operators in Definition 1.3 may be expressed as

$$\bar{a}_\hbar := \frac{q_\hbar + ip_\hbar}{\sqrt{2}} = \sqrt{\frac{\hbar}{2}}\left(M_x + \frac{d}{dx}\right) \text{ and} \tag{2.14}$$
$$a_\hbar^* := \frac{q_\hbar - ip_\hbar}{\sqrt{2}} = \sqrt{\frac{\hbar}{2}}\left(M_x - \frac{d}{dx}\right). \tag{2.15}$$

# 3 Weyl Operator

**Proposition 2.4.** *Let $\alpha := (\xi + i\pi)/\sqrt{2} \in \mathbb{C}$, $\hbar > 0$, and $U(\alpha)$ and $U_\hbar(\alpha)$ be as in Definition 1.6. Then*

$$(U(\alpha) f)(x) = \exp\left(i\pi\left(x - \frac{1}{2}\xi\right)\right) f(x - \xi) \ \forall\ f \in L^2(m), \tag{2.16}$$

$U(\alpha)\mathcal{S} = \mathcal{S}$,

$$U_\hbar(\alpha)^* a_\hbar U_\hbar(\alpha) = a_\hbar + \alpha, \text{ and} \tag{2.17}$$

$$U_\hbar(\alpha)^* a_\hbar^\dagger U_\hbar(\alpha) = a_\hbar^\dagger + \bar{\alpha}, \tag{2.18}$$

*as identities on* $\mathcal{S}$.

**Proof.** Given $f \in \mathcal{S}$ let $F(t,x) := (U(t\alpha) f)(x)$ so that

$$\frac{\partial}{\partial t} F(t,x) = \left(i\pi x - \xi \frac{\partial}{\partial x}\right) F(t,x) \text{ with } F(0,x) = f(x). \tag{2.19}$$

Solving this equation by the method of characteristics then gives Eq. (2.16). [Alternatively one easily verifies directly that

$$F(t,x) := \exp(it\pi(x - \frac{1}{2}t\xi)) f(x - t\xi)$$

solves Eq. (2.19).] It is clear from Eq. (2.16) that $U(\alpha)\mathcal{S} \subset \mathcal{S}$ and $U(-\alpha)U(\alpha) = I$ for all $\alpha \in \mathbb{C}$. Therefore $\mathcal{S} \subset U(-\alpha)\mathcal{S}$. Replacing $\alpha$ by $-\alpha$ in this last inclusion allows us to conclude that $U(\alpha)\mathcal{S} = \mathcal{S}$. The formula in Eq. (2.19) also directly extends to $L^2(m)$ where it defines a unitary operator. The identities in Eqs. (2.17) and (2.18) for $\hbar = 1$ follows by simple direct calculations using Eq. (2.16). The case of general $\hbar > 0$ then follows by simple scaling arguments.

■

**Remark 2.5.** *Another way to prove Eq. (2.17) is to integrate the identity,*

$$\frac{d}{dt} U_\hbar(t\alpha)^* a_\hbar U_\hbar(t\alpha) = -U_\hbar(t\alpha)^* \left[\frac{1}{\hbar}\left(\alpha \cdot a_\hbar^\dagger - \bar{\alpha} \cdot a_\hbar\right), a_\hbar\right] U_\hbar(t\alpha) = \alpha,$$

*with respect to* $t$ *on* $\mathcal{S}$ *and the initial condition* $U(0) = I$.

**Definition 2.6.** *Suppose that* $\{W(t)\}_{t\in\mathbb{R}}$ *is a one parameter family of (possibly) unbounded operators on a Hilbert space* $\langle \mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}} \rangle$. *Given a dense subspace,* $D \subset \mathcal{K}$, *we say $W(t)$ is **strongly** $\|\cdot\|_{\mathcal{K}}$-norm **differentiable** on $D$ if 1) $D \subset D(W(t))$ for all $t \in \mathbb{R}$ and 2) for all $\psi \in D$, $t \to W(t)\psi$ is $\|\cdot\|_{\mathcal{K}}$-norm differentiable. For notational simplicity we will write $\dot{W}(t)\psi$ for $\frac{d}{dt}[W(t)\psi]$.*

**Proposition 2.7.** *If $\mathbb{R} \ni t \to \alpha(t) \in \mathbb{C}$ is a $C^1$ function and $\mathcal{N} := \mathcal{N}_\hbar|_{\hbar=1}$ the number operator defined in Eq. (1.11), then $\{U(\alpha(t))\}_{t\in\mathbb{R}}$ is strongly $L^2(m)$-norm differentiable on $D\left(\sqrt{\mathcal{N}}\right)$ as in the Definition 2.6 and for all $f \in D\left(\sqrt{\mathcal{N}}\right)$ we have*

$$\begin{aligned}\frac{d}{dt}(U(\alpha(t))f) &= \left(\dot{\alpha}(t)a^* - \overline{\dot{\alpha}(t)}\bar{a} + i\,\mathrm{Im}\left(\alpha(t)\overline{\dot{\alpha}(t)}\right)\right)U(\alpha(t))f \\ &= U(\alpha(t))\left(\dot{\alpha}(t)a^* - \overline{\dot{\alpha}(t)}\bar{a} - i\,\mathrm{Im}\left(\alpha(t)\overline{\dot{\alpha}(t)}\right)\right)f.\end{aligned}$$

*Moreover, $U(\alpha(t))$ preserves $D\left(\sqrt{\mathcal{N}}\right)$, $C_c(\mathbb{R})$, and $\mathcal{S}$.*

**Proof.** From Corollary 3.26 below we know $D(\partial_x) \cap D(M_x) = D\left(\sqrt{\mathcal{N}}\right)$. Using this fact, the proposition is a straightforward verification based on Eq. (2.16). The reader not wishing to carry out these computations may find it instructive to give a formal proof based on the algebraic fact that $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ where $A$ and $B$ are operators such that the commutator, $[A,B] := AB - BA$, commutes with both $A$ and $B$.

■

As we do not wish to make any particular choice of quantization scheme, in Part I we will describe all operators as a non-commutative polynomial functions of $a_\hbar$ and $a_\hbar^\dagger$. This is the topic of the next chapter.

# 4 Non-commutative Polynomial Expansions

**Notation 2.8.** *Let $\mathbb{C}\langle\theta,\theta^*\rangle$ be the space of non-commutative polynomials in the non-commutative indeterminates. That is to say $\mathbb{C}\langle\theta,\theta^*\rangle$ is the vector space over*

*$\mathbb{C}$ whose basis consists of words in the two letter alphabet, $\Lambda_1 = \{\theta, \theta^*\}$, cf. Eq. (1.21). The general element, $P(\theta, \theta^*)$, of $\mathbb{C}\langle\theta, \theta^*\rangle$ may be written as*

$$P(\theta, \theta^*) = \sum_{k=0}^{d} \sum_{\mathbf{b}=(b_1,\ldots,b_k)\in\Lambda_1^k} c_k(\mathbf{b})\, b_1 \ldots b_k, \tag{2.20}$$

*where $d \in \mathbb{N}_0$ and*

$$\left\{c_k(\mathbf{b}) : 0 \le k \le d \text{ and } \mathbf{b} = (b_1, \ldots, b_k) \in \Lambda_1^k\right\} \subset \mathbb{C}.$$

*If $c_d : \Lambda_1^d \to \mathbb{C}$ is not the zero function, we say $d =: \deg_\theta P$ is the degree of $P$.*

It is sometimes convenient to decompose $P(\theta, \theta^*)$ in Eq. (2.20) as

$$P(\theta, \theta^*) = \sum_{k=0}^{d} P_k(\theta, \theta^*) \tag{2.21}$$

where

$$P_k(\theta, \theta^*) = \sum_{b_1,\ldots,b_k\in\Lambda_1} c_k(b_1, \ldots, b_k)\, b_1 \ldots b_k. \tag{2.22}$$

Polynomials of the form in Eq. (2.22) are said to be **homogeneous** of degree $k$. By convention, $P_0 := P_0(\theta, \theta^*)$ is just an element of $\mathbb{C}$. We endow $\mathbb{C}\langle\theta, \theta^*\rangle$ with its $\ell^1$ – norm, $|\cdot|$, defined for $P$ as in Eq. (2.20) by

$$|P| := \sum_{k=0}^{d} |P_k| \text{ where } |P_k| = \sum_{\mathbf{b}=(b_1,\ldots,b_k)\in\Lambda_1^k} |c_k(\mathbf{b})|. \tag{2.23}$$

**Definition 2.9** (Monomials)**.** *For $\mathbf{b} = (b_1, \ldots, b_k) \in \{\theta, \theta^*\}^k$ let $u_\mathbf{b} \in \mathbb{C}\langle\theta, \theta^*\rangle$ be the monomial,*

$$u_\mathbf{b}(\theta, \theta^*) = b_1 \ldots b_k \tag{2.24}$$

*with the convention that for $k = 0$ we associate the unit element $u_0 = 1$.*

As usual, we make $\mathbb{C}\langle\theta,\theta^*\rangle$ into a non-commutative algebra with its natural multiplication determined on the word basis elements $\cup_{k=0}^{\infty}\left\{u_{\mathbf{b}}:\mathbf{b}\in\{\theta,\theta^*\}^k\right\}$ by concatenation of words, i.e. $u_{\mathbf{b}}u_{\mathbf{d}}=u_{(\mathbf{b},\mathbf{d})}$ where if $\mathbf{d}=(d_1,\ldots,d_l)\in\{\theta,\theta^*\}^l$

$$(\mathbf{b},\mathbf{d}):=(b_1,\ldots,b_k,d_1,\ldots,d_l)\in\{\theta,\theta^*\}^{k+l}.$$

For example, $\theta\theta\theta^*\cdot\theta^*\theta=\theta\theta\theta^*\theta^*\theta$. We also define a **natural involution** on $\mathbb{C}\langle\theta,\theta^*\rangle$ determined by $(\theta)^*=\theta^*$, $(\theta^*)^*=\theta$, $z^*=\bar{z}$ for $z\in\mathbb{C}$, and $(\alpha\cdot\beta)^*=\beta^*\alpha^*$ for $\alpha,\beta\in\mathbb{C}\langle\theta,\theta^*\rangle$. Formally, if $\mathbf{b}=(b_1,\ldots,b_k)\in\{\theta,\theta^*\}^k$, then

$$u_{\mathbf{b}}^*=b_k^*b_{k-1}^*\ldots b_1^*=u_{\mathbf{b}^*}\text{ where }\mathbf{b}^*:=\left(b_k^*,b_{k-1}^*,\ldots,b_1^*\right). \tag{2.25}$$

In what follows we will often denote an $P\in\mathbb{C}\langle\theta,\theta^*\rangle$ by $P(\theta,\theta^*)$.

**Definition 2.10** (Symmetric Polynomials)**.** *We say $P\in\mathbb{C}\langle\theta,\theta^*\rangle$ is **symmetric** provided $P=P^*$.*

If $\mathcal{A}$ is any unital algebra equipped with an involution, $\xi\to\xi^\dagger$, and $\xi$ is any fixed element of $\mathcal{A}$, then there exists a unique algebra homomorphism

$$P(\theta,\theta^*)\in\mathbb{C}\langle\theta,\theta^*\rangle\to P\left(\xi,\xi^\dagger\right)\in\mathcal{A}$$

determined by substituting $\xi$ for $\theta$ and $\xi^\dagger$ for $\theta^*$. Moreover, the homomorphism preserves involutions, i.e. $\left[P\left(\xi,\xi^\dagger\right)\right]^\dagger=P^*\left(\xi,\xi^\dagger\right)$. The two special cases of this construction that we need here are contained in the following two definitions.

**Definition 2.11** (Classical Symbols)**.** *The symbol (or **classical** residue) of $P\in\mathbb{C}\langle\theta,\theta^*\rangle$ is the function $P^{\mathrm{cl}}\in\mathbb{C}[z,\bar{z}]$ (= the commutative polynomials in $z$ and $\bar{z}$ with complex coefficients) defined by $P^{\mathrm{cl}}(\alpha):=P(\alpha,\bar{\alpha})$ where we view $\mathbb{C}$ as a commutative algebra with an involution given by complex conjugation.*

**Definition 2.12** (Polynomial Operators)**.** *If $P(\theta, \theta^*) \in \mathbb{C}\langle\theta, \theta^*\rangle$ is a non-commutative polynomial and $\hbar > 0$, then $P\left(a_\hbar, a_\hbar^\dagger\right)$ is a differential operator on $L^2(m)$ whose domain is $\mathcal{S}$. [Notice that $P\left(a_\hbar, a_\hbar^\dagger\right)$ preserves $\mathcal{S}$, i.e. $P\left(a_\hbar, a_\hbar^\dagger\right)\mathcal{S} \subset \mathcal{S}$.] We further let $P_\hbar := \overline{P\left(a_\hbar, a_\hbar^\dagger\right)}$ be the closure of $P\left(a_\hbar, a_\hbar^\dagger\right)$. Any linear differential operator of the form $P\left(a_\hbar, a_\hbar^\dagger\right)$ for some $P(\theta, \theta^*) \in \mathbb{C}\langle\theta, \theta^*\rangle$ will be called a **polynomial operator**.*

We introduce the following notation in order to write out $P\left(a_\hbar, a_\hbar^\dagger\right)$ more explicitly.

**Notation 2.13.** *For any $\hbar > 0$ let $\Xi_\hbar : \{\theta, \theta^*\} \to \left\{a_\hbar, a_\hbar^\dagger\right\}$ be define by*

$$\Xi_\hbar(b) = \begin{cases} a_\hbar & \textit{if} \quad b = \theta \\ a_\hbar^\dagger & \textit{if} \quad b = \theta^* \end{cases}. \tag{2.26}$$

*In the special case where $\hbar = 1$ we will simply denote $\Xi_1$ by $\Xi$.*

With this notation if $P \in \mathbb{C}\langle\theta, \theta^*\rangle$ is as in Eq. (2.20), then $P\left(a_\hbar, a_\hbar^\dagger\right)$ may be written as,

$$P\left(a_\hbar, a_\hbar^\dagger\right) = \sum_{k=0}^{d} \sum_{\mathbf{b}=(b_1,\dots,b_k)\in\Lambda_1^k} c_k(\mathbf{b})\,\Xi_\hbar(b_1)\dots\Xi_\hbar(b_k) \tag{2.27}$$

or as

$$P\left(a_\hbar, a_\hbar^\dagger\right) = \sum_{k=0}^{d} \sum_{\mathbf{b}=(b_1,\dots,b_k)\in\Lambda_1^k} \hbar^{k/2} c_k(\mathbf{b})\, u_{\mathbf{b}}\left(a, a^\dagger\right) \tag{2.28}$$

**Definition 2.14** (Monomial Operators)**.** *Any linear differential operator of the form $u_{\mathbf{b}}\left(a, a^\dagger\right) = \Xi_1(b_1)\dots\Xi_1(b_k)$ for some $\mathbf{b} = (b_1, \dots, b_k) \in \{\theta, \theta^*\}^k$ and $k \in \mathbb{N}_0$ will be called a **monomial operator.***

**Remark 2.15.** *If $H(\theta, \theta^*) \in \mathbb{C}\langle\theta, \theta^*\rangle$ is symmetric (i.e. $H = H^*$), then;*

*1.* $H\left(a_\hbar, a_\hbar^\dagger\right)$ *is a symmetric operator on* $\mathcal{S}$ *(i.e.* $\left[H\left(a_\hbar, a_\hbar^\dagger\right)\right]^\dagger = H\left(a_\hbar, a_\hbar^\dagger\right)$*) for any* $\hbar > 0$ *and*

*2.* $H^{\mathrm{cl}}(z) := H(z, \bar{z})$ *is a real valued function on* $\mathbb{C}$.

*Indeed,*

$$\left[H\left(a_\hbar, a_\hbar^\dagger\right)\right]^\dagger = H^*\left(a_\hbar, a_\hbar^\dagger\right) = H\left(a_\hbar, a_\hbar^\dagger\right)$$

*and*

$$\overline{H^{\mathrm{cl}}(\alpha)} := \overline{H(\alpha, \bar{\alpha})} = H^*(\alpha, \bar{\alpha}) = H(\alpha, \bar{\alpha}) = H^{\mathrm{cl}}(\alpha).$$

*The main point of Part I is to show under Assumption 1.11 on* $H$ *that classical Hamiltonian dynamics associated to* $H^{\mathrm{cl}}$ *determine the limiting quantum mechanical dynamics determined by* $H_\hbar := \overline{H\left(a_\hbar, a_\hbar^\dagger\right)}$.

We have analogous definitions and statements for the non-commutative algebra, $\mathbb{C}\langle\theta_1, \dots, \theta_n, \theta_1^*, \dots, \theta_n^*\rangle$, of non-commuting polynomials in $2n$ – indeterminants, $\Lambda_n = \{\theta_1, \dots, \theta_n, \theta_1^*, \dots, \theta_n^*\}$, as in Eq. (1.21).

**Notation 2.16.** *Let* $\mathbb{C}[x]\langle\theta, \theta^*\rangle$ *and* $\mathbb{C}[\alpha, \bar{\alpha}]\langle\theta, \theta^*\rangle$ *denote the non-commutative polynomials in* $\{\theta, \theta^*\}$ *with coefficients in the commutative polynomial rings,* $\mathbb{C}[x]$ *and* $\mathbb{C}[\alpha, \bar{\alpha}]$ *respectively. For* $P \in \mathbb{C}[x]\langle\theta, \theta^*\rangle$ *or* $P \in \mathbb{C}[\alpha, \bar{\alpha}]\langle\theta, \theta^*\rangle$ *we will write* $\deg_\theta P$ *to indicate that we are computing the degree relative to* $\{\theta, \theta^*\}$ *and not relative to* $x$ *or* $\{\alpha, \bar{\alpha}\}$.

For any $\alpha \in \mathbb{C}$ and $P(\theta, \theta^*) \in \mathbb{C}\langle\theta, \theta^*\rangle$ with $d = \deg_\theta P$, let $\{P_k(\alpha : \theta, \theta^*)\}_{k=0}^d \subset \mathbb{C}[\alpha, \bar{\alpha}]\langle\theta, \theta^*\rangle$ denote the unique homogeneous polynomials in $\mathbb{C}\langle\theta, \theta^*\rangle$ with coefficients which are polynomials in $\alpha$ and $\bar{\alpha}$ such that $\deg_\theta P_k(\alpha : \theta, \theta^*) = k$ and

$$P(\theta + \alpha, \theta^* + \bar{\alpha}) = \sum_{k=0}^{d} P_k(\alpha : \theta, \theta^*). \tag{2.29}$$

**Example 2.17.** *If*

$$P(\theta,\theta^*) = \theta\theta^*\theta + \theta^*\theta\theta^*$$

*then*

$$\begin{aligned} P(\theta+\alpha,\theta^*+\bar{\alpha}) &= (\theta+\alpha)(\theta^*+\bar{\alpha})(\theta+\alpha) + (\theta^*+\bar{\alpha})(\theta+\alpha)(\theta^*+\bar{\alpha}) \\ &= P_0 + P_1 + P_2 + P_{\geq 3} \end{aligned}$$

*where*

$$\begin{aligned} P_0(\alpha,\theta,\theta^*) &= \alpha^2\bar{\alpha} + \bar{\alpha}^2\alpha = P^{\mathrm{cl}}(\alpha) \\ P_1(\alpha,\theta,\theta^*) &= \left(2|\alpha|^2 + \overline{\alpha}^2\right)\theta + \left(2|\alpha|^2 + \alpha^2\right)\theta^* \\ &= \frac{\partial P^{\mathrm{cl}}}{\partial \alpha}(\alpha)\theta + \frac{\partial P^{\mathrm{cl}}}{\partial \bar{\alpha}}(\alpha)\theta^* \\ P_2(\alpha,\theta,\theta^*) &= \overline{\alpha}\theta^2 + \alpha\theta^{*2} + (\alpha+\overline{\alpha})\theta^*\theta + (\alpha+\overline{\alpha})\theta\theta^* \\ &= \frac{1}{2}\left(\frac{\partial^2 P^{\mathrm{cl}}}{\partial \alpha^2}(\alpha)\theta^2 + \frac{\partial^2 P^{\mathrm{cl}}}{\partial \bar{\alpha}^2}(\alpha)\theta^{*2}\right) + \frac{d}{dt}|_{t=0}\frac{d}{ds}|_{s=0}P(s\theta+\alpha,t\theta^*+\bar{\alpha}) \\ P_{\geq 3}(\alpha,\theta,\theta^*) &= \theta\theta^*\theta + \theta^*\theta\theta^*. \end{aligned}$$

*This example is generalized in the following theorem.*

**Theorem 2.18.** *Let $P(\theta,\theta^*) \in \mathbb{C}\langle\theta,\theta^*\rangle$ and $\alpha \in \mathbb{C}$, then*

$$\begin{aligned} P_0(\alpha:\theta,\theta^*) &= P^{\mathrm{cl}}(\alpha) \\ P_1(\alpha:\theta,\theta^*) &= \left[\frac{\partial P^{\mathrm{cl}}}{\partial \alpha}(\alpha)\theta + \frac{\partial P^{\mathrm{cl}}}{\partial \bar{\alpha}}(\alpha)\theta^*\right] \text{ and} \\ P_2(\alpha:\theta,\theta^*) &= \frac{1}{2}\left(\frac{\partial^2 P^{\mathrm{cl}}}{\partial \alpha^2}(\alpha)\theta^2 + \frac{\partial^2 P^{\mathrm{cl}}}{\partial \bar{\alpha}^2}(\alpha)\theta^{*2}\right) \\ &\quad + \frac{d}{dt}|_{t=0}\frac{d}{ds}|_{s=0}P(s\theta+\alpha,t\theta^*+\bar{\alpha}). \end{aligned} \tag{2.30}$$

*where*

$$\frac{d}{dt}|_{t=0}\frac{d}{ds}|_{s=0}P\left(s\theta+\alpha,t\theta^{*}+\bar{\alpha}\right)=\frac{\partial^{2}P^{\mathrm{cl}}}{\partial\alpha\partial\bar{\alpha}}\left(\alpha\right)\theta^{*}\theta \mod \theta^{*}\theta=\theta\theta^{*}$$

*for all* $\alpha\in\mathbb{C}$*. So we have*

$$\begin{aligned}&P\left(\theta+\alpha,\theta^{*}+\bar{\alpha}\right)\\&\quad=P^{\mathrm{cl}}\left(\alpha\right)+\left[\frac{\partial P^{\mathrm{cl}}}{\partial\alpha}\left(\alpha\right)\theta+\left(\frac{\partial}{\partial\bar{\alpha}}P^{\mathrm{cl}}\right)\left(\alpha\right)\theta^{*}\right]+P_{2}\left(\alpha:\theta,\theta^{*}\right)+P_{\geq3}\left(\alpha:\theta,\theta^{*}\right)\end{aligned} \tag{2.31}$$

*where the remainder term,* $P_{\geq3}$ *is a sum of homogeneous terms of degree* 3 *or more. Moreover if* $P=P^{*}$*, then* $P_{2}^{*}=P_{2}$ *and* $P_{\geq3}^{*}=P_{\geq3}$*.*

**Proof.** If $p=\deg_{\theta}P$, then

$$P\left(t\theta+\alpha,t\theta^{*}+\bar{\alpha}\right)=\sum_{k=0}^{p}t^{k}P_{k}\left(\alpha:\theta,\theta^{*}\right)\ \forall\ t\in\mathbb{R},$$

and it follows (by Taylor's theorem) that

$$P_{k}\left(\alpha:\theta,\theta^{*}\right)=\frac{1}{k!}\left(\frac{d}{dt}\right)^{k}|_{t=0}P\left(t\theta+\alpha,t\theta^{*}+\bar{\alpha}\right). \tag{2.32}$$

From Eq. (2.32),

$$\begin{aligned}P_{0}\left(\alpha:\theta,\theta^{*}\right)&=P\left(\alpha,\bar{\alpha}\right)=P^{\mathrm{cl}}\left(\alpha\right)\ \text{ and}\\P_{1}\left(\alpha:\theta,\theta^{*}\right)&=\frac{d}{dt}|_{t=0}P\left(t\theta+\alpha,t\theta^{*}+\bar{\alpha}\right)\\&=\frac{d}{dt}|_{t=0}P\left(t\theta+\alpha,\bar{\alpha}\right)+\frac{d}{dt}|_{t=0}P\left(\alpha,t\theta^{*}+\bar{\alpha}\right)\\&=\frac{\partial P^{\mathrm{cl}}}{\partial\alpha}\left(\alpha\right)\theta+\frac{\partial P^{\mathrm{cl}}}{\partial\bar{\alpha}}\left(\alpha\right)\theta^{*}.\end{aligned}$$

Similarly from Eq. (2.32),

$$\begin{aligned}
P_2(\alpha:\theta,\theta^*) =& \frac{1}{2}\left(\frac{d}{dt}\right)^2|_{t=0}P(t\theta+\alpha,t\theta^*+\bar{\alpha}) \\
=& \frac{1}{2}\left(\frac{d}{dt}\right)^2|_{t=0}\left[P(t\theta+\alpha,\bar{\alpha})+P(\alpha,t\theta^*+\bar{\alpha})\right] \\
&+\frac{d}{dt}|_{t=0}\frac{d}{ds}|_{s=0}P(s\theta+\alpha,t\theta^*+\bar{\alpha}) \\
=& \frac{1}{2}\left(\frac{\partial^2 P^{\mathrm{cl}}}{\partial\alpha^2}(\alpha)\,\theta^2+\frac{\partial^2 P^{\mathrm{cl}}}{\partial\bar{\alpha}^2}(\alpha)\,\theta^{*2}\right) \\
&+\frac{d}{dt}|_{t=0}\frac{d}{ds}|_{s=0}P(s\theta+\alpha,t\theta^*+\bar{\alpha})\,.
\end{aligned}$$

If $P(\theta,\theta^*)\in\mathbb{C}\langle\theta,\theta^*\rangle$ is symmetric, then $P(t\theta+\alpha,t\theta^*+\bar{\alpha})\in\mathbb{C}\langle\theta,\theta^*\rangle$ is symmetric and hence from Eq. (2.32) it follows that $P_k(\alpha:\theta,\theta^*)\in\mathbb{C}\langle\theta,\theta^*\rangle$ is still symmetric and therefore so is the remainder term,

$$P_{\geq 3}(\alpha:\theta,\theta^*)=\sum_{k=3}^{p}P_k(\alpha:\theta,\theta^*)\,.$$

■

# Chapter 3

# Polynomial Operators

## 1 Algebra of Polynomial Operators

**Notation 3.1.** *For* $\mathbf{b} = (b_1, \ldots, b_k) \in \{\theta, \theta^*\}^k$, $p(\mathbf{b})$, $q(\mathbf{b})$, *and* $\ell(\mathbf{b})$ *be the* $\mathbb{Z}$ – *valued functions defined by*

$$p(\mathbf{b}) := \#\{i : b_i = \theta\}, \ q(\mathbf{b}) := \#\{i : b_i = \theta^*\}, \ \text{and} \tag{3.1}$$

$$\ell(\mathbf{b}) := \sum_{i=1}^{k} \left(1_{b_i=\theta^*} - 1_{b_i=\theta}\right) = q(\mathbf{b}) - p(\mathbf{b}). \tag{3.2}$$

*Thus* $p(\mathbf{b})$ $(q(\mathbf{b}))$ *is the number of* $\theta$*'s* ($\theta^*$*'s*) *in* $\mathbf{b}$ *and* $\ell(\mathbf{b})$ *counts the excess number of* $\theta^*$*'s over* $\theta$*'s in* $\mathbf{b}$.

**Lemma 3.2** (Normal Ordering)**.** *If* $P(\theta, \theta^*) \in \mathbb{C}\langle\theta, \theta^*\rangle$ *with* $d = \deg_\theta P$, *then there exists* $R(\hbar : \theta, \theta^*) \in \mathbb{C}[\hbar]\langle\theta, \theta^*\rangle$ *(a non-commutative polynomial in* $\{\theta, \theta^*\}$ *with polynomial coefficients in* $\hbar$*) such that* $\deg_\theta R(\hbar : \theta, \theta^*) \le d - 2$ *and*

$$P\left(a_\hbar, a_\hbar^\dagger\right) = \sum_{0 \le k,l;\ k+l \le d} \frac{1}{k! \cdot l!} \left(\frac{\partial^{k+l} P^{\mathrm{cl}}}{\partial \overline{\alpha}^k \partial \alpha^l}\right)(0)\, a_\hbar^{\dagger k} a_\hbar^l + \hbar R\left(\hbar : a_\hbar, a_\hbar^\dagger\right) \ \forall\ \hbar > 0.$$

**Proof.** By linearity it suffices to consider the case here $P(\theta, \theta^*)$ is a homogeneous polynomial of degree $d$ which may be written as

$$P(\theta,\theta^*) = \sum_{\mathbf{b}\in\{\theta,\theta^*\}^d} c(\mathbf{b})\, u_{\mathbf{b}}(\theta,\theta^*) = \sum_{p=0}^{d} \sum_{\mathbf{b}\in\{\theta,\theta^*\}^d} 1_{p(\mathbf{b})=p}\, c(\mathbf{b})\, u_{\mathbf{b}}(\theta,\theta^*). \tag{3.3}$$

Since

$$P(\alpha,\bar{\alpha}) = \sum_{p=0}^{d} \left[\sum_{\mathbf{b}\in\{\theta,\theta^*\}^d} 1_{p(\mathbf{b})=p}\, c(\mathbf{b})\right] \alpha^p \bar{\alpha}^{d-p}$$

it follows that

$$\frac{1}{(d-p)!\cdot p!}\left(\frac{\partial^d P^{cl}}{\partial\overline{\alpha}^{d-p}\partial\alpha^p}\right)(0) = \sum_{\mathbf{b}\in\{\theta,\theta^*\}^d} 1_{p(\mathbf{b})=p}\, c(\mathbf{b}).$$

On the other hand, if $\mathbf{b}\in\{\theta,\theta^*\}^d$ and $p := p(\mathbf{b})$, then making use of the CCRs of Eq. (1.7) it is easy to show there exists $R_{\mathbf{b}}(\hbar,\theta,\theta^*) \in \mathbb{C}[\hbar]\langle\theta,\theta^*\rangle$ such that $\deg_\theta R_{\mathbf{b}}(\hbar,\theta,\theta^*) \le d-2$ such that

$$u_{\mathbf{b}}\left(a_\hbar, a_\hbar^\dagger\right) = a_\hbar^{\dagger(d-p)} a_\hbar^p + \hbar R_{\mathbf{b}}\left(\hbar, a_\hbar, a_\hbar^\dagger\right). \tag{3.4}$$

Replacing $\theta$ by $a_\hbar$ and $\theta^*$ by $a_\hbar^\dagger$ in Eq. (3.3) and using Eq. (3.4) we find,

$$\begin{aligned}
P\left(a_\hbar, a_\hbar^\dagger\right) &= \sum_{p=0}^{d} \sum_{\mathbf{b}\in\{\theta,\theta^*\}^d} 1_{p(\mathbf{b})=p}\, c(\mathbf{b})\, u_{\mathbf{b}}\left(a_\hbar, a_\hbar^\dagger\right) \\
&= \sum_{p=0}^{d} \left[\sum_{\mathbf{b}\in\{\theta,\theta^*\}^d} 1_{p(\mathbf{b})=p}\, c(\mathbf{b})\right] a_\hbar^{\dagger(d-p)} a_\hbar^p + \hbar \sum_{\mathbf{b}\in\{\theta,\theta^*\}^d} c(\mathbf{b})\, R_{\mathbf{b}}\left(\hbar, a_\hbar, a_\hbar^\dagger\right) \\
&= \sum_{p=0}^{d} \frac{1}{(d-p)!\cdot p!}\left(\frac{\partial^d P^{\mathrm{cl}}}{\partial\overline{\alpha}^{d-p}\partial\alpha^p}\right)(0)\, a_\hbar^{\dagger(d-p)} a_\hbar^p + \hbar R\left(\hbar, a_\hbar, a_\hbar^\dagger\right)
\end{aligned}$$

where

$$R(\hbar,\theta,\theta^*) = \sum_{\mathbf{b}\in\{\theta,\theta^*\}^d} c(\mathbf{b})\, R_{\mathbf{b}}(\hbar,\theta,\theta^*).$$

■

**Corollary 3.3.** *If $P(\theta, \theta^*)$ and $Q(\theta, \theta^*)$ are non-commutative polynomials such that $P^{\mathrm{cl}} = Q^{\mathrm{cl}}$, then there exists $R(\hbar : \theta, \theta^*) \in \mathbb{C}[\hbar]\langle\theta, \theta^*\rangle$ with $\deg_\theta R(\hbar : \theta, \theta^*) \leq \deg_\theta(P - Q)(\theta, \theta^*) - 2$ such that*

$$P\left(a_\hbar, a_\hbar^\dagger\right) = Q\left(a_\hbar, a_\hbar^\dagger\right) + \hbar R\left(\hbar, a_\hbar, a_\hbar^\dagger\right).$$

**Proof.** Apply Lemma 3.2 to the non-commutative polynomial, $P(\theta, \theta^*) - Q(\theta, \theta^*)$. ■

**Proposition 3.4.** *For all $H \in \mathbb{C}\langle\theta, \theta^*\rangle$, there exists a polynomial, $p_H \in \mathbb{C}[z, \bar{z}]$ such that*

$$\begin{aligned} &H_2\left(\alpha : a, a^\dagger\right) \\ &\quad = \frac{1}{2}\left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\alpha^2}\right)(\alpha)\, a^2 + \frac{1}{2}\left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\overline{\alpha}^2}\right)(\alpha)\, a^{\dagger 2} + \left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\alpha\partial\overline{\alpha}}\right)(\alpha)\, a^\dagger a + p_H(\alpha, \bar{\alpha})\, I \end{aligned}$$

*for all $\alpha \in \mathbb{C}$ where $H_2(\alpha : \theta, \theta^*)$ is defined in Eq. (2.29).*

**Proof.** As we have seen the structure of $H_2(\alpha : \theta, \theta^*)$ implies there exists $\rho, \gamma, \delta \in \mathbb{C}[\alpha, \bar{\alpha}]$ such that

$$2H_2(\alpha : \theta, \theta^*) = \rho(\alpha, \bar{\alpha})\,\theta^2 + \overline{\rho(\alpha, \bar{\alpha})}\theta^{*2} + \gamma(\alpha, \bar{\alpha})\,\theta^*\theta + \delta(\alpha, \bar{\alpha})\,\theta\theta^*.$$

From this equation we find,

$$2H_2(\alpha : z, \bar{z}) = \rho(\alpha, \bar{\alpha})\, z^2 + \overline{\rho(\alpha, \bar{\alpha})}\bar{z}^2 + [\gamma(\alpha, \bar{\alpha}) + \delta(\alpha, \bar{\alpha})]\, z\bar{z}$$

while form Eq. (2.30) we may conclude that

$$2H_2(\alpha : z, \bar{z}) = \left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\alpha^2}\right)(\alpha)\, z^2 + \left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\overline{\alpha}^2}\right)(\alpha)\, \bar{z}^2 + 2\left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\alpha\partial\overline{\alpha}}\right)(\alpha)\, \bar{z}z. \qquad (3.5)$$

Comparing these last two equations shows,

$$\left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\alpha^2}\right)(\alpha) = \rho\left(\alpha,\bar{\alpha}\right), \quad \left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\overline{\alpha}^2}\right)(\alpha) = \overline{\rho\left(\alpha,\bar{\alpha}\right)}, \text{ and}$$
$$\left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\alpha\partial\overline{\alpha}}\right)(\alpha) = \frac{1}{2}\left[\gamma\left(\alpha,\bar{\alpha}\right) + \delta\left(\alpha,\bar{\alpha}\right)\right].$$

Using these last identities and the canonical commutations relations we find,

$$\begin{aligned}
&2H_2\left(\alpha : a, a^\dagger\right)\\
&= \rho\left(\alpha,\bar{\alpha}\right)a^2 + \overline{\rho\left(\alpha,\bar{\alpha}\right)}a^{\dagger 2} + \gamma\left(\alpha,\bar{\alpha}\right)a^\dagger a + \delta\left(\alpha,\bar{\alpha}\right)aa^\dagger\\
&= \rho\left(\alpha,\bar{\alpha}\right)a^2 + \overline{\rho\left(\alpha,\bar{\alpha}\right)}a^{\dagger 2} + \left[\gamma\left(\alpha,\bar{\alpha}\right) + \delta\left(\alpha,\bar{\alpha}\right)\right]a^\dagger a + \delta\left(\alpha,\bar{\alpha}\right)I\\
&= \left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\alpha^2}\right)(\alpha)\,a^2 + \left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\overline{\alpha}^2}\right)(\alpha)\,a^{\dagger 2} + 2\left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\alpha\partial\overline{\alpha}}\right)(\alpha)\,a^\dagger a + p_H\left(\alpha,\bar{\alpha}\right)I
\end{aligned}$$

with $p_H\left(\alpha,\bar{\alpha}\right) = \delta\left(\alpha,\bar{\alpha}\right)$. ■

Proposition 3.4 and the following simple commutator formulas,

$$\left[a^\dagger a, a\right] = -a, \quad \left[a^{\dagger 2}, a\right] = -2a^\dagger,$$
$$\left[a^\dagger a, a^\dagger\right] = a^\dagger, \text{ and } \left[a^2, a^\dagger\right] = 2a,$$

immediately give the following corollary.

**Corollary 3.5.** *If $H \in \mathbb{C}\left\langle\theta,\theta^*\right\rangle$ and $\alpha \in \mathbb{C}$, then*

$$\begin{aligned}
\left[H_2\left(\alpha : a, a^\dagger\right), a\right] &= -\left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\alpha\partial\overline{\alpha}}\right)(\alpha)\,a - \left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\overline{\alpha}^2}\right)(\alpha)\,a^\dagger\\
\left[H_2\left(\alpha : a, a^\dagger\right), a^\dagger\right] &= \left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\alpha^2}\right)(\alpha)\,a + \left(\frac{\partial^2 H^{\mathrm{cl}}}{\partial\alpha\partial\overline{\alpha}}\right)(\alpha)\,a^\dagger.
\end{aligned}$$

# 2 Expectations and variances for translated states

The next result is a fairly easy consequence of Proposition 2.4 and the expansion of non-commutative polynomials into their homogeneous components.

**Corollary 3.6** (Concentrated states)**.** *Let $P(\theta, \theta^*) \in \mathbb{C}\langle \theta, \theta^* \rangle$, $\psi \in \mathcal{S}$, $\hbar > 0$, and $\alpha \in \mathbb{C}$, then*

$$\left\langle P\left(a_\hbar, a_\hbar^\dagger\right)\right\rangle_{U_\hbar(\alpha)\psi} = P(\alpha, \bar{\alpha}) + O\left(\sqrt{\hbar}\right) \tag{3.6}$$

$$\mathrm{Var}_{U_\hbar(\alpha)\psi}\left(P\left(a_\hbar, a_\hbar^\dagger\right)\right) = O\left(\sqrt{\hbar}\right), \tag{3.7}$$

*and*

$$\lim_{\hbar \downarrow 0}\left\langle P\left(\frac{a_\hbar - \alpha}{\sqrt{\hbar}}, \frac{a_\hbar^\dagger - \bar{\alpha}}{\sqrt{\hbar}}\right)\right\rangle_{U_\hbar(\alpha)\psi} = \left\langle P\left(a, a^\dagger\right)\right\rangle_\psi \tag{3.8}$$

*where $\langle \cdot \rangle_{U_\hbar(\alpha)\psi}$ is defined in Definition 1.7. [In fact, the equality in the last equation holds before taking the limit as $\hbar \to 0$.]*

**Proof.** From Proposition 2.4 and Eq. (2.29),

$$U_\hbar(\alpha)^* P\left(a_\hbar, a_\hbar^\dagger\right) U_\hbar(\alpha) = P\left(a_\hbar + \alpha, a_\hbar^\dagger + \bar{\alpha}\right) = \sum_{k=0}^{d} P_k\left(\alpha : a_\hbar, a_\hbar^\dagger\right) \tag{3.9}$$

and hence

$$\begin{aligned}\left\langle P\left(a_\hbar, a_\hbar^\dagger\right)\right\rangle_{U_\hbar(\alpha)\psi} &= \left\langle U_\hbar(\alpha)^* P\left(a_\hbar, a_\hbar^\dagger\right) U_\hbar(\alpha)\right\rangle_\psi = \left\langle P\left(a_\hbar + \alpha, a_\hbar^\dagger + \bar{\alpha}\right)\right\rangle_\psi \\ &= \left\langle \sum_{k=0}^{d} P_k\left(\alpha : a_\hbar, a_\hbar^\dagger\right)\right\rangle_\psi = P_0(\alpha) + \sum_{k=1}^{d} \hbar^{k/2}\left\langle P_k\left(\alpha : a, a^\dagger\right)\right\rangle_\psi\end{aligned}$$

from which Eq. (3.6) follows where $P_0(\alpha)$ is defined in Notation 2.8. Similarly, making use of the fact that $(P^2)_0(\alpha) = (P_0^2)(\alpha)$

$$\left\langle P^2\left(a_\hbar, a_\hbar^\dagger\right)\right\rangle_{U_\hbar(\alpha)\psi} = \left(P_0^2\right)(\alpha) + \sum_{k=1}^{2d} \hbar^{k/2} \left\langle \left(P^2\right)_k \left(\alpha : a, a^\dagger\right)\right\rangle_\psi \tag{3.10}$$

and hence

$$\begin{aligned}\operatorname{Var}_{U_\hbar(\alpha)\psi}\left(P\left(a_\hbar, a_\hbar^\dagger\right)\right) &= \left(P_0^2\right)(\alpha) + \sum_{k=1}^{2d} \hbar^{k/2} \left\langle \left(P^2\right)_k \left(\alpha : a, a^\dagger\right)\right\rangle_\psi \\ &\quad - \left[\left(P_0(\alpha) + \sum_{k=0}^{d} \hbar^{k/2} \left\langle P_k\left(\alpha : a, a^\dagger\right)\right\rangle_\psi\right)\right]^2 \\ &= O\left(\sqrt{\hbar}\right).\end{aligned}$$

Lastly, using Eq. (3.9) one shows,

$$\left\langle P\left(\frac{a_\hbar - \alpha}{\sqrt{\hbar}}, \frac{a_\hbar^\dagger - \bar{\alpha}}{\sqrt{\hbar}}\right)\right\rangle_{U_\hbar(\alpha)\psi} = \left\langle P\left(\frac{a_\hbar + \alpha - \alpha}{\sqrt{\hbar}}, \frac{a_\hbar^\dagger + \bar{\alpha} - \bar{\alpha}}{\sqrt{\hbar}}\right)\right\rangle_\psi = \left\langle P\left(a, a^\dagger\right)\right\rangle_\psi$$

which certainly implies Eq. (3.8). ■

**Remark 3.7.** *If $\psi \in \mathcal{S}$ and $\alpha \in \mathbb{C}$, Eqs. (3.6) and (3.7) should be interpreted to say that for small $\hbar > 0$, $U_\hbar(\alpha)\psi$ is a state which is concentrated in phase space near $\alpha$. Consequently, these are good initial states for discussing the classical $(\hbar \to 0)$ limit of quantum mechanics.*

The next result shows that, under Assumption 1.11, the classical equations of motions in Eq. (1.1) have global solutions which remain bounded in time.

**Proposition 3.8.** *If $C$ and $C_1$ are the constants appearing in Eq. (1.14) of Assumption 1.11, $\alpha_0 \in \mathbb{C}$, and $\alpha(t) \in \mathbb{C}$ is the maximal solution of Hamilton's ordinary differential equations (1.1), then $\alpha(t)$ is defined for all time $t$ and moreover,*

$$|\alpha(t)|^2 \le C_1\left(H^{cl}(\alpha(0)) + C\right), \tag{3.11}$$

*where* $H^{cl}(\alpha) := H(\alpha, \bar{\alpha})$ *.*

**Proof.** Equation (1.14) with $\beta = 1$ implies

$$\langle \mathcal{N}_\hbar \rangle_\psi \leq C_1 \langle H_\hbar + C \rangle_\psi \text{ for all } \psi \in \mathcal{S}. \tag{3.12}$$

Replacing $\psi$ by $U_\hbar(\alpha)\psi$ in Eq. (3.12) and then letting $\hbar \downarrow 0$ gives (with the aid of Corollary 3.6) the estimate,

$$|\alpha|^2 \leq C_1 \left( H^{\mathrm{cl}}(\alpha) + C \right) \text{ for all } \alpha \in \mathbb{C}. \tag{3.13}$$

If $\alpha(t)$ solves Hamilton's Eq. (1.1) then $H^{\mathrm{cl}}(\alpha(t)) = H^{\mathrm{cl}}(\alpha(0))$ for all $t$. As the level sets of $H^{\mathrm{cl}}$ are compact because of the estimate in Eq. (3.13) there is no possibility for $\alpha(t)$ to explode and hence solutions will exist for all times $t$ and moreover must satisfy the estimate in Eq. (3.11). ■

# 3 Analysis of Monomial Operators of $a$ and $a^\dagger$

In this section, recall that $a = a_1$ and $a^\dagger = a_1^\dagger$ as in Definition 1.3. Let

$$\Omega_0(x) := \frac{1}{\sqrt[4]{4\pi}} \exp\left( -\frac{1}{2} x^2 \right) \text{ and } \left\{ \Omega_n := \frac{1}{\sqrt{n!}} a^{\dagger n} \Omega_0 \right\}_{n=0}^{\infty}. \tag{3.14}$$

**Convention:** $\Omega_n \equiv 0$ for all $n \in \mathbb{Z}$ with $n < 0$.

The following theorem summarizes the basic well known and easily verified properties of these functions which essentially are all easy consequences of the canonical commutation relations, $\left[a, a^\dagger\right] = I$ on $\mathcal{S}$. We will provide a short proof of these well known results for the readers convenience.

**Theorem 3.9.** *The functions* $\{\Omega_n\}_{n=0}^{\infty} \subset \mathcal{S}$ *form an orthonormal basis for* $L^2(m)$

*which satisfy for all* $n \in \mathbb{N}_0$,

$$a\Omega_n = \sqrt{n}\Omega_{n-1}, \tag{3.15}$$

$$a^\dagger\Omega_n = \sqrt{n+1}\Omega_{n+1} \text{ and} \tag{3.16}$$

$$a^\dagger a\Omega_n = n\Omega_n. \tag{3.17}$$

**Proof.** First observe that $\Omega_n(x)$ is a polynomial $(p_n(x))$ of degree $n$ times $\Omega_0(x)$. Therefore the span of $\{\Omega_n\}_{n=0}^\infty$ are all functions of the form $p(x)\Omega_0(x)$ where $p \in \mathbb{C}[x]$. As $\mathbb{C}[x]$ is dense in $L^2(\Omega_0^2(x)\,dx)$ it follows that $\{\Omega_n\}_{n=0}^\infty$ is total in $L^2(m)$.

For the remaining assertions let us recall, if $A$ and $B$ are operators on some vector space (like $\mathcal{S}$) and $ad_A B := [A, B]$, then $ad_A$ acts as a derivation, i.e.

$$ad_A(BC) = (ad_A B)C + B(ad_A C). \tag{3.18}$$

Combining this observation with $ad_a a^\dagger = I$ then shows $ad_a a^{\dagger n} = n a^{\dagger n-1}$ so that

$$a\Omega_n = a\frac{1}{\sqrt{n!}}a^{\dagger n}\Omega_0 = \frac{1}{\sqrt{n!}}\left(ad_a a^{\dagger n}\right)\Omega_0 = \frac{n}{\sqrt{n!}}a^{\dagger(n-1)}\Omega_0 = \sqrt{n}\Omega_{n-1}$$

which proves Eq. (3.15). Equation (3.16) is obvious from the definition of $\{\Omega_n\}_{n=0}^\infty$ and Eq. (3.17) follows from Eqs. (3.15) and (3.16). As $\{\Omega_n\}_{n=0}^\infty$ are eigenvectors of the symmetric operator $a^\dagger a$ with distinct eigenvalues it follows that $\langle\Omega_n, \Omega_m\rangle = 0$ if $m \neq n$. So it only remains to show $\|\Omega_n\|^2 = 1$ for all $n$. However, taking the $L^2(m)$ -norm of Eq. (3.16) gives

$$\begin{aligned}(n+1)\|\Omega_{n+1}\|^2 &= \left\|a^\dagger\Omega_n\right\|^2 = \left\langle\Omega_n, aa^\dagger\Omega_n\right\rangle = \left\langle\Omega_n, \left(a^\dagger a + I\right)\Omega_n\right\rangle \\ &= (n+1)\|\Omega_n\|^2,\end{aligned}$$

i.e. $n \to \|\Omega_n\|^2$ is constant in $n$. As we normalized $\Omega_0$ to be a unit vector, the proof is complete.

■

**Notation 3.10.** *For* $N \in \mathbb{N}_0$, *let* $\mathcal{P}_N$ *denote orthogonal projection of* $L^2(m)$ *onto* $\operatorname{span}\{\Omega_n : 0 \le n \le N\}$, *i.e.*

$$\mathcal{P}_N f := \sum_{n=0}^{N} \langle f, \Omega_n \rangle \, \Omega_n \text{ for all } f \in L^2(m). \tag{3.19}$$

**Notation 3.11** (Standing Notation)**.** *For the remainder of this chapter let* $k, j \in \mathbb{N}$, $\mathbf{b} = (b_1, \ldots, b_k) \in \{\theta, \theta^*\}^k$, $q := q(\mathbf{b})$, $l := \ell(\mathbf{b})$, $\mathbf{d} = (d_1, \ldots, d_j) \in \{\theta, \theta^*\}^j$, *and* $\ell(\mathbf{d})$ *be as in Notation 3.1. We further let* $\mathcal{A}$ *and* $\mathcal{D}$ *be the two monomial operators,*

$$\begin{aligned} \mathcal{A} &:= u_{\mathbf{b}}\left(a, a^\dagger\right) = \Xi(b_1) \ldots \Xi(b_k) \text{ and} \\ \mathcal{D} &:= u_{\mathbf{d}}\left(a, a^\dagger\right) = \Xi(d_1) \ldots \Xi(d_j). \end{aligned}$$

**Lemma 3.12.** *To each monomial operator* $\mathcal{A} = u_{\mathbf{b}}\left(a, a^\dagger\right)$ *as in Notation 3.11, there exists* $c_{\mathcal{A}} : \mathbb{N}_0 \to [0, \infty)$ *such that*

$$\mathcal{A}\Omega_n = c_{\mathcal{A}}(n) \cdot \Omega_{n+l} \text{ for all } n \in \mathbb{N}_0 \tag{3.20}$$

*where (as above)* $\Omega_m := 0$ *if* $m < 0$. *Moreover,* $c_{\mathcal{A}}$ *satisfies* $c_{\mathcal{A}^\dagger}(n) = c_{\mathcal{A}}(n - l)$ *(where by convention* $c_{\mathcal{A}}(n) \equiv 0$ *if* $n < 0$*),*

$$0 \le c_{\mathcal{A}}(n) \le (n + q)^{\frac{k}{2}} \text{ and } c_{\mathcal{A}}(n) \asymp n^{k/2} \text{ (i.e. } \lim_{n\to\infty} \frac{c_{\mathcal{A}}(n)}{n^{k/2}} = 1\text{)}. \tag{3.21}$$

**Proof.** Since $a$ and $a^\dagger$ shift $\Omega_n$ to its adjacent $\Omega_{n-1}$ and $\Omega_{n+1}$ respectively from Theorem 3.9, it is easy to see that Eq. (3.20) holds for some constants $c_{\mathcal{A}}(n) \in \mathbb{R}$. Moreover a simple induction argument on $k$ shows there exists $\delta_i \in \mathbb{Z}$

with $\delta_i \leq q$ such that

$$c_{\mathcal{A}}(n) = \left(\sqrt{\prod_{i=1}^{k}(n+\delta_i)}\right) \geq 0. \tag{3.22}$$

The estimate and the limit statement in Eq. (3.21) now follows directly from the Eq. (3.22).

