UNIVERSITY OF CALIFORNIA, SAN DIEGO

A Finite Dimensional Approximation to Pinned Wiener Measure on Symmetric Spaces

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The dissertation of Zhehua Li is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

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DEDICATION

To my parents.

EPIGRAPH

I... a universe of atoms, an atom in the universe. —Richard Feynman

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

Five years sounds like a long time, but when you are hearing the countdown clock ticking, all the memories frame in minds like a series of flashbacks. My five–year Ph.D. life in San Diego is filled with every kind of emotions, among all of them is my appreciation to all people who have influenced me one way or another.

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VITA

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ABSTRACT OF THE DISSERTATION

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Professor Bruce K. Driver, Chair

Let M be a Riemannian manifold, $o \in M$ be a fixed base point, $W_o(M)$ be the space of continuous paths from $[0,1]$ to M starting at $o \in M$, and let ν_x denote Wiener measure on $W_o(M)$ conditioned to end at $x \in M$. The goal of this thesis is to give a rigorous interpretation of the informal path integral expression for ν_x;

$$d\nu_x(\sigma) \text{ ``}=\text{''} \delta_x(\sigma(1)) \frac{1}{Z} e^{-\frac{1}{2}E(\sigma)} \mathcal{D}\sigma \ , \ \sigma \in W_o(M).$$

In this expression $E(\sigma)$ is the "energy" of the path σ, δ_x is the δ – function based

at x, $\mathcal{D}\sigma$ is interpreted as an infinite dimensional volume "measure" and Z is a certain "normalization" constant. We will interpret the above path integral expression as a limit of measures, $\nu^1_{\mathcal{P},x}$, indexed by partitions, $\mathcal{P}$ of $[0,1]$. The measures $\nu^1_{\mathcal{P},x}$ are constructed by restricting the above path integral expression to the finite dimensional manifolds, $H_{\mathcal{P},x}(M)$, of piecewise geodesics in $W_o(M)$ which are allowed to have jumps in their derivatives at the partition points and end at x. The informal volume measure, $\mathcal{D}\sigma$, is then taken to be a certain Riemannian volume measure on $H_{\mathcal{P},x}(M)$. When M is a symmetric space of non–compact type, we show how to naturally interpret the pinning condition, i.e. the δ – function term, in such a way that $\nu^1_{\mathcal{P},x}$, are in fact well defined finite measures on $H_{\mathcal{P},x}(M)$. The main theorem of the this thesis then asserts that $\nu^1_{\mathcal{P},x} \to \nu_x$ (in a weak sense) as the mesh size of $\mathcal{P}$ tends to zero. Along the way we develop a number of integration–by–parts arguments for the approximate measures, $\nu^1_{\mathcal{P},x}$, which are analogous to those known for the measures, ν_x.

Chapter 1

Overview

Throughout this dissertation, we fix $\left(M^d, g, \nabla, o\right)$ to be a pointed complete Riemannian manifold of dimension d with Riemannian metric g, Levi-Civita covariant derivative ∇ and base point $o \in M$. We further let

$$W_o(M) := \{\sigma \in C([0,1] \mapsto M) \mid \sigma(0) = o\}$$

be the **Wiener space** on M and let ν be the **Wiener measure** on $W_o(M)$—i.e. the law of the M–valued Brownian motion which starts at $o \in M$.

Richard Feynman, in his groundbreaking 1942 thesis, offered a path integral representation of the quantum particle state based on the *principle of least action*. In quantum physics, the state of a quantum particle is described by a wave function ϕ which satisfies the Schrödinger equation,

$$i\frac{\partial}{\partial t}\phi = H\phi$$

where $H = -\frac{1}{2}\Delta_g + V$ is the Schrödinger operator, Δ_g is the Laplace-Beltrami operator on (M, g, o), $V : M \to \mathbb{R}$ is an external potential and i is the imaginary unit. For our purpose, a slight modification is considered: after an analytic continuation (roughly change $t \to it$), one can reproduce Feynman's expression for the solution to the heat equation, which is usually considered as Schrödinger equation's imaginary–time counterpart,

$$\frac{\partial}{\partial t}\phi = -H\phi \ , \ \phi\left(x,0\right) = f\left(x\right). \tag{1.1}$$

Let e^{-tH} be the solution operator of heat equation (1.1), meaning $e^{-tH}f$ solves heat equation (1.1) when such a solution exists. Under modest regularity conditions, this operator admits an integrable kernel $p_t^H\left(\cdot,\cdot\right)$. In the physics literature one frequently finds Feynman type informal identities of the form,

$$p_1^H\left(o,x\right) = \text{``}\frac{1}{Z}\int_{W_o(M)} \delta_x\left(\sigma\left(1\right)\right) e^{-\int_0^1\left[\frac{1}{2}|\dot{\sigma}(\tau)|^2 + V(\sigma(\tau))\right]d\tau}\mathcal{D}\sigma\text{''} \tag{1.2}$$

and

$$\left(e^{-H}f\right)\left(o\right) = \text{``}\frac{1}{Z}\int_{W_o(M)} f\left(\sigma\left(t\right)\right) e^{-\int_0^1\left[\frac{1}{2}|\dot{\sigma}(\tau)|^2 + V(\sigma(\tau))\right]d\tau}\mathcal{D}\sigma\text{''} \tag{1.3}$$

Variants of these informal path integrals are often used as the basis for "defining" and making computations in quantum-field theories. From a mathematical perspective, making sense of such path integrals is thought to be a necessary step to developing a rigorous definition of interacting quantum field theories, (see for example; Glimm and Jaffe [18], Barry Simon [33], the Clay Mathematics Institute's Millennium problem involving Yang-Mills and Mass Gap). In general, path integrals like those appearing in (1.2) suffer from at least five distinct flaws;

1. The normalizing constant Z should typically be interpreted as either 0 or ∞ depending on the context.

2. The energy function

$$E\left(\sigma\right) := \frac{1}{2}\int_0^1 \left|\dot{\sigma}\left(\tau\right)\right|^2 d\tau$$

appearing in the exponent in (1.2) requires σ to be appropriately differentiable;

this is at odds with the fact that sample paths of Wiener measure ν are almost surely nowhere differentiable.

3. There is no Lebesgue measure $\mathcal{D}\sigma$ on infinite dimensional path spaces.

4. δ_x is a distribution so pointwise evaluation does not make sense.

5. It is generally not permissible to multiply a distribution δ_x with a measure $\frac{1}{Z}\exp\left(-\frac{1}{2}\int_0^1 |\dot{\sigma}(\tau)|^2 d\tau\right)\mathcal{D}\sigma$.

Various attempts to use path integrals to rigorously construct solutions to the Schrödinger (heat) equation have been made, out of which we highlight two routes. One is to approximate the path integral through piecewise "linear" paths or polygonal paths, which evolves as a finite dimensional approximation scheme that will be discussed more in Section 1.1. Another route, pioneered by Kac, is the realization of taking Wiener measure as the framework of integration over path spaces. Roughly speaking, when $V = 0$, one should interprets

$$\text{``}\frac{1}{Z}e^{-\frac{1}{2}\int_0^1|\dot{\sigma}(\tau)|^2 d\tau}\mathcal{D}\sigma\text{''} := d\nu(\sigma) \tag{1.4}$$

and

$$\text{``}\int_{W_o(M)} \delta_x(\sigma(1))\,\frac{1}{Z}e^{-\frac{1}{2}\int_0^1|\dot{\sigma}(\tau)|^2 d\tau}\mathcal{D}\sigma\text{''} := p_1(x,y) \tag{1.5}$$

where $p_t(x,y)$ is the heat kernel on M, (also the fundamental solution to the heat equation if viewed from partial differential equation point of view). In particular, if $M = \mathbb{R}$,

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}}e^{\frac{-(x-y)^2}{2t}}$$

is the well known density function of a normal random variable with mean 0 and variance t. In general, if V possesses some integrability or regularity, one can prove rigorously the following results which are usually categorized as Feynman–Kac–type formula

$$p^{H}(x,y) = p_1(x,y)\int_{W_o(M)} e^{-\int_0^1 V(\Sigma_s)ds} d\nu_x$$

and

$$e^{-H} f(x) = \int_M p_1(x,y) f(y)\, dy \int_{W_o(M)} e^{-\int_0^1 V(\Sigma_s)ds} d\nu_x$$

where $\Sigma_s : \sigma \ni W_o(M) \to \sigma(s) \in M$ is the coordinate function. Interested readers may refer to [31] and references therein for a thorough summary of this field in Euclidean space with a flavor of rigorous quantum field theory and may refer to [5] for a survey of results in general Riemannian manifolds.

1.1 Finite Dimensional Approximation Scheme for Path Integrals

The central idea behind finite dimensional approximation scheme is to define a path integral as a limit of the same integrands restricted to "natural" approximate path spaces, for example, piecewise linear paths, broken lines, polygonal paths and so on. The ill–defined expression under these finite dimensional approximations usually becomes well–defined or has better interpretations, see ([17], [24]). Not suprisingly, thanks to Kac, Wiener measure is found to share a similar finite dimensional approximation scheme. For example, when $M = \mathbb{R}^d$, it is known that Wiener measure on $W\left(\mathbb{R}^d\right)$ may be approximated by Gaussian measures on piecewise linear path spaces. More specifically, Eq. (1.4) restricted to a finite dimensional subspace of piecewise linear paths based on a partition of $[0,1]$ has a natural interpretation as Gaussian probability measure resulting from the canonical isometry between the piecewise linear path space and $\mathbb{R}^{dn}$, where n is the number of partition points. By combining Wiener's theorem on the existence of Wiener measure with the dominated convergence theorem, one can see that these Gaussian measures converge weakly to ν as the mesh of partition tends to zero, (see for example [15, Proposition 6.17] for details). An analogous theory on general manifolds was also developed, see for example [32], Atiyah [4], Bismut [6], Andersson and Driver [3] and references therein. In [3], followed by [30] and [29], the finite dimensional approximation problem is viewed in its full geometric form by restricting the expression in Eq. (1.4) to finite dimensional sub-manifolds of piecewise geodesic paths on M. Unlike the flat case ($M = \mathbb{R}^d$) where the choice of translation invariant Riemannian metric on path spaces is irrelevant, various Riemannian metrics on approximate path spaces are explored. Based on these metrics, different approximate measures are constructed which lead to different limiting measures on $W_o(M)$. Some limits agree with the results seen elsewhere (for example, from the Feynman–Kac formula or in physics experiments, see [3], [29]) while others are mathematically intriguing in their own right (see [30]). In this thesis we adopt a so–called $G^1_{\mathcal{P}}$ metric on the piecewise geodesic space. In [3], the finite dimensional approximation result based on this metric is shown to agree with the classical result in Euclidean space.

In the remainder of this section, we briefly summarize some results in [3] to give reader a better understanding of how the finite dimensional approximation scheme goes as well as establishing some necessary notations used in this thesis.

Definition 1.1 (Cameron-Martin space on (M, o)) *Let*

$$H(M) := \left\{ \sigma \in C([0,1] \mapsto M) : \sigma(0) = o \ , \ \sigma \text{ is a.c. and } \int_0^1 |\sigma'(s)|^2 \, ds < \infty \right\}$$

*be the **Cameron-Martin space** on (M, o). (Here a.c. means absolutely continuous.)*

Notation 1.2 *Denote $\Gamma(TM)$ to be differentiable sections of TM and $\Gamma_\sigma(TM)$ to be differentiable sections of TM along $\sigma \in H(M)$.*

The space, $H(M)$, is an infinite dimensional Hilbert manifold which is a central object in problems related to variation of continuous paths. Roughly speaking, it specifies the directions to which we are allowed to take directional direvatives for random variables on $(W_o(M), \nu)$. [26] contains a good exposition of the manifold of paths. For example, Theorem 1.2.9 in [26] presents its differentiable structure in terms of atlases. Compared to the local structure of $H(M)$, we are more interested in its Riemannian structure.

The following metric is a commonly used Riemannian metric on $H(M)$.

Definition 1.3 *For any $\sigma \in H(M)$ and $X, Y \in \Gamma_\sigma^{a.c.}(TM)$,*

$$G^1(X,Y) = \int_0^1 \left\langle \frac{\nabla X}{ds}(s), \frac{\nabla Y}{ds}(s) \right\rangle_g ds$$

where $\Gamma_\sigma^{a.c.}(TM)$ is the set of absolutely continuous vector fields along σ with finite energy, i.e. $\int_0^1 \left\langle \frac{\nabla X}{ds}(s), \frac{\nabla X}{ds}(s) \right\rangle_g ds < \infty$.

Remark 1.4 *To see that G^1 is a metric on $H(M)$, we identify the tangent space $T_\sigma H(M)$ with $\Gamma_\sigma^{a.c.1}(TM)$. To motivate this identification, consider a differentiable one-parameter family of curves σ_t in $H(M)$ such that $\sigma_0 = \sigma$. By definition of tangent vector, $\frac{d}{dt}|_0 \sigma_t(s)$ should be viewed as a tangent vector at σ. This is actually the case, for detailed proof, see Theorem 1.3.1 in [26].*

Definition 1.5 (Piecewise geodesic space) *Given a partition*

$$\mathcal{P} := \{0 = s_0 < \cdots < s_n = 1\} \text{ of } [0,1],$$

define:

$$H_{\mathcal{P}}(M) := \left\{\sigma \in H(M) \cap C^2([0,1] \setminus \mathcal{P}) : \nabla\sigma'(s)/ds = 0 \text{ for } s \notin \mathcal{P}\right\}. \qquad (1.6)$$

The piecewise geodesic space $H_{\mathcal{P}}(M)$ can be viewed as a finite dimensional embedded submanifold of $H(M)$. As for its tangent space, following the argument of

Theorem 1.3.1 in [26], for any $\sigma \in H_{\mathcal{P}}(M)$, the tangent space $T_\sigma H_{\mathcal{P}}(M)$ may be identified with vector-fields along σ of the form $X(s) \in T_{\sigma(s)}M$ where $s \to X(s)$ is piecewise C^1 and satisfies Jacobi equation for $s \notin \mathcal{P}$, i.e.

$$\frac{\nabla^2 X}{ds^2}(s) = R(\dot{\sigma}(s), X(s))\,\dot{\sigma}(s),$$

where R is the curvature tensor. (See Theorem 2.41 below for a more detailed description of $TH_{\mathcal{P}}(M)$). After specifying the tangent space of $H_{\mathcal{P}}(M)$, we can define the $G^1_{\mathcal{P}}$ metric as follows,

Definition 1.6 *For any $\sigma \in H_{\mathcal{P}}(M)$ and $X, Y \in T_\sigma H_{\mathcal{P}}(M)$, let*

$$G^1_{\mathcal{P}}\langle X, Y\rangle := \sum_{j=1}^{n} \left\langle \frac{\nabla X}{ds}(s_{j-1}+), \frac{\nabla Y}{ds}(s_{j-1}+) \right\rangle_g \Delta_j \tag{1.7}$$

where $\Delta_j = s_j - s_{j-1}$ and $\frac{\nabla Y}{ds}(s_{j-1}+) = \lim_{s \downarrow s_{j-1}} \frac{\nabla Y}{ds}(s)$.

Endowed with the Riemannian metric $G^1_{\mathcal{P}}$, $H_{\mathcal{P}}(M)$ becomes a finite dimensional Riemannian manifold and the left hand side of (1.4) is now well–defined on $H_{\mathcal{P}}(M)$ if $\mathcal{D}\sigma$ is interpreted as the volume measure induced from this Riemannian metric. This motivates the following approximate measure definition.

Definition 1.7 (Approximate measure on $H_{\mathcal{P}}(M)$) *Let $\nu^1_{\mathcal{P}}$ be the probability measure on $H_{\mathcal{P}}(M)$ defined by;*

$$d\nu^1_{\mathcal{P}}(\sigma) = \frac{1}{Z^1_{\mathcal{P}}} e^{-\frac{1}{2}\int_0^1 \langle \sigma'(s), \sigma'(s)\rangle ds} dvol_{G^1_{\mathcal{P}}}(\sigma), \tag{1.8}$$

where $dvol_{G^1_{\mathcal{P}}}$ is the volume measure on $H_{\mathcal{P}}(M)$ induced from the metric $G^1_{\mathcal{P}}$ and $Z^1_{\mathcal{P}}$ is the normalization constant.

Further, Andesson and Driver proved that these measures converge weakly to Wiener measure.

Theorem 1.8 (Anderson-Driver, Theorem 1.8. [3]) *Suppose $f : W(M) \to \mathbb{R}$ is bounded and continuous, then*

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma)\, d\nu^1_{\mathcal{P}}(\sigma) = \int_{W_o(M)} f(\sigma)\, d\nu(\sigma).$$

1.2 Main Theorems

In this section we state the main results of this thesis while avoiding many technical details.

Definition 1.9 (Pinned piecewise geodesic space) *For any $x \in M$,*

$$H_{\mathcal{P},x}(M) := \{\sigma \in H_{\mathcal{P}}(M) : \sigma(1) = x\}.$$

We prove below in Proposition 3.8 that when M has non–positive sectional curvature, $H_{\mathcal{P},x}(M)$ is an embedded submanifold of $H_{\mathcal{P}}(M)$.

Theorem 1.10 *If M is a Hadamard manifold with bounded sectional curvature and $\mathcal{P} = \{k/n\}_{k=0}^n$ are equally-spaced partitions, then there exists a finite measure $\nu^1_{\mathcal{P},x}$ supported on $H_{\mathcal{P},x}(M)$, such that for any bounded continuous function f on $H_{\mathcal{P}}(M)$,*

$$\lim_{m\to\infty} \int_{H_{\mathcal{P}}(M)} \delta_x^{(m)}(\sigma(1))\, f(\sigma)\, d\nu^1_{\mathcal{P}}(\sigma) = \int_{H_{\mathcal{P}}(M)} f(\sigma)\, d\nu^1_{\mathcal{P},x}(\sigma).$$

where $\delta_x^{(m)}$ is an approximating sequence of δ_x in $C_0^\infty(M)$.

Recall that a Hadamard manifold is a simply connected complete Riemannian manifold with non-positive sectional curvature.

Theorem 1.10 can be viewed as a finite dimensional version of (1.5). A rigorous theory explaning (1.5) is Watanabe's theory of generalized Wiener functionals.

In [37], Watanabe considers the following expression

$$\mathbb{E}\left[\delta_x \circ E_1 \cdot \Phi\right]$$

where E_1 is the **end point evaluation map**, i.e. for any $\sigma \in W_o(M)$, $E_1(\sigma) = \sigma(1)$ and Φ are some “nice ”Wiener functionals (test functions). As was shown by Airault–Malliavin [1] and Sugita [36], if $M = \mathbb{R}^d$ is a Euclidean space, there exists a modification of Φ, called quasi–continuous modification (denoted by $\tilde{\Phi}$), such that the following identity holds:

$$\mathbb{E}_{\nu_x}\left[\tilde{\Phi}\right] = \mathbb{E}_{\nu}\left[\delta_x \circ E_1 \cdot \Phi\right].$$

The point of this theorem is that it represents a generalized Wiener functional $\delta_x \circ E_1$ as a measure ν_x supported on a ”hypersurface” $\mathcal{S}_x := \left\{\sigma \in W\left(\mathbb{R}^d\right) : E_1(\sigma) = x\right\}$. Theorem 1.10 represents $\delta_x \circ E_1$ as a measure $\nu^1_{\mathcal{P},x}$ (See Definition 3.10) in the ”hypersurface” $H_{\mathcal{P},x}(M)$, which can be viewed as a finite dimensional analog of Identity 1.2.

The next theorem asserts, under additional geometric restrictions, that the measure $\nu^1_{\mathcal{P},x}$ we obtained from Theorem 1.10 serves as a good approximation to pinned Wiener measure ν_x.

Theorem 1.11 *If M is a Hadamard manifold with constant sectional curvature, then*

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma)\, d\nu^1_{\mathcal{P},x}(\sigma) = \int_{W(M)} f(\sigma)\, d\nu_x(\sigma)$$

for $f \in \mathcal{FC}^\infty_{1-}$ (see Notation 7.11) and ν_x is pinned Wiener measure, see Theorem 2.17 below.

1.3 Structure of the Thesis

For the guidance to the reader, we give a brief summary of the contents of this thesis.

In Chapter 2 we set up some notation and preliminaries in probability and geometry. In particular we present the Eells-Elworthy-Malliavin construction of Brownian motion on manifolds.

In Chapter 3 we define explicitly the pinned approximate meausre $\nu^1_{\mathcal{P},x}$ and study its properties. In Theorem 3.12, we prove that $\nu^1_{\mathcal{P},x}$ is a finite measure and that $x \to \nu^1_{\mathcal{P},x}(H_{\mathcal{P},x}(M))$ is a continuous function on M. This property is the key ingredient in proving Theorem 1.10, which is given in Chapter 3.

In Chapter 4 we develop the so–called orthogonal lift of a vector field X on M to a vector field $\tilde{X}(\cdot)$ on $W_o(M)$. We define $\tilde{X}(\cdot)$ first on $H(M)$ by minimizing a norm of $\tilde{X}(\cdot)$ which is induced from a “damped ”metric related to the Ricci curvature of M. This lift is then “stochastically ”extended to $W_o(M)$. Some tools from Malliavin calculus are reviewed as needed in order to define $\tilde{X}(\cdot)$ as an anticipating differential opearator on $W_o(M)$. We then establish integration–by–parts formula for $\tilde{X}(\cdot)$.

In Chapter 5 we focus on the finite dimensional manifold $H_{\mathcal{P}}(M)$. In Section 5.1 a parametrization of the tangent space of $H_{\mathcal{P}}(M)$ is given. Using this parametrization and some linear algebra we obtain a formula for the orthogonal lift $\tilde{X}_{\mathcal{P}}$ of $X \in \Gamma(TM)$ relative to the norm induced from the $G^1_{\mathcal{P}}$ metrc on $H_{\mathcal{P}}(M)$.

In Chapter 6, (using the development maps introduced in Chapter 2), we view $\tilde{X}_{\mathcal{P}}$ as defined on all of $W_o(M)$ and show that for any bounded cylinder function f (also introduced in Chapter 2), $\tilde{X}_{\mathcal{P}}f \to \tilde{X}f$ in $L^{\infty-}(W_o(M))$ and more challengingly, we show $\tilde{X}^{tr,\nu}f - \tilde{X}^{tr,\nu^1_{\mathcal{P}}}_{\mathcal{P}}f \to 0$, where $\tilde{X}^{tr,\nu}$ is the adjoint of $\tilde{X}$ with respect to ν and $\tilde{X}^{tr,\nu^1_{\mathcal{P}}}_{\mathcal{P}}$ is the adjoint of $\tilde{X}_{\mathcal{P}}$ with respect to $\nu^1_{\mathcal{P}}$.

In Chapter 7 We combines all the tools that are developed from previous chapters to prove the main Theorem (1.11) of this thesis.

Chapter 2

Background and Notation

For the remainder of the thesis, let $u_0 : \mathbb{R}^d \to T_oM$ be a fixed orthonormal frame at $o \in M$ which we add to the standard setup (M, g, o, u_0, ∇). We will first introduce the orthonormal frame bundle $\mathcal{O}(M)$ which is crucial in the Eells-Elworthy-Malliavin construction of Brownian motion. A connection is then defined on $\mathcal{O}(M)$. The reader may refer to Appendix $A.2$ for a more detailed exposition of principal bundles ($\mathcal{O}(M)$ is a special case of a principal bundle) and connections on them.

Definition 2.1 (Orthonormal Frame Bundle $(\mathcal{O}(M), \pi)$**)** *For any $x \in M$, denote by $\mathcal{O}(M)_x$ the space of orthonormal frames on T_xM, i.e. the space of linear isometries from $\mathbb{R}^d$ to T_xM. Denote $\mathcal{O}(M) := \cup_{x \in M} \mathcal{O}(M)_x$ and let $\pi : \mathcal{O}(M) \to M$ be the (fiber) projection map, i.e. for each $u \in \mathcal{O}(M)_x$, $\pi(u) = x$. The pair $(\mathcal{O}(M), \pi)$ is the orthonormal frame bundle over M whose structure group is the orthogonal group $O(d)$–the $d \times d$ real orthogonal matrices.*

Definition 2.2 (Connection on $\mathcal{O}(M)$**)** *A connection on $\mathcal{O}(M)$ is uniquely specified by the $\mathfrak{so}(d)$–valued connection form ω^{∇} on $\mathcal{O}(M)$ determined by ∇;*

for any $u \in \mathcal{O}(M)$ *and* $X \in T_u\mathcal{O}(M)$,

$$\omega_u^{\nabla}(X) := u^{-1}\frac{\nabla u(s)}{ds}\mid_{s=0}$$

where $u(\cdot)$ *is a differentiable curve on* $\mathcal{O}(M)$ *such that* $u(0) = u$ *and* $\frac{du(s)}{ds}\mid_{s=0} = X$. *For any* $\xi \in \mathbb{R}^d$, $\frac{\nabla u(s)}{ds}\mid_{s=0}\xi := \frac{\nabla u(s)\xi}{ds}\mid_{s=0}$ *is the covariant derivative of* $u(\cdot)\xi$ *along* $\pi(u(\cdot))$ *at* $\pi(u)$.

ω^{∇} *determines a decomposition of* $T\mathcal{O}(M)$. *We will call the kernel of* ω^{∇} *the horizontal vector space (denoted by* $HT\mathcal{O}(M)$*) and call the compliment space the vertical vector space (denoted by* $VT\mathcal{O}(M)$*).*

Definition 2.3 *For any* $a \in \mathbb{R}^d$, *define the horizontal lift* $B_a \in \Gamma(T\mathcal{O}(M))$ *of* a *in the following way: for any* $u \in \mathcal{O}(M)$,

- $\omega_u^{\nabla}(B_a(u)) = 0$
- $\pi_*(B_a(u)) = ua$

Remark 2.4 *By the rank-nullity theorem, it is easy to see that the above conditions determine uniquely the horizontal lift.*

Recall that we have defined the Cameron-Martin space on M:

$$H(M) := \left\{\sigma \in C([0,1], M) : \sigma(0) = o, \sigma \text{ is a.c. and } \int_0^1 |\sigma'(s)|_g^2 \, ds < \infty\right\} \tag{2.1}$$

Similarly we define $H_0(\mathbb{R}^d)$ and $H_{u_0}(\mathcal{O}(M))$ by changing the state spaces to be $\mathbb{R}^d$, $\mathcal{O}(M)$, reference points to be 0, u_0 and using the usual metric for g on the Euclidean spaces $\mathbb{R}^d$, $\mathbb{R}^{d\times d}$.

Definition 2.5 (Horizontal lift of a path) *For any* $\sigma \in H(M)$, *a curve* $u : [0,1] \to \mathcal{O}(M)$ *is said to be a horizontal lift of* σ *if* $\pi \circ u = \sigma$ *and the tangent vector to* $u(s)$ *always belongs to* $HT_{u(s)}\mathcal{O}(M)$.