Since $\mathcal{A}^{\dagger}\Omega_n = c_{\mathcal{A}^{\dagger}}(n)\,\Omega_{n-l}$, we find

$$c_{\mathcal{A}^{\dagger}}(n) = \left\langle \mathcal{A}^{\dagger}\Omega_n, \Omega_{n-l}\right\rangle = \left\langle \Omega_n, \mathcal{A}\Omega_{n-l}\right\rangle = \left\langle \Omega_n, c_{\mathcal{A}}(n-l)\,\Omega_n\right\rangle = c_{\mathcal{A}}(n-l)\,.$$

■

**Example 3.13.** *Suppose that $p, q \in \mathbb{N}_0$, $k = p+q$, $\ell = q-p$, and $\mathcal{A} = a^p a^{\dagger q}$. Then*

$$\begin{aligned}\mathcal{A}\Omega_n &= a^p a^{\dagger q}\Omega_n = a^p\sqrt{\prod_{i=1}^{q}(n+i)}\cdot\Omega_{n+q}\\ &= \sqrt{\prod_{i=1}^{q}(n+i)}\cdot a^p\Omega_{n+q} = \sqrt{\prod_{i=1}^{q}(n+i)}\sqrt{\prod_{j=0}^{p-1}(n+q-j)}\Omega_{n+\ell}\end{aligned}$$

*where*

$$0 \leq c_{\mathcal{A}}(n) = \sqrt{\prod_{i=1}^{q}(n+i)}\cdot\sqrt{\prod_{j=0}^{p-1}(n+q-j)} \leq (n+q)^{\frac{k}{2}}\,. \tag{3.23}$$

**Definition 3.14.** *For $\beta \geq 0$, let*

$$D_\beta := \left\{ f \in L^2(\mathbb{R}) : \sum_{n=0}^{\infty} \left|\langle f, \Omega_n\rangle\right|^2 n^{2\beta} < \infty \right\}.$$

*[We will see shortly that $D_\beta = D\left(\mathcal{N}^\beta\right)$, see Example 3.19.]*

**Theorem 3.15.** *Let $k = \deg_\theta u_{\mathbf{b}}(\theta, \theta^*)$, $\mathcal{A} = u_{\mathbf{b}}\left(a, a^{\dagger}\right)$, $l = \ell(\mathbf{b}) \in \mathbb{Z}$ be as in Notations 3.11 and 3.1 and $c_{\mathcal{A}}(n)$ be coefficients in Lemma 3.12. Then $\mathcal{A}$ and $\mathcal{A}^{\dagger}$*

*are closable operators satisfying;*

1. $\bar{\mathcal{A}} = \mathcal{A}^{\dagger *}$ *and* $\overline{\mathcal{A}^\dagger} = \mathcal{A}^*$ *where we write* $\mathcal{A}^{\dagger *}$ *for* $\left(\mathcal{A}^\dagger\right)^*$.

2. $D\left(\bar{\mathcal{A}}\right) = D_{k/2} = D\left(\overline{\mathcal{A}^\dagger}\right)$ *and if* $g \in D_{k/2}$, *then*

$$\mathcal{A}^* g = \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle \mathcal{A}^\dagger \Omega_n = \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle c_{\mathcal{A}}(n-l)\, \Omega_{n-l} \text{ and} \tag{3.24}$$

$$\mathcal{A}^{\dagger *} g = \bar{\mathcal{A}} g = \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle \mathcal{A} \Omega_n = \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle c_{\mathcal{A}}(n)\, \Omega_{n+l} \tag{3.25}$$

*with the conventions that* $c_{\mathcal{A}}(n)$ *and* $\Omega_n = 0$ *if* $n < 0$.

3. *The subspace,*

$$\mathcal{S}_0 := \operatorname{span}\{\Omega_n\}_{n=0}^{\infty} \subset \mathcal{S} \subset L^2(m) \tag{3.26}$$

*is a core of both* $\bar{\mathcal{A}}$ *and* $\overline{\mathcal{A}^\dagger}$. *More explicitly if* $g \in D_{k/2}$, *then*

$$\bar{\mathcal{A}} g = \lim_{N \to \infty} \mathcal{A} \mathcal{P}_N g \text{ and } \overline{\mathcal{A}^\dagger} g = \lim_{N \to \infty} \mathcal{A}^\dagger \mathcal{P}_N g$$

*where* $\mathcal{P}_N$ *is the orthogonal projection operator onto* $\operatorname{span}\{\Omega_k\}_{k=0}^{n}$ *as in Notation 3.10.*

**Proof.** Since $\langle \mathcal{A} f, g \rangle = \left\langle f, \mathcal{A}^\dagger g \right\rangle$ for all $f, g \in \mathcal{S} = D(\mathcal{A}) = D\left(\mathcal{A}^\dagger\right)$, it follows that $\mathcal{A} \subset \mathcal{A}^{\dagger *}$ and $\mathcal{A}^\dagger \subset \mathcal{A}^*$ and therefore both $\mathcal{A}$ and $\mathcal{A}^\dagger$ are closable (see Theorem VIII.1 on p.252 of [32]) and

$$\overline{\mathcal{A}^\dagger} \subset \mathcal{A}^* \text{ and } \bar{\mathcal{A}} \subset \mathcal{A}^{\dagger *}. \tag{3.27}$$

If $g \in D(\mathcal{A}^*) \subset L^2(m)$ ,then from Theorem 3.9 and Lemma 3.12, we have

$$\begin{aligned}
\mathcal{A}^* g &= \sum_{n=0}^{\infty} \langle \mathcal{A}^* g, \Omega_n \rangle \Omega_n = \sum_{n=0}^{\infty} \langle g, \mathcal{A}\Omega_n \rangle \Omega_n \\
&= \sum_{n=0}^{\infty} \langle g, c_{\mathcal{A}}(n) \Omega_{n+l} \rangle \Omega_n = \sum_{n=0}^{\infty} \langle g, \Omega_{n+l} \rangle c_{\mathcal{A}}(n) \Omega_n \\
&= \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle c_{\mathcal{A}}(n-l) \Omega_{n-l} = \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle \mathcal{A}^{\dagger} \Omega_n, \qquad (3.28)
\end{aligned}$$

wherein we have used the conventions stated after Eq. (3.25) repeatedly. Since, by Lemma 3.12, $\left\{\mathcal{A}^{\dagger}\Omega_n = c_{\mathcal{A}}(n-l)\Omega_{n-l}\right\}_{n=0}^{\infty}$ is an orthogonal set such that

$$\left\|\mathcal{A}^{\dagger}\Omega_n\right\|_2^2 = |c_{\mathcal{A}}(n-l)|^2 \asymp n^k,$$

it follows that the last sum in Eq. (3.28) is convergent iff

$$\sum_{n=0}^{\infty} |\langle g, \Omega_n \rangle|^2 n^k < \infty \iff g \in D_{k/2}.$$

Conversely if $g \in D_{k/2}$ and $f \in \mathcal{S} = D(\mathcal{A})$ we have,

$$\begin{aligned}
\left\langle \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle \mathcal{A}^{\dagger}\Omega_n, f \right\rangle &= \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle \left\langle \mathcal{A}^{\dagger}\Omega_n, f \right\rangle \\
&= \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle \langle \Omega_n, \mathcal{A} f \rangle = \langle g, \mathcal{A} f \rangle
\end{aligned}$$

from which it follows that $g \in D(\mathcal{A}^*)$ and $\mathcal{A}^* g$ is given as in Eq. (3.28).

In summary, we have shown $D(\mathcal{A}^*) = D_{k/2}$ and $\mathcal{A}^* g$ is given by Eq. (3.28). Moreover, from Eq. (3.28), if $g \in D_{k/2}$ then

$$\mathcal{A}^* g = \lim_{N\to\infty} \sum_{n=0}^{N} \langle g, \Omega_n \rangle \mathcal{A}^{\dagger}\Omega_n = \lim_{N\to\infty} \mathcal{A}^{\dagger}\mathcal{P}_N g$$

which implies $g \in D\left(\overline{\mathcal{A}^\dagger}\right)$ and $\mathcal{A}^* g = \overline{\mathcal{A}^\dagger} g$, i.e. $\mathcal{A}^* \subset \overline{\mathcal{A}^\dagger}$. Combining this last assertion with the first inclusion in Eq. (3.27) implies and $\mathcal{A}^* = \overline{\mathcal{A}^\dagger}$. This proves all of the assertions involving $\mathcal{A}^*$ and $\overline{\mathcal{A}^\dagger}$. We may now complete the proof by applying these assertions with $\mathcal{A} = u_{\mathbf{b}}\left(a, a^\dagger\right)$ replaced by $\mathcal{A}^\dagger = u_{\mathbf{b}^*}\left(a, a^\dagger\right)$ and using the facts that $\mathcal{A}^{\dagger\dagger} = \mathcal{A}$, $\ell\left(\mathbf{b}^*\right) = -\ell\left(\mathbf{b}\right) = -l$, and $c_{\mathcal{A}^\dagger}\left(n\right) = c_{\mathcal{A}}\left(n - l\right)$. ■

**Theorem 3.16.** *Let* $k = \deg_\theta u_{\mathbf{b}}\left(\theta, \theta^*\right)$, $j = \deg_\theta u_{\mathbf{d}}\left(\theta, \theta^*\right)$, $\mathcal{A} = u_{\mathbf{b}}\left(a, a^\dagger\right)$, $\mathcal{D} = u_{\mathbf{d}}\left(a, a^\dagger\right)$, $\ell\left(\mathbf{b}\right)$, *and* $\ell\left(\mathbf{d}\right)$ *be as in Notations 3.11 and 3.1. Then;*

1. $\overline{\mathcal{A}\mathcal{D}} = \bar{\mathcal{A}}\bar{\mathcal{D}}$,

2. $\left(\mathcal{A}\mathcal{D}\right)^* = \mathcal{D}^*\mathcal{A}^*$, *and*

3. $\bar{\mathcal{A}} := \overline{u_{\mathbf{b}}\left(a, a^\dagger\right)} = u_{\mathbf{b}}\left(\bar{a}, a^*\right)$, *i.e. if* $\mathcal{A}$ *is a monomial operator in* $a$ *and* $a^\dagger$, *then* $\bar{\mathcal{A}}$ *is the operator resulting from replacing* $a$ *by* $\bar{a}$ *and* $a^\dagger$ *by* $a^*$ *everywhere in* $\mathcal{A}$.

**Proof.** Because of the conventions described after Eq. (3.25), in the argument below it will be easier to view all sums over $n \in \mathbb{Z}$ instead of $n \in \mathbb{N}_0$. We will denote all of these infinite sums simply as $\sum_n$. We now prove each item in turn.

1. Since $\mathcal{A}\mathcal{D}$ is a monomial operator of degree $k + j$ it follows from Theorem 3.15 that $D\left(\overline{\mathcal{A}\mathcal{D}}\right) = D_{(k+j)/2}$. On the other hand, $f \in D\left(\bar{\mathcal{A}}\bar{\mathcal{D}}\right)$ iff $f \in D\left(\bar{\mathcal{D}}\right) = D_{j/2}$ and $\overline{\mathcal{D}} f \in D\left(\bar{\mathcal{A}}\right) = D_{k/2}$. Moreover, $\overline{\mathcal{D}} f = \mathcal{D}^{\dagger *} f \in D\left(\bar{\mathcal{A}}\right) = D_{k/2}$ iff

$$\begin{aligned}
\infty > \sum_n \left|\left\langle \overline{\mathcal{D}} f, \Omega_n \right\rangle\right|^2 n^k &= \sum_n \left|\left\langle f, \mathcal{D}^\dagger \Omega_n \right\rangle\right|^2 n^k \\
&= \sum_n \left|\left\langle f, \Omega_{n-\ell(\mathbf{d})} \right\rangle\right|^2 \left|c_{\mathcal{D}^\dagger}\left(n\right)\right|^2 n^k. \qquad (3.29)
\end{aligned}$$

However, by Lemma 3.12 we know $\left|c_{\mathcal{D}^\dagger}\left(n\right)\right|^2 \asymp n^j$ and so the condition in Eq. (3.29) is the same as saying $f \in D_{(k+j)/2}$. Thus we have shown

$D\left(\bar{\mathcal{A}}\bar{\mathcal{D}}\right) = D\left(\overline{\mathcal{A}\mathcal{D}}\right)$. Moreover, if $f \in D_{(k+j)/2}$, then by Theorem 3.15 and Lemma 3.12 we find,

$$\begin{aligned}\bar{\mathcal{A}}\bar{\mathcal{D}}f &= \sum_n \left\langle \overline{\mathcal{D}}f, \Omega_n \right\rangle \mathcal{A}\Omega_n = \sum_n \left\langle f, \mathcal{D}^{\dagger}\Omega_n \right\rangle \mathcal{A}\Omega_n \\ &= \sum_n \left\langle f, c_{\mathcal{D}}\left(n - \ell\left(\mathbf{d}\right)\right)\Omega_{n-\ell(\mathbf{d})} \right\rangle \mathcal{A}\Omega_n \\ &= \sum_n \left\langle f, \Omega_n \right\rangle \mathcal{A}c_{\mathcal{D}}\left(n\right)\Omega_{n+\ell(\mathbf{d})} \\ &= \sum_n \left\langle f, \Omega_n \right\rangle \mathcal{A}\mathcal{D}\Omega_n = \overline{\mathcal{A}\mathcal{D}}f. \end{aligned} \tag{3.30}$$

2. By item 1. of Theorem 3.15 and item 1. of this theorem,

$$\left(\mathcal{A}\mathcal{D}\right)^* = \overline{\left(\mathcal{A}\mathcal{D}\right)^{\dagger}} = \overline{\mathcal{D}^{\dagger}\mathcal{A}^{\dagger}} = \overline{\mathcal{D}^{\dagger}}\,\overline{\mathcal{A}^{\dagger}} = \mathcal{D}^*\mathcal{A}^*.$$

3. This follows by induction on $k = \deg_{\theta} u_{\mathbf{b}}$ making use of item 1. of Theorem 3.15 and item 1.

■

**Corollary 3.17** (Diagonal form of the Number Operator)**.** *If* $\mathcal{N} = u_{(\theta^*,\theta)}\left(\bar{a}, a^*\right) = a^*\bar{a}$ *as in Definition 1.4, then by* $\mathcal{N} = \overline{a^{\dagger}a}$,

$$D\left(\mathcal{N}\right) = D_1 = \left\{ f \in L^2\left(m\right) : \sum_{n=0}^{\infty} n^2 \left|\left\langle f, \Omega_n\right\rangle\right|^2 < \infty \right\},$$

*and for* $f \in D\left(\mathcal{N}\right)$,

$$\mathcal{N}f = \sum_{n=0}^{\infty} n \left\langle f, \Omega_n \right\rangle \Omega_n.$$

**Proof.** Since $\mathcal{N} = u_{(\theta^*,\theta)}\left(\bar{a}, a^*\right)$, it follows by Theorem 3.16 that

$$\mathcal{N} = \overline{u_{(\theta^*,\theta)}\left(a, a^{\dagger}\right)} = \overline{a^{\dagger}a}$$

and then by Theorem 3.15 that $D(\mathcal{N}) = D_1$. Moreover, by items 1 and 2 in the Theorem 3.15, if $f \in D(\mathcal{N})$, then

$$\mathcal{N} f = \sum_{n=0}^{\infty} \langle f, \Omega_n \rangle a^{\dagger} a \Omega_n = \sum_{n=0}^{\infty} n \langle f, \Omega_n \rangle \Omega_n.$$

■

**Definition 3.18** (Functional Calculus for $\mathcal{N}$)**.** *Given a function $G : \mathbb{N}_0 \to \mathbb{C}$ let $G(\mathcal{N})$ be the unique closed operator on $L^2(m)$ such that $G(\mathcal{N})\Omega_n := G(n)\Omega_n$ for all $n \in \mathbb{N}_0$. In more detail,*

$$D(G(\mathcal{N})) := \left\{ u \in L^2(m) : \sum_{n=0}^{\infty} |G(n)|^2 |\langle u, \Omega_n \rangle|^2 < \infty \right\} \tag{3.31}$$

*and for $u \in D(G(\mathcal{N}))$,*

$$G(\mathcal{N}) u := \sum_{n=0}^{\infty} G(n) \langle u, \Omega_n \rangle \Omega_n.$$

**Example 3.19.** *If $\beta \geq 0$, then $D(\mathcal{N}^{\beta}) = D_{\beta}$ where $D_{\beta}$ was defined in Definition 3.14.*

**Notation 3.20.** *If $J \subset \mathbb{N}_0$ and*

$$\mathbf{1}_J(n) := \begin{cases} 1 & \text{if } n \in J \\ 0 & \text{otherwise} \end{cases},$$

*then*

$$\mathbf{1}_J(\mathcal{N}) f = \sum_{n \in J} \langle f, \Omega_n \rangle \Omega_n \tag{3.32}$$

*is orthogonal projection onto $\overline{\operatorname{span}\{\Omega_n : n \in J\}}$. When $J = \{0, 1, \ldots, N\}$, then $\mathbf{1}_J(\mathcal{N})$ (or also write $1_{\mathcal{N} \leq N}$) is precisely the orthogonal projection operator already defined in Eq. (3.19) above.*

At this point it is convenient to introduce a scale of Sobolev type norms on $L^2(m)$.

**Notation 3.21** ($\beta$ – Norms)**.** *For $\beta \geq 0$ and $f \in L^2(m)$, let*

$$\|f\|_\beta^2 := \sum_{n=0}^{\infty} |\langle f, \Omega_n\rangle|^2 (n+1)^{2\beta}. \tag{3.33}$$

**Remark 3.22.** *From Definition 3.18 and Notation 3.21, it is readily seen that*

$$\begin{aligned} D_\beta = D\left(\mathcal{N}^\beta\right) &= \left\{f \in L^2(m) : \|f\|_\beta^2 < \infty\right\}, \\ \|f\|_\beta^2 &= \left\|(\mathcal{N}+I)^\beta f\right\|_{L^2(m)}^2 \;\; \forall\, f \in D\left(\mathcal{N}^\beta\right), \\ D\left(\mathcal{N}^\beta\right) &= D\left((\mathcal{N}+1)^\beta\right) \text{ for all } \beta \geq 0, \text{ and} \\ \|\cdot\|_{\beta_1} &\leq \|\cdot\|_{\beta_2} \text{ and } D\left(\mathcal{N}^{\beta_2}\right) \subseteq D\left(\mathcal{N}^{\beta_1}\right) \text{ for all } 0 \leq \beta_1 \leq \beta_2. \end{aligned}$$

*The normed space, $\left(D\left(\mathcal{N}^\beta\right), \|\cdot\|_\beta\right)$, is a Hilbertian space which is isomorphic to $\ell^2(\mathbb{N}_0, \mu_\beta)$ where $\mu_\beta(n) := (1+n)^{2\beta}$. The isomorphism is given by the unitary map,*

$$f \in D\left(\mathcal{N}^\beta\right) \to \{\langle f, \Omega_n\rangle\}_{n=0}^\infty \in \ell^2(\mathbb{N}_0, \mu_\beta).$$

It is well known (see for example, Theorem 1 of [37]) that

$$\mathcal{S} = \bigcap_{n=0}^{\infty} D(\mathcal{N}^n) = \bigcap_{\beta \geq 0} D\left(\mathcal{N}^\beta\right). \tag{3.34}$$

The inclusion $\mathcal{S} \subset \bigcap_{n=0}^\infty D(\mathcal{N}^n)$ is easy to understand since if $n \in \mathbb{N}_0$, $\left(a^\dagger a + I\right)^n$ is symmetric on $\mathcal{S}$ and therefore if $f \in \mathcal{S}$ we have,

$$\begin{aligned} \|f\|_n^2 &= \sum_{n=0}^{\infty} |\langle f, \Omega_n\rangle|^2 (n+1)^{2n} = \sum_{n=0}^{\infty} \left|\left\langle f, \left(a^\dagger a + I\right)^n \Omega_n\right\rangle\right|^2 \\ &= \sum_{n=0}^{\infty} \left|\left\langle \left(a^\dagger a + I\right)^n f, \Omega_n\right\rangle\right|^2 = \left\|\left(a^\dagger a + I\right)^n f\right\|_{L^2(m)}^2 < \infty. \end{aligned}$$

The following related result will be useful in the sequel.

**Proposition 3.23.** *The subspace $\mathcal{S}_0$ in Eq. (3.26) is dense (and so is $\mathcal{S}$) in $\left(D\left(\mathcal{N}^\beta\right), \|\cdot\|_\beta\right)$ for all $\beta \geq 0$. Moreover, if $f \in D\left(\mathcal{N}^\beta\right)$, then $\mathcal{P}_N f \in \mathcal{S}_0$ and $\|f - \mathcal{P}_N f\|_\beta \to 0$ as $N \to \infty$.*

**Proof.** If $f \in D\left(\mathcal{N}^\beta\right)$, then

$$\sum_{n=0}^{\infty} |\langle f, \Omega_n \rangle|^2 (1+n)^{2\beta} = \|f\|_\beta^2 < \infty$$

and hence

$$\|f - \mathcal{P}_N f\|_\beta^2 = \sum_{n=N+1}^{\infty} |\langle f, \Omega_n \rangle|^2 (1+n)^{2\beta} \to 0 \text{ as } N \to \infty.$$

■

**Remark 3.24.** *The zero norm, $\|\cdot\|_0$, is just a standard $L^2(m)$-norm and we will typically drop the subscript $0$ and simply write $\|\cdot\|$ for $\|\cdot\|_0 = \|\cdot\|_{L^2(m)}$.*

**Remark 3.25.** *If $\mathcal{A} = u_{\mathbf{b}}\left(a, a^\dagger\right)$ and $k = \deg_\theta u_{\mathbf{b}}(\theta, \theta^*)$, then by Eq. (3.33) and the Theorem 3.15, we have*

$$D\left(\bar{\mathcal{A}}\right) = D_{k/2} = D\left(\mathcal{N}^{\frac{k}{2}}\right). \tag{3.35}$$

**Corollary 3.26.** *The following domain statement holds;*

$$D(\bar{a}) = D(a^*) = D\left(\mathcal{N}^{1/2}\right) = D(M_x) \cap D(\partial_x). \tag{3.36}$$

*Moreover for* $f \in D\left(\mathcal{N}^{1/2}\right)$,

$$\bar{a}f = \sum_{n=1}^{\infty} \sqrt{n}\,\langle f, \Omega_n\rangle\,\Omega_{n-1} \text{ and} \tag{3.37}$$

$$a^* f = \sum_{n=0}^{\infty} \sqrt{n+1}\,\langle f, \Omega_n\rangle\,\Omega_{n+1}. \tag{3.38}$$

**Proof.** $D(\bar{a}) = D(a^*) = D\left(\mathcal{N}^{1/2}\right)$ is followed by the Eq. (3.35) in the Remark 3.25. Eqs (3.37) and (3.38) are consequence from Theorem 3.15. The only new statement to prove here is that $D\left(\mathcal{N}^{1/2}\right) = D(M_x) \cap D(\partial_x)$. If $f \in D(M_x) \cap D(\partial_x)$ we have

$$\begin{aligned}\sqrt{n}\,\langle f, \Omega_n\rangle = \left\langle f, a^\dagger \Omega_{n-1}\right\rangle &= \frac{1}{\sqrt{2}}\langle f, (M_x - \partial_x)\,\Omega_{n-1}\rangle \\ &= \frac{1}{\sqrt{2}}\langle (M_x + \partial_x)\,f, \Omega_{n-1}\rangle\end{aligned}$$

from which it follows that

$$\sum_{n=1}^{\infty} \left|\sqrt{n}\,\langle f, \Omega_n\rangle\right|^2 = \frac{1}{2}\left\|(M_x + \partial_x)\,f\right\|^2 < \infty$$

and therefore $f \in D(\bar{a}) = D\left(N^{1/2}\right)$. Conversely if $f \in D\left(\mathcal{N}^{1/2}\right)$ and we let $f_m := \sum_{k=0}^{m} \langle f, \Omega_k\rangle\,\Omega_k$ for all $m \in \mathbb{N}$, then $f_m \to f$, $\bar{a}f_m \to \bar{a}f$ and $a^* f_m \to a^* f$ in $L^2$. Thus it follows that in the limit as $m \to \infty$,

$$\begin{aligned}M_x f_m &= \frac{1}{\sqrt{2}}(\bar{a} + a^*)\,f_m \to \frac{1}{\sqrt{2}}(\bar{a} + a^*)\,f \text{ and} \\ \partial_x f_m &= \frac{1}{\sqrt{2}}(\bar{a} - a^*)\,f_m \to \frac{1}{\sqrt{2}}(\bar{a} - a^*)\,f.\end{aligned}$$

As $M_x$ and $\partial_x$ are closed operators, it follows that $f \in D(M_x) \cap D(\partial_x)$. ■

# 4 Operator Inequalities

**Notation 3.27** ($\beta_1, \beta_2$ – Operator Norms)**.** *Let $\beta_1, \beta_2 \geq 0$. If $T : D\left(\mathcal{N}^{\beta_1}\right) \to D\left(\mathcal{N}^{\beta_2}\right)$ is a linear map, let*

$$\|T\|_{\beta_1 \to \beta_2} := \sup_{0 \neq \psi \in D\left(\mathcal{N}^{\beta_1}\right)} \frac{\|T\psi\|_{\beta_2}}{\|\psi\|_{\beta_1}}. \tag{3.39}$$

*denote the corresponding operator norm. We say that $T$ is $\beta_1 \to \beta_2$ bounded if $\|T\|_{\beta_1 \to \beta_2} < \infty$. In the special case when $\beta_1 = \beta_2 = \beta$, let $\left(B\left(D\left(\mathcal{N}^{\beta}\right)\right), \|\cdot\|_{\beta \to \beta}\right)$ denote the Banach space of all $\beta \to \beta$ bounded linear operators, $T : D\left(\mathcal{N}^{\beta}\right) \to D\left(\mathcal{N}^{\beta}\right)$.*

**Remark 3.28.** *Let $\beta_1, \beta_2, \beta_3 \geq 0$. As usual, if $T : D\left(\mathcal{N}^{\beta_1}\right) \to D\left(\mathcal{N}^{\beta_2}\right)$ and $S : D\left(\mathcal{N}^{\beta_2}\right) \to D\left(\mathcal{N}^{\beta_3}\right)$ are any linear operators, then*

$$\|ST\|_{\beta_1 \to \beta_3} \leq \|S\|_{\beta_2 \to \beta_3} \|T\|_{\beta_1 \to \beta_2}. \tag{3.40}$$

**Proposition 3.29.** *Let $k = \deg_\theta u_{\mathbf{b}}(\theta, \theta^*)$ and $\mathcal{A} = u_{\mathbf{b}}\left(a, a^\dagger\right)$ be as in Notations 3.11 and 3.1. If $\beta \geq 0$, then $\bar{\mathcal{A}} D\left(\mathcal{N}^{\beta + k/2}\right) \subset D\left(\mathcal{N}^{\beta}\right)$ and*

$$\left\|\bar{\mathcal{A}}\right\|^2_{\beta + \frac{k}{2} \to \beta} \leq k^k (k+1)^{2\beta} \leq (k+1)^{2\beta + k}. \tag{3.41}$$

*Moreover,*

$$\left\|\bar{\mathcal{A}} f\right\|_\beta \leq \left\|(\mathcal{N} + k)^{k/2} (\mathcal{N} + k + 1)^\beta f\right\| \ \forall \ f \in D\left(\mathcal{N}^{\beta + k/2}\right). \tag{3.42}$$

**Proof.** Let $f \in D\left(\mathcal{N}^{\beta + k/2}\right) \subset D\left(\mathcal{N}^{k/2}\right)$ and recall from Lemma 3.12 that $c^\dagger_{\mathcal{A}}(n) = c_{\mathcal{A}}(n - l)$ and $|c_{\mathcal{A}}(n)|^2 \leq (n+k)^k$. Using these facts and the fact that

$\bar{\mathcal{A}} = \mathcal{A}^{\dagger *}$ (see Theorem 3.15), we find,

$$\begin{aligned}
\left\|\bar{\mathcal{A}} f\right\|_{\beta}^{2} &= \sum_{n} \left|\left\langle \bar{\mathcal{A}} f, \Omega_{n}\right\rangle\right|^{2} (1+n)^{2\beta} = \sum_{n} \left|\left\langle f, \mathcal{A}^{\dagger}\Omega_{n}\right\rangle\right|^{2} (1+n)^{2\beta} 1_{n\geq 0} \\
&= \sum_{n} \left|\left\langle f, \Omega_{n-l}\right\rangle\right|^{2} \left|c_{\mathcal{A}}(n-l)\right| (1+n)^{2\beta} 1_{n\geq 0} \\
&= \sum_{n} \left|\left\langle f, \Omega_{n}\right\rangle\right|^{2} (1+n+l)^{2\beta} 1_{n+l\geq 0} \left|c_{\mathcal{A}}(n)\right|^{2} \\
&\leq \sum_{n} \left|\left\langle f, \Omega_{n}\right\rangle\right|^{2} (n+k+1)^{2\beta} (n+k)^{k} \qquad (3.43) \\
&= \left\| (\mathcal{N}+k)^{k/2} (\mathcal{N}+k+1)^{\beta} f \right\|_{0}^{2}
\end{aligned}$$

which proves Eq. (3.42). Using

$$n + a \leq a(n+1) \ \text{ for } a \geq 1 \text{ and } n \in \mathbb{N}_0 \qquad (3.44)$$

in Eq. (3.43) with $a = k$ and $a = k+1$ shows,

$$\left\|\bar{\mathcal{A}} f\right\|_{\beta}^{2} \leq k^{k} (k+1)^{2\beta} \sum_{n} \left|\left\langle f, \Omega_{n}\right\rangle\right|^{2} (1+n)^{2\beta+k} = k^{k} (k+1)^{2\beta} \|f\|_{\beta+k/2}^{2}.$$

The previous inequality proves Eq. (3.41) and also $\bar{\mathcal{A}} D\left(\mathcal{N}^{\beta+k/2}\right) \subset D\left(\mathcal{N}^{\beta}\right)$. ∎

**Corollary 3.30.** *If $P(\theta, \theta^{*}) \in \mathbb{C}\langle \theta, \theta^{*}\rangle$ and $d = \deg_{\theta} P$, then $D\left(\mathcal{N}^{d/2}\right) = D\left(P(\bar{a}, a^{*})\right)$, $P(\bar{a}, a^{*}) \subseteq \overline{P(a, a^{\dagger})}$, and*

$$\|P(\bar{a}, a^{*})\|_{\beta+d/2\to\beta} \leq \sum_{k=0}^{d} k^{k/2} (k+1)^{\beta} |P_k| \ \text{ for all } \beta \geq 0. \qquad (3.45)$$

**Proof.** The operator $P(\bar{a}, a^{*})$ is a linear combination of operators of the form $u_{\mathbf{b}}(\bar{a}, a^{*})$ where $k = \deg_{\theta} u_{\mathbf{b}}(\theta, \theta^{*}) \leq d$. By Theorem 3.15, it follows that $D\left(u_{\mathbf{b}}(\bar{a}, a^{*})\right) = D\left(\mathcal{N}^{k/2}\right) \supseteq D\left(\mathcal{N}^{d/2}\right)$ and hence $D\left(\mathcal{N}^{d/2}\right) = D\left(P(\bar{a}, a^{*})\right)$.

Furthermore, Proposition 3.29 shows

$$\|u_{\mathbf{b}}(\bar{a},a^*)\|_{\beta+d/2\to\beta} \leq \|u_{\mathbf{b}}(\bar{a},a^*)\|_{\beta+k/2\to\beta} \leq k^{k/2}(k+1)^{\beta}.$$

This estimate, the triangle inequality, and the definition of $|P_k|$ in Eq. (2.23) leads directly to the inequality in Eq. (3.45).

If $f \in D\left(\mathcal{N}^{d/2}\right)$, it follows from Eq. (3.45) and Proposition 3.23 that

$$P(\bar{a},a^*)f = \lim_{N\to\infty} P(\bar{a},a^*)\mathcal{P}_N f = \lim_{N\to\infty} P\left(a,a^\dagger\right)\mathcal{P}_N f$$

which shows $f \in D\left(\overline{P(a,a^\dagger)}\right)$ and $\overline{P(a,a^\dagger)}f = P(\bar{a},a^*)f$. ■

**Notation 3.31.** *For $x\in\mathbb{R}$ let $(x)_+ := \max(x,0)$.*

**Lemma 3.32.** *If $\mathcal{A} = u_{\mathbf{b}}\left(a,a^\dagger\right)$, $k = \deg_\theta u_{\mathbf{b}}(\theta,\theta^*)$, $l = \ell(\mathbf{b}) \in \mathbb{Z}$ are as in Notations 3.11 and 3.1, then for all $\beta \geq 0$ we have,*

$$(\mathcal{N}+1)^{\beta}\bar{\mathcal{A}}f = \bar{\mathcal{A}}\left((\mathcal{N}+l)_+ + 1\right)^{\beta} f \text{ for all } f\in D\left(\mathcal{N}^{\beta+\frac{k}{2}}\right). \tag{3.46}$$

**Proof.** Using Proposition 3.29 and Remark 3.28 it is readily verified that the operators on both sides of Eq. (3.46) are bounded linear operators from $D\left(\mathcal{N}^{\beta+\frac{k}{2}}\right)$ to $L^2(m)$. Since $\mathcal{S}_0$ is dense in $D\left(\mathcal{N}^{\beta+\frac{k}{2}}\right)$ (see Proposition 3.23) it suffices to verify Eq. (3.46) for $f=\Omega_n$ for all $n\in\mathbb{N}_0$ which is trivial. Indeed, $\bar{\mathcal{A}}\Omega_n = c_{\mathcal{A}}(n)\Omega_{n+l}$ which is zero if $n+l<0$ and hence

$$\begin{aligned}(\mathcal{N}+1)^{\beta}\bar{\mathcal{A}}\Omega_n &= \left((n+l)_+ + 1\right)^{\beta}\bar{\mathcal{A}}\Omega_n = \bar{\mathcal{A}}\left((n+l)_+ + 1\right)^{\beta}\Omega_n \\ &= \bar{\mathcal{A}}\left((\mathcal{N}+l)_+ + 1\right)^{\beta}\Omega_n.\end{aligned}$$

■

**Proposition 3.33.** *Let $k\in\mathbb{N}$, $\mathbf{b}\in\{\theta,\theta^*\}^k$, $\mathcal{A}$, and $\ell(\mathbf{b})$ be as in Notation 3.1.*

*For any* $\beta \geq 0$, *it gets*

$$\begin{aligned}
&\left\| \left[ (\mathcal{N}+1)^{\beta}, \bar{\mathcal{A}} \right] (\mathcal{N}+1)^{-\beta} \varphi \right\| \\
&\qquad \leq \beta k^{k/2} \left| \ell(\mathbf{b}) \right| \left(1+\left| \ell(\mathbf{b}) \right|\right)^{|\beta-1|} \left\| (\mathcal{N}+1)^{k/2-1} \mathbf{1}_{\mathcal{N}\geq -l} \varphi \right\| && (3.47) \\
&\qquad \leq \beta k^{k/2} \left| \ell(\mathbf{b}) \right| \left(1+\left| \ell(\mathbf{b}) \right|\right)^{|\beta-1|} \left\| (\mathcal{N}+1)^{k/2-1} \varphi \right\| && (3.48)
\end{aligned}$$

*for all* $\varphi \in D\left(\mathcal{N}^{k/2}\right)$.

**Proof.** Let $l := \ell(\mathbf{b})$. By Lemma 3.32 and the identity, $\bar{\mathcal{A}} = \bar{\mathcal{A}} \mathbf{1}_{\mathcal{N}+l\geq 0}$, for all $\psi \in D\left(\mathcal{N}^{k/2+\beta}\right)$ we have,

$$\begin{aligned}
\left[ (\mathcal{N}+1)^{\beta}, \bar{\mathcal{A}} \right] \psi &= \left[ (\mathcal{N}+1)^{\beta} \bar{\mathcal{A}} - \bar{\mathcal{A}} (\mathcal{N}+1)^{\beta} \right] \psi \\
&= \bar{\mathcal{A}} \left[ \left( (\mathcal{N}+l)_{+} + 1 \right)^{\beta} - (\mathcal{N}+1)^{\beta} \right] \psi \\
&= \bar{\mathcal{A}} \mathbf{1}_{\mathcal{N}+l\geq 0} \left[ \left( (\mathcal{N}+l)_{+} + 1 \right)^{\beta} - (\mathcal{N}+1)^{\beta} \right] \psi \\
&= \bar{\mathcal{A}} \left[ (\mathcal{N}+l+1)^{\beta} - (\mathcal{N}+1)^{\beta} \right] \mathbf{1}_{\mathcal{N}+l\geq 0} \psi \\
&= \bar{\mathcal{A}} \left[ \beta \int_0^l (\mathcal{N}+1+r)^{\beta-1} dr \right] \mathbf{1}_{\mathcal{N}+l\geq 0} \psi.
\end{aligned}$$

Combining this equation with Eq. (3.42) of Proposition 3.29 shows,

$$\begin{aligned}
\left\| \left[ (\mathcal{N}+1)^{\beta}, \bar{\mathcal{A}} \right] \psi \right\| &\leq \left\| (\mathcal{N}+k)^{k/2} \left[ \beta \int_0^l (\mathcal{N}+1+r)^{\beta-1} dr \right] \mathbf{1}_{\mathcal{N}\geq -l} \psi \right\| \\
&\leq \beta \left| \int_0^l \left\| (\mathcal{N}+k)^{k/2} (\mathcal{N}+1+r)^{\beta-1} \mathbf{1}_{\mathcal{N}\geq -l} \psi \right\| dr \right|. \\
&\leq \beta k^{k/2} \left| \int_0^l \left\| (\mathcal{N}+1)^{k/2} (\mathcal{N}+1+r)^{\beta-1} \mathbf{1}_{\mathcal{N}\geq -l} \psi \right\| dr \right|.
\end{aligned}$$

For $x \geq \max(0, -l)$ and $r$ between 0 and $l$, one shows

$$(x+1+r)^{\beta-1} \leq (1+|l|)^{|\beta-1|} (x+1)^{\beta-1}$$

which combined with the previously displayed equation implies,

$$\left\| \left[ (\mathcal{N}+1)^{\beta}, \bar{\mathcal{A}} \right] \psi \right\| \leq \beta k^{k/2} (1+|l|)^{|\beta-1|} |l| \left\| (\mathcal{N}+1)^{\frac{k}{2}+\beta-1} \mathbf{1}_{\mathcal{N} \geq -l} \psi \right\|. \qquad (3.49)$$

Finally, Eq. (3.47) easily follows by replacing $\psi$ by $(\mathcal{N}+1)^{-\beta} \varphi \in D\left(\mathcal{N}^{k/2}\right)$ in Eq. (3.49).

■

# 5 Truncated Estimates

**Notation 3.34** (Operator Truncation)**.** *If $Q = P\left(a, a^{\dagger}\right)$ is a polynomial operator on $L^2(m)$ and $M > 0$, let*

$$Q_M := \mathbf{1}_{\mathcal{N} \leq M} Q \mathbf{1}_{\mathcal{N} \leq M} = \mathcal{P}_M Q \mathcal{P}_M. \qquad (3.50)$$

*and refer to $Q_M$ as the **level-$M$ truncation of** $Q$. [Recall that $\mathcal{P}_M = \mathbf{1}_{\mathcal{N} \leq M}$ are as in Notations 3.10 and 3.20.]*

**Proposition 3.35.** *If $k \in \mathbb{N}$, $\beta \geq 0$, $0 < M < \infty$, $\mathbf{b} \in \{\theta, \theta^*\}^k$, $\mathcal{A} = u_{\mathbf{b}}\left(a, a^{\dagger}\right)$, and $\ell(\mathbf{b})$ are as in Notation 3.1, then*

$$\|\mathcal{A}_M\|_{\beta \to \beta} \leq (M+k)^{k/2} (1+|\ell(\mathbf{b})|)^{\beta} \leq k^{k/2} (1+|\ell(\mathbf{b})|)^{\beta} (M+1)^{k/2}. \qquad (3.51)$$

*Consequently if $P \in \mathbb{C}\langle \theta, \theta^* \rangle$ with $d = \deg_{\theta} P$, then*

$$\left\| \left[ P\left(a, a^{\dagger}\right) \right]_M \right\|_{\beta \to \beta} \leq \sum_{k=0}^{d} (M+k)^{k/2} (1+k)^{\beta} |P_k| \qquad (3.52)$$

*which in particular implies that the map,*

$$P \in \mathbb{C}\langle \theta, \theta^* \rangle \to \left[ P\left(a, a^{\dagger}\right) \right]_M \in \left( B\left(D\left(\mathcal{N}^{\beta}\right)\right), \|\cdot\|_{\beta \to \beta} \right),$$

*depends continuously on the coefficients of $P$.*

**Proof.** With $l = \ell(\mathbf{b})$, we have for all $n \in \mathbb{N}_0$,

$$\begin{aligned}
\mathcal{A}_M^* \Omega_n &= (\mathcal{P}_M \mathcal{A} \mathcal{P}_M)^* \Omega_n = \mathcal{P}_M \mathcal{A}^* \mathcal{P}_M \Omega_n \\
&= 1_{n \le M} \mathcal{P}_M \mathcal{A}^\dagger \Omega_n = 1_{n \le M} c_{\mathcal{A}}(n-l) \mathcal{P}_M \Omega_{n-l} \\
&= 1_{n \le M} 1_{n-l \le M} c_{\mathcal{A}}(n-l) \Omega_{n-l}. \qquad (3.53)
\end{aligned}$$

From this identity and simple estimates using Eq. (3.44) repeatedly we find, for $f \in D\left(\mathcal{N}^\beta\right)$,

$$\begin{aligned}
\|\mathcal{A}_M f\|_\beta^2 &= \sum_n |\langle \mathcal{A}_M f, \Omega_n \rangle|^2 (1+n)^{2\beta} \\
&= \sum_n 1_{0 \le n \le M} 1_{0 \le n-l \le M} |\langle f, \Omega_{n-l} \rangle|^2 |c_{\mathcal{A}}(n-l)|^2 (1+n)^{2\beta} \\
&= \sum_n 1_{0 \le n+l \le M} 1_{0 \le n \le M} |\langle f, \Omega_n \rangle|^2 |c_{\mathcal{A}}(n)|^2 (1+n+l)^{2\beta} \\
&\le \sum_n 1_{0 \le n+l \le M} 1_{0 \le n \le M} |\langle f, \Omega_n \rangle|^2 (k+n)^k (1+n+|l|)^{2\beta} \\
&\le (M+k)^k (1+|l|)^{2\beta} \sum_n 1_{0 \le n+l \le M} 1_{0 \le n \le M} |\langle f, \Omega_n \rangle|^2 (1+n)^{2\beta} \\
&\le (M+k)^k (1+|\ell(\mathbf{b})|)^{2\beta} \|f\|_\beta^2 \le k^k (M+1)^k (1+|\ell(\mathbf{b})|)^{2\beta} \|f\|_\beta^2 .
\end{aligned}$$

■

**Theorem 3.36.** *Let $k \in \mathbb{N}$, $\beta \ge 0$, $\mathbf{b} \in \{\theta, \theta^*\}^k$, and $\mathcal{A} = u_{\mathbf{b}}\left(a, a^\dagger\right)$ be as in Notation 3.1. If $\alpha \ge \beta + k/2$, then*

$$\left\|\bar{\mathcal{A}} - \mathcal{A}_M\right\|_{\alpha \to \beta} \le (M-k+2)^{(\beta+k/2-\alpha)} \text{ for all } M \ge k. \qquad (3.54)$$

*Consequently, if $\alpha > \beta + k/2$, then*

$$\lim_{M\to\infty} \left\| \left(\bar{\mathcal{A}} - \mathcal{A}_M\right) \varphi \right\|_\beta^2 = 0 \ \forall \ \varphi \in D\left(\mathcal{N}^\alpha\right). \tag{3.55}$$

**Proof.** Let $M \geq k$. From Proposition 3.29, $\bar{\mathcal{A}} - \mathcal{A}_M$ is a bounded operator from $\left(D\left(\mathcal{N}^\alpha\right), \|\cdot\|_\alpha\right)$ to $\left(D\left(\mathcal{N}^\beta\right), \|\cdot\|_\beta\right)$. Making use of Eq. (3.53) we find

$$\begin{aligned}\left(\mathcal{A}^\dagger - \mathcal{P}_M \mathcal{A}^\dagger \mathcal{P}_M\right) \Omega_n &= c_{\mathcal{A}}\left(n-l\right)\left[1 - 1_{n\leq M} \cdot 1_{n-l\leq M}\right] \Omega_{n-l} \\ &= c_{\mathcal{A}}\left(n-l\right) 1_{n>M\wedge(M+l)} \Omega_{n-l} \text{ for all } n \in \mathbb{Z}.\end{aligned}$$

Hence, if $\varphi \in D\left(\mathcal{N}^\alpha\right) \subset D\left(\mathcal{N}^{k/2}\right) = D\left(\bar{\mathcal{A}}\right)$, then

$$\begin{aligned}\left\| \left(\bar{\mathcal{A}} - \mathcal{A}_M\right) \varphi \right\|_\beta^2 &= \sum_n \left| \left\langle \left(\bar{\mathcal{A}} - \mathcal{A}_M\right) \varphi, \Omega_n \right\rangle \right|^2 (n+1)^{2\beta} \\ &= \sum_n \left| \left\langle \varphi, \left(\mathcal{A}^\dagger - \mathcal{P}_M \mathcal{A}^\dagger \mathcal{P}_M\right) \Omega_n \right\rangle \right|^2 (n+1)^{2\beta} \\ &= \sum_n 1_{n>M\wedge(M+l)} (n+1)^{2\beta} \left|\left\langle \varphi, \Omega_{n-l}\right\rangle\right|^2 \left|c_{\mathcal{A}}(n-l)\right|^2 \\ &= \sum_n 1_{n+l>M\wedge(M+l)} (n+l+1)^{2\beta} \left|\left\langle \varphi, \Omega_n\right\rangle\right|^2 \left|c_{\mathcal{A}}(n)\right|^2 \\ &= \sum_n \rho(n) (n+1)^{2\alpha} \left|\left\langle \varphi, \Omega_n\right\rangle\right|^2 \leq \max_n \rho(n) \|\varphi\|_\alpha^2\end{aligned}$$

where

$$\rho(n) := 1_{n+l>M\wedge(M+l)} \frac{(n+l+1)^{2\beta}}{(n+1)^{2\alpha}} \left|c_{\mathcal{A}}(n)\right|^2.$$

This completes the proof since simple estimates using Lemma 3.12 and the fact that $n \geq M - k + 1$ shows,

$$\rho(n) \leq k^k (k+1)^{2\beta} (M-k+2)^{2(\beta+k/2-\alpha)}.$$

$\blacksquare$

**Corollary 3.37.** *If $P(\theta,\theta^*) \in \mathbb{C}\langle \theta,\theta^*\rangle$, $d = \deg_\theta P$, $\beta \geq 0$, and $\alpha \geq \beta + d/2$, then for any $M \geq d$,*

$$\left\| \left[ P\left(a, a^\dagger\right) \right]_M - P\left(\bar{a}, a^*\right) \right\|_{\alpha\to\beta} \leq \sum_{k=0}^{d} |P_k| \left(M-k+2\right)^{(\beta+k/2-\alpha)}$$
$$\leq \left(M-d+2\right)^{(\beta+d/2-\alpha)} |P| . \tag{3.56}$$

**Proof.** This result a simple consequence of Theorem 3.36, the triangle inequality, and the elementary estimate,

$$\left(M-k+2\right)^{(\beta+k/2-\alpha)} \leq \left(M-d+2\right)^{(\beta+d/2-\alpha)} \text{ for } 0 \leq k \leq d.$$

■

**Proposition 3.38.** *If $P(\theta,\theta^*) \in \mathbb{C}\langle \theta,\theta^*\rangle$ is as in Eq. (2.20) and $|P_k|$ is as in Eq. (2.23), then for all $\beta \geq 0$,*

$$\left\| \left[ \left(\mathcal{N}+1\right)^{\beta}, P\left(a, a^\dagger\right)_M \right] \left(\mathcal{N}+1\right)^{-\beta} \right\|_{0\to 0}$$
$$\leq \sum_{k=1}^{d} \beta k^{k/2} k \left(1+k\right)^{|\beta-1|} \left(M+1\right)^{(k/2-1)_+} |P_k| \tag{3.57}$$
$$\leq K(\beta, d) \cdot \sum_{k=1}^{d} \left(M+1\right)^{(k/2-1)_+} |P_k| \tag{3.58}$$

*where*

$$K(\beta, d) := \beta d^{1+\frac{d}{2}} \left(1+d\right)^{|\beta-1|} . \tag{3.59}$$

**Proof.** If $f \in L^2(m)$, $\mathbf{b} \in \{\theta,\theta^*\}^k$ and $\mathcal{A}_{\mathbf{b}} := u_{\mathbf{b}}\left(a, a^\dagger\right)$, then by Proposition 3.33,

$$\begin{aligned}\left\| \left[ (\mathcal{N}+1)^{\beta}, [\mathcal{A}_{\mathbf{b}}]_M \right] (\mathcal{N}+1)^{-\beta} f \right\| &= \left\| \left[ (\mathcal{N}+1)^{\beta}, \mathcal{P}_M \mathcal{A}_{\mathbf{b}} \mathcal{P}_M \right] (\mathcal{N}+1)^{-\beta} f \right\| \\ &= \left\| \mathcal{P}_M \left[ (\mathcal{N}+1)^{\beta}, \mathcal{A}_{\mathbf{b}} \right] (\mathcal{N}+1)^{-\beta} \mathcal{P}_M f \right\| \\ &\leq \beta k^{k/2} k (1+k)^{|\beta-1|} \left\| (\mathcal{N}+1)^{k/2-1} \mathcal{P}_M f \right\| \\ &\leq \beta k^{k/2} k (1+k)^{|\beta-1|} (M+1)^{(k/2-1)_+} \|f\| .\end{aligned}$$

Hence $P \in \mathbb{C}\langle \theta, \theta^* \rangle$ with $d = \deg_\theta P$ is given as in Eq. (2.20) (so that $P\left(a, a^\dagger\right)$ is as in Eq. (2.28) with $\hbar = 1$), then by the triangle inequality we find,

$$\begin{aligned}&\left\| \left[ (\mathcal{N}+1)^{\beta}, P\left(a, a^\dagger\right)_M \right] (\mathcal{N}+1)^{-\beta} \right\|_{0\to 0} \\ &\quad \leq \sum_{k=1}^{d} \left\| \left[ (\mathcal{N}+1)^{\beta}, P_k\left(a, a^\dagger\right)_M \right] (\mathcal{N}+1)^{-\beta} \right\|_{0\to 0} \\ &\quad \leq \sum_{k=1}^{d} \beta k^{k/2} k (1+k)^{|\beta-1|} (M+1)^{(k/2-1)_+} |P_k|\end{aligned}$$

where the absence of the $k = 0$ term is a consequence $P_0\left(a, a^\dagger\right)_M$ is proportional to $\mathcal{P}_M$ and hence commutes with $(\mathcal{N}+1)^{\beta}$.