Theorem 2.6 *Given $\sigma \in H(M)$ and $u_0 \in \pi^{-1}(\sigma(0))$, there exists a unique horizontal lift $u(s)$ such that $u(0) = u_0$. We will denote this map by ψ.*

Proof. The condition of existence of horizontal lift u of σ is equivalent to:

$$\begin{matrix} \pi(u(s)) = \sigma(s) \\ \omega^{\nabla}(u'(s)) = 0 \end{matrix} \quad \text{for } s \in [0,1]$$

For any $s \in [0,1]$, there exists U_α in the open cover of M and $\epsilon > 0$ such that $\sigma(\tau) \in U_\alpha$ for $\tau \in (s-\epsilon, s+\epsilon) \cap [0,1]$. Denote by ω_α the restriction of the connection one-form ω on $\pi^{-1}(U_\alpha)$ and $\phi_\alpha \circ u(\tau) = (\sigma(\tau), g(\tau)) \in U_\alpha \times G$, where $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is the local trivialization. Then after identifying $T(U_\alpha \times G)$ with $TU_\alpha \times TG$, the condition $\omega^{\nabla}(u'(\tau)) = 0$ is equivalent to $A_{\sigma(\tau)}\sigma'(\tau) + C_{\sigma(\tau)}g'(\tau) = 0$, where A and C are two $\mathfrak{g}-$valued one forms on U_α and G. Since $\sigma(\tau)$ is fixed, this gives rise to a linear system of ODEs of $g(\tau)$, since the initial condition is specified, there is a unique solution $g(\tau)$ and hence the unique $u(\tau)$.

■

Notation 2.7 *A path $u \in H_{u_0}(\mathcal{O}(M))$ is said to be horizontal if the tangent vector to $u(s)$ always belongs to $HT_{u(s)}\mathcal{O}(M)$. We denote the set of horizontal paths by $HH_{u_0}(\mathcal{O}(M))$.*

Fact 2.8 *Notice that $u(\sigma, s)\, u_0^{-1}$ is the parallel translation $//_s(\sigma)$ along σ.*

Remark 2.9 *From Theorem 2.6 we can see that there is a one to one correspondence between $HH_{u_0}(\mathcal{O}(M))$ and $H_o(M)$. More explicitly, ψ is a diffeomorphism from $H(M)$ to $HH_{u_0}(\mathcal{O}(M))$ whose inverse is π.*

Definition 2.10 (Development map) *Given $w \in H_0(\mathbb{R}^d)$, the solution to the ordinary differential equation*

$$du(s) = \sum_{i=1}^{d} B_{e_i}(u(s))\, dw^i(s), u(0) = u_0$$

is defined to be the development of w *to* $H_{u_0}(\mathcal{O}(M))$ *and we will denote this map* $w \to u$ *by* η*, i.e.* $\eta(w) = u$. η *is said to be the development map to* $H_{u_0}(\mathcal{O}(M))$*. Here* $\{e_i\}_{i=1}^d$ *is the standard basis of* $\mathbb{R}^d$*.*

Remark 2.11 *From Definition 2.10 and the smooth dependence of driving path in ODE systems we can see that* η *is a diffeomorphism from* $H_0(\mathbb{R}^d)$ *to* $HH_{u_0}(\mathcal{O}(M))$*.*

Definition 2.12 (Rolling map) $\phi = \pi \circ \eta : H_0(\mathbb{R}^d) \to H(M)$ *is said to be the development map to* $H(M)$*.*

Remark 2.13 *From Remark 2.9 and 2.11 one can see that* ϕ *has a smooth inverse* ϕ^{-1}*, which can be defined explicitly as follows:*

Definition 2.14 (Anti-rolling map) *Given* $\sigma \in H(M)$ *with* $u = \psi(\sigma)$*. The anti-development of* σ *is a curve* $w \in H_0(\mathbb{R}^d)$ *defined by:*

$$w_t = \int_0^t u_s^{-1} \sigma_s' ds$$

It is not hard to see $w = \phi^{-1}(\sigma)$*.*

The following diagram illustrate the one–to–one correspondence between $H(M)$, $H_0(\mathbb{R}^d)$ and $H_{u_0}(\mathcal{O}(M))$. The Eells-Elworthy-Malliavin construction of Brownian motion depends in essence on a stochastic version of the maps defined above. Since the development maps on the smooth category are defined through ordinary differential equations, a natural way to introduce probability is to replace ODEs by (Stratonovich) stochastic diffrerential equations.

First we set up some measure theoretic notation and conventions. Suppose that $(\Omega, \{\mathcal{G}_s\}, \mathcal{G}, P)$ is a filtered measurable space with a finite measure P. For any $\mathcal{G}$—measurable function f, we use $P(f)$ and $\mathbb{E}_P[f]$ (if P is a probability measure) to denote the integral $\int_\Omega f dP$. Given two filtered measurable spaces $(\Omega, \{\mathcal{G}_s\}, \mathcal{G}, P)$ and $(\Omega', \{\mathcal{G}_s'\}, \mathcal{G}', P')$ and a $\mathcal{G}/\mathcal{G}'$ measurable map $f : \Omega \to \Omega'$, the law of f under

P is the push-forward measure $f_* P(\cdot) := P(f^{-1}(\cdot))$. We are mostly interested in the path spaces $W_o(M)$, $W_0(\mathbb{R}^d)$ and $W_{u_0}(\mathcal{O}(M))$, where the following notation is being used.

Notation 2.15 *If (Y, y) is a pointed manifold, then $W(Y) := C([0,1], Y)$ is the space of all continuous paths in Y equipped with the uniform topology. $W_y(Y) := \{w \in W(Y) \mid w(0) = y\}$ refers to the subset of continuous paths that start at y.*

Definition 2.16 *For any $s \in [0,1]$ let $\Sigma_s : W_y(Y) \to Y$ be the* ***coordinate functions*** *given by $\Sigma_s(\sigma) = \sigma(s)$.*

We will often view Σ as a map from $W_y(Y)$ to $W_y(Y)$ in the following way: for any $\sigma \in W_y(Y)$ and $s \in [0,1]$, $\Sigma(\sigma)(s) = \Sigma_s(\sigma)$. Let $\mathcal{F}_s^o$ be the $\sigma-$algebra generated by $\{\Sigma_\tau : \tau \leq s\}$. We use $\mathcal{F}_1^o$ as the raw $\sigma-$algebra and $\{\mathcal{F}_s^o\}_{0 \leq s \leq 1}$ as the filtration on $W_y(Y)$. The next theorem defines the Wiener measure ν and pinned Wiener measure ν_x on $(W_y(Y), \mathcal{F}_1^o)$.

Theorem 2.17 *There exist two finite measures ν and ν_x on $(W_y(Y), \mathcal{F}_1^o)$ which are uniquely determined by their finite dimensional distributions as follows. For any partition $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = 1$ of $[0,1]$ and bounded functions $f : Y^n \to \mathbb{R}$;*

$$\nu(f(\Sigma_{s_1}, \ldots, \Sigma_{s_n})) = \int_{Y^n} f(x_1, \ldots, x_n) \Pi_{i=1}^n p_{\Delta s_i}(x_{i-1}, x_i) \, dx_1 \cdots dx_n \tag{2.2}$$

and

$$\nu_x(f(\Sigma_{s_1}, \ldots, \Sigma_{s_n})) = \int_{Y^{n-1}} f(x_1, \ldots, x_n) \Pi_{i=1}^n p_{\Delta s_i}(x_{i-1}, x_i) \, dx_1 \cdots dx_{n-1} \tag{2.3}$$

where $p_t(\cdot,\cdot)$ is the heat kernel on Y, $\Delta_i = s_i - s_{i-1}$, $x_0 \equiv y$ and $x_n \equiv x$ in (2.3).

Fact 2.18 *From Theorem 2.17 it is clear that the law of the adapted process* $\Sigma : W_y(Y) \to W_y(Y)$ *is* ν *and* Σ *is said to be the canonical Brownian motion on* Y*.*

Definition 2.19 (Brownian motion) *A stochastic process* $X : (\Omega, \mathcal{G}_s, \{\mathcal{G}\}, P) \to (W_y(Y), \nu)$ *is said to be a standard Brownian motion on* Y *if the law of* X *is* ν *i.e.* $X_* P := P \circ X^{-1} = \nu$*.*

Remark 2.20 *Using Theorem 2.17, we can construct Wiener measure and pinnned Wiener measure on* $W_0\left(\mathbb{R}^d\right)$*,* $W_o(M)$ *and* $W_{u_0}(\mathcal{O}(M))$ *respectively. In order to avoid ambiguity from moving between* $W_0\left(\mathbb{R}^d\right)$ *and* $W_o(M)$*, we fix the symbol* $\mu\,(\mu_x)$ *as the Wiener (pinned Wiener) measure on* $W_0\left(\mathbb{R}^d\right)$ *and reserve the symbol* $\nu\,(\nu_x)$ *as the Wiener (pinned Wiener) measure on* $W_o(M)$*. Meanwhile we reserve* Σ *as the canonical Brownian motion on* M*.*

Definition 2.21 ($Y-$**valued semimartingale)** *Let* Y *be a differentiable manifold and* $(\Omega, \mathcal{G}_s, \{\mathcal{G}\}, P)$ *be a filtered probability space. A random map* $X : (\Omega, \mathcal{G}_s, \{\mathcal{G}\}, P) \to W_y(Y)$ *is called a* $Y-$*valued semimartingale if* $f(X)$ *is a* $\mathbb{R}-$*valued semimartingale for all* $f \in C^\infty(Y)$*.*

Proposition 2.22 X *is a* $Y-$*valued semimartingale iff there exist d vector fields* $\{V_i\}_{i=1}^d$ *and a* $\mathbb{R}^d-$*valued driving semimartingale* w *such that* X *is the solution to the stochastic differential equation:*

$$\delta X_t = \sum_{i=1}^{d} V_i(X_t)\,\delta w_t^i,\ X(0) = o,\ 0 \le t \le 1. \tag{2.4}$$

where δ *is the Stratonovich differential and Eq.*(2.4) *means for any* $f \in C^\infty(M)$*,*

$$f(X_t) = f(o) + \int_0^t \sum_{i=1}^{d} V_i f(X_t)\,\delta w_t^i\ ,\ 0 \le t \le 1. \tag{2.5}$$

Proof. Refer to Theorem 1.1 in [25] for a proof using local coordinate charts or a more direct proof in Theorem 1.2.9 in [22] using Whitney's imbedding theorem to imbed M in a Euclidean space. ■

Notation 2.23 *Let $\mathfrak{S}\left(\mathbb{R}^d\right)$, $\mathfrak{S}(M)$ and $\mathfrak{S}(H\mathcal{O}(M))$ be the space of based semimartingales on $\mathbb{R}^d$ starting at 0, based semimartingales on M starting at $o \in M$ and based semimartingales on the horizontal subbundle $H\mathcal{O}(M)$ starting at u_0 respectively.*

Definition 2.24 (Stochastic development map) *For any $w \in \mathfrak{S}\left(\mathbb{R}^d\right)$, the solution to the following SDE*

$$\delta u_s = \sum_{i=1}^{d} B_{e_i}(u_s)\, \delta w_s \ , \ u(0) = u_0$$

is said to be the development of w. This map $w \in \mathfrak{S}\left(\mathbb{R}^d\right) \to u \in \mathfrak{S}(H\mathcal{O}(M))$ is said to be the stochastic development map on $\mathfrak{S}(H\mathcal{O}(M))$.

Remark 2.25 *The stochastic development map defined in 2.24 is actually an equivalence class of maps with respect to ν. In this thesis we will fix a version and denote it by $\tilde{\eta}$.*

Lemma 2.26 *Given two manifolds M, N and a smooth map $f : M \to N$, if w is a semimartingale on M, then $f(w)$ is a semimartingale on N.*

Remark 2.27 *Using Lemma 2.26, we can easily see that for any $w \in \mathfrak{S}\left(\mathbb{R}^d\right)$, $\pi(\tilde{\eta}(w))$ is a semimartingale on M. We will call $\tilde{\phi} := \pi \circ \tilde{\eta}$ the **stochastic development map** on $\mathfrak{S}(M)$.*

Theorem 2.28 (Horizontal lift of semimartingale) *Given $\tilde{\sigma} \in \mathfrak{S}(M)$, there exists a unique (up to $\nu-$equivalence) $\tilde{u} \in \mathfrak{S}(H\mathcal{O}(M))$ such that*

$$\pi(\tilde{u}_s) = \tilde{\sigma}_s \tag{2.6}$$

After fixing a version, we will call this map $\mathfrak{S}(M) \ni \tilde{\sigma} \mapsto \tilde{u} \in \mathfrak{S}(H\mathcal{O}(M))$ *the (Stochastic) horizontal lift, simply denoted by* $\tilde{\psi}$.

Proof. See Theorem 2.3.5 in [22] ■

Definition 2.29 (Stochastic anti-development map) *Given* $\tilde{\sigma} \in \mathfrak{S}(M)$, *the stochastic anti-development of* σ *is* $w \in \mathfrak{S}(\mathbb{R}^d)$ *defined by:*

$$\delta w_s = \tilde{u}_s^{-1} \delta \tilde{\sigma}_s \ , \ w_0 = 0 \tag{2.7}$$

Denote this map by $\tilde{\Phi}$.

Fact 2.30 *We state the following fact that are frequently used in the thesis. The proof can be found in Appendix A.*

- ϕ *is a diffeomorphism from* $H_0(\mathbb{R}^d)$ *to* $H(M)$,
- $\phi\mid_{H_{\mathcal{P}}(\mathbb{R}^d)}$ *is a diffeomorphism from* $H_{\mathcal{P}}(\mathbb{R}^d)$ *to* $H_{\mathcal{P}}(M)$,
- $\tilde{\phi}^{-1}(\Sigma)$ *is a Brownian motion on* $(W_0(\mathbb{R}^d), \mu)$,
- $\pi \circ \tilde{\eta} = I_{W(M)}$–$\nu$ *a.s.*
- $\tilde{\phi} \circ \tilde{\Phi} = I_{W(M)}$–$\nu$ *a.s.*

Notation 2.31 *From now on some notations are fixed for the conveniance of consistency. For any* $\sigma \in H(M)$, $u.(\sigma) \in H_{u_0}(\mathcal{O}(M))$ *is its horizontal lift and* $b.(\sigma) \in H_0(\mathbb{R}^d)$ *is its anti-rolling. Recall that* $\{\Sigma\}$ *is fixed to be the standard Brownian motion on* $(W(M), \nu)$. *We also fix* $\beta := \tilde{\Phi}(\Sigma)$ *to be the anti-rolling of* Σ, *which is a Brownian motion on* $W_0(\mathbb{R}^d)$. $\tilde{u}. := \tilde{\eta}(\Sigma)$ *is the (stochastic) horizontal lift of* Σ.

Notation 2.32 (Path approximation map) $\pi_{\mathcal{P}} : W\left(\mathbb{R}^d\right) \to H_{\mathcal{P}}\left(\mathbb{R}^d\right)$ *is the path approximation map: i.e. if* $s \in [s_{i-1}, s_i]$, $\sigma \in W\left(\mathbb{R}^d\right)$,

$$\pi_{\mathcal{P}}(\sigma)(s) := \sigma(s_{i-1}) + \frac{\Delta_i \sigma}{\Delta_i s}(s - s_{i-1}).$$

where $\Delta_i \sigma = \sigma_{s_i} - \sigma_{s_i}$ *and* $\Delta_i s = s_{s_i} - s_{s_i}$.

Notation 2.33 $\beta_{\mathcal{P}}$ *is the piecewise Brownian Motion on* $\mathbb{R}^d$ *defined explicitly below: If* $s \in [s_{i-1}, s_i]$,

$$\beta_{\mathcal{P}}(s) := \beta(s_{i-1}) + \frac{\Delta_i \beta}{\Delta_i}(s - s_{i-1})$$

where $\Delta_i \beta = \beta(s_i) - \beta(s_{i-1})$*and* $\Delta_i = s_i - s_{i-1}$.

Notation 2.34 (Geometric preliminary)

- *For any* $\sigma \in H(M)$, *define* $R_{u(\sigma,s)}(\cdot,\cdot)\cdot$ *to be a map from* $\mathbb{R}^d \otimes \mathbb{R}^d$ *to* $\mathbb{R}^d$ *as follows:*

$$R_{u(\sigma,s)}(\cdot,\cdot)\cdot = u(\sigma,s)^{-1} R(u(\sigma,s)\cdot, u(\sigma,s)\cdot)\, u(\sigma,s) \tag{2.8}$$

  where R *is the curvature tensor of* M. *Similarly define* $R_{\tilde{u}(\sigma,s)}(\cdot,\cdot)\cdot$ *to be a random map (up to* ν*-equivalence) from* $\mathbb{R}^d \otimes \mathbb{R}^d$ *to* $\mathbb{R}^d$ *as follows:*

$$R_{\tilde{u}(\sigma,s)}(\cdot,\cdot)\cdot = \tilde{u}(\sigma,s)^{-1} R(\tilde{u}(\sigma,s)\cdot, \tilde{u}(\sigma,s)\cdot)\, \tilde{u}(\sigma,s) \tag{2.9}$$

- $Ric(\cdot) := \sum_{i=1}^d R(v_i,\cdot)\, v_i$ *is the Ricci curvature tensor on* M. *Here* $\{v_i\}_{i=1}^d$ *is an orthonormal basis of proper tangent space. Using* $u(\sigma,s)$ *or* $\tilde{u}(\sigma,s)$ *to pull back* R, *we can define* $Ric_{u(\sigma,s)}$ *and* $Ric_{\tilde{u}(\sigma,s)}$ *to be maps (Random maps) from* $\mathbb{R}^d$ *to* $\mathbb{R}^d$.

- *For any* $p \in M$, $exp_p : T_pM \to M$ *is the Riemannian exponential map, i.e.*

for any $\xi \in$ *domain of* exp_p,

$$exp_p(\xi) = \gamma\left(|\xi|, \frac{\xi}{|\xi|}\right)$$

where $\gamma(t,v)$ *is the unique geodesic of* M *with* $\gamma(0) = p$ *and* $\gamma'(0) = v$

Remark 2.35 *The existence of unique local geodesic* $\gamma(t,v)$ *is a standard result in differential geometry, see Proposition 2.17 in [9].*

Remark 2.36 *Sometimes in the thesis we will suppress* σ*, sometimes even* s *in* $u(\sigma, s)$ *when there is no confusion.*

Remark 2.37 *In this thesis the partition* $\mathcal{P}$ *is always equally spaced, so* $|\mathcal{P}| \equiv \Delta_i \equiv \frac{1}{n}$ *for* $i = 1, ..., n$.

We introduce two commonly used test function spaces on $W_o(M)$

Definition 2.38 $f : W_o(M) \mapsto \mathbb{R}$ *is a* ***smooth restricted cylinder function*** *if there exists a partition*

$$\mathcal{P} := \{0 < s_1 < \cdots < s_n \leq 1\}$$

of $[0,1]$ *and a smooth function* $F : M^n \to \mathbb{R}$ *such that:*

$$f = F(\Sigma_{s_1}, \Sigma_{s_2}, \ldots, \Sigma_{s_n})$$

Denote this space by $\mathcal{RFC}^\infty$.

Definition 2.39 $f : W_o(M) \mapsto \mathbb{R}$ *is a* ***smooth cylinder function*** *iff there exists a partition*

$$\mathcal{P} := \{0 < s_1 < \cdots < s_n \leq 1\}$$

of $[0,1]$ *and a smooth function* $F : \mathcal{O}(M)^n \to \mathbb{R}$ *such that:*

$$f = F(\tilde{u}_{s_1}, \tilde{u}_{s_2}, \ldots, \tilde{u}_{s_n})$$

Denote this space by $\mathcal{FC}^\infty$.

Definition 2.40 (Jacobi equation) *For* $\sigma \in H(M)$, $Y \in \Gamma_\sigma(TM)$, *we say* $Y(s) \in T_{\sigma(s)}M$ *satisfies Jacobi equation if:*

$$\frac{\nabla^2}{ds^2} Y(s) = R(\sigma'(s), Y(s))\, \sigma'(s).$$

Further if the horizontal lift $u(s)$ *of* σ *is used, we let* $y(s) := u^{-1}(s) Y(s)$. *It then follows that* $y(s)$ *satisfies the pulled back Jacobi equation,*

$$y''(s) = R_{u(s)}(b'(s), y(s))\, b'(s), \tag{2.10}$$

where $b'(s) = u(s)^{-1} \sigma'(s)$. *Once we have Jacobi equation, we can describe the tangent space* $TH_{\mathcal{P}}(M)$ *of* $H_{\mathcal{P}}(M)$*:*

We formalize the tangent space of $H_{\mathcal{P}}(M)$ mentioned in Definition 1.5.

Theorem 2.41 (Tangent space to $H_{\mathcal{P}}(M)$) *For all* $\sigma \in H_{\mathcal{P}}(M)$,

$$T_\sigma H_{\mathcal{P}}(M) = \left\{ u(s) J(s) \mid J(s) \in C\left([0,1], \mathbb{R}^d\right), \ J \in H_{\mathcal{P},\sigma} \text{ with } J(0) = 0 \right\}. \tag{2.11}$$

Here $J \in H_{\mathcal{P},\sigma}$ *means*

$$J''(s) = R_{u(s)}(b'(s_{i-1}+), J(s))\, b'(s_{i-1}+) \text{ for } s \in [s_{i-1}, s_i) \ i = 1, ..., n.$$

Proof. See Theorem 1.3.1 in [26]. ∎

Notation 2.42 **($\{C_{\mathcal{P},i}(\sigma,s)\}_{i=1}^{n}$ and $\{S_{\mathcal{P},i}(\sigma,s)\}_{i=1}^{n}$)** *Let*

$$\mathcal{P} := \{0 = s_0 < s_1 < \cdots < s_n = 1\}$$

be a partition of $[0,1]$, $K_i := [s_{i-1}, s_i]$ *and* $\Delta_i := s_i - s_{i-1}$ *for* $1 \leq i \leq n$, *and say that* $f(s)$ *satisfies the* i *–Jacobi's equation if*

$$f''(s) = R_{u(s)}\left(u^{-1}\sigma'(s_{i-1}+), f(s)\right) u^{-1}\sigma'(s_{i-1}+) \text{ for } s \in K_i. \tag{2.12}$$

where $u^{-1}\sigma'(s) := u(\sigma,s)^{-1}\sigma'(s) \in \mathbb{R}^d$ *and* $u(s)$ *is the horizontal lift of* σ.

We now let $C_{\mathcal{P},i}(\sigma,s)$ *and* $S_{\mathcal{P},i}(\sigma,s) \in \mathrm{End}(\mathbb{R}^d)$ *denote the solution to Eq. (2.12) with initial conditions,*

$$C_{\mathcal{P},i}(s_{i-1}) = I,\ C'_{\mathcal{P},i}(s_{i-1}) = 0,\ S_{\mathcal{P},i}(s_{i-1}) = 0 \text{ and } S'_{\mathcal{P},i}(s_{i-1}) = I$$

and we further let

$$C_{\mathcal{P},i}(\sigma) := C_{\mathcal{P},i}(\sigma, s_i) \text{ and } S_{\mathcal{P},i}(\sigma) := S_{\mathcal{P},i}(\sigma, s_i).$$

Here we view $C_{\mathcal{P},i}(s)$ *and* $S_{\mathcal{P},i}(s)$ *as maps from* $H_{\mathcal{P}}(M)$ *to* $\mathrm{End}(\mathbb{R}^d)$.

Definition 2.43 *Define for all* $i = 1, \cdots, n$,

$$f_{\mathcal{P},i}(\sigma,s) = \begin{cases} 0 & s \in [0, s_{i-1}] \\ \frac{S_{\mathcal{P},i}(\sigma,s)}{\Delta_i} & s \in [s_{i-1}, s_i] \\ \frac{C_{\mathcal{P},j}(\sigma,s)C_{\mathcal{P},j-1}(\sigma)\cdots\cdots C_{\mathcal{P},i+1}(\sigma)S_{\mathcal{P},i}(\sigma)}{\Delta_i} & s \in [s_{j-1}, s_j] \text{ for } j = i+1, \cdots, n \end{cases}$$

with the convention that $S_{\mathcal{P},0} \equiv |\mathcal{P}|\, I$ *and* $f_{\mathcal{P},0} \equiv I$.

Remark 2.44 *The functions* $\{f_{\mathcal{P},i}\}_{i=0}^{n}$ *encode the functions* $S_{\mathcal{P},j}(s)$, $C_{\mathcal{P},j}(s)$, *for* *example,*

$$S_{\mathcal{P},j}\left(s\right) = \Delta_j f_{\mathcal{P},j}\left(s\right)$$

$$C_{\mathcal{P},j}\left(s\right) = f_{\mathcal{P},j-1}\left(s\right) f_{\mathcal{P},j-1}^{-1}\left(s_j\right)$$

Chapter 3

Approximate Pinned Measure $\nu^1_{\mathcal{P},x}$

3.1 Representation of δ − function

Given X a smooth manifold M or $\mathbb{R}^d$ or open subset of the first two, we will denote the distribution on X by $\mathcal{D}'(X)$ and, compactly supported distribution by $\mathcal{E}'(X)$. For a matrix A, $eig(A)$ is refered to as the set of eigenvalues of A. For a fixed point $x \in M$, we consider δ_x on M, for any $f \in C^{\infty}(M)$,

$$\delta_x(f) = f(x) \, . \, ((\delta_x \in \mathcal{E}'(M)))$$

First of all, we give a representation of δ_x on $\mathbb{R}^d$.

Lemma 3.1 (Representation of δ − function on flat space) *There exist functions $\{g_i\}_{i=0}^d$ such that*

$$\delta_0 = g_0 + \sum_{j=1}^{d} \frac{\partial g_j}{\partial x_j} \text{ in } \mathcal{E}'\left(\mathbb{R}^d\right) \tag{3.1}$$

i.e. for any $f \in C_0^\infty\left(\mathbb{R}^d\right)$*,*

$$f(0) = \int_{\mathbb{R}^d} \left(g_0 + \sum_{j=1}^{d} \frac{\partial g_j}{\partial x_j} \right) f dx = \int_{\mathbb{R}^d} \left(g_0 f - \sum_{j=1}^{d} \frac{\partial f}{\partial x_j} g_j \right) dx \tag{3.2}$$

where $g_0 \in C_0^\infty\left(\mathbb{R}^d\right)$, $\{g_j\}_{j=1}^d \subset C^\infty\left(\mathbb{R}^d / \{0\}\right)$ *with compact support and satisfies*

$$|g_j(x)| \leq c\,|x|^{1-d} \text{ for } j = 1, \cdots, d. \tag{3.3}$$

This lemma can be derived from Lemma 10.10 in [34]. Here we provide another proof using the fundamental solution to the Laplace's equation.