■

# Chapter 4

# Basic Linear ODE Results

**Notation 4.1.** *If $(X, \|\cdot\|)$ is a Banach space, then $B(X)$ is notated as a collection of bounded linear operators from $X$ to itself and $\|\cdot\|_{B(X)}$ is denoted as an operator norm. (e.g. $\left(B\left(D\left(\mathcal{N}^{\beta}\right)\right), \|\cdot\|_{\beta\to\beta}\right)$ in Notation 3.27. )*

**Lemma 4.2** (Basic Linear ODE Theorem)**.** *Suppose that $(X, \|\cdot\|)$ is a Banach space and $t \to C(t) \in B(X)$ is an operator norm continuous map. Then to each $s \in \mathbb{R}$ there exists a unique solution, $U(t,s) \in B(X)$, to the ordinary differential equation,*

$$\frac{d}{dt}U(t,s) = C(t)\,U(t,s) \;\; \text{with } U(s,s) = I. \tag{4.1}$$

*Moreover, the function $(t,s) \to U(t,s) \in B(X)$ is operator norm continuously differentiable in each of its variables and $(t,s) \to \partial_t U(t,s)$ and $(t,s) \to \partial_s U(t,s)$ are operator norm continuous functions into $B(X)$,*

$$\begin{aligned} \partial_s U(t,s) &= -U(t,s)\,C(s) \;\; \text{with } U(t,t) = I, \;\; \text{and} \\ U(t,s)\,U(s,\sigma) &= U(t,\sigma) \;\; \text{for all } s,\sigma,t \in \mathbb{R}. \end{aligned}$$

**Proof.** Let $V(t)$ and $W(t)$ in $B(X)$ solve the ordinary differential equations,

$$\frac{d}{dt}V(t) = C(t)V(t) \text{ with } V(0) = I \text{ and}$$
$$\frac{d}{dt}W(t) = -W(t)C(t) \text{ with } W(0) = I.$$

We then have

$$\frac{d}{dt}[W(t)V(t)] = -W(t)C(t)V(t) + W(t)C(t)V(t) = 0$$

so that $W(t)V(t) = I$ for all $t$. Moreover, $Z(t) := V(t)W(t)$ solves the differential equation,

$$\begin{aligned}\frac{d}{dt}Z(t) &= -V(t)W(t)C(t) + C(t)V(t)W(t)\\ &= [C(t), Z(t)] \text{ with } Z(0) = V(0)W(0) = I.\end{aligned}$$

The unique solution to this differential equation is $Z(t) = I$ from which we conclude $V(t)W(t) = I$ for all $t \in \mathbb{R}$. In summary, we have shown $W(t)$ and $V(t)$ are inverses of one another. It is now easy to check that

$$U(t,s) = V(t)V(s)^{-1} = V(t)W(s)$$

from which all of the rest of the stated results easily follow. ■

**Proposition 4.3** (Operator Norm Bounds)**.** *Suppose that $(K, \langle\cdot,\cdot\rangle)$ is a Hilbert space, $A$, is a self-adjoint operators on $K$ with $A \geq I$, and make $D(A)$ into a Hilbert space using the inner product, $\langle\cdot,\cdot\rangle_A$, defined by*

$$\langle\psi,\varphi\rangle_A := \langle A\psi, A\varphi\rangle \text{ for all } \varphi,\psi \in D(A).$$

*Further suppose that $t \to C(t) \in B(K)$[see Notation 4.1] is a $\|\cdot\|_K$-operator* *norm continuous map such that* $C(t)\,D(A) \subset D(A)$ *for all* $t$ *and the map* $t \to C(t)\,|_{D(A)} \in B(D(A))$ *is* $\|\cdot\|_A$*-operator norm continuous. Let* $U(t,s) \in B(K)$ *be as in Lemma 4.2. Then,*

1. $U(t,s)\,D(A) \subset D(A)$ *for all* $s, t \in \mathbb{R}$, *and*

$$U(t,s)\,U(s,\sigma) = U(t,\sigma).$$

2. $U(t,s)\,|_{D(A)}$ *solves*

$$\frac{d}{dt}U(t,s)\,|_{D(A)} = C(t)\,|_{D(A)}U(t,s)\,|_{D(A)} \text{ with } U(s,s)\,|_{D(A)} = I_{D(A)}$$

*where the derivative on the left side of this equation is taken relative to the operator norm on the Hilbert space,* $(D(A), \langle\cdot,\cdot\rangle_A)$.

3. *For all* $s, t \in \mathbb{R}$,

$$\|U(t,s)\|_{B(K)} \le \exp\left(\frac{1}{2}\left|\int_s^t \|C(\tau) + C^*(\tau)\|_{B(K)}\, d\tau\right|\right) \tag{4.2}$$

*where* $\|\cdot\|_{B(K)}$ *is as in Notation 4.1. Moreover,* $U(t,s)$ *is unitary on* $K$ *if* $C(t)$ *is skew adjoint for all* $t \in \mathbb{R}$.

4. *For all* $s, t \in \mathbb{R}$,

$$\begin{aligned}&\|U(t,s)\|_{B(D(A))}\\ &\quad\le \exp\left(\left|\int_s^t \left[\frac{1}{2}\|C(\tau) + C^*(\tau)\|_{B(K)} + \left\|[A, C(\tau)]\,A^{-1}\right\|_{B(K)}\right] d\tau\right|\right).\end{aligned} \tag{4.3}$$

*and*

$$\|U(t,s)\|_{B(D(A))} \geq \exp\left(-\left|\int_s^t \left[\frac{1}{2}\|C(\tau)+C^*(\tau)\|_{B(K)} + \left\|[A,C(\tau)]\,A^{-1}\right\|_{B(K)}\right] d\tau\right|\right) \tag{4.4}$$

*where* $\|[A,C(\tau)]\,A^{-1}\|_{B(K)}$ *is defined to be* $\infty$ *if* $[A,C(\tau)]\,A^{-1}$ *is an unbounded operator on* $K$.

**Proof.** Let $U(t,s)$ be as in Lemma 4.2 when $X = K$ and $U_A(t,s)$ be as in Lemma 4.2 when $X = D(A)$. Further suppose that $\psi_0 \in D(A)$ and let $\psi(t) := U(t,s)\,\psi_0$ and $\psi_A(t) := U_A(t,s)\,\psi_0$. We now prove each item in turn.

1. Since $\psi(t)$ and $\psi_A(t)$ both solve the differential equation (in the $K$ – norm) [Note: $\|\cdot\|_A \geq \|\cdot\|_K$]

$$\dot{\varphi}(t) = C(t)\,\varphi(t) \text{ with } \varphi(s) = \psi_0, \tag{4.5}$$

it follows by the uniqueness of solutions to ODE that

$$U(t,s)\,\psi_0 = \psi(t) = \psi_A(t) = U_A(t,s)\,\psi_0 \in D(A).$$

The results of items 1. and 2. now easily follow.

2. It is well known and easily verified that $U(t,s)$ is unitary on $K$ if $C(t)$ is skew adjoint. The estimate in Eq. (4.2) is a special case of the estimate in Eq. (4.3) when $A = I$ so it suffices to prove the latter estimate.

3. With $\psi(t) = U(t,s)\psi_0 = U_A(t,s)\psi_0 \in D(A)$ as above we have,

$$\begin{aligned}\frac{d}{dt}\|\psi\|_A^2 &= 2\operatorname{Re}\langle C\psi,\psi\rangle_A = 2\operatorname{Re}\langle AC\psi, A\psi\rangle \\ &= 2\operatorname{Re}\left[\langle CA\psi, A\psi\rangle + \langle [A,C]\psi, A\psi\rangle\right] \\ &= \langle (C+C^*)A\psi, A\psi\rangle + 2\operatorname{Re}\left\langle [A,C]A^{-1}A\psi, A\psi\right\rangle\end{aligned}$$

and therefore,

$$\left|\frac{d}{dt}\|\psi\|_A^2\right| \le \left(\|C+C^*\|_{B(K)} + 2\left\|[A,C]A^{-1}\right\|_{B(K)}\right)\|\psi\|_A^2.$$

This last inequality may be integrated to find,

$$\left(\frac{\|\psi(t)\|_A^2}{\|\psi_0\|_A^2}\right)^{\pm 1} \le \exp\left(\left|\int_s^t \left[\|C(\tau)+C^*(\tau)\|_{B(K)} + 2\left\|[A,C(\tau)]A^{-1}\right\|_{B(K)}\right]d\tau\right|\right)$$

from which Eqs. (4.3) and (4.4) easily follow.

■

# 1 Truncated Evolutions

Let $P(t:\theta,\theta^*) \in \mathbb{C}\langle\theta,\theta^*\rangle$ with $\deg_\theta P(t:\theta,\theta^*) = d \in \mathbb{N}$ be a one parameter family of **symmetric** non-commutative polynomials whose coefficients depend continuously on $t$. In more detail we may write $P(t:\theta,\theta^*)$ as;

$$P(t:\theta,\theta^*) = \sum_{k=0}^{d} P_k(t:\theta,\theta^*) \text{ where} \tag{4.6}$$

$$P_k(t:\theta,\theta^*) = \sum_{\mathbf{b}\in\{\theta,\theta^*\}^k} c_k(t,\mathbf{b})\, u_{\mathbf{b}}(\theta,\theta^*) \tag{4.7}$$

and all coefficients, $t \to c_k(t, \mathbf{b})$ are continuous in $t$. Let $Q(t) := P\left(t : a, a^\dagger\right)$ and for any $M > 0$ let $Q_M(t) = \mathcal{P}_M Q(t) \mathcal{P}_M$ be the truncation of $Q(t)$ as in Notation 3.34. Applying Lemma 4.2 with $C(t) = -iQ_M(t)$ shows, for each $M \in \mathbb{N}$ there exists $U^M(t,s) \in B(L^2(m))$ such that for all $s \in \mathbb{R}$,

$$i\frac{d}{dt}U^M(t,s) = Q_M(t) U^M(t,s) \text{ with } U^M(s,s) = I. \tag{4.8}$$

**Theorem 4.4.** *Let $M > 0$ and $U^M(t,s)$ be defined as in Eq. (4.8). Then;*

1. $(t,s) \to U^M(t,s) \in B(L^2(m))$ *are jointly operator norm continuous in* $(t,s)$ *and* $U^M(t,s)$ *is unitary on* $L^2(m)$ *for each* $t, s \in \mathbb{R}$.

2. *If* $\sigma, s, t \in \mathbb{R}$, *then*

$$U^M(t,s) U^M(s,\sigma) = U^M(t,\sigma). \tag{4.9}$$

3. *If* $\beta \geq 0$ *and* $s, t \in \mathbb{R}$, *then* $U^M(t,s) D\left(\mathcal{N}^\beta\right) = D\left(\mathcal{N}^\beta\right)$, $U^M(t,s)|_{D\left(\mathcal{N}^\beta\right)}$ *is continuous in* $(t,s)$ *in the* $\|\cdot\|_\beta$*-operator norm topology,* $\partial_t U^M(t,s)|_{D\left(\mathcal{N}^\beta\right)}$, *and* $\partial_s U^M(t,s)|_{D\left(\mathcal{N}^\beta\right)}$ *exists in the* $\|\cdot\|_\beta$*-operator norm topology (see Notation 3.21) and again are continuous functions of* $(t,s)$ *in this topology and satisfy*

$$i\frac{d}{dt}U^M(t,s)\varphi = Q_M(t) U^M(t,s)\varphi \tag{4.10}$$

$$i\frac{d}{ds}U^M(t,s)\varphi = -U^M(t,s) Q_M(s)\varphi. \tag{4.11}$$

4. *If* $\beta \geq 0$ *and* $t, s \in \mathbb{R}$, *then with* $K(\beta, d) < \infty$ *as in Eq. (3.59) we have*

$$\left\|U^M(t,s)\right\|_{\beta\to\beta} \leq \exp\left(K(\beta,d)\sum_{k=1}^{d}(M+1)^{(k/2-1)_+}\int_{J_{st}}|P_k(\tau,\theta,\theta^*)|\,d\tau\right). \tag{4.12}$$

*where* $J_{st} = [\min(s,t), \max(s,t)]$, *and* $\|\cdot\|_{\beta\to\beta}$ *is as in Notation 3.27,* $P_k$ *as* *in Eq. (4.7) and $K(\beta, d)$ is as in Eq. (3.59).*

**Remark 4.5.** *Taking $t = \sigma$ in Eq. (4.9) and using the fact that $U^M(t,s)$ is unitary on $L^2(m)$, it follows that*

$$U^M(t,s)^{-1} = U^M(s,t) = U^M(t,s)^* . \tag{4.13}$$

**Remark 4.6.** *From the item 3 of the Theorem and Eq. (3.34), we can conclude that $U^M(t,s)\mathcal{S} = \mathcal{S}$.*

**Proof.** The continuity of $U^M$ in the item 1. and the identity in Eq. (4.9) both follow from Lemma 4.2. Since $Q_M(t)^* = Q_M(t)$ it follows that $C(t) := -iQ_M(t)$ is skew-adjoint and so the unitary property in the first item is a consequence of item 3. of Proposition 4.3. The remaining item 3. and 4. follow from Proposition 4.3 with $A := (\mathcal{N} + I)^\beta$ and $C(t) := -iQ_M(t)$. The hypothesis that $C(t)D(A) \subset D(A)$ and $t \to C(t) \in B(D(A))$ is $\|\cdot\|_\beta$-operator norm continuous in $t$ has been verified in Proposition 3.35. Moreover, from Eq. (3.58) of Proposition 3.38 we know

$$\left\| [A, C(\tau)] A^{-1} \right\|_{B(L^2(m))} \leq K(\beta, d) \sum_{k=1}^{d} (M+1)^{(k/2-1)_+} |P_k(\tau, \theta, \theta^*)| .$$

Equation (4.12) now follows directly from Eq. (4.3) and the fact that $C(t)$ is skew adjoint. Finally, the inclusion, $U^M(t,s) D\left(\mathcal{N}^\beta\right) \subseteq D\left(\mathcal{N}^\beta\right)$, follows by Proposition 4.3. The opposite inclusion is then deduced using $U^M(t,s)^{-1} = U^M(s,t)$ which follows from Eq. (4.9). ■

**Corollary 4.7.** *Recall $P(t : \theta, \theta^*)$ as in Eq. (4.6). Let $\hbar > 0$, $M > 0$, $U_\hbar^M(t,s)$ denotes the solution to the ordinary differential equation,*

$$i\hbar \frac{d}{dt} U_\hbar^M(t,s) = \left[ P\left(t : a_\hbar, a_\hbar^\dagger\right)\right]_M U_\hbar^M(t,s) \text{ with } U_\hbar^M(s,s) = I,$$

*If $\beta \geq 0$ and $s, t \in \mathbb{R}$, then*

$$\left\| U_{\hbar}^{M}(t,s) \right\|_{\beta \to \beta} \leq e^{K(\beta,d) \sum_{k=1}^{d} \hbar^{k/2-1} (M+1)^{(k/2-1)_{+}} \int_{J_{s,t}} |P_k(\tau:\theta,\theta^*)| d\tau}, \tag{4.14}$$

*where $K(\beta,d) < \infty$ is as in Eq. (3.59). In particular if $P_1(t:\theta,\theta^*) \equiv 0$, $\eta \in (0,1]$, and $0 < \hbar \leq \eta \leq 1$, then*

$$\left\| U_{\hbar}^{M}(t,s) \right\|_{\beta \to \beta} \leq e^{K(\beta,d)(\hbar M+1)^{\frac{d}{2}-1} \sum_{k=2}^{d} \int_{J_{s,t}} |P_k(\tau:\theta,\theta^*)| d\tau}. \tag{4.15}$$

**Proof.** Since

$$\frac{1}{\hbar} P_k\left(t : a_{\hbar}, a_{\hbar}^{\dagger}\right) = \frac{1}{\hbar} \hbar^{k/2} P_k\left(t : a, a^{\dagger}\right) = \hbar^{k/2-1} P_k\left(t : a, a^{\dagger}\right),$$

Eq. (4.14) follows from Theorem 4.4 after making the replacement,

$$P(t :, \theta, \theta^*) \longrightarrow \sum_{k=0}^{d} \hbar^{k/2-1} P_k(t : \theta, \theta^*).$$

Equation (4.15) then follows from Eq. (4.14) since for $2 \leq k \leq d$ and $0 < \hbar \leq \eta \leq 1$,

$$\hbar^{k/2-1} (M+1)^{(k/2-1)_{+}} = (\hbar M + \hbar)^{(k/2-1)} \leq (\hbar M + 1)^{\frac{d}{2}-1}.$$

■

# Chapter 5

# Quadratically Generated Unitary Groups

Let $P(t:\theta,\theta^*) \in \mathbb{C}\langle\theta,\theta^*\rangle$ be a continuously varying one parameter family of **symmetric** polynomials with $d = \deg_\theta P(t:\theta,\theta^*) \le 2$. Then $Q(t) := P\left(t:a,a^\dagger\right)$ may be decomposed as;

$$Q(t) = \sum_{j=0}^{6} c_j(t)\,\mathcal{A}^{(j)} \tag{5.1}$$

where $\mathcal{A}^{(j)}$ is a monomial in $a$ and $a^\dagger$ of degree no bigger than 2 and $c_j(\cdot)$ is continuous for each $0 \le j \le 6$ and $\mathcal{A}^{(0)} = 1$ by convention. The main goal of this chapter is to record the relevant information we need about solving the following time dependent Schrödinger equation;

$$i\dot{\psi}(t) = \overline{Q(t)}\psi(t) \text{ with } \psi(s) = \varphi, \tag{5.2}$$

where $s \in \mathbb{R}$ and $\varphi \in D(\mathcal{N})$ and the derivative is taken in $L^2(m)$.

**Theorem 5.1** (Uniqueness of Solutions)**.** *If* $\mathbb{R} \ni t \to \psi(t) \in D(\mathcal{N})$ *solves Eq. (5.2) then* $\|\psi(t)\| = \|\varphi\|$ *for all* $t \in \mathbb{R}$. *Moreover, there is at most one solution to Eq. (5.2).*

**Proof.** If $\psi(t)$ solves Eq. (5.2), then because $\overline{Q(t)}$ is symmetric on $D(\mathcal{N})$,

$$\frac{d}{dt}\|\psi(t)\|^2 = 2\operatorname{Re}\left\langle \dot{\psi}(t), \psi(t)\right\rangle = 2\operatorname{Re}\left\langle -i\overline{Q(t)}\psi(t), \psi(t)\right\rangle = 0.$$

Therefore it follows that $\|\psi(t)\|^2 = \|\psi(s)\|^2 = \|\varphi\|^2$ which proves the isometry property and because the equation (5.2) is linear this also proves uniqueness of solutions. ■

Theorem 5.5 below (among other things) guarantees the existence of solutions to Eq. (5.2). This result may be in fact be viewed as an aspect of the well known metaplectic representation. Nevertheless, we will provide a full proof as we need some detailed bounds on the solutions to Eq. (5.2).

In order to prove existence to Eq. (5.2) we are going to construct the evolution operator $U(t,s)$ associated to Eq. (5.2) as a limit of the truncated evolution operators, $U^M(t,s)$, defined by Eq. (4.8) with $Q_M(t) = \mathcal{P}_M Q(t)\mathcal{P}_M$ where $Q(t)$ is as in Eq. (5.1). The next estimate provides uniform bounds on $U^M(t,s)$.

**Corollary 5.2** (Uniform Bounds). *Continuing the notation above if $\beta \geq 0$, $-\infty < S < T < \infty$, and $M \in \mathbb{N}$, then*

$$\left\|U^M(t,s)\right\|_{\beta\to\beta} \leq \exp\left(K(\beta,S,T,P)\,|t-s|\right) \text{ for all } S < s,t \leq T \tag{5.3}$$

*where*

$$K(\beta,S,T,P) = \beta 4\cdot 3^{|\beta-1|}\sum_{j=1}^{6}\max_{\tau\in[S,T]}|c_j(\tau)| < \infty. \tag{5.4}$$

**Proof.** This result follows directly from Theorem 4.4 and the assumed continuity of the coefficients of $P(t:\theta,\theta^*)$ along with the assumption that $d = \deg_\theta P(t:\theta,\theta^*) \leq 2$. ■

The next proposition will be a key ingredient in the proof of Proposition 5.4 below which guarantees that $\lim_{M\to\infty} U^M(t,s)$ exists.

**Proposition 5.3.** *If $\beta \in \mathbb{R}$ and $\psi \in D\left(\mathcal{N}^{\beta+1}\right)$, then for all $-\infty < S < T < \infty$*

$$\lim_{M\to\infty} \sup_{K<\infty} \sup_{S\le s,\tau\le T} \left\| \left[\overline{Q(\tau)} - Q_M(\tau)\right] U^K(\tau, s)\, \psi \right\|_\beta = 0 \text{ and} \tag{5.5}$$

$$\lim_{M\to\infty} \sup_{K<\infty} \sup_{S\le s,\tau\le T} \left\| U^K(\tau, s) \left[\overline{Q(s)} - Q_M(s)\right] \psi \right\|_\beta = 0. \tag{5.6}$$

**Proof.** Let us express $Q(t)$ as in Eq. (5.1). Since

$$Q_M(t) = \sum_{j=0}^{6} c_j(t)\, \mathcal{A}_M^{(j)} \tag{5.7}$$

where $\mathcal{A}_M^{(j)}$ is the truncation of $\mathcal{A}^{(j)}$ as in Notation 3.34, to complete the proof it suffices to show,

$$\lim_{M\to\infty} \sup_{K<\infty} \sup_{S\le s,\tau\le T} \left\| \left[\bar{\mathcal{A}} - \mathcal{A}_M\right] U^K(\tau, s)\, \psi \right\|_\beta = 0 \text{ and} \tag{5.8}$$

$$\lim_{M\to\infty} \sup_{K<\infty} \sup_{S\le s,\tau\le T} \left\| U^K(\tau, s) \left[\bar{\mathcal{A}} - \mathcal{A}_M\right] \psi \right\|_\beta = 0 \tag{5.9}$$

where $\mathcal{A}$ is a monomial in $a$ and $a^\dagger$ with degree 2 or less.

According to Theorem 3.36 and Corollary 5.2, if $\psi \in D(\mathcal{N}^\alpha)$ with $\alpha \ge \beta+1$, then

$$\begin{aligned}
\left\| \left[\bar{\mathcal{A}} - \mathcal{A}_M\right] U^K(\tau, s)\, \psi \right\|_\beta &\le \left\| \left[\bar{\mathcal{A}} - \mathcal{A}_M\right] U^K(\tau, s) \right\|_{\alpha\to\beta} \|\psi\|_\alpha \\
&\le \left\| \left[\bar{\mathcal{A}} - \mathcal{A}_M\right] \right\|_{\alpha\to\beta} \left\| U^K(\tau, s) \right\|_{\alpha\to\alpha} \|\psi\|_\alpha \\
&\le C(\alpha, \beta, S, T, P)\,(M+1)^{\beta+1-\alpha} \|\psi\|_\alpha
\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\left\|U^K(\tau,s)\left[\bar{\mathcal{A}}-\mathcal{A}_M\right]\psi\right\|_\beta &\le \left\|U^K(\tau,s)\right\|_{\beta\to\beta}\left\|\left[\bar{\mathcal{A}}-\mathcal{A}_M\right]\psi\right\|_\beta\\ &\le \left\|U^K(\tau,s)\right\|_{\beta\to\beta}\left\|\left[\bar{\mathcal{A}}-\mathcal{A}_M\right]\right\|_{\alpha\to\beta}\|\psi\|_\alpha\\ &\le \tilde{C}(\alpha,\beta,S,T,P)(M+1)^{\beta+1-\alpha}\|\psi\|_\alpha \qquad (5.11)\end{aligned}$$

from which Eqs. (5.8) and (5.9) follow if $\psi\in D(\mathcal{N}^\alpha)$ with $\alpha>\beta+1$.

The general case, $\alpha=\beta+1$, follows by a standard "$3\varepsilon$" argument, the uniform (in $M>0$) estimates in Eq. (5.10) and (5.11) and the density of $\mathcal{S}_0\subset\mathcal{S}\subset D\left(\mathcal{N}^{\beta+1}\right)$ from Proposition 3.23.

■

**Proposition 5.4.** *If $\beta\ge 0$, $-\infty<S<T<\infty$ and $\psi\in D\left(\mathcal{N}^\beta\right)$, then it follows that*

$$\lim_{M,K\to\infty}\sup_{S\le s,t\le T}\left\|\left[U^K(t,s)-U^M(t,s)\right]\psi\right\|_\beta=0. \qquad (5.12)$$

**Proof.** By item 3 in Theorem 4.4, we have

$$i\frac{d}{dt}\left[U^M(s,t)\,U^K(t,s)\right]=U^M(s,t)\left[Q_K(t)-Q_M(t)\right]U^K(t,s) \qquad (5.13)$$

in the sense of $\|\cdot\|_\beta$-operator norm. Integrating the identity Eq. (5.13) gives

$$U^M(s,t)\,U^K(t,s)=I-i\int_s^t U^M(s,\tau)\left[Q_K(\tau)-Q_M(\tau)\right]U^K(\tau,s)\,d\tau. \qquad (5.14)$$

Using Eq. (4.9) in Theorem 4.4 and multiplying this identity by $U^M(t,s)$ then shows,

$$U^K(t,s)-U^M(t,s)=-i\int_s^t U^M(t,\tau)\left[Q_K(\tau)-Q_M(\tau)\right]U^K(\tau,s)\,d\tau.$$

Applying this equation to $\psi\in D\left(\mathcal{N}^{\beta+1}\right)$ and then making use of Corollary 5.2 and the triangle inequality for integrals shows,

$$\begin{aligned}
&\left\| \left[U^K(t,s) - U^M(t,s)\right]\psi\right\|_\beta \\
&\quad\le \left| \int_s^t \left\|U^M(t,\tau)\left[Q_K(\tau) - Q_M(\tau)\right]U^K(\tau,s)\psi\right\|_\beta d\tau\right| \\
&\quad\le \int_s^t \left\|U^M(t,\tau)\right\|_{\beta\to\beta}\left\|\left[Q_K(\tau) - Q_M(\tau)\right]U^K(\tau,s)\psi\right\|_\beta d\tau \\
&\quad\le K(\beta,S,T)\left| \int_s^t \left\|\left[Q_K(\tau) - Q_M(\tau)\right]U^K(\tau,s)\psi\right\|_\beta d\tau\right| \\
&\quad\le K(\beta,S,T)\left| \int_s^t \left\|\left[Q_K(\tau) - \bar{Q}(\tau)\right]U^K(\tau,s)\psi\right\|_\beta d\tau\right| \\
&\qquad\quad + K(\beta,S,T)\left| \int_s^t \left\|\left[\bar{Q}(\tau) - Q_M(\tau)\right]U^K(\tau,s)\psi\right\|_\beta d\tau\right|
\end{aligned}$$

and the latter expression tends to zero locally uniformly in $(t,s)$ as $K, M \to \infty$ by Proposition 5.3. This proves Eq. (5.12) for $\psi \in D\left(\mathcal{N}^{\beta+1}\right)$. Note that $\mathcal{S}$ is dense in $\left(D\left(\mathcal{N}^\beta\right), \|\cdot\|_\beta\right)$ from Proposition 3.23. The uniform estimate in Eq. (5.3) of Corollary 5.2 along with a standard density argument shows Eq. (5.12) holds for $\psi \in D\left(\mathcal{N}^\beta\right)$. ■

**Theorem 5.5.** *Let $Q(t) := P\left(t : a, a^\dagger\right)$ be as above, i.e. $P$ is a symmetric non-commutative polynomial of $\{\theta, \theta^*\}$ of $\deg_\theta P \le 2$ and having coefficients depending continuously on $t \in \mathbb{R}$. Then there exists a unique strongly continuous family of unitary operators $\{U(t,s)\}_{t,s\in\mathbb{R}}$ on $L^2(m)$ such that for all $\varphi \in D(\mathcal{N})$, $\psi(t) := U(t,s)\varphi$ solves Eq. (5.2). Furthermore $\{U(t,s)\}_{t,s\in\mathbb{R}}$ satisfies the following properties;*

1. *For all $s, t, \tau \in \mathbb{R}$ we have*

$$U(t,s) = U(t,\tau)U(\tau,s). \tag{5.15}$$

2. *For all $\beta \ge 0$ and $s, t \in \mathbb{R}$, $U(t,s)D\left(\mathcal{N}^\beta\right) = D\left(\mathcal{N}^\beta\right)$ and $(t,s) \to U(t,s)\varphi$*

*are jointly* $\|\cdot\|_{\beta}$*-norm continuous for all* $\varphi \in D\left(\mathcal{N}^{\beta}\right)$.

3. *If* $-\infty < S < T < \infty$, *then*

$$C\left(\beta, S, T\right) := \sup_{S \leq s, t \leq T} \left\|U\left(t, s\right)\right\|_{\beta \rightarrow \beta} < \infty. \tag{5.16}$$

4. *For* $\beta \geq 0$ *and* $\varphi \in D\left(\mathcal{N}^{\beta+1}\right)$, $t \rightarrow U\left(t, s\right)\varphi$ *and* $s \rightarrow U\left(t, s\right)\varphi$ *are strongly* $\|\cdot\|_{\beta}$ *–differentiable (see Definition 2.6) and satisfy*

$$i\frac{d}{dt}U\left(t, s\right)\varphi = \bar{Q}\left(t\right)U\left(t, s\right)\varphi \text{ with } U\left(s, s\right)\varphi = \varphi \tag{5.17}$$

*and*

$$i\frac{d}{ds}U\left(t, s\right)\varphi = -U\left(t, s\right)\bar{Q}\left(s\right)\varphi \text{ with } U\left(s, s\right)\varphi = \varphi \tag{5.18}$$

*where the derivatives are taken relative to the* $\beta$ *– norm,* $\|\cdot\|_{\beta}$.

**Proof. Item 1.** Let $\varphi \in D\left(\mathcal{N}^{\beta}\right)$. From Proposition 5.4 we know that $L_{\varphi}\left(t, s\right) := \lim_{M\rightarrow\infty} U^{M}\left(t, s\right)\varphi$ exists locally uniformly in $\left(t, s\right)$ in the $\beta$ – norm and therefore $\left(t, s\right) \rightarrow L_{\varphi}\left(t, s\right) \in D\left(\mathcal{N}^{\beta}\right)$ is $\beta$ – norm continuous jointly in $\left(t, s\right)$. In particular, this observation with $\beta = 0$ allows us to define

$$U\left(t, s\right) = s - \lim_{M\rightarrow\infty} U^{M}\left(t, s\right)$$

where the limit is taken in the strong $L^{2}\left(m\right)$ - operator topology. Since the operator product is continuous under strong convergence, by taking the strong limit of Eq. (4.9) shows the first equality in Eq. (5.15) holds. By taking $s = t$ in Eq. (5.15) we conclude that $U\left(t, s\right)$ is invertible and hence is unitary on $L^{2}\left(m\right)$ as it is already known to be an isometry because it is the strong limit of unitary operators. This proves the item 1. of the theorem.

**Items 2.** As we have just seen, for any $\varphi \in D\left(\mathcal{N}^{\beta}\right)$ we know that

$(t,s) \to U(t,s)\varphi = L_\varphi(t,s) \in D\left(\mathcal{N}^\beta\right)$ is $\|\cdot\|_\beta$ – continuous which proves item 2. Along the way we have shown $U(t,s)D\left(\mathcal{N}^\beta\right) \subset D\left(\mathcal{N}^\beta\right)$ and equality then follows using Eq. (5.15).

**Item 3** follows by the Eq. (5.3) in Corollary 5.2 where the bounds are independent of $M$.

So it only remains to prove item 4. of the theorem. We begin with proving the following claim.

**Claim.** If $\varphi \in D\left(\mathcal{N}^{\beta+1}\right)$, then

$$Q_M(\tau)U^M(\tau,s)\varphi \to \bar{Q}(\tau)U(\tau,s)\varphi \text{ as } M \to \infty \text{ and} \tag{5.19}$$

$$U^M(\tau,s)Q_M(s)\varphi \to U(\tau,s)\bar{Q}(s)\varphi \text{ as } M \to \infty \tag{5.20}$$

locally uniformly in $(\tau,s)$ in the $\|\cdot\|_\beta$ – topology.

**Proof of the claim.** Using $\sup_{\tau\in[S,T]}\left\|\bar{Q}(\tau)\right\|_{\beta+1\to\beta} < \infty$ (see Corollary 3.30) and the simple estimate,

$$\begin{aligned}
&\left\|Q_M(\tau)U^M(\tau,s)\varphi - \bar{Q}(\tau)U(\tau,s)\varphi\right\|_\beta \\
&\quad\le \left\|\left[Q_M(\tau)-\bar{Q}(\tau)\right]U^M(\tau,s)\varphi\right\|_\beta + \left\|\bar{Q}(\tau)\left[U^M(\tau,s)-U(\tau,s)\right]\varphi\right\|_\beta \\
&\quad\le \left\|\left[Q_M(\tau)-\bar{Q}(\tau)\right]U^M(\tau,s)\varphi\right\|_\beta + \left\|\bar{Q}(\tau)\right\|_{\beta+1\to\beta}\left\|\left[U^M(\tau,s)-U(\tau,s)\right]\varphi\right\|_{\beta+1},
\end{aligned}$$

the local uniform convergence in Eq. (5.19) is now a consequence of Propositions 5.3 and 5.4. The local uniform convergence in Eq. (5.20) holds by the same methods now based on the simple estimate,

$$\begin{aligned}
&\left\|U^M(\tau,s)Q_M(\tau)\varphi - U(\tau,s)\bar{Q}(\tau)\varphi\right\|_\beta \\
&\quad\le \left\|U^M(\tau,s)\left[Q_M(\tau)-\bar{Q}(\tau)\right]\varphi\right\|_\beta + \left\|\left[U^M(\tau,s)-U(\tau,s)\right]\bar{Q}(\tau)\varphi\right\|_\beta
\end{aligned} \tag{5.21}$$

along with Propositions 5.3 and 5.4. Since (see Eq. (5.1)) $\bar{Q}(t)\varphi = \sum_{j=0}^{6} c_j\left(t\right)\bar{\mathcal{A}}^{(j)} \in D\left(\mathcal{N}^{\beta}\right)$ where each $c_j\left(t\right)$ is continuous in $t$, the latter term in Eq. (5.21) is estimated by a sum of 7 terms resulting from the estimates in Proposition 5.4 with $\psi = \bar{\mathcal{A}}^{(j)}\varphi$ for $0 \leq j \leq 6$. This completes the proof of the claim.

**Item 4.** By integrating Eqs. (4.10) and (4.11) on $t$ we find,

$$U^M\left(t,s\right)\varphi = \varphi - i\int_s^t Q_M\left(\tau\right)U^M\left(\tau,s\right)\varphi d\tau \text{ and} \tag{5.22}$$

$$U^M\left(t,s\right)\varphi = \varphi + i\int_t^s U^M\left(t,\sigma\right)Q_M\left(\sigma\right)\varphi d\sigma \tag{5.23}$$

where the integrands are $\left\|\cdot\right\|_\beta$ – continuous and the integrals are taken relative to the $\left\|\cdot\right\|_\beta$ – topology. As a consequence of the above claim, we may let $M \to \infty$ in Eqs. (5.22) and (5.23) to find

$$U\left(t,s\right)\varphi = \varphi - i\int_s^t \bar{Q}\left(\tau\right)U\left(\tau,s\right)\varphi d\tau \text{ and}$$

$$U\left(t,s\right)\varphi = \varphi + i\int_t^s U\left(t,\sigma\right)\bar{Q}\left(\sigma\right)\varphi d\sigma$$

where again the integrands are $\left\|\cdot\right\|_\beta$ – continuous and the integrals are taken relative to the $\left\|\cdot\right\|_\beta$ – topology. Equations (5.17) and (5.18) follow directly from the previously displayed equations along with the fundamental theorem of calculus. ■

**Remark 5.6.** *By taking $t = s$ in Eq. (5.15) and using the fact that $U\left(t,s\right)$ is unitary on $L^2\left(m\right)$, it follows that*

$$U\left(t,\tau\right)^{-1} = U\left(\tau,t\right) = U^*\left(t,\tau\right), \tag{5.24}$$

*where $U^*\left(t,\tau\right)$ is the $L^2\left(m\right)$ - adjoint of $U\left(\tau,t\right)$. Also observe from Item 2. of Theorem 5.5 and Eq. (3.34) that*

$$U\left(t,s\right)\mathcal{S} = \mathcal{S} \text{ for all } s,t \in \mathbb{R}. \tag{5.25}$$

**Remark 5.7.** *Recall that if $X$ is a Banach space, $\psi(h) \in X$, $T(h) \in B(X)$ for $0 < |h| < 1$, and $\psi(h) \to \psi \in X$ and $T(h) \xrightarrow{s} T \in B(X)$ as $h \to 0$, then $T(h)\psi(h) \to T\psi$ as $h \to 0$.*

**Theorem 5.8.** *Let $Q(t)$ and $U(t,s)$ be as in Theorem 5.5 and set $W(t) := U(t,0)$. If $\varphi \in \mathcal{S}$, $R \in \mathbb{C}\langle\theta,\theta^*\rangle$, and $\mathcal{R} := R\left(a,a^\dagger\right)$, then*

$$\frac{d}{dt}W(t)^* \mathcal{R}W(t)\varphi = iW(t)^*[Q(t),\mathcal{R}]W(t)\varphi$$

*where the derivative may be taken relative to the $\|\cdot\|_\beta$ – topology for any $\beta \geq 0$.*

**Proof.** Let $d = \deg_\theta R$, $\psi(t) = \mathcal{R}W(t)\varphi$ and

$$f(t) := W(t)^* \mathcal{R}W(t)\varphi = W(t)^*\psi(t) = U(0,t)\psi(t).$$

In the proof we will write $\|\cdot\|_\beta\text{-}\frac{d}{dt}\psi(t)$ to indicate that we are taking the derivative relative to the $\beta$ – norm topology.

Using the result of Theorem 5.5 and the fact that $\|\mathcal{R}\|_{\beta+d/2\to\beta} < \infty$ (Corollary 3.30) it easily follows that

$$\|\cdot\|_\beta\text{-}\frac{d}{dt}\psi(t) = -i\mathcal{R}Q(t)W(t)\varphi. \tag{5.26}$$

Combining this assertion with Remark 5.7 and the $\beta$ – norm strong continuity of $W(t)^*$ (again Theorem 5.5) we may conclude that

$$\|\cdot\|_\beta\text{-}\lim_{h\to 0} W(t+h)^* \frac{\psi(t+h)-\psi(t)}{h} = W(t)^*\dot{\psi}(t) = -iW(t)^*\mathcal{R}Q(t)W(t)\varphi.$$

Hence, as

$$\frac{f(t+h)-f(t)}{h} = W(t+h)^*\frac{\psi(t+h)-\psi(t)}{h} + \frac{W(t+h)^*-W(t)^*}{h}\psi(t),$$

we may conclude

$$\begin{aligned}\|\cdot\|_\beta\text{-}\frac{d}{dt}f(t) &= \|\cdot\|_\beta\text{-}\lim_{h\to 0}\frac{f(t+h)-f(t)}{h} \\ &= -iW(t)^*\mathcal{R}Q(t)W(t)\varphi + \dot{W}^*(t)\psi(t) \\ &= -iW(t)^*\mathcal{R}Q(t)W(t)\varphi + iW(t)^*Q(t)\mathcal{R}W(t)\varphi\end{aligned}$$

which completes the proof. ■

# 1 Consequences of Theorem 5.5

**Notation 5.9.** *Let $H \in \mathbb{C}\langle\theta,\theta^*\rangle$ be a symmetric non-commutative polynomial in $\theta$ and $\theta^*$. Let $\alpha \in \mathbb{C}$ and $H_2(\alpha : \theta,\theta^*)$ as in Eq. (2.21) be the degree 2 homogeneous component of $H(\theta+\alpha,\theta^*+\bar{\alpha})$. From Remark 2.15 and Theorem 2.18, $H^{\mathrm{cl}}(\alpha)$ is real-valued and $H_2(\alpha : \theta,\theta^*)$ is still symmetric.*

**Corollary 5.10.** *Let $H \in \mathbb{C}\langle\theta,\theta^*\rangle$ be a symmetric non-commutative polynomial in $\theta$ and $\theta^*$, $H_2(\alpha : \theta,\theta^*)$ be as in Notation 5.9, and suppose that $\mathbb{R} \ni t \to \alpha(t) \in \mathbb{C}$ is a given continuous function. Then there exists a unique one parameter strongly continuous family of unitary operators $\{W_0(t)\}_{t\in\mathbb{R}}$ on $L^2(m)$ such that (with $W_0^*(t)$ being the $L^2$ - adjoint of $W_0(t)$);*

1. *$W_0(t)\mathcal{S} = \mathcal{S}$ and $W_0^*(t)\mathcal{S} = \mathcal{S}$.*
2. *$W_0(t)D(\mathcal{N}^\beta) = D(\mathcal{N}^\beta)$, $W_0(t)^*D(\mathcal{N}^\beta) = D(\mathcal{N}^\beta)$, and for all $0 \le T < \infty$, there exists $C_{T,\beta} = C_{T,\beta}(\alpha) < \infty$ such that*

$$\sup_{|t|\le T}\|W_0(t)\|_{\beta\to\beta} \vee \|W_0(t)^*\|_{\beta\to\beta} \le C_{T,\beta}. \tag{5.27}$$

3. *The maps $t \to W_0(t)\psi$ and $t \to W_0^*(t)\psi$ are $\|\cdot\|_\beta$-norm continuous for all $\psi \in D(\mathcal{N}^\beta)$.*

4. *For each* $\beta \geq 0$ *and* $\psi \in D\left(\mathcal{N}^{\beta+1}\right)$;

$$i\left(\|\cdot\|_\beta \text{-}\frac{\partial}{\partial t}\right) W_0(t)\psi = \overline{H_2\left(\alpha(t) : a, a^\dagger\right)} W_0(t)\psi \text{ with } W_0(0)\psi = \psi \quad (5.28)$$

*and*

$$-i\left(\|\cdot\|_\beta \text{-}\frac{\partial}{\partial t}\right) W_0(t)^*\psi = W_0(t)^*\,\overline{H_2\left(\alpha(t) : a, a^\dagger\right)}\psi \text{ with } W_0(0)^*\psi = \psi. \quad (5.29)$$

*[In Eqs. (5.28) and (5.29), one may replace* $\overline{H_2\left(\alpha(t) : a, a^\dagger\right)}$ *by* $H_2\left(\alpha(t) : \overline{a}, a^*\right)$ *as both operators are equal on* $D(\mathcal{N})$ *by Corollary 3.30.]*

**Proof.** The stated results follow from Theorem 5.5 and Remark 5.6 with $Q(t) := H_2\left(\alpha(t) : a, a^\dagger\right)$ after setting $W_0(t) = U(t,0)$ in which case that $W_0(t)^* = U(t,0)^* = U(0,t)$. ■

**Corollary 5.11.** *If* $\alpha \in \mathbb{C}$, $U(\alpha)$ *is as in Definition 1.6, and* $U(\alpha)^*$ *is the* $L^2(m)$-*adjoint of* $U(\alpha)$, *then for any* $\beta \geq 0$;

1. $U(\alpha)\mathcal{S} = \mathcal{S}$ *and* $U(\alpha)^*\mathcal{S} = \mathcal{S}$ *(also seen in Proposition 2.4),*
2. $U(\alpha) D\left(\mathcal{N}^\beta\right) = D\left(\mathcal{N}^\beta\right)$ *and* $U(\alpha)^* D\left(\mathcal{N}^\beta\right) = D\left(\mathcal{N}^\beta\right)$, *and*
3. *the following operator norm bounds hold,*

$$\|U(\alpha)\|_{\beta\to\beta} \vee \|U(\alpha)^*\|_{\beta\to\beta} \leq \exp\left(8\beta \cdot 3^{|\beta-1|}|\alpha|\right). \quad (5.30)$$

**Proof.** Let $\alpha(t) = t\alpha$,

$$H(t:\theta,\theta^*) = \dot{\alpha}(t)\theta^* - \overline{\dot{\alpha}(t)}\theta + i\,\mathrm{Im}\left(\alpha(t)\overline{\dot{\alpha}(t)}\right) = \alpha\theta^* - \overline{\alpha}\theta$$

so that

$$Q(t) = \alpha a^\dagger - \overline{\alpha}a + i\,\mathrm{Im}(t\alpha\overline{\alpha}) = \alpha a^\dagger - \overline{\alpha}a.$$

By Proposition 2.7, if $\varphi \in D(\mathcal{N})$, $\psi(t) := U(t\alpha)\, U(s\alpha)^* \varphi$, then $\psi$ satisfies Eq. (5.2) and therefore items 1. and 2. follow Theorem 5.5 and Remark 5.6. To get the explicit upper bound in Eq. (5.30), we apply Corollary 5.2 with $S = 0$, $T = 1$, $P(t, \theta, \theta^*) = \alpha\theta^* - \bar{\alpha}\theta$ in order to conclude, for any $M \in (0, \infty)$, that

$$\left\|U^M(\alpha)\right\|_{\beta \to \beta} \le \exp\left(\beta 4 \cdot 3^{|\beta-1|}\left[|\alpha| + |\bar{\alpha}|\right]\right) = \exp\left(8\beta \cdot 3^{|\beta-1|}\,|\alpha|\right)$$

Letting $M \to \infty$ (as in the proof of Theorem 5.5) then implies

$$\|U(\alpha)\|_{\beta \to \beta} \le \exp\left(8\beta \cdot 3^{|\beta-1|}\,|\alpha|\right).$$

Using $U(\alpha)^* = U(-\alpha)$, the previous equation is sufficient to prove the estimated in Eq. (5.30). ■

**Corollary 5.12.** *Let $U(\alpha)$ be as in Definition 1.6, $U(\alpha)^*$ be the $L^2(m)$-adjoint of $U(\alpha)$, $\mathbb{R} \ni t \to \alpha(t) \in \mathbb{C}$ be a $C^1$ function, and*

$$Q(t) := \dot{\alpha}(t)\, a^\dagger - \overline{\dot{\alpha}(t)} a + i \operatorname{Im}\left(\alpha(t)\, \overline{\dot{\alpha}(t)}\right).$$

*Then for any $\beta \ge 0$;*

1. *the maps $t \to U(\alpha(t))\,\psi$ and $t \to U(\alpha(t))^*\,\psi$ are $\|\cdot\|_\beta$-continuous for all $\psi \in D\left(\mathcal{N}^\beta\right)$, and*
2. *for each $\beta \ge 0$ and $\psi \in D\left(\mathcal{N}^{\beta+1}\right)$;*

$$i\left(\|\cdot\|_\beta\text{-}\frac{\partial}{\partial t}\right) U(\alpha(t))\,\psi = \overline{Q(t)} U(\alpha(t))\,\psi \tag{5.31}$$

*and*

$$-i\left(\|\cdot\|_\beta\text{-}\frac{\partial}{\partial t}\right) U(\alpha(t))^*\,\psi = U(\alpha(t))^*\,\overline{Q(t)}\psi. \tag{5.32}$$

**Proof.** Let

$$H\left(t:\theta,\theta^{*}\right):=\dot{\alpha}\left(t\right)\theta^{*}-\overline{\dot{\alpha}\left(t\right)}\theta+i\operatorname{Im}\left(\alpha\left(t\right)\overline{\dot{\alpha}\left(t\right)}\right)$$

so that $Q\left(t\right)=H\left(t:a,a^{\dagger}\right)$. By Proposition 2.7 if $\varphi\in D\left(\mathcal{N}\right)$, $\psi\left(t\right):=U\left(\alpha\left(t\right)\right)U\left(\alpha\left(s\right)\right)^{*}\varphi$, then $\psi$ satisfies Eq. (5.2) and therefore the corollary again follows from Theorem 5.5 and Remark 5.6. ■

**Theorem 5.13** (Properties of $a\left(t\right)$)**.** *Let $H\in\mathbb{C}\left\langle\theta,\theta^{*}\right\rangle$ be symmetric and $H^{cl}\in\mathbb{C}\left[z,\bar{z}\right]$ be the symbol of $H$, ($H^{cl}$ is necessarily real valued by Remark 2.15.) Further suppose that $\alpha\left(t\right)\in\mathbb{C}$ satisfying Hamilton's equations of motion (see Eq. (2.3) has global solutions, $a\left(t\right)$ and $a^{\dagger}\left(t\right)$ are the operators on $\mathcal{S}$ as described in Eqs. (1.8), and (1.9), and $W_{0}\left(t\right)$ is the unitary operator in Corollary 5.10. Then for all $t\in\mathbb{R}$ the following identities hold;*

$$W_{0}\left(t\right)^{*}aW_{0}\left(t\right)=a\left(t\right),\qquad W_{0}\left(t\right)^{*}a^{\dagger}W_{0}\left(t\right)=a^{\dagger}\left(t\right),\tag{5.33}$$

$$W_{0}\left(t\right)^{*}\overline{a}W_{0}\left(t\right)=\overline{a\left(t\right)},\qquad W_{0}\left(t\right)^{*}a^{*}W_{0}\left(t\right)=a^{*}\left(t\right),\tag{5.34}$$

$$W_{0}\left(t\right)^{*}\overline{a^{\dagger}}W_{0}\left(t\right)=\overline{a^{\dagger}\left(t\right)}\tag{5.35}$$

$$D\left(\overline{a\left(t\right)}\right)=D\left(\sqrt{\mathcal{N}}\right)=D\left(a^{*}\left(t\right)\right)\tag{5.36}$$

$$a^{*}\left(t\right)=\overline{a^{\dagger}\left(t\right)},\tag{5.37}$$

$$\overline{a\left(t\right)}=\gamma\left(t\right)\overline{a}+\delta\left(t\right)a^{*},\quad \textit{and}\tag{5.38}$$

$$a^{*}\left(t\right)=\overline{\delta\left(t\right)}\bar{a}+\overline{\gamma\left(t\right)}a^{*},\tag{5.39}$$

*where the closures and adjoints are taken relative to the $L^{2}\left(m\right)$-inner product.*

**Proof.** Recall from Proposition 2.2 that

$$v\left(t\right):=\frac{\partial^{2}H^{\mathrm{cl}}}{\partial\alpha\partial\overline{\alpha}}\left(\alpha\left(t\right)\right)\in\mathbb{R}\text{ and }u\left(t\right):=\frac{\partial^{2}H^{\mathrm{cl}}}{\partial\overline{\alpha}^{2}}\left(\alpha\left(t\right)\right)\in\mathbb{C}.$$

With this notation, the commutator formulas in Corollary 3.5 with $\alpha=\alpha\left(t\right)$ may be written as,

$$\begin{aligned}\left[H_2\left(\alpha(t):a,a^\dagger\right),a\right] &= -v(t)\,a-u(t)\,a^\dagger\\ \left[H_2\left(\alpha(t):a,a^\dagger\right),a^\dagger\right] &= \bar{u}(t)\,a+v(t)\,a^\dagger.\end{aligned}$$

For $\varphi\in\mathcal{S}$, let

$$\psi(t):=W_0(t)^*\,aW_0(t)\,\varphi \text{ and } \psi^\dagger(t):=W_0(t)^*\,a^\dagger W_0(t)\,\varphi.$$

From Theorem 5.8 with $W(t)=W_0(t)$, $Q(t)=H_2\left(\alpha(t):a,a^\dagger\right)$, and $\mathcal{R}=a$ and $\mathcal{R}=a^\dagger$, we find

$$\begin{aligned}i\frac{d}{dt}\psi(t) &= W_0(t)^*\left[v(t)\,a+u(t)\,(\alpha(t))\,a^\dagger\right]W_0(t)\,\varphi\\ &= v(t)\,\psi(t)+u(t)\,\psi^\dagger(t)\\ i\frac{d}{dt}\psi^\dagger(t) &= -W_0(t)^*\left[\bar{u}(t)\,a+v(t)\,a^\dagger\right]W_0(t)\,\varphi\\ &= -\bar{u}(t)\,\psi(t)+v(t)\,\psi^\dagger(t).\end{aligned}$$

In other words,

$$i\frac{d}{dt}\begin{bmatrix}\psi(t)\\ \psi^\dagger(t)\end{bmatrix}=\begin{bmatrix}v(t) & u(t)\\ -\bar{u}(t) & -\bar{v}(t)\end{bmatrix}\begin{bmatrix}\psi(t)\\ \psi^\dagger(t)\end{bmatrix}\in L^2(m)\times L^2(m).$$

This linear differential equation has a unique solution which, using Proposition 2.2, is given by

$$\begin{bmatrix}\psi(t)\\ \psi^\dagger(t)\end{bmatrix}=\Lambda(t)\begin{bmatrix}\psi(0)\\ \psi^\dagger(0)\end{bmatrix}=\Lambda(t)\begin{bmatrix}a\varphi\\ a^\dagger\varphi\end{bmatrix}$$

where $\Lambda(t)$ is the $2\times 2$ matrix given in Eq. (2.6). This completes the proof of Eq.