Proof of Lemma 3.1. Define the Newtonian kernel $\Gamma(x)$ on $\mathbb{R}^d$ $(d > 2)$:

$$\Gamma(x) = \frac{|x|^{2-d}}{d(2-d)\,w_d}$$

where w_d is the volume of unit ball on $\mathbb{R}^d$. Then it is well-known $\Gamma(x)$ is the fundamental solution of Laplace's equation, i.e. for any $y \in \mathbb{R}^d$, denote by Δ the Laplacian on $\mathbb{R}^d$:

$$\Delta\Gamma(\cdot - y) = \delta_y(\cdot) \text{ in } \mathcal{E}'\left(\mathbb{R}^d\right).$$

where δ_y is the delta function at y and the equality is interpreted in the distributional sense. In particular if $y = 0$, we get:

$$\Delta\Gamma(\cdot) = \delta_0(\cdot).$$

Denote $\nabla\Gamma$ by Z, then $Z \in C^\infty\left(\mathbb{R}^d / \{0\}\right)$ and we have:

$$|Z| = \left| \frac{x\,|x|^{-d}}{d w_d} \right| \leq C_d\,|x|^{1-d}$$

where C_d is a constant depending only on d and

$$\nabla \cdot Z = \delta_0 \text{ in } \mathcal{E}'\left(\mathbb{R}^d\right).$$

In order to get compact support, we construct a cutoff function $\phi \in C_0^\infty\left(\mathbb{R}^d\right)$ such that $\phi \equiv 1$ on $B(0,1)$ and $\phi \equiv 0$ on $\mathbb{R}^d / B(0,2)$, where $B(x,r)$ is the ball on $\mathbb{R}^d$ centered at x with radius r. Then we have:

$$\nabla \cdot (\phi Z) = \nabla \phi \cdot Z + \phi \nabla \cdot Z \text{ in } \mathcal{E}'\left(\mathbb{R}^d\right).$$

Since the support of δ_0 is $\{0\}$, we get:

$$\delta_0 = \nabla \cdot Z = \phi \nabla \cdot Z = \nabla \cdot (\phi Z) - \nabla \phi \cdot Z$$

where $-\nabla \phi \cdot Z \in C_0^\infty\left(\mathbb{R}^d\right)$ and $\{\phi Z_{x_i}\}_{i=1}^d \subset C^\infty\left(\mathbb{R}^d / \{0\}\right)$ with compact support and $|\phi Z_{x_i}| \leq c\,|x|^{1-d}$ for some $c > 0$. ■

Based on this representation we can get a representation of δ_p for any $p \in M$. Before we get to the representation of δ_p we state a smooth Urysohn lemma.

Lemma 3.2 (Smooth Urysohn Lemma) *If M is a smooth manifold, then for any two disjoint closed sets V_1 and V_2, there exists a function $f \in C^\infty(M, [0,1])$ such that $f^{-1}(\{0\}) = V_1$ and $f^{-1}(\{1\}) = V_2$.*

Theorem 3.3 (Representation of $\delta -$ function on manifold) *For any $p \in M$, there exist functions $\{g_j\}_{j=0}^d \subset C^\infty(M/\{p\}) \cap L^{\frac{d}{d-1}}(M)$ with compact support and smooth vector fields $\{X_j\}_{j=1}^d \subset \Gamma^\infty(TM)$ with compact support such that*

$$\delta_p = g_0 + \sum_{j=1}^{d} X_j g_j \text{ in } \mathcal{E}'(M). \tag{3.4}$$

Proof. Pick a chart $\{U, x\}$ near $p \in M$ such that $x(p) = 0$. Since $x(U) = \mathbb{R}^d$, one can apply Lemma 3.1 on $x(U) \simeq \mathbb{R}^d$ and get:

$$\delta_0 = \tilde{g}_0 - \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \tilde{g}_j$$

where δ_0 is the delta mass on $x(U)$ supported at the origin. So for any $h \in C^\infty(U)$

$$\begin{aligned} h(p) &= h \circ x^{-1}(0) \\ &= \int_{\mathbb{R}^d} \left(\tilde{g}_0 - \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \tilde{g}_j \right) h \circ x^{-1} d\lambda \\ &= \int_{\mathbb{R}^d} \left(\tilde{g}_0 + \sum_{j=1}^{d} \tilde{g}_j \frac{\partial}{\partial x_j} \right) h \circ x^{-1} d\lambda \end{aligned}$$

where $d\lambda$ is the Lebesque measure on $\mathbb{R}^d$. Consider $\left\{ \frac{\tilde{g}_j}{\sqrt{\det g}} \circ x \right\}_{j=0}^{d}$ where $g = (g_{ij})_{1 \leq i,j \leq d}$ is the metric matrix, i.e. $g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \right\rangle_g$. From Lemma 3.1 we know that $\frac{\tilde{g}_j}{\sqrt{\det g}} \circ x$ has compact support in U and therefore $K := \cup_{j=1}^{d} supp \left(\frac{\tilde{g}_j}{\sqrt{\det g}} \circ x \right)$ is compact in U. Using Lemma 3.2 we can construct a smooth function $\phi \in C^\infty(M \to [0,1])$ such that $\phi^{-1}(\{0\}) = M/U$ and $\phi^{-1}(\{1\}) = K$. Define

$$\hat{g}_0 = \phi \frac{\tilde{g}_0}{\sqrt{\det g}} \circ x$$

and

$$\hat{g}_j = \phi \frac{\tilde{g}_j}{\sqrt{\det g}} \circ x, \;\; X_j = \phi \cdot \left(x^{-1} \right)_* \frac{\partial}{\partial x_j} \text{ for } j = 1, \ldots, d$$

Then for any $f \in C^{\infty}(M)$,

$$\begin{aligned}
&\int_M \left(\hat{g}_0 + \sum_{j=1}^{d} \hat{g}_j X_j\right) f dvol \\
&\qquad = \int_U \left(\hat{g}_0 + \sum_{j=1}^{d} \hat{g}_j X_j\right) f dvol \\
&\qquad = \int_U \frac{\tilde{g}_0}{\sqrt{\det g}} \circ x \cdot \phi f dvol \\
&\qquad + \sum_{j=1}^{d} \int_U \phi^2 \frac{\tilde{g}_j}{\sqrt{\det g}} \circ x \left(\left(x^{-1}\right)_* \frac{\partial \phi f}{\partial x_j} - \left(x^{-1}\right)_* \frac{\partial \phi}{\partial x_j} f\right) dvol
\end{aligned}$$

Here $dvol$ is the volume measure on M.

Since $\phi \cdot (x^{-1})_* \frac{\partial \phi}{\partial x_j} \equiv 0$ and $\phi \equiv 1$ on K, we have:

$$\begin{aligned}
\int_M \left(\hat{g}_0 + \sum_{j=1}^{d} \hat{g}_j X_j\right) f dvol &= \int_U \left(\frac{\tilde{g}_0}{\sqrt{\det g}} \circ x + \sum_{j=1}^{d} \frac{\tilde{g}_j}{\sqrt{\det g}} \circ x \left(x^{-1}\right)_* \frac{\partial}{\partial x_j}\right) f dvol \\
&= \int_{\mathbb{R}^d} \left(\frac{\tilde{g}_0}{\sqrt{\det g}} + \sum_{j=1}^{d} \frac{\tilde{g}_j}{\sqrt{\det g}} \frac{\partial}{\partial x_j}\right) f \circ x^{-1} \sqrt{\det g} d\lambda \\
&= \int_{\mathbb{R}^d} \left(\tilde{g}_0 + \sum_{j=1}^{d} \tilde{g}_j \frac{\partial}{\partial x_j}\right) f \circ x^{-1} d\lambda \\
&= f \circ x^{-1}(0) \\
&= f(p)
\end{aligned}$$

Therefore, by the Divergence Theorem, formally (in distributional sense) we can write down δ_p as

$$\delta_p = g_0 + \sum_{j=1}^{d} X_j g_j$$

where

$$g_0 = \hat{g}_0 - \sum_{j=1}^{d} \hat{g}_j \cdot div X_j$$

and for $j = 1, \ldots, n$,

$$g_j = -\hat{g}_j.$$

From the construction one can see that $X_j \in \Gamma^\infty(TM)$ and $\{g_j\}_{j=0}^d \subset C^\infty(M/\{p\}) \cap L^{\frac{d}{d-1}}(M)$ with compact support. ■

Lemma 3.4 *$C_0^\infty(M)$ is dense in $L^p(M)$ for any $1 \leq p < \infty$.*

Proof. Recall that simple functions on M are finite linear combinations of indicator functions 1_E where $vol(E) < \infty$. Since simple functions are dense in $L^p(M)$. It suffices to show that $C_0^\infty(M)$ is dense in the space of simple functions with respect to L^p-norm. Given a simple function 1_E,

$$\int_M 1_E dvol = vol(E)$$

Since the volume measure is regular, there exists a compact set K and open set U such that

$$K \subset E \subset U$$

and

$$vol(K) \geq vol(U) - \epsilon.$$

Now apply Lemma 3.2 we can find a cutoff function $f \in C_0^\infty(M)$ such that $f^{-1}(\{0\}) = M/U$ and $f^{-1}(\{1\}) = K$. It follows that

$$\|f - 1_E\|_{L^p(M)}^p = \int_M |f - 1_E|^p \, dvol \leq vol(U - K) \leq \epsilon,$$

which proves the denseness of $C_0^\infty(M)$ in the space of simple functions and thus in $L^p(M)$.

■

Remark 3.5 *Using Lemma 3.4 and Theorem 3.3, for any $g_j, j = 1, \cdots, d$, we can* *find a sequence* $\left\{g_j^{(m)}\right\}_m \subset C_0^\infty(M)$ *such that*

$$g_j^{(m)} \to g_j \text{ in } L^{\frac{d}{d-1}}(M)$$

In particular, since g_j *has compact support, we can make* $\cup_m suppg_j^{(m)}$ *to be compact.*

Corollary 3.6 *Define*

$$\delta_x^{(m)} := g_0^{(m)} + \sum_{j=1}^{d} X_j g_j^{(m)} \in C_0^\infty(M).$$

Then $\left\{\delta_x^{(m)}\right\}_m$ *is an approximating sequence of delta mass* δ_x*, i.e.*

$$\delta_x^{(m)} \to \delta_x \text{ in } \mathcal{D}'(M).$$

Proof. Using integration by parts, we have for any $f \in C(M)$,

$$\int_M f\delta_x^{(m)} d\lambda = \int_M \left(g_0^{(m)} + \sum_{j=1}^{d} X_j g_j^{(m)}\right) f d\lambda \tag{3.5}$$

$$= \int_M \left(g_0^{(m)} f + \sum_{j=1}^{d} g_j^{(m)} X_j^* f\right) d\lambda \tag{3.6}$$

Since $K := \cup_m suppg_j^{(m)}$ is compact, $f \cdot 1_K$ and $X_j^* f \cdot 1_K \in L^{\infty-}(M)$, then 3.6 easily follows by Holder's inequality. ■

3.2 Definition of $\nu_{\mathcal{P},x}^1$

In this section we will give the explicit definition of $\nu_{\mathcal{P},x}^1$ proposed in Theorem 1.10. Recall from Definition 3.14 that

$$H_{\mathcal{P},x}(M) := \{\sigma \in H_{\mathcal{P}}(M) \mid \sigma(1) = x\}.$$

This set can be viewed as the pre-image of x under the **end point evaluation map** E_1. In general, it is not guaranteed that $H_{\mathcal{P},x}(M)$ is an embedded submanifold of $H_{\mathcal{P}}(M)$, or equivalently, E_1 is an submersion. The following is an easy, yet illuminating, example showing what can go wrong:

Example 3.7 *If* $M = \mathbb{S}^2$ *and* $\mathcal{P} := \{0,1\}$ *with starting point being the North pole, then* $\dim H_{\mathcal{P}}(M) = 2$*. Consider*

$$X(\sigma, s) := (0, \pi \sin s\pi, 0) \in T_\sigma H_{\mathcal{P}}(M)$$

where

$$\sigma(s) = (\sin s\pi, 0, \cos s\pi).$$

An one parameter family realizing $X(\sigma, s)$ *would be*

$$\sigma_t(s) = (\sin s\pi \cos t\pi, \sin s\pi \sin t\pi, \cos s\pi)$$

From which one can easily see that:

$$E_{1*\sigma}(X) = \frac{d}{dt}|_0 E_1(\sigma_t) = \frac{d}{dt}|_0 \sigma_t(1) = X(\sigma, 1) = 0.$$

So by Rank-Nullity theorem, $E_{1*\sigma}$ *is not surjective.*

The problem comes from the conjugate points on M. Two points p and q are conjugate points along a geodesic σ if there exists non-zero Jacobi field (smooth vector field along σ satisfying Jacobi equation) that vanishes at p and q. This fact will allow the kernel of E_{1*} to be "overly large "(more accurately dimension exceeds $(n-1)d$), so by Rank-nullity theorem, E_{1*} can not be surjective. In this thesis we consider manifolds with non–positive sectional curvature. These manifolds do not have conjugate points. From the next proposition we will see that E_1 is a submersion on these manifolds.

Proposition 3.8 *If M is complete with non-positive sectional curvature, then for any $x \in M$, $H_{\mathcal{P},x}(M) := E_1{}^{-1}(\{x\})$ is an embedded submanifold of $H_{\mathcal{P}}(M)$.*

Proof. It suffices to show $E_1 : H_{\mathcal{P}}(M) \to M$ is a submersion. Since M is complete, for any $y \in M$, there exists a geodesic σ parametrized on $[0,1]$ and connecting o and y. So E_1 is surjective. To show E_{1*} is surjective, we use a class of vector fields $\left\{X^{h_{\alpha,n}}\right\}_{\alpha=1}^{d}$ in Definition 3.20. Notice that

$$E_{1*}\left(X^{h_{\alpha,n}}\right) = X_1^{h_{\alpha,n}} = \sqrt{n}u(1)\,S_{\mathcal{P},n}e_\alpha$$

where $u(\cdot) = u(\sigma,\cdot)$ is the horizontal lift of $\sigma \in H_{\mathcal{P}}(M)$. From Proposition B.1 we know $S_{\mathcal{P},n}$ is invertible, therefore $\left\{E_{1*}\left(X^{h_{\alpha,n}}\right)\right\}_{\alpha=1}^{d}$ spans $T_{E_1(\sigma)}M$. So E_{1*} is surjective. ■ Since $H_{\mathcal{P},x}(M)$ is an embedded submanifold of $H_{\mathcal{P}}(M)$, we can restrict the Riemannian metric $G_{\mathcal{P}}^1$ on $TH_{\mathcal{P}}(M)$ in Eq. (1.7) to a Riemannian metric on $TH_{\mathcal{P},x}(M)$.

Definition 3.9 *Assuming M has non-positive sectional curvature, for any $x \in M$, let $G_{\mathcal{P},x}^1$ be the restriction of $G_{\mathcal{P}}^1$ to $T_\sigma H_{\mathcal{P},x}(M) \subset T_\sigma H_{\mathcal{P}}(M)$. Further, let $\mathrm{vol}_{G_{\mathcal{P},x}^1}$ be the associated volume measure on $H_{\mathcal{P},x}(M)$.*

Based on the "Lebesgue measure" $vol_{G_{\mathcal{P},x}^1}$ on $H_{\mathcal{P},x}(M)$, we can construct the pinned approximate measure $\nu_{\mathcal{P},x}^1$:

Definition 3.10 *Let $\nu_{\mathcal{P},x}^1$ be the measure on $H_{\mathcal{P},x}(M)$ defined by*

$$d\nu_{\mathcal{P},x}^1(\sigma) = \frac{1}{J_{\mathcal{P}}(\sigma)}\frac{1}{Z_{\mathcal{P}}^1}e^{\frac{-E(\sigma)}{2}}d\,\mathrm{vol}_{G_{\mathcal{P},x}^1}(\sigma) \tag{3.7}$$

where $J_{\mathcal{P}}(\sigma) := \sqrt{\det\left(E_{1*\sigma}E_{1*\sigma}^{tr}\right)}$ depends on $\mathcal{P}$ since the domain of E_1 is $H_{\mathcal{P}}M$.

3.3 Continuous Dependence on the Parameter $x \in M$

Recall that a Hadamard manifold is a simply connected complete manifold with non–positive sectional curvature. Throughout this section we assume M is a Hadamard manifold whose sectional curvature is bounded below by $-N$. The following theorem illustrates that measures $\nu^1_{\mathcal{P},x}$ are finite and "continuously varying"with respect to x.

Notation 3.11 *We will denote by $C_b(X)$ bounded continuous functions on a topological space X.*

Theorem 3.12 *For any $f \in C_b\left(H_{\mathcal{P},x}\left(M\right)\right), x \in M$, define:*

$$h_{\mathcal{P}}\left(x\right) := \int_{H_{\mathcal{P},x}(M)} f\left(\sigma\right) d\nu^1_{\mathcal{P},x}\left(\sigma\right).$$

Then $h_{\mathcal{P}}\left(x\right) \in C\left(M\right)$.

Remark 3.13 *Set $f \equiv 1$, one can see that $\nu^1_{\mathcal{P},x}\left(H_{\mathcal{P},x}\left(M\right)\right) < \infty$. So Theorem 3.12 implies that $\nu^1_{\mathcal{P},x}$ is a finite measure and thus any bounded measurable function on $H_{\mathcal{P},x}\left(M\right)$ is integrable with respect to $\nu^1_{\mathcal{P},x}$.*

Before proving this theorem, we need to set up some notations and auxiliary results.

Notation 3.14 *We fix $n \in \mathbb{N}$ and let $s_i := \frac{i}{n}$ with $\tau := 1 - \frac{1}{n} = s_{n-1}$. We further define $\mathcal{K} := H_{\mathcal{P}}\left(\left[0,\tau\right], M\right)$ be the space of piecewise geodesic paths, $\sigma : \left[0,\tau\right] \to M$ such that $\sigma\left(0\right) = o \in M$.*

Lemma 3.15 *For $x, y \in M$, we can choose an unique element $\log_x\left(y\right) \in T_xM$ so that*

$$\gamma_{y,x}\left(t\right) := \exp_x\left(\left(t-\tau\right)\frac{1}{n}\log_x\left(y\right)\right),$$

is the unique minimal-lengh-geodesic connecting x *to* y *such that* $\gamma_{y,x}(\tau) = x$ *and* $\gamma_{y,x}(1) = y$.

Proof. Since M is a Hadamard manifold, by the Theorem of Hadamard (See Theorem A.2 in Appendix A), $\exp_x : T_xM \to M$ is a diffeomorphism. Therefore we can see that $\log_x(y) = exp_x^{-1}(y)$ is unique and it follows that the geodesic $\gamma_{y,x}$ is unique. ■

Definition 3.16 *For any given* $y \in M$, *let* $\psi_y : \mathcal{K} \to H_{\mathcal{P},y}(M) := E_1^{-1}(\{y\})$ *defined as in Proposition 3.8 be defined by*

$$\psi_y(\sigma) := \gamma_{y,\sigma(\tau)} * \sigma$$

where

$$\left(\gamma_{y,\sigma(\tau)} * \sigma\right)(t) = \begin{cases} \sigma(t) & \text{if } 0 \le t \le \tau \\ \gamma_{y,\sigma(\tau)}(t) & \text{if } \tau \le t \le 1 \end{cases}.$$

Notation 3.17 *For any* $\sigma \in H_{\mathcal{P},y}(M)$, *denote* $u(\sigma,\tau)^{-1}\log_{\sigma(\tau)}(y)$ *by* $\xi_{y,\sigma}$, *then* $\xi_{y,\sigma} \in T_oM$. *Denote by* $V(\sigma,s) := (C_y(\sigma,s), S_y(\sigma,s))^t \in \mathbb{R}^{2d\times d}$ *the fundamental solution to the ODE:*

$$V'(\sigma,s) = \begin{pmatrix} 0 & I_{d\times d} \\ A_{\xi_y}(\sigma,s) & 0 \end{pmatrix} V(\sigma,s)$$

where $A_{\xi_y}(\sigma,s) = R_{u(\sigma,1-s)}(\xi_{y,\sigma},\cdot)\,\xi_{y,\sigma}$.

The next lemma characterizes the differential of ψ_y:

Lemma 3.18 *Let* $\sigma \in \mathcal{K}$, *recall from Theorem 2.41 that* $X^h(\sigma,\cdot) = u(\sigma,\cdot)\,h(\sigma,\cdot) \in T_\sigma\mathcal{K}$ *iff* $h(\sigma,\cdot)$ *satisfies the piecewise Jacobi equation as in 2.42. Then*

$$\psi_{y*}\left(X^h(\sigma,\cdot)\right) = X^{\hat{h}}(\psi_y(\sigma),\cdot) := u(\psi_y(\sigma),\cdot)\,\hat{h}(\psi_y(\sigma),\cdot)$$

where

$$\hat{h}(\psi_y(\sigma), s) = \begin{cases} h(\psi_y(\sigma), s) & s \in [0,\tau] \\ S_y(\psi_y(\sigma), 1-s)\, S_y\left(\psi_y(\sigma), \frac{1}{n}\right)^{-1} h(\sigma,\tau) & s \in [\tau, 1] \end{cases}. \tag{3.8}$$

Proof. From now on we will suppress the path argument $\psi_y(\sigma)$ in $\hat{h}$. Suppose that $t \to \sigma_t \in \mathcal{K}$ is an one-parameter family of curves in $\mathcal{K}$ such that $\sigma_0 = \sigma$ and $\frac{d}{dt}|_0 \sigma_t = X^h(\sigma)$. Then we have

$$\psi_{y*}\left(X^h(\sigma)\right) = \frac{d}{dt}|_0 \psi_y(\sigma_t) = \frac{d}{dt}|_0 \gamma_{y,\sigma_t(\tau)} * \sigma_t.$$

If $s \in [0,\tau]$, then

$$\frac{d}{dt}|_0 \left(\gamma_{y,\sigma_t(\tau)} * \sigma_t\right)(s) = \frac{d}{dt}|_0 \sigma_t(s) = X_s^h(\sigma).$$

While if $s \in [\tau, 1]$ we have

$$\frac{d}{dt}|_0 \left(\gamma_{y,\sigma_t(\tau)} * \sigma_t\right)(s) = \frac{d}{dt}|_0 \gamma_{y,\sigma_t(\tau)}(t) =: X_s^{\hat{h}}(\psi_y(\sigma))$$

We know that $X_s^{\hat{h}}$ is determined by,

1. $\hat{h}$ satisfies Jacobi's equation,
2. $\hat{h}(\tau) = h(\tau)$ and $\hat{h}(1) = 0$.

Denote $\hat{h}(s)$ by $g(1-s)$ for $s \in [\tau, 1]$, the above conditions are equivalent to g being the solution to the following boundary value problem:

$$\begin{cases} g''(s) = A_{\xi_y}(s)\, g(s) \\ g(0) = 0 \\ g\left(\frac{1}{n}\right) = h(\tau) \end{cases}.$$

Then we use $S_y(\cdot)$ to express the solution. Here we use a result that for each $s \in \left[0, \frac{1}{n}\right]$, $S_y(s)$ is invertible. It can be derived from Proposition B.1 applied to $A_{\xi_y}(s)$.

$$g(s) = S_y(s)\, S_y\left(\frac{1}{n}\right)^{-1} h(\tau) \text{ for } s \in [0, \tau]$$

and thus

$$\hat{h}(s) = g(1-s) = S_y(1-s)\, S_y\left(\frac{1}{n}\right)^{-1} h(\tau) \text{ for } s \in [\tau, 1].$$

■

Corollary 3.19 *For any $y \in M$, ψ_y is a diffeomorphism.*

Proof. From Lemma 3.18 it is easy to see that the push forward $(\psi_y)_*$ of ψ_y is one to one and thus an isomorphism since $\dim(\mathcal{K}) = \dim(H_{\mathcal{P},y}(M))$. Therefore the inverse function theorem implies that ψ_y is a local diffeomorphism. Furthermore, M being a Hadamard manifold implies that ψ_y is bijective, so ψ_y is actually a diffeomorphism. ■

Notation 3.20 *We construct an orthonormal basis $\left\{X^{h_{\alpha,i}} : 1 \le \alpha \le d, 1 \le i \le n\right\}$ of $H_{\mathcal{P}}(M)$ as follows:*

$$h_{\alpha,i} \in H_{\mathcal{P},\sigma} \text{ and } h'_{\alpha,i}(s_j+) = \frac{\delta_{i-1,j} e_\alpha}{\sqrt{\Delta_{j+1}}} \text{ for } j = 0, ..., n-1 \tag{3.9}$$

where the definition of $H_{\mathcal{P},\sigma}$ can be found in Definition 2.11. An orthonormal basis $\left\{X^{h_{\alpha,i}} : 1 \le \alpha \le d, 1 \le i \le n-1\right\}$ of $H_{\mathcal{P}}(M)$ of $\mathcal{K}$ can be constructed similarly:

$$h_{\alpha,i} \in H_{\mathcal{P},\sigma} \text{ and } h'_{\alpha,i}(s_j+) = \frac{\delta_{i-1,j} e_\alpha}{\sqrt{\Delta_{j+1}}} \text{ for } j = 0, ..., n-2.$$

In this chapter we will use the same notation for both these two sets of orthonormal basis and it shouldn't cause confusion from the context.

Remark 3.21 *It is not hard to see using Proposition 5.1 that*

$$h_{\alpha,i}(s) = \frac{1}{\sqrt{n}} f_{\mathcal{P},i}(s)\, e_\alpha \tag{3.10}$$

where $\{f_{\mathcal{P},i}(s)\}$ *is given in Definition 2.43.*

Definition 3.22 $f : M \to N$ *is a differentiable map between two Riemannian manifolds* M, N*. The* ***Normal Jacobian*** *of* f *is defined to be* $\sqrt{\det (f_* f_*^{tr})}$.

We will use the orthonormal basis $\left\{X^{h_{\alpha,i}} : 1 \le \alpha \le d, 1 \le i \le n-1\right\}$ of $\mathcal{K}$ to estimate the Normal Jacobian $J_{\mathcal{P}}$ in Lemma 3.23 and the "volume change "V_x (See precise definition in Lemma 3.25) brought by the diffeomorphism ψ_x in Lemma 3.25 and 3.26.