(5.33) since

$$\begin{bmatrix} W_0(t)^* aW_0(t)\varphi \\ W_0(t)^* a^\dagger W_0(t)\varphi \end{bmatrix} = \begin{bmatrix} \psi(t) \\ \psi^\dagger(t) \end{bmatrix} \text{ and } \Lambda(t) \begin{bmatrix} a\varphi \\ a^\dagger\varphi \end{bmatrix} = \begin{bmatrix} a(t)\varphi \\ a^\dagger(t)\varphi \end{bmatrix}.$$

The statements in Eqs. (5.34), (5.35) and (5.36) are easy consequences of the fact that $W_0(t)$ is a unitary operator on $L^2(m)$ which preserves $D(\mathcal{N})$ (see Corollary 5.10). Using Eqs. (5.34) and (5.35) along with Theorem 3.15 shows,

$$\overline{a^\dagger(t)} = W_0(t)^* \overline{a^\dagger} W_0(t) = W_0(t)^* a^* W_0(t) = a(t)^*$$

which gives Eq. (5.37).

If $\varphi \in D(\mathcal{N})$, using item 3. of Theorem 3.15 and the formula for $a(t)$ and $a^\dagger(t)$ in Eqs. (1.8) and (1.9) we find

$$\begin{aligned} \lim_{M\to\infty} a(t)\mathcal{P}_M\varphi &= \lim_{M\to\infty}\left[\gamma(t)a\mathcal{P}_M\varphi + \delta(t)a^\dagger\mathcal{P}_M\varphi\right] \\ &= \gamma(t)\bar{a}\varphi + \delta(t)a^*\varphi \end{aligned}$$

$$\begin{aligned} \lim_{M\to\infty} a^\dagger(t)\mathcal{P}_M\varphi &= \lim_{M\to\infty}\left[\delta(t)a\mathcal{P}_M\varphi + \overline{\gamma(t)}a^\dagger\mathcal{P}_M\varphi\right] \\ &= \overline{\delta(t)}\bar{a}\varphi + \overline{\gamma(t)}a^*\varphi. \end{aligned}$$

The above two equations along with Corollary 3.30 show Eqs. (5.38) and (5.39). ∎

# Chapter 6

# Bounds on the Quantum Evolution

Throughout this chapter and the rest of Part I, let $H \in \mathbb{R}\langle\theta,\theta^*\rangle$ be a non-commutative polynomial satisfying Assumption 1.11. Before getting to the proof of the main theorems we need to address some domain issues. Recall as in Assumption 1.11 we let $H_\hbar := \overline{H\left(a_\hbar, a_\hbar^\dagger\right)}$.

The following abstract proposition (Stone's theorem) is a routine application of the spectral theorem, see on p.265 of [32] for details.

**Proposition 6.1.** *Supposed $H$ is a self-adjoint operator on a separable Hilbert space, $\mathcal{K}$, and there is a $C \in \mathbb{R}$ and $\varepsilon > 0$ such that $H + CI \geq \varepsilon I$. For any $\beta \geq 0$ let $\|\cdot\|_{(H+CI)^\beta}$ $(\geq \varepsilon \|\cdot\|_{\mathcal{K}})$ be the Hilbertian norm on $D\left((H+CI)^\beta\right)$ defined by,*

$$\|f\|_{(H+CI)^\beta} = \left\|(H+CI)^\beta f\right\|_{\mathcal{K}} \ \forall\ f \in D\left((H+CI)^\beta\right).$$

*Then for all $t \in \mathbb{R}$ and $\beta \geq 0$,*

$$e^{-itH} D((H+CI)^{\beta}) = D((H+CI)^{\beta}) \text{ and}$$
$$\left\| e^{-itH}\psi \right\|_{(H+CI)^{\beta}} = \|\psi\|_{(H+CI)^{\beta}} \;\; \forall\, \psi \in D((H+CI)^{\beta}).$$

*Moreover, if $\beta \geq 0$ and $\varphi \in D((H+CI)^{\beta+1})$, then*

$$\|\cdot\|_{(H+CI)^{\beta}} - \frac{d}{dt} e^{-iHt}\varphi = -iHe^{-iHt}\varphi = -ie^{-iHt}H\varphi.$$

In this chapter we are going to show, as a consequence of Proposition 6.3 below, that

$$e^{iH_{\hbar}t/\hbar}\bar{a}e^{-iH_{\hbar}t/\hbar}\mathcal{S} \text{ and } e^{iH_{\hbar}t/\hbar}a^{*}e^{-iH_{\hbar}t/\hbar}\mathcal{S} \subseteq \mathcal{S}. \tag{6.1}$$

**Lemma 6.2.** *For any unbounded operator $T$ and constant $C \in \mathbb{R}$, then for any $n \in \mathbb{N}_0$,*

$$D\left((T+C)^{n}\right) = D\left(T^{n}\right).$$

**Proof.** We first show by induction that $D\left((T+C)^{n}\right) \subset D\left(T^{n}\right)$ for all $n \in \mathbb{N}$. The case $n = 1$ is trivial. Then the induction step is

$$\begin{aligned} f \in D\left((T+C)^{n+1}\right) &\implies f \in D\left((T+C)^{n}\right) \text{ and } (T+C)^{n} f \in D\left(T+C\right) \\ &\implies f \in D\left((T+C)^{n}\right) \text{ and } (T+C)^{n} f \in D\left(T\right) \\ &\implies f \in D\left(T^{n}\right) \text{ and } (T+C)^{n} f \in D\left(T\right) \end{aligned}$$

But

$$(T+C)^{n} f = T^{n} f + \sum_{k=0}^{n-1} \binom{n}{k} C^{n-k} T^{k} f = T^{n} f + g$$

where $g \in D\left(T\right)$ and hence

$$T^{n} f = (T+C)^{n} f - g \in D\left(T\right) \implies f \in D\left(T^{n+1}\right).$$

finishing the inductive step.

To finish the proof, we replace $T$ by $T-C$ above to learn

$$D\left(T^{n}\right)=D\left((T-C+C)^{n}\right)\subset D\left((T-C)^{n}\right)$$

and then replace $C$ by $-C$ to find $D\left(T^{n}\right)\subset D\left((T+C)^{n}\right)$. ■

**Proposition 6.3.** *Let $H\left(\theta,\theta^{*}\right)$ and $\eta>0$ be as in Assumption 1.11, then* $\exp\left(-iH_{\hbar}t\right)$ *leaves $\mathcal{S}$ invariant and more explicitly, it is* $\exp\left(-iH_{\hbar}t\right)\mathcal{S}=\mathcal{S}$ *for all $t\in\mathbb{R}$.*

**Proof.** The fact that $\mathcal{S}\subseteq H_{\hbar}^{n}$ for all $n\in\mathbb{N}$ along with Eq. (1.14) in the Assumption 1.11 and Eq. (3.51), we learn that

$$\mathcal{S}(\mathbb{R})\subset\bigcap_{n=1}^{\infty}D\left(H_{\hbar}^{n}\right)\subseteq\bigcap_{n=1}^{\infty}D(\mathcal{N}_{\hbar}^{n})=\mathcal{S}(\mathbb{R})$$

This shows $\mathcal{S}(\mathbb{R})=\bigcap_{n=1}^{\infty}D\left(H_{\hbar}^{n}\right)$ and this finishes the proof since, see Proposition 6.1, $\exp\left(-iH_{\hbar}t\right)$ leaves $\bigcap_{n=1}^{\infty}D\left(H_{\hbar}^{n}\right)$ invariant, i.e.,$\exp\left(-iH_{\hbar}t\right)\mathcal{S}\subseteq\mathcal{S}$ for all $t\in\mathbb{R}$. By multiplying $\exp\left(iH_{\hbar}t\right)$ on both sides, we yield $\mathcal{S}\subseteq\exp\left(iH_{\hbar}t\right)\mathcal{S}$. Therefore, $\exp\left(-iH_{\hbar}t\right)\mathcal{S}=\mathcal{S}$ is resulted if we replacing $t$ to $-t$. ■

**Lemma 6.4.** *If $P\in\mathbb{C}\left\langle\theta,\theta^{*}\right\rangle$, $\delta:=\deg_{\theta}P\in\mathbb{N}_{0}$, and $C\left(P\right):=\sum_{k=0}^{\delta}\left|P_{k}\right|k^{k/2}$, then*

$$\left\|P\left(\bar{a}_{\hbar},a_{\hbar}^{*}\right)\psi\right\|\leq C\left(P\right)\left\|\left(I+\mathcal{N}_{\hbar}\right)^{\delta/2}\psi\right\|\;\;\forall\;0<\hbar\leq1\text{ and }\psi\in D\left(\mathcal{N}^{\delta/2}\right).\quad(6.2)$$

**Proof.** Let $P_{k}$ be the degree $k$ homogeneous component of $P$ as in Eq.

(2.22). Then according to Corollary 3.30 with $\beta = 0$ and $d = k$ we have,

$$\begin{aligned}
\|P_k(\bar{a}_\hbar, a^*_\hbar)\psi\| &= \hbar^{k/2}\|P_k(\bar{a}, a^*)\psi\| \\
&\le |P_k| k^{k/2}\hbar^{k/2}\|\psi\|_{k/2} \\
&= |P_k| k^{k/2}\hbar^{k/2}\left\|(I+\mathcal{N})^{k/2}\psi\right\| \\
&= |P_k| k^{k/2}\left\|(\hbar I+\mathcal{N}_\hbar)^{k/2}\psi\right\| \\
&\le |P_k| k^{k/2}\left\|(I+\mathcal{N}_\hbar)^{k/2}\psi\right\| \le |P_k| k^{k/2}\left\|(I+\mathcal{N}_\hbar)^{\delta/2}\psi\right\|.
\end{aligned}$$

Summing this inequality on $k$ using $P = \sum_{k=0}^{\delta} P_k$ and the triangle inequality leads directly to Eq. (6.2). ■

Let us recall the Löwner-Heinz inequality in Theorem 1.36 so that we can compare $\mathcal{N}_\hbar$ and $H_\hbar$ for all non-negative power $\beta$ by using this inequality.

**Theorem** (Löwner-Heinz inequality)**.** *Let $A$ and $B$ be non-negative self-adjoint operators on a Hilbert space. If $A \le B$ (see Notation 1.10), then $A^r \le B^r$ for $0 \le r \le 1$.*

**Corollary 6.5.** *Let $H(\theta, \theta^*) \in \mathbb{R}\langle\theta, \theta^*\rangle$, $1 > \eta > 0$, and $C$ be as in Assumption 1.11 and set $\tilde{C} := C + 1$. Then for each $\beta \ge 0$, there exists constants $\widetilde{C}_\beta < \infty$ and $\widetilde{D}_\beta < \infty$ such that, for all $0 \le \hbar < \eta$,*

$$(\mathcal{N}_\hbar + I)^\beta \le \widetilde{C}_\beta \left(H_\hbar + \widetilde{C}\right)^\beta \quad \textit{and} \tag{6.3}$$

$$\left(H_\hbar + \widetilde{C}\right)^\beta \le \widetilde{D}_\beta (\mathcal{N}_\hbar + I)^{\beta d/2}. \tag{6.4}$$

**Proof.** Using the simple estimate,

$$(x+1)^\beta \le 2^{(\beta-1)_+}\left(x^\beta + 1\right) \;\; \forall\; x, \beta \ge 0, \tag{6.5}$$

along with Eq. (1.14) implies,

$$\begin{aligned}(\mathcal{N}_\hbar + I)^\beta &\preceq 2^{(\beta-1)_+}\left(\mathcal{N}_\hbar^\beta + I\right) \preceq 2^{(\beta-1)_+}\left(C_\beta\left(H_\hbar + C\right)^\beta + I\right) \\ &\preceq 2^{(\beta-1)_+} C_\beta \left(H_\hbar + C + I\right)^\beta, \qquad (6.6)\end{aligned}$$

wherein we have assumed $C_\beta \geq 1$ without loss of generality. Lemma 10.10 on p.230 of [34] asserts, if $A$ and $B$ are non-negative self-adjoint operators and $A \preceq B$, then $A \leq B$. Therefore we can deduce from Eq. (6.6) that

$$(\mathcal{N}_\hbar + I)^\beta \leq 2^{(\beta-1)_+} C_\beta \left(H_\hbar + C + I\right)^\beta$$

which gives Eq. (6.3).

We now turn to the proof of Eq. (6.4). For $n \in \mathbb{N}$, let $P^{(n)} \in \mathbb{C}\langle\theta, \theta^*\rangle$ be defined by

$$P^{(n)}(\theta, \theta^*) := \left(H(\theta, \theta^*) + \tilde{C}\right)^n$$

so that $\deg_\theta P^{(n)} = dn$ and for $\psi \in D\left(\mathcal{N}^{dn/2}\right)$, we have

$$\left(H_\hbar + \widetilde{C}\right)^n \psi = P^{(n)}\left(\bar{a}_\hbar, a_\hbar^*\right)\psi.$$

With these observations, we may apply Lemma 6.4 to find for any $0 < \hbar < \eta \leq 1$ that

$$\left\|\left(H_\hbar + \widetilde{C}\right)^n \psi\right\| \leq C\left(P^{(n)}\right)\left\|(I + \mathcal{N}_\hbar)^{\frac{dn}{2}} \psi\right\| \;\forall\; \psi \in D\left(\mathcal{N}^{dn/2}\right).$$

The last displayed equation is equivalent (see Notation 1.10) to the operator inequality,

$$\left(H_\hbar + \widetilde{C}\right)^{2n} \leq C\left(P^{(2n)}\right)(I + \mathcal{N}_\hbar)^{dn}.$$

Hence if $0 \leq \beta \leq 2n$, we may apply the Löwner-Heinz inequality with $r = \beta/2n$ to conclude

$$\left(H_\hbar+\widetilde{C}\right)^\beta \leq \left[C\left(P^{(n)}\right)\right]^{\beta/2n}\left(I+\mathcal{N}_\hbar\right)^{\beta d/2}.$$

As $n\in\mathbb{N}$ was arbitrary, the proof is complete. ■

**Theorem 6.6.** *Let $H(\theta,\theta^*)\in\mathbb{R}\langle\theta,\theta^*\rangle$, $d=\deg_\theta H$, and $1>\eta>0$ be as in Assumption 1.11 and suppose $0<\hbar<\eta\leq 1$.*

1. *If $\beta\geq 0$ then*
$$e^{-iH_\hbar t/\hbar}D\left(\mathcal{N}^{\beta d/2}\right)\subseteq D\left(\mathcal{N}^\beta\right). \tag{6.7}$$
*and there exists $C_\beta<\infty$ such that*
$$\left\|e^{-iH_\hbar t/\hbar}\right\|_{\beta d/2\to\beta}\leq C_\beta\hbar^{-\beta} \text{ for all } t\in\mathbb{R}. \tag{6.8}$$

2. *If $\beta\geq 0$ and $\psi\in D\left(\mathcal{N}^{(\beta+1)d/2}\right)\subset D\left(H_\hbar^{\beta+1}\right)$, then*
$$e^{-iH_\hbar t/\hbar}\psi,\ H_\hbar e^{-iH_\hbar t/\hbar}\psi,\ \text{and } e^{-iH_\hbar t/\hbar}H_\hbar\psi$$
*are all in $D\left(\mathcal{N}^\beta\right)$ for all $t\in\mathbb{R}$ and moreover,*
$$i\hbar\left(\|\cdot\|_\beta\text{-}\frac{d}{dt}\right)e^{-iH_\hbar t/\hbar}\psi=H_\hbar e^{-iH_\hbar t/\hbar}\psi=e^{-iH_\hbar t/\hbar}H_\hbar\psi, \tag{6.9}$$
*where, as before, $\|\cdot\|_\beta$-$\frac{d}{dt}$ indicates the derivative is taken in $\beta$ – norm topology.*

**Proof.** If $\beta\geq 0$, it follows from Corollary 6.5 (with $\beta$ replaced by $2\beta$) that

$$D\left(\mathcal{N}^{\beta d/2}\right)=D\left(\mathcal{N}_\hbar^{\beta d/2}\right)\subset D\left(\left(H_\hbar+\widetilde{C}\right)^\beta\right)\subset D\left(\mathcal{N}_\hbar^\beta\right)=D\left(\mathcal{N}^\beta\right) \tag{6.10}$$

and

$$\|\psi\|_{(\mathcal{N}_\hbar+I)^\beta}\leq\sqrt{\widetilde{C}_{2\beta}}\,\|\psi\|_{\left(H_\hbar+\widetilde{C}\right)^\beta}\ \forall\ \psi\in D\left(\left(H_\hbar+\widetilde{C}\right)^\beta\right).$$

Moreover if $0 < \hbar < \eta \leq 1$, a simple calculus inequality shows

$$\hbar^\beta \|\psi\|_\beta = \hbar^\beta \|\psi\|_{(\mathcal{N}+I)^\beta} \leq \|\psi\|_{(\mathcal{N}_\hbar+I)^\beta}$$

and hence

$$\|\psi\|_\beta \leq \hbar^{-\beta}\sqrt{\widetilde{C}_{2\beta}}\,\|\psi\|_{\left(H_\hbar+\widetilde{C}\right)^\beta} \;\; \forall\, \psi \in D\left(\left(H_\hbar + \widetilde{C}\right)^\beta\right). \tag{6.11}$$

From Proposition 6.1 we know for all $t \in \mathbb{R}$ that

$$e^{-iH_\hbar t/\hbar} D\left(\left(H_\hbar + \widetilde{C}\right)^\beta\right) = D\left(\left(H_\hbar + \widetilde{C}\right)^\beta\right) \text{ and}$$
$$\left\|e^{-iH_\hbar t/\hbar}\psi\right\|_{\left(H_\hbar+\widetilde{C}\right)^\beta} = \|\psi\|_{\left(H_\hbar+\widetilde{C}\right)^\beta}.$$

Combining these statements with Eqs. (6.10) and (6.11) respectively shows,

$$e^{-iH_\hbar t/\hbar} D\left(\mathcal{N}^{\beta d/2}\right) \subset e^{-iH_\hbar t/\hbar} D\left(\left(H_\hbar + \widetilde{C}\right)^\beta\right) = D\left(\left(H_\hbar + \widetilde{C}\right)^\beta\right) \subset D\left(\mathcal{N}^\beta\right).$$

Moreover, if $\varphi \in D\left(\mathcal{N}^{\beta d/2}\right) \subset D\left(\left(H_\hbar + \widetilde{C}\right)^\beta\right)$, then

$$\left\|e^{-iH_\hbar t/\hbar}\varphi\right\|_\beta \leq \hbar^{-\beta}\sqrt{\widetilde{C}_{2\beta}}\left\|e^{-iH_\hbar t/\hbar}\varphi\right\|_{\left(H_\hbar+\widetilde{C}\right)^\beta} = \hbar^{-\beta}\sqrt{\widetilde{C}_{2\beta}}\,\|\varphi\|_{\left(H_\hbar+\widetilde{C}\right)^\beta}.$$

However, from Eq. (6.4) (again with $\beta \to 2\beta$) we also know

$$\|\varphi\|_{\left(H_\hbar+\widetilde{C}\right)^\beta} \leq \sqrt{\widetilde{D}_{2\beta}} \cdot \|\varphi\|_{(\mathcal{N}_\hbar+I)^{\beta d/2}} \leq \sqrt{\widetilde{D}_{2\beta}} \cdot \|\varphi\|_{(\mathcal{N}+I)^{\beta d/2}}.$$

Combining the last two displayed equations proves the estimate in Eq. (6.8) with $C_\beta := \sqrt{\widetilde{C}_{2\beta} \cdot \widetilde{D}_{2\beta}}$.

If we now further assume that $\psi \in D\left(\mathcal{N}^{(\beta+1)d/2}\right)$, then $\psi \in D\left(H_\hbar^{\beta+1}\right)$ by

Eq. (6.10) then, by Proposition 6.1, it follows that

$$H_\hbar e^{-iH_\hbar t/\hbar}\psi = e^{-iH_\hbar t/\hbar} H_\hbar \psi \in D\left(\left(H_\hbar + \widetilde{C}\right)^\beta\right) \subset D\left(\mathcal{N}^\beta\right)$$

and

$$i\hbar\left(\|\cdot\|_{H_\hbar^\beta}\text{-}\frac{d}{dt}\right)\psi\left(t\right) = H_\hbar\psi\left(t\right) = e^{-iH_\hbar t/\hbar} H_\hbar \psi_0. \tag{6.12}$$

Owing to Eq. (6.11) the $\beta$ – norm is weaker than $\|\cdot\|_{H_\hbar^\beta}$ – norm and hence Eq. (6.12) directly implies the weaker Eq. (6.9).

■

# Chapter 7

# A Key One Parameter Family of Unitary Operators

In this chapter (except for Lemma 7.2) we will always suppose that $H(\theta, \theta^*)$ and $1 \geq \eta > 0$ are as in Assumption 1.11, $\alpha_0 \in \mathbb{C}$, and $\alpha(t)$ denotes the solution to Hamilton's classical equations (1.1) of motion with $\alpha(0) = \alpha_0$. From Corollary 3.6, $U_\hbar(\alpha_0)\psi$ is a state on $L^2(m)$ which has position and momentum concentrated at $\xi_0 + i\pi_0 = \sqrt{2}\alpha_0$ in the limit as $\hbar \downarrow 0$. Thus if quantum mechanics is to limit to classical mechanics as $\hbar \downarrow 0$, one should expect that the quantum evolution, $\psi_\hbar(t) := e^{-iH_\hbar t/\hbar} U_\hbar(\alpha_0)\psi$, of the state, $U_\hbar(\alpha_0)\psi$, should be concentrated near $\alpha(t)$ in phase space as $\hbar \downarrow 0$. One possible candidate for these approximate states would be $U_\hbar(\alpha(t))\psi$ or more generally any state of the form, $U_\hbar(\alpha(t)) W_0(t)\psi$, where $\{W_0(t) : t \in \mathbb{R}\}$ are unitary operators on $L^2(m)$ which preserve $\mathcal{S}$. All states of this form concentrate their position and momentum expectations near $\sqrt{2}\alpha(t)$, see Remark 3.7. These remarks then motivate us to consider the one parameter family of unitary operators $V_\hbar(t)$ defined by,

$$V_\hbar(t) := U_\hbar(-\alpha(t)) e^{-iH_\hbar t/\hbar} U_\hbar(\alpha_0) = U_\hbar(\alpha(t))^* e^{-iH_\hbar t/\hbar} U_\hbar(\alpha_0). \tag{7.1}$$

Because of Propositions 2.4 and 6.3, we know $V_\hbar(t)\mathcal{S} = \mathcal{S}$ for all $0 < \hbar < \eta$ and in particular, $V_\hbar(t)\mathcal{S} = \mathcal{S} \subset D\left(P\left(a, a^\dagger\right)\right)$ for any $P(\theta, \theta^*) \in \mathbb{C}\langle\theta, \theta^*\rangle$. The main point of this chapter is to study the basic properties of this family of unitary operators with an eye towards showing that $\lim_{\hbar\downarrow 0} V_\hbar(t)$ exists (modulo a phase factor). Our first task is to differentiate $V_\hbar(t)$ for which we will need the following differentiation lemma.

**Lemma 7.1** (Product Rule)**.** *Let $P(\theta, \theta^*) \in \mathbb{C}\langle\theta, \theta^*\rangle$, $k := \deg_\theta P(\theta, \theta^*) \in \mathbb{N}_0$, and $P := P\left(a, a^\dagger\right)$. Suppose that $U(t)$ and $T(t)$ are unitary operators on $L^2(m)$ which preserve $\mathcal{S}$. We further assume;*

1. *for each $\varphi \in \mathcal{S}$, $t \to U(t)\varphi$ and $t \to T(t)\varphi$ are $\|\cdot\|_\beta$ – differentiable for all $\beta \geq 0$. We denote the derivative by $\dot{U}(t)\varphi$ and $\dot{T}(t)\varphi$ respectively. [Notice that $\dot{U}(t)\varphi$ and $\dot{T}(t)\varphi$ are all in $\cap_{\beta\geq 0} D\left(\mathcal{N}^\beta\right) = \mathcal{S}$, see Eq. (3.34) for the last equality, i.e. $\dot{U}(t)$ and $\dot{T}(t)$ preserves $\mathcal{S}$.]*

2. *For each $\beta \geq 0$ there exists $\alpha \geq 0$ and $\varepsilon > 0$ such that*

$$K := \sup_{|\Delta|\leq\varepsilon} \|U(t+\Delta)\|_{\alpha\to\beta} < \infty.$$

*Then for any $\beta \geq 0$,*

$$\|\cdot\|_\beta\text{-}\frac{d}{dt}\left[U(t)PT(t)\varphi\right] = \dot{U}(t)PT(t)\varphi + U(t)P\dot{T}(t)\varphi. \tag{7.2}$$

**Proof.** Let $\varphi \in \mathcal{S}$ and then define $\varphi(t) = U(t)PT(t)\varphi$. To shorten notation let $\Delta f$ denote $f(t+\Delta) - f(t)$. We then have,

$$\frac{\Delta\varphi}{\Delta} = \left[U(t+\Delta)P\frac{\Delta T}{\Delta} + \frac{\Delta U}{\Delta}PT(t)\right]\varphi$$

and so

$$\frac{\Delta\varphi}{\Delta} - U(t)\,P\dot{T}(t)\,\varphi - \dot{U}(t)\,PT(t)\,\varphi$$
$$= U(t+\Delta)\,P\left[\frac{\Delta T}{\Delta} - \dot{T}(t)\right]\varphi + [\Delta U]\,P\dot{T}(t)\,\varphi + \left[\frac{\Delta U}{\Delta} - \dot{U}(t)\right]PT(t)\,\varphi. \tag{7.3}$$

Using the assumptions of the theorem it follows that for each $\beta < \infty$, since $P\dot{T}(t)\,\varphi \in \mathcal{S}$, we may conclude that

$$\left\|[\Delta U]\,P\dot{T}(t)\,\varphi\right\|_{\beta} \to 0 \text{ as } \Delta \to 0, \text{ and}$$

$$\left\|\left[\frac{\Delta U}{\Delta} - \dot{U}(t)\right]PT(t)\,\varphi\right\|_{\beta} \to 0 \text{ as } \Delta \to 0.$$

Furthermore, using the assumptions along with Eq. (3.41) in the Proposition 3.29, it follows that when $\triangle \to 0$,

$$\left\|U(t+\Delta)\,P\left[\frac{\Delta T}{\Delta} - \dot{T}(t)\right]\varphi\right\|_{\beta}$$
$$\leq \|U(t+\Delta)\|_{\alpha\to\beta}\,\|P\|_{\alpha+\frac{k}{2}\to\alpha}\left\|\left[\frac{\Delta T}{\Delta} - \dot{T}(t)\right]\varphi\right\|_{\alpha+\frac{k}{2}} \to 0.$$

which combined with Eq. (7.3) shows $\varphi(t) = U(t)\,PT(t)\,\varphi$ is $\|\cdot\|_{\beta}$ – differentiable and the derivative is given as in Eq. (7.2).

$\blacksquare$

**Lemma 7.2.** *If $\alpha : \mathbb{R} \to \mathbb{C}$ is* ***any*** *$C^1$ – function and $V_\hbar(t)$ is defined as in Eq. (7.1), then for all $\psi \in \mathcal{S}$, $t \to V_\hbar(t)\,\psi$ and $t \to V_\hbar^*(t)\,\psi$ are $\|\cdot\|_{\beta}$-norm differentiable* *for all* $\beta < \infty$ *and moreover,*

$$\frac{d}{dt} V_\hbar(t)\psi = \Gamma_\hbar(t) V_\hbar(t)\psi \text{ and} \tag{7.4}$$

$$\frac{d}{dt} V_\hbar^*(t)\psi = -V_\hbar^*(t)\Gamma_\hbar(t)\psi \tag{7.5}$$

*where*

$$\Gamma_\hbar(t) := \frac{1}{\hbar}\left(\overline{\dot{\alpha}(t)} a_\hbar - \dot{\alpha}(t) a_\hbar^\dagger + i \operatorname{Im}\left(\alpha(t)\overline{\dot{\alpha}(t)}\right) - iH\left(a_\hbar + \alpha(t), a_\hbar^\dagger + \bar{\alpha}(t)\right)\right). \tag{7.6}$$

**Proof.** Let $U(t) := U_\hbar(-\alpha(t)) = U\left(-\alpha(t)/\sqrt{\hbar}\right)$, $T(t) := e^{-iH_\hbar t/\hbar}$ and $\varphi := U_\hbar(\alpha_0)\psi$. From Propositions 2.4 and 2.7 we know $U(t)\mathcal{S} = \mathcal{S}$ and

$$i\frac{d}{dt}U(t)f = Q(t)U(t)f \text{ for } f \in \mathcal{S}. \tag{7.7}$$

where

$$Q(t) = i\left(-\frac{\dot{\alpha}(t)}{\sqrt{\hbar}}a^\dagger + \frac{\overline{\dot{\alpha}(t)}}{\sqrt{\hbar}}a\right) - \frac{1}{\hbar}\operatorname{Im}\left(\alpha(t)\overline{\dot{\alpha}(t)}\right). \tag{7.8}$$

As $Q(t)$ is linear in $a$ and $a^\dagger$, we may apply Corollaries 5.11 and 5.12 in order to conclude that $U(t)$ satisfies the hypothesis in Lemma 7.1. Moreover, by Proposition 6.3 and the item 2 in Theorem 6.6, we also know that $T(t)\mathcal{S} = \mathcal{S}$ and it satisfies the hypothesis of Lemma 7.1. Therefore by taking $P(\theta, \theta^*) = 1$ (so $P = I$) in

Lemma 7.1, we learn

$$\begin{aligned}\frac{d}{dt}V_\hbar(t)\psi =&\dot{U}(t)T(t)\varphi + U(t)\dot{T}(t)\varphi\\=&\left[\left(-\frac{\dot{\alpha}(t)}{\sqrt{\hbar}}a^\dagger + \frac{\overline{\dot{\alpha}(t)}}{\sqrt{\hbar}}a\right) + \frac{i}{\hbar}\operatorname{Im}\left(\alpha(t)\overline{\dot{\alpha}(t)}\right)\right]U(t)T(t)\varphi\\&+U(t)\frac{H_\hbar}{i\hbar}T(t)\varphi\\=&\frac{1}{\hbar}\left[\left(-\dot{\alpha}(t)a_\hbar^\dagger + \overline{\dot{\alpha}(t)}a_\hbar^\dagger\right) + i\operatorname{Im}\left(\alpha(t)\overline{\dot{\alpha}(t)}\right)\right]V_\hbar(t)\psi\\&+U_\hbar(-\alpha(t))\frac{H_\hbar}{i\hbar}U_\hbar(\alpha(t))U_\hbar(-\alpha(t))T(t)\varphi\\=&\Gamma_\hbar(t)V_\hbar(t)\psi,\end{aligned}$$

wherein the last equality we have used Proposition 2.4 to conclude,

$$U_\hbar(-\alpha(t))H\left(a_\hbar, a_\hbar^\dagger\right)U_\hbar(\alpha(t)) = H\left(a_\hbar + \alpha(t), a_\hbar^\dagger + \bar{\alpha}(t)\right).$$

This completes the proof of Eq. (7.4). We now turn to the proof of Eq. (7.5).

Now let $U(t) = U_\hbar^*(\alpha_0)e^{iH_\hbar t/\hbar}$ and $T(t) := U_\hbar(\alpha(t))$ and observe by taking adjoint of Eq. (7.1) that

$$V_\hbar^*(t) := U_\hbar^*(\alpha_0)e^{iH_\hbar t/\hbar}U_\hbar(\alpha(t)) = U(t)T(t).$$

Working as above, we again easily show that both $U(t)$ and $T(t)$ satisfy the hypothesis of Lemma 7.1 and moreover by replacing $\alpha$ by $-\alpha$ in Eq. (7.8) we know

$$i\frac{d}{dt}T(t)\psi = T(t)\left[i\left(\frac{\dot{\alpha}(t)}{\sqrt{\hbar}}a^\dagger - \frac{\overline{\dot{\alpha}(t)}}{\sqrt{\hbar}}a\right) + \frac{1}{\hbar}\operatorname{Im}\left(\alpha(t)\overline{\dot{\alpha}(t)}\right)\right]\psi.$$

We now apply Lemma 7.1 with $P(\theta,\theta^*) = 1$ and $\varphi = \psi$ along with some basic algebraic manipulations to show Eq. (7.5) is also valid.

■

Specializing our choice of $\alpha(t)$ in Lemma 7.2 leads to the following important result.

**Theorem 7.3.** *Let $\Gamma_\hbar(t)$ be as in Eq. (7.6). If $\alpha(t)$ satisfies Hamilton's equations of motion (Eq. (1.1), $V_\hbar(t)$ is defined as in Eq. (7.1), then*

$$\begin{aligned}\Gamma_\hbar(t) =& \frac{i}{\hbar}\operatorname{Im}\left(\alpha(t)\overline{\dot{\alpha}(t)}\right) - \frac{i}{\hbar}H^{cl}(\alpha(t)) \\ & - iH_2\left(\alpha(t) : a, a^\dagger\right) - \frac{i}{\hbar}H_{\geq 3}\left(\alpha(t) : a_\hbar, a_\hbar^\dagger\right),\end{aligned} \tag{7.9}$$

*on $\mathcal{S}$ where $H^{cl}$, $H_2$ and $H_{\geq 3}$ are as in Eq. (2.31) by replacing $P$ by $H$.*

**Proof.** From the expansion of $H(\theta+\alpha,\theta^*+\bar{\alpha})$ described in Eq. (2.29) and Theorem 2.18 we have

$$\begin{aligned}& H\left(a_\hbar + \alpha(t), a_\hbar^\dagger + \bar{\alpha}(t)\right) \\ & = H^{\mathrm{cl}}(\alpha(t)) + \left(\frac{\partial H^{\mathrm{cl}}}{\partial \alpha}\right)(\alpha(t))\, a_\hbar + \left(\frac{\partial H^{\mathrm{cl}}}{\partial \overline{\alpha}}\right)(\alpha(t))\, a_\hbar^\dagger \\ & + H_2\left(\alpha(t) : a_\hbar, a_\hbar^\dagger\right) + H_{\geq 3}\left(\alpha(t) : a_\hbar, a_\hbar^\dagger\right).\end{aligned} \tag{7.10}$$

So if $\alpha(t)$ satisfies Hamilton's equations of motion,

$$i\dot{\alpha}(t) = \left(\frac{\partial}{\partial \bar{\alpha}}H^{\mathrm{cl}}\right)(\alpha(t)) \text{ with } \alpha(0) = \alpha_0, \tag{7.11}$$

it follows using Eq. (7.10) in Eq. (7.6) that we may cancel all the terms linear in $a_\hbar$ or $a_\hbar^\dagger$ in which case $\Gamma_\hbar(t)$ in Eq. (7.6) may be written as in Eq. (7.9). ■

In order to remove a (non-essential) highly oscillatory phase factor[1] from $V_\hbar(t)$ let

$$f(t) := \int_0^t \left(H^{\mathrm{cl}}(\alpha(\tau)) - \operatorname{Im}\left(\alpha(\tau)\overline{\dot{\alpha}(\tau)}\right)\right) d\tau \tag{7.12}$$

[1]As usual in quantum mechanics, the overall phase factor will not affect the expected values of observables and so we may safely ignore it in this introductory description.

and then define

$$W_{\hbar}(t)=e^{\frac{i}{\hbar}f(t)}V_{\hbar}(t)=e^{\frac{i}{\hbar}f(t)}U_{\hbar}(-\alpha(t))\,e^{-iH_{\hbar}t/\hbar}U_{\hbar}(\alpha_0)\,. \tag{7.13}$$

More generally for $s,t\in\mathbb{R}$, let

$$W_{\hbar}(t,s)=W_{\hbar}(t)\,W_{\hbar}^{*}(s)=e^{\frac{i}{\hbar}[f(t)-f(s)]}U_{\hbar}(-\alpha(t))\,e^{-iH_{\hbar}(t-s)/\hbar}U_{\hbar}(\alpha(s))\,. \tag{7.14}$$

**Proposition 7.4.** *Let $H(\theta,\theta^{*})\in\mathbb{R}\langle\theta,\theta^{*}\rangle$ and $\eta>0$ satisfy Assumption 1.11, $d=\deg_{\theta}H$, and $W_{\hbar}(t,s)$ be as in Eq. (7.14). Then*

$$W_{\hbar}(t,s)\,D\left(\mathcal{N}^{\beta\frac{d}{2}}\right)\subseteq D\left(\mathcal{N}^{\beta}\right)\ \ \forall\ s,t\in\mathbb{R}\text{ and }\beta\geq 0. \tag{7.15}$$

*Moreover, we have $W_{\hbar}(t,s)\,\mathcal{S}=\mathcal{S}$ for all $s,t\in\mathbb{R}$.*

**Proof.** Eq. (7.15) is a direct consequence from $U_{\hbar}(\alpha(\cdot))\,\mathcal{N}^{\beta}=\mathcal{N}^{\beta}$ in Corollary 5.11 and $e^{-iH_{\hbar}t/\hbar}D\left(\mathcal{N}^{\beta\frac{d}{2}}\right)\subseteq D\left(\mathcal{N}^{\beta}\right)$ from the item 1 in Theorem 6.6. Then, by Eq. (3.34), it follows that $W_{\hbar}(t,s)\,\mathcal{S}\subseteq\mathcal{S}$. By multiplying $W_{\hbar}(t,s)^{-1}=W_{\hbar}(s,t)$ on both sides of the last inclusion, we can conclude that $W_{\hbar}(t,s)\,\mathcal{S}=\mathcal{S}$. ■

**Definition 7.5.** *For $\hbar>0$ and $t\in\mathbb{R}$, $L_{\hbar}(t)$ be the operator on $\mathcal{S}$ defined as,*

$$\begin{aligned}L_{\hbar}(t)&=\frac{1}{\hbar}\left(H\left(a_{\hbar}+\alpha(t)\,,a_{\hbar}^{\dagger}+\bar{\alpha}(t)\right)-H^{cl}(\alpha(t))-H_{1}\left(\alpha(t):a_{\hbar},a_{\hbar}^{\dagger}\right)\right)\\&=H_{2}\left(\alpha(t):a,a^{\dagger}\right)+\frac{1}{\hbar}H_{\geq 3}\left(\alpha(t):a_{\hbar},a_{\hbar}^{\dagger}\right).\end{aligned} \tag{7.16}$$

**Theorem 7.6.** *Both $t\to W_{\hbar}(t,s)$ and $s\to W_{\hbar}(t,s)$ are strongly continuous on*

$L^2(m)$. *Moreover, if* $\psi \in \mathcal{S}$ *and* $\beta \geq 0$, *then*

$$i\left(\|\cdot\|_\beta \text{-}\partial_t\right) W_\hbar(t,s)\psi = L_\hbar(t) W_\hbar(t,s)\psi, \text{ and} \tag{7.17}$$

$$i\left(\|\cdot\|_\beta \text{-}\partial_s\right) W_\hbar(t,s)\psi = -W_\hbar(t,s) L_\hbar(s)\psi. \tag{7.18}$$

**Proof.** The strong continuity of $W_\hbar(t,s)$ in $s$ and in $t$ follows from the strong continuity of both $U(\alpha(t))$ and $e^{-iH_\hbar t/\hbar}$, see Corollary 5.10 and Proposition 6.1. The derivative formulas in Eqs. (7.17) and (7.18) follow directly from Lemma 7.2 and Theorem 7.3 along with the an additional term coming from the product rule involving the added scalar factor, $e^{\frac{i}{\hbar}[f(t)-f(s)]}$. ■

For the rest of Part I the following notation will be in force.

**Notation 7.7.** *Let* $\alpha_0 \in \mathbb{C}$, $H(\theta,\theta) \in \mathbb{R}\langle\theta,\theta^*\rangle$ *satisfy the Assumption 1.11,* $t \to \alpha(t)$ *solve the Hamiltonian' s equation Eq. (1.1) with* $\alpha(0) = \alpha_0$, *and* $H_2(\alpha(\tau):\theta,\theta^*)$ *be the degree* 2 *homogeneous component of* $H(\theta+\alpha(\tau),\theta^*+\overline{\alpha}(\tau))$ *as in Proposition 3.4. Further let*

$$W_0(t,s) := W_0(t) W_0^*(s) \tag{7.19}$$

*where* $W_0(t)$ *is the unique one parameter strongly continuous family of unitary operators satisfying,*

$$i\frac{\partial}{\partial t}W_0(t) = \overline{H_2(\alpha(t):a,a^\dagger)}W_0(t) \text{ with } W_0(0) = I \tag{7.20}$$

*as described in Corollary 5.10.*

**Remark 7.8.** *Since*

$$\frac{i}{\hbar}H_{\geq 3}\left(\alpha(t):a_\hbar,a_\hbar^\dagger\right) = i\sqrt{\hbar}\sum_{l\geq 3}\hbar^{(l-3)/2}H_l\left(\alpha(t),a,a^\dagger\right),$$

*it follows that $L_\hbar(t)$ in Eq. (7.16) satisfies,*

$$\lim_{\hbar\downarrow 0} L_\hbar(t)\,\psi = H_2\left(\alpha(t) : a, a^\dagger\right)\psi \text{ for all } \psi \in \mathcal{S}.$$

*From this observation it is reasonable to expect $W_\hbar(t) \to W_0(t)$ where $W_0(t)$ is as in Notation 7.7. This is in fact the key content of Part I, see Theorem 9.3 below. To complete the proof we will still need a fair number of preliminary results.*

# 1 Crude Bounds on $W_\hbar$

**Theorem 7.9.** *Suppose that $H(\theta,\theta^*) \in \mathbb{R}\langle\theta,\theta^*\rangle$ and $0 < \hbar < \eta \le 1$ satisfy Assumption 1.11, $d = \deg_\theta H$, and $W_\hbar(t,s)$ is as in Eq. (7.14). Then for all $\beta \ge 0$, there exists $C_{\beta,H} < \infty$ depending only on $\beta \ge 0$ and $H$ such that, for all $s,t \in \mathbb{R}$,*

$$\begin{aligned} &W_\hbar(t,s)\, D\left(\mathcal{N}^{\beta d/2}\right) \subset D\left(\mathcal{N}^\beta\right) \text{ and} \\ &\left\|\mathcal{N}^\beta W_\hbar(t,s)\,\psi\right\| \le \hbar^{-\beta} C_{\beta,H}\,\|\psi\|_{\frac{\beta d}{2}}. \end{aligned} \tag{7.21}$$

*[This bound is crude in the sense that $\hbar^{-\beta}C_{\beta,H} \uparrow \infty$ as $\hbar \downarrow 0$. We will do much better later in Theorem 9.1.]*

**Proof.** Let $\beta \ge 0$. From Proposition 7.4 it follows that $W_\hbar(t,s)\,D\left(\mathcal{N}^{\beta d/2}\right) \subseteq D\left(\mathcal{N}^\beta\right)$. Moreover,

$$\begin{aligned} \left\|\mathcal{N}^\beta W_\hbar(t,s)\,\psi\right\| &\le \|W_\hbar(t,s)\,\psi\|_\beta \\ &= \left\|U_\hbar(-\alpha(t))\, e^{-iH_\hbar(t-s)/\hbar} U_\hbar(\alpha(s))\,\psi\right\|_\beta \\ &\le \|U_\hbar^*(\alpha(t))\|_{\beta\to\beta} \left\|e^{-iH_\hbar(t-s)/\hbar}\right\|_{\beta d/2\to\beta} \|U_\hbar(\alpha(s))\|_{\beta d/2\to\beta d/2} \|\psi\|_{\beta d/2}. \end{aligned}$$

Note that $\kappa := \sup_{t\in\mathbb{R}} |\alpha(t)| < \infty$ from Proposition 3.8 , then by the Corollary

5.11, there exists a constant $C = C(\beta, d, \kappa)$ such that

$$\sup_{t\in\mathbb{R}} \|U_\hbar^*(\alpha(t))\|_{\beta\to\beta} \vee \sup_{s\in\mathbb{R}} \|U_\hbar(\alpha(s))\|_{\beta d/2\to\beta d/2} \leq C(\beta, d, \kappa).$$

Then, combing all above inequalities along with Eq. (6.8) in Theorem 6.6, we have

$$\left\|\mathcal{N}^\beta W_\hbar(t,s)\psi\right\| \leq C_{\beta,H}\hbar^{-\beta}\|\psi\|_{\beta d/2}$$

and therefore, Eq. (7.21) follows immediately. ■

# Chapter 8

# Asymptotics of the Truncated Evolutions

As in Chapter 7, we assume that $H(\theta,\theta^*) \in \mathbb{R}\langle\theta,\theta^*\rangle$ and $\eta > 0$ are as in Assumption 1.11, $\alpha_0 \in \mathbb{C}$, and $\alpha(t)$ denotes the solution to Eq. (1.1) with $\alpha(0) = \alpha_0$. Further let $L_\hbar(t)$ be as in Eq. (7.16), i.e.

$$L_\hbar(t) = \sum_{k=2}^{d} \hbar^{\frac{k}{2}-1} H_k\left(\alpha(t) : a, a^\dagger\right). \tag{8.1}$$

**Definition 8.1** (Truncated Evolutions)**.** *For $0 \leq M < \infty$ and $0 < \hbar < \infty$, let $L_\hbar^M(t) = \mathcal{P}_M L_\hbar(t) \mathcal{P}_M$ be the level $M$ truncation of $L_\hbar(t)$ (see Notation 3.34) and let $W_\hbar^M(t,s)$ be the associated **truncated evolution** defined to be the solution to the ordinary differential equation,*

$$i\frac{d}{dt}W_\hbar^M(t,s) = L_\hbar^M(t)\, W_\hbar^M(t,s) \;\; \text{with } W^M(s,s) = I \tag{8.2}$$

*as in Section 1 in Chapter 4. We further let $W_\hbar^M(t) = W_\hbar^M(t,0)$.*

From the results of Theorem 4.4 with $Q_M(t) = L_\hbar^M(t)$ and $U^M(t,s) =$

$W_\hbar^M(t,s)$, we know that $W_\hbar^M(t,s)$ is unitary on $L^2(m)$ and

$$W_\hbar^M(t,s) = W_\hbar^M(t,0)\, W_\hbar^M(0,s) = W_\hbar^M(t)\, W_\hbar^M(s)^*$$

and in particular, $W_\hbar^M(t)^* = W_\hbar^M(0,t)$.

**Proposition 8.2.** *Suppose that $H(\theta,\theta^*) \in \mathbb{R}\langle\theta,\theta^*\rangle$ and $\eta > 0$ satisfy Assumption 1.11, $d = \deg_\theta H > 0 \in 2\mathbb{N}$, and further let $W_\hbar(t,s)$, $W_0(t,s)$ and $W_\hbar^M(t,s)$ be as in Eq. (7.14), Notation 7.7, and Definition 8.1 respectively. If $\psi \in D\left(\mathcal{N}^{\frac{d}{2}}\right)$ and $0 < \hbar < \eta$, then*

$$W_\hbar(t,s)\,\psi - W_\hbar^M(t,s)\,\psi = i\int_s^t W_\hbar(t,\tau)\left[L_\hbar^M(\tau) - \overline{L_\hbar(\tau)}\right] W_\hbar^M(\tau,s)\,\psi d\tau \quad (8.3)$$

*and*

$$W_\hbar(t,s)\,\psi - W_0(t,s)\,\psi = i\int_s^t W_\hbar(t,\tau)\left[H_2(\alpha(\tau):\bar{a},a^*) - \overline{L_\hbar(\tau)}\right] W_0(\tau,s)\,\psi d\tau \quad (8.4)$$

*where $L_\hbar(t)$ and $H_2(\alpha(\tau):\bar{a},a^*)$ are as in Eqs. (7.16) and (7.20) and $L_\hbar^M(\tau) = \mathcal{P}_M L_\hbar(t)\,\mathcal{P}_M$ as in Definition 8.1. [The integrands in Eqs. (8.3) and (8.4) are $L^2(m)$-norm continuous functions of $\tau$ and therefore the integrals above are well defined.]*

**Proof.** Let $B\left(D\left(\mathcal{N}^{\frac{d}{2}}\right), L^2(m)\right)$ denote the space of bounded linear operators from $D\left(\mathcal{N}^{\frac{d}{2}}\right)$ to $L^2(m)$. The integrals in Eq. (8.3) and (8.4) may be interpreted as $L^2(m)$ – valued Riemann integrals because their integrands are $L^2(m)$ – continuous functions of $\tau$. This is consequence of the observations that both

$$F(\tau) := W_\hbar(t,\tau)\left[L_\hbar^M(\tau) - \overline{L_\hbar(\tau)}\right] W_\hbar^M(\tau,s) \text{ and}$$
$$G(\tau) := W_\hbar(t,\tau)\left[H_2(\alpha(\tau):\bar{a},a^*) - \overline{L_\hbar(\tau)}\right] W_0(\tau,s)$$

are strongly continuous $B\left(D\left(\mathcal{N}^{\frac{d}{2}}\right), L^2(m)\right)$ – valued functions of $\tau$. To verify this assertion recall that;

1. $\tau \to W_\hbar^M(\tau, s)$ is $\|\cdot\|_{d/2\to d/2}$ continuous by Item 3. of Theorem 4.4 and $\tau \to W_0(\tau, s)\psi$ is $\|\cdot\|_{\frac{d}{2}}$ – continuous by Corollary 5.10.

2. Both $L_\hbar^M(\tau) - \overline{L_\hbar(\tau)}$ and $H_2(\alpha(\tau) : \bar{a}, a^*) - \overline{L_\hbar(\tau)}$ are easily seen to be strongly continuous as functions of $\tau$ with values in $B\left(D\left(\mathcal{N}^{\frac{d}{2}}\right), L^2(m)\right)$ by using Corollary 3.30 and noting that the coefficients of the four operators depend continuously on $\tau$.

3. The map, $\tau \to W_\hbar(t, \tau)$ is strongly continuous on $L^2(m)$ by Theorem 7.6.

As strong continuity is preserved under operator products, it follows that both $F(\tau)$ and $G(\tau)$ are strongly continuous.

By Remark 4.6 and Proposition 7.4 we know that $W_\hbar^M(t, s)\mathcal{S} = \mathcal{S}$ and $W_\hbar(t, s)\mathcal{S} = \mathcal{S}$. Moreover, from item 3. of Theorem 4.4 and Theorem 7.6, if $\varphi \in \mathcal{S}$, then both $t \to W_\hbar^M(t, s)\varphi$ and $t \to W_\hbar(t, s)\varphi$ and are $\|\cdot\|_\beta$-differentiable for $\beta \geq 0$. Since $W_\hbar(t, s)$ is unitary (see Eq. (7.14)), it follows that $\sup_{t,s\in\mathbb{R}} \|W_\hbar(t, s)\|_{0\to 0} = 1$. Therefore, by applying Lemma 7.1 with $U(\tau) = W_\hbar(t, \tau)$, $P(\theta, \theta^*) = 1$, and $T(\tau) = W_\hbar^M(\tau, s)$ while making use of Eqs. (7.18) and (8.2) to find,

$$i\frac{d}{d\tau} W_\hbar(t, \tau) W_\hbar^M(\tau, s)\varphi = F(\tau)\varphi.$$

A similar arguments using Corollary 5.10 in place of Theorem 4.4 shows,

$$i\frac{d}{d\tau} W_\hbar(t, \tau) W_0(\tau, s)\varphi = G(\tau)\varphi.$$

Equations (8.3) and (8.4) now follow for $\psi = \varphi \in \mathcal{S}$ by integrating the last two displayed equations and making use of the fundamental theorem of calculus.

By the uniform boundedness principle (or by direct estimates already provided), it follows that

$$\sup_{\tau\in J_{s,t}} \|F(\tau)\|_{\frac{d}{2}\to 0} < \infty \text{ and } \sup_{\tau\in J_{s,t}} \|G(\tau)\|_{\frac{d}{2}\to 0} < \infty,$$

where $J_{s,t} := [\min(s,t), \max(s,t)]$. Because of these observation and the fact that $\mathcal{S}$ is dense in $D\left(\mathcal{N}^{\frac{d}{2}}\right)$, it follows that by a standard "$\varepsilon/3$ – argument" that Eqs. (8.3) and (8.4) are valid for all $\psi \in D\left(\mathcal{N}^{\frac{d}{2}}\right)$. ■

**Theorem 8.3.** *Let $0 < \eta \leq 1$, $H(\theta,\theta^*) \in \mathbb{R}\langle\theta,\theta^*\rangle$ be a polynomial of degree $d$ satisfying Assumption 1.11 and $d \geq 2$ be an even number. Then for all $\beta \geq d/2$ and $-\infty < S < T < \infty$, there exists a constant, $K(\beta,\alpha_0,H,S,T) < \infty$ such that*

$$\sup_{S<s,t<T} \left\| W_\hbar(t,s) - W_\hbar^{\hbar^{-1}}(t,s) \right\|_{\beta\to 0} \leq K(\beta,\alpha_0,H,S,T)\,\hbar^{\beta-1} \;\forall\; 0 < \hbar < \eta. \quad (8.5)$$

**Proof.** Since $W_\hbar(t,s)$ and $W_\hbar^{\hbar^{-1}}(t,s)$ are unitary from Theorem 4.4 and Eq. (7.14) and $\|\cdot\|_\beta \geq \|\cdot\|_0$ in Remark 3.22, it follows

$$\sup_{S<s,t<T} \left\| W_\hbar(t,s) - W_\hbar^{\hbar^{-1}}(t,s) \right\|_{\beta\to 0} \leq 1, \quad (8.6)$$

and hence Eq. (8.5) holds if $\eta \wedge d^{-1} \leq \hbar < \eta$. The remaining thing to show is Eq.(8.5) still holds for $0 < \hbar < \eta \wedge d^{-1}$.

Let $\psi \in D\left(\mathcal{N}^\beta\right) \subset D\left(\mathcal{N}^{d/2}\right)$. Taking the $L^2(m)$ – norm of Eq. (8.3) implies,

$$\left\| \left[W_\hbar(t,s) - W_\hbar^M(t,s)\right]\psi \right\| \leq \int_{J_{s,t}} \left\| W_\hbar(t,\tau)\left[L_\hbar^M(\tau) - \overline{L_\hbar(\tau)}\right] W_\hbar^M(\tau,s)\,\psi \right\| d\tau, \quad (8.7)$$

where

$$\begin{aligned}
&\left\|W_\hbar(t,\tau)\left[L_\hbar^M(\tau)-\overline{L_\hbar(\tau)}\right]W_\hbar^M(\tau,s)\psi\right\| \\
&\quad=\left\|\left[L_\hbar^M(\tau)-\overline{L_\hbar(\tau)}\right]W_\hbar^M(\tau,s)\psi\right\| \\
&\quad\le\left\|L_\hbar^M(\tau)-\overline{L_\hbar(\tau)}\right\|_{\beta\to 0}\left\|W_\hbar^M(\tau,s)\right\|_{\beta\to\beta}\|\psi\|_\beta . \qquad (8.8)
\end{aligned}$$

In order to simplify this estimate further, let

$$P(\hbar,t:\theta,\theta^*)=\sum_{k=2}^{d}\hbar^{\frac{k}{2}-1}H_k(\alpha(t):\theta,\theta^*),$$

in which case, $L_\hbar(t)=P\left(\hbar,t:a,a^\dagger\right)$. It follows from Corollary 3.37 with $\beta=0$ and $\alpha\to\beta$ that (for $M\ge d$)

$$\begin{aligned}
\left\|L_\hbar^M(\tau)-\overline{L_\hbar(\tau)}\right\|_{\beta\to 0}&\le\sum_{k=2}^{d}\hbar^{\frac{k}{2}-1}\left|H_k(\alpha(t):\theta,\theta^*)\right|(M-k+2)^{k/2-\beta} \\
&\le K(\alpha_0,H)\,\hbar^{-1}\sum_{k=2}^{d}(\hbar M-k\hbar+2\hbar)^{k/2}(M-k+2)^{-\beta}
\end{aligned}$$

and from Eq. (4.15) that

$$\begin{aligned}
\left\|W_\hbar^M(\tau,s)\right\|_{\beta\to\beta}&\le e^{K(\beta,d)(\hbar M+1)^{\frac{d}{2}-1}\sum_{k=2}^{d}\int_{J_{s,\tau}}\left|\hbar^{\frac{k}{2}-1}H_k(\alpha(\sigma):\theta,\theta^*)\right|d\sigma} \\
&\le e^{\tilde K(\beta,d,H)(\hbar M+1)^{\frac{d}{2}-1}|t-s|}.
\end{aligned}$$

Thus reducing to the case where $M=\hbar^{-1}$ (i.e. $M\hbar=1$) we see there exists $\tilde K(\beta,\alpha_0,H,S,T)<\infty$ such that

$$\left\|L_\hbar^{\hbar^{-1}}(\tau)-\overline{L_\hbar(\tau)}\right\|_{\beta\to 0}\left\|W_\hbar^{\hbar^{-1}}(\tau,s)\right\|_{\beta\to\beta}\le\tilde K(\beta,\alpha_0,H,S,T)\,\hbar^{\beta-1}$$

which combined with Eqs. (8.7) and (8.8) implies Eq. (8.5) with $K(\beta,\alpha_0,H,S,T)=$

$$\tilde{K}\left(\beta, \alpha_0, H, S, T\right)\left[T - S\right].$$

■

# Chapter 9

# Proof of the main Theorems

The next theorem combines the crude bound in Theorem 7.9 with the asymptotics of the truncated evolutions in Theorem 8.3 in order to give a much improved version of Theorem 7.9.

**Theorem 9.1** ($N$ – Sobolev Boundedness of $W_\hbar(t)$). *Suppose that $H(\theta, \theta^*) \in \mathbb{R}\langle\theta, \theta^*\rangle$ and $\eta > 0$ satisfy Assumption 1.11, $d = \deg_\theta H > 0 \in 2\mathbb{N}$, and $W_\hbar(t, s)$ and $W_\hbar(t)$ be as in Eqs. (7.14) and (7.13) respectively. Then for each $\beta \geq 0$, $-\infty < S < T < \infty$, there exists $K_\beta(S, T) < \infty$ such that for all $\psi \in D\left(\mathcal{N}^{(2\beta+1)d}\right)$, all $0 < \hbar < \eta \leq 1$, and all $S \leq s, t \leq T$ we have*

$$\left\|\mathcal{N}^\beta W_\hbar(t, s)\psi\right\| \leq K_\beta(S, T)\|\psi\|_{(2\beta+1)d}, \tag{9.1}$$

*and*

$$\sup_{S \leq s, t \leq T}\|W_\hbar(t, s)\|_{(2\beta+1)d \to \beta} \leq \tilde{K}_\beta(S, T), \tag{9.2}$$

*where*

$$\tilde{K}_\beta(S, T) := (1 + K_\beta(S, T))\, 2^{(\beta-1)_+}. \tag{9.3}$$

*In particular this estimate implies, for* $0<\hbar<\eta\leq 1$,

$$\sup_{S\leq t\leq T}\left[\left\Vert W_{\hbar}\left(t\right)\right\Vert_{\left(2\beta+1\right)d\rightarrow\beta}\vee\left\Vert W_{\hbar}^{\ast}\left(t\right)\right\Vert_{\left(2\beta+1\right)d\rightarrow\beta}\right]\leq\tilde{K}_{\beta}\left(S,T\right). \tag{9.4}$$

*[The bound in Eq. (9.2) improves on the crude bound in Eq. (8.5) in that the bound now does not blow up as* $\hbar\downarrow 0$.*]*

**Remark 9.2.** *The bound in Eq.(9.1) is not tight in that the index,* $\left(2\beta+1\right)d$*, of the norm on the right side of this equation is not claimed to be optimal.*

**Proof.** The case $\beta=0$ is a trivial and so we now assume $\beta>0$. If $\psi\in D\left(\mathcal{N}^{\left(2\beta+1\right)d}\right)$, then by Proposition 7.4 $W_{\hbar}\left(t,s\right)\psi\in D\left(\mathcal{N}^{2\left(2\beta+1\right)}\right)$. Some simple algebra then shows $\left\langle W_{\hbar}\left(t,s\right)\psi,\mathcal{N}^{2\beta}W_{\hbar}\left(t,s\right)\psi\right\rangle=A+B$, where

$$\begin{aligned}
A&:=\left\langle W_{\hbar}^{\hbar^{-1}}\left(t,s\right)\psi,\mathcal{N}^{2\beta}W_{\hbar}^{\hbar^{-1}}\left(t,s\right)\psi\right\rangle\text{ and}\\
B&:=\left\langle\left[W_{\hbar}\left(t,s\right)-W_{\hbar}^{\hbar^{-1}}\left(t,s\right)\right]\psi,\mathcal{N}^{2\beta}W_{\hbar}\left(t,s\right)\psi\right\rangle\\
&\quad+\left\langle\mathcal{N}^{2\beta}W_{\hbar}^{\hbar^{-1}}\left(t,s\right)\psi,\left[W_{\hbar}\left(t,s\right)-W_{\hbar}^{\hbar^{-1}}\left(t,s\right)\right]\psi\right\rangle.
\end{aligned}$$

The $\left\vert B\right\vert$ term is bounded by the following two terms.

$$\begin{aligned}
\left\vert B\right\vert&\leq\left\Vert\left[W_{\hbar}\left(t,s\right)-W_{\hbar}^{\hbar^{-1}}\left(t,s\right)\right]\psi\right\Vert\cdot\left\Vert\mathcal{N}^{2\beta}W_{\hbar}\left(t,s\right)\psi\right\Vert\\
&\quad+\left\Vert\left[W_{\hbar}\left(t,s\right)-W_{\hbar}^{\hbar^{-1}}\left(t,s\right)\right]\psi\right\Vert\cdot\left\Vert\mathcal{N}^{2\beta}W_{\hbar}^{\hbar^{-1}}\left(t,s\right)\psi\right\Vert.
\end{aligned}$$

Therefore, using Eq. (4.15) in Corollary 4.7, Theorem 8.3 with $\beta$ replaced by $\frac{d}{2}+2\beta$, and Theorem 7.9, it follows that

$$\begin{aligned}|B| &\leq \left\| \left[ W_{\hbar}(t,s) - W_{\hbar}^{\hbar^{-1}}(t,s) \right] \psi \right\| \cdot \left( \left\| \mathcal{N}^{2\beta} W_{\hbar}(t,s)\psi \right\| + \left\| \mathcal{N}^{2\beta} W_{\hbar}^{\hbar^{-1}}(t,s)\psi \right\| \right) \\ &\leq C\hbar^{2\beta+\frac{d}{2}-1} \|\psi\|_{\frac{d}{2}+2\beta} \cdot \left( \hbar^{-2\beta} \left\| (\mathcal{N}+I)^{\beta d} \psi \right\| + \left\| (\mathcal{N}+I)^{2\beta} \psi \right\| \right) \\ &\leq C\hbar^{\frac{d}{2}-1} \|\psi\|_{\frac{d}{2}+2\beta} \left( \|\psi\|_{\beta d} + \hbar^{2\beta} \|\psi\|_{2\beta} \right) \\ &\leq C\hbar^{\frac{d}{2}-1} \|\psi\|_{(2\beta+1)d}^2 < \infty \text{ for all } S \leq s,t \leq T \text{ and } 0 < \hbar < \eta. \end{aligned} \tag{9.5}$$

In the last inequality we have used, $\frac{d}{2} + 2\beta \leq (2\beta+1)\,d$ when $\beta > 0$ and $d \geq 2$. Corollary 4.7 directly implies there exists $C > 0$ such that

$$|A| = \left\| \mathcal{N}^{\beta} W_{\hbar}^{\hbar^{-1}}(t,s)\psi \right\|_{\beta}^2 \leq C \|\psi\|_{\beta}^2 \leq C \|\psi\|_{(2\beta+1)d}^2$$

for all $S \leq s,t \leq T$ and therefore, we get

$$\left\| \mathcal{N}^{\beta} W_{\hbar}(t,s)\psi \right\|^2 = \left\langle W_{\hbar}(t,s)\psi, \mathcal{N}^{2\beta} W_{\hbar}(t,s)\psi \right\rangle \leq (K_{\beta}(S,T))^2 \|\psi\|_{(2\beta+1)d}^2 \tag{9.6}$$

for an appropriate constant $K_{\beta}(S,T)$. Equation (9.1) is proved and Eq. (9.2) is a consequence of Eq. (9.1) and the inequality in Eq. (6.5). Equation (9.2) also implies Eq. (9.4) because $W_{\hbar}(t) = W_{\hbar}(t,0)$and $W_{\hbar}^{*}(t) = W_{\hbar}(0,t)$. ■

**Theorem 9.3.** *Suppose that $H(\theta,\theta^*) \in \mathbb{R}\langle\theta,\theta^*\rangle$ and $0 < \eta \leq 1$ satisfy Assumptions 1.11. Let $d = \deg_{\theta} H \in 2\mathbb{N}$, $W_{\hbar}(t,s)$, and $W_0(t,s)$ be as in Eq. (7.14) and Notation 7.7 respectively. Then $W_{\hbar}(t,s) \xrightarrow{s} W_0(t,s)$ as $\hbar \downarrow 0$. Moreover for all $\beta \geq 0$ and $-\infty < S < T < \infty$ there exists $K = K_{\beta}(S,T) < \infty$ such that, for $0 < \hbar < \eta \leq 1$,*

$$\sup_{S \leq s,t \leq T} \left\| \mathcal{N}^{\beta} \left( W_0(t,s) - W_{\hbar}(t,s) \right) \psi \right\| \leq K\sqrt{\hbar}\, \|\psi\|_{\frac{d}{2}(4\beta+3)} \;\; \forall\, \psi \in D\left( \mathcal{N}^{\frac{d}{2}(4\beta+3)} \right) \tag{9.7}$$

*and, with* $\tilde{K} := (1+K)\, 2^{(\beta-1)_+}$,

$$\sup_{s,t\in[S,T]} \|W_0(t,s) - W_\hbar(t,s)\|_{\frac{d}{2}(4\beta+3)\to\beta} \le \tilde{K}\sqrt{\hbar}. \tag{9.8}$$

*In particular, for* $0<\hbar<\eta\le 1$,

$$\sup_{S\le t\le T} \|W_0(t) - W_\hbar(t)\|_{\frac{d}{2}(4\beta+3)\to\beta} \vee \|W_0^*(t) - W_\hbar^*(t)\|_{\frac{d}{2}(4\beta+3)\to\beta} \le \tilde{K}\sqrt{\hbar}. \tag{9.9}$$

**Proof.** The claimed strong convergence now follows from Eq. (9.7) with $\beta = 0$ along with a standard density argument. To simplify notation, let

$$p = d(2\beta+1) \text{ and } q = \frac{d}{2}(4\beta+3) = p + \frac{d}{2}.$$

If $\psi \in D(\mathcal{N}^q) \subseteq D\left(\mathcal{N}^{\frac{d}{2}}\right)$, then by Eq. (8.4) in Proposition 8.2, Eq. (7.16), and Corollary 3.30,

$$\begin{aligned} W_\hbar(t,s)\psi - W_0(t,s)\psi &= i\int_s^t W_\hbar(t,\tau)\left[H_2(\alpha(\tau):\bar{a},a^*) - \bar{L}_\hbar(\tau)\right] W_0(\tau,s)\psi d\tau \\ &= i\int_s^t W_\hbar(t,\tau)\left[\frac{1}{\hbar}H_{\ge 3}(\alpha(\tau):\bar{a}_\hbar,a_\hbar^*)\right] W_0(\tau,s)\psi d\tau. \end{aligned}$$

Then, by using theorem 9.1, we find for all $0<\hbar<\eta\le 1$ and $S\le s,t\le T$ (with

$d = \deg_\theta H)$ that

$$
\begin{aligned}
&\left\|\left(W_\hbar(t,s) - W_0(t,s)\right)\psi\right\|_\beta \\
&\leq \int_{J_{s,t}} \left\| W_\hbar(t,\tau)\left[\frac{1}{\hbar}H_{\geq 3}\left(\alpha(\tau) : \bar{a}_\hbar, a_\hbar^*\right)\right] W_0(\tau,s)\,\psi \right\|_\beta d\tau \\
&\leq \int_S^T \left\|W_\hbar(t,\tau)\right\|_{p\to\beta} \left\| \left[\frac{1}{\hbar}H_{\geq 3}\left(\alpha(\tau) : \bar{a}_\hbar, a_\hbar^*\right)\right] W_0(\tau,s)\,\psi \right\|_p d\tau \\
&\leq K \int_S^T \left\| \frac{1}{\hbar}H_{\geq 3}\left(\alpha(\tau) : \bar{a}_\hbar, a_\hbar^*\right) \right\|_{q\to p} \left\|W_0(t,\tau)\right\|_{q\to q} \|\psi\|_q \, d\tau \\
&\leq K\sqrt{\hbar} \int_S^T \left\| H_{\geq 3}\left(\alpha(\tau), \sqrt{\hbar} : \bar{a}, a^*\right) \right\|_{q\to p} \left\|W_0(t,\tau)\right\|_{q\to q} d\tau \, \|\psi\|_q , \qquad (9.10)
\end{aligned}
$$

where $H_{\geq 3}\left(\alpha(\tau), \sqrt{\hbar} : \theta, \theta^*\right) \in \mathbb{R}\left[\alpha(\tau), \sqrt{\hbar}\right]\langle\theta,\theta^*\rangle$ is a polynomial in $\left(\alpha(\tau), \sqrt{\hbar}, \theta, \theta^*\right)$ which is a sum of terms homogeneous of degree three or more in the $\{\theta,\theta^*\}$ – grading. By Eq. (3.45) in Corollary 3.30 and Eq. (5.27) in Corollary 5.10,

$$
\sup_{S\leq t\leq T} \int_S^T \left\| H_{\geq 3}\left(\alpha(\tau) : \bar{a}_\hbar, a_\hbar^*\right) \right\|_{q\to p} \left\|W_0(t,\tau)\right\|_{q\to q} d\tau < \infty
$$

which along with Eq. (9.10) completes the proof of Eq. (9.7). Equation (9.8) follows directly from Eq. (9.7) after making use of Eq. (6.5). Equation (9.9) is a special case of Eq. (9.8) because of the identities; $W_\hbar(t) = W_\hbar(t,0)$, $W_\hbar^*(t) = W_\hbar(0,t)$, $W_0(t) = W_0(t,0)$ and $W_0(t)^* = W_0(0,t)$. ■

# 1 Proof of Theorem 1.17

We now finish Part I by showing that Eqs. (9.4) and (9.9) can be used to prove the main theorems of Part I, namely Theorem 1.17 and Corollaries 1.19 and 1.21. For the rest of Chapter 9, we always assume that $H \in \mathbb{R}\langle\theta,\theta^*\rangle$ and $1 \geq \eta > 0$ satisfy Assumption 1.11, $d = \deg_\theta H > 0 \in 2\mathbb{N}$, $W_\hbar(t)$ is defined as in Eq. (7.13), and $W_0(t)$ is as in Notation 7.7.

**Notation 9.4.** *For $\hbar \geq 0$, let*

$$a(\hbar : t) := W_\hbar^*(t)\, a W_\hbar(t) \;\; \text{and } a^\dagger(\hbar : t) := W_\hbar^*(t)\, a^\dagger W_\hbar(t) \tag{9.11}$$

*as operator on $\mathcal{S}$. It should be noted that under Assumption 1.11 we have $a^\dagger(\hbar : t) = a(\hbar : t)^\dagger$ for $0 \leq \hbar < \eta$.*

According to Theorem 5.13, if $a(t)$ and $a^\dagger(t)$ are as in Eqs. (1.8) and (1.9) respectively then satisfies,

$$a(t) = W_0^*(t)\, a W_0(t) = a(0 : t) \;\; \text{and} \tag{9.12}$$

$$a^\dagger(t) = W_0^*(t)\, a^\dagger W_0(t) = a^\dagger(0 : t) \tag{9.13}$$

as operators on $\mathcal{S}$. For this reason we will typically write $a(t)$ and $a^\dagger(t)$ for $a(0 : t)$ and $a(0 : t)$ respectively.

By Proposition 2.4 and Eq.(7.13), the operator $A_\hbar(t)$ defined in Eq. (1.22) satisfies,

$$\begin{aligned} U_\hbar^*(\alpha_0)\, A_\hbar(t)\, U_\hbar(\alpha_0) &= U_\hbar^*(\alpha_0)\, e^{itH_\hbar/\hbar} a_\hbar e^{-itH_\hbar/\hbar} U_\hbar(\alpha_0) \\ &= W_\hbar^*(t)\left(a_\hbar + \alpha(t)\right) W_\hbar(t) \\ &= \alpha(t) + \sqrt{\hbar} W_\hbar^*(t)\, a W_\hbar(t) \\ &= \alpha(t) + \sqrt{\hbar}\; a(\hbar : t) \;\; \text{on } \mathcal{S}. \end{aligned} \tag{9.14}$$

**Notation 9.5.** *For $t \in \mathbb{R}$ and $0 \leq \hbar < \eta$, let*

$$B_\theta(\hbar : t) := \overline{a(\hbar : t)} = W_\hbar^*(t)\, \bar{a} W_\hbar(t) \;\; \text{and}$$

$$B_{\theta^*}(\hbar : t) := a(\hbar : t)^* = W_\hbar^*(t)\, a^* W_\hbar(t)\,.$$

*When $\hbar = 0$ we will denote $B_b(0 : t)$ more simply as $B_b(t)$ for $b \in \{\theta, \theta^*\}$.*

**Lemma 9.6.** *Let $\eta > 0$ and $d > 0 \in 2\mathbb{N}$ be as in Theorem 9.1, $b \in \{\theta, \theta^*\}$, $t \in [S,T]$, and $B_b(\hbar : t)$ be as in Notation 9.5. Then, for any $\beta \geq 0$, there exists a constant $C(\beta, S, T) > 0$ such that*

$$\sup_{t\in[S,T]} \max_{b\in\{\theta,\theta^*\}} \|B_b(\hbar : t)\|_{g(\beta)\to\beta} \leq C(\beta, S, T) \text{ for } 0 < \hbar < \eta \tag{9.15}$$

*where $g(\beta) = 4d^2\beta + 2d(d+1)$.*

**Proof.** For definiteness, suppose that $b = \theta^*$ as the case $b = \theta$ is proved analogously. If $q = (2\beta + 1)d$ and

$$p = \left[2\left(q + \frac{1}{2}\right) + 1\right] d = 4d^2\beta + 2d(d+1),$$

then

$$\|B_b(\hbar : t)\|_{p\to\beta} \leq \|W_\hbar^*(t)\|_{q\to\beta} \|a^*\|_{q+\frac{1}{2}\to q} \|W_\hbar(t)\|_{p\to q+\frac{1}{2}}$$

which combined with the estimates in Eqs. (3.41) and (9.4) gives the estimate in Eq. (9.15). ■

**Lemma 9.7.** *Let $\beta \geq 0$, $b \in \{\theta, \theta^*\}$, $-\infty < S < T < \infty$, $\eta > 0$, and $d > 0 \in 2\mathbb{N}$ be the same as Lemma 9.6. Then there exists a constant $C(\beta, S, T) > 0$ such that*

$$\sup_{t\in[S,T]} \|B_b(\hbar : t) - B_b(t)\|_{r(\beta)\to\beta} \leq C(\beta, S, T)\sqrt{\hbar} \text{ for } 0 \leq \hbar < \eta \tag{9.16}$$

*where $r(\beta) = (4d^2)\beta + (3d+2)d$.*

**Proof.** Let us suppose that $b = \theta$ as the proof for $b = \theta^*$ is very similar.

Given $p \geq \beta$ (to be chosen later) we have,

$$\begin{aligned}
&\left\| B_b\left(\hbar : t\right) - B_b\left(t\right) \right\|_{p\rightarrow\beta} \\
&\quad = \left\| W_\hbar^*\left(t\right) \bar{a} W_\hbar\left(t\right) - W_0^*\left(t\right) \bar{a} W_0\left(t\right) \right\|_{p\rightarrow\beta} \\
&\quad \leq \left\| \left[W_\hbar^*\left(t\right) - W_0^*\left(t\right)\right] \bar{a} W_\hbar\left(t\right) \right\|_{p\rightarrow\beta} + \left\| W_0^*\left(t\right) \bar{a} \left[W_\hbar\left(t\right) - W_0\left(t\right)\right] \right\|_{p\rightarrow\beta}. \quad (9.17)
\end{aligned}$$

Using Eqs. (3.41), (9.4), and (9.9), there exists a constant $C_1 := C_1\left(\beta, S, T\right)$ such that the first term will become

$$\begin{aligned}
&\left\| \left[W_\hbar^*\left(t\right) - W_0^*\left(t\right)\right] \bar{a} W_\hbar\left(t\right) \right\|_{p_1\rightarrow\beta} \\
&\qquad \leq \left\| \left[W_\hbar^*\left(t\right) - W_0^*\left(t\right)\right] \right\|_{q_1\rightarrow\beta} \left\| \bar{a} \right\|_{q_1+\frac{1}{2}\rightarrow q_1} \left\| W_\hbar\left(t\right) \right\|_{p_1\rightarrow q_1+\frac{1}{2}} \leq C_1\sqrt{\hbar}
\end{aligned}$$

where

$$q_1 = \frac{d}{2}\left(4\beta+3\right) \text{ and } p_1 = \left(2\left(q_1+\frac{1}{2}\right)+1\right)d = \left(4d^2\right)\beta + \left(3d+2\right)d.$$

Likewise, using Eqs. (3.41), (5.27) and (9.9), there exists a constant $C_2 := C_2\left(\beta, S, T\right)$ such that the second term will become

$$\begin{aligned}
&\left\| W_0^*\left(t\right) \bar{a} \left[W_\hbar\left(t\right) - W_0\left(t\right)\right] \right\|_{p_2\rightarrow\beta} \\
&\qquad \leq \left\| W_0^*\left(t\right) \right\|_{q_2\rightarrow\beta} \left\| \bar{a} \right\|_{q_2+\frac{1}{2}\rightarrow q_2} \left\| W_\hbar\left(t\right) - W_0\left(t\right) \right\|_{p_2\rightarrow q_2+\frac{1}{2}} \leq C_2\sqrt{\hbar}
\end{aligned}$$

where

$$q_2 = \beta \text{ and } p_2 = \frac{d}{2}\left(4\left(q_2+\frac{1}{2}\right)+3\right) = \left(2d\right)\beta + \frac{5d}{2}.$$

Since $d \geq 2$ and $\beta \geq 0$, it follows that $p_2 \leq p_1$ and so taking $p = p_1$ in Eq. (9.17) and making use of the previous estimates proves Eq. (9.16). ■

**Notation 9.8.** *For $n \in \mathbb{N}$, let $d = \deg_\theta H > 0$ and*

$$\sigma_n := \left(4d^2\right) 2d\,(d+1)\,\frac{\left(4d^2\right)^n - 1}{4d^2 - 1} + (3d+2)\,d. \tag{9.18}$$

**Lemma 9.9.** *Let $S$, $T$, $d$ and $\eta$ be the same as Lemma 9.6 and $\sigma_n$ be as in Notation 9.5 for $n \in \mathbb{N}$. Then there exists $C_n(S,T) < \infty$ such that for any $\mathbf{b} = (b_1, \dots, b_n) \in \{\theta, \theta^*\}^n$, $0 \le \hbar < \eta$, and $(t_1, \dots, t_n) \in [S,T]$ we have*

$$\left\| B_1(\hbar) \dots B_n(\hbar) - B_1 \dots B_n \right\|_{\sigma_n \to 0} \le C_n(S,T)\sqrt{\hbar}, \tag{9.19}$$

*where $B_i(\hbar) := B_{b_i}(\hbar : t_i)$ and $B_i := B_i(0) = B_{b_i}(t_i)$ for $1 \le i \le n$, see Notation 9.5.*

**Proof.** By a telescoping series arguments,

$$\begin{aligned} &B_1(\hbar) \dots B_n(\hbar) - B_1 \dots B_n \\ &\qquad = \sum_{i=1}^{n} \left[ B_1(\hbar) \dots B_i(\hbar) B_{i+1} \dots B_n - B_1(\hbar) \dots B_{i-1}(\hbar) B_i \dots B_n \right] \\ &\qquad = \sum_{i=1}^{n} B_1(\hbar) \dots B_{i-1}(\hbar) \left[ B_i(\hbar) - B_i \right] B_{i+1} \dots B_n \end{aligned}$$

and therefore

$$\begin{aligned} &\left\| B_1(\hbar) \dots B_n(\hbar) - B_1 \dots B_n \right\|_{\sigma_n \to 0} \\ &\qquad \le \sum_{i=1}^{n} \left\| B_1(\hbar) \dots B_{i-1}(\hbar) \left[ B_i(\hbar) - B_i \right] B_{i+1} \dots B_n \right\|_{\sigma_n \to 0}. \end{aligned} \tag{9.20}$$

To finish the proof it suffices to show for $1 \le i \le n$ that

$$\left\| B_1(\hbar) \dots B_{i-1}(\hbar) \left[ B_i(\hbar) - B_i \right] B_{i+1} \dots B_n \right\|_{\sigma_n \to 0} \le C\sqrt{\hbar}.$$

Now

$$\begin{aligned}
&\left\|B_1(\hbar)\ldots B_{i-1}(\hbar)\left[B_i(\hbar)-B_i\right]B_{i+1}\ldots B_n\right\|_{\sigma_n\to 0}\\
&\qquad\leq \left\|B_1(\hbar)\ldots B_{i-1}(\hbar)\right\|_{v\to 0}\left\|B_i(\hbar)-B_i\right\|_{u\to v}\left\|B_{i+1}\ldots B_n\right\|_{\sigma_n\to u}
\end{aligned}$$

where we will choose all $\sigma_n$, $u$, and $v\geq 0$ appropriately. First off if $\beta\geq 0$ and $\mathcal{A}=\overline{a}$ or $a^*$, then (see Proposition 3.29) $\mathcal{A}:D\left(\mathcal{N}^{\beta+\frac{1}{2}}\right)\to D\left(\mathcal{N}^{\beta}\right)$ and (see Corollary 5.10) $W_0(t):\mathcal{N}^{\beta}\to\mathcal{N}^{\beta}$ are bounded operators and therefore,

$$\left\|B_{i+1}\ldots B_n\right\|_{\sigma_n\to u}<\infty \text{ if } \sigma_n=u+\frac{1}{2}(n-i). \tag{9.21}$$

Also, with $r(v)$ as in Lemma 9.7, there exists $C$ such that, for $0<\hbar<\eta$,

$$\left\|B_i(\hbar)-B_i\right\|_{u\to v}\leq C\sqrt{\hbar} \text{ if } u=r(v). \tag{9.22}$$

Using Lemma 9.6, there exists $C>0$ such that, for $0<\hbar<\eta$,

$$\left\|B_1(\hbar)\ldots B_{i-1}(\hbar)\right\|_{v\to 0}\leq C$$

provided that

$$v=g^{i-1}(0)=2d(d+1)\frac{\left(4d^2\right)^i-1}{4d^2-1}. \tag{9.23}$$

If we let $1\leq i\leq n$ and

$$\begin{aligned}
\sigma_n(i)&=r\left(g^{i-1}(0)\right)+\frac{1}{2}(n-i)\\
&=\left(4d^2\right)2d(d+1)\frac{\left(4d^2\right)^i-1}{4d^2-1}+(3d+2)d+\frac{1}{2}(n-i),
\end{aligned}$$

then the by the above bounds it follows that

$$\left\| B_1(\hbar) \dots B_{i-1}(\hbar) \left[ B_i(\hbar) - B_i \right] B_{i+1} \dots B_n \right\|_{\sigma_n(i) \to 0} < \infty. \tag{9.24}$$

One shows $\sigma_n(i)$ is increasing in $i$ and therefore $\max_{1 \le i \le n} \sigma_n(i) = \sigma_n(n) = \sigma_n$ where $\sigma_n$ is as in Notation 9.8. Equation (9.19) now follows from Eqs. (9.20) and (9.24) with $\sigma_n(i)$ increased to $\sigma_n$. ■

We finish the proof of Theorem 1.17 with Lemma 9.9.

**Proof of Theorem 1.17.** Note that we have already shown that $A_\hbar(t_i)$ and $A_\hbar^\dagger(t_i)$ preserve $\mathcal{S}$ from Eq. (6.1) and $U_\hbar(\alpha_0)\mathcal{S} = \mathcal{S}$ and $U_\hbar(\alpha_0)^* \mathcal{S} = \mathcal{S}$ from Proposition 2.4. To show Eq.(1.23), for $\psi \in \mathcal{S}$, we have

$$\begin{aligned}
&\left\langle P\left(\left\{A_\hbar(t_i) - \alpha(t_i), A_\hbar^\dagger(t_i) - \overline{\alpha}(t_i)\right\}_{i=1}^n\right)\right\rangle_{U_\hbar(\alpha_0)\psi} \\
&= \left\langle P\left(\left\{U_\hbar^*(\alpha_0) A_\hbar(t_i) U_\hbar(\alpha_0) - \alpha(t_i), U_\hbar^*(\alpha_0) A_\hbar^\dagger(t_i) U_\hbar(\alpha_0) - \overline{\alpha}(t_i)\right\}_{i=1}^n\right)\right\rangle_\psi \\
&= \left\langle P\left(\left\{\sqrt{\hbar} a(\hbar : t_i), \sqrt{\hbar} a^\dagger(\hbar : t_i)\right\}_{i=1}^n\right)\right\rangle_\psi
\end{aligned} \tag{9.25}$$

where $\langle \cdot \rangle_\psi$ is defined in Definition 1.7 and the last step is asserted by Eq. (9.14). Supposed $p = \deg\left(P\left(\{\theta, \theta^*\}_{i=1}^n\right)\right)$ and $p_{\min}$ is then minimum degree of each non-constant term in $P\left(\{\theta_i, \theta_i^*\}_{i=1}^n\right)$. As $p = 0$ is a trivial case, we assume $p > 0$. Then, it follows

$$P\left(\{\theta_i, \theta_i^*\}_{i=1}^n\right) = P_0 + \sum_{k=p_{min}}^{p} P_k\left(\{\theta_i, \theta_i^*\}_{i=1}^n\right) \tag{9.26}$$

where $P_0 \in \mathbb{C}$ and

$$P_k\left(\{\theta_i, \theta_i^*\}_{i=1}^n\right) = \sum_{b_1, \dots, b_k \in \left\{\theta_i, \theta_i^*\right\}_{i=1}^n} c(b_1, \dots, b_k)\, b_1 \dots b_k$$

is a homogeneous polynomial of $\{\theta_i, \theta_i^*\}_{i=1}^n$ with degree $k$. Plugging Eq.(9.26) into

Eq.(9.25) gives,

$$\begin{aligned}&\left\langle P\left(\left\{\sqrt{\hbar}a\left(\hbar:t_i\right),\sqrt{\hbar}a^\dagger\left(\hbar:t_i\right)\right\}_{i=1}^n\right)\right\rangle_\psi\\ =&P_0+\sum_{k=p_{\min}}^{p}\hbar^{\frac{k}{2}}\left\langle P_k\left(\left\{a\left(\hbar:t_i\right),a^\dagger\left(\hbar:t_i\right)\right\}_{i=1}^n\right)\right\rangle_\psi\end{aligned}\tag{9.27}$$

wherein we have used the fact that $P_k$ is a homogeneous polynomial of degree $k$ in $\{\theta_i,\theta_i^*\}_{i=1}^n$. By Lemma 9.9, for $0<\hbar<\eta$, we have

$$\left\|P_k\left(\left\{a\left(\hbar:t_i\right),a^\dagger\left(\hbar:t_i\right)\right\}_{i=1}^n\right)\psi\right\|=\left\|P_k\left(\left\{a\left(t_i\right),a^\dagger\left(t_i\right)\right\}_{i=1}^n\right)\psi\right\|+O\left(\sqrt{\hbar}\right).$$

Therefore, for $k\geq 1$, we have

$$\begin{aligned}&\hbar^{\frac{k}{2}}\left\langle P_k\left(\left\{a\left(\hbar:t_i\right),a^\dagger\left(\hbar:t_i\right)\right\}_{i=1}^n\right)\right\rangle_\psi\\ &\qquad=\hbar^{\frac{k}{2}}\left\langle P_k\left(\left\{a\left(t_i\right),a^\dagger\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_\psi+O\left(\hbar^{\frac{k+1}{2}}\right).\end{aligned}\tag{9.28}$$

Applying Eq.(9.28) to Eq.(9.27), we have

$$\begin{aligned}&\left\langle P\left(\left\{\sqrt{\hbar}a\left(\hbar:t_i\right),\sqrt{\hbar}a^\dagger\left(\hbar:t_i\right)\right\}_{i=1}^n\right)\right\rangle_\psi\\ &\quad=P_0+\sum_{k=p_{\min}}^{p}\hbar^{\frac{k}{2}}\left\langle P_k\left(\left\{a\left(t_i\right),a^\dagger\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_\psi+O\left(\hbar^{\frac{k+1}{2}}\right)\\ &\quad=\left\langle P\left(\left\{\sqrt{\hbar}a\left(t_i\right),\sqrt{\hbar}a^\dagger\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_\psi+O\left(\hbar^{\frac{p_{\min}+1}{2}}\right).\end{aligned}$$

Therefore, Eq.(1.23) follows immediately. ■

# 2 Proof of Corollary 1.19

Let $P\left(\{\theta_i,\theta_i^*\}_{i=1}^n\right)\in\mathbb{C}\left\langle\{\theta_i,\theta_i^*\}_{i=1}^n\right\rangle$ be a non-commutative polynomial, $\psi\in\mathcal{S}$ and $\{t_1,\ldots,t_n\}\subseteq\mathbb{R}$. With out loss of generality, we assume $\deg\left(P\right)\geq 1$. We define, (may see Notation 2.16),

$$\begin{aligned}\widetilde{P}\left(\{\alpha(t_i):\theta_i,\theta_i^*\}_{i=1}^n\right) &= P\left(\{\theta_i+\alpha(t_i),\theta_i^*+\overline{\alpha}(t_i)\}_{i=1}^n\right)\\ &\in \mathbb{C}\left[\left\{\alpha(t_i),\overline{\alpha(t_i)}\right\}_{i=1}^n\right]\langle\{\theta_i,\theta_i^*\}_{i=1}^n\rangle.\end{aligned}$$

Note that $\deg_\theta\left(\widetilde{P}\right)=\deg(P)$ (see Notation 2.16) and $\widetilde{p}_{\min}\geq 1$ because $\deg\left(\widetilde{P}\right)\geq 1$. By Theorem 1.17 , for $0<\hbar<\eta$, we have

$$\begin{aligned}&\left\langle P\left(\left\{A_\hbar(t_i),A_\hbar^\dagger(t_i)\right\}_{i=1}^n\right)\right\rangle_{U_\hbar(\alpha_0)\psi}\\ &=\left\langle\widetilde{P}\left(\left\{\alpha(t_i):A_\hbar(t_i)-\alpha(t_i),A_\hbar^\dagger(t_i)-\overline{\alpha}(t_i)\right\}_{i=1}^n\right)\right\rangle_{U_\hbar(\alpha_0)\psi}\\ &=\left\langle\widetilde{P}\left(\left\{\alpha(t_i):\sqrt{\hbar}a(t_i),\sqrt{\hbar}a^\dagger(t_i)\right\}_{i=1}^n\right)\right\rangle_\psi+O\left(\hbar^{\frac{\widetilde{p}_{\min}+1}{2}}\right)\\ &=\left\langle P\left(\left\{\alpha(t_i)+\sqrt{\hbar}a(t_i),\overline{\alpha}(t_i)+\sqrt{\hbar}a^\dagger(t_i)\right\}_{i=1}^n\right)\right\rangle_\psi+O\left(\hbar^{\frac{\widetilde{p}_{\min}+1}{2}}\right)\\ &=\left\langle P\left(\left\{\alpha(t_i)+\sqrt{\hbar}a(t_i),\overline{\alpha}(t_i)+\sqrt{\hbar}a^\dagger(t_i)\right\}_{i=1}^n\right)\right\rangle_\psi+O(\hbar).\end{aligned}$$

The last equality is because $\widetilde{p}_{\min}$ is at least 1. Therefore, Eq. (1.25) follows.

# 3 Proof of Corollary 1.21

By Eqs. (1.8) and (1.9) in Definition 1.3, the term $\left\langle P_1\left(\left\{\alpha(t_i):a(t_i),a^\dagger(t_i)\right\}_{i=1}^n\right)\right\rangle_\psi$ in Eq.(1.26) is bounded independent of $\hbar$ for $\psi\in\mathcal{S}$. Therefore, by setting $\hbar\to 0$ in Eq.(1.26), Eq.(1.28) follows. To show Eq.(1.29), let $p_{\min}$ be the minimum degree of all non constant terms in $P\left(\{\theta_i,\theta_i^*\}_{i=1}^n\right)$. We assume $p_{\min}\geq 1$ as usual. Otherwise, it means $P$ is a constant polynomial which is a trivial case in Eq. (1.29). With the same notations as in Eq. (9.26), we have

$$P\left(\{\theta_i,\theta_i^*\}_{i=1}^n\right)=P_0+\sum_{k=p_{\min}}^{p}P_k\left(\{\theta_i,\theta_i^*\}_{i=1}^n\right).$$

Then, we apply Eq.(1.23) on each term $P_k$ where $k \geq 1$, and get

$$\begin{aligned}
&\left\langle P_k\left(\left\{A_\hbar\left(t_i\right)-\alpha\left(t_i\right), A_\hbar^\dagger\left(t_i\right)-\overline{\alpha}\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_{U_\hbar(\alpha_0)\psi} \\
=&\left\langle P_k\left(\left\{\sqrt{\hbar}a\left(t_i\right), \sqrt{\hbar}a^\dagger\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_\psi + O\left(\hbar^{\frac{k+1}{2}}\right) \\
=&\hbar^{\frac{k}{2}}\left(\left\langle P_k\left(\left\{a\left(t_i\right), a^\dagger\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_\psi + O\left(\hbar^{\frac{1}{2}}\right)\right).
\end{aligned} \tag{9.29}$$

By applying Eq.(9.29), we have

$$\begin{aligned}
&\left\langle P\left(\left\{\frac{A_\hbar\left(t_i\right)-\alpha\left(t_i\right)}{\sqrt{\hbar}}, \frac{A_\hbar^\dagger\left(t_i\right)-\overline{\alpha}\left(t_i\right)}{\sqrt{\hbar}}\right\}_{i=1}^n\right)\right\rangle_{U_\hbar(\alpha_0)\psi} \\
&= P_0 + \sum_{k=p_{\min}}^{p} \frac{1}{\hbar^{\frac{k}{2}}}\left\langle P_k\left(\left\{A_\hbar\left(t_i\right)-\alpha\left(t_i\right), A_\hbar^\dagger\left(t_i\right)-\overline{\alpha}\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_{U_\hbar(\alpha_0)\psi} \\
&= P_0 + \sum_{k=p_{\min}}^{p} \left\langle P_k\left(\left\{a\left(t_i\right), a^\dagger\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_\psi + O\left(\hbar^{\frac{1}{2}}\right) \\
&= \left\langle P\left(\left\{a\left(t_i\right), a^\dagger\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_\psi + O\left(\hbar^{\frac{1}{2}}\right).
\end{aligned}$$

Eq.(1.29) follows.

# Part II

# Powers of Symmetric Differential Operators

# Chapter 10

# A Structure Theorem for Symmetric Differential Operators

**Remark 10.1.** *It is useful to observe if $A$ and $B$ are two linear transformation from a vector space, $V$, to itself, then*

$$AB^2 + B^2 A = 2BAB + [B, [B, A]],$$

*where $[A, B] := AB - BA$ denotes the commutator of $A$ and $B$.*

**Proposition 10.2.** *Suppose $\{a_k\}_{k=0}^d \subset C^\infty(\mathbb{R}, \mathbb{R})$ and $L$ is the $d^{th}$ -order differential operator on $C^\infty(\mathbb{R})$ as defined in Eq. (1.30). If $L$ is symmetric according to Definition 1.26 (i.e. $L = L^\dagger$ where $L^\dagger$ is as in Eq. (1.32)), then $d$ is even (let $m = d/2$) and there exists $\{b_l\}_{l=0}^m \subset C^\infty(\mathbb{R}, \mathbb{R})$ such that*

$$L = \sum_{l=0}^{m} (-1)^l \partial^l M_{b_l} \partial^l \tag{10.1}$$

*where $M_{b_l}$ is as in Notation 1.24. Moreover, $b_m = (-1)^m a_{2m} = (-1)^m a_d$.*

**Proof.** Since $L = L^\dagger$, we have

$$L = \frac{1}{2}(L + L^\dagger) = \frac{1}{2}\sum_{k=0}^{d}[a_k\partial^k + (-1)^k\partial^k M_{a_k}] \tag{10.2}$$

$$= \frac{1}{2}[a_d\partial^d + (-1)^d\partial^d M_{a_d}] + [\text{diff. operator of order } d-1]. \tag{10.3}$$

If $d$ were odd, then $(-1)^d = -1$ and hence (using the product rule),

$$\begin{aligned}\frac{1}{2}[a_d\partial^d + (-1)^d\partial^d M_{a_d}] &= \frac{1}{2}[M_{a_d}, \partial^d] \\ &= [\text{diff. operator of order } d-1]\end{aligned}$$

which combined with Eq. (10.3) would imply that $L$ was in fact a differential operator of order no greater than $d-1$. This shows that $L$ must be an even order operator.

Now knowing that $d$ is even, let $m := d/2 \in \mathbb{N}$. From Eq. (10.2), we learn that

$$\begin{aligned}L &= \frac{1}{2}\sum_{k=0}^{2m}\left[a_k\partial^k + (-1)^k\partial^k M_{a_k}\right] \\ &= \frac{1}{2}\left[a_{2m}\partial^{2m} + \partial^{2m}M_{a_{2m}}\right] + R\end{aligned}$$

where $R$ is given by

$$R = \frac{1}{2}\sum_{k=0}^{2m-1}\left[M_{a_k}\partial^k + (-1)^k\partial^k M_{a_k}\right].$$

Moreover by Remark 1.27, $R$ is still symmetric. As in the previous paragraph $R$ is in fact an even order differential operator and its order is at most $2m-2$. Using

Remark 10.1 with $A = M_{a_{2m}}$, $B = \partial^m$, and $V = C^\infty(\mathbb{R})$, we learn that

$$\frac{1}{2}\left[a_{2m}\partial^{2m} + \partial^{2m}M_{a_{2m}}\right] = \partial^m M_{a_{2m}}\partial^m + \frac{1}{2}\left[\partial^m, \left[\partial^m, M_{a_{2m}}\right]\right]$$
$$= \partial^m M_{a_{2m}}\partial^m + [\text{diff. operator of order at most } 2m-2].$$

Combining the last three displayed equations together shows

$$L = \partial^m a_{2m}\partial^m + S$$

where $S = L - \partial^m a_{2m}\partial^m$ is a symmetric (by Remark 1.27) even order differential operator or at most $2m-2$. It now follows by the induction hypothesis that

$$S = \sum_{l=0}^{m-1}(-1)^l\,\partial^l M_{b_l}\partial^l \implies L = \sum_{l=0}^{m}(-1)^l\,\partial^l M_{b_l}\partial^l$$

where $b_m := (-1)^m a_{2m}$. ■

Our next goal is to record the explicit relationship between $\{a_k\}_{k=0}^{2m}$ in Eq. (1.30) and $\{b_k\}_{k=0}^{m}$ in Eq. (10.1).

**Convention 10.3.** *To simplify notation in what follows, for $k$, $l \in \mathbb{Z}$, let*

$$\binom{l}{k} := \begin{cases} \frac{l!}{k!(l-k)!} & \text{if } 0 \le k \le l \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 10.4.** *If $\{a_k\}_{k=0}^{2m} \cup \{b_l\}_{l=0}^{m} \subset C^\infty(\mathbb{R},\mathbb{R})$ and*

$$\sum_{l=0}^{m}(-1)^l\,\partial^l b_l(x)\,\partial^l = \sum_{k=0}^{2m} a_k(x)\,\partial^k, \tag{10.4}$$

*then*

$$a_k := \sum_{l=0}^{m}\binom{l}{k-l}(-1)^l\,\partial^{2l-k} b_l. \tag{10.5}$$

**Proof.** By the product rule,

$$\partial^l M_{b_l} = \sum_{r=0}^{l} \binom{l}{r} \left(\partial^{l-r} b_l\right) \partial^r,$$

and therefore,

$$\begin{aligned}
\sum_{l=0}^{m} (-1)^l \, \partial^l M_{b_l} \partial^l &= \sum_{l=0}^{m} \sum_{r=0}^{l} \binom{l}{r} (-1)^l \left(\partial^{l-r} b_l\right) \partial^{l+r} \\
&= \sum_{k=0}^{2m} \left[ \sum_{l=0}^{m} \sum_{r=0}^{m} \binom{l}{r} (-1)^l \left(\partial^{l-r} b_l\right) 1_{k=l+r} \right] \partial^k \\
&= \sum_{k=0}^{2m} \left[ \sum_{l=0}^{m} \binom{l}{k-l} (-1)^l \left(\partial^{2l-k} b_l\right) \right] \partial^k.
\end{aligned}$$

Combining this result with Eq. (10.4) gives the identities in Eq. (10.5). ■

Let us observe that the binomial coefficient of $a_l$ is zero unless $0 \le k-l \le l$, i.e. $l \le k \le 2l$. To emphasize this restriction, we may write Eq. (10.5) as

$$a_k = \sum_{l=0}^{m} 1_{l \le k \le 2l} \binom{l}{k-l} (-1)^l \, \partial^{2l-k} b_l. \tag{10.6}$$

Taking $k = 2p$ in Eq. (10.6) and multiplying the result by $(-1)^p = (-1)^{-p}$ leads to the following corollary.

**Corollary 10.5.** *For* $0 \le p \le m$,

$$(-1)^p \, a_{2p} = \sum_{l=0}^{m} 1_{p \le l \le 2p} \binom{l}{2p-l} (-1)^{(l-p)} \, \partial^{2(l-p)} b_l. \tag{10.7}$$

We will see in Theorem 10.7 below that the relations in Eq. (10.7) may be used to uniquely write the $\{b_l\}_{l=0}^m$ in terms of linear combinations for the $\{a_{2k}\}_{k=0}^m$. In particular this shows if the operator $L$ described in Eq. (1.30) is symmetric then $\{b_l\}_{l=0}^m$ is completely determined by the $a_k$ with $k$ even.