Lemma 3.23 *For any* $\sigma \in H_{\mathcal{P}}(M)$*, define* $J_{\mathcal{P}}(\sigma) := \sqrt{\det E_{1*\sigma} (E_{1*\sigma})^{tr}}$*. Then*

$$J_{\mathcal{P}}(\sigma) = \sqrt{\det \left(\frac{1}{n} \sum_{i=1}^{n} f_{\mathcal{P},i}(\sigma, 1)\, f_{\mathcal{P},i}^{tr}(\sigma, 1) \right)}$$

Proof. Recall that E_1 is the end point evalutation map. Notice that

$$E_{1*\sigma} X^h(\sigma) = X^h(\sigma, 1)\,. \tag{3.11}$$

So for any $v \in T_{E_1(\sigma)}M$, here we suppress σ:

$$\left\langle (E_{1*})^{tr} v, X^h \right\rangle_{G^1_{\mathcal{P}}} = \left\langle v, E_{1*} X^h \right\rangle_{T_{E_1(\sigma)}M} = \left\langle u(1)^{-1} v, h(1) \right\rangle_{\mathbb{R}^d},$$

from which we get an expansion of $(E_{1*})^{tr} v$ using the orthonormal basis

$$\left\{X^{h_{\alpha,i}} : 1 \le \alpha \le d, 1 \le i \le n\right\} \text{ of } TH_{\mathcal{P}}(M)$$

$$(E_{1*})^{tr} v = \sum_{i,\alpha} \left\langle (E_{1*})^{tr} v, X^{h_{\alpha,i}} \right\rangle_{G^1_{\mathcal{P}}} X^{h_{\alpha,i}} = \sum_{i,\alpha} \left\langle u(1)^{-1} v, h_{\alpha,i}(1) \right\rangle_{\mathbb{R}^d} X^{h_{\alpha,i}}.$$

So choosing an orthonormal basis $\{u(1)\,e_\alpha\}_{\alpha=1}^d$ of $T_{E_1(\sigma)}M$ and notice that

$$h_{\gamma,i}(1) = \frac{1}{\sqrt{n}} f_{\mathcal{P},i}(1)\,e_\gamma,$$

we can compute:

$$\begin{aligned}
\det\left(E_{1*}(E_{1*})^{tr}\right) &= \det\left\{\left\langle (E_{1*})^{tr} u(1)\,e_\alpha, (E_{1*})^{tr} u(1)\,e_\beta\right\rangle_{T_{E_1(\sigma)}M}\right\}_{\alpha,\beta} \\
&= \det\left\{\sum_{i=1}^n \sum_{\gamma=1}^d \langle h_{\gamma,i}(1), e_\alpha\rangle \langle h_{\gamma,i}(1), e_\beta\rangle\right\}_{\alpha,\beta} \\
&= \det\left\{\sum_{i=1}^n \sum_{\gamma=1}^d \frac{1}{n}\left\langle e_\gamma, f_{\mathcal{P},i}^{tr}(1)\,e_\alpha\right\rangle \left\langle e_\gamma, f_{\mathcal{P},i}^{tr}(1)\,e_\beta\right\rangle\right\}_{\alpha,\beta} \\
&= \det\left\{\sum_{i=1}^n \frac{1}{n}\left\langle f_{\mathcal{P},i}^{tr}(1)\,e_\alpha, f_{\mathcal{P},i}^{tr}(1)\,e_\beta\right\rangle\right\}_{\alpha,\beta} \\
&= \det\left(\frac{1}{n}\sum_{i=1}^n f_{\mathcal{P},i}(1)\, f_{\mathcal{P},i}^{tr}(1)\right).
\end{aligned}$$

■

Using the expression of $J_{\mathcal{P}}$ in Lemma 3.23, we can easily derive the following estimate.

Corollary 3.24

$$J_{\mathcal{P}}(\sigma) \geq 1.$$

Proof. For any $v \in \mathbb{C}^d$, using Proposition B.1, we have:

$$\begin{aligned}
\left\langle \frac{1}{n}\sum_{i=1}^n f_{\mathcal{P},i}(\sigma,1)\, f_{\mathcal{P},i}^{tr}(\sigma,1)\,v, v\right\rangle &= \frac{1}{n}\sum_{i=1}^n \left\| f_{\mathcal{P},i}^{tr}(\sigma,1)\,v\right\|^2 \\
&\geq \frac{1}{n}\sum_{i=1}^n \|v\|^2 \\
&= \|v\|^2.
\end{aligned}$$

So by Min-max theorem, $eig\left(\frac{1}{n}\sum_{i=1}^{n} f_{\mathcal{P},i}(\sigma,1)\, f_{\mathcal{P},i}^{tr}(\sigma,1)\right) \subset [1,+\infty)$ and therefore:

$$J_{\mathcal{P}}(\sigma) = \sqrt{\det\left(\frac{1}{n}\sum_{i=1}^{n} f_{\mathcal{P},i}(\sigma,1)\, f_{\mathcal{P},i}^{tr}(\sigma,1)\right)} \geq 1.$$

■

Lemma 3.25 *For any $\sigma \in \mathcal{K}$, define*

$$V_x(\sigma) := \sqrt{\det\left((\psi_{x*\sigma})^{tr}\, \psi_{x*\sigma}\right)} \tag{3.12}$$

Then

$$V_x(\sigma) = \sqrt{\det\left(I + L_x(\sigma)\, F_{\mathcal{P}}(\sigma)\, L_x(\sigma)^{tr}\right)} \tag{3.13}$$

where

$$L_x(\sigma) := C_x\left(\sigma, \frac{1}{n}\right) S_x\left(\sigma, \frac{1}{n}\right)^{-1}$$

and

$$F_{\mathcal{P}}(\sigma) := \frac{1}{n^2}\sum_{i=0}^{n-2} f_{\mathcal{P},i}(\sigma,\tau)\, f_{\mathcal{P},i}(\sigma,\tau)^{tr}.$$

Proof. Using 3.8 and differentiating $\hat{h}$ with respect to s, we get:

$$\hat{h}'(\sigma,\tau+) = -C_x\left(\sigma, \frac{1}{n}\right) S_x\left(\sigma, \frac{1}{n}\right)^{-1} h(\sigma,\tau) := -L_x(\sigma)\, h(\sigma,\tau) \tag{3.14}$$

Also notice that from Proposition 5.1,

$$h(\sigma,\tau) = \frac{1}{n}\sum_{i=0}^{n-1} f_{\mathcal{P},i+1}(\sigma,\tau)\, h'(\sigma, s_i+),$$

so we have

$$\hat{h}'(\sigma,\tau+) = -L_x(\sigma)\frac{1}{n}\sum_{i=0}^{n-1} f_{\mathcal{P},i+1}(\sigma,\tau)\, h'(\sigma, s_i+). \tag{3.15}$$

For any $\alpha, \beta \in \{1, ..., d\}$ and $i, j \in \{1, ..., n-1\}$:

$$\left\langle \psi_{x*}\left(X^{h_{\alpha,i}}(\sigma)\right), \psi_{x*}\left(X^{h_{\beta,j}}(\sigma)\right)\right\rangle_{T_{\psi_x(\sigma)}H_{\mathcal{P},x}(M)} \tag{3.16}$$

$$= \frac{1}{n}\sum_{k=0}^{n-2}\left\langle h'_{\alpha,i}(s_{k+}), h'_{\beta,j}(s_{k+})\right\rangle + \frac{1}{n}\left\langle \hat{h}'_{\alpha,i}(\tau+), \hat{h}'_{\beta,j}(\tau+)\right\rangle \tag{3.17}$$

$$= \delta_{(\alpha,i)}^{(\beta,j)} + \frac{1}{n}\left\langle L_x(\sigma)\frac{1}{n}\frac{f_{\mathcal{P},i}(\tau)e_\alpha}{\sqrt{\frac{1}{n}}}, L_x(\sigma)\frac{1}{n}\frac{f_{\mathcal{P},j}(\tau)e_\beta}{\sqrt{\frac{1}{n}}}\right\rangle \tag{3.18}$$

$$= \delta_{(\alpha,i)}^{(\beta,j)} + \left\langle L_x(\sigma)\frac{1}{n}f_{\mathcal{P},i}(\tau)e_\alpha, L_x(\sigma)\frac{1}{n}f_{\mathcal{P},j}(\tau)e_\beta\right\rangle, \tag{3.19}$$

where $\delta_{(\alpha,i)}^{(\beta,j)} = \begin{cases} 1 & \alpha = \beta, i = j \\ 0 & \text{otherwise} \end{cases}.$

It follows that the volume change

$$V_x(\sigma) = \sqrt{\det\left(I_{\left(\mathbb{R}^d\right)^{n-1}} + \hat{T}_x(\sigma)\right)} \tag{3.20}$$

where

$$\hat{T}_x(\sigma) \in End\left(\left(\mathbb{R}^d\right)^{n-1}\right)$$

with

$$\left(\hat{T}_x(\sigma)\right)_{d(i-1)+\alpha, d(j-1)+\beta} = \left\langle L_x(\sigma)\frac{1}{n}f_{\mathcal{P},i}(\sigma,\tau)e_\alpha, L_x(\sigma)\frac{1}{n}f_{\mathcal{P},j}(\sigma,\tau)e_\beta\right\rangle.$$

Notice that

$$I_{\left(\mathbb{R}^d\right)^{n-1}} + \hat{T}_x(\sigma) = S_\sigma^{tr}S_\sigma$$

where

$$S_\sigma = \begin{pmatrix} I_{\left(\mathbb{R}^d\right)^{n-1}} \\ A_x(\sigma) \end{pmatrix} \in M_{nd\times(n-1)d}$$

and

$$A_x(\sigma) = \left(\frac{1}{n}L_x(\sigma) f_{\mathcal{P},0}(\sigma,\tau) e_1, \cdots, \frac{1}{n}L_x(\sigma) f_{\mathcal{P},n-2}(\sigma,\tau) e_d\right) \in M_{d\times(n-1)d}$$

Apply Lemma D.1 we get:

$$\begin{aligned}
\det\left(I_{\left(\mathbb{R}^d\right)^{n-1}} + \hat{T}_x(\sigma)\right) &= \det\left(I_{\left(\mathbb{R}^d\right)} + A_x(\sigma) A_x(\sigma)^{tr}\right) \\
&= \det\left(I + \frac{1}{n^2}\sum_{i=0}^{n-2}\sum_{\alpha=1}^{d} L_x(\sigma) f_{\mathcal{P},i}(\tau) e_\alpha e_\alpha^{tr} f_{\mathcal{P},i}(\tau)^{tr} L_x(\sigma)^{tr}\right) \\
&= \det\left(I + L_x(\sigma) F_{\mathcal{P}}(\sigma) L_x(\sigma)^{tr}\right)
\end{aligned}$$

where

$$F_{\mathcal{P}}(\sigma) := \frac{1}{n^2}\sum_{i=0}^{n-2} f_{\mathcal{P},i}(\sigma,\tau) f_{\mathcal{P},i}(\sigma,\tau)^{tr}.$$

■

Lemma 3.26 *For any $\sigma \in \mathcal{K}$,*

$$V_x(\sigma) \le \sum_{k=0}^{d}\binom{d}{k} n^{\frac{k}{2}} e^{\frac{Nk}{2}d^2(\sigma(\tau),x)} \Pi_{j=0}^{n-2} e^{kNd^2(\sigma(s_j),\sigma(s_{j+1}))} \tag{3.21}$$

Proof. From Lemma 3.25 and D, one can see, after suppressing σ,

$$\begin{aligned}
\det\left(I_{\left(\mathbb{R}^d\right)^{n-1}} + \hat{T}_x\right) &= \det\left(I + L_x F_{\mathcal{P}} L_x^{tr}\right) \\
&= \Pi_{i=1}^{d}\left(1 + \lambda_{i,x}\right) \\
&\le \left(1 + \max_{1\le i\le d}\lambda_{i,x}\right)^d
\end{aligned}$$

where $\{\lambda_{i,x}\} = eig\left(L_x F_{\mathcal{P}} L_x^{tr}\right)$.

Notice that

$$\max_{1\le i\le d}\lambda_{i,x}=\left\|L_x(\sigma)\,F_{\mathcal{P}}L_x(\sigma)^{tr}\right\|\le\|L_x(\sigma)\|^2\,\|F_{\mathcal{P}}\|$$
$$\le\frac{1}{n}\|L_x(\sigma)\|^2\sup_{0\le i\le n-2}\|f_{\mathcal{P},i}(\tau)\|^2$$

Apply Proposition B.3, we get:

$$\left\|C_x\left(\sigma,\frac{1}{n}\right)\right\|\le e^{\frac{N}{2}d^2(\sigma(\tau),x)}$$

where for any $x,y\in M$, $d(x,y)$ is the geodesic distance between x and y. and

$$\left\|S_x^{-1}\left(\sigma,\frac{1}{n}\right)\right\|\le n,$$

so

$$\|L_x(\sigma)\|^2\le n^2e^{Nd^2(\sigma(\tau),x)}$$

and

$$\max_{1\le i\le d}\lambda_{i,x}\le ne^{Nd^2(\sigma(\tau),x)}\sup_{0\le i\le n-2}\|f_{\mathcal{P},i}(\sigma,\tau)\|^2.$$

Therefore

$$V_x(\sigma)=\left(1+\max_{1\le i\le d}\lambda_{i,x}\right)^{\frac{d}{2}}\le\left(1+ne^{Nd^2(\sigma(\tau),x)}\sup_{0\le i\le n-2}\|f_{\mathcal{P},i}(\sigma,\tau)\|^2\right)^{\frac{d}{2}}$$
$$\le\left(1+n^{\frac{1}{2}}e^{\frac{N}{2}d^2(\sigma(\tau),x)}\sup_{0\le i\le n-2}\|f_{\mathcal{P},i}(\sigma,\tau)\|\right)^{d}$$
$$=\sum_{k=0}^{d}\binom{d}{k}n^{\frac{k}{2}}e^{\frac{Nk}{2}d^2(\sigma(\tau),x)}\sup_{0\le i\le n-2}\|f_{\mathcal{P},i}(\sigma,\tau)\|^k.\quad(3.22)$$

Apply Proposition B.3 again to $f_{\mathcal{P},i}(\sigma,\tau)$, we get:

$$\begin{aligned}
\|f_{\mathcal{P},i}(\tau)\| &\leq \|C_{\mathcal{P},n-1}\| \cdots \|C_{\mathcal{P},i+1}\| \left\| \frac{S_i}{\Delta_i} \right\| \\
&\leq e^{\frac{1}{2}Nd^2(\sigma(s_{n-2}),\sigma(s_{n-1}))} \cdot \cdots \cdot e^{\frac{1}{2}Nd^2(\sigma(s_{i-1}),\sigma(s_i))} \left(1 + \frac{Nd^2(\sigma(s_{i-1}),\sigma(s_i))}{6}\right) \\
&\leq \Pi_{j=i-1}^{n-2} e^{\frac{1}{2}Nd^2(\sigma(s_j),\sigma(s_{j+1}))} \cdot e^{\frac{Nd^2(\sigma(s_{i-1}),\sigma(s_i))}{6}} \\
&\leq \Pi_{j=i-1}^{n-2} e^{Nd^2(\sigma(s_j),\sigma(s_{j+1}))} \\
&\leq \Pi_{j=0}^{n-2} e^{Nd^2(\sigma(s_j),\sigma(s_{j+1}))}
\end{aligned}$$

Taking supremum over i, we get:

$$\sup_{0 \leq i \leq n-2} \|f_{\mathcal{P},i}(\sigma,\tau)\| \leq \Pi_{j=0}^{n-2} e^{Nd^2(\sigma(s_j),\sigma(s_{j+1}))}. \tag{3.23}$$

and 3.21 follows. ■

Definition 3.27 *For any X, $Y \in T\mathcal{K}$(the tangent bundle of $\mathcal{K}$), define $G^0_{\mathcal{P},\tau}$, $G^1_{\mathcal{P},\tau}$ to be:*

$$G^0_{\mathcal{P},\tau}(X,Y) = \sum_{i=1}^{n-1} \langle X(s_i), Y(s_i) \rangle \Delta_i$$

and

$$G^1_{\mathcal{P},\tau}(X,Y) = \sum_{i=1}^{n-1} \left\langle \frac{\nabla X}{ds}(s_{i-1}), \frac{\nabla Y}{ds}(s_{i-1}) \right\rangle \Delta_i$$

Lemma 3.28 *$G^0_{\mathcal{P},\tau}$ is a metric on $\mathcal{K}$.*

Proof. The only non–trivial part is to check $G^1_{\mathcal{P},\tau}(X,X) = 0 \implies X = 0$. Since M has non–positive curvature, there are no conjugate points. For each $0 \leq i \leq n-1$, there is a unique Jacobi field X connecting $\sigma(s_i)$ and $\sigma(s_{i+1})$ with specified $X(s_i)$ and $\frac{\nabla Y}{ds}(s_i)$. $G^1_{\mathcal{P},\tau}(X,X) = 0 \implies \frac{\nabla Y}{ds}(s_i) = 0$ for any $1 \leq i \leq n$. Notice that $X(0) = 0$, so by the uniqueness of Jacobi field, $X \equiv 0$. ■

Remark 3.29 *Since M has non-positive curvatures, $G^0_{\mathcal{P},\tau}$ is indeed a metric on $\mathcal{K}$ since the only one-paramater family of geodesics with fixed end points is a constant* *family consisting of the unique geodesic connecting the starting point and the ending point.*

Definition 3.30 *Based on the metric $G^0_{\mathcal{P},\tau}$ and $G^1_{\mathcal{P},\tau}$, we define measures $\nu^0_{\mathcal{P},\tau}$ and $\nu^1_{\mathcal{P},\tau}$ on $\mathcal{K}$ as follows:*

$$\nu^0_{\mathcal{P},\tau} := \frac{n^{(n-1)d}}{(2\pi)^{(n-1)\frac{d}{2}}} e^{-\frac{1}{2}E} dvol_{G^0_{\mathcal{P},\tau}}$$

and

$$\nu^1_{\mathcal{P},\tau} = \frac{1}{(2\pi)^{(n-1)\frac{d}{2}}} e^{-\frac{1}{2}E} dvol_{G^1_{\mathcal{P},\tau}}$$

Lemma 3.31 *Let*

$$\rho_{\mathcal{P}}(\sigma) = \Pi_{i=1}^{n-1} \det\left(\frac{S_{\mathcal{P},i}(\sigma)}{n}\right)$$

then $\nu^0_{\mathcal{P},\tau} = \rho_{\mathcal{P}} \nu^1_{\mathcal{P},\tau}$. What's more:

$$\rho_{\mathcal{P}}(\sigma) \geq 1.$$

Proof. The argument to show $\rho_{\mathcal{P}}$ is the density of $\nu^0_{\mathcal{P},\tau}$ with respect to $\nu^1_{\mathcal{P},\tau}$ is almost exactly the same as Theorem 5.9 in [3] with a slight change of ending point from 1 to τ. Here we focus on the lower bound estimate of $\rho_{\mathcal{P}}(\sigma)$. Since for any $v \in \mathbb{C}^d$,

$$\left\| \frac{S_{\mathcal{P},i}}{n} v \right\| \geq \|v\|,$$

we know from propsition B.1 that for any $\lambda \in eig\left(\frac{S_{\mathcal{P},i}}{n}\right)$,

$$|\lambda| \geq 1$$

And from which we know:

$$\rho_{\mathcal{P}}(\sigma) = \Pi_{i=1}^{n-1} \det\left(\frac{S_{\mathcal{P},i}(\sigma)}{n}\right) \geq 1.$$

■

Proof of Theorem 3.12. Since ψ_x is a diffeomorphism, apply Theorem C.1 and we have:

$$h_{\mathcal{P}}(x) = \int_{H_{\mathcal{P},x}(M)} \frac{1}{Z^1_{\mathcal{P}}} \frac{f}{J_{\mathcal{P}}}(\sigma) e^{-\frac{1}{2}E(\sigma)} dvol_{G^1_{\mathcal{P},x}}(\sigma) \tag{3.24}$$

$$= \int_{\mathcal{K}} \frac{1}{Z^1_{\mathcal{P}}} \frac{f}{J_{\mathcal{P}}} \circ \psi_x(\sigma) e^{-\frac{1}{2}E\circ\psi_x(\sigma)} V_x(\sigma) dvol_{G^1_{\mathcal{P},\tau}}(\sigma) \tag{3.25}$$

Notice that

$$\frac{1}{Z^1_{\mathcal{P}}} e^{-\frac{1}{2}E\circ\psi_x(\sigma)} = \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{(2\pi)^{(n-1)\frac{d}{2}}} e^{-\frac{1}{2}E(\sigma)} e^{-\frac{n}{2}d^2(\sigma(\tau),x)}, \tag{3.26}$$

So

$$h_{\mathcal{P}}(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathcal{K}} \frac{f}{J_{\mathcal{P}}} \circ \psi_x(\sigma) e^{-\frac{n}{2}d^2(\sigma(\tau),x)} V_x(\sigma) d\nu_{G^1_{\mathcal{P},\tau}}(\sigma) \tag{3.27}$$

Combine (3.22), (3.23) we know that:

$$e^{-\frac{n}{2}d^2(\sigma(\tau),x)} V_x(\sigma) \le \sum_{k=0}^{d} \binom{d}{k} n^{\frac{k}{2}} e^{\frac{Nk-n}{2}d^2(\sigma(\tau),x)} \Pi_{j=0}^{n-2} e^{Nd^2(\sigma(s_j),\sigma(s_{j+1}))} \tag{3.28}$$

So

$$\sup_{x\in M} e^{-\frac{n}{2}d^2(\sigma(\tau),x)} V_x(\sigma) \le \sup_{x\in M} e^{-\frac{n-Nk}{2}d^2(\sigma(\tau),x)} \sum_{k=0}^{d} \binom{d}{k} n^{\frac{k}{2}} \Pi_{j=0}^{n-2} e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))} \tag{3.29}$$

When n is large enough, $n - Nk > 0$. Therefore $e^{-\frac{n-Nk}{2}d^2(\sigma(\tau),x)} \le 1$ and it suffices to show

$$\mathbb{E}_{\nu_{G^1_{\mathcal{P},\tau}}} \left[\sum_{k=0}^{d} \binom{d}{k} n^{\frac{k}{2}} \Pi_{j=0}^{n-2} e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))} \right] < \infty. \tag{3.30}$$

For each $k \leq d$ we have:

$$\mathbb{E}_{\nu_{G^1_{\mathcal{P},\tau}}}\left[\binom{d}{k} n^{\frac{k}{2}} \Pi_{j=0}^{n-2} e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))}\right] = C_n \mathbb{E}_\mu\left[\Pi_{j=0}^{n-2} e^{Nk|\Delta_{j+1}\beta|^2}\right] \tag{3.31}$$

$$= C_n \Pi_{j=0}^{n-2} e^{\frac{Nk}{n}} \tag{3.32}$$

$$= C_n \tag{3.33}$$

where C_n is a generic constant.

Since for any $\sigma \in \mathcal{K}$, $\frac{f}{J_{\mathcal{P}}} \circ \psi_x(\sigma) e^{-\frac{n}{2}d^2(\sigma(\tau),x)} V_x(\sigma)$ is continuous with respect to $x \in M$, so by dominated convergence theorem, $h_{\mathcal{P}}(x) \in C(M)$. ■

Not only can we show that $h_{\mathcal{P}}(x)$ is a continuous function, it is bounded uniformly in $x \in M$ and partition $\mathcal{P}$, as is shown in the following proposition.

Proposition 3.32 $\sup_{\mathcal{P}} h_{\mathcal{P}}(x) < \infty$.

Proof. Based on Equation 3.27,

$$h_{\mathcal{P}}(x) \leq C_d \int_{\mathcal{K}} e^{-\frac{n}{2}d^2(\sigma(\tau),x)} V_x(\sigma)\, d\nu_{G^1_{\mathcal{P},\tau}}(\sigma) \tag{3.34}$$

Combine 3.22, 3.23 we know that:

$$e^{-\frac{n}{2}d^2(\sigma(\tau),x)} V_x(\sigma) \leq \sum_{k=0}^{d} \binom{d}{k} n^{\frac{k}{2}} e^{\frac{Nk-n}{2}d^2(\sigma(\tau),x)} \Pi_{j=0}^{n-2} e^{Nd^2(\sigma(s_j),\sigma(s_{j+1}))} \tag{3.35}$$

For each $k \leq d$, apply Lemma 3.31, we have:

$$\mathbb{E}_{\nu_{G^1_{\mathcal{P},\tau}}}\left[e^{-\frac{n-Nk}{2}d^2(\sigma(\tau),x)} \binom{d}{k} n^{\frac{k}{2}} \Pi_{j=0}^{n-2} e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))}\right] \tag{3.36}$$

$$= \binom{d}{k} n^{\frac{k}{2}} \int_{\mathcal{K}} e^{-\frac{n-Nk}{2}d^2(\sigma(\tau),x)} \Pi_{j=0}^{n-2} e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))} d\nu_{G^1_{\mathcal{P},\tau}}(\sigma) \tag{3.37}$$

$$= \binom{d}{k} n^{\frac{k}{2}} \int_{\mathcal{K}} e^{-\frac{n-Nk}{2}d^2(\sigma(\tau),x)} \Pi_{j=0}^{n-2} e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))} \frac{1}{\rho_{\mathcal{P}}(\sigma)} d\nu^0_{\mathcal{P},\tau}(\sigma) \tag{3.38}$$

$$\leq \binom{d}{k} n^{\frac{k}{2}} \int_{\mathcal{K}} e^{-\frac{n-Nk}{2}d^2(\sigma(\tau),x)} \Pi_{j=0}^{n-2} e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))} d\nu^0_{\mathcal{P},\tau}(\sigma) \tag{3.39}$$

Now define the projection map $\pi_{\mathcal{P}}:\mathcal{K}\to M^{n-1}$, for any $\sigma\in\mathcal{K}$,

$$\pi_{\mathcal{P}}(\sigma):=(\sigma(s_1),\ldots,\sigma(s_{n-1})).$$

Since M is a Hadamard manifold, $\pi_{\mathcal{P}}$ is a diffeomorphism. From there one can get:

$$\binom{d}{k}n^{\frac{k}{2}}\int_{\mathcal{K}}e^{-\frac{n-Nk}{2}d^2(\sigma(\tau),x)}\Pi_{j=0}^{n-2}e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))}d\nu^0_{\mathcal{P},\tau}(\sigma) \tag{3.40}$$

$$=\frac{\binom{d}{k}n^{\frac{k+(n-1)d}{2}}}{(2\pi)^{\frac{(n-1)d}{2}}}\int_{M^{n-1}}e^{-\frac{n-Nk}{2}d^2(x_{n-1},x)}\Pi_{j=0}^{n-2}e^{-\frac{1}{2}(n-2Nk)d^2(x_j,x_{j+1})}dx_1\cdots dx_{n-1} \tag{3.41}$$

Corollary 4.2 in [35] gives a lower bound of heat kernels of manifold M such that $Ric\geq(1-d)N$:

$$p_t(x,y)\geq(2\pi t)^{-\frac{d}{2}}e^{-\frac{\rho^2}{2t}}\left(\frac{\sinh\sqrt{N}\rho}{\sqrt{N}\rho}\right)^{\frac{1-d}{2}}e^{-Ct}$$

where N is the curvature bound and C is some constant depending only on d and N and $\rho=d(x,y)$. Using the fact that:

$$\frac{\sinh\sqrt{N}\rho}{\sqrt{N}\rho}\leq e^{\frac{N\rho^2}{2}}$$

It follows that

$$p_t(x,y)\geq(2\pi t)^{-\frac{d}{2}}e^{-\frac{1}{2}\left(\frac{1}{t}+\frac{N(d-1)}{2}\right)\rho^2}e^{-Ct}$$

let

$$t=\frac{1}{n-N_1}$$

where $N_1=2Nd+\frac{N(d-1)}{2}$.

We have, for any $j \in \{0, \ldots, n-1\}$:

$$e^{-\frac{1}{2}(n-2Nd)d^2(x_j,x_{j+1})} \le e^{Ct} p_t\left(x_j, x_{j+1}\right)(2\pi t)^{\frac{d}{2}}.$$

So

$$\begin{aligned}
&\frac{\binom{d}{k} n^{\frac{k+(n-1)d}{2}}}{(2\pi)^{\frac{(n-1)d}{2}}} \int_{M^{n-1}} \sup_{x\in M} e^{-\frac{n-2Nk}{2} d^2(x_{n-1},x)} \Pi_{j=0}^{n-2} e^{-\frac{1}{2}(n-2Nd)d^2(x_j,x_{j+1})} dx_1 \cdots dx_{n-1} \\
&\le \frac{\binom{d}{k} n^{\frac{k+(n-1)d}{2}}}{(n-N_1)^{\frac{nd}{2}}} e^{C\frac{n}{n-N_1}} \int_{M^{n-1}} p_{\frac{1}{n-N_1}}\left(x_{n-1}, x\right) \Pi_{j=0}^{n-2} p_{\frac{1}{n-N_1}}\left(x_j, x_{j+1}\right) dx_1 \cdots dx_{n-1} \\
&= \frac{\binom{d}{k} e^{\frac{Cn}{n-N_1}}}{n^{\frac{d-k}{2}}\left(1-\frac{N_1}{n}\right)^{\frac{nd}{2}}} \int_M p_{\frac{1}{n-N_1}}\left(x_{n-1}, x\right) p_{\frac{n-1}{n-N_1}}\left(0, x_{n-1}\right) dx_{n-1} \qquad (3.42)
\end{aligned}$$

Since the heat kernel is continuous w.r.t. to time, combine (3.39) ,(3.41) and (3.42), we get

$$\frac{\binom{d}{k} e^{\frac{Cn}{n-N_1}}}{n^{\frac{d-k}{2}}\left(1-\frac{N_1}{n}\right)^{\frac{nd}{2}}} p_{\frac{n}{n-N_1}}(0, x) \le C.$$

and hence

$$h_{\mathcal{P}}(x) \le C.$$

where C is a generic constants depending only on d and N. ■

Theorem 3.12 shows that the class of approximate pinned measures $\left\{\nu^1_{\mathcal{P},x}\right\}$ are finite measures and using the continuity result for $h_{\mathcal{P}}(x)$, one can see that $\nu^1_{\mathcal{P},x}$ is deserved to be formally expressed as $\delta_x(\sigma(1))\nu^1_{\mathcal{P}}$ and it should be interpreted in the following sense:

Corollary 3.33 *Denote by $\delta_x \in \mathcal{E}'(M)$ the delta mass at $x \in M$, for any $\left\{\delta_x^{(m)}\right\} \subset C_0^\infty(M)$ such that*

$$\delta_x^{(m)} \to \delta_x \text{ in } \mathcal{E}'(M)$$

i.e. for any $h \in C^{\infty}(M)$, *we have:*

$$\lim_{m\to\infty}\int_M h(y)\,\delta_x^{(m)}(y)\,dy = \int_M h(y)\,\delta_x(y)\,dy =: h(x)$$

where dy *is the volume measure on* M. *Then for any* $f \in C_b^{\infty}(H_{\mathcal{P}}(M))$,

$$\lim_{m\to\infty}\int_{H_{\mathcal{P}}(M)} \delta_x^{(m)}(\sigma(1))\,f(\sigma)\,d\nu_{\mathcal{P}}^1(\sigma) = \int_{H_{\mathcal{P},x}(M)} f(\sigma)\,d\nu_{\mathcal{P},x}^1(\sigma).$$

Proof. Apply the co-area formula in Theorem 2.3 in [11], we have:

$$\begin{aligned}\int_{H_{\mathcal{P}}(M)} \delta_x^{(m)}(\sigma(1))\,f(\sigma)\,d\nu_{\mathcal{P}}^1(\sigma) &= \int_M \delta_x^{(m)}(y)\,dy\int_{H_{\mathcal{P},y}(M)} f(\sigma)\,d\nu_{\mathcal{P},y}^1(\sigma)\\ &= \int_M h_{\mathcal{P}}(y)\,\delta_x^{(m)}(y)\,dy\end{aligned}$$

From Theorem 3.12 we know $h_{\mathcal{P}}(x) \in C(M)$, therefore:

$$\begin{aligned}\lim_{m\to\infty}\int_{H_{\mathcal{P}}(M)} \delta_x^{(m)}(\sigma(1))\,f(\sigma)\,d\nu_{\mathcal{P}}^1(\sigma) &= \lim_{m\to\infty}\int_M h_{\mathcal{P}}(y)\,\delta_x^{(m)}(y)\,dy\\ &= h_{\mathcal{P}}(x)\\ &= \int_{H_{\mathcal{P},x}(M)} f(\sigma)\,d\nu_{\mathcal{P},x}^1(\sigma).\end{aligned}$$

■

Chapter 4

The Orthogonal Lift $\tilde{X}$ of X on $H(M)$ and Its Stochastic Extension

4.1 Damped Metrics and Adjoints

Definition 4.1 (α–inner product) *Let $\alpha(t) \in \operatorname{End}\left(\mathbb{R}^d\right)$ be a continuously varying matrix valued function. For $h, k \in H_0\left(\mathbb{R}^d\right)$ let*

$$\langle h, k\rangle_\alpha := \int_0^1 \left(\frac{d}{dt} h(t) + \alpha(t) h(t)\right) \cdot \left(\frac{d}{dt} k(t) + \alpha(t) k(t)\right) dt.$$

Remark 4.2 *We denote the norm induced by α–inner product by $\|\cdot\|_\alpha$, differenting from the notation $\|\cdot\|_{H_0\left(\mathbb{R}^d\right)}$ for the norm induced by the H^1– inner product: $\langle h, l\rangle_{H^1} = \int_0^1 h'(s) \cdot l'(s)\, ds$.*

For the moment, let $E_1 : H_0\left(\mathbb{R}^d\right) \to \mathbb{R}^d$ be the end point evaluation map in the case where $M = \mathbb{R}^d$. Let ${E_1}^* : \mathbb{R}^d \to H_0\left(\mathbb{R}^d\right)$ be the adjoint of E_1 with respect to the α–inner product, i.e. for any $a \in \mathbb{R}^d$ and $h \in H_0\left(\mathbb{R}^d\right)$,

$$\langle E_1 h, a \rangle_{\mathbb{R}^d} = \langle h, (E_1{}^*)\, a \rangle_\alpha .$$

The next theorem computes E_1^* which is crucial in constructing the orthogonal lift in Section 4.2.

Theorem 4.3 *Let $a \in \mathbb{R}^d$ and $\alpha(t)$ is positive semi–definite for any $0 \le t \le 1$, then $E_1^* a \in H_0\left(\mathbb{R}^d\right)$ is given by*

$$(E_1^* a)(t) = \left(S(t) \int_0^t [S(s)^* S(s)]^{-1} S(1)^* \, ds \right) a. \tag{4.1}$$

where $S(t) \in \mathrm{Aut}\left(\mathbb{R}^d\right)$ solves

$$\frac{d}{dt} S(t) + \alpha(t) S(t) = 0 \text{ with } S(0) = I$$

and

$$v(t) = \left(\int_0^t [S(s)^* S(s)]^{-1} S(1)^* \, ds \right) a.$$

Proof. Notice that if $h(t) = S(t) w(t)$ with $w(\cdot) \in H_0\left(\mathbb{R}^d\right)$, then

$$\begin{aligned} \left(\frac{d}{dt} + \alpha(t)\right) h(t) &= \left(\frac{d}{dt} + \alpha(t)\right) [S(t) w(t)] \\ &= \left[\left(\frac{d}{dt} + \alpha(t)\right) S(t)\right] w(t) + S(t) \dot{w}(t) \\ &= S(t) \dot{w}(t) . \end{aligned}$$

And in particular,

$$\langle Sv, Sw \rangle_\alpha = \int_0^1 S(t) \dot{v}(t) \cdot S(t) \dot{w}(t) \, dt.$$

Notice that $\frac{d}{dt}\langle S(t)a,a\rangle = -\langle \alpha(t)a,a\rangle \geq 0$, so

$$\langle S(t)a,a\rangle \geq \langle S(0)a,a\rangle = \|a\|^2$$

This implies $S(t) \in Aut(\mathbb{R}^d)$. Given $a \in \mathbb{R}^d$, let $w(t) = E_1^* a$ and define $v(t) := S(t)^{-1} w(t)$ so that $E_1^* a = S(t)v(t)$. Then by the definition of the adjoint we find,

$$\begin{aligned}\int_0^1 S(t)\dot{v}(t)\cdot S(t)\dot{w}(t)\,dt &= \langle Sv, Sw\rangle_\alpha = \langle E_1^* a, Sw\rangle_\alpha = a\cdot E_1(Sw) \\ &= a\cdot S(1)w(1) = \int_0^1 S(1)^* a\cdot \dot{w}(t)\,dt\end{aligned}$$

As $w \in H_0(\mathbb{R}^d)$ is arbitrary we may conclude that

$$S(t)^* S(t)\dot{v}(t) = S(1)^* a \implies v(t) = \int_0^t [S(s)^* S(s)]^{-1} S(1)^* a ds$$

which proves (4.1). ■

Theorem 4.4 *If $a \in \mathbb{R}^d$, then $h(\cdot) \in H_0(\mathbb{R}^d)$ defined by*

$$h(t) := S(t)\left(\int_0^t [S(s)^* S(s)]^{-1} ds\right)\left(\int_0^1 [S(s)^* S(s)]^{-1} ds\right)^{-1} S(1)^{-1} a, \quad (4.2)$$

is the minimal length element of $H_0(\mathbb{R}^d)$ such that $E_1 h = a$.

i.e.

$$\|h\|_\alpha = \inf\left\{\|k\|_\alpha \mid k(\cdot) \in H_0(\mathbb{R}^d),\ E_1 k = a\right\}.$$

Proof. Since $H_0(\mathbb{R}^d) = \mathrm{Nul}(E_1)^\perp \oplus \mathrm{Nul}(E_1)$, we have $E_1 h = a \implies E_1 h_k = a$ and $\|h\|_\alpha \geq \|h_k\|_\alpha$ where h_k is the orthogonal projection of h onto $\mathrm{Nul}(E_1)^\perp$. So we are looking for the element, $h \in H_0(\mathbb{R}^d)$, such that $E_1 h = a$ and $h \in \mathrm{Nul}(E_1)^\perp = \mathrm{Ran}(E_1{}^*)$. In other words we should have $h = E_1^* v$ for some $v \in \mathbb{R}^d$. Thus, using

(4.1), we need to demand that

$$a = E_1 E_1{}^* v = (E_1{}^* v)(1) = \left(S(1) \int_0^1 [S(s)^* S(s)]^{-1} S(1)^* \, ds \right) v,$$

i.e.

$$v = \left(S(1) \int_0^1 [S(s)^* S(s)]^{-1} S(1)^* \, ds \right)^{-1} a.$$

It then follows that

$$\begin{aligned} h(t) &= E_1{}^* \left(S(1) \int_0^1 [S(s)^* S(s)]^{-1} S(1)^* \, ds \right)^{-1} a \\ &= \left(S(t) \int_0^t [S(s)^* S(s)]^{-1} S(1)^* \, ds \right) \left(S(1) \int_0^1 [S(s)^* S(s)]^{-1} S(1)^* \, ds \right)^{-1} a \end{aligned}$$

which is equivalent to (4.2).

Alternative proof: Let $h := E_1{}^* a \in H_0(\mathbb{R}^d)$ and $k \in H_0(\mathbb{R}^d)$, then

$$\begin{aligned} a \cdot k(1) = a \cdot E_1(k) = \langle E_1{}^* a, k \rangle_\alpha &= \langle h, k \rangle_\alpha \\ &= \int_0^1 \left(\frac{d}{dt} h(t) + \alpha(t) h(t) \right) \cdot z(t) \, dt \end{aligned} \tag{4.3}$$

where

$$\frac{d}{dt} k(t) + \alpha(t) k(t) =: z(t).$$

Solving the previous equation for k in terms of z gives,

$$k(t) = S(t) \int_0^t S(s)^{-1} z(s) \, ds.$$

Using this result with $t = 1$ back in (4.3) shows

$$\int_0^1 \left(\frac{d}{dt}h(t) + \alpha(t)h(t)\right) \cdot z(t)\,dt = a \cdot S(1)\int_0^1 S(s)^{-1} z(s)\,ds$$
$$= \int_0^1 S^*(s)^{-1} S(1)^* a \cdot z(s)\,ds.$$

As $z(s)$ is arbitrary in $L^2\left([0,1], \mathbb{R}^d\right)$ we may conclude that

$$\frac{d}{dt}h(t) + \alpha(t)h(t) = S^*(t)^{-1} S(1)^* a.$$

Solving this equation for h then shows,

$$\left(E_1{}^* a\right)(t) = h(t) = S(t)\int_0^t S(s)^{-1} S^*(s)^{-1} S(1)^* a ds$$
$$= \left(S(t)\left[\int_0^t S(s)^{-1} S^*(s)^{-1} ds\right] S(1)^*\right) a$$

and so we again recover (4.1). ∎

Remark 4.5 *The expression in (4.2) matches the well known result for damped metrics where* $\alpha = \frac{1}{2}\operatorname{Ric}_u$. *Further observe that if* $\alpha(t) = 0$ *(i.e. we are in the flat case) then* $S(t) = I$ *and the above expression reduces to* $h(t) = ta$ *as we know to be the correct result.*

Definition 4.6 *Let* $\langle \cdot, \cdot \rangle_{Ric_u}$ *be the* ***damped metric on*** $TH(M)$ *defined by*

$$\langle X, Y\rangle_{Ric_u} := \int_0^1 \left\langle \left[\frac{\nabla}{ds} + \frac{1}{2}Ric\right] X(s), \left[\frac{\nabla}{ds} + \frac{1}{2}Ric\right] Y(s)\right\rangle ds \qquad (4.4)$$

for all $X, Y \in \Gamma_\sigma(TM) = T_\sigma H(M)$ *and* $\sigma \in H(M)$.

If $X = X^{J_1}$ and $Y = X^{J_2}$ with that $J_1, J_2 \in H_0\left(\mathbb{R}^d\right)$, then we have

$$\left\langle X^{J_1}, X^{J_2}\right\rangle_{Ric_u} = \int_0^1 \left\langle \left[\frac{d}{ds} + \frac{1}{2}Ric_{u_s}\right] J_1(s), \left[\frac{d}{ds} + \frac{1}{2}Ric_{u_s}\right] J_2(s)\right\rangle ds. \qquad (4.5)$$

4.2 The Orthogonal Lift $\tilde{X}$ on $H(M)$

In this section we construct the orthogonal lift $\tilde{X} \in \Gamma(TH(M))$ of $X \in \Gamma(TM)$ which is defined to be the minimal length element in $\Gamma(TH(M))$ relative the damped metric introduced in Definition 4.6.

Definition 4.7 *For each $\sigma \in H(M)$, recall that $u_s(\sigma)$ is the horizontal lift of σ. Denote by T_s the solution to the following initial value problem:*

$$\begin{cases} \frac{d}{ds}T_s + \frac{1}{2}Ric_{u_s}T_s = 0 \\ T_0 = I \end{cases} \tag{4.6}$$

Lemma 4.8 *For all $s \in [0,1]$, T_s is invertible. Further both $\sup_{0\leq s\leq 1} \|T_s\|$ and $\sup_{0\leq s\leq 1} \|T_s^{-1}\|$ are bounded by $e^{\frac{1}{2}(d-1)N}$, where $(d-1)N$ is a bound of $\|\mathrm{Ric}\|$.*

Proof. Let U_s solve the ODE,

$$\begin{cases} \frac{d}{ds}U_s = \frac{1}{2}U_s \operatorname{Ric}_{u_s} \\ U_0 = I. \end{cases} \tag{4.7}$$

Then one easily shows that

$$\frac{d}{ds}[U_sT_s] = 0 \implies U_sT_s = U_0T_0 = I$$

and this shows that U_s is a left inverse to T_s. As we are in finite dimensions it follows that T_s^{-1} exists and is equal to U_s. The stated bounds now follow by Gronwall's inequality. ■

Definition 4.9 *Let*

$$\mathbf{K}_s := T_s \left[\int_0^s T_r^{-1}\left(T_r^{-1}\right)^* dr\right] T_1^*. \tag{4.8}$$

Remark 4.10 *A simple computation shows that* $\mathbf{K}_s$ *satisfies the following initial value problem:*

$$\begin{cases} \mathbf{K}_s' = & -\frac{1}{2}\operatorname{Ric}_{u_s}\mathbf{K}_s + \left(T_1 T_s^{-1}\right)^* \\ \mathbf{K}_0 = & 0. \end{cases} \tag{4.9}$$

Conversely, from Duhamel's principle and (4.6) it is easy to deduce the formula in Definition 4.9.

Lemma 4.11 *With* $\mathbf{K}_s$ *as in Definition 4.9,* $\mathbf{K}_1$ *is invertible and* $\left\|\mathbf{K}_1^{-1}\right\| \leq e^{(d-1)N}$*, provided* $\|\operatorname{Ric}\| \leq (d-1)\,N$*.*

Proof. Since

$$\mathbf{K}_1 := \int_0^1 \left(T_1 T_r^{-1}\right)\left(T_1 T_r^{-1}\right)^* dr$$

is a symmetric positive semi-definite operator such that

$$\langle \mathbf{K}_1 v, v\rangle = \int_0^1 \left\|\left(T_1 T_r^{-1}\right)^* v\right\|^2 dr \;\forall v \in \mathbb{C}^d.$$

Apply Lemma 4.8 to the expression given;

$$\begin{aligned} \langle \mathbf{K}_1 v, v\rangle &\geq \int_0^1 e^{-(d-1)N}\left\|\left(T_r^{-1}\right)^* v\right\|^2 dr \\ &\geq \int_0^1 e^{-2(d-1)N}\|v\|^2 dr \\ &= e^{-2(d-1)N}\|v\|^2 \end{aligned}$$

From which it follows that $eig\left(\mathbf{K}_1\right) \subset [e^{-(d-1)N}, \infty)$. ■

Definition 4.12 *Let* $X \in \Gamma(TM)$*, define two maps* $H : H(M) \to \mathbb{R}^d$ *and* $J : [0,1] \times H(M) \to \mathbb{R}^d$ *as follows,*

$$\tilde{H} = u_1^{-1}(\sigma)\, X \circ E_1(\sigma) \tag{4.10}$$

and

$$J(\sigma, s) := J_s(\sigma) := \mathbf{K}_s(\sigma)\,\mathbf{K}_1^{-1}(\sigma)\,H(\sigma)\,. \tag{4.11}$$

Theorem 4.13 *Given $X \in \Gamma(TM)$ the minimal length lift, $\tilde{X}$, relative to the damped metric in Definition 4.6 of X to $\Gamma(TH(M))$ is given by $\tilde{X} = X^J$. Further we know that J_s is the solution to the following ODE:*

$$J_s' = -\frac{1}{2}Ric_{u_s}J_s + \phi_s,\;\; J_0 = 0$$

where $\phi_s = (T_1T_s^{-1})^*\mathbf{K}_1^{-1}H = (T_s^{-1})^*\left[\int_0^1 T_r^{-1}(T_r^{-1})^*\,dr\right]^{-1}T_1^{-1}H.$

Proof. Apply Theorem 4.4 with $\alpha_s = \frac{1}{2}Ric_{u_s}$. ■

Following the construction above, one can define an similar object (still denoted by $\tilde{X}$) on $W_o(M)$. Recall from Notation 2.31 that $\tilde{u}$ is the stochastic horizontal lift of the canonical Brownian motion Σ on M.

Definition 4.14 *Define $\tilde{T}_s$ to be the solution to the following (random) initial value problem:*

$$\begin{cases} \frac{d}{ds}\tilde{T}_s + \frac{1}{2}Ric_{\tilde{u}_s}\tilde{T}_s = 0 \\ \tilde{T}_0 = I \end{cases} \tag{4.12}$$

Definition 4.15 *Define*

$$\tilde{\mathbf{K}}_s := \tilde{T}_s\left[\int_0^s \tilde{T}_r^{-1}\left(\tilde{T}_r^{-1}\right)dr\right]\tilde{T}_1^*. \tag{4.13}$$

Remark 4.16 *Following the same arguments used in Lemma 4.8 and 4.11, one can see the bounds obtained there still hold for $\tilde{T}$ and $\tilde{\mathbf{K}}$.*

Definition 4.17 *For any $X \in \Gamma(TM)$ define $\tilde{H}$ and $\tilde{J}_{(.)}$ on $W_o(M)$,*

$$\tilde{H} = \tilde{u}_1^{-1}X \circ E_1 \tag{4.14}$$

and

$$\tilde{J}_s := \tilde{\mathbf{K}}_s \tilde{\mathbf{K}}_1^{-1} \tilde{H}. \tag{4.15}$$

Notation 4.18 *Given a measurable function $h : W_o(M) \to H_0(\mathbb{R}^d)$, let $Z_h : W_o(M) \to H_0(\mathbb{R}^d)$ be the solution to the following ODE:*

$$\begin{cases} Z_h{}'(s) = -\frac{1}{2} Ric_{\tilde{u}_s} Z_h(s) + h'_s \\ Z_h(0) = 0. \end{cases}$$

Definition 4.19 *For any $X \in \Gamma(TM)$, define*

$$\tilde{X}_s = X_s^{Z_\Phi} := \tilde{u}_s Z_\Phi(s) \;\; \text{for } 0 \leq s \leq 1$$

where

$$\Phi_s = \int_0^s \left(\tilde{T}_\tau^{-1}\right)^* \left[\int_0^1 \left(\tilde{T}_r^* \tilde{T}_r\right)^{-1} dr\right]^{-1} \tilde{T}_1^{-1} \tilde{H} d\tau.$$

4.3 Review of Calculus on Wiener Space

In this section we interpret X^{Z_Φ} as a first order differential operator on some geometric Wiener functionals (see Definition 4.36). The main difficulty there is the non-adaptedness of Φ. To overcome this difficulty, we express X^{Z_Φ} in terms of geometric vector field (see Definition 4.27) with non–adapted coefficients. However, these coefficients are differentiable Wiener functionals in "Malliavin calculus "sense. Based on this observation we derive an integration–by–parts formula for X^{Z_Φ} which naturally shows X^{Z_Φ} is a closable first order differential operator on $L^2(W_o(M))$. The integration–by–parts formula will also be one of our main tool of dealing with δ—function pinning in this thesis. We begin with a brief review of the classical theory of calculus on Wiener space that is needed in our work.

The first order differential geometry on path spaces that we will use can be traced back to the famous Cameron-Martin Theorem (see [7]).

Theorem 4.20 (Cameron-Martin) *For any $h \in H_0\left(\mathbb{R}^d\right)$, consider the flow ϕ_t^h generated by h, i.e. for any $w \in W_0\left(\mathbb{R}^d\right)$, $\phi_t^h(w) = w + th$. Notice that ϕ_t^h is the flow of the vector field $D_h := \frac{\partial}{\partial h}$. Then the pull–back measure $\mu^h(\cdot) := \left(\phi_1^h\right)_* \mu(\cdot) = \mu(\cdot - h)$ and Wiener measure μ are mutually absolutely continuous.*

The map ϕ_t^h is usually called Cameron-Martin shift and the phenomenon described in Theorem 4.20 is called quasi-invariance of μ under the Cameron-Martin shift. The generalization of Cameron-Martin Theorem to path spaces on a manifold came quite a while later in 1990s. Driver initiated the geometric Cameron-Martin theory in [12] and [13] where he considered the "vector field "X^h(or more precisely an equivalence class of vector fields) on $W_o(M)$ defined as follows,

$$X_s^h(\sigma) = \tilde{u}_s(\sigma)\, h_s$$

where $h \in \left\{f \in C^1([0,1]) : f(0) = 0\right\} \subset H_0\left(\mathbb{R}^d\right)$.

Theorem 4.21 *Let (M, g, o, ∇) be a compact manifold and h be as above, then for any $\sigma \in W_o(M)$, there exists a unique flow ϕ_t^h of X^h, i.e. $\phi_t^h : W_o(M) \mapsto W_o(M)$ satisfying:*

$$\frac{d}{dt}\phi_t^h(\sigma) = X^h\left(\phi_t^h(\sigma)\right) \text{ with } \phi_0^h = I$$

and $\nu_t^h(\cdot) := \left(\phi_t^h\right)_* \nu$ is equivalent to ν.

The existence of the flow and the quasi-invariance of the Wiener measure were later extended to all Cameron-Martin vector field X^h, $h \in H_0\left(\mathbb{R}^d\right)$ in [20] and [16] and then to a geometrically and stochastically complete Riemannian manifold in [21] and [23]. Owing to the facts that Cameron-Martin vector fields do not form a Lie Algrbra and more general vector fields naturally appreared in practice, it is useful to introduce a broader class of so called "adapted vector fields ", see [14] and [8].

Definition 4.22 (Vector valued Brownian semimartingales) *V is a finite dimensional vector space. A function $f : W_o(M) \times [0,1] \to V$ is called a Brownian semimartingale if f has the following representation:*

$$f(s) = \int_0^s Q_\tau d\beta_\tau + \int_0^s r_\tau d\tau$$

where (Q_s, r_s) is a predictable process with values in $Hom\left(\mathbb{R}^d, V\right) \times V$, V is a vector space. We will call (Q_s, r_s) the kernels of f.

Definition 4.23 ($\mathcal{H}^q$ space) *For each $q \geq 1$, $f : W_o(M) \times [0,1] \to V$ jointly measurable, we define the root mean square norm in $L^q(W_o(M), \nu)$ to be:*

$$\|f\|_{R^q(V)} \equiv \left\| \left(\int_0^1 |f(\cdot, s)|_V^2 \, ds \right)^{\frac{1}{2}} \right\|_{L^q(W_o(M),\nu)}$$

Let $\mathcal{H}^q$ be the space of all Brownian semimartingales such that

$$\|f\|_{\mathcal{H}^q} := \left\|Q^f\right\|_{R^q} + \left\|r^f\right\|_{R^q} < \infty$$

Definition 4.24 ($\mathcal{B}^q$ space) *For each $q \geq 1$, $f : W_o(M) \times [0,1] \to V$ jointly measurable, we define the supremum norm in $L^q(W_o(M), \nu)$ to be:*

$$\|f\|_{S^q(V)} \equiv \|f^*\|_{L^q(W_o(M),\nu)}$$

where f^* is the essential supremum of $s \to f(\cdot, s)$ relative to Lebesque measure on $[0,1]$. *Let $\mathcal{B}^q$ be the space of all Brownian semimartingales such that*

$$\|f\|_{\mathcal{B}^q} := \left\|Q^f\right\|_{S^q} + \left\|r^f\right\|_{S^q} < \infty$$

Definition 4.25 (Adapted vector field) *An adapted vector field on $W_0\left(\mathbb{R}^d\right)$ is an $\mathbb{R}^d$–valued Brownian semimartingale with predictable kernels $Q_\cdot \in \mathfrak{so}(d)$*

and $r_{\cdot} \in L^2[0,1]$ $\nu - a.s.$ *We denote the space of adapted vector fields by* $\mathcal{V}$ *and* $\mathcal{V}^q := \mathcal{V} \cap \mathcal{H}^q$.

Notation 4.26 *We will use the following notations in this dissertation:*$\mathcal{H}^{\infty-} := \cap_{q\geq 1}\mathcal{H}^q$, $\mathcal{B}^{\infty-} = \cap_{q\geq 1}\mathcal{B}^q$ *and* $\mathcal{V}^{\infty-} = \mathcal{V} \cap \mathcal{H}^{\infty-}$.