# 1 The divergence form of $L$

**Notation 10.6.** *For $r, s, n \in \mathbb{N}_0$ and $0 \le r, s \le m$, let*

$$C_n(r,s) = \sum \binom{k_1}{2r-k_1}\binom{k_2}{2k_1-k_2}\cdots\binom{k_{n-1}}{2k_{n-2}-k_{n-1}}\binom{s}{2k_{n-1}-s}$$

*where the sum is over $r < k_1 < k_2 < \cdots < k_{n-1} < s$. We also let*

$$K_m(r,s) = \sum_{n=1}^{m-1} (-1)^n C_n(r,s). \tag{10.8}$$

*In particular, $C_n(0,s) = C_n(m,s) = K_m(0,s) = K_m(m,s) = 0$ for all $0 \le s \le m$.*

**Theorem 10.7.** *If Eq. (10.4) holds then*

$$(-1)^r b_r = a_{2r} + \sum_{r<s\le m} K_m(r,s)\partial^{2(s-r)} a_{2s} \ \forall\ 0 \le r \le m. \tag{10.9}$$

**Proof.** For $x \in \mathbb{R}$ let $\mathbf{b}(x)$ and $\mathbf{a}(x)$ denote the column vectors in $\mathbb{R}^{m+1}$ defined by

$$\begin{aligned}
\mathbf{b}(x) &= \left((-1)^0 b_0(x), (-1)^1 b_1(x), \ldots, (-1)^m b_m(x)\right)^{\mathrm{tr}} \text{ and} \\
\mathbf{a}(x) &= \left(a_0(x), a_2(x), a_4(x), \ldots, a_{2m}(x)\right)^{tr}.
\end{aligned}$$

Further let $U$ be the $(m+1)\times(m+1)$ matrix with entries $\{U_{r,k}\}_{r,k=0}^m$ which are linear constant coefficient differential operators given by

$$U_{r,k} := 1_{r<k\le 2r}\binom{k}{2r-k}\partial^{2(k-r)}.$$

Notice that by definition, $U_{r,k} = 0$ unless $k > r$ and $U_{0,k} = 0$ for $0 \le k \le m$. Hence

$U$ is nilpotent and $U^m = 0$. Further observe that Eq. (10.7) may be written as

$$\begin{aligned} a_{2r} &= (-1)^r\, b_r + \sum_{r<k\le m} \binom{k}{2r-k} (-1)^k\, \partial^{2(k-r)} b_k \\ &= (-1)^r\, b_r + \sum_{r<k\le m} U_{r,k}\, (-1)^k\, b_k \end{aligned}$$

or equivalently stated $\mathbf{a}= (I+U)\mathbf{b}$. As $U$ is nilpotent with $U^m = 0$, this last equation may be solved for $\mathbf{b}$ using

$$\mathbf{b} = (I+U)^{-1}\mathbf{a} = \mathbf{a} + \sum_{n=1}^{m-1} (-U)^n \mathbf{a}. \tag{10.10}$$

In components this equation reads,

$$(-1)^r\, b_r = a_r + \sum_{n=1}^{m-1} (-1)^n \sum_{s=0}^{m} U^n_{r,s} a_{2s} \tag{10.11}$$

However, with the aid of Lemma 10.8 below and the definition of $K_m(r,s)$ in Eq. (10.8) it follows that

$$\sum_{n=1}^{m-1} (-1)^n U^n_{r,s} = \sum_{n=1}^{m-1} (-1)^n C_n(r,s) \partial^{2(s-r)} = K_m(r,s) \partial^{2(s-r)}$$

which combined with Eq. (10.11) and the fact that $K_m(r,s) = 0$ unless $0 < r < s \le m$ proves Eq. (10.9). ■

**Lemma 10.8.** *Let $1 \le n \le m$ and $0 \le r, s \le m$, then $U^m = 0$ and*

$$U^n_{r,s} = C_n(r,s) \partial^{2(s-r)}. \tag{10.12}$$

**Proof.** By definition of matrix multiplication,

$$
\begin{aligned}
U^n_{r,s} = & \sum_{k_1,\ldots,k_{n-1}=1}^{m} 1_{r<k_1\leq 2r}\binom{k_1}{2r-k_1}\partial^{2(k_1-r)}1_{k_1<k_2\leq 2k_1}\binom{k_2}{2k_1-k_2}\partial^{2(k_2-k_1)}\ldots \\
& \qquad \ldots 1_{k_{n-1}<k_n\leq 2k_{n-1}}\binom{k_n}{2k_{n-1}-k_n}\partial^{2(k_n-k_{n-1})}1_{k_n=s} \\
= & \sum_{r<k_1<k_2<\cdots<k_{n-1}<s} \binom{k_1}{2r-k_1}\left[\prod_{j=1}^{n-2}\binom{k_{j+1}}{2k_j-k_{j+1}}\right]\binom{s}{2k_{n-1}-s}\partial^{2(s-r)} \\
= & C_n(r,s)\partial^{2(s-r)}.
\end{aligned}
$$

■

# Chapter 11

# The structure of $L^n$

In this chapter let us fix a $2m$ – order symmetric differential operator, $L$, acting on $C^\infty(\mathbb{R})$ which can be written as in both of the equations (1.30) and (10.1) where the coefficients, $\{a_k\}_{k=0}^{2m}$ and $\{b_l\}_{l=0}^{m}$ are all real valued smooth functions on $\mathbb{R}$. If $n \in \mathbb{N}$, $L^n$ is a $2mn$ – order symmetric linear differential operator on $C^\infty(\mathbb{R})$ and hence there exists $\{A_k\}_{k=0}^{2mn} \subset C^\infty(\mathbb{R}, \mathbb{R})$ and (using Proposition 10.2) $\{B_l\}_{l=0}^{mn} \subset C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$L^n = \sum_{k=0}^{2mn} A_k \partial^k = \sum_{\ell=0}^{mn} (-1)^\ell \partial^\ell B_\ell \partial^\ell. \tag{11.1}$$

Our goal in this chapter is to compute the coefficients $\{A_k\}_{k=0}^{2mn}$ in terms of the coefficients $\{b_l\}_{l=0}^{m}$ defining $L$ as in Eq. (10.1). It turns out that it is useful to compare $L^n$ to the operators which is constructed by writing out $L^n$ while pretending that the coefficients $\{a_k\}_{k=0}^{2m}$ or $\{b_l\}_{l=0}^{m}$ are constant. This is explained in the following notations.

**Notation 11.1.** *For $n \in \mathbb{N}$ and $m \in \mathbb{N}$, let $\Lambda_m := \{0, 1, \ldots, m\} \subset \mathbb{N}_0$ and for $\mathbf{j} = (j_1, \ldots, j_n) \in \Lambda_m^n$, let $|\mathbf{j}| = j_1 + j_2 + \cdots + j_n$. If $\mathbf{k} = (k_1, \ldots, k_n) \in \Lambda_m^n$ is another multi-index, we write $\mathbf{k} \leq \mathbf{j}$ to mean $k_i \leq j_i$ for $1 \leq i \leq n$. [We will use this* *notation when $m=\infty$ as well in which case $\Lambda_\infty = \mathbb{N}_0$.]*

**Notation 11.2.** *Given $n \in \mathbb{N}$, and $L$ as in Eq. (10.1), let $\{\mathcal{B}_\ell\}_{\ell=0}^{mn}$ be $C^\infty(\mathbb{R},\mathbb{R})$ – functions defined by*

$$\mathcal{B}_\ell := \sum_{\mathbf{j}\in\Lambda_m^n} 1_{|\mathbf{j}|=\ell} b_{j_1} \dots b_{j_n} \tag{11.2}$$

*and $\mathcal{L}_{\mathcal{B}}^{(n)}$ be the differential operator given by*

$$\mathcal{L}_{\mathcal{B}}^{(n)} := \sum_{\ell=0}^{nm} (-1)^\ell \partial^\ell \mathcal{B}_\ell \partial^\ell. \tag{11.3}$$

*It will also be convenient later to set $\mathcal{B}_{k/2} \equiv 0$ when $k$ is an odd integer.*

**Example 11.3.** *If $m=1$ and $n=2$, then*

$$\begin{aligned} L &= -\partial b_1 \partial + b_0 \text{ and} \\ L^2 &= \partial b_1 \partial^2 b_1 \partial - \partial b_1 \partial b_0 - b_0 \partial b_1 \partial + b_0^2. \end{aligned}$$

*To put $L^2$ into divergence form we repeatedly use the product rule, $\partial V = V\partial + V'$. Thus*

$$\begin{aligned} \partial b_1 \partial b_0 + b_0 \partial b_1 \partial &= \partial b_1 b_0 \partial + \partial b_1 b_0' + \partial b_0 b_1 \partial - b_0' b_1 \partial \\ &= 2\partial b_1 b_0 \partial + (b_1 b_0')' \end{aligned}$$

*and*

$$\begin{aligned} \partial b_1 \partial^2 b_1 \partial &= \partial^2 b_1 \partial b_1 \partial - \partial b_1' \partial b_1 \partial \\ &= \partial^2 b_1 b_1 \partial^2 + \partial^2 b_1 b_1' \partial - \partial b_1' b_1' \partial - \partial b_1' b_1 \partial^2 \\ &= \partial^2 b_1 b_1 \partial^2 + \partial (b_1 b_1')' \partial - \partial b_1' b_1' \partial. \end{aligned}$$

*Combining the last three displayed equations together shows*

$$\begin{aligned} L^2 &= \partial^2 b_1^2 \partial^2 + \partial \left[ -2b_1 b_0 + (b_1 b_1')' - (b_1')^2 \right] \partial + b_0^2 - (b_1 b_0')' \\ &= \partial^2 b_1^2 \partial^2 + \partial \left[ -2b_1 b_0 + b_1 b_1'' \right] \partial + b_0^2 - (b_1 b_0')' . \end{aligned} \tag{11.4}$$

*Dropping all terms in Eq. (11.4) which contain a derivative of* $b_1$ *or* $b_0$ *shows*

$$\mathcal{L}_{\mathcal{B}}^{(2)} = \partial^2 b_1^2 \partial^2 - \partial \left[ 2b_0 b_1 \right] \partial + b_0^2. \tag{11.5}$$

**Notation 11.4.** *For* $\mathbf{j} \in \mathbb{N}_0^n$ *and* $\mathbf{k} \in \mathbb{N}_0^n$, *let*

$$\binom{\mathbf{k}}{\mathbf{j}} := \prod_{i=1}^{n} \binom{k_i}{j_i}$$

*where the binomial coefficients are as in Convention 10.3.*

**Lemma 11.5.** *If* $L$ *is as in Eq. (10.1),*

$$M_{e^{-i\xi(\cdot)}} L M_{e^{i\xi(\cdot)}} = \sum_{l=0}^{m} (-1)^l (\partial + i\xi)^l M_{b_l(\cdot)} (\partial + i\xi)^l . \tag{11.6}$$

**Proof.** If $f \in C^\infty(\mathbb{R})$, the product rule gives,

$$\partial_x \left[ e^{i\xi x} f(x) \right] = e^{i\xi x} (\partial_x + i\xi) f(x) ,$$

which is to say,

$$M_{e^{-i\xi(\cdot)}} \partial M_{e^{i\xi(\cdot)}} = (\partial + i\xi) . \tag{11.7}$$

Combining Eq. (11.7) with the fact that

$$M_{e^{-i\xi(\cdot)}} M_{b_l} M_{e^{i\xi(\cdot)}} = M_{b_l}$$

allows us to conclude,

$$\begin{aligned}M_{e^{-i\xi(\cdot)}}LM_{e^{i\xi(\cdot)}} &= \sum_{l=0}^{m}(-1)^l M_{e^{-i\xi(\cdot)}}\partial^l M_{b_l}\partial^l M_{e^{i\xi(\cdot)}} \\ &= \sum_{l=0}^{m}(-1)^l M_{e^{-i\xi(\cdot)}}\partial^l M_{e^{i\xi(\cdot)}} M_{b_l} M_{e^{-i\xi(\cdot)}}\partial^l M_{e^{i\xi(\cdot)}} \\ &= \sum_{l=0}^{m}(-1)^l (\partial + i\xi)^l M_{b_{l}(\cdot)} (\partial + i\xi)^l .\end{aligned}$$

■

**Notation 11.6.** *For* $\mathbf{q},\mathbf{l},\mathbf{j}\in\Lambda_m^n$, *let*

$$C_k(\mathbf{q},\mathbf{l},\mathbf{j}) := (-1)^{|\mathbf{q}|}\binom{\mathbf{q}}{\mathbf{l}}\binom{\mathbf{q}}{\mathbf{j}} 1_{j_1=0} 1_{2|\mathbf{q}|-k=|\mathbf{l}|+|\mathbf{j}|>0}, \tag{11.8}$$

*and for* $k\in\Lambda_{2m}$ *let*

$$T_k := \sum_{\mathbf{q},\mathbf{l},\mathbf{j}\in\Lambda_m^n} C_k(\mathbf{q},\mathbf{l},\mathbf{j})\left(\partial^{l_n} M_{b_{q_n}}\partial^{j_n}\right)\left(\partial^{l_{n-1}} M_{b_{q_{n-1}}}\partial^{j_{n-1}}\right)\dots\left(\partial^{l_1} M_{b_{q_1}}\partial^{j_1}\right)\mathbf{1} \tag{11.9}$$

*where* $\mathbf{1}$ *is a function constantly equal to* 1. *We will often abuse notation and write this last equation as,*

$$T_k(x) := \sum_{\mathbf{q},\mathbf{l},\mathbf{j}\in\Lambda_m^n} C_k(\mathbf{q},\mathbf{l},\mathbf{j})\left(\partial_x^{l_n} b_{q_n}(x)\,\partial_x^{j_n}\right)\dots\left(\partial_x^{l_2} b_{q_2}\partial_x^{j_2}\right)\partial_x^{l_1} b_{q_1}(x).$$

**Proposition 11.7** ($A_k = A_k(\{b_l\}_{l=0}^m)$)**.** *If* $L$ *is given as in Eq. (10.1), then coefficients* $\{A_k\}_{k=0}^{2mn}$ *of* $L^n$ *in Eq. (11.1) are given by*

$$A_k = 1_{k\in 2\mathbb{N}_0}\cdot(-1)^{k/2}\mathcal{B}_{k/2} + T_k, \tag{11.10}$$

*where* $\mathcal{B}_\ell$ *and* $T_k$ *are as in Notations 11.2 and 11.6 respectively. Moreover, if we* *further assume* $\{b_l\}_{l=0}^{m}$ *are polynomial functions such that*

$$\deg(b_l) \le \max\{\deg(b_0), 0\} \;\; \textit{for } 1 \le \ell \le m, \tag{11.11}$$

*then* $\{\mathcal{B}_\ell\}_{\ell=0}^{mn}$ *and* $\{T_k\}_{k=0}^{2mn}$ *are polynomials such that*

$$\deg(T_k) < \max\{n \deg(b_0), 0\} = \max\{\deg(\mathcal{B}_0), 0\} \;\; \textit{for } 0 \le k \le 2mn.$$

**Proof.** First observe that if $L^n$ is described as in Eq. (11.1), then

$$\sum_{k=0}^{2mn} A_k(x)(i\xi)^k = \sigma_n(x,\xi) = e^{-i\xi x} L_x^n \left(e^{i\xi x}\right)$$

where $\sigma_n := \sigma_{L^n}$ is a symbol of $L^n$ defined in Eq. (1.31) and $L_x^n$ is a differential operator with respect to $x$. To compute the right side of this equation, take the $n^{\text{th}}$ – power of Eq. (11.6) to learn

$$\begin{aligned} &M_{e^{-i\xi(\cdot)}} L^n M_{e^{i\xi(\cdot)}} = \left(M_{e^{-i\xi(\cdot)}} L M_{e^{i\xi(\cdot)}}\right)^n \\ &= \sum_{q_1,\dots,q_n=0}^{m} (\partial + i\xi)^{q_n} (-1)^{q_n} M_{b_{q_n}} (\partial + i\xi)^{q_n} \dots (\partial + i\xi)^{q_1} (-1)^{q_1} M_{b_{q_1}} (\partial + i\xi)^{q_1} \\ &= \sum_{\mathbf{q} \in \mathbf{\Lambda}_m} (-1)^{|\mathbf{q}|} (\partial + i\xi)^{q_n} M_{b_{q_n}} (\partial + i\xi)^{q_n} \dots (\partial + i\xi)^{q_1} M_{b_{q_1}} (\partial + i\xi)^{q_1}. \end{aligned}$$

Applying this result to the constant function $\mathbf{1}$ then shows

$$\begin{aligned} \sigma_n(x,\xi) &= e^{-i\xi x} L_x^n \left(e^{i\xi x}\right) = M_{e^{-i\xi x}} L_x^n M_{e^{i\xi x}} \mathbf{1} \\ &= \sum_{\mathbf{q} \in \mathbf{\Lambda}_m} (-1)^{|\mathbf{q}|} (\partial + i\xi)^{q_n} M_{b_{q_n}} (\partial + i\xi)^{q_n} \dots (\partial + i\xi)^{q_1} M_{b_{q_1}} (\partial + i\xi)^{q_1} \mathbf{1}. \end{aligned}$$

Making repeatedly used of the binomial formula to expand out all the terms

$(\partial + i\xi)^q$ appearing above then gives,

$$\sum_{k=0}^{2mn} A_k(x)(i\xi)^k = \sigma_n(x,\xi)$$
$$= \sum_{\mathbf{q},\mathbf{l},\mathbf{j}\in\mathbf{\Lambda}_m} (-1)^{|\mathbf{q}|}\binom{\mathbf{q}}{\mathbf{l}}\binom{\mathbf{q}}{\mathbf{j}}(i\xi)^{2|\mathbf{q}|-|\mathbf{l}|-|\mathbf{j}|}\,\partial_x^{l_n} b_{q_n}(x)\,\partial_x^{j_n}\dots\partial_x^{l_1} b_{q_1}(x)\,\partial_x^{j_1}\mathbf{1}.$$

Looking the coefficient of $(i\xi)^k$ on the right side of this expression shows,

$$\begin{aligned} A_k(x) &:= \sum_{\mathbf{q},\mathbf{l},\mathbf{j}\in\mathbf{\Lambda}_m}\binom{\mathbf{q}}{\mathbf{l}}\binom{\mathbf{q}}{\mathbf{j}}(-1)^{|\mathbf{q}|}\,1_{2|\mathbf{q}|-|\mathbf{l}|-|\mathbf{j}|=k}\partial_x^{l_n} b_{q_n}(x)\,\partial_x^{j_n}\dots\partial_x^{l_1} b_{q_1}(x)\,\partial_x^{j_1}\mathbf{1} \\ &= \sum_{\mathbf{q},\mathbf{l},\mathbf{j}\in\mathbf{\Lambda}_m}\binom{\mathbf{q}}{\mathbf{l}}\binom{\mathbf{q}}{\mathbf{j}}(-1)^{|\mathbf{q}|}\,1_{j_1=0}1_{2|\mathbf{q}|-|\mathbf{l}|-|\mathbf{j}|=k}\partial_x^{l_n} b_{q_n}(x)\,\partial_x^{j_n}\dots\partial_x^{l_1} b_{q_1}(x) \\ &= \sum_{\mathbf{q}\in\mathbf{\Lambda}_m} 1_{2|\mathbf{q}|=k}(-1)^{|\mathbf{q}|}\,b_{q_n}(x)\dots b_{q_1}(x) \\ &\quad + \sum_{\mathbf{q},\mathbf{l},\mathbf{j}\in\mathbf{\Lambda}_m}\binom{\mathbf{q}}{\mathbf{l}}\binom{\mathbf{q}}{\mathbf{j}}(-1)^{|\mathbf{q}|}\,1_{j_1=0}1_{2|\mathbf{q}|-k=|\mathbf{l}|+|\mathbf{j}|>0}\partial_x^{l_n} b_{q_n}(x)\,\partial_x^{j_n}\dots\partial_x^{l_1} b_{q_1}(x) \end{aligned}$$

which completes the proof of Eq. (11.10). The remaining assertions now easily follow from the formulas for $\{\mathcal{B}_\ell\}_{\ell=0}^{mn}$ and $\{T_k\}_{k=0}^{2mn}$ in Notations 11.2 and 11.6 and the assumption in Eq. (11.11). ■

As we can see from Example 11.3, computing the coefficients $\{B_\ell\}_{\ell=0}^{nm}$ in Eq. (11.1) can be tedious in terms of the coefficients $\{b_l\}_{l=0}^{m}$ defining $L$ as in Eq. (10.1). Although we do not need the explicit formula for the $\{B_\ell\}_{\ell=0}^{nm}$, we will need some general properties of these coefficients which we develop below.

**Proposition 11.8.** *Given $B_\ell = B_\ell(\{b_l\}_{l=0}^{m})$ of $L^n$ in Eq. (11.1). There are constants $\hat{C}(n,\ell,\mathbf{k},\mathbf{p})$ for $n\in\mathbb{N}_0$, $0\le\ell\le mn$, $\mathbf{k}\in\Lambda_m^n$ and $\mathbf{p}\in\Lambda_{2m}^n$ such that;*

1. *$\hat{C}(n,\ell,\mathbf{k},\mathbf{p}) = 0$ unless $0<|\mathbf{p}| = 2|\mathbf{k}| - 2\ell$ and*

*2. if* $L = \sum_{l=0}^{m} (-1)^l \partial^l b_l \partial^l$ *then* $L^n = \sum_{\ell=0}^{mn} (-1)^\ell \partial^\ell B_\ell \partial^\ell$, *where*

$$B_\ell = \mathcal{B}_\ell + R_\ell \tag{11.12}$$

*with* $\mathcal{B}_\ell$ *as in Eq. (11.2) and* $R_\ell$ *is defined by*

$$R_\ell = \sum_{\mathbf{k}\in\Lambda_m^n, \mathbf{p}\in\Lambda_{2m}^n} \hat{C}(n,\ell,\mathbf{k},\mathbf{p}) \left(\partial^{p_1} b_{k_1}\right)\left(\partial^{p_2} b_{k_2}\right)\ldots\left(\partial^{p_n} b_{k_n}\right). \tag{11.13}$$

*[Notice that* $2|\mathbf{k}| - 2\ell \le 2mn - 2\ell$ *and so if* $\ell = mn$, *we must have* $|\mathbf{p}| = 2|\mathbf{k}| - 2\ell = 0$ *and so* $\hat{C}(n,\ell,\mathbf{k},\mathbf{p}) = 0$. *This shows that* $R_{mn} = 0$ *which can easily be verified independently if the reader so desires.]*

**Proof.** By Theorem 10.7, we know that $L^n = \sum_{\ell=0}^{mn} (-1)^\ell \partial^\ell B_\ell \partial^\ell$ where

$$(-1)^\ell B_\ell = A_{2\ell} + \sum_{\ell<s\le mn} K_{mn}(\ell,s)\partial^{2(s-\ell)} A_{2s} \ \forall\ 0 \le \ell \le mn \tag{11.14}$$

and $\{A_k\}_{k=0}^{2mn}$ are the coefficients in Eq. (11.1). Using the formula for the $\{A_k\}$ from Proposition 11.7 in Eq. (11.14) implies,

$$(-1)^\ell B_\ell = (-1)^\ell \mathcal{B}_\ell + T_{2\ell} + \sum_{\ell<s\le mn} K_{mn}(\ell,s)\partial^{2(s-\ell)}\left[(-1)^s \mathcal{B}_s + T_{2s}\right],$$

i.e. $B_\ell = \mathcal{B}_\ell + R_\ell$ where

$$R_\ell = (-1)^\ell T_{2\ell} + \sum_{\ell<s\le mn} (-1)^\ell K_{mn}(\ell,s)\partial^{2(s-\ell)}\left[(-1)^s \mathcal{B}_s + T_{2s}\right].$$

It now only remains to see that this remainder term may be written as in Eq. (11.13).

Recall from Eq. (11.2) that $\mathcal{B}_s := \sum_{\mathbf{j}\in\Lambda_m^n} 1_{|\mathbf{j}|=s} b_{j_1} \dots b_{j_n}$ and so

$$\partial^{2(s-\ell)}\mathcal{B}_s = \sum_{\mathbf{j}\in\Lambda_m^n} 1_{|\mathbf{j}|=s}\partial^{2(s-\ell)}\left[b_{j_1}\dots b_{j_n}\right].$$

For $s > \ell$, $\partial^{2(s-\ell)}\left[b_{j_1}\dots b_{j_n}\right]$ is a linear combination of terms of the form,

$$\left(\partial^{p_1}b_{j_1}\right)\left(\partial^{p_2}b_{j_2}\right)\dots\left(\partial^{p_n}b_{j_n}\right)$$

where $0 < |\mathbf{p}| = 2(s-\ell) = 2\,|\mathbf{j}| - 2\ell$ as desired. Similarly, from Eq. (11.9), $T_{2s}$ is a linear combination of monomials of the form,

$$\partial^{l_n}M_{b_{q_n}}\partial^{j_n}\dots\partial^{l_2}M_{b_{q_2}}\partial^{j_2}\partial^{l_1}b_{q_1} \text{ with } 2\,|\mathbf{q}| - 2s = |\mathbf{l}| + |\mathbf{j}| > 0 \text{ and } j_1 = 0.$$

It then follows that

$$\partial^{2(s-\ell)}\partial^{l_n}M_{b_{q_n}}\partial^{j_n}\dots\partial^{l_2}M_{b_{q_2}}\partial^{j_2}\partial^{l_1}b_{q_1}$$

is a linear combination of monomials of the form,

$$\left(\partial^{p_1}b_{q_1}\right)\left(\partial^{p_2}b_{q_2}\right)\dots\left(\partial^{p_n}b_{q_n}\right)$$

where

$$0 < |\mathbf{p}| = 2(s-\ell) + |\mathbf{l}| + |\mathbf{j}| = 2(s-\ell) + 2\,|\mathbf{q}| - 2s = 2\,|\mathbf{q}| - 2\ell.$$

Putting all of these comments together completes the proof. ■

# Chapter 12

# The Essential Self Adjointness Proof

This chapter is devoted to the proof of Theorem 1.30. Lemma 12.1 records a simple sufficient condition for showing a symmetric operator on a Hilbert space is in fact essentially self-adjoint. For the remainder of Part II, we assume that the coefficients, $\{a_k\}_{k=0}^{d}$, of $L$ in Eq. (1.30) are all in $\mathbb{R}[x]$ and we now restrict $L$ to $\mathcal{S}$ as described in Notation 1.29. The operators, $L^n$, are then defined for all $n \in \mathbb{N}$ and we still have $\mathcal{D}(L^n) = \mathcal{S}$, see Remark 1.28.

**Lemma 12.1** (Self-Adjointness Criteria)**.** *Let $\mathbb{L} : \mathcal{K} \to \mathcal{K}$ be a densely defined symmetric operator on a Hilbert space $\mathcal{K}$ and let $S = \mathcal{D}(\mathbb{L})$ be the domain of $\mathbb{L}$. Assume there exists linear operators $T_\mu : S \to S$ and bounded operators $R_\mu : \mathcal{K} \to \mathcal{K}$ for $\mu \in \mathbb{R}$ such that;*

1. $(\mathbb{L} + i\mu)T_\mu u = (I + R_\mu)u$ *for all* $u \in S$, *and*

2. *there exists* $M > 0$ *such that* $\|R_\mu\|_{op} < 1$ *for* $|\mu| > M$.

*Under these assumptions,* $\mathbb{L}|_S$ *is essentially self-adjoint.*

**Proof.** $\|R_\mu\|_{op} < 1$ for $|\mu| > M$ is assumed in condition 2 which implies $I + R_\mu$ is invertible. Therefore, if $f \in \mathcal{K}$, then $g := (I + R_\mu)^{-1} f \in \mathcal{K}$ satisfies $(I + R_\mu) g = f$. We may then choose $\{g_n\}_{n=1}^{\infty} \subset S$ such that $g_n \to g$ in $\mathcal{K}$. Let $s_n = T_\mu g_n \in S$. We have, by condition 1, that

$$\begin{aligned}\|(\mathbb{L} + i\mu) s_n - f\| &= \|(I + R_\mu) g_n - f\| \\ &\leq ||(I + R_\mu)(g_n - g)|| \\ &\leq \|I + R_\mu\|_{op} ||g_n - g|| \to 0 \text{ as } n \to \infty.\end{aligned}$$

We have thus verified that $\mathrm{Ran}(\mathbb{L} + i\mu)|_S$ is dense in $K$ for all $|\mu| > M$ from which it follows that $\mathbb{L}|_S$ is essentially self-adjoint, see for example the corollary on p.257 in Reed and Simon [32]. ■

**Notation 12.2.** *Let $\{\mathcal{B}_\ell(x)\}_{\ell=0}^{mn}$ be the coefficients defined in Eq. (11.2) and define*

$$\Sigma(x, \xi) := \sum_{\ell=0}^{mn} \mathcal{B}_\ell(x) \xi^{2\ell}. \tag{12.1}$$

From Eqs. (11.10) and (1.31) the symbol, $\sigma_n(x, \xi) := \sigma_{L^n}(x, \xi)$, of $L^n$ presented as in Eq. (11.1) may be written as

$$\sigma_n(x, \xi) = \sum_{\ell=0}^{mn} \left[\mathcal{B}_\ell(x) + (-1)^\ell T_{2\ell}(x)\right] \xi^{2\ell} - i \sum_{\ell=1}^{mn} (-1)^\ell T_{2\ell-1}(x) \xi^{2\ell-1} \tag{12.2}$$

$$= \Sigma(x, \xi) + \sum_{\ell=0}^{mn} \left[(-1)^\ell T_{2\ell}(x)\right] \xi^{2\ell} - i \sum_{\ell=1}^{mn} (-1)^\ell T_{2\ell-1}(x) \xi^{2\ell-1}, \tag{12.3}$$

where the coefficients $\{T_k\}_{k=0}^{2mn}$ are as in Eq. (11.9). More importantly, for our purposes,

$$\mathrm{Re}\,\sigma_n(x, \xi) = \Sigma(x, \xi) + \sum_{\ell=0}^{mn} \left[(-1)^\ell T_{2\ell}(x)\right] \xi^{2\ell}. \tag{12.4}$$

The following lemma will be useful in estimating all of these functions of $(x, \xi)$.

**Lemma 12.3.** *Let* $0 \le k_1 < k_2 < \infty$ *and* $p(x)$, $q(x)$ *and* $r(x)$ *be real polynomials such that* $\deg p \le \deg q$, $q > 0$, *and* $r$ *is bounded from below.*

1. *If* $\deg(p) < \deg(r)$ *or* $p$ *is a constant function, then, for any* $k_1 < k_2$ *and* $\lambda > 0$, *there exists* $c_\lambda$ *such that*

$$\left|p(x)\,\xi^{k_1}\right| \le \lambda\Big(q(x)\,|\xi|^{k_2} + r(x)\Big) + c_\lambda. \tag{12.5}$$

2. *If* $\deg(p) \le \deg(r)$, *then for any* $k_1 < k_2$ *and* $\lambda > 0$, *there exists constants* $c_\lambda$ *and* $d_\lambda$ *such that*

$$\left|p(x)\,\xi^{k_1}\right| \le \lambda q(x)\,|\xi|^{k_2} + c_\lambda r(x) + d_\lambda. \tag{12.6}$$

**Proof.** Since $\deg p \le \deg q$ and $q > 0$, $\deg q \in 2\mathbb{N}_0$ and $K := \sup_{x\in\mathbb{R}} |p(x)|/q(x) < \infty$, i.e. $p(x) \le Kq(x)$. One also has for every $\tau > 0$, there exists $0 < a_\tau < \infty$ such that $\left|\xi^{k_1}\right| \le \tau\,|\xi|^{k_2} + a_\tau$. Combining these estimates shows,

$$\begin{aligned}\left|p(x)\,\xi^{k_1}\right| &\le \tau\,|p(x)|\,|\xi|^{k_2} + a_\tau\,|p(x)| &(12.7)\\ &\le \tau K q(x)\,|\xi|^{k_2} + a_\tau\,|p(x)|. &(12.8)\end{aligned}$$

If $\deg p < \deg r$, for every $\delta > 0$ there exists $0 < b_\delta < \infty$ such that $|p(x)| \le \delta r(x) + b_\delta$ which combined with Eq. (12.8) implies

$$\left|p(x)\,\xi^{k_1}\right| \le \tau K q(x)\,|\xi|^{k_2} + a_\tau\left(\delta r(x) + b_\delta\right)$$

and Eq. (12.5) follows by choosing $\tau = \lambda/K$ and then $\delta = \lambda/a_\tau$ so that $c_\lambda = a_\tau b_\delta$.

If $\deg p \le \deg r$, then there exists $C_1, C_2 < \infty$ such that $|p(x)| \le C_1 r(x) + C_2$ which combined with Eq. (12.8) with $\tau = \lambda/K$ shows

$$\left|p(x)\,\xi^{k_1}\right| \le \lambda q(x)\,|\xi|^{k_2} + a_{\lambda/K}\left(C_1 r(x) + C_2\right)$$

from which Eq. (12.6) follows. ■

With the use of Lemma 12.3, the following Lemma helps us to estimate the growth of $T_k(x)$ (see Notation 11.6) and its derivatives of $L^n$ in Eq. (11.1) for $0 \leq k \leq 2mn$

**Lemma 12.4.** *Suppose that $\{b_l\}_{l=0}^m$ are polynomials satisfying the assumptions in Theorem 1.30. Then for each $0 \leq k \leq 2mn$, $\beta \in \mathbb{N}_0$, and $\delta > 0$, there exists $C = C(k, \beta, \delta) < \infty$ such that*

$$\left(1 + |\xi|^k\right)\left(1 + |x|^\beta\right)\left|\partial_x^\beta T_k(x)\right| \leq \delta\Sigma(x, \xi) + C(k, \beta, \delta). \tag{12.9}$$

**Proof.** If $\deg b_0 \leq 0$, then by condition 2 in Theorem 1.30 it follows that $\{b_l\}_{l=0}^m$ are all constants in which case $T_k \equiv 0$ and the Lemma is trivial. So for the rest of the proof we assume $\deg b_0 > 0$.

According to Eq. (11.9), $T_k$ may be expressed as a linear combination of terms of the form, $(\partial^{p_1} b_{j_1}) \ldots (\partial^{p_n} b_{j_n})$, where $\mathbf{j}$ and $\mathbf{p}$ are multi-indices such that $2|\mathbf{j}| - k = |\mathbf{p}| > 0$. If $\mathbf{j}$ and $\mathbf{p}$ are multi-indices such that $2|\mathbf{j}| - k = |\mathbf{p}| > 0$ and $(\partial^{p_1} b_{j_1}) \ldots (\partial^{p_n} b_{j_n}) \neq 0$, then $b_{j_1} \ldots b_{j_n}$ is strictly positive and

$$\deg(b_{j_1} \ldots b_{j_n}) \geq \deg[(\partial^{p_1} b_{j_1}) \ldots (\partial^{p_n} b_{j_n})] + |\mathbf{p}| > 0.$$

Given the term, $b_{j_1} \ldots b_{j_n}$, appears in $\mathcal{B}_{|\mathbf{j}|}$, we conclude that $\mathcal{B}_{|\mathbf{j}|}$ is strictly positive and

$$\deg((\partial^{p_1} b_{j_1}) \ldots (\partial^{p_n} b_{j_n})) < \deg(b_{j_1} \ldots b_{j_n}) \leq \deg\left(\mathcal{B}_{|\mathbf{j}|}\right).$$

Moreover from condition 2 in Theorem 1.30, $\deg b_j \leq \deg b_0$ for all $j$ and therefore we also have

$$\deg((\partial^{p_1} b_{j_1}) \ldots (\partial^{p_n} b_{j_n})) < \deg(b_{j_1} \ldots b_{j_n}) \leq \deg(b_0^n) = \deg \mathcal{B}_0.$$

Moreover, for any $r, \beta \in \mathbb{N}_0$ with $r \leq \beta$ we still have

$$\begin{aligned}\deg\left\{x^r \partial_x^\beta \left[(\partial^{p_1} b_{j_1}) \ldots (\partial^{p_n} b_{j_n})\right]\right\} &\leq \deg\left((\partial^{p_1} b_{j_1}) \ldots (\partial^{p_n} b_{j_n})\right) \\ &< \min\left\{\deg\left(\mathcal{B}_{|\mathbf{j}|}\right), \deg\left(\mathcal{B}_0\right)\right\}.\end{aligned}$$

Hence by substituting

$$p(x) = x^r \partial_x^\beta \left[(\partial^{p_1} b_{j_1}) \ldots (\partial^{p_n} b_{j_n})\right], \; q(x) = \mathcal{B}_{|\mathbf{j}|}(x) \text{ and } r(x) = \mathcal{B}_0(x)$$

in Lemma 12.3, for every $\lambda > 0$ there exists $C_\lambda < \infty$ such that

$$\begin{aligned}\left|x^r \partial_x^\beta \left[(\partial^{p_1} b_{j_1}) \ldots (\partial^{p_n} b_{j_n})\right] \xi^k\right| &\leq \lambda\left[\mathcal{B}_{|\mathbf{j}|}(x)\, \xi^{2|\mathbf{j}|} + \mathcal{B}_0(x)\right] + C_\lambda \\ &\leq \lambda \cdot \Sigma(x, \xi) + C_\lambda\end{aligned}$$

and similarly,

$$\left|x^r \partial_x^\beta \left[(\partial^{p_1} b_{j_1}) \ldots (\partial^{p_n} b_{j_n})\right]\right| \leq \lambda \Sigma(x, \xi) + C_\lambda.$$

These last two equations with $r = 0$ and $r = \beta$ combine to show, for all $\lambda > 0$, there exists $C_\lambda < \infty$ such that

$$\left(1 + |\xi|^k\right)\left(1 + |x|^\beta\right)\left|\partial_x^\beta \left[(\partial^{p_1} b_{j_1}) \ldots (\partial^{p_n} b_{j_n})\right]\right| \leq 4\left(\lambda \Sigma(x, \xi) + C_\lambda\right).$$

By using this result in Eq. (11.9), one then sees there is a constant $K < \infty$ such that

$$\left(1 + |\xi|^k\right)\left(1 + |x|^\beta\right)\left|\partial_x^\beta T_k(x)\right| \leq K \lambda \Sigma(x, \xi) + K C_\lambda.$$

Equation (12.9) now follows by replacing $\lambda$ by $\delta/K$ in the above equation. ∎

The following Lemma is to study the growth of $\mathcal{B}_\ell(x)$ (see Notation 11.2) and its derivatives of $L^n$ in Eq. (11.1) for $0 \leq l \leq mn$

**Lemma 12.5.** *Again suppose that $\{b_l\}_{l=0}^m$ are polynomials satisfying the assump-*

*tions in Theorem 1.30. For all $\ell \in \Lambda_{mn}$, and $\beta \in \mathbb{N}_0$, there exists $C < \infty$ such that*

$$\left|\partial_x^\beta \mathcal{B}_\ell(x)\right| \left(|\xi|^{2\ell} + 1\right) \left(|x|^\beta + 1\right) \leq C\Sigma(x,\xi) + C. \tag{12.10}$$

*Moreover, if we assume $b_0$ is not the zero polynomial, then we may drop the second $C$ in Eq. (12.10), i.e. there exists $C < \infty$ such that*

$$\left|\partial_x^\beta \mathcal{B}_\ell(x)\right| \left(|\xi|^{2\ell} + 1\right) \left(|x|^\beta + 1\right) \leq C\Sigma(x,\xi). \tag{12.11}$$

**Proof.** Case 1. If $b_0 = 0$ then by the assumption 2 of Theorem 1.30 each $b_l$ is a constant for $1 \leq l \leq m$ and therefore $\partial_x^\beta \mathcal{B}_\ell(x) = 0$ for all $\beta > 0$, i.e. $\mathcal{B}_\ell$ are constant for all $\ell$. Moreover, if $\beta = 0$, from the definition of $\Sigma(x,\xi)$ in Eq. (12.1) it follows that $\mathcal{B}_\ell \xi^{2\ell} \leq \Sigma(x,\xi)$ and hence

$$|\mathcal{B}_\ell(x)| \left(|\xi|^{2\ell} + 1\right) = \mathcal{B}_\ell\left(\xi^{2\ell} + 1\right) \leq \Sigma(x,\xi) + \mathcal{B}_\ell$$

and so Eq. (12.10) holds with $C = \max_{1 \leq \ell \leq mn} \max(1, \mathcal{B}_\ell)$.

Case 2. If $b_0 \neq 0$, let us assume $\mathcal{B}_\ell$ is not the zero polynomial for otherwise there is nothing to prove. Since $x^\beta \partial_x^\beta \mathcal{B}_\ell(x)$ and $\partial_x^\beta \mathcal{B}_\ell(x)$ both have degree no more than $\deg \mathcal{B}_\ell$ and $\mathcal{B}_\ell > 0$, we may conclude there exists $C < \infty$ such that

$$\left(1 + |x|^\beta\right) \left|\partial_x^\beta \mathcal{B}_\ell(x)\right| = \left|\partial_x^\beta \mathcal{B}_\ell(x)\right| + \left|x^\beta \partial_x^\beta \mathcal{B}_\ell(x)\right| \leq C\mathcal{B}_\ell(x). \tag{12.12}$$

Multiplying this equation by $\xi^{2\ell}$ then shows,

$$\left(1 + |x|^\beta\right) \left|\partial_x^\beta \mathcal{B}_\ell(x)\right| \xi^{2\ell} \leq C\mathcal{B}_\ell(x)\, \xi^{2\ell} \leq C\Sigma(x,\xi) \tag{12.13}$$

while $\deg(\mathcal{B}_\ell) \leq \deg(\mathcal{B}_0)$ and $\mathcal{B}_0 > 0$ , then there exists $C_1 < \infty$ such that

$$\mathcal{B}_\ell(x) \leq C_1 \mathcal{B}_0(x) \leq C_1 \Sigma(x,\xi)$$

which combined with Eq. (12.12) shows

$$\left(1+|x|^{\beta}\right)\left|\partial_x^{\beta}\mathcal{B}_{\ell}(x)\right| \le C_1\Sigma(x,\xi).$$

This estimate along with Eq. (12.13) then completes the proof of Eq. (12.10) with no second $C$. ■

**Notation 12.6.** *For any non-negative real-valued functions $f$ and $g$ on some domain $U$, we write $f \lesssim g$ to mean there exists $C>0$ such that $f(y) \le Cg(y)$ for all $y \in U$.*

The following result is an immediate corollary of Proposition 11.7 and Lemmas 12.4 and 12.5.

**Corollary 12.7.** *Suppose that $\{b_l\}_{l=0}^{m}$ are polynomials satisfying the assumptions in Theorem 1.30. If $\{A_k\}_{k=0}^{2mn}$ are the coefficients of $L^n$ as in Eq. (11.1), then for all $\beta \in \mathbb{N}_0$ and $0 \le k \le 2mn$,*

$$\left|\partial_x^{\beta}A_k(x)\right|\left(1+|\xi|^{k}\right)\left(1+|x|^{\beta}\right) \lesssim \Sigma(x,\xi)+1. \tag{12.14}$$

The next lemma is a direct consequence of Lemmas 12.4.

**Lemma 12.8.** *Let $L$ be the operator in Eq. (1.34), where we now assume that $\{b_l\}_{l=0}^{m}$ are polynomials satisfying the assumptions of Theorem 1.30. Then there exists $c>0$ such that the following hold:*

$$\sum_{\ell=0}^{mn}\left|T_{2\ell}(x)\xi^{2\ell}\right| \le \sum_{k=0}^{2mn}\left|T_k(x)\xi^{k}\right| \le \frac{1}{2}\left(\Sigma(x,\xi)+c\right), \text{ and} \tag{12.15}$$

$$\frac{3}{2}\Sigma(x,\xi)+\frac{1}{2}c \ge \operatorname{Re}\sigma_n(x,\xi) \ge \frac{1}{2}\Sigma(x,\xi)-\frac{1}{2}c. \tag{12.16}$$

*Alternatively, adding c to both sides of Eq. (12.16) shows*

$$\frac{3}{2}\left(\Sigma\left(x,\xi\right)+c\right)\geq \operatorname{Re}\sigma_{L^n+c}\left(x,\xi\right)\geq \frac{1}{2}\left(\Sigma\left(x,\xi\right)+c\right). \tag{12.17}$$

*A key point is that* $\operatorname{Re}\sigma_n\left(x,\xi\right):=\operatorname{Re}\sigma_{L^n}\left(x,\xi\right)$ *(see Notation 12.2) is bounded from below.*

**Notation 12.9.** *For the rest of this chapter, we fix a* $c>0$ *as in Lemma 12.8 and then define* $\mathbb{L}:=L^n+c$ *where* $\mathcal{D}\left(\mathbb{L}\right)=\mathcal{S}$ *and* $\{b_l\}_{l=1}^m$ *in Eq. (1.34) satisfies the assumptions in Theorem 1.30. According to the definition of symbol in Eq. (1.31),*

$$\sigma_{\mathbb{L}}\left(x,\xi\right):=\sigma_{L^n+c}\left(x,\xi\right)=\sigma_n\left(x,\xi\right)+c$$

Because of our choice of $c>0$ we know that

$$\kappa:=\inf_{(x,\xi)}\operatorname{Re}\sigma_{\mathbb{L}}\left(x,\xi\right)>0. \tag{12.18}$$

**Corollary 12.10.** *For all* $l,k\in\mathbb{N}_0$,

$$\left|\partial_\xi^l\partial_x^k\sigma_{\mathbb{L}}\left(x,\xi\right)\right|\left(1+|\xi|\right)^l\left(1+|x|\right)^k\lesssim\Sigma\left(x,\xi\right)+1 \tag{12.19}$$

*or equivalently,*

$$\left|\frac{\partial_\xi^l\partial_x^k\sigma_{\mathbb{L}}\left(x,\xi\right)}{\operatorname{Re}\sigma_{\mathbb{L}}\left(x,\xi\right)}\right|\lesssim\frac{1}{\left(1+|\xi|\right)^l\left(1+|x|\right)^k} \tag{12.20}$$

*if Eq. (12.17) is applied.*

**Proof.** We have

$$\partial_\xi^l\partial_x^k\sigma_{\mathbb{L}}\left(x,\xi\right)=\sum_{j=l}^{2mn}\left(i\right)^j\frac{j!}{(j-l)!}\partial_x^kA_j\left(x\right)\xi^{j-l}$$

and therefore,

$$\begin{aligned}
&\left|\partial_\xi^l \partial_x^k \sigma_{\mathbb{L}}(x,\xi)\right| (1+|\xi|)^l (1+|x|)^k \\
&\qquad \le \sum_{j=l}^{2mn} \frac{j!}{(j-l)!} \left|\partial_x^k A_j(x)\right| \left|\xi^{j-l}\right| (1+|\xi|)^l (1+|x|)^k \\
&\qquad \lesssim \sum_{j=l}^{2mn} \left|\partial_x^k A_j(x)\right| \left(1+|\xi|^j\right)\left(1+|x|^k\right) \\
&\qquad \lesssim \Sigma(x,\xi) + 1.
\end{aligned}$$

The last step is asserted by Corollary 12.7. Equation (12.20) follows directly from Eqs. (12.17) and (12.19). ■

By the Fourier inversion formula, if $\psi \in \mathcal{S}$, then

$$\psi(x) = \int_{\mathbb{R}} \widehat{\psi}(\xi)\, e^{ix\xi} d\xi, \tag{12.21}$$

where $\widehat{\psi}$ is the Fourier transform of $\psi$ defined by

$$\widehat{\psi}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iy\xi} \psi(y) dy. \tag{12.22}$$

Recall that, with these normalizations, that

$$\|\psi\| = \sqrt{2\pi} \left\|\hat{\psi}\right\| \ \ \forall\ \psi \in L^2(\mathbb{R}). \tag{12.23}$$

Letting $\mu \in \mathbb{R}$ and then applying $\mathbb{L} + i\mu$ to Eq. (12.21) gives the following pseudo-differential operator representation of $(\mathbb{L}+i\mu)\psi$,

$$(\mathbb{L}+i\mu)\psi(x) = \int_{\mathbb{R}} [\sigma_{\mathbb{L}}(x,\xi) + i\mu]\, e^{ix\xi} \widehat{\psi(\xi)} d\xi. \tag{12.24}$$

Let $\kappa$ be as in Eq. (12.18), it follows that for any $\mu \in \mathbb{R}$,

$$|\sigma_{\mathbb{L}}(x,\xi) + i\mu| \geq |\mathrm{Re}\,\sigma_{\mathbb{L}}(x,\xi)| \geq \kappa > 0$$

for all $(x,\xi) \in \mathbb{R}^2$. Therefore, the following integrand in Eq. (12.25) is integrable for $u \in \mathcal{S}$ and we may define

$$(T_\mu u)(x) = \int_{\mathbb{R}} \frac{1}{\sigma_{\mathbb{L}}(x,\xi) + i\mu} e^{ix\xi} \hat{u}(\xi) d\xi. \tag{12.25}$$

Furthermore, we will show that $T_\mu$ actually preserves $\mathcal{S}$ later in this chapter (see Proposition 12.15).

**Notation 12.11.** *If $\{q_k(x)\}_{k=0}^{j}$ is a collection of smooth functions and*

$$q(x,\theta) = \sum_{k=0}^{j} q_k(x)\,\theta^k, \tag{12.26}$$

*then $q(x,\partial)$ is defined to be the $j^{th}$ – order differential operator given by*

$$q(x,\partial) := \sum_{k=0}^{j} q_k(x)\,\partial_x^k. \tag{12.27}$$

*Similarly, for $\xi \in \mathbb{R}$, we let*

$$q\left(x, \frac{1}{i}\partial_x + \xi\right) := \sum_{k=0}^{j} q_k(x)\left(\frac{1}{i}\partial_x + \xi\right)^k. \tag{12.28}$$

For the proofs below, recall from Eq. (11.7) that

$$q(\partial_x)\, M_{e^{i\xi\cdot x}} = M_{e^{i\xi\cdot x}}\, q(\partial_x + i\xi). \tag{12.29}$$

whenever $q(\theta)$ is a polynomial in $\theta$.