A class of vector field called geometric vector field can be constructed using adapted vector fields.

Definition 4.27 (Geometric vector field) *For any* $h \in \mathcal{V}$,

$$X_s^h := \tilde{u}_s h_s \ 0 \leq s \leq 1$$

is said to be a ***geometric vector field****.*

Theorem 4.28 (Approximate Flow of Geometric Vector Field) *Let* X^h *be a geometric vector field as above with* $h \in \mathcal{V} \cap \mathcal{S}^\infty \cap \mathcal{B}^\infty$, $t \in \mathbb{R}$, *there exists a funcion* $E\left(tX^h\right) : W_o(M) \to W_o(M)$ *such that*

$$\frac{d}{dt} |_0 \, E\left(tX^h\right) = X^h \text{ in } \mathcal{B}^{\infty-}.$$

Proof. See Corollary 4.6 in [10]. ■

For a geometric vector field, one can not construct a real flow as is constructed for Cameron–Martin vector field in Theorem 4.21. However the theorem above gurantees we can view them as vector fields from the natural definition. In the next definition we specify a domain of these operators.

Notation 4.29 *In this chapter, we fix* $\mathcal{D}(L)$ *to be the domain of an operator* L.

Definition 4.30 *Given a geometric vector field* X^h, *let* $\mathcal{D}\left(X^h\right)$ *denote the domain of* X^h *given by*

$$\mathcal{D}\left(X^h\right) := \left\{ f : W_o(M) \to \mathbb{R} \mid X^h f := \frac{d}{dt} |_0 \, f\left(E\left(tX^h\right)\right) \in L^{\infty-}\left(W_o(M)\right) \right\}.$$

Notation 4.31 *Recall from Notation 4.18 that Z_h satisfies the following ODE,*

$$Z_h'(s) = -\frac{1}{2} Ric_{\tilde{u}_s} Z_h(s) + h_s' \text{ with } Z_h(0) = 0. \tag{4.16}$$

We will use Z_α as the shorthand of Z_h where $h_s = \int_0^s \left(\tilde{T}_r^{-1}\right)^* e_\alpha dr$, $1 \leq \alpha \leq d$.

Lemma 4.32 *Let X^{Z_α} be given above, then X^{Z_α} is a geometric vector field with $Z_\alpha \in \mathcal{V}^\infty \cap \mathcal{B}^\infty$.*

Proof. Recall that Z_α satisfies the following ODE:

$$Z_\alpha'(s) = -\frac{1}{2} Ric_{\tilde{u}_s} Z_\alpha(s) + \left(\tilde{T}_s^{-1}\right)^* e_\alpha \text{ with } Z_\alpha(0) = 0. \tag{4.17}$$

Since $\left(\tilde{T}_s^{-1}\right)^* e_\alpha$ is adapted, Z_α' is adapted. So Z_α is a Brownian semimartingale with $Q \equiv 0$ and $r = Z_\alpha'$. Gronwall inequality implies that Z_α is bounded, and the bound is independent of $\sigma \in W_o(M)$ and $s \in [0,1]$. Therefore $Z_\alpha \in \mathcal{V}^\infty \cap \mathcal{B}^\infty$. ■

The next theorem shows how to differentiate a cylinder function $f \in \mathcal{FC}$ along a geometric vector field.

Notation 4.33 *Given $k : W_o(M) \to H_0\left(\mathbb{R}^d\right)$, denote $\int_0^s R_{\tilde{u}_r}(k_r, \delta\beta_r)$ by $A_s\langle k \rangle$, where δ is the stratonovich differential.*

Notation 4.34 *Suppose $F \in C\left(\mathcal{O}(M)^n\right)$ and $\mathcal{P} = \{0 < s_1 < \cdots < s_n \leq 1\}$ is a partition of $[0,1]$, set*

$$F(u) = F\left(u_{s_1}, \ldots, u_{s_n}\right),$$

then for $A : [0,1] \to \mathfrak{so}(d)$ and $h : [0,1] \to \mathbb{R}^d$, set

$$F'(u)\langle A + h \rangle := \frac{d}{dt}|_0 F\left(ue^{tA}\right) + \frac{d}{dt}|_0 F\left(e^{tB_h}(u)\right)$$

where $ue^{tA}(s) = u_s e^{tA_s} \in \mathcal{O}(M)$ and $e^{tB_h}(u)(s) = e^{tB_{h_s}}(u_s) \in \mathcal{O}(M)$.

Theorem 4.35 *For all* $h \in \mathcal{V}^2$*,*

$$X^{Z_h} f := F'(\tilde{u}) \langle -A \langle Z_h \rangle + Z_h \rangle \tag{4.18}$$

is well defined. That is to say, $\mathcal{FC} \subset \mathcal{D}\left(X^{Z_h}\right)$*. Moreover, if* $g \in \mathcal{FC}^{\infty}$*, then*

$$\mathbb{E}\left[X^{Z_h} f \cdot g\right] = \mathbb{E}\left[f \cdot \left(X^{Z_h}\right)^{tr,\nu} g\right] \tag{4.19}$$

where $\left(X^{Z_h}\right)^{tr,\nu} := -X^{Z_h} + \int_0^1 \langle h'_s, d\beta_s \rangle$*.*

Proof. See Proposition 4.10 in [10] . ■

The following lemma gives an anticipating expansion of $\tilde{X}$ in terms of $\{X^{Z_h}\}_{h \in H(M)}$.

Definition 4.36 (Orthogonal lift on $W_o(M)$**)** *For any* $f \in \mathcal{FC}^{\infty}$*, define*

$$\tilde{X} f := \sum_{\alpha=1}^{d} \left\langle \tilde{C} \tilde{H}, e_\alpha \right\rangle X^{Z_\alpha} f$$

where $\tilde{C} = \left[\int_0^1 \left(\tilde{T}_r^* \tilde{T}_r\right)^{-1} dr\right]^{-1} \tilde{T}_1^{-1}$ *and by the previous notation (Notation 4.18),*

$$X_s^{Z_\alpha} = \tilde{u}_s Z_\alpha(s)$$

Remark 4.37 *To motivate this definition, recall that we have obtained a lift* $\tilde{X} = X^{Z_\Phi} := \tilde{u}_s Z_\Phi(s)$ *of* $X \in \Gamma(TM)$*, where*

$$\Phi_s = \int_0^s \left(\tilde{T}_\tau^{-1}\right)^* \left[\int_0^1 \left(\tilde{T}_r^* \tilde{T}_r\right)^{-1} dr\right]^{-1} \tilde{T}_1^{-1} \tilde{H} d\tau.$$

It is clear that $\Phi \in H_0\left(\mathbb{R}^d\right)$ *is not adapted. Therefore we cannot apply the theory for geometric vector field. Alternatively we can expand* Φ *in terms of adapted vector* *fields,*

$$\Phi_s = \sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \int_0^s \left(\tilde{T}_r^{-1} \right)^* e_\alpha dr. \tag{4.20}$$

By superposition principle,

$$Z_\Phi(s) = \sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle Z_\alpha(s)$$

and further

$$X^{Z_\Phi} = \sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle X^{Z_\alpha}. \tag{4.21}$$

Remark 4.38 *From the construction above, one can see that if X has compact support (which is case we care about, see the Representation Formula 3.4), then $\left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle$ is bounded and the domain of the operator $\tilde{X}$, denoted by $\mathcal{D}\left(\tilde{X}\right)$ can be defined as*

$$\mathcal{D}\left(\tilde{X}\right) := \cap_{\alpha=1}^{d} \mathcal{D}\left(X^{Z_\alpha}\right).$$

This actually has already implied that $\tilde{X}$ is a closable operator on $L^2(W_o(M))$. However, it is still necessary and intereting to develop an integration–by–parts formula for this operator $\tilde{X}$.

4.4 Computing $\tilde{X}^{tr,\nu}$

This section is devoted to studying of the existence of $\tilde{X}^{tr,\nu}$ (The adjoint operator of $\tilde{X}$ with respect to ν). The crucial step to show existence is checking the anticipating coefficients (6.68) are differentiable in the Malliavin sense reviewed in Section 4.3. What is more, an explicit formula which has clearer structure as indicated in Corollary C.3 is given under the condition that the covariant derivative of the curvature tensor is bounded, which includes manifold with non–positive constant sectional curvature.

Proposition 4.39 *If Ric and ∇Ric are bounded and $h \in \mathcal{V}^{\infty}$, then for any $s \in [0,1]$, $Ric_{\tilde{u}_s} \in \mathcal{D}\left(X^h\right)$. What is more, for any $q \geq 1$,*

$$\sup_{s\in[0,1]} \left\| X^h Ric_{\tilde{u}_s} \right\|_{L^q(W_o(M))} < \infty.$$

Proof. Since $Ric_{\tilde{u}_s} \in \mathcal{FC}^{\infty}$, from Theorem 4.35 we know $Ric_{\tilde{u}_s} \in \mathcal{D}\left(X^h\right)$ and

$$X^h Ric_{\tilde{u}_s} = \nabla_{X_s^h} Ric + \left[A_s\langle h\rangle, Ric_{\tilde{u}_s}\right].$$

Since $h \in \mathcal{V}^{\infty}$, $X^h \in L^{\infty-}(W_o(M))$. Then by Burkholder's inequality, $A_s\langle h\rangle \in L^{\infty-}(W_o(M))$, since Ric and ∇Ric are bounded, we have

$$\sup_{s\in[0,1]} \left\| X^h Ric_{\tilde{u}_s} \right\|_{L^q(W_o(M))} < \infty.$$

■

Theorem 4.40 *Let $\tilde{T}_s$ be as defined in Definition 4.14, then*

$$\tilde{T}_s \in \mathcal{D}\left(X^{Z_\alpha}\right) \text{ for } 1 \leq \alpha \leq d.$$

Proof. For each X^{Z_α}, recall from Lemma 4.32 that $Z_\alpha \in \mathcal{V}^{\infty-}$, so we can apply Theorem 4.39 and get $\left\| X^{Z_\alpha} Ric_{\tilde{u}_s} \right\| \in L^{\infty-}(W_o(M))$. Denote by G_s the solution to the following ODE

$$G_s' = -\frac{1}{2} Ric_{\tilde{u}_s} G_s - \frac{1}{2}\left(X^{Z_\alpha} Ric_{\tilde{u}_s}\right)\tilde{T}_s \text{ with } G_0 = 0 \tag{4.22}$$

Denote by $G_s(t) = \frac{\tilde{T}_s(t)-\tilde{T}_s}{t}$ where $\tilde{T}_s(t) = \tilde{T}_s \circ E\left(tX^{Z_\alpha}\right)$ and $E\left(tX^{Z_\alpha}\right)$ is the approximate flow of X^{Z_α} defined in Definition 4.28. It is easy to see that $G_s(t)$

satisfies the following ODE:

$$G_s'(t) = -\frac{1}{2}Ric_{\tilde{u}_s}G_s(t) - \frac{1}{2t}\left(Ric_{\tilde{u}_s(t)} - Ric_{\tilde{u}_s}\right)\tilde{T}_s \text{ with } G_0(t) = 0 \tag{4.23}$$

Then let $H_s(t)$ be $H_s(t) := G_s(t) - G_s$, we know $H_s(t)$ satisfies

$$H_s'(t) = -\frac{1}{2}Ric_{\tilde{u}_s}H_s(t) - \frac{1}{2}\left(\frac{Ric_{\tilde{u}_s(t)} - Ric_{\tilde{u}_s}}{t}\tilde{T}_s(t) + \left(X^{Z_\alpha}Ric_{\tilde{u}_s}\right)\tilde{T}_s\right), \; H_0(t) = 0. \tag{4.24}$$

By definition $\tilde{T}_s \in \mathcal{D}\left(X^{Z_\alpha}\right) \iff H_s(t) \to 0$ in $L^{\infty-}(W_o(M))$.

By Gronwall's inequality, we have

$$|H_s(t)| \le \int_0^s \left|\frac{Ric_{\tilde{u}_r(t)} - Ric_{\tilde{u}_r}}{t}\tilde{T}_r(t) + X^{Z_\alpha}Ric_{\tilde{u}_r}\tilde{T}_r\right| dr e^{\frac{d(N-1)}{2}} \tag{4.25}$$

Following Theorem 4.4 in [10], we know

$$\frac{Ric_{\tilde{u}_r(t)} - Ric_{\tilde{u}_r}}{t} \to X^{Z_\alpha}Ric_{\tilde{u}_r}$$

and

$$\tilde{T}_r(t) \to \tilde{T}_r \to 0$$

uniformly on $r \in [0,1]$ in $L^{\infty-}(W_o(M))$ as $t \to 0$. So we have $H_s(t) \to 0$ in $L^{\infty-}(W_o(M))$ as $t \to 0$. ■

Corollary 4.41 *Recall that we have defined* $\tilde{C} = \left[\int_0^1 \left(\tilde{T}_r^*\tilde{T}_r\right)^{-1} dr\right]^{-1} \tilde{T}_1^{-1}$ *in Definition 4.36, then*

$$\tilde{C} \in \mathcal{D}\left(X^{Z_\alpha}\right) \text{ for } 1 \le \alpha \le d.$$

Proof. By the product rule, for any $s \in [0,1]$,

$$X^{Z_\alpha}\left(\tilde{T}_s^{-1}\right) = -\tilde{T}_s\left(X^{Z_\alpha}\tilde{T}_s\right)\tilde{T}_s \in L^{\infty-}(W_o(M)),$$

so $\tilde{T}_s^{-1} \in \mathcal{D}\left(X^{Z_\alpha}\right)$ and thus $\int_0^1 \left(\tilde{T}_r^* \tilde{T}_r\right)^{-1} dr \in \mathcal{D}\left(X^{Z_\alpha}\right)$. Then apply the product rule again we get $\tilde{C} \in \mathcal{D}\left(X^Z \alpha\right)$. ■

Lemma 4.42 *Given $X \in \Gamma(TM)$, recall from Definition 4.36 that $\tilde{X}$ is its orthogonal lift on $W_o(M)$, then*

$$\tilde{X}^{\mathrm{tr},\nu} = -\tilde{X} + \sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1}\right)^* e_\alpha, d\beta_s \right\rangle + \sum_{\alpha=1}^{d} \left\langle -X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right), e_\alpha \right\rangle.$$

In other words we are claiming that

$$\mathbb{E}\left[\tilde{X}f \cdot g\right] = \mathbb{E}\left[f \cdot \tilde{X}^{tr,\nu} g\right]$$

for all $f, g \in \mathcal{D}\left(\tilde{X}\right)$.

Proof. Since $\tilde{T}$ is adapted and uniformly bounded,

$$Z_{\int_0^\cdot \left(\tilde{T}_r^{-1}\right)^* e_\alpha dr} \in \mathcal{V}^{\infty-} := \cap_{q \geq 1} \mathcal{V}^q.$$

By Theorem 4.35, for all $f, g \in \mathcal{FC}^\infty \subset \mathcal{D}\left(\tilde{X}\right)$,

$$X^{Z_\alpha} f = F'(\tilde{u}) \langle -(A\langle Z_\alpha \rangle) + Z_\alpha \rangle$$

and

$$\mathbb{E}\left[X^{Z_\alpha} f \cdot g\right] = \mathbb{E}\left[f \cdot \left(-X^{Z_\alpha} + \int_0^1 \left\langle \left(T_s^{-1}\right)^* e_\alpha, d\beta_s \right\rangle\right) g\right].$$

Therefore we formally have

$$\mathbb{E}\left[\tilde{X} f \cdot g\right] = \mathbb{E}\left[\sum_{\alpha\,=\,1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle X^{Z_\alpha} f \cdot g\right] \tag{4.26}$$

$$= \sum_{\alpha=1}^{d} \mathbb{E}\left[X^{Z_\alpha} f \cdot \left(g \cdot \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle\right)\right] \tag{4.27}$$

$$= I + II + III \tag{4.28}$$

where

$$I = \mathbb{E}\left[f \cdot \left(-\tilde{X}\right) g\right]$$

$$II = \mathbb{E}\left[f \cdot g \cdot \sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1}\right)^* e_\alpha, d\beta_s \right\rangle\right]$$

$$III = \mathbb{E}\left[f \cdot g \cdot \sum_{\alpha=1}^{d} \left\langle -X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right), e_\alpha \right\rangle\right].$$

Therefore by the Chain rule, $\tilde{T}_s^{-1} \in \mathcal{D}\left(X^{Z_{\int_0^{\cdot}\left[\tilde{T}_r^{-1}\right]^* e_\alpha dr}}\right)$ and it follows that

$$F := \int_0^1 \left(\tilde{T}_r^* \tilde{T}_r\right)^{-1} dr \in \mathcal{D}\left(X^{Z_{\int_0^{\cdot}\left[\tilde{T}_r^{-1}\right]^* e_\alpha dr}}\right).$$

or more explicitly,

$$\dot{F} = \int_0^1 \left[\left(\dot{\tilde{T}_r^{-1}}\right)\left(\left[\tilde{T}_r^{-1}\right]^*\right) + \tilde{T}_r^{-1}\left(\dot{\tilde{T}_r^{-1}}\right)^*\right] dr$$

Notice that $\tilde{C}\tilde{T}_1 = F^{-1}$. Since $\tilde{T}_1 \in \mathcal{D}\left(X^{Z_\alpha}\right)$, by product rule again it suffices to show that

$$F^{-1} \in \mathcal{D}\left(X^{Z_{\int_0^{\cdot}\left[\tilde{T}_r^{-1}\right]^* e_\alpha dr}}\right).$$

Apply the Chain rule again, we know

$$\dot{F^{-1}} = -F^{-1} \cdot \dot{F} \cdot F^{-1} \tag{4.29}$$

So $F^{-1} \in \mathcal{D}\left(X^{Z_{\int_0^{\cdot}\left[\tilde{T}_r^{-1}\right]^* e_\alpha dr}}\right)$.

The following lemma gives a more explicit expression of the last term

$$\sum_{\alpha=1}^{d}\left\langle -X^{Z_{\int_0^{\cdot}\left[\tilde{T}_r^{-1}\right]^* e_\alpha dr}}\left(\tilde{C}\tilde{H}\right), e_\alpha\right\rangle$$

under the constant sectional curvature condition. ■

Lemma 4.43 *If further the covariant differential of the curvature tensor is 0, i.e.* $\nabla R \equiv 0$, *then*

$$-\sum_{\alpha=1}^{d}\left\langle X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right), e_\alpha\right\rangle = divX \circ E_1 - \sum_{\alpha=1}^{d}\left\langle \tilde{C}A_1\left\langle Z_\alpha\right\rangle \tilde{H}, e_\alpha\right\rangle. \tag{4.30}$$

Proof. Since M has constant sectional curvature,

$$Ric_{\tilde{u}}(\cdot) := \sum_i R_{\tilde{u}}(e_i, \cdot) e_i = \sum_i NQ_{e_i} = N(d-1)I = N'I.$$

where $N' = N(d-1)$.

From there we know$\tilde{T}_s = e^{-\frac{1}{2}N'sI}$ is deterministic. Therefore

$$\tilde{C} = \left(\int_0^1 \left[\tilde{T}_r^*\tilde{T}_r\right]^{-1} dr\right)^{-1} \tilde{T}_1^{-1} \text{ is deterministic.}$$

Notice that $\tilde{H} = \tilde{u}_1^{-1}X(\pi \circ \tilde{u}_1) \in \mathcal{FC}^\infty$. So we can apply Theorem 4.35 to $\tilde{H}$,

$$\begin{aligned}\sum_{\alpha=1}^{d}\left\langle X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right), e_\alpha\right\rangle &= \sum_{\alpha=1}^{d}\left\langle \tilde{C}X^{Z_\alpha}\tilde{H}, e_\alpha\right\rangle \\ &= I + II\end{aligned}$$

where

$$I = -\sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{u}_1^{-1}\nabla_{Z_\alpha(1)}X, e_\alpha \right\rangle$$

and

$$II = \sum_{\alpha=1}^{d} \left\langle \tilde{C}A_1 \langle Z_\alpha \rangle \tilde{H}, e_\alpha \right\rangle.$$

Claim: $I = -divX \circ E_1$.

Proof of Claim:

$$\begin{aligned} I &= -\sum_{\alpha=1}^{d} \left\langle \tilde{u}_1\tilde{C}\tilde{u}_1^{-1}\nabla_{\tilde{u}_1\tilde{C}^{-1}\tilde{u}_1^{-1}\tilde{u}_1 e_\alpha}X, \tilde{u}_1 e_\alpha \right\rangle \\ &= -\sum_{\alpha=1}^{d} \left\langle A^{-1}\nabla_{Af_\alpha}X, f_\alpha \right\rangle \\ &= -\sum_{\alpha=1}^{d} \left\langle \nabla_{Af_\alpha}X, \left(A^{-1}\right)^* f_\alpha \right\rangle \end{aligned}$$

where $A = \tilde{u}_1\tilde{C}^{-1}\tilde{u}_1^{-1} \in End\left(T_{E_1(\sigma)}M\right)$ and $\{f_\alpha\} = \{\tilde{u}_1 e_\alpha\}$ is an orthonormal basis of $T_{E_1(\sigma)}M$. Since $\langle \nabla.X, \cdot \rangle$ is bilinear on $T_{E_1(\sigma)}M$, by the Universal property of tensor product we know there exists a linear map $l : T_{E_1(\sigma)}M \otimes T_{E_1(\sigma)}M \mapsto \mathbb{R}$ such that

$$\left\langle \nabla_{Af_\alpha}X, \left(A^{-1}\right)^* f_\alpha \right\rangle = l\left(Af_\alpha \otimes \left(A^{-1}\right)^* f_\alpha\right)$$

and therefore:

$$\sum_{\alpha=1}^{d} \left\langle \nabla_{Af_\alpha}X, \left(A^{-1}\right)^* f_\alpha \right\rangle = l\left(\sum_{\alpha=1}^{d} Af_\alpha \otimes \left(A^{-1}\right)^* f_\alpha\right) \tag{4.31}$$

Using the isomorphism between $T_1^1(V) \mapsto End(V)$:$(a \otimes b)\, v = a \cdot \langle b, v \rangle$ one can easily see:

$$\sum_{\alpha=1}^{d} Af_{\alpha} \otimes \left(A^{-1}\right)^{*} f_{\alpha} = \sum_{\alpha=1}^{d} f_{\alpha} \otimes f_{\alpha} \tag{4.32}$$

Combine (4.31) and (4.32) we have

$$I = -\sum_{\alpha=1}^{d} \langle \nabla_{f_{\alpha}} X, f_{\alpha} \rangle = -divX \circ E_1$$

and (4.30). ■

Chapter 5

The Orthogonal Lift $\tilde{X}_{\mathcal{P}}$ on $H_{\mathcal{P}}(M)$

5.1 A Parametrization of $T_\sigma H_{\mathcal{P}}(M)$

Recall from Theorem 2.41 that for each $\sigma \in H_{\mathcal{P}}(M)$, $X(\sigma, s) \in T_\sigma H_{\mathcal{P}}(M)$ iff $J(\sigma, s) := u(\sigma, s)^{-1} X(\sigma, s)$ satisfies

$$J''(s) = R_{u(s)}\left(b'(s_{i-1}+), J(s)\right) b'(s_{i-1}+) \text{ for } s \in [s_{i-1}, s_i) \; i = 1, ..., n.$$

where $b = \phi(\sigma)$ is the anti–rolling of σ.

From above we observe that J can be parametrized by

$$\{J'(s_i+) = k_i\}_{i=0}^{n-1}$$

where $(k_0, k_1, \ldots, k_{n-1})$ is an arbitrary element of $\left(\mathbb{R}^d\right)^n$. Proposition 5.1 explains this parametrization.

Proposition 5.1 *Given* $(k_0, k_1, \ldots, k_{n-1}) \in (\mathbb{R}^d)^n$, *the associated* $J(\cdot)$ *is given by*

$$J(s) = \frac{1}{n} \sum_{i=0}^{l-1} f_{\mathcal{P},i+1}(s)\, k_i \text{ for } s \in [s_{l-1}, s_l]\ ,\ 1 \leq l \leq n. \tag{5.1}$$

Proof. From the definition of $f_{\mathcal{P},i+1}$ (see Definition 2.43), It is equivalent to show

$$J(s) = C_{\mathcal{P},l}(s) \left[\sum_{i=0}^{l-2} C_{\mathcal{P},l-1} \ldots C_{\mathcal{P},i+2} S_{\mathcal{P},i+1} k_i \right] + S_{\mathcal{P},l}(s)\, k_{l-1} \text{ when } s \in [s_{l-1}, s_l].$$

We will show it as follows,

$$\begin{aligned} J(s) &= C_{\mathcal{P},1}(s)\, 0 + S_{\mathcal{P},1}(s)\, k_0 = S_{\mathcal{P},1}(s)\, k_0 \text{ for } s \in [s_0, s_1] \text{ and} \\ J(s_1) &= S_{\mathcal{P},1} k_0, \end{aligned}$$

$$\begin{aligned} J(s) &= C_{\mathcal{P},2}(s)\, S_{\mathcal{P},1} k_0 + S_{\mathcal{P},2}(s)\, k_1 \text{ for } s \in [s_1, s_2] \text{ and} \\ J(s_2) &= C_{\mathcal{P},2} S_{\mathcal{P},1} k_0 + S_{\mathcal{P},2} k_1 \end{aligned}$$

$$\begin{aligned} J(s) &= C_{\mathcal{P},3}(s) \left[C_{\mathcal{P},2} S_{\mathcal{P},1} k_0 + S_{\mathcal{P},2} k_1 \right] + S_{\mathcal{P},3}(s)\, k_2 \\ &= C_{\mathcal{P},3}(s)\, C_{\mathcal{P},2} S_1 k_0 + C_{\mathcal{P},3}(s)\, S_{\mathcal{P},2} k_1 + S_{\mathcal{P},3}(s)\, k_2 \text{ for } s \in [s_2, s_3] \text{ and} \\ J(s_3) &= C_{\mathcal{P},3} C_{\mathcal{P},2} S_{\mathcal{P},1} k_0 + C_{\mathcal{P},3} S_{\mathcal{P},2} k_1 + S_{\mathcal{P},3} k_2. \end{aligned}$$

Continuing this way inductively we learn for $s \in [s_{l-1}, s_l]$ that

$$\begin{aligned}
J(s) &= C_{\mathcal{P},l}(s)\, C_{\mathcal{P},l-1} \ldots C_{\mathcal{P},2} S_{\mathcal{P},1} k_0 + C_{\mathcal{P},l}(s)\, C_{\mathcal{P},l-1} \ldots C_{\mathcal{P},3} S_2 k_1 + \\
&+ \cdots + C_{\mathcal{P},l}(s)\, S_{\mathcal{P},l-1} k_{l-2} + S_{\mathcal{P},l}(s)\, k_{l-1} \\
&= \sum_{i=0}^{l-2} C_{\mathcal{P},l}(s)\, C_{\mathcal{P},l-1} \ldots C_{\mathcal{P},i+2} S_{\mathcal{P},i+1} k_i + S_{\mathcal{P},l}(s)\, k_{l-1} \\
&= C_{\mathcal{P},l}(s) \left[\sum_{i=0}^{l-2} C_{\mathcal{P},l-1} \ldots C_{\mathcal{P},i+2} S_{\mathcal{P},i+1} k_i \right] + S_{\mathcal{P},l}(s)\, k_{l-1}
\end{aligned}$$

as desired. ■

Definition 5.2 *For each* $s \in [0,1]$*, define* $\mathbf{L}_s : \left(\mathbb{R}^d\right)^n \to \mathbb{R}^d$ *as follows: for* $s \in [s_{l-1}, s_l]$*,*

$$\mathbf{L}_s (k_0, \ldots, k_{n-1}) = \frac{1}{n} \sum_{i=0}^{l-1} f_{\mathcal{P},i+1}(s)\, k_i. \tag{5.2}$$

What we care most is when $s = 1$, then

$$\mathbf{L}_1 (k_0, \ldots, k_{n-1}) = \frac{1}{n} \sum_{i=0}^{n-1} f_{\mathcal{P},i+1}(1)\, k_i \tag{5.3}$$

We now compute the adjoint of $\mathbf{L}_1$.

Lemma 5.3 *For any* $v \in \mathbb{R}^d$*,*

$$\mathbf{L}_1^* v = \frac{1}{n} \left(f_{\mathcal{P},1}^*(1)\, v, f_{\mathcal{P},2}^*(1)\, v, \ldots, f_{\mathcal{P},n}^*(1)\, v \right). \tag{5.4}$$

Proof.

$$\langle \mathbf{L}(1)(k_0, \ldots, k_{n-1}), v \rangle = \sum_{i=0}^{n-1} \left\langle \frac{1}{n} f_{\mathcal{P},i+1}(1)\, k_i, v \right\rangle = \sum_{i=0}^{n-1} \left\langle k_i, \frac{1}{n} f_{\mathcal{P},i+1}^*(1)\, v \right\rangle. \tag{5.5}$$

From which it follows that

$$\mathbf{L}_1^* v = \frac{1}{n}\left(f^*_{\mathcal{P},1}(1)\, v, f^*_{\mathcal{P},2}(1)\, v, \ldots, f^*_{\mathcal{P},n}(1)\, v\right). \tag{5.6}$$

■

Definition 5.4 *We now define*

$$\mathbf{K}_{\mathcal{P}}(s)\, v := n\mathbf{L}(s)\left(\mathbf{L}(1)^* v\right) \tag{5.7}$$

In particular,

$$\mathbf{K}_{\mathcal{P}}(1)\, v = \frac{1}{n}\sum_{i=0}^{n-1} f_{\mathcal{P},i+1}(1)\, f^*_{\mathcal{P},i+1}(1)\, v \tag{5.8}$$

Recall that given a matrix A, $eig(A)$ denotes the eigenvalues of A.

Lemma 5.5 (Invertibility of $\mathbf{K}_{\mathcal{P}}(1)$) *If M has non-positive sectional curvature, then*

$$eig\left(\mathbf{K}_{\mathcal{P}}(1)\right) \subset [1, \infty) \tag{5.9}$$

and thus $\mathbf{K}_{\mathcal{P}}(1)$ is invertible.

Proof. Denote $R_{u_s}\left(b'(s_{i-1}+), \cdot\right) b'(s_{i-1}+)$ by $A_{\mathcal{P},i}(s) : H_{\mathcal{P}}(M) \to End\left(\mathbb{R}^d\right)$. Notice that M having non-positive sectional curvature guarantees $A_{\mathcal{P},i}(s)$ is non-negative. Then apply Proposition B.1 we get, for any $i = 1, \cdots, n, v \in \mathbb{C}^d$,

$$\|C_{\mathcal{P},i} v\| \geq \|v\| \text{ and } \|S_{\mathcal{P},i} v\| \geq \frac{1}{n}\|v\|$$

From which it follows that:

$$\begin{aligned}\|f_{\mathcal{P},i}(1)\,v\| &= n\,\|C_{\mathcal{P},n}C_{\mathcal{P},n-1}\cdots C_{\mathcal{P},i+1}S_{\mathcal{P},i}v\| \\ &\geq n\cdot\frac{1}{n}\,\|v\| \\ &= \|v\|\end{aligned}$$

Notice that from the min-max theorem, $\|C_{\mathcal{P},i}v\|^2 \geq \|v\|^2 \iff eig\left(C^*_{\mathcal{P},i}C_{\mathcal{P},i}\right) \subset [1,\infty)$, and since all the eigenvalues of $C^*_{\mathcal{P},i}C_{\mathcal{P},i}$ are non-zero, we have

$$eig\left(C^*_{\mathcal{P},i}C_{\mathcal{P},i}\right) = eig\left(C_{\mathcal{P},i}C^*_{\mathcal{P},i}\right)$$

Therefore $\left\|C^*_{\mathcal{P},i}v\right\|^2 \geq \|v\|^2$ for all $v \in \mathbb{C}^d$. Similarly, we can obtain $\left\|S^*_{\mathcal{P},i}v\right\|^2 \geq \frac{1}{n^2}\|v\|^2$.

So for any $b \in \mathbb{C}^d$,

$$\begin{aligned}\langle \mathbf{K}_{\mathcal{P}}(1)\,b,b\rangle &= \frac{1}{n}\sum_{i=0}^{n-1}\left\langle f_{\mathcal{P},i+1}(1)\,f^*_{\mathcal{P},i+1}(1)\,b,b\right\rangle \\ &= \frac{1}{n}\sum_{i=0}^{n-1}\left\|f^*_{\mathcal{P},i+1}(1)\,b\right\|^2 \\ &\geq \frac{1}{n}\cdot n\,\|b\|^2 \\ &= \|b\|^2\end{aligned}$$

This implies that

$$eig\left(\mathbf{K}_{\mathcal{P}}(1)\right) \subset [1,\infty)$$

In particular, $\mathbf{K}_{\mathcal{P}}(1)$ is invertible. ■

5.2 Existence and Uniqueness of Orthogonal Lift $\tilde{X}_{\mathcal{P}}$

In this section we lift a vector field $X \in \Gamma(TM)$ onto a vector field $\tilde{X}_{\mathcal{P}} \in \Gamma(TH_{\mathcal{P}}(M))$ based on the "least square"spirit.

Theorem 5.6 (Orthogonal lift) *For all $X \in \Gamma(TM)$, we can find an orthogonal lift $\tilde{X}_{\mathcal{P}} \in \Gamma(TH_{\mathcal{P}}(M))$ in the sense that:*

1. *For all $h \in C^1(M)$,*

$$\tilde{X}_{\mathcal{P}}(h \circ E_1)(\sigma) = (Xh)(E_1(\sigma)) \tag{5.10}$$

2. *For all $\sigma \in H_{\mathcal{P}}(M)$,*

$$\left\| \tilde{X}_{\mathcal{P}}(\sigma) \right\|_{G^1_{\mathcal{P}}} = \inf\{\|Y(\sigma)\|_{G^1_{\mathcal{P}}} : Y \in \Gamma(TH_{\mathcal{P}}(M)), Y \text{ satisfies } (5.10)\}. \tag{5.11}$$

Since $T_\sigma H_{\mathcal{P}}(M) = \mathrm{Nul}\left(E_{1*,\sigma}\right) \oplus \left\{\mathrm{Nul}\left(E_{1*,\sigma}\right)\right\}^{\perp}$. So a general $X^J \in \Gamma(TH_{\mathcal{P}}(M))$ has minimal length iff $X^J \in \left\{\mathrm{Nul}\left(E_{1*,\sigma}\right)\right\}^{\perp}$. The following lemma characterize $\left\{\mathrm{Nul}\left(E_{1*,\sigma}\right)\right\}^{\perp}$.

Lemma 5.7 *$X^k \in \left\{\mathrm{Nul}\left(E_{1*,\sigma}\right)\right\}^{\perp}$ iff*

$$(k'(s_0+), ..., k'(s_{n-1}+)) \in (\mathrm{Nul}\,\mathbf{L}_1)^{\perp} = \mathrm{Ran}\left(\mathbf{L}_1^*\right).$$

Proof. Notice that for all $X^J, X^k \in TH_{\mathcal{P}}(M)$,

$$\begin{aligned} \left\langle X^J, X^k \right\rangle_{G^1_{\mathcal{P}}} = 0 &\iff \sum_{i=0}^{n-1} \langle J'(s_i+), k'(s_i+)\rangle \Delta_{i+1} = 0 \\ &\iff \sum_{i=0}^{n-1} \langle J'(s_i+), k'(s_i+)\rangle = 0. \end{aligned}$$

and

$$X^J(\sigma) \in \mathrm{Nul}\left(E_{1*,\sigma}\right) \iff E_{1*,\sigma}\left(X^{J_1}\right) = u_1(\sigma)\, J(\sigma, 1) = 0 \iff J_1(\sigma) = 0. \tag{5.12}$$

Recall that $J_1 = \mathbf{L}_1\left(J'(s_0+), ..., J'(s_{n-1}+)\right)$, so

$$J_1 = 0 \iff \left(J'(s_0+), ..., J'(s_{n-1}+)\right) \in \mathrm{Nul}\left(\mathbf{L}_1\right)$$

Notice that

$$\sum_{i=0}^{n-1} \langle J'(s_i+), k'(s_i+)\rangle = \langle \left(J'(s_0+), ..., J'(s_{n-1}+)\right), \left(k'(s_0+), ..., k'(s_{n-1}+)\right)\rangle .$$

So $X^k \in \left\{\mathrm{Nul}\left(E_{1*,\sigma}\right)\right\}^{\perp}$ iff

$$\left(k'(s_0+), ..., k'(s_{n-1}+)\right) \in \left\{\mathrm{Nul}\left(\mathbf{L}_1\right)\right\}^{\perp} = \mathrm{Ran}\left(\mathbf{L}_1^*\right).$$

■

Remark 5.8 *According to (5.4),*

$$\mathrm{Ran}\left(\mathbf{L}_1^*\right) = \left\{ \left(\frac{1}{n} f^*_{\mathcal{P},1}(1)\, v, \frac{1}{n} f^*_{\mathcal{P},2}(1)\, v, \ldots, \frac{1}{n} f^*_{\mathcal{P},n}(1)\, v \right), \quad \forall\ v \in \mathbb{R}^d \right\},$$

and for all $\left(J'(s_0+), ..., J'(s_{n-1}+)\right) \in \mathrm{Nul}(\mathbf{L}_1)$,

$$\begin{aligned}
&\left\langle \left(\frac{1}{n} f^*_{\mathcal{P},1}(1)\, v, \frac{1}{n} f^*_{\mathcal{P},2}(1)\, v, \ldots, \frac{1}{n} f^*_{\mathcal{P},n}(1)\, v \right), \left(J'(s_0+), ..., J'(s_{n-1}+)\right) \right\rangle \\
&= \frac{1}{n} \sum_{i=0}^{n-1} \left\langle f^*_{\mathcal{P},i+1}(1)\, v, J'(s_i+) \right\rangle = \left\langle v, \sum_{i=0}^{n-1} \frac{1}{n} f^*_{\mathcal{P},i+1}(1)\, J_1'(s_i+) \right\rangle \\
&= \langle v, J_1 \rangle \\
&= 0.
\end{aligned}$$

Therefore we know in order to have the minimal length, X^J must have the following form:

$$J_s = \mathbf{K}_{\mathcal{P}}(s)\, v$$

for some $v \in \mathbb{R}^d$ to be determined.

Definition 5.9 *Define $\tilde{X}_{\mathcal{P}} \in \Gamma(TH_{\mathcal{P}}(M))$ to be $\tilde{X}_{\mathcal{P}}(\cdot) = u.J_{\mathcal{P}}(\cdot)$ where*

$$J_{\mathcal{P}}(s) := \mathbf{K}_{\mathcal{P}}(s)\, \mathbf{K}_{\mathcal{P}}(1)^{-1} {u_1}^{-1} X \circ E_1.$$

Proof of Theorem 5.6. Firstly, we show that for all $h \in C^1(M)$, $\sigma \in H_{\mathcal{P}}(M)$,

$$\tilde{X}_{\mathcal{P}}(h \circ E_1)(\sigma) = (Xh)(E_1(\sigma)).$$

Since $\tilde{X}_{\mathcal{P}}(\sigma) = \frac{d}{dt}|_0 \sigma_t$ for some one parameter family $\{\sigma_t\}$,

$$\begin{aligned}\tilde{X}_{\mathcal{P}}(h \circ E_1)(\sigma) &= \frac{d}{dt}|_0 (h \circ E_1)(\sigma_t) = \frac{d}{dt}|_0 h(\sigma_t(1)) = \left(\tilde{X}_{\mathcal{P}}(\sigma, 1)\, h\right)(\sigma(1)) \\ &= \left(u_1(\sigma)\, \mathbf{K}_{\mathcal{P}}(\sigma, 1)\, \mathbf{K}_{\mathcal{P}}(\sigma, 1)^{-1} u_1^{-1}(\sigma)\, X(\sigma(1))\, h\right)(\sigma(1)) = (Xh)(E_1(\sigma)).\end{aligned}$$

So condition (5.10) holds. The fact that condition (5.11) is valid is easily seen from Remark 5.8.

The uniqueness of $\tilde{X}_{\mathcal{P}}$ can always be concluded from the following argument: From condition (5.10), we need:

$$\tilde{X}_{\mathcal{P}}(\sigma, 1) = X(\sigma(1)).$$

This implies $\mathbf{K}_{\mathcal{P}}(1)\, v = u_1^{-1} X \circ E_1$. Since $\mathbf{K}_{\mathcal{P}}(1)$ is invertible, we can just pick v to be $\mathbf{K}_{\mathcal{P}}(1)^{-1} u_1^{-1} X \circ E_1$.

We will explore the limit of the orthogonal lift $\tilde{X}_{\mathcal{P}}$ in Chapter 6. ■

5.3 Finite Dimensional Adjoint $\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1}$

In this section we study $\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1}$—the adjoint of $\tilde{X}_{\mathcal{P}}$ with respect to $\nu_{\mathcal{P}}^1$.

Lemma 5.10

$$\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} = -\tilde{X}_{\mathcal{P}} + M_{\int_0^1 \left\langle J'_{\mathcal{P}}(s), b'(s)\right\rangle ds} - M_{div\tilde{X}_{\mathcal{P}}} \tag{5.13}$$

where $M_{\cdot}$ is the multiplication operator and $div\tilde{X}_{\mathcal{P}}$ is the divergence of $\tilde{X}_{\mathcal{P}}$ with respect to $vol_{G_{\mathcal{P}}^1}$.

Proof. Therein we identify the measure $\nu_{\mathcal{P}}^1$ with the associated nd—form. So by Cartan's magic formula, for all $f \in \Omega^0 (H_{\mathcal{P}}(M)) \simeq C^\infty (H_{\mathcal{P}}(M))$,

$$\mathcal{L}_{\tilde{X}_{\mathcal{P}}} \left(f\nu_{\mathcal{P}}^1\right) = d\left(i_{\tilde{X}_{\mathcal{P}}}\left(f\nu_{\mathcal{P}}^1\right)\right) + i_{\tilde{X}_{\mathcal{P}}}\left(d\left(f\nu_{\mathcal{P}}^1\right)\right).$$

Since $f\nu_{\mathcal{P}}^1$ is an nd–form, $d\left(f\nu_{\mathcal{P}}^1\right) = 0$. By Stokes' theorem, $\int_{H_{\mathcal{P}}M} d\left(i_{\tilde{X}_{\mathcal{P}}}\left(f\nu_{\mathcal{P}}^1\right)\right) = 0$. Therefore we have:

$$\int_{H_{\mathcal{P}}(M)} \mathcal{L}_{\tilde{X}_{\mathcal{P}}}\left(f\nu_{\mathcal{P}}^1\right) = 0$$

and

$$\begin{aligned}\int_{H_{\mathcal{P}}(M)} \left(\tilde{X}_{\mathcal{P}} f\right) d\nu_{\mathcal{P}}^1 &= \int_{H_{\mathcal{P}}(M)} \mathcal{L}_{\tilde{X}_{\mathcal{P}}}\left(f\nu_{\mathcal{P}}^1\right) - \int_{H_{\mathcal{P}}(M)} f\mathcal{L}_{\tilde{X}_{\mathcal{P}}}\left(\nu_{\mathcal{P}}^1\right) \\ &= -\int_{H_{\mathcal{P}}(M)} f\mathcal{L}_{\tilde{X}_{\mathcal{P}}}\left(\nu_{\mathcal{P}}^1\right).\end{aligned} \tag{5.14}$$

Recall that $\nu_{\mathcal{P}}^1 = \frac{1}{Z_{\mathcal{P}}^1} e^{-\frac{1}{2}E} vol_{G_{\mathcal{P}}^1}$, so

$$\mathcal{L}_{\tilde{X}_{\mathcal{P}}}\left(\nu_{\mathcal{P}}^1\right) = \left[\tilde{X}_{\mathcal{P}}\left(\frac{1}{Z_{\mathcal{P}}^1} e^{-\frac{1}{2}E}\right)\right] vol_{G_{\mathcal{P}}^1} + \left(div\tilde{X}_{\mathcal{P}}\right)\nu_{\mathcal{P}}^1. \tag{5.15}$$

In there,

$$\tilde{X}_{\mathcal{P}}\left(\frac{1}{Z_{\mathcal{P}}^{1}}e^{-\frac{1}{2}E}\right) = -\frac{1}{2}\tilde{X}_{\mathcal{P}}(E)\frac{1}{Z_{\mathcal{P}}^{1}}e^{-\frac{1}{2}E}$$
$$= -\int_0^1\left\langle \sigma'(s+), \frac{\nabla \tilde{X}_{\mathcal{P}}}{ds}(s+)\right\rangle ds\frac{1}{Z_{\mathcal{P}}^{1}}e^{-\frac{1}{2}E}$$
$$= -\int_0^1\langle b'(s+), J'_{\mathcal{P}}(s+)\rangle ds\frac{1}{Z_{\mathcal{P}}^{1}}e^{-\frac{1}{2}E}. \tag{5.16}$$

Combine (5.14), (5.15) and (5.16) we get (5.13). ■

5.4 Computing $div\tilde{X}_{\mathcal{P}}$

Recall from Definition 3.20 that

$$X^{h_{\alpha,i}}(\sigma,s) = u(\sigma,s)\frac{1}{\sqrt{n}}f_{\mathcal{P},i}(s)e_{\alpha}\ ,\ 1\leq\alpha\leq d\ ,\ 1\leq i\leq n$$

is an orthonormal frame on $(TH_{\mathcal{P}}(M), G_{\mathcal{P}}^{1})$. Using this orthonormal frame, one can get an expression of $div\tilde{X}_{\mathcal{P}}$.

Proposition 5.11

$$div\tilde{X}_{\mathcal{P}} = \sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle X^{h_{\alpha,j}}J'_{\mathcal{P}}(s_{j-1}+), e_{\alpha}\right\rangle\sqrt{\Delta_j}$$

Proof. By definition

$$div\tilde{X}_{\mathcal{P}} = \sum_{\alpha=1}^{d}\sum_{j=1}^{n}G_{\mathcal{P}}^{1}\left\langle\left[X^{h_{\alpha,j}},\tilde{X}_{\mathcal{P}}\right], X^{h_{\alpha,j}}\right\rangle.$$

Now fix j and α, notice that $\tilde{X}_{\mathcal{P}} = X^{J_{\mathcal{P}}}$, apply Theorem 3.5 in [3] to the computation of the Lie bracket $\left[X^{h_{\alpha,j}}, \tilde{X}_{\mathcal{P}}\right]$, we have:

$$\left[X^{h_{\alpha,j}}, \tilde{X}_{\mathcal{P}}\right] = X^{f(h_{\alpha,j},J_{\mathcal{P}})},$$

where

$$f_s\left(h_{\alpha,j}, J_{\mathcal{P}}\right) = \left(X^{h_{\alpha,j}} J_{\mathcal{P}}\right)(s) - \left(X^{J_{\mathcal{P}}} h_{\alpha,j}\right)(s) + \\ q_s\left(X^{h_{\alpha,j}}\right) J_{\mathcal{P}}(s) - q_s\left(X^{J_{\mathcal{P}}}\right) h_{\alpha,j}(s)$$

and

$$q_s\left(X^f\right) = \int_0^s R_{u_r}\left(b'(r+), f(r)\right) dr.$$

Therefore

$$G^1_{\mathcal{P}}\left\langle \left[X^{h_{\alpha,j}}, \tilde{X}_{\mathcal{P}}\right], X^{h_{\alpha,j}} \right\rangle = \sum_{i=1}^n \left\langle f', h'_{\alpha,j}\right\rangle_{s_{i-1}+} \Delta_i \tag{5.17}$$

$$= \sum_{i=1}^n \left\langle \left(X^{h_{\alpha,j}} J_{\mathcal{P}}\right)' - \left(X^{J_{\mathcal{P}}} h_{\alpha,j}\right)', h'_{\alpha,j}\right\rangle_{s_{i-1}+} \Delta_i$$
$$+ \sum_{i=1}^n \left\langle \left(q_s\left(X^{h_{\alpha,j}}\right) J_{\mathcal{P}}(s)\right)' - \left(q_s\left(X^{J_{\mathcal{P}}}\right) h_{\alpha,j}(s)\right)', h'_{\alpha,j}\right\rangle_{s_{i-1}+} \Delta_i$$

Here $'$ is the derivative with respect to time s.

Since the manifold is torsion free and $h'_{\alpha,j}(s_{i-1}+)$ is independent of σ,

$$\left(X^{J_{\mathcal{P}}} h_{\alpha,j}\right)'(s_{i-1}+) = \left(X^{J_{\mathcal{P}}} h'_{\alpha,j}\right)(s_{i-1}+) = 0.$$

Then we look at

$$\left(q_s\left(X^{h_{\alpha,j}}\right) J_{\mathcal{P}}(s)\right)' = q'_s\left(X^{h_{\alpha,j}}\right) J_{\mathcal{P}}(s) + q_s\left(X^{h_{\alpha,j}}\right) J'_{\mathcal{P}}(s).$$

Notice that

$$h'_{\alpha,j}\left(s_{i-1}+\right)\neq 0 \text{ iff } i=j$$

and when $i=j$,

$$h_{\alpha,j}\left(s\right)=0 \text{ for } s\leq s_{i-1}$$

so both $q'_{s_{i-1}}\left(X^{h_{\alpha,j}}\right)=0$ and $q_{s_{i-1}}\left(X^{h_{\alpha,j}}\right)=0$.

From which it follows:

$$\sum_{i=1}^{n}\left\langle\left(q_s\left(X^{h_{\alpha,j}}\right)J_{\mathcal{P}}\left(s\right)\right)',h'_{\alpha,j}\right\rangle_{s_{i-1}+}\Delta_i=0 \tag{5.18}$$

and

$$\sum_{i=1}^{n}\left\langle q'_s\left(X^{J_{\mathcal{P}}}\right)h_{\alpha,j}\left(s\right),h'_{\alpha,j}\right\rangle_{s_{i-1}+}\Delta_i=0 \tag{5.19}$$

Lastly because $q_s\left(X^{J_{\mathcal{P}}}\right)$ is skew-symmetric,

$$\sum_{i=1}^{n}\left\langle q_s\left(X^{J_{\mathcal{P}}}\right)h'_{\alpha,j},h'_{\alpha,j}\right\rangle_{s_{i-1}+}\Delta_i=0 \tag{5.20}$$

Therefore we have

$$G^1_{\mathcal{P}}\left\langle\left[X^{h_{\alpha,j}},\tilde{X}_{\mathcal{P}}\right],X^{h_{\alpha,j}}\right\rangle=\sum_{i=1}^{n}\left\langle X^{h_{\alpha,j}}J'_{\mathcal{P}},h'_{\alpha,j}\right\rangle_{s_{i-1}+}\Delta_i \tag{5.21}$$

$$=\left\langle X^{h_{\alpha,j}}J'_{\mathcal{P}}\left(s_{j-1}+\right),e_\alpha\right\rangle\sqrt{\Delta_j}. \tag{5.22}$$

Then sum over α and j, we have

$$div\tilde{X}_{\mathcal{P}}=\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle X^{h_{\alpha,j}}J'_{\mathcal{P}}\left(s_{j-1}+\right),e_\alpha\right\rangle\sqrt{\Delta_j}.$$

■

Chapter 6

Convergence Result

Notation 6.1 *Recall that* $\beta := \tilde{\Phi} \circ \Sigma : W_o(M) \mapsto W_0(\mathbb{R}^d)$ *is a Brownian motion on* $\mathbb{R}^d$*. Here* Σ *is the canonical Brownian motion on* M *and* $\tilde{\Phi}$ *is the stochastic anti-development map. We also define* $b_{\mathcal{P}} := \pi_{\mathcal{P}} \circ \beta : W_o(M) \mapsto H_{\mathcal{P}}(\mathbb{R}^d)$ *to be the piecewise Brownian motion on* $\mathbb{R}^d$ *and* $u_{\mathcal{P}} := \eta \circ b_{\mathcal{P}}$ *to be the horizontal lift of* $b_{\mathcal{P}}$*. What's more, notice that* $\phi \circ b_{\mathcal{P}} \in H_{\mathcal{P}}(M)$*, here* ϕ *is the development map onto* $H(M)$*, so after identifying* $C_{\mathcal{P},i}$*,* $S_{\mathcal{P},i}$ *and hence* $f_{\mathcal{P},i}$ *with* $C_{\mathcal{P},i} \circ \phi \circ b_{\mathcal{P}}$*,* $S_{\mathcal{P},i} \circ \phi \circ \phi \circ b_{\mathcal{P}}$ *and* $f_{\mathcal{P},i} \circ \phi \circ \phi \circ b_{\mathcal{P}}$*, we can view them as matrix valued random variables on* $W_o(M)$*. The point here is to make the notations short and it should not cause confusions after this explanation.*

6.1 Wong-Zakai Approximation Scheme

Wong-Zakai approximation scheme are types of theorems that try to approximate solutions of a stochastic differential equations (SDE) by solutions of (random) ordinary differential equations driven by smooth approximations of the semimartingale that drives the SDE. Wong and Zakai [38], [39] first studied this problem in the case of one dimensional Brownian motion and there are a lot of generalizations that follow, which are partially listed in here : [2], [19] and so on.

We record a Wong-Zakai type theorem in the form that fits our need.