**Lemma 12.12.** *Let $q(x,\theta)$ be as in Eq. (12.26) where the coefficients $\{q_k(x)\}_{k=0}^{j}$ are now assumed to be polynomials in $x$. Further let*

$$(Su)(x) := \int_{\mathbb{R}} \Gamma(x,\xi)\, e^{ix\xi}\hat{u}(\xi)d\xi \;\forall\; u \in \mathcal{S}, \tag{12.30}$$

*where $\Gamma(x,\xi)$ is a smooth function such that $\Gamma(x,\xi)$ and all of its derivatives in both $x$ and $\xi$ have at most polynomial growth in $\xi$ for any fixed $x$. Then*

$$(q(x,\partial_x)\, Su)(x) = \int_{\mathbb{R}} e^{i\xi\cdot x} q(i\partial_\xi, \partial_x + i\xi)\left[\Gamma(x,\xi)\,\hat{u}(\xi)\right] d\xi, \tag{12.31}$$

*where*

$$q(i\partial_\xi, \partial_x + i\xi) := \sum_{k=0}^{j} q_k(i\partial_\xi)\left(\partial_x + iM_\xi\right)^k. \tag{12.32}$$

**Proof.** Using Eq. (12.29) we find,

$$\begin{aligned}
(q(x,\partial)\, Su)(x) &= \int_{\mathbb{R}} q(x,\partial_x)\left[\Gamma(x,\xi)\,\hat{u}(\xi)\, e^{i\xi\cdot x}\right] d\xi \\
&= \sum_{k=0}^{j} \int_{\mathbb{R}} q_k(x)\, \partial_x^k \left[\Gamma(x,\xi)\,\hat{u}(\xi)\, e^{i\xi\cdot x}\right] d\xi \\
&= \sum_{k=0}^{j} \int_{\mathbb{R}} q_k(x)\, e^{i\xi\cdot x} \left(\partial_x + i\xi\right)^k \left[\Gamma(x,\xi)\,\hat{u}(\xi)\right] d\xi \\
&= \sum_{k=0}^{j} \int_{\mathbb{R}} \left[q_k(-i\partial_\xi)\, e^{i\xi\cdot x}\right] \left(\partial_x + i\xi\right)^k \left[\Gamma(x,\xi)\,\hat{u}(\xi)\right] d\xi \\
&= \sum_{k=0}^{j} \int_{\mathbb{R}} e^{i\xi\cdot x} q_k(i\partial_\xi) \left(\partial_x + i\xi\right)^k \left[\Gamma(x,\xi)\,\hat{u}(\xi)\right] d\xi \\
&= \int_{\mathbb{R}} e^{i\xi\cdot x} q(i\partial_\xi, \partial_x + i\xi)\left[\Gamma(x,\xi)\,\hat{u}(\xi)\right] d\xi.
\end{aligned}$$

We have used the assumptions on $\Gamma$ to show; (1) that $\partial_x$ commutes with the integral giving the first equality above, and (2) that

$$\xi \to \left(\partial_x + i\xi\right)^k \left[\Gamma(x,\xi)\,\hat{u}(\xi)\right] \in \mathcal{S}$$

which is used to justify the integration by parts used in the second to last equality. ■

**Lemma 12.13.** *Suppose that* $f \in C^{\infty}\left(\mathbb{R}^j, (0,\infty)\right),$ *then for every multi-index,* $\alpha = (\alpha_1, \ldots, \alpha_j) \in \mathbb{N}_0^j$ *with* $\alpha \neq 0$ *there exists a polynomial function,* $P_\alpha$*, with no constant term such that*

$$\partial^\alpha \frac{1}{f} = \frac{1}{f} P_\alpha \left( \left\{ \frac{\partial^\beta f}{f} : 0 < \beta \leq \alpha \right\} \right)$$

*where* $\partial^\alpha := \partial_1^{\alpha_1} \ldots \partial_j^{\alpha_j}$*. Moreover,* $P_\alpha \left( \left\{ \frac{\partial^\beta f}{f} : 0 < \beta \leq \alpha \right\} \right)$ *is a linear combination of monomials of the form* $\prod_{i=1}^k \frac{\partial^{\beta^{(i)}} f}{f}$ *where* $\beta^{(i)} \in \mathbb{N}_0^j$ *for* $1 \leq i \leq k$ *and* $1 \leq k \leq |\alpha|$ *such that* $\sum_{i=1}^k \beta^{(i)} = \alpha$*, and* $\beta^{(i)} \neq 0$ *for all* $i$*.*

**Proof.** The proof is a straight forward induction argument which will be left to the reader. However, by way of example one easily shows,

$$\partial_1 \partial_2 \frac{1}{f} = \frac{1}{f} \cdot \left[ \frac{\partial_1 f}{f} \frac{\partial_2 f}{f} - \frac{\partial_1 \partial_2 f}{f} \right].$$

■

**Corollary 12.14.** *Let* $\mu \in \mathbb{R}$*. If*

$$\Gamma(x,\xi) := \frac{1}{\sigma_{\mathbb{L}}(x,\xi) + i\mu}, \tag{12.33}$$

*then*

$$|\Gamma(x,\xi)| \leq \frac{1}{|\operatorname{Re} \sigma_{\mathbb{L}}(x,\xi)|} \lesssim \frac{1}{b_m^n(x)\,\xi^{2mn} + b_0^n(x) + c} \lesssim 1 \tag{12.34}$$

*and for any* $\alpha, \beta \in \mathbb{N}_0$ *with* $\alpha + \beta > 0$*, there exists a constant* $c_{\alpha,\beta} > 0$ *such that*

$$\left| \partial_x^\alpha \partial_\xi^\beta \Gamma(x,\xi) \right| \leq |\Gamma(x,\xi)| \cdot c_{\alpha,\beta} \left(1 + |\xi|\right)^{-\beta} \left(1 + |x|\right)^{-\alpha}. \tag{12.35}$$

**Proof.** The estimate in Eq. (12.34) is elementary from Eq.(12.17) in Lemma

12.8 and will be left to the reader. From Lemma 12.13,

$$\begin{aligned}\partial_x^\alpha \partial_\xi^\beta \Gamma(x,\xi) &= \partial_x^\alpha \partial_\xi^\beta \frac{1}{\sigma_{\mathbb{L}}(x,\xi) + i\mu} = \frac{1}{\sigma_{\mathbb{L}}(x,\xi) + i\mu} \cdot \Theta_{(\alpha,\beta)}(x,\xi) \\ &= \Gamma(x,\xi) \cdot \Theta_{(\alpha,\beta)}(x,\xi)\end{aligned}$$

where $\Theta_{(\alpha,\beta)}(x,\xi)$ is a linear combination of the following functions,

$$\prod_{j=1}^{J} \frac{\partial_x^{k_j} \partial_\xi^{l_j} \sigma_{\mathbb{L}}(x,\xi)}{\sigma_{\mathbb{L}}(x,\xi) + i\mu}$$

where $0 \le k_j \le \alpha,\ 0 \le l_j \le \beta,\ k_j + l_j > 0,\ \sum_{j=1}^{J} k_j = \alpha,\ \sum_{j=1}^{J} l_j = \beta$ and $1 \le J \le \alpha + \beta$. The estimate in Eq. (12.20) implies,

$$\begin{aligned}\prod_{j=1}^{J} \left| \frac{\partial_x^{k_j} \partial_\xi^{l_j} \sigma_{\mathbb{L}}(x,\xi)}{\sigma_{\mathbb{L}}(x,\xi) + i\mu} \right| &\le \prod_{j=1}^{J} \left| \frac{\partial_x^{k_j} \partial_\xi^{l_j} \sigma_{\mathbb{L}}(x,\xi)}{\operatorname{Re} \sigma_{\mathbb{L}}(x,\xi)} \right| \\ &\lesssim \prod_{j=1}^{J} \frac{1}{(1+|\xi|)^{l_j} (1+|x|)^{k_j}} \lesssim (1+|\xi|)^{-\beta} (1+|x|)^{-\alpha}\end{aligned}$$

which altogether gives the estimated in Eq. (12.35). ■

**Proposition 12.15** ($T_\mu$ preserves $\mathcal{S}$). *If $T_\mu$ is as defined in Eq. (12.25), then $T_\mu(\mathcal{S}) \subset \mathcal{S}$ for all $\mu \in \mathbb{R}$.*

**Proof.** Let $\Gamma$ be as in Eq. (12.33) so that $T_\mu = S$ where $S$ is as in Lemma 12.12. According Corollary 12.14, for all $\alpha, \beta \in \mathbb{N}_0$, we know that $\left|\partial_x^\alpha \partial_\xi^\beta \Gamma(x,\xi)\right| \le C_{\alpha,\beta}$ for some constants $C_{\alpha,\beta}$ and hence, from Lemma 12.12, if $q(x,\theta)$ is as in Eq. (12.26), then

$$|q(x,\partial_x) T_\mu u(x)| \le \int_{\mathbb{R}} |q(i\partial_\xi, \partial_x + i\xi)[\Gamma(x,\xi)\hat{u}(\xi)]|\, d\xi. \tag{12.36}$$

The integrand in Eq. (12.36) may be bounded by a finite linear combination of terms of the form

$$\left|\partial_x^\alpha \partial_\xi^\beta \Gamma(x,\xi)\right| \cdot \left|\xi^j \partial_\xi^l \hat{u}\right|(\xi) \lesssim \left|\xi^j \partial_\xi^l \hat{u}\right|(\xi).$$

Since $\hat{u} \in \mathcal{S}$, $\left|\xi^j \partial_\xi^l \hat{u}\right|(\xi)$ is integrable and therefore we may conclude that

$$\sup_{x\in\mathbb{R}} \left|\left(q(x,\partial_x) T_\mu u\right)(x)\right| < \infty.$$

As $q(x,\theta)$ was an arbitrary polynomial in $(x,\theta)$ we conclude that $T_\mu u \in \mathcal{S}$. ■

We assume $\Gamma$ is as in Eq. (12.33) for the remainder of Part II.

**Lemma 12.16.** *For all $\mu \in \mathbb{R}$ and $u \in \mathcal{S}$,*

$$[\mathbb{L} + i\mu] T_\mu u = [I + R_\mu] u \tag{12.37}$$

*where*

$$(R_\mu u)(x) = \int_{\mathbb{R}} \rho_\mu(x,\xi) \hat{u}(\xi) e^{ix\xi} d\xi, \tag{12.38}$$

$$\rho_\mu(x,\xi) := \left(\left[\sigma_{\mathbb{L}}\left(x, \frac{1}{i}\partial_x + \xi\right) - \sigma_{\mathbb{L}}(x,\xi)\right] \frac{1}{\sigma_{\mathbb{L}}(x,\xi) + i\mu}\right), \tag{12.39}$$

*and $\sigma_{\mathbb{L}}\left(x, \frac{1}{i}\partial_x + \xi\right)$ is as in Eq. (12.28).*

**Proof.** As $\sigma_{\mathbb{L}}(x,\xi)$ is a polynomial in the $\xi$ – variable with smooth coefficients in the $x$ – variable, there is no problem justifying the identity,

$$\left([\mathbb{L} + i\mu] T_\mu u\right)(x) = \int_{\mathbb{R}} [\mathbb{L}_x + i\mu] \left(\Gamma(x,\xi) e^{ix\xi}\right) \hat{u}(\xi) d\xi, \tag{12.40}$$

where the subscript $x$ on $\mathbb{L}$ indicates that $\mathbb{L}$ acts on $x$ – variable only. Using

$\mathbb{L} = \sigma_{\mathbb{L}}\left(x, \frac{1}{i}\partial_x\right)$ along with Eq. (12.29) shows,

$$\begin{aligned}
&[\mathbb{L}_x + i\mu]\left(\Gamma(x,\xi)\,e^{ix\xi}\right) \\
&\quad = \left[\sigma_{\mathbb{L}}\left(x, \frac{1}{i}\partial_x\right) + i\mu\right]\left(e^{ix\xi}\Gamma(x,\xi)\right) \\
&\quad = e^{ix\xi}\left[\sigma_{\mathbb{L}}\left(x, \frac{1}{i}\partial_x + \xi\right) + i\mu\right]\Gamma(x,\xi) \\
&\quad = e^{ix\xi}\left[\sigma_{\mathbb{L}}\left(x, \frac{1}{i}\partial_x + \xi\right) - \sigma_{\mathbb{L}}(x,\xi) + (\sigma_{\mathbb{L}}(x,\xi) + i\mu)\right]\Gamma(x,\xi) \\
&\quad = e^{ix\xi}\left[\sigma_{\mathbb{L}}\left(x, \frac{1}{i}\partial_x + \xi\right) - \sigma_{\mathbb{L}}(x,\xi)\right]\Gamma(x,\xi) + e^{ix\xi}
\end{aligned}$$

which combined with Eq. (12.40) gives Eq. (12.37). ■

**Lemma 12.17.** *$\rho_\mu(x,\xi)$ in Eq. (12.39) can be explicitly written as*

$$\rho_\mu(x,\xi) = \sum_{k=1}^{2mn}\sum_{j=1}^{k}\binom{k}{j}A_k(x)\,(i\xi)^{k-j}\,\partial_x^j\Gamma(x,\xi)\,. \tag{12.41}$$

*where $A_k(x)$ and $\Gamma(x,\xi)$ as in Eq. (11.1) and Eq. (12.33) respectively. Moreover, there exists $C < \infty$ independent of $\mu$ so that*

$$|\rho_\mu(x,\xi)| \le C\frac{1}{1+|x|}\cdot\frac{1}{1+|\xi|}. \tag{12.42}$$

**Proof.** Using Eq. (1.35) and the formula of $\sigma_{\mathbb{L}}$ in Notation 12.9, we may write Eq. (12.39) more explicitly as,

$$\begin{aligned}
\rho_\mu(x,\xi) &= \sum_{k=0}^{2mn}A_k(x)\left[(\partial_x + i\xi)^k - (i\xi)^k\right]\Gamma(x,\xi) \\
&= \sum_{k=1}^{2mn}\sum_{j=1}^{k}\binom{k}{j}A_k(x)\,(i\xi)^{k-j}\,\partial_x^j\Gamma(x,\xi)\,.
\end{aligned}$$

Therefore, using the estimate in Eq. (12.35) of Corollary 12.14 with $\beta = 0$ and

$\alpha = j$, we learn

$$\begin{aligned} |\rho_\mu(x,\xi)| &\le \sum_{k=1}^{2mn}\sum_{j=1}^{k}\binom{k}{j}|A_k(x)|\,|\xi|^{k-j}\left|\partial_x^j\Gamma(x,\xi)\right| \\ &\lesssim \sum_{k=1}^{2mn}\sum_{j=1}^{k}\binom{k}{j}|A_k(x)|\,|\xi|^{k-j}|\Gamma(x,\xi)|\frac{1}{1+|x|}. \end{aligned}$$

Moreover, for any $1 \le j \le k$,

$$\begin{aligned} |A_k(x)|\,|\xi|^{k-j}|\Gamma(x,\xi)| &\lesssim \frac{\Sigma(x,\xi)+1}{1+|\xi|^k}|\xi|^{k-j}|\Gamma(x,\xi)| \\ &\le \frac{1}{1+|\xi|^j}\frac{\Sigma(x,\xi)+1}{|\operatorname{Re}\sigma_{\mathbb{L}}(x,\xi)|} \\ &\lesssim \frac{1}{1+|\xi|^j} \lesssim \frac{1}{1+|\xi|}, \end{aligned}$$

wherein we have used the estimates in Eq. (12.14) with $\beta = 0$ in the first step, and the left inequality in Eq. (12.34) in the second step, and Eq. (12.17) in the third step. ■

We are now prepared to complete the proof of Theorem 1.30. The following notation will be used in the proof.

**Notation 12.18.** *If $g : \mathbb{R}^2 \to \mathbb{C}$ is a measurable function we let*

$$\begin{aligned} \|g(x,\xi)\|_{L^2(d\xi)} &:= \left(\int_{\mathbb{R}}|g(x,\xi)|^2\,d\xi\right)^{1/2} \quad \textit{and} \\ \|g(x,\xi)\|_{L^2(dx\otimes d\xi)} &:= \left(\int_{\mathbb{R}^2}|g(x,\xi)|^2\,dxd\xi\right)^{1/2}. \end{aligned}$$

**Proof of Theorem 1.30.** The only thing left to show is that condition 2 in Lemma 12.1 is verified. Thus we have to estimate the operator norm of the error term,

$$(R_\mu u)(x) = \int_{\mathbb{R}}\rho_\mu(x,\xi)\,e^{ix\xi}\hat{u}(\xi)d\xi.$$

Using the Cauchy–Schwarz inequality and the isometry property (see Eq. (12.23) of the Fourier transform it follows that

$$\|R_\mu u\|_{L^2(dx)} \leq \frac{1}{\sqrt{2\pi}} \|\rho_\mu\|_{L^2(dx\otimes d\xi)} \cdot \|u\|_{L^2(dx)}$$

where $\rho_\mu$ is the symbol of $R_\mu$ as defined in Eq. (12.39). Since, by Lemma 12.13 and Eq. (12.41), $\lim_{\mu\to\pm\infty} \rho_\mu (x, \xi) = 0$ and, from Eq. (12.42), $\rho_\mu$ is dominated by

$$C (1 + |x|)^{-1} (1 + |\xi|)^{-1} \in L^2 (dx \otimes d\xi) ,$$

it follows that $\|R_\mu\|_{op} \leq \frac{1}{\sqrt{2\pi}} \|\rho_\mu\|_{L^2(dx\otimes d\xi)} \to 0$ as $\mu \to \pm\infty$ and in particular, $\|R_\mu\|_{op} < 1$ when $|\mu|$ is sufficiently large. Therefore, $\mathbb{L}|_{\mathcal{S}}$ is essentially self-adjoint from Lemma 12.1 and hence $L^n|_{\mathcal{S}} = (\mathbb{L} - c)|_{\mathcal{S}}$ ($c$ from Notation 12.9) is also essentially self-adjoint. ■

# Chapter 13

# The Divergence Form of $L^n$ and $L_\hbar^n$

Suppose now that $L$ in Eq. (10.1) with polynomial coefficients $\{b_l\}_{l=0}^m$ is a symmetric differential operator on $\mathcal{S}$. In Chapter 11, we have expressed the symmetric differential operators on $\mathcal{S}$, $L^n$, in the divergence form with the polynomial coefficients $\{B_\ell\}$ as in Eq. (11.1) for $n \in \mathbb{N}$. The goal of this chapter is to derive some basic properties of the polynomial coefficients $\{B_\ell\}$ and generalize coefficients properties for a scaled version $L_\hbar^n$ where $L_\hbar$ is in Eq. (1.39).

**Proposition 13.1.** *Suppose that $\{b_l\}_{l=0}^m$ are real polynomials. Let $\mathcal{B}_\ell$ and $R_\ell$ are in Eqs. (11.2) and (11.13) respectively.*

1. *If $\deg b_l \le \deg b_{l-1}$ for $1 \le l \le m$, then*

$$\deg(R_\ell) \le \deg \mathcal{B}_\ell - 2 \text{ and } \deg B_\ell = \deg \mathcal{B}_\ell \text{ for } 0 \le \ell \le mn.$$

2. *If we only assume that $\deg b_l \le \deg b_{l-1} + 2$ for $1 \le l \le m$, then*

$$\deg R_\ell \le \deg \mathcal{B}_\ell \quad \text{for } 0 \le \ell \le mn.$$

**Proof.** From Eq. (11.12), $\deg(B_\ell) = \deg \mathcal{B}_\ell$ follows automatically if

$$\deg(R_\ell) \le \deg \mathcal{B}_\ell - 2 \tag{13.1}$$

holds. Therefore, the only thing to prove in the item 1 is Eq. (13.1). From Proposition 11.8, $R_\ell$ is a linear combination of $(\partial^{p_1} b_{k_1})(\partial^{p_2} b_{k_2})\dots(\partial^{p_n} b_{k_n})$ with $0 < |\mathbf{p}| = 2|\mathbf{k}| - 2\ell$. For each index $\mathbf{k}$, there exists $\mathbf{j}$ with $\mathbf{j} \le \mathbf{k}$ such that $|\mathbf{j}| = \ell$ and for this $\mathbf{j}$ we have

$$\begin{aligned}\deg\left((\partial^{p_1} b_{k_1})(\partial^{p_2} b_{k_2})\dots(\partial^{p_n} b_{k_n})\right) &\le \sum_{i=1}^{n} \deg(b_{k_i}) - |\mathbf{p}| \\ &\le \sum_{i=1}^{n} \deg(b_{j_i}) - |\mathbf{p}| \\ &\le \deg(\mathcal{B}_\ell) - 2,\end{aligned}$$

wherein we have used $|\mathbf{p}| \ge 2$ ($|\mathbf{p}|$ is positive even) and

$$\deg(\mathcal{B}_\ell) = \max_{|\mathbf{j}|=\ell} \deg(b_{j_1}\dots b_{j_n}) = \max_{|\mathbf{j}|=\ell} \sum_{i=1}^{n} \deg(b_{j_i}).$$

Now suppose that we only assume $\deg(b_{k+1}) \le \deg(b_k) + 2$ (which then implies $\deg(b_{k+r}) \le \deg(b_k) + 2r$ for $0 \le r \le m - k$). Working as above and remember that $0 < |\mathbf{p}| = 2|\mathbf{k}| - 2\ell$ and $|j| = \ell$ we find

$$\begin{aligned}\sum_{i=1}^{n} \deg(b_{k_i}) - |\mathbf{p}| &\le \sum_{i=1}^{n} \left[\deg(b_{j_i}) + 2(k_i - j_i)\right] - |\mathbf{p}| \\ &= \sum_{i=1}^{n} \deg(b_{j_i}) + 2(|\mathbf{k}| - |\mathbf{j}|) - |\mathbf{p}| \\ &= \sum_{i=1}^{n} \deg(b_{j_i}) \le \deg(\mathcal{B}_\ell).\end{aligned}$$

■

# 1 Scaled Version of Divergence Form

We now take $\hbar > 0$ and let $L_\hbar$ be defined as in Eq. (1.39) where the $\hbar$ – dependent coefficients, $\{b_{l,\hbar}(x)\}_{l=0}^m$, satisfy Assumption 1.34. To apply the previous formula already developed (for $\hbar = 1$) we need only make the replacements,

$$b_l(x) \to \hbar^l b_{l,\hbar}\left(\sqrt{\hbar}x\right) \text{ for } 0 \le l \le m. \tag{13.2}$$

The result of this transformation on $L^n$ is recorded in the following lemma.

**Notation 13.2.** *Let $x_1, \ldots, x_j$ be variables on $\mathbb{R}$. We denote $\mathbb{R}[x_1, \ldots, x_j]$ be a collection of polynomials in $x_1, \ldots, x_j$ with real-valued coefficients.*

**Proposition 13.3.** *Let $n \in \mathbb{N}$, $\hbar > 0$, and $\{b_{l,\hbar}(x)\}_{l=0}^m \subset \mathbb{R}[x]$ and let $L_\hbar$ be as in Eq. (1.39). Then $L_\hbar^n$ is an operator on $\mathcal{S}$ and*

$$L_\hbar^n = \sum_{\ell=0}^{mn} (-\hbar)^\ell \partial^\ell B_{\ell,\hbar}\left(\sqrt{\hbar}(\cdot)\right)\partial^\ell, \tag{13.3}$$

*where*[1]

$$B_{\ell,\hbar} := \mathcal{B}_{\ell,\hbar} + R_{\ell,\hbar} \in \mathbb{R}[x], \tag{13.4}$$

$$\mathcal{B}_{\ell,\hbar} = \sum_{\mathbf{k}\in\Lambda_m^n} 1_{|\mathbf{k}|=\ell} b_{k_1,\hbar} b_{k_2,\hbar} \ldots b_{k_n,\hbar}, \tag{13.5}$$

$$R_{\ell,\hbar} = \sum_{\substack{\mathbf{k}\in\Lambda_m^n,\\ \mathbf{p}\in\Lambda_{2m}^n}} \hat{C}(n,\ell,\mathbf{k},\mathbf{p})\,\hbar^{|\mathbf{p}|}\left(\partial^{p_1} b_{k_1,\hbar}\right)\ldots\left(\partial^{p_n} b_{k_n,\hbar}\right). \tag{13.6}$$

**Proof.** Making the replacements $b_l(x) \to \hbar^l b_{l,\hbar}\left(\sqrt{\hbar}x\right)$ in Eqs. (11.2) and

[1] Below, we use $\hat{C}(n,\ell,\mathbf{k},\mathbf{p}) = 0$ unless $0 < |\mathbf{p}| = 2|\mathbf{k}| - 2\ell$, i.e. $|\mathbf{k}| = \ell + |\mathbf{p}|/2$.

(11.13) shows

$$\mathcal{B}_{\ell}\left(x\right)\to\sum_{\mathbf{j}\in\Lambda_m^n}1_{|\mathbf{j}|=\ell}\left[\hbar^{j_1}b_{j_1,\hbar}\left(\sqrt{\hbar}x\right)\dots\hbar^{j_n}b_{j_n,\hbar}\left(\sqrt{\hbar}x\right)\right]=\hbar^{\ell}\mathcal{B}_{\ell,\hbar}\left(\sqrt{\hbar}x\right) \tag{13.7}$$

and

$$\begin{aligned}
R_{\ell}\left(x\right)\to&\sum_{\mathbf{k}\in\Lambda_m^n,\mathbf{p}\in\Lambda_{2m}^n}\hat{C}\left(n,\ell,\mathbf{k},\mathbf{p}\right)(-1)^{|\mathbf{k}|}\hbar^{|\mathbf{k}|+\frac{|\mathbf{p}|}{2}}\left[\left(\partial^{p_1}b_{k_1,\hbar}\right)\dots\left(\partial^{p_n}b_{k_n,\hbar}\right)\right]\left(\sqrt{\hbar}x\right)\\
&=\sum_{\mathbf{k}\in\Lambda_m^n,\mathbf{p}\in\Lambda_{2m}^n}\hat{C}\left(n,\ell,\mathbf{k},\mathbf{p}\right)(-1)^{|\mathbf{k}|}\hbar^{\ell+|\mathbf{p}|}\left[\left(\partial^{p_1}b_{k_1,\hbar}\right)\dots\left(\partial^{p_n}b_{k_n,\hbar}\right)\right]\left(\sqrt{\hbar}x\right)\\
&=\hbar^{\ell}\sum_{\mathbf{k}\in\Lambda_m^n,\mathbf{p}\in\Lambda_{2m}^n}\hat{C}\left(n,\ell,\mathbf{k},\mathbf{p}\right)(-1)^{|\mathbf{k}|}\hbar^{|\mathbf{p}|}\left[\left(\partial^{p_1}b_{k_1,\hbar}\right)\dots\left(\partial^{p_n}b_{k_n,\hbar}\right)\right]\left(\sqrt{\hbar}x\right)\\
&=\hbar^{\ell}R_{\ell,\hbar}\left(\sqrt{\hbar}x\right).
\end{aligned} \tag{13.8}$$

Therefore it follows that

$$B_{\ell}\left(x\right)=\mathcal{B}_{\ell}\left(x\right)+R_{\ell}\left(x\right)\to\hbar^{\ell}\left[\mathcal{B}_{\ell,\hbar}+R_{\ell,\hbar}\right]\left(\sqrt{\hbar}x\right)=\hbar^{\ell}B_{\ell,\hbar}\left(\sqrt{\hbar}x\right)$$

where $L_{\hbar}^{n}$ is then given as in Eq. (13.3). ■

**Notation 13.4.** *Let*

$$\mathcal{L}_{\hbar}^{(n)}=\sum_{\ell=0}^{mn}\left(-\hbar\right)^{\ell}\partial^{\ell}\mathcal{B}_{\ell,\hbar}\left(\sqrt{\hbar}\left(\cdot\right)\right)\partial^{\ell},\ \textit{and} \tag{13.9}$$

$$\mathcal{R}_{\hbar}^{(n)}=\sum_{\ell=0}^{mn-1}\left(-\hbar\right)^{\ell}\partial^{\ell}R_{\ell,\hbar}\left(\sqrt{\hbar}\left(\cdot\right)\right)\partial^{\ell}. \tag{13.10}$$

*as operators on $\mathcal{S}$. Then $L_{\hbar}^{n}$ can also be written as*

$$L_{\hbar}^{n}=\mathcal{L}_{\hbar}^{(n)}+\mathcal{R}_{\hbar}^{(n)}\ \textit{on}\ \mathcal{S}. \tag{13.11}$$

# Chapter 14

# Operator Comparison

The main purpose of this chapter is to prove Theorem 1.37. First off, since the inequality symbol $\preceq_{\mathcal{S}}$ appears very often in this chapter which is defined in Notation 1.10, let us recall its definition. If $A$ and $B$ are symmetric operators on $\mathcal{S}$ (see Definition 1.9), then we say $A \preceq_{\mathcal{S}} B$ if

$$\langle A\psi, \psi\rangle \preceq_{\mathcal{S}} \langle B\psi, \psi\rangle \text{ for all } \psi \in \mathcal{S}$$

where $\langle \cdot, \cdot \rangle$ is the usual $L^2(m)$ – inner product as in Eq. (1.4).

**Lemma 14.1.** *Let $\{c_\ell\}_{\ell=0}^{M-1} \subset \mathbb{R}$ be given constants. Then for any $\delta > 0$, there exists $C_\delta < \infty$ such that*

$$\sum_{\ell=0}^{M-1} c_\ell \left(-\hbar\partial^2\right)^\ell \preceq_{\mathcal{S}} \delta \left(-\hbar\partial^2\right)^M + C_\delta I \ \forall\ \hbar > 0. \tag{14.1}$$

**Proof.** By conjugating Eq. (14.1) by the Fourier transform in Eq.(12.22) (so that $\frac{1}{i}\partial \to \xi$) and using Eq.(12.23), we may reduce Eq. (14.1) to the easily verified statement; for all $\delta > 0$, there exists $C_\delta < \infty$ such that

$$\sum_{\ell=0}^{M-1} c_\ell w^\ell \le \delta w^M + C_\delta \; \forall \; w \ge 0. \tag{14.2}$$

Here, $w$ is shorthand for $\hbar\xi^2$. ■

**Lemma 14.2.** *Let $I \subset \mathbb{R}$ be a compact interval. Suppose $\{p_k(\cdot)\}_{k=0}^{m_p}$ and $\{q_k(\cdot)\}_{k=0}^{m_q} \subset C(I,\mathbb{R})$ where $m_p \in 2\mathbb{N}_0$ and $m_q \in \mathbb{N}_0$ such that*

$$p(x,y) = \sum_{k=0}^{m_p} p_k(y)\, x^k \text{ and } q(x,y) = \sum_{k=0}^{m_q} q_k(y)\, x^k$$

*and $\delta := \min_{y\in I} p_{m_p}(y) > 0$. If we further assume $m_p > m_q$ then for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that we have*

$$|q(x,y)| \le \epsilon p(x,y) + C_\epsilon \text{ for all } y \in I \text{ and } x \in \mathbb{R}. \tag{14.3}$$

*If $m_p = m_q$ there exists $D$ and $E > 0$ such that we have*

$$|q(x,y)| \le Dp(x,y) + E \text{ for all } y \in I \text{ and } x \in \mathbb{R}. \tag{14.4}$$

**Proof.** Let $M$ be an upper bound for $|p_k(y)|$ and $|q_l(y)|$ for all $y \in I$, $0 \le k \le m_p$ and $0 \le l \le m_q$. Then for any $D > 0$ we have,

$$|q(x,y)| - Dp(x,y) \le \rho_D(x) \tag{14.5}$$

where

$$\rho_D(x) := M \sum_{k=0}^{m_q} |x|^k - D\delta\, |x|^{m_p} + DM \sum_{k=0}^{m_p-1} |x|^k .$$

If $m_p > m_q$ we see $\lim_{x\to\pm\infty} \rho_D(x) = -\infty$ for all $D = \varepsilon > 0$ and hence $C_\varepsilon := \max_{x\in\mathbb{R}} \rho_\varepsilon(x) < \infty$ which combined with Eq. (14.5) proves Eq. (14.3). If $m_p = m_q$

and $D$ is chosen so that $D\delta > M$, we again will have $\lim_{x\to\pm\infty} \rho_D(x) = -\infty$ and so $E := \max_{x\in\mathbb{R}} \rho_D(x) < \infty$ which combined with Eq. (14.5) proves Eq. (14.4). ■

**Lemma 14.3.** *Suppose that* $\{b_{l,\hbar}(\cdot)\}_{l=0}^m$ *and* $\eta > 0$ *satisfy Assumption 1.34 and* $c_{b_m} > 0$ *is the constant in Eq. (1.41). Let* $n \in \mathbb{N}$ *and* $\{B_{\ell,\hbar}\}_{\ell=0}^{mn}$ *and* $\{\mathcal{B}_{\ell,\hbar}\}_{\ell=0}^{mn}$ *be the polynomials defined in Eqs. (13.4) and (13.5) respectively. Then* $\{B_{\ell,\hbar}\}_{\ell=0}^{mn}$ *and* $\{\mathcal{B}_{\ell,\hbar}\}_{\ell=0}^{mn}$ *satisfy items 1 and 3 of Assumption 1.34 and in particular,*

$$B_{mn,\hbar} = \mathcal{B}_{mn,\hbar} = b_{m,\hbar}^n \geq (c_{b_m})^n . \tag{14.6}$$

*Moreover, if* $R_{\ell,\hbar}$ *is the polynomial in Eq. (13.6), then for any* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *such that*

$$|R_{\ell,\hbar}(x)| \leq \epsilon \mathcal{B}_{\ell,\hbar}(x) + C_\epsilon \ \forall \ x \in \mathbb{R}, \ 0 < \hbar < \eta, \ \& \ \ 0 \leq \ell \leq mn. \tag{14.7}$$

**Proof.** From Eq. (13.5)

$$\mathcal{B}_{\ell,\hbar} = \sum_{\mathbf{k}\in\Lambda_m^n} 1_{|\mathbf{k}|=\ell} b_{k_1,\hbar} b_{k_2,\hbar} \dots b_{k_n,\hbar}$$

from which it easily follows that $\mathcal{B}_{\ell,\hbar}$ is a real polynomial with real valued coefficients depending continuously on $\hbar$. Thus we have verified that the $\{\mathcal{B}_{\ell,\hbar}\}_{\ell=0}^{mn}$ satisfy item 1. of Assumption 1.34.

The highest order coefficient of the polynomial $\mathcal{B}_{\ell,\hbar}$ is a linear combination of $n$-fold products among the highest order coefficients of $\{b_{l,\hbar}(x)\}_{l=0}^m$ and hence is still bounded from below by a positive constant independent of $\hbar \in (0,\eta)$. This observation along with the estimate, $\mathcal{B}_{mn,\hbar} = b_{m,\hbar}^n \geq c_{b_m}^n$, shows $\{\mathcal{B}_{\ell,\hbar}\}_{\ell=0}^{mn}$ also satisfies item 3 of Assumption 1.34.

Applying Proposition 13.1 with $b_l(x) \to \hbar^l b_{l,\hbar}\left(\sqrt{\hbar}x\right)$, it follows (making use of Eqs. (13.7) and (13.8)) that

$$\deg R_{\ell,\hbar} \leq \deg \mathcal{B}_{\ell,\hbar} - 2 \text{ for } 0 \leq \ell \leq mn \text{ and } 0 < \hbar < \eta. \tag{14.8}$$

Since the leading order coefficient of $B_{\ell,\hbar}$ is a continuous function of $\hbar$ which satisfies condition 3 of Assumption 1.34, we may conclude that the degree estimate above also holds at $\hbar = 0$ and $\hbar = \eta$. We now apply Lemma 14.2 with $p(x,\hbar) = \mathcal{B}_{\ell,\hbar}(x)$ (note $\mathcal{B}_{\ell,\hbar}$ satisfies items 1 and 3 of Assumption 1.34), $q(x,\hbar) = R_{\ell,\hbar}(x)$, and $I = [0,\eta]$ to conclude Eq. (14.7) holds.

Finally, let $0 < \hbar < \eta$ be fixed. We learn $B_{\ell,\hbar} = \mathcal{B}_{\ell,\hbar} + R_{\ell,\hbar}$ from Eq.(13.4). It is clear that $\{B_{\ell,\hbar}\}_{\ell=0}^{mn}$ satisfies the item 1 of Assumption 1.34. From Eq. (14.8), the highest order coefficient of $B_{\ell,\hbar}$ and $\mathcal{B}_{\ell,\hbar}$ are the same and Proposition 11.8 shows that $R_{mn,\hbar} = 0$ which implies $B_{mn,\hbar} = \mathcal{B}_{mn,\hbar}$. Therefore $\{B_{\ell,\hbar}\}_{\ell=0}^{mn}$ also satisfies the item 3 of Assumption 1.34. ■

# 1 Estimating the quadratic form associated to $L_{\hbar}^{n}$

**Theorem 14.4.** *Supposed $\{b_{l,\hbar}(x)\}$ and $\eta > 0$ satisfies Assumption 1.34 and let $L_\hbar$ and $\mathcal{L}_\hbar^{(n)}$ be the operators in Eqs. (1.39) and (13.9) respectively. Then for any $n \in \mathbb{N}$, there exists $C_n < \infty$ so that for all $0 < \hbar < \eta$ and $c > C_n$;*

$$\frac{3}{2}\left(\mathcal{L}_\hbar^{(n)} + c\right) \succeq_{\mathcal{S}} L_\hbar^n + c \succeq_{\mathcal{S}} \frac{1}{2}\left(\mathcal{L}_\hbar^{(n)} + c\right)$$

*and both $\mathcal{L}_\hbar^{(n)} + c$ and $L_\hbar^n + c$ are positive operators.*

**Proof.** Let $\psi \in \mathcal{S}$ and $0 < \hbar < \eta$. From Eqs. (13.10) and (13.11) we can conclude

$$|\langle \left(L_\hbar^n - \mathcal{L}_\hbar^{(n)}\right)\psi, \psi\rangle| = \left|\left\langle \mathcal{R}_\hbar^{(n)}\psi, \psi\right\rangle\right| \leq \sum_{\ell=0}^{nm-1} \hbar^\ell \left|\left\langle R_{\ell,\hbar}\left(\sqrt{\hbar}\,(\cdot)\right)\partial^\ell\psi, \partial^\ell\psi\right\rangle\right|.$$

From Eq. (14.7) in Lemma 14.3 by taking $\epsilon = \frac{1}{2}$ and $C_\epsilon = C_{\frac{1}{2}}$ we have

$$|R_{\ell,\hbar}(x)| \leq \frac{1}{2}\mathcal{B}_{\ell,\hbar}(x) + C_{\frac{1}{2}} \tag{14.9}$$

for all $0 \leq \ell \leq mn-1$ and $\hbar \in (0,\eta)$. With the use of Eq.(14.9), we learn

$$\begin{aligned}
&|\langle \left(L_\hbar^n - \mathcal{L}_\hbar^{(n)}\right)\psi,\psi\rangle| \\
&\quad \leq \sum_{\ell=0}^{nm-1} \hbar^\ell \left\langle \left(\frac{1}{2}\mathcal{B}_{\ell,\hbar}\left(\sqrt{\hbar}\,(\cdot)\right) + C_{\frac{1}{2}}\right)\partial^\ell\psi, \partial^\ell\psi \right\rangle \\
&\quad = \frac{1}{2}\sum_{\ell=0}^{nm-1} \left\langle (-\hbar)^\ell\, \partial^\ell \mathcal{B}_{\ell,\hbar}\left(\sqrt{\hbar}\,(\cdot)\right)\partial^\ell\psi, \psi \right\rangle + C_{\frac{1}{2}} \left\langle \sum_{\ell=0}^{nm-1} (-\hbar)^\ell\, \partial^{2\ell}\psi, \psi \right\rangle. \quad (14.10)
\end{aligned}$$

By Eq. (14.6) in Lemma 14.3, we have

$$\mathcal{B}_{mn,\hbar} = b_{m,\hbar}^n \geq c_{b_m}^n > 0.$$

So making use of Lemma 14.1 by taking $\delta = c_{b_m}^n/2$ and $c_\ell = C_{\frac{1}{2}}$, there exists $C_\delta < \infty$ such that for all $0 < \hbar < \eta$ and $\psi \in \mathcal{S}$,

$$\begin{aligned}
&C_{\frac{1}{2}} \left\langle \sum_{\ell=0}^{nm-1} (-\hbar)^\ell\, \partial^{2\ell}\psi, \psi \right\rangle \\
&\qquad \leq (-\hbar)^{mn} \langle \delta \partial^{mn}\psi, \partial^{mn}\psi\rangle + \frac{1}{2}C_\delta \langle \psi,\psi\rangle \\
&\qquad \leq \frac{1}{2}(-\hbar)^{mn} \left\langle \mathcal{B}_{mn,\hbar}\left(\sqrt{\hbar}\,(\cdot)\right)\partial^{mn}\psi, \partial^{mn}\psi \right\rangle + \frac{1}{2}C_\delta \langle \psi,\psi\rangle.
\end{aligned}$$

By combining Eqs. (14.10) and (14.1), we get

$$
\begin{aligned}
&|\langle\psi,\left(L_\hbar^n-\mathcal{L}_\hbar^{(n)}\right)\psi\rangle| \\
&\quad\leq \frac{1}{2}\left\langle\left(\sum_{\ell=0}^{nm-1}(-\hbar)^\ell\,\partial^\ell\mathcal{B}_{\ell,\hbar}\left(\sqrt{\hbar}\,(\cdot)\right)\partial^\ell+(-\hbar)^{mn}\,\partial^{mn}\mathcal{B}_{mn,\hbar}\left(\sqrt{\hbar}\,(\cdot)\right)\partial^{mn}\right)\psi,\psi\right\rangle \\
&\qquad+\frac{1}{2}C_\delta\left\langle\psi,\psi\right\rangle \\
&\quad=\frac{1}{2}\left\langle\left(\mathcal{L}_\hbar^{(n)}+C_\delta\right)\psi,\psi\right\rangle.
\end{aligned}
\tag{14.11}
$$

It is easy to conclude that

$$
\langle\left(\mathcal{L}_\hbar^{(n)}+C_\delta\right)\psi,\psi\rangle\geq 0.
$$

As a result, for all $c>C_\delta$, $0<\hbar<\eta$, by Eq. (14.11), we get

$$
\left|\langle\left[(L_\hbar^n+c)-\left(\mathcal{L}_\hbar^{(n)}+c\right)\right]\psi,\psi\rangle\right|=\left|\langle\left(L_\hbar^n-\mathcal{L}_\hbar^{(n)}\right)\psi,\psi\rangle\right|\leq\frac{1}{2}\langle\left(\mathcal{L}_\hbar^{(n)}+c\right)\psi,\psi\rangle
$$

and the desired result follows. ■

# 2 Proof of the operator comparison Theorem 1.37

The purpose of this section is to prove Theorem 1.37. We begin with a preparatory lemma whose proof requires the following notation.

**Notation 14.5.** *For any divergence form differential operator $L$ on $\mathcal{S}$ described as in Eq. (10.1) we may decompose $L$ into its top order and lower order pieces, $L=L^{top}+L^{<}$ where*

$$
L^{top}:=(-1)^m\,\partial^m M_{b_m}\partial^m \text{ and } L^{<}:=\sum_{l=0}^{m-1}(-1)^l\,\partial^l M_{b_l}\partial^l. \tag{14.12}
$$

**Lemma 14.6.** *Let* $\{\mathcal{B}_{\ell,\hbar}(x)\}_{\ell=0}^{M}$ *be polynomial functions depending continuously on* $\hbar$ *which satisfies the conditions* 1 *and* 3 *of Assumption 1.34 and so in particular,*

$$c_{\mathcal{B}_M} := \inf\left\{\mathcal{B}_{M,\hbar}\left(\sqrt{\hbar}x\right) : x \in \mathbb{R} \text{ and } 0 < \hbar < \eta\right\} > 0. \tag{14.13}$$

*If*

$$K_\hbar = \sum_{\ell=0}^{M} (-\hbar)^\ell \partial^{2\ell} \text{ and } \mathcal{L}_\hbar = \sum_{\ell=0}^{M} (-\hbar)^\ell \partial^\ell M_{\mathcal{B}_{\ell,\hbar}\left(\sqrt{\hbar}(\cdot)\right)} \partial^\ell$$

*are operators on* $\mathcal{S}$ *then for all* $\gamma > \frac{1}{c_{\mathcal{B}_M}}$, *there exists* $C_\gamma < \infty$ *such that*

$$K_\hbar \preceq_{\mathcal{S}} \gamma \mathcal{L}_\hbar + C_\gamma I. \tag{14.14}$$

**Proof.** Using the conditions 1 and 3 of Assumption 1.34 on $\{\mathcal{B}_{\ell,\hbar}(x)\}_{\ell=0}^{M}$ where $b_{l,\hbar}$ is replaced by $\mathcal{B}_{\ell,\hbar}$, we may choose $E > 0$ such that for all $0 \le \ell < M$,

$$c_\ell := \inf\left\{\mathcal{B}_{\ell,\hbar}\left(\sqrt{\hbar}x\right) + E : x \in \mathbb{R} \text{ and } 0 < \hbar < \eta\right\} > 0$$

and therefore,

$$(-\hbar)^\ell \partial^\ell M_{\mathcal{B}_{\ell,\hbar}\left(\sqrt{\hbar}(\cdot)\right)} \partial^\ell + E(-\hbar)^\ell \partial^{2\ell} = (-\hbar)^\ell \partial^\ell M_{\left[\mathcal{B}_{\ell,\hbar}\left(\sqrt{\hbar}(\cdot)\right)+E\right]} \partial^\ell \ge 0 \ \forall \ \ell$$

and in particular $\mathcal{L}_\hbar^< + EK_\hbar^< \succeq_{\mathcal{S}} 0$ which in turn implies

$$\mathcal{L}_\hbar^{top} \preceq_{\mathcal{S}} \mathcal{L}_\hbar^{top} + \mathcal{L}_\hbar^< + EK_\hbar^< = \mathcal{L}_\hbar + EK_\hbar^<.$$

Using this observation and Eq. (14.13) we find,

$$K_\hbar^{top} = (-\hbar)^M \partial^{2M} \preceq_{\mathcal{S}} \frac{1}{c_{\mathcal{B}_M}} \mathcal{L}_\hbar^{top} \preceq_{\mathcal{S}} \frac{1}{c_{\mathcal{B}_M}} \left(\mathcal{L}_\hbar + EK_\hbar^<\right). \tag{14.15}$$

By Lemma 14.1 and Eq. (14.15), for any $\delta > 0$, there exists $C_\delta < \infty$ such that for all $\hbar > 0$,

$$K_{\hbar}^{<} \preceq_{\mathcal{S}} \delta K_{\hbar}^{top} + C_{\delta} I \preceq_{\mathcal{S}} \delta \frac{1}{c_{\mathcal{B}_M}} \left(\mathcal{L}_{\hbar} + E K_{\hbar}^{<}\right) + C_{\delta} I. \tag{14.16}$$

Given $\varepsilon > 0$ small we may use the previous equation with $\delta > 0$ chosen so that $\varepsilon \geq \frac{\delta}{c_{\mathcal{B}_M} - \delta E}$ to learn there exists $C'_{\varepsilon} < \infty$ such that

$$K_{\hbar}^{<} \preceq_{\mathcal{S}} \varepsilon \mathcal{L}_{\hbar} + C'_{\varepsilon} I. \tag{14.17}$$

Combining this inequality with Eq. (14.15) then shows,

$$K_{\hbar} = K_{\hbar}^{top} + K_{\hbar}^{<} \preceq_{\mathcal{S}} \frac{1}{c_{\mathcal{B}_M}} \left(\mathcal{L}_{\hbar} + E\left(\varepsilon \mathcal{L}_{\hbar} + C'_{\varepsilon} I\right)\right) + \varepsilon \mathcal{L}_{\hbar} + C'_{\varepsilon} I.$$

Thus choosing $\varepsilon > 0$ sufficiently small in this inequality allows us to conclude for every $\gamma > \frac{1}{c_{\mathcal{B}_M}}$ there exists $C_{\gamma} < \infty$ such that Eq. (14.14) holds. ■

We are now ready to give the proof of Theorem 1.37.

**Proof of Theorem 1.37.** Recall $\eta := \min\left\{\eta_{\tilde{L}}, \eta_L\right\}$ defined in Theorem 1.37. By the assumption in Eq. (1.44) of Theorem 1.37,

$$\left|\tilde{b}_{l,\hbar}(x)\right| \leq c_1 \left(b_{l,\hbar}(x) + c_2\right) \ \forall\ 0 \leq l \leq m_{\tilde{L}} \text{ and } 0 < \hbar < \eta.$$

Moreover, using items 1 and 3 of Assumption 1.34, by increasing the size of $c_2$ if necessary, we may further assume that $b_{l,\hbar}(x) + c_2 \geq 0$ for all $x \in \mathbb{R}$, $0 \leq l \leq m_L$, and $0 < \hbar < \eta$. With out loss of generality, we may define $\tilde{b}_{l,\hbar}(\cdot) \equiv 0$ for all $l > m_{\tilde{L}}$ and hence $\tilde{\mathcal{B}}_{\ell,\hbar}(\cdot) = 0$ for all $\ell > m_{\tilde{L}} n$. It then follows that there exists $E_1, E_2 < \infty$

such that for $0 \le \ell \le m_L n$,

$$\begin{aligned}
\tilde{\mathcal{B}}_{\ell,\hbar} \le \left|\tilde{\mathcal{B}}_{\ell,\hbar}\right| &\le \sum_{\mathbf{k}\in\Lambda^n_{m_{\tilde{L}}}} 1_{|\mathbf{k}|=\ell} \left|\tilde{b}_{k_1,\hbar}\dots\tilde{b}_{k_n,\hbar}\right| \\
&\le \sum_{\mathbf{k}\in\Lambda^n_{m_{\tilde{L}}}} 1_{|\mathbf{k}|=\ell} c_1^n (b_{k_1,\hbar}+c_2)\dots(b_{k_n,\hbar}+c_2) \\
&\le \sum_{\mathbf{k}\in\Lambda^n_{m_L}} 1_{|\mathbf{k}|=\ell} c_1^n (b_{k_1,\hbar}+c_2)\dots(b_{k_n,\hbar}+c_2) \\
&= E_1\mathcal{B}_{\ell,\hbar} + E_2,
\end{aligned} \tag{14.18}$$

wherein we have used Eq.(14.4) in Lemma 14.2 for the last inequality by taking

$$p\,(x,\hbar) = \mathcal{B}_{\ell,\hbar} = \sum_{\mathbf{k}\in\Lambda^n_{m_L}} 1_{|\mathbf{k}|=\ell}(b_{k_1,\hbar}\dots b_{k_n,\hbar}) \text{ and}$$
$$q\,(x,\hbar) = \sum_{\mathbf{k}\in\Lambda^n_{m_L}} 1_{|\mathbf{k}|=\ell} c_1^n (b_{k_1,\hbar}+c_2)\dots(b_{k_n,\hbar}+c_2),$$

where (by Lemma 14.3) $\mathcal{B}_{\ell,\hbar}$ is an even degree polynomial with a positive leading order coefficient. Hence if we let $\tilde{\mathcal{L}}^{(n)}_\hbar$ and $\mathcal{L}^{(n)}_\hbar$ be as in Eq. (13.9), i.e.

$$\tilde{\mathcal{L}}^{(n)}_\hbar = \sum_{\ell=0}^{m_{\tilde{L}}n} (-\hbar)^\ell\, \partial^\ell \widetilde{\mathcal{B}}_{\ell,\hbar}\left(\sqrt{\hbar}\,(\cdot)\right)\partial^\ell \text{ and } \mathcal{L}^{(n)}_\hbar = \sum_{\ell=0}^{m_L n} (-\hbar)^\ell\, \partial^\ell \mathcal{B}_{\ell,\hbar}\left(\sqrt{\hbar}\,(\cdot)\right)\partial^\ell,$$

then it follows directly from Eq.(14.18) that

$$\tilde{\mathcal{L}}^{(n)}_\hbar \preceq_{\mathcal{S}} E_1\mathcal{L}^{(n)}_\hbar + E_2K_\hbar \text{ where } K_\hbar := \sum_{\ell=0}^{nm_L} (-\hbar)^\ell\,\partial^{2\ell}.$$

Because of Lemma 14.3, we may apply Lemma 14.6 with $M = nm_L$ and $\mathcal{L}_\hbar = \mathcal{L}^{(n)}_\hbar$ to conclude there exists $\gamma > 0$ and $C < \infty$ such that $K_\hbar \preceq_{\mathcal{S}} \gamma\mathcal{L}^{(n)}_\hbar + CI$ and thus,

$$\tilde{\mathcal{L}}^{(n)}_\hbar \preceq_{\mathcal{S}} (E_1 + \gamma E_2)\,\mathcal{L}^{(n)}_\hbar + E_2CI.$$

By Theorem 14.4, there exists $C_L$ and $C_{\tilde{L}}$ such that

$$\begin{aligned}&\frac{1}{2}\mathcal{L}_\hbar^{(n)} \preceq_{\mathcal{S}} L_\hbar^n + C_L \text{ and}\\ &\quad\tilde{L}_\hbar^n \preceq_{\mathcal{S}} \frac{3}{2}\left(\tilde{\mathcal{L}}_\hbar^{(n)} + C_{\tilde{L}}\right) \preceq_{\mathcal{S}} \frac{3}{2}\left(\left(E_1 + \gamma E_2\right)\mathcal{L}_\hbar^{(n)} + E_2 CI + C_{\tilde{L}}\right).\end{aligned}$$

From these last two inequalities, it follows that $\tilde{L}_\hbar^n \preceq_{\mathcal{S}} C_1\left(L_\hbar^n + C_2\right)$ for appropriately chosen constants $C_1$ and $C_2$. ■

# 3 Proof of Corollary 1.39

For the reader's convenience let us restated Corollary 1.39 here.

**Corollary** (1.39)**.** *Suppose $\{b_{l,\hbar}(x)\}_{l=0}^m \subset \mathbb{R}[x]$ and $\eta > 0$ satisfies Assumption 1.34, $L_\hbar$ is the operator in the Eq. (1.39), and suppose that $C \geq 0$ has been chosen so that $0 \preceq_{\mathcal{S}} L_\hbar + CI$ for all $0 < \hbar < \eta$. (The existence of $C$ is guaranteed by Corollary 1.38.) Then for any $0 < \hbar < \eta$, $\bar{L}_\hbar + CI$ is a non-negative self-adjoint operator on $L^2(m)$ and $\mathcal{S}$ is a core for $\left(\bar{L}_\hbar + C\right)^r$ for all $r \geq 0$.*

Before proving this corollary we need to develop a few more tools. From Lemma 14.3, $\{B_{\ell,\hbar}\}_{\ell=0}^{mn} \subset \mathbb{R}[x]$ in Eq. (11.1) satisfies both items 1 and 3 of Assumption 1.34. Therefore, $B_{\ell,\hbar}$ is bounded below for $0 \leq \ell \leq mn-1$ and $B_{mn,\hbar} > 0$. We may choose $C > 0$ sufficiently large so that

$$B_{\ell,\hbar} + C > 0 \text{ for } 0 \leq \ell \leq mn-1 \text{ and } 0 < \hbar < \eta. \tag{14.19}$$

**Notation 14.7.** *Let $C > 0$ be chosen so that Eq. (14.19) holds and then define* *the operator,* $\hat{L}_\hbar$*, by*

$$\begin{aligned}\hat{L}_\hbar &:= \sum_{\ell=0}^{mn} (-\hbar)^\ell \partial^\ell \left(B_{\ell,\hbar}\left(\sqrt{\hbar}\,(\cdot)\right) + C1_{\ell<mn}\right)\partial^\ell \\ &= (-\hbar)^{mn}\partial^{mn} B_{mn,\hbar}\left(\sqrt{\hbar}\,(\cdot)\right)\partial^{mn} + \sum_{\ell=0}^{mn-1} (-\hbar)^\ell \partial^\ell \left(B_{\ell,\hbar}\left(\sqrt{\hbar}\,(\cdot)\right) + C\right)\partial^\ell\end{aligned}$$

*with domain,* $\mathcal{D}\left(\hat{L}_\hbar\right) = \mathcal{S}$.

**Lemma 14.8.** *There exists* $\tilde{C}_1$ *and* $\tilde{C}_2 > 0$ *such that*

$$\left\|\hbar\partial^{2M}\psi\right\| \le \tilde{C}_1 \left\|\hat{L}_\hbar\psi\right\| + \tilde{C}_2 \left\|\psi\right\|$$

*holds for all* $0 \le M \le mn$, $0 < \hbar < \eta$, *and* $\psi \in \mathcal{S}$.

**Proof.** As in Eq. (12.22), let $\hat{\psi}$ denote the Fourier transform of $\psi \in \mathcal{S}$ and recall that $\|\psi\|_2 = \sqrt{2\pi}\left\|\hat{\psi}\right\|_2$. Hence it follows,

$$\begin{aligned}\left\|\hbar^M\partial^{2M}\psi\right\| &= \sqrt{2\pi}\left\|\hbar^M\xi^{2M}\hat{\psi}(\xi)\right\| \\ &\le \sqrt{2\pi}\left\|\left(\sum_{\ell=0}^{M}\hbar^\ell\xi^{2\ell}\right)\hat{\psi}(\xi)\right\| = \left\|\left(\sum_{\ell=0}^{M}(-\hbar)^\ell\partial^{2\ell}\right)\psi\right\|. \qquad (14.20)\end{aligned}$$

With the same $C$ in Notation 14.7 and using Eq. (14.19), we can see that

$$1 \le (B_{\ell,\hbar} + C1_{\ell<mn}) + 1 \;\forall\; 0 \le \ell \le mn \;\&\; 0 < \hbar < \eta.$$

Therefore applying the operator comparison Theorem 1.37 with $\widetilde{L}_\hbar = \sum_{\ell=0}^{M}(-\hbar)^\ell\partial^{2\ell}$, $L_\hbar = \hat{L}_\hbar$, and $n = 2$, there exists $C_1$ and $C_2 > 0$ such that for

$$\left\langle\left(\sum_{\ell=0}^{M}(-\hbar)^\ell\partial^{2\ell}\right)^2\psi,\psi\right\rangle \le C_1\left\langle\hat{L}_\hbar^2\psi,\psi\right\rangle + C_2\langle\psi,\psi\rangle \;\;\forall\;\; \psi\in\mathcal{S} \;\&\; 0<\hbar<\eta.$$

Combining this inequality with Eq. (14.20) shows there exists other constants $\widetilde{C}_1$ and $\widetilde{C}_2 > 0$ such that

$$\left\|\hbar^M \partial^{2M} \psi\right\| \leq \left\| \left( \sum_{\ell=0}^{M} (-\hbar)^\ell \partial^{2\ell} \right) \psi \right\| \leq \widetilde{C}_1 \left\| \hat{L}_\hbar \psi \right\| + \widetilde{C}_2 \|\psi\| .$$

■

**Lemma 14.9.** *Let $A$ and $B$ be closed operators on a Hilbert space $\mathcal{K}$ and suppose there exists a subspace, $S \subseteq \mathcal{D}(A) \cap \mathcal{D}(B)$, such that $S$ is dense and $S$ is a core of $B$. If there exists a constant $C > 0$ such that*

$$\|A\psi\| \leq \|B\psi\| + C \|\psi\| \ \ \forall\ \psi \in S, \tag{14.21}$$

*then $\mathcal{D}(B) \subseteq \mathcal{D}(A)$ and*

$$\|A\psi\| \leq \|B\psi\| + C \|\psi\| \ \ \forall\ \psi \in \mathcal{D}(B). \tag{14.22}$$

**Proof.** If $\psi \in \mathcal{D}(B)$, there exists $\psi_k \in S$ such that $\psi_k \to \psi$ and $B\psi_k \to B\psi$ as $k \to \infty$. Because of Eq. (14.21) $\{A\psi_k\}_{k=1}^{\infty}$ is Cauchy in $\mathcal{K}$ and hence convergent. As $A$ is closed we may conclude that $\psi \in \mathcal{D}(A)$ and that $\lim_{k\to\infty} A\psi_k = A\psi$. Therefore Eq. (14.22) holds by replacing $\psi$ in Eq. (14.21) by $\psi_k$ and then passing to the limit as $k \to \infty$. ■

**Proposition 14.10.** *Suppose $\{b_{l,\hbar}(x)\}_{l=0}^{m} \subset \mathbb{R}[x]$ and $\eta > 0$ satisfies Assumption 1.34 and $L_\hbar$ is defined by Eq. (1.39) with $\mathcal{D}(L_\hbar) = \mathcal{S}$ for $0 < \hbar < \eta$. Then $\bar{L}_\hbar$ is self-adjoint and $\mathcal{S}$ is a core for $\bar{L}_\hbar^n$ for all $n \in \mathbb{N}$ and $0 < \hbar < \eta$. [Note $\bar{L}_\hbar^n$ is a well defined self-adjoint operator by the spectral theorem.]*

**Proof.** Recall that $L_\hbar^n$ may be written in divergence form as in Eq. (13.3) where $B_{\ell,\hbar} = \mathcal{B}_{\ell,\hbar} + R_{\ell,\hbar}$ and $\mathcal{B}_{\ell,\hbar} \in \mathbb{R}[x]$ and $R_{\ell,\hbar} \in \mathbb{R}[x]$ are as in Eqs. (13.5) and (13.6) respectively. By Assumption 1.34, $\deg(b_{l-1}) \leq \deg(b_l)$, which used in combination with the item 1 in Proposition 13.1 and the definition of $\mathcal{B}_\ell$ in Eq. (13.5) implies,

$$\begin{aligned}\deg\left(B_{\ell,\hbar}\right) = \deg\left(\mathcal{B}_{\ell,\hbar}\right) &\leq \max\left\{\deg\left(b_{0,\hbar}^{n}\right), 0\right\} \\ &\leq \max\left\{\deg\left(\mathcal{B}_{0,\hbar}\right), 0\right\} = \max\left\{\deg\left(B_{0,\hbar}\right), 0\right\}.\end{aligned}$$

Each term in $\hat{L}_\hbar$ defined in Notation 14.7 is a positive operator and by Theorem 1.30, $\overline{\hat{L}_\hbar}$ is self-adjoint. [Recall that $\mathcal{D}\left(\hat{L}_\hbar\right) := \mathcal{S}$.] Moreover by Lemma 14.1, for all $\delta > 0$ there exists $C_\delta < \infty$ such that

$$\left(\hat{L}_\hbar - L_\hbar^n\right)^2 = \left(\sum_{\ell=0}^{nm-1} (-\hbar)^\ell C \partial^{2\ell}\right)^2 \preceq_{\mathcal{S}} \delta\left(-\hbar\right)^{mn} \partial^{4mn} + C_\delta I \tag{14.23}$$

which implies,

$$\left\|\left(\hat{L}_\hbar - L_\hbar^n\right)\psi\right\| \leq \delta\left\|(\hbar)^{mn}\partial^{2nm}\psi\right\| + C_\delta\left\|\psi\right\| \ \forall\ \psi \in \mathcal{S}.$$

This inequality along with Lemma 14.8 then gives

$$\begin{aligned}\left\|\left(\hat{L}_\hbar - L_\hbar^n\right)\psi\right\| &\leq \delta\left(C_1\left\|\hat{L}_\hbar\psi\right\| + C_2\left\|\psi\right\|\right) + C_\delta\left\|\psi\right\| \\ &\leq \delta C_1\left\|\hat{L}_\hbar\psi\right\| + \left(\delta C_2 + C_\delta\right)\left\|\psi\right\| \ \forall\ \psi \in \mathcal{S}.\end{aligned}$$

Therefore for any $a > 0$ we may take $\delta > 0$ so that $a := \delta C_1$ and then let $C_a := (\delta C_2 + C_\delta) < \infty$ in the previous estimate in order to show,

$$\left\|\left(\hat{L}_\hbar - L_\hbar^n\right)\psi\right\| \leq a\left\|\hat{L}_\hbar\psi\right\| + C_a\left\|\psi\right\| \ \forall\ \psi \in \mathcal{S}. \tag{14.24}$$

As a consequence of this inequality with $a < 1$ and a variant of the Kato-Rellich theorem (see Theorem X.13 on p. 163 of [31]), we may conclude $\overline{L_\hbar^n}$ is self-adjoint. As this holds for $n = 1$, we conclude that $\bar{L}_\hbar$ is self-adjoint. By the spectral theorem,

$\bar{L}^n_\hbar$ is also self-adjoint. Since $L^n_\hbar \subset \bar{L}^n_\hbar$, we know that $\overline{L^n_\hbar} \subset \bar{L}^n_\hbar$ and therefore $\overline{L^n_\hbar} = \bar{L}^n_\hbar$ as both operators are self-adjoint. Finally, $L^n_\hbar = \bar{L}^n_\hbar|_{\mathcal{S}}$ and hence $\overline{\bar{L}^n_\hbar|_{\mathcal{S}}} = \overline{L^n_\hbar} = \bar{L}^n_\hbar$ which shows $\mathcal{S}$ is a core for $\bar{L}^n_\hbar$.

■

**Lemma 14.11.** *If $A$ is any essentially self-adjoint operator on a Hilbert space $K$ and $q : \mathbb{R} \to \mathbb{C}$ is a measurable function such that, for some constants $C_1$ and $C_2$,*

$$|q(x)| \le C_1 |x| + C_2 \ \forall \ x \in \mathbb{R},$$

*then $\mathcal{D}(A)$ is a core for $q\left(\bar{A}\right)$.*

**Proof.** To prove this we may assume by the spectral theorem that $\mathcal{K} = L^2(\Omega, \mathcal{B}, \mu)$ and $\bar{A} = M_f$ where $(\Omega, \mathcal{B}, \mu)$ is a $\sigma -$finite measure space and $f : \Omega \to \mathbb{R}$ is a measurable function. Of course in this model, $q\left(\bar{A}\right) = M_{q\circ f}$. In this case, $\mathcal{D} := \mathcal{D}(A) \subset \mathcal{D}(M_f)$ is a dense subspace of $L^2(\mu)$ such that for all $g \in \mathcal{D}(M_f)$ there exists $g_n \in \mathcal{D}$ such that $g_n \to g$ and $fg_n \to fg$ in $L^2(\mu)$ as $n \to \infty$. For this same sequence we have

$$\left\|q\left(\bar{A}\right) g_n - q\left(\bar{A}\right) g\right\|_2 = \|q(f)[g_n - g]\|_2 \le C_1 \|f[g_n - g]\|_2 + C_2 \|g_n - g\| \to 0$$

as $n \to \infty$. This shows that

$$q\left(\bar{A}\right)|_{\mathcal{D}\left(M_f\right)} \subset \overline{q\left(\bar{A}\right)|_{\mathcal{D}}} \subset q\left(\bar{A}\right). \tag{14.25}$$

For $g \in \mathcal{D}\left(q\left(\bar{A}\right)\right)$ (i.e. both $g$ and $g \cdot q \circ f$ are in $\in L^2(\mu)$ ), let $g_n := g 1_{|f|\le n} \in \mathcal{D}(M_f)$. Then $g_n \to g$ in $L^2(\mu)$ as $n \to \infty$ by DCT. Moreover

$$|g_n q \circ f - gq \circ f| = \left(g 1_{|f|\le n} - g\right) q \circ f \le 2|g| |q \circ f| \in L^2(\mu),$$

and so $\|g_n q \circ f - gq \circ f\|_2 \to 0$ as $n \to \infty$ by DCT as well. This shows that

$q\left(\bar{A}\right)=\overline{q\left(\bar{A}\right)|_{\mathcal{D}\left(M_f\right)}}$ and hence it now follows from Eq. (14.25) that

$$q\left(\bar{A}\right)=\overline{q\left(\bar{A}\right)|_{\mathcal{D}\left(M_f\right)}}\subset\overline{q\left(\bar{A}\right)|_{\mathcal{D}}}\subset q\left(\bar{A}\right).$$

■

**Lemma 14.12.** *Let $B$ be a non-negative self-adjoint operator on a Hilbert space, $\mathcal{K}$. If $S$ is a core for $B^n$ for some $n\in\mathbb{N}_0$, then $S$ is a core for $B^r$ for any $0\le r\le n$. [By the spectral theorem, $B^r$ is again a non-negative self-adjoint operator on $\mathcal{K}$ for any $0\le r<\infty$.]*

**Proof.** Let $A=B^n|_S$ so that by assumption $\bar{A}=B^n$, i.e. $A$ is essentially self-adjoint. The proof is then finished by applying Lemma 14.11 with $q\left(x\right)=\left|x\right|^{r/n}$ upon noticing, $q\left(\bar{A}\right)=q\left(B^n\right)=\left|B^n\right|^{r/n}=B^r$. ■

**Proof of Corollary 1.39.** Let $C\ge 0$ be the constant in the statement of Corollary 1.39. It is simple to verify that $\{b_{l,\hbar}+C1_{l=0}\}_{l=0}^m$ and $\eta>0$ satisfies Assumption 1.34, and therefore applying Proposition 14.10 with $\{b_{l,\hbar}\}_{l=0}^m$ replaced by $\{b_{l,\hbar}+C1_{l=0}\}_{l=0}^m$ shows $\bar{L}_\hbar+C$ is self-adjoint and $\mathcal{S}$ is core for $\left(\bar{L}_\hbar+C\right)^n$ for all $n\in\mathbb{N}$ and $0<\hbar<\eta$. It then follows from Lemma 14.12 that $S=\mathcal{S}$ is a core for $\left(\bar{L}_\hbar+C\right)^r$ for all $0\le r\le n$ and $0<\hbar<\eta$. As $n\in\mathbb{N}$ was arbitrary, the proof is complete. ■

# 4 Proof of Corollary 1.40

In order to prove Corollary 1.40, we will need a lemma below.

**Lemma 14.13.** *Let $A$ and $B$ be non-negative self-adjoint operators on a Hilbert Space $\mathcal{K}$. Suppose $S$ is a dense subspace of $\mathcal{K}$ so that $S\subseteq\mathcal{D}\left(A\right)\cap\mathcal{D}\left(B\right)$, $AS\subseteq S$ and $BS\subseteq S$. If we further assume that for each $n\in\mathbb{N}_0$, $S$ is a core of $B^n$. Then the following are equivalent:*

1. *For any $n \in \mathbb{N}_0$ there exists $C_n > 0$ such that $A^n \preceq_S C_n B^n$.*

2. *For each $r \geq 0$, there exists $C_r$ such that $A^r \leq C_r B^r$.*

3. *For each $v \geq 0$, there exists $C_v$ such that $A^v \preceq C_v B^v$.*

*Recall the different operator inequality notations, $\preceq_S$, $\preceq$ and $\leq$, were defined in Notation 1.10.*

**Proof.** ($1 \Rightarrow 2$) $A^n \preceq_{\mathcal{S}} C_n B^n$ implies for all $\psi \in S$ we have

$$\left\|\sqrt{A^n}\psi\right\|^2 = \langle A^n\psi, \psi\rangle \leq C_n \langle B^n\psi, \psi\rangle = \left\|\sqrt{C_n B^n}\psi\right\|^2.$$

Note $S$ is a core of $C_n B^n$ and hence $S$ is also a core of $\sqrt{C_n B^n}$ by taking $q(x) = \sqrt{|x|}$ in Lemma 14.11. By using Lemma 14.9 with $C = 0$ we have $\mathcal{D}\left(\sqrt{B^n}\right) = \mathcal{D}\left(\sqrt{C_n B^n}\right) \subseteq \mathcal{D}\left(\sqrt{A^n}\right)$ and

$$\left\|\sqrt{A^n}\psi\right\| \leq \left\|\sqrt{C_n B^n}\psi\right\| \text{ for all } \psi \in \mathcal{D}\left(\sqrt{C_n B^n}\right),$$

i.e. $A^n \leq C_n B^n$. It then follows by the Löwner-Heinz inequality (Theorem 1.36) that $A^{nr} \leq C_n^r B^{nr}$ for all $0 \leq r \leq 1$. Since $n \in \mathbb{N}$ was arbitrary, we have verified the truth of item 2.

($2 \Rightarrow 3$) Given item 2, it is easy to verify that $\mathcal{D}(B^v) = \mathcal{D}(C_v B^v) \subseteq \mathcal{D}(A^v)$ for all $v \geq 0$. In particularly, we have $\mathcal{D}(B^v) \subseteq \mathcal{D}(A^v) \cap \mathcal{D}\left(\sqrt{B^v}\right)$ for any $v \geq 0$. Hence, by taking $r = v$ in item 2,

$$\langle A^v\psi, \psi\rangle = \left\|\sqrt{A^v}\psi\right\|^2 \leq \left\|\sqrt{C_v B^v}\psi\right\|^2 = \langle C_v B^v\psi, \psi\rangle \ \forall \ \psi \in \mathcal{D}(B^v),$$

i.e. $A^v \preceq C_v B^v$.

($3 \Rightarrow 1$) The assumption that $S \subseteq \mathcal{D}(A) \cap \mathcal{D}(B)$, $AS \subseteq S$ and $BS \subseteq S$ follows that $S \subseteq \mathcal{D}(B^n) \cap \mathcal{D}(A^n)$ for all $n \in \mathbb{N}_0$. By taking $v = n$, we learn that $A^n \preceq C_n B^n$ which certainly implies $A^n \preceq_S C_n B^n$. ■

**Proof of the Corollary 1.40.** We first observe that the coefficients,

$$\{b_{l,\hbar}(\cdot)+C1_{l=0}\}_{l=0}^{m_L} \text{ and } \left\{\tilde{b}_{l,\hbar}(\cdot)+\tilde{C}1_{l=0}\right\}_{l=0}^{m_{\tilde{L}}}$$

still satisfy Assumption 1.34. Using this observation along with the inequalities, $L_\hbar+C\succeq_{\mathcal{S}} I$ and $\tilde{L}_\hbar+\tilde{C}\succeq_{\mathcal{S}} 0$, we may use Corollary 1.39 to conclude both $\bar{L}_\hbar+C$ and $\overline{\tilde{L}_\hbar}+\tilde{C}$ are non-negative self adjoint operators and $\mathcal{S}$ is a core for $\left(\bar{L}_\hbar+C\right)^r$ for all $r\geq 0$ and all $0<\hbar<\eta$. By the operator comparison Theorem 1.37 with $b_{l,\hbar}$ replaced by $b_{l,\hbar}+C1_{l=0}$ and $\tilde{b}_{l,\hbar}$ replaced by $\tilde{b}_{l,\hbar}+\tilde{C}1_{l=0}$, for any $n\in\mathbb{N}_0$, there exists $C_1$ and $C_2>0$ such that

$$\left(\tilde{L}_\hbar+\tilde{C}\right)^n\preceq_{\mathcal{S}} C_1\left(\left(L_\hbar+C\right)^n+C_2\right). \tag{14.26}$$

Because $\left(L_\hbar+C\right)^n\succeq_{\mathcal{S}} I$, we may conclude from Eq. (14.26) that

$$\left(\tilde{L}_\hbar+\tilde{C}\right)^n\preceq_{\mathcal{S}} C_n\left(L_\hbar+C\right)^n \quad \forall \quad n\in\mathbb{N}_0,$$

where $C_n=C_1\left(C_2+1\right)$. By taking $A=\overline{\tilde{L}_\hbar}+\tilde{C}$ and $B=\left(\bar{L}_\hbar+C\right)$ and $S=\mathcal{S}$ in Lemma 14.13, we may conclude that for any $v\geq 0$, there exists $C_v>0$ such that Eq. (1.46) holds, i.e.$\left(\overline{\tilde{L}_\hbar}+\tilde{C}\right)^r\preceq C_r\left(\bar{L}_\hbar+C\right)^r$ $\forall\ 0<\hbar<\eta$. ■

# Chapter 15

# Discussion of the 2nd condition in Assumption 1.34

We try to relax conditions 2 in Assumption 1.34. The degree restriction Eq. (1.40) allows the choice of $\eta$ independent of a power $n$ in both Theorem 14.4 and Theorem 1.37. If a weaker condition of the degree restriction is assumed, which is

$$\deg(b_{l,\hbar}) \leq \deg(b_{l-1,\hbar}) + 2 \text{ for all } 0 < \hbar < \eta \text{ and } 0 \leq l \leq m,$$

then Theorems 15.2 and 15.3 are resulted where now $\eta$ does depend on $n$.

**Lemma 15.1.** *Supposed there exists $\eta > 0$ such that $\deg(b_{l,\hbar}) \leq \deg(b_{l-1,\hbar}) + 2$ for all $0 < \hbar < \eta$ and $0 \leq k \leq m$. Let $\mathcal{B}_{\ell,\hbar}(x)$ and $R_{\ell,\hbar}(x)$ be in Eqs. (13.5) and (13.6) respectively. Then we have*

$$\deg_x(R_{\ell,\hbar}) \leq \deg_x(\mathcal{B}_{\ell,\hbar}) \text{ for } \hbar \in (0,\eta) \text{ and } 0 \leq \ell \leq mn. \tag{15.1}$$

**Proof.** Eq. (15.1) follows immediately if we apply the item 2 in Proposition 13.1 with $b_l(x) = \hbar^l b_{l,\hbar}\left(\sqrt{\hbar}x\right)$ with $\hbar$ fixed. ■

**Theorem 15.2.** *Let $L_\hbar$ be an operator in the Eq. (1.39). Supposed $b_{l,\hbar}(x)$ satisfies the conditions* 1 *and* 3 *in Assumption 1.34 and we assume*

$$\deg(b_{l,\hbar}) \leq \deg(b_{l-1,\hbar}) + 2 \text{ for all } 0 < \hbar < \eta \text{ and } 0 \leq l \leq m, \tag{15.2}$$

*where $\eta$ is the $\eta$ in Assumption 1.34. Then for any $n \in \mathbb{N}_0$, there exists $C_n$ and $\eta_n$ such that for all $0 < \hbar < \eta_n$ and $c > C_n$*

$$\frac{3}{2}\left(\mathcal{L}_\hbar^{(n)} + c\right) \succeq_{\mathcal{S}} L_\hbar^n + c \succeq_{\mathcal{S}} \frac{1}{2}\left(\mathcal{L}_\hbar^{(n)} + c\right).$$

**Proof.** Let $\psi \in \mathcal{S}$ and $0 < \hbar < \eta$, we have

$$|\langle\left(L_\hbar^n - \mathcal{L}_\hbar^{(n)}\right)\psi, \psi\rangle| = \left|\left\langle \mathcal{R}_\hbar^{(n)}\psi, \psi\right\rangle\right| \leq \sum_{\ell=0}^{nm-1} \left|\left\langle R_{\ell,\hbar}\partial^\ell\psi, \partial^\ell\psi\right\rangle\right|.$$

where $\mathcal{R}_\hbar^{(n)}$ and $R_{\ell,\hbar}$ are still defined in the same way as Eqs. (13.10) and (13.6) respectively. From Lemma 15.1, $\deg_x(\mathcal{B}_{\ell,\hbar}) \geq \deg_x(R_{\ell,\hbar})$ where $\mathcal{B}_{\ell,\hbar}$ is defined in Eq. (13.5). Note $|p| > 0$ in $R_{\ell,\hbar}$ from Eq. (13.6). Although $\deg(R_{\ell,\hbar})$ can be the same as $\deg(\mathcal{B}_{\ell,\hbar})$, the extra $\hbar^{|\mathbf{p}|}$ factor in the $R_\ell(\hbar)$ makes $|R_{\ell,\hbar}|$ decrease more rapidly than $\mathcal{B}_{\ell,\hbar}$ as $\hbar$ decrease to 0. As a result, there exist constants $\eta_n > 0$ and $C$ such that

$$|R_{\ell,\hbar}(x)| \leq \frac{1}{2}\mathcal{B}_{\ell,\hbar}(x) + C$$

for all $0 \leq \ell \leq mn - 1$ and $0 < \hbar < \eta_n$. Therefore

$$\begin{aligned}|\langle\left(L_\hbar^n - \mathcal{L}_\hbar^{(n)}\right)\psi, \psi\rangle| &\leq \sum_{\ell=0}^{nm-1} \hbar^\ell \left\langle \left(\frac{1}{2}\mathcal{B}_{\ell,\hbar}\left(\sqrt{\hbar}(\cdot)\right) + C\right)\partial^\ell\psi, \partial^\ell\psi\right\rangle \\ &= \frac{1}{2}\sum_{\ell=0}^{nm-1}\left\langle(-\hbar)^\ell\,\partial^\ell\mathcal{B}_{\ell,\hbar}\partial^\ell\psi, \psi\right\rangle + C\left\langle\sum_{\ell=0}^{nm-1}(-\hbar)^\ell\,\partial^{2\ell}\psi, \psi\right\rangle.\end{aligned} \tag{15.3}$$

Then by following the argument in Theorem 14.4, we can conclude that there exists $C_n > 0$ such that for all $0 < \hbar < \eta_n$ and $c > C_n$ we have

$$\frac{1}{2}\sum_{\ell=0}^{nm-1}\left\langle \partial^\ell \mathcal{B}_{\ell,\hbar}\partial^\ell \psi, \psi\right\rangle + C\left\langle \sum_{\ell=0}^{nm-1}(-\hbar)^\ell\,\partial^{2\ell}\psi, \psi\right\rangle \leq \frac{1}{2}\left\langle \left(\mathcal{L}_\hbar^{(n)} + c\right)\psi, \psi\right\rangle.$$

The result follows immediately by combing the above inequality and Eq. (15.3). ■

As a result, the operator comparison theorem now have choice of $\eta$ depending on a power $n$.

**Theorem 15.3.** *Let*

$$\tilde{L}_\hbar = \sum_{\ell=0}^{m_{\tilde{L}}}(-\hbar)^k\partial^k\tilde{b}_{k,\hbar}(\sqrt{\hbar}x)\partial^k \text{ and } L_\hbar = \sum_{\ell=0}^{m_L}(-\hbar)^k\partial^k b_{k,\hbar}(\sqrt{\hbar}x)\partial^k$$

*be operators on $\mathcal{S}$ satisfying conditions in Theorem 15.2. Denote $\eta_{\tilde{L}}$ and $\eta_L$ as the $\eta$ of $\tilde{L}_\hbar$ and $L_\hbar$ in Assumption 1.34 respectively. If $m_{\tilde{L}} \leq m_L$ and there exists $c_1$ and $c_2$ such that*

$$|\tilde{b}_{l,\hbar}(x)| \leq c_1\left(b_{l,\hbar}(x) + c_2\right) \text{ for all } 0 \leq \ell \leq m_{\tilde{L}} \text{ and } 0 < \hbar < \min\{\eta_{\tilde{L}}, \eta_L\},$$

*then for any $n$, there exists $C_1$, $C_2$ and $\eta_n$ such that*

$$\left(\tilde{L}_\hbar\right)^n \preceq_{\mathcal{S}} C_1\left(L_\hbar^n + C_2\right)$$

*for all $0 < \hbar < \eta_n$.*

**Proof.** The exact same proof as Theorem 1.37 with the use of Theorem 15.2 instead of Theorem 14.4. ■

# Part III

# Appendix

# Appendix A

# Main Theorems in terms of the standard CCRs

Let

$$\hat{a}_\hbar = \frac{1}{\sqrt{2}}\left(M_x + \hbar\frac{d}{dx}\right) \text{ and } \hat{a}_\hbar^\dagger = \frac{1}{\sqrt{2}}\left(M_x - \hbar\frac{d}{dx}\right)$$

(as an operator on $\mathcal{S}$) be the more standard representation for the annihilation and creation operators form of the CCRs used in the physics literature. We will reformulate Theorem 1.17, Corollaries 1.19 and 1.21 in the standard CCRs. The following lemma (whose proof is left to the reader) implements the equivalence of our representation of the canonical commutation relations (CCRs) to the standard representation of the CCRs.

**Lemma A.1.** *For $\rho > 0$, let $S_\rho : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the unitary map defined by*

$$(S_\rho f)(x) := \sqrt{\rho} f(\rho x) \text{ for } x \in \mathbb{R}.$$

*Then $S_\rho \mathcal{S} = \mathcal{S}$ and it follows that*

$$\hat{a}_\hbar = S_{\hbar^{-1/2}} a_\hbar S_{\hbar^{1/2}} \text{ and } \hat{a}_\hbar^\dagger = S_{\hbar^{-1/2}} a_\hbar^\dagger S_{\hbar^{1/2}}$$

**Definition A.2.** *For $\hbar > 0$ and $\alpha := (\xi + i\pi)/\sqrt{2}$, let*

$$\hat{U}_\hbar(\alpha) = \exp\left(\frac{1}{\hbar}\left(\overline{\alpha\hat{a}_\hbar^\dagger - \bar{\alpha}\hat{a}_\hbar}\right)\right)$$

*be the unitary operator on $L^2(\mathbb{R})$ which implements translation by $(\xi, \pi)$ in phase space.*

Using the more standard representation of the CCRs instead, we have an immediate corollary from Theorem 1.17.

**Theorem A.3.** *Suppose $H(\theta, \theta^*) \in \mathbb{R}\langle\theta, \theta^*\rangle$ is a non-commutative polynomial in two indeterminates, $d = degH > 0$ and $0 < \eta \leq 1$ satisfying the same assumptions in Theorem 1.17. Let $\hat{H}_\hbar := \overline{H\left(\hat{a}_\hbar, \hat{a}_\hbar^\dagger\right)}$. We define*

$$\hat{A}_\hbar(t) := e^{i\hat{H}_\hbar t/\hbar}\hat{a}_\hbar e^{-i\hat{H}_\hbar t/\hbar}$$

*denote $\hat{a}_\hbar$ in the Heisenberg picture. Furthermore for all $\psi \in \mathcal{S}$, $\alpha_0 \in \mathbb{C}$, $0 < \hbar < \eta$, real numbers $\{t_i\}_{i=1}^n \subset \mathbb{R}$, and non-commutative polynomial, $P(\{\theta_i, \theta_i^*\}_{i=1}^n) \in \mathbb{C}\langle\{\theta_i, \theta_i^*\}_{i=1}^n\rangle$, in $2n$ – indeterminants where $p_{\min}$ be the minimum degree of all non constant terms in $P(\{\theta_i, \theta_i^*\}_{i=1}^n)$, the following weak limits (in the sense of non-commutative probability) hold;*

$$\begin{aligned}\left\langle P\left(\left\{\hat{A}_\hbar(t_i) - \alpha(t_i), \hat{A}_\hbar^\dagger(t_i) - \overline{\alpha}(t_i)\right\}_{i=1}^n\right)\right\rangle_{\hat{U}_\hbar(\alpha_0)S_{\hbar^{-1/2}}\psi} \\ = \left\langle P\left(\left\{\sqrt{\hbar}a(t_i), \sqrt{\hbar}a^\dagger(t_i)\right\}_{i=1}^n\right)\right\rangle_\psi + O\left(\hbar^{\frac{p_{\min}+1}{2}}\right).\end{aligned} \tag{A.1}$$

*where $a(t)$ and $a^\dagger(t)$ are as in Eqs. (1.8) and (1.9).*

**Proof.** By the Lemma A.1, we have

$$A_\hbar(t_i) = S_{\hbar^{1/2}}\hat{A}_\hbar(t_i)S_{\hbar^{-1/2}} \text{ and } U_\hbar(\alpha_0) = S_{\hbar^{1/2}}\hat{U}_\hbar(\alpha_0)S_{\hbar^{-1/2}}$$

on $\mathcal{S}$. Therefore,

$$\begin{aligned}&\left\langle P\left(\left\{\hat{A}_\hbar\left(t_i\right)-\alpha\left(t_i\right),\hat{A}_\hbar^\dagger\left(t_i\right)-\overline{\alpha}\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_{\hat{U}_\hbar(\alpha_0)S_{\hbar^{-1/2}}\psi}\\=&\left\langle P\left(\left\{S_{\hbar^{\frac{1}{2}}}\hat{A}_\hbar\left(t_i\right)S_{\hbar^{-\frac{1}{2}}}-\alpha\left(t_i\right),S_{\hbar^{\frac{1}{2}}}\hat{A}_\hbar^\dagger\left(t_i\right)S_{\hbar^{-\frac{1}{2}}}-\overline{\alpha}\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_{S_{\hbar^{\frac{1}{2}}}\hat{U}_\hbar(\alpha_0)S_{\hbar^{-\frac{1}{2}}}\psi}\\=&\left\langle P\left(\left\{A_\hbar\left(t_i\right)-\alpha\left(t_i\right),A_\hbar^\dagger\left(t_i\right)-\overline{\alpha}\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_{U_\hbar(\alpha_0)\psi}.\end{aligned}$$

Then, Eq.(A.1) follows by applying Eq.(1.23).. ■

Likewise we can show two corollaries of Theorem A.3 below which behave like Corollaries 1.19 and 1.21.

**Corollary A.4.** *Under the same notations and assumptions in Theorem A.3, then ,for* $0<\hbar<\eta$*, we have*

$$\begin{aligned}&\left\langle P\left(\left\{\hat{A}_\hbar\left(t_i\right),\hat{A}_\hbar^\dagger\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_{\hat{U}_\hbar(\alpha_0)S_{\hbar^{-1/2}}\psi}\\&\quad=\left\langle P\left(\left\{\alpha\left(t_i\right)+\sqrt{\hbar}a\left(t_i\right),\overline{\alpha}\left(t_i\right)+\sqrt{\hbar}a^\dagger\left(t_i\right)\right\}\right)\right\rangle_\psi+O\left(\hbar\right).\end{aligned}\tag{A.2}$$

**Proof.** It is a similar proof as Theorem A.3. Using Lemma A.1, we can conclude

$$\left\langle P\left(\left\{\hat{A}_\hbar\left(t_i\right),\hat{A}_\hbar^\dagger\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_{\hat{U}_\hbar(\alpha_0)S_{\hbar^{-1/2}}\psi}=\left\langle P\left(\left\{A_\hbar\left(t_i\right),A_\hbar^\dagger\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_{U_\hbar(\alpha_0)\psi}.$$

Then, the rest of the proof is simply to apply Eq.(1.25) and hence, Eq.(A.2) follows. ■

**Corollary A.5.** *Under the same notations and assumptions in Theorem A.3, let* $\hat{\psi}_\hbar=\hat{U}_\hbar\left(\alpha_0\right)S_{\hbar^{-1/2}}\psi$*. As* $\hbar\to 0^+$*, we have*

$$\left\langle P\left(\left\{\hat{A}_\hbar\left(t_i\right),\hat{A}_\hbar^\dagger\left(t_i\right)\right\}_{i=1}^n\right)\right\rangle_{\hat{\psi}_\hbar}\to P\left(\left\{\alpha\left(t_i\right),\overline{\alpha}\left(t_i\right)\right\}_{i=1}^n\right).$$

*and*

$$\left\langle P\left(\left\{\frac{\hat{A}_{\hbar}\left(t_i\right)-\alpha\left(t_i\right)}{\sqrt{\hbar}}, \frac{\hat{A}_{\hbar}^{\dagger}\left(t_i\right)-\bar{\alpha}\left(t_i\right)}{\sqrt{\hbar}}\right\}_{i=1}^{n}\right)\right\rangle_{\hat{\psi}_{\hbar}} \rightarrow \left\langle P\left(\left\{a\left(t_i\right), a^{\dagger}\left(t_i\right)\right\}_{i=1}^{n}\right)\right\rangle_{\psi}. \tag{A.3}$$

*We abbreviate this convergence by saying*

$$\mathrm{Law}_{\hat{\psi}_{\hbar}}\left(\left\{\frac{\hat{A}_{\hbar}\left(t_i\right)-\alpha\left(t_i\right)}{\sqrt{\hbar}}, \frac{\hat{A}_{\hbar}^{\dagger}\left(t_i\right)-\bar{\alpha}\left(t_i\right)}{\sqrt{\hbar}}\right\}_{i=1}^{n}\right) \rightarrow \mathrm{Law}_{\psi}\left(\left\{a\left(t_i\right), a^{\dagger}\left(t_i\right)\right\}_{i=1}^{n}\right).$$

**Proof.** Similar to the proof in Theorem A.3, by using Lemma A.1, we have

$$\left\langle P\left(\left\{\hat{A}_{\hbar}\left(t_i\right), \hat{A}_{\hbar}^{\dagger}\left(t_i\right)\right\}_{i=1}^{n}\right)\right\rangle_{\hat{\psi}_{\hbar}} = \left\langle P\left(\left\{A_{\hbar}\left(t_i\right), A_{\hbar}^{\dagger}\left(t_i\right)\right\}_{i=1}^{n}\right)\right\rangle_{U_{\hbar}(\alpha_0)\psi},$$

and

$$\begin{aligned}
&\left\langle P\left(\left\{\frac{\hat{A}_{\hbar}\left(t_i\right)-\alpha\left(t_i\right)}{\sqrt{\hbar}}, \frac{\hat{A}_{\hbar}^{\dagger}\left(t_i\right)-\bar{\alpha}\left(t_i\right)}{\sqrt{\hbar}}\right\}_{i=1}^{n}\right)\right\rangle_{\hat{\psi}_{\hbar}} \\
=&\left\langle P\left(\left\{\frac{A_{\hbar}\left(t_i\right)-\alpha\left(t_i\right)}{\sqrt{\hbar}}, \frac{A_{\hbar}^{\dagger}\left(t_i\right)-\bar{\alpha}\left(t_i\right)}{\sqrt{\hbar}}\right\}_{i=1}^{n}\right)\right\rangle_{U_{\hbar}(\alpha_0)\psi}.
\end{aligned}$$

Therefore, the corollary is a direct consequence of Corollary 1.21. ■

# Appendix B

# Operators Associated to Quantization

Let $\mathcal{A}$ denote the algebra of linear differential operator on $\mathcal{S}$ which have polynomial coefficients. Remark 1.27 shows that the $\dagger$ operation on $\mathcal{A}$ defined in Eq. (1.32) is an involution of $\mathcal{A}$. For $\hbar > 0$ , let $a_\hbar \in \mathcal{A}$ and its formal adjoint, $a_\hbar^\dagger$, be the annihilation and creation operators respectively as in Definition 1.3 given by

$$a_\hbar = \sqrt{\frac{\hbar}{2}}\left(M_x + \partial_x\right) \text{ and } a_\hbar^\dagger := \sqrt{\frac{\hbar}{2}}\left(M_x - \partial_x\right) \text{ on } \mathcal{S}. \tag{B.1}$$

These operators satisfy the commutation relation $\left[a_\hbar, a_\hbar^\dagger\right] = \hbar I$ on $\mathcal{S}$.

Let $\mathbb{R}\left\langle \theta, \theta^*\right\rangle$ be the space of non-commutative polynomials over $\mathbb{R}$ in two indeterminants $\left\{\theta, \theta^*\right\}$. Thus, given $H\left(\theta, \theta^*\right) \in \mathbb{R}\left\langle \theta, \theta^*\right\rangle$, there exists $d \in \mathbb{N}$ (the degree of $H\left(\theta, \theta^*\right)$ in $\theta$ and $\theta^*$) and coefficients,

$$\cup_{k=0}^{d}\left\{C_k\left(\mathbf{b}\right) \in \mathbb{R} : \mathbf{b} \in \left\{\theta, \theta^*\right\}^k\right\}$$

such that $H(\theta,\theta^*) = \sum_{k=0}^{d} H_k(\theta,\theta^*)$ where

$$H_k(\theta,\theta^*) := \sum_{\mathbf{b}=(b_1,\ldots,b_k)\in\left\{\theta,\theta^\dagger\right\}^k} C_k(\mathbf{b})\, b_1\ldots b_k \in \mathbb{R}\langle\theta,\theta^*\rangle\,. \tag{B.2}$$

We let $H(\theta,\theta^*)^* \in \mathbb{R}\langle\theta,\theta^*\rangle$ be defined by $H(\theta,\theta^*)^* = \sum_{k=0}^{d} H_k(\theta,\theta^*)^*$ where

$$H_k(\theta,\theta^*)^* := \sum_{\mathbf{b}=(b_1,\ldots,b_k)\in\{\theta,\theta^*\}^k} C_k(\mathbf{b})\, b_k^*\ldots b_1^* \tag{B.3}$$

and for $b \in \{\theta,\theta^*\}$,

$$b^* := \begin{cases} \theta^* & \text{if} \quad b=\theta \\ \theta & \text{if} \quad b=\theta^* \end{cases}.$$

The operation, $H(\theta,\theta^*) \to H(\theta,\theta^*)^*$ defines an involution on $\mathbb{R}\langle\theta,\theta^*\rangle$ and we say that $H(\theta,\theta^*) \in \mathbb{R}\langle\theta,\theta^*\rangle$ is **symmetric** if $H(\theta,\theta^*) = H(\theta,\theta^*)^*$. If $H(\theta,\theta^*) \in \mathbb{R}\langle\theta,\theta^*\rangle$ is symmetric, then $H\left(a_\hbar, a_\hbar^\dagger\right)$ is a symmetric linear differential operator with polynomial coefficients as in Definition 1.26.

In the following lemmas and theorem let $\mathbb{R}[x]$ and $\mathbb{R}\left[\sqrt{\hbar},x\right]$ be as in Notation 13.2.

**Lemma B.1.** *If $\hbar > 0$ and $H \in \mathbb{R}\langle\theta,\theta^*\rangle$ is a noncommutative polynomial with degree $d$, then $H\left(a_\hbar, a_\hbar^\dagger\right)$ can be written as a linear differential operator*

$$H\left(a_\hbar, a_\hbar^\dagger\right) = \sum_{l=0}^{d} \hbar^{\frac{l}{2}} G_l\left(\sqrt{\hbar}, \sqrt{\hbar}x\right)\partial_x^l$$

*where $G_l\left(\sqrt{\hbar},x\right) \in \mathbb{R}\left[\sqrt{\hbar},x\right]$ is a polynomial of $\sqrt{\hbar}$ and $x$ for $0 \le l \le d$.*

**Proof.** Let $H(\theta,\theta^*) = \sum_{k=0}^{d} H_k(\theta,\theta^*)$ with $H_k(\theta,\theta^*)$ be as in Eq. (B.2).

We then have $H\left(a_{\hbar},a_{\hbar}^{\dagger}\right)=\sum_{k=0}^{d}H_{k}\left(a_{\hbar},a_{\hbar}^{\dagger}\right)$ where

$$H_{k}\left(a_{\hbar},a_{\hbar}^{\dagger}\right)=(\hbar)^{k/2}\sum_{\mathbf{b}\in\{\theta,\theta^{*}\}^{k}}C_{k}\left(\mathbf{b}\right)\hat{b}_{1}\dots\hat{b}_{k},$$

and for $b\in\{\theta,\theta^{*}\}$,

$$\hat{b}:=\begin{cases}a & \text{if} \quad b=\theta\\ a^{\dagger} & \text{if} \quad b=\theta^{*}.\end{cases}$$

Using the definition of $a_{\hbar}$ and $a_{\hbar}^{\dagger}$ in Eq. (B.1), there exists

$$\left\{\tilde{C}_{k}\left(\varepsilon\right)\in\mathbb{R}:\varepsilon=\left(\varepsilon_{1},\dots,\varepsilon_{k}\right)\in\{\pm1\}^{k}\right\}$$

such that

$$H_{k}\left(a_{\hbar},a_{\hbar}^{\dagger}\right)=(\hbar)^{k/2}\sum_{\varepsilon\in\{\pm1\}^{k}}C\left(\varepsilon\right)\left(x+\varepsilon_{1}\partial_{x}\right)\dots\left(x+\varepsilon_{k}\partial_{x}\right).$$

From the previous equation it is easy to see

$$H_{k}\left(a_{\hbar},a_{\hbar}^{\dagger}\right)=(\hbar)^{k/2}\sum_{l=0}^{k}g_{l,k}\left(x\right)\partial_{x}^{l} \tag{B.4}$$

where $g_{l,k}\in\mathbb{R}\left[x\right]$ with

$$\deg_{x}\left(g_{l,k}\right)\leq k-l. \tag{B.5}$$

Summing Eq. (B.4) on $k$ and then switching two sums shows

$$H\left(a_{\hbar},a_{\hbar}^{\dagger}\right)=\sum_{k=0}^{d}\sum_{l=0}^{k}(\hbar)^{k/2}g_{l,k}\left(x\right)\partial_{x}^{l}=\sum_{l=0}^{d}\hbar^{\frac{l}{2}}\left(\sum_{k=l}^{d}\hbar^{\frac{k-l}{2}}g_{l,k}\left(x\right)\right)\partial_{x}^{l}. \tag{B.6}$$

There exists $G_l\left(\sqrt{\hbar},x\right)\in\mathbb{R}\left[\sqrt{\hbar},x\right]$ such that

$$G_l\left(\sqrt{\hbar},\sqrt{\hbar}x\right)=\sum_{k=l}^{d}\hbar^{\frac{k-l}{2}}g_{l,k}\left(x\right)$$

because each monomial of $x$ in $g_{l,k}\left(x\right)$ can be multiplied with enough $\sqrt{\hbar}$ from $\hbar^{\frac{k-l}{2}}$ by using Eq. (B.5). ■

**Theorem B.2.** *If $\hbar>0$ and $H\in\mathbb{R}\langle\theta,\theta^*\rangle$ is a **symmetric** noncommutative polynomial with degree $d$, then there exits $m\in\mathbb{N}_0$ and $\{f_l\}_{l=0}^{m}\subset\mathbb{R}\left[\sqrt{\hbar},x\right]$ such that*

$$H\left(a_\hbar,a_\hbar^\dagger\right)=\sum_{l=0}^{m}\left(-\hbar\right)^l\partial^l f_l\left(\sqrt{\hbar},\sqrt{\hbar}x\right)\partial^l\ \text{ on }\mathcal{S}.$$

**Proof.** Since $H$ is symmetric, $H\left(a_\hbar,a_\hbar^\dagger\right)$ is symmetric, see Definition 1.26. So by Proposition 10.2, $d=2m$ for some $m\in\mathbb{N}_0$ and $H\left(a_\hbar,a_\hbar^\dagger\right)$ in Eq. (B.6) may be written in a divergence form

$$H\left(a_\hbar,a_\hbar^\dagger\right)=\sum_{l=0}^{m}\left(-1\right)^l\partial_x^l M_{b_l}\partial_x^l. \tag{B.7}$$

By substituting $a_l\left(x\right)=h^{\frac{l}{2}}G_l\left(\sqrt{\hbar},\sqrt{\hbar}x\right)$ for $0\le l\le d=2m$ and $r=l$ in Eq. (10.9) from Theorem 10.7, for all $0\le l\le m$, we have

$$\begin{aligned}\left(-1\right)^l b_l\frac{1}{\hbar^l}&=\left[h^l G_{2l}\left(\sqrt{\hbar},\sqrt{\hbar}x\right)+\sum_{l<s\le m}K_m(l,s)h^s\partial^{2(s-l)}G_{2s}\left(\sqrt{\hbar},\sqrt{\hbar}x\right)\right]\times\frac{1}{\hbar^l}\\&=G_{2l}\left(\sqrt{\hbar},\sqrt{\hbar}x\right)+\sum_{l<s\le m}K_m(l,s)\hbar^{2(s-l)}\left(\partial^{2(s-l)}G_{2s}\right)\left(\sqrt{\hbar},\sqrt{\hbar}x\right)\end{aligned}$$

By using Lemma B.1, it follows that the R.H.S. in the above equation is a polynomial of $\sqrt{\hbar}$ and $\sqrt{\hbar}x$ . Therefore, there exists $f_l\left(\sqrt{\hbar},x\right)\in\mathbb{R}\left[\sqrt{\hbar},x\right]$ such that

$$\left(-1\right)^l b_l=\hbar^l f_l\left(\sqrt{\hbar},\sqrt{\hbar}x\right)$$

and hence, using the above equation along with Eq. (B.7), we can conclude that

$$H\left(a_\hbar, a_\hbar^\dagger\right) = \sum_{l=0}^{m} (-1)^l \partial_x^l M_{b_l} \partial_x^l = \sum_{l=0}^{m} \hbar^l \partial_x^l f_l\left(\sqrt{\hbar}, \sqrt{\hbar} x\right) \partial_x^l.$$

■

**Remark B.3.** *The functions, $f_l\left(\sqrt{\hbar}, x\right)$, in Theorem B.2 are examples of the functions, $b_{l,\hbar}(x)$, appearing in Eq. (1.38).*

**Example B.4.** *Let $H^{cl}(x,\xi) = x^2\xi^2$ be a classical Hamiltonian where $x$ is position and $\xi$ is momentum on a state space $\mathbb{R}^2$. We would like to lift this to a symmetric polynomial in two symmetric indeterminate $\hat{q} = \frac{\theta+\theta^*}{\sqrt{2}}$ and $\hat{p} = \frac{\theta-\theta^*}{i\sqrt{2}}$. The Weyl lift of $x^2\xi^2$ is given by*

$$\begin{aligned} H\left(\theta, \theta^\dagger\right) =& \frac{1}{4!}\left(\hat{q}^2\hat{p}^2 + \text{ all permutations}\right) \\ &= \frac{1}{4!} \cdot 2! \cdot 2! \begin{bmatrix} \hat{q}^2\hat{p}^2 + \hat{q}\hat{p}^2\hat{q} + \hat{p}^2\hat{q}^2 \\ +\hat{p}\hat{q}^2\hat{p} + \hat{p}\hat{q}\hat{p}\hat{q} + \hat{q}\hat{p}\hat{q}\hat{p} \end{bmatrix} \\ &= \frac{1}{3!} \begin{bmatrix} \hat{q}^2\hat{p}^2 + \hat{q}\hat{p}^2\hat{q} + \hat{p}^2\hat{q}^2 \\ +\hat{p}\hat{q}^2\hat{p} + \hat{p}\hat{q}\hat{p}\hat{q} + \hat{q}\hat{p}\hat{q}\hat{p} \end{bmatrix} \in \mathbb{R}\left\langle \theta, \theta^* \right\rangle. \end{aligned}$$

*Making the substitutions*

$$\hat{q} \to \frac{a_\hbar + a_\hbar^\dagger}{\sqrt{2}} = \sqrt{\hbar} M_x \text{ and } \hat{p} \to \frac{a_\hbar - a_\hbar^\dagger}{i\sqrt{2}} = \frac{\sqrt{\hbar}}{i}\partial$$

*above gives the Weyl quantization of $x^2\xi^2$ to be*

$$H\left(a_\hbar, a_\hbar^\dagger\right) = -\frac{\hbar^2}{3!}\left(x^2\partial^2 + \partial^2 x^2 + x\partial x\partial + \partial x\partial x + x\partial^2 x + \partial x^2\partial\right) \text{ on } \mathcal{S}$$

*which after a little manipulation using the product rule repeatedly may be written as*

$$\begin{aligned} H\left(a_{\hbar}, a_{\hbar}^{\dagger}\right) &= -\hbar^2 \partial x^2 \partial - \frac{1}{2}\hbar^2 \\ &= -\hbar \partial b_{1,\hbar}\left(\sqrt{\hbar}x\right)\partial + b_{0,\hbar}\left(\sqrt{\hbar}x\right) \end{aligned}$$

*where* $b_{1,\hbar}(x) = x^2$ *and* $b_{0,\hbar}(x) = -\frac{1}{2}\hbar^2$.

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