Theorem 6.2 (Theorem 4.14 in [3]) *Let* $f : \mathbb{R}^d \times \mathbb{R}^n \rightarrow End(\mathbb{R}^d, \mathbb{R}^n)$ *and* $f_0 : \mathbb{R}^d \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ *be twice differentiable with bounded continuous derivatives. Let* $\xi_0 \in \mathbb{R}^n$ *and* $\mathcal{P}$ *be a partition of* $[0,1]$. *Further let* β *and* $\beta_{\mathcal{P}}$ *be as in Notation 3.14 and* $\xi_{\mathcal{P}}(s)$ *denote the solution to the ordinary differential equation:*

$$\xi_{\mathcal{P}}'(s) = f(\xi_{\mathcal{P}}(s))b_{\mathcal{P}}'(s) + f_0(\xi_{\mathcal{P}}(s)), \qquad \xi_{\mathcal{P}}(0) = \xi_0 \tag{6.1}$$

and ξ *denote the solution to the Stratonovich stochastic differential equation,*

$$d\xi(s) = f(\xi(s))\delta\beta(s) + f_0(\xi(s))ds, \qquad \xi(0) = \xi_0. \tag{6.2}$$

Then, for any $\gamma \in (0, \frac{1}{2})$, $p \in [1, \infty)$, *there is a constant* $C(p,\gamma) < \infty$ *depending only on* f *and* M, *so that*

$$\lim_{|\mathcal{P}| \to 0} \mathbb{E}\left[\sup_{s \leq 1} |\xi_{\mathcal{P}}(s) - \xi(s)|^p\right] \leq C(p,\gamma)|\mathcal{P}|^{\gamma p}. \tag{6.3}$$

Corollary 6.3 $\sup_{0 \leq s \leq 1} |u_{\mathcal{P}}(s) - \tilde{u}(s)| \rightarrow 0$ *in* $L^{\infty-}(W_o(M))$.

6.2 Convergence of $\tilde{X}_{\mathcal{P}}$ to $\tilde{X}$

6.2.1 Some Useful Estimates for $\{C_{\mathcal{P},i}\}_{i=1}^n$ and $\{S_{\mathcal{P},i}\}_{i=1}^n$

We apply Proposition B.1 to get the following estimates: Lemma 6.4 to Lemma 6.7.

Lemma 6.4 *For any* $i \in \{1, ..., n\}$ *and* $s \in [s_{i-1}, s_i]$, *we have*

$$|C_{\mathcal{P},i}(s)| \leq \cosh\left(\sqrt{N}\,|\Delta_i \beta|\right) \leq e^{\frac{1}{2}N|\Delta_i\beta|^2}.$$

Lemma 6.5 *For any $i \in \{1, ..., n\}$ and $s \in [s_{i-1}, s_i]$, we have*

$$\begin{aligned}|S_{\mathcal{P},i}(s)| &\leq \sqrt{N}\,|\Delta_i\beta|\,\frac{\sinh\left(\sqrt{N}\,|\Delta_i\beta|\right)}{\sqrt{N}\,|\Delta_i\beta|} \\ &\leq \cosh\left(\sqrt{N}\,|\Delta_i\beta|\right)\sqrt{N}\,|\Delta_i\beta| \leq \sqrt{N}\,|\Delta_i\beta|\,e^{\frac{1}{2}N|\Delta_i\beta|^2}.\end{aligned}$$

Lemma 6.6 *For any $i \in \{1, ..., n\}$, we have*

$$|S_{\mathcal{P},i} - \Delta_i I| \leq \frac{N\,|\Delta_i\beta|^2\,\Delta_i}{6} e^{\frac{1}{2}N|\Delta_i\beta|^2}$$

Lemma 6.7 *For any $i \in \{1, ..., n\}$, we have*

$$|C_{\mathcal{P},i} - I| \leq \frac{N\,|\Delta_i\beta|^2}{2} e^{\frac{1}{2}N|\Delta_i\beta|^2}$$

Lemma 6.8 *For all $\gamma \in \left(0, \frac{1}{2}\right)$, define $K_\gamma := \sup_{s,t\in[0,1],s\neq t}\left\{\frac{|\beta_t - \beta_s|}{|t-s|^\gamma}\right\}$, then there exists an $\epsilon_\gamma > 0$ such that $\mathbb{E}\left[e^{\epsilon K_\gamma^2}\right] < \infty$.*

Proof. See Fernique's Theorem (Theorem 3.2) in [28]. ■

Remark 6.9 *From Lemma 6.8, it is easy to see any polynomial of ϵK_γ has finite moments of all orders.*

6.2.2 Size Estimates of $f_{\mathcal{P},i}(s)$

Recall from Definition 2.43 that $f_{\mathcal{P},i} : W_o(M) \times [0, 1] \to End\left(\mathbb{R}^d\right)$ $0 \leq i \leq n$ is given by

$$f_{\mathcal{P},i}(s) = \begin{cases} 0 & s \in [0, s_{i-1}] \\ \frac{S_{\mathcal{P},i}(s)}{\Delta_i} & s \in [s_{i-1}, s_i] \\ \frac{C_{\mathcal{P},j}(s)C_{\mathcal{P},j-1}\cdots\cdots C_{\mathcal{P},i+1}S_{\mathcal{P},i}}{\Delta_i} & s \in [s_{j-1}, s_j] \text{ for } j = i+1, \cdots, n \end{cases}$$

with the convention that $S_{\mathcal{P},0} \equiv |\mathcal{P}|\, I$ and $f_{\mathcal{P},0} \equiv I$.

Using the estimates in Subsection 6.2.1, it is easy to get a control over the size of $f_{\mathcal{P},i}(s)$.

Lemma 6.10 *For any $q \geq 1$, there exists a constant C_q such that*

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\}}\sup_{s\in\mathcal{P}}|f_{\mathcal{P},i}(s)|^q\right] \leq C_q$$

Proof. For all $i, j \in \{0, \cdots, n\}$, we only need to consider the case when $j \geq i$, since if $j < i$, $f_{\mathcal{P},i}(s_j) \equiv 0$. Since

$$f_{\mathcal{P},i}(s_j) = \frac{C_{\mathcal{P},j}C_{\mathcal{P},j-1}\cdot\cdots\cdot C_{\mathcal{P},i+1}S_{\mathcal{P},i}}{\Delta_i},$$

so

$$|f_{\mathcal{P},i}(s_j)|^q \leq |C_{\mathcal{P},j}|^q\,|C_{\mathcal{P},j-1}|^q\cdot\cdots\cdot|C_{\mathcal{P},i+1}|^q\left|\frac{S_{\mathcal{P},i}}{\Delta_i}\right|^q.$$

Apply Lemma 6.4 and 6.6, we get

$$|f_{\mathcal{P},i}(s_j)|^q \leq e^{\frac{1}{2}qN\sum_{k=i}^{j}|\Delta_k\beta|^2}\left(e^{-\frac{N}{2}|\Delta_i\beta|^2} + \frac{N|\Delta_i\beta|^2}{6}\right)^q \tag{6.4}$$

$$\leq e^{\frac{1}{2}qN\sum_{k=i}^{j}|\Delta_k\beta|^2}\left(1 + \frac{N|\Delta_i\beta|^2}{6}\right)^q \tag{6.5}$$

$$\leq e^{\frac{1}{2}qN\sum_{k=i}^{j}|\Delta_k\beta|^2}e^{\frac{Nq|\Delta_i\beta|^2}{6}} \tag{6.6}$$

$$\leq e^{qN\sum_{k=1}^{n}|\Delta_k\beta|^2}. \tag{6.7}$$

Since $e^{qN\sum_{k=1}^{n}|\Delta_k\beta|^2}$ is independent of i and j, we have

$$\sup_{i\in\{1,\cdots,n\}}\sup_{s\in\mathcal{P}}|f_{\mathcal{P},i}(s)|^q \leq e^{qN\sum_{k=1}^{n}|\Delta_k\beta|^2}. \tag{6.8}$$

Therefore

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\}}\sup_{s\in\mathcal{P}}|f_{\mathcal{P},i}(s)|^q\right]\leq\mathbb{E}\left[e^{qN\sum_{k=1}^n|\Delta_k\beta|^2}\right] \tag{6.9}$$

$$=\Pi_{k=1}^n\mathbb{E}\left[e^{qN|\Delta_k\beta|^2}\right]=\Pi_{k=1}^n e^{\frac{qN}{n}}=e^{qN}. \tag{6.10}$$

■

Notation 6.11 *Given $n\in\mathbb{N}$ and $s\in[0,1]$, let $\underline{s}=s_{k-1}$ when $s\in[s_{k-1},s_k)$, $|\mathcal{P}|=\frac{1}{n}$ is the mesh size of the partition $\mathcal{P}$ and also let*

$$A_{\mathcal{P},k}(s):=R_{u_{\mathcal{P}}(s)}\left(\beta'_{\mathcal{P}}(s_{k-1}+),\cdot\right)\beta'_{\mathcal{P}}(s_{k-1}+).$$

Lemma 6.12 *If $q\geq1,\gamma\in\left(0,\frac{1}{2}\right)$, then*

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\},s\in[0,1]}|f_{\mathcal{P},i}(s)-f_{\mathcal{P},i}(\underline{s})|^q\right]\leq C_{q,\gamma}|\mathcal{P}|^{2q\gamma}.$$

Proof. Taylor's expansion gives

$$f_{\mathcal{P},i}(s)-f_{\mathcal{P},i}(\underline{s})=\int_{\underline{s}}^s A_{\mathcal{P},k}(r)f_{\mathcal{P},i}(r)(s-r)\,dr \tag{6.11}$$

$$=\int_{\underline{s}}^s A_{\mathcal{P},k}(r)\left(f_{\mathcal{P},i}(r)-f_{\mathcal{P},i}(\underline{r})\right)(s-r)\,dr+\int_{\underline{s}}^s A_{\mathcal{P},k}(r)f_{\mathcal{P},i}(\underline{r})(s-r)\,dr. \tag{6.12}$$

Since $|A_{\mathcal{P},k}(s)|\leq N\left|\frac{\Delta_k\beta}{\Delta_k}\right|^2$, we have

$$|f_{\mathcal{P},i}(s)-f_{\mathcal{P},i}(\underline{s})|\leq\frac{N}{\Delta_k}|\Delta_k\beta|^2\int_{\underline{s}}^s|f_{\mathcal{P},i}(r)-f_{\mathcal{P},i}(\underline{r})|\,dr+\frac{1}{2}N|\Delta_k\beta|^2\sup_{s\in\mathcal{P}}|f_{\mathcal{P},i}(s)|.$$

By Gronwall's inequality, we have:

$$\begin{aligned}\left|f_{\mathcal{P},i}(s)-f_{\mathcal{P},i}(\underline{s})\right| &\leq \frac{1}{2}N\left|\Delta_k\beta\right|^2\sup_{s\in\mathcal{P}}\left|f_{\mathcal{P},i}(s)\right|e^{\frac{N}{\Delta_k}\left|\Delta_k\beta\right|^2(s-\underline{s})}\\ &\leq \frac{1}{2}N\left|\Delta_k\beta\right|^2\sup_{s\in\mathcal{P}}\left|f_{\mathcal{P},i}(s)\right|e^{N\left|\Delta_k\beta\right|^2}\end{aligned}$$

Use estimate (6.8), we have

$$\begin{aligned}\left|f_{\mathcal{P},i}(s)-f_{\mathcal{P},i}(\underline{s})\right|^q &\leq \frac{N^q}{2^q}\left|\Delta_k\beta\right|^{2q}e^{qN\left|\Delta_k\beta\right|^2}e^{qN\sum_{j=1}^n\left|\Delta_j\beta\right|^2} &(6.13)\\ &\leq C_q\left|\mathcal{P}\right|^{2q\gamma}e^{2qN\sum_{k=1}^n\left|\Delta_k\beta\right|^2}K_\gamma^{2q}. &(6.14)\end{aligned}$$

Notice that $e^{2qN\sum_{k=1}^n\left|\Delta_k\beta\right|^2}$ has finite moments of all orders based on a computation exactly the same as (6.10) and so is K_γ^{2q} following Remark 6.8, using the Holder's inequality and we get

$$\mathbb{E}\left[\sup_{s\in[0,1]}\left|f_{\mathcal{P},i}(s)-f_{\mathcal{P},i}(\underline{s})\right|^q\right]\leq C_{q,\gamma}\left|\mathcal{P}\right|^{2q\gamma}. \tag{6.15}$$

■

Theorem 6.13 *For all $q\geq 1,\gamma\in\left(0,\frac{1}{2}\right)$ there exist a constant $C_{q,\gamma}$ such that*

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\}}\sup_{s\in[s_i,1]}\left|f_{\mathcal{P},i}(s)-\tilde{T}_s\tilde{T}_{s_i}^{-1}\right|^q\right]\leq C_{q,\gamma}\left|\mathcal{P}\right|^{\gamma q}. \tag{6.16}$$

In order to prove Theorem 6.13, we need the following results.

Lemma 6.14 *For all $q\geq 1,\gamma\in\left(0,\frac{1}{2}\right)$ there exist a constant $C_{q,\gamma}$ such that:*

$$\begin{aligned}&\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\}}\sup_{s\in\mathcal{P}/\{s_0,\cdots,s_{i-1}\}}\left|f_{\mathcal{P},i}(s)-\left(f_{\mathcal{P},i}(s_i)-\int_{s_i}^s Ric_{u_{\mathcal{P}}(\underline{r})}f_{\mathcal{P},i}(\underline{r})\,dr\right)\right|^q\right] &(6.17)\\ &\leq C_{q,\gamma}\left|\mathcal{P}\right|^{\gamma q}. &(6.18)\end{aligned}$$

Proof. For all $s_j \in \mathcal{P}$ with $j \geq i+1$ and for $k = i, \cdots, j-1$, we have

$$f_{\mathcal{P},i}(s_{k+1}) = f_{\mathcal{P},i}(s_k) + \frac{1}{\Delta_{k+1}^2}\int_{s_k}^{s_{k+1}} R_{u_{\mathcal{P}}(r)}(\Delta_{k+1}\beta, f_{\mathcal{P},i}(r))\,\Delta_{k+1}\beta\,(s_{k+1}-r)\,dr \tag{6.19}$$

$$= f_{\mathcal{P},i}(s_k) + \frac{1}{2}R_{u_{\mathcal{P}}(s_k)}(\Delta_{k+1}\beta, f_{\mathcal{P},i}(s_k))\,\Delta_{k+1}\beta + e_{i,k}$$

where

$$\begin{aligned} e_{i,k} = &\frac{1}{\Delta_{k+1}^2}\int_{s_k}^{s_{k+1}} R_{u_{\mathcal{P}}(r)}(\Delta_{k+1}\beta, f_{\mathcal{P},i}(r))\,\Delta_{k+1}\beta\,(s_{k+1}-r)\,dr \\ &- \frac{1}{\Delta_{k+1}^2}\int_{s_k}^{s_{k+1}} R_{u_{\mathcal{P}}(s_k)}(\Delta_{k+1}\beta, f_{\mathcal{P},i}(s_k))\,\Delta_{k+1}\beta\,(s_{k+1}-r)\,dr \end{aligned}$$

Since $\{f_{\mathcal{P},i}(s_j)\}_j$is adapted, by Ito's lemma, is adapted,

$$\begin{aligned} \frac{1}{2}R_{u_{\mathcal{P}}(s_k)}(\Delta_{k+1}\beta, f_{\mathcal{P},i}(s_k))\,\Delta_{k+1}\beta = &\frac{1}{2}\int_{s_k}^{s_{k+1}} R_{u_{\mathcal{P}}(s_k)}(\beta_r - \beta_{s_k}, f_{\mathcal{P},i}(s_k))\,d\beta_r \\ &+ \frac{1}{2}\int_{s_k}^{s_{k+1}} R_{u_{\mathcal{P}}(s_k)}(d\beta_r, f_{\mathcal{P},i}(s_k))\,(\beta_r - \beta_{s_k}) \\ &- \frac{1}{2}Ric_{u_{\mathcal{P}}(s_k)} f_{\mathcal{P},i}(s_k)\,\Delta_k \end{aligned}$$

Summing (6.19) over k from i to $j-1$, we have

$$f_{\mathcal{P},i}(s_j) = f_{\mathcal{P},i}(s_i) - \frac{1}{2}\int_{s_i}^{s_j} Ric_{u_{\mathcal{P}}(\underline{r})} f_{\mathcal{P},i}(\underline{r})\,dr + M_{\mathcal{P},s_j} + \sum_{k=i}^{j-1} e_{i,k}$$

where

$$M_{\mathcal{P},s} := \frac{1}{2}\int_{s_i}^{s} R_{u_{\mathcal{P}}(\underline{r})}(\beta_r - \beta_{\underline{r}}, f_{\mathcal{P},i}(\underline{r}))\,d\beta_r + \frac{1}{2}\int_{s_i}^{s} R_{u_{\mathcal{P}}(\underline{r})}(d\beta_r, f_{\mathcal{P},i}(\underline{r}))\,(\beta_r - \beta_{\underline{r}})$$

is a $\mathbb{R}^d$-valued martingale starting from s_i. By the Burkholder-Davis-Gundy inequality, for $q \geq 1$,

$$\mathbb{E}\left[\sup_{s\in[s_i,1]} |M_{\mathcal{P},s}|^q\right] \leq C_q \mathbb{E}\left[\langle M_{\mathcal{P}}\rangle_1^{\frac{q}{2}}\right] \tag{6.20}$$

where $\langle M_{\mathcal{P}}\rangle$ is the quadratic variation process of $M_{\mathcal{P}}$. An estimate of $\langle M_{\mathcal{P}}\rangle$ gives

$$\langle M_{\mathcal{P}}\rangle_1 \leq dN^2 \int_{s_i}^1 |\beta_r - \beta_{\underline{r}}|^2 |f_{\mathcal{P},i}(\underline{r})|^2 dr \leq dN^2 \int_0^1 |\beta_r - \beta_{\underline{r}}|^2 |f_{\mathcal{P},i}(\underline{r})|^2 dr,$$

and by Jensen's inequality,

$$\langle M_{\mathcal{P}}\rangle_1^{\frac{q}{2}} \leq d^{\frac{q}{2}} N^q \int_0^1 |\beta_r - \beta_{\underline{r}}|^q |f_{\mathcal{P},i}(\underline{r})|^q dr.$$

Since $\{f_{\mathcal{P},i}(\underline{r})\}_{r\in[0,1]}$ is adapted to the filtration generated by β, using the independence of $|\beta_r - \beta_{\underline{r}}|^q$ and $f_{\mathcal{P},i}(\underline{r})$ we have:

$$\begin{aligned}\mathbb{E}\left[\langle M_{\mathcal{P}}\rangle_1^{\frac{q}{2}}\right] &\leq d^{\frac{q}{2}} N^q \int_0^1 \mathbb{E}\left[|\beta_r - \beta_{\underline{r}}|^q\right] \mathbb{E}\left[|f_{\mathcal{P},i}(\underline{r})|^q\right] dr \\ &= C_q \sup_{s\in\mathcal{P}} \mathbb{E}\left[|f_{\mathcal{P},i}(s)|^q\right] |\mathcal{P}|^{\frac{q}{2}}.\end{aligned}$$

By Lemma 6.10, we know

$$\mathbb{E}\left[\langle M_{\mathcal{P}}\rangle_1^{\frac{q}{2}}\right] \leq C_q |\mathcal{P}|^{\frac{q}{2}} \tag{6.21}$$

Then to prove Lemma 6.14, it suffices to show:

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\}, j\in\{i+1,\cdots,n\}} \left|\sum_{k=i}^{j-1} e_{i,k}\right|^q\right] \leq C_q |\mathcal{P}|^{\gamma q} \tag{6.22}$$

Since $|e_{i,k}| \leq I + II$, where

$$I = \frac{1}{\Delta_{k+1}^2} \left| \int_{s_k}^{s_{k+1}} R_{u_{\mathcal{P}}(r)}\left(\Delta_{k+1}\beta, f_{\mathcal{P},i}(r) - f_{\mathcal{P},i}(s_k)\right) \Delta_{k+1}\beta\,(s_{k+1} - r)\, dr \right|$$

$$II = \frac{1}{\Delta_{k+1}^2} \left| \int_{s_k}^{s_{k+1}} \left(R_{u_{\mathcal{P}}(s_k)} - R_{u_{\mathcal{P}}(r)} \right) \left(\Delta_{k+1}\beta, f_{\mathcal{P},i}(s_k) \right) \Delta_{k+1}\beta \left(s_{k+1} - r \right) dr \right|$$

use (6.14) , we know

$$\begin{aligned} I &\le \frac{N}{2} \sup_{i \in \{1,\cdots,n\}, r \in [0,1]} \left| f_{\mathcal{P},i}(r) - f_{\mathcal{P},i}(\underline{r}) \right| \left| \Delta_{k+1}\beta \right|^2 \\ &\le C K_\gamma^4 \left| \mathcal{P} \right|^{4\gamma} e^{2N^2 \sum_{k=1}^n \left| \Delta_k \beta \right|^2} \end{aligned}$$

Since

$$\left| R_{u_{\mathcal{P}}(s_k)} - R_{u_{\mathcal{P}}(r)} \right| \le \int_{s_k}^{s_{k+1}} \left| \beta_{\mathcal{P}}'(s) \right| ds = \left| \Delta_{k+1}\beta \right| \le K_\gamma \left| \mathcal{P} \right|^\gamma$$

and use (6.8), we have

$$\begin{aligned} II &\le \frac{N}{2} \sup_{i \in \{1,\cdots,n\}, r \in \mathcal{P}} \left| f_{\mathcal{P},i}(r) \right| \left| \Delta_{k+1}\beta \right|^2 \sup_{r \in [s_k, s_{k+1}]} \left| R_{u_{\mathcal{P}}(s_k)} - R_{u_{\mathcal{P}}(r)} \right| \\ &\le C K_\gamma^3 \left| \mathcal{P} \right|^{3\gamma} e^{N^2 \sum_{k=1}^n \left| \Delta_k \beta \right|^2}. \end{aligned}$$

So

$$\left| \sum_{k=i}^{j-1} e_{i,k} \right| \le \frac{1}{|\mathcal{P}|} (I + II) \le C \left(K_\gamma^4 \left| \mathcal{P} \right|^{4\gamma - 1} + K_\gamma^3 \left| \mathcal{P} \right|^{3\gamma - 1} \right) e^{2N^2 \sum_{k=1}^n \left| \Delta_k \beta \right|^2}.$$

Since if γ approaches $\frac{1}{2}$, $3\gamma - 1$ approaches $\frac{1}{2}$, so use Lemma 6.8 we get

$$\mathbb{E} \left[\sup_{i \in \{0,\cdots,n\}, j \in \{i+1,\cdots,n\}} \left| \sum_{k=i}^{j-1} e_{i,k} \right|^q \right] \le C_q \left| \mathcal{P} \right|^{\gamma q}.$$

Combine (6.21) and (6.22) we obtain (6.18). ■

Proof of Theorem 6.13. Define

$$\hat{f}_{\mathcal{P},i}(s) = f_{\mathcal{P},i}(s_i) - \frac{1}{2} \int_{s_i}^{s} Ric_{u_{\mathcal{P}}(r)} f_{\mathcal{P},i}(r) \, dr. \tag{6.23}$$

Then

$$\left|\hat{f}_{\mathcal{P},i}(s_j) - f_{\mathcal{P},i}(s_j)\right| \leq \left|\frac{1}{2}\int_{s_i}^{s}\left(Ric_{u_{\mathcal{P}}(r)} - Ric_{u_{\mathcal{P}}(\underline{r})}\right) f_{\mathcal{P},i}(\underline{r})\, dr\right| + \left|\frac{1}{2}\int_{s_i}^{s} Ric_{u_{\mathcal{P}}(r)}\left(f_{\mathcal{P},i}(r) - f_{\mathcal{P},i}(\underline{r})\right) dr\right|.$$

Since

$$\left|Ric_{u_{\mathcal{P}}(r)} - Ric_{u_{\mathcal{P}}(\underline{r})}\right| \leq CK_\gamma \left|\mathcal{P}\right|^\gamma,$$

using Lemma 6.10 and (6.8), we know:

$$\left|\int_{s_i}^{s}\left(Ric_{u_{\mathcal{P}}(r)} - Ric_{u_{\mathcal{P}}(\underline{r})}\right) f_{\mathcal{P},i}(\underline{r})\, dr\right|^q \leq C_q K_\gamma^q \left|\mathcal{P}\right|^{\gamma q} \tag{6.24}$$

and

$$\mathbb{E}\left[\left|\int_{s_i}^{s}\left(Ric_{u_{\mathcal{P}}(r)} - Ric_{u_{\mathcal{P}}(\underline{r})}\right) f_{\mathcal{P},i}(\underline{r})\, dr\right|^q\right] \leq C_q \left|\mathcal{P}\right|^{\gamma q}.$$

Then consider

$$\left|\int_{s_i}^{s} Ric_{u_{\mathcal{P}}(r)}\left(f_{\mathcal{P},i}(r) - f_{\mathcal{P},i}(\underline{r})\right) dr\right|,$$

apply Lemma 6.12, we can easily see:

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\}}\left|\int_{s_i}^{s} Ric_{u_{\mathcal{P}}(r)}\left(f_{\mathcal{P},i}(r) - f_{\mathcal{P},i}(\underline{r})\right) dr\right|^q\right] \leq C_q \left|\mathcal{P}\right|^q \tag{6.25}$$

Combine (6.24) and (6.25) we get:

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\},j\geq i}\left|\hat{f}_{\mathcal{P},i}(s_j) - f_{\mathcal{P},i}(s_j)\right|^q\right] \leq C_q \left|\mathcal{P}\right|^{\gamma q} \tag{6.26}$$

Then define $\tilde{f}_{\mathcal{P},i}(s)$ to be the solution to the following ODE:

$$\begin{cases} \frac{d}{ds}\tilde{f}_{\mathcal{P},i}(s) + \frac{1}{2}Ric_{u_{\mathcal{P}}(s)}\tilde{f}_{\mathcal{P},i}(s) = 0 \\ \tilde{f}_{\mathcal{P},i}(s_i) = I. \end{cases}$$

Consider $\left|\tilde{f}_{\mathcal{P},i}(s) - \hat{f}_{\mathcal{P},i}(s)\right|$, since

$$\tilde{f}_{\mathcal{P},i}(s) = I - \frac{1}{2}\int_{s_i}^{s} Ric_{u_{\mathcal{P}}(r)}\tilde{f}_{\mathcal{P},i}(r)\,dr$$

so

$$\left|\tilde{f}_{\mathcal{P},i}(s) - \hat{f}_{\mathcal{P},i}(s)\right| \leq \left|f_{\mathcal{P},i}(s_i) - I\right| + \frac{1}{2}\int_{s_i}^{s} N\left|\tilde{f}_{\mathcal{P},i}(r) - \hat{f}_{\mathcal{P},i}(r)\right| dr.$$

By Gronwall's inequality, we have:

$$\left|\tilde{f}_{\mathcal{P},i}(s) - \hat{f}_{\mathcal{P},i}(s)\right| \leq \left|f_{\mathcal{P},i}(s_i) - I\right| e^{\frac{1}{2}N}.$$

Use Lemma 6.6, we have

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\},s\geq s_i}\left|\tilde{f}_{\mathcal{P},i}(s) - \hat{f}_{\mathcal{P},i}(s)\right|^q\right] \leq C_q\left|\mathcal{P}\right|^q. \tag{6.27}$$

Lastly, we look at $\tilde{f}_{\mathcal{P},i}(s) - \tilde{T}_s\tilde{T}_{s_i}^{-1}$. Notice that $\tilde{T}_s\tilde{T}_{s_i}^{-1}$ satisfies the following ODE,

$$\begin{cases} \left(\tilde{T}_s\tilde{T}_{s_i}^{-1}\right)' + \frac{1}{2}Ric_{\tilde{u}_s}\left(\tilde{T}_s\tilde{T}_{s_i}^{-1}\right) = 0 \\ \left(\tilde{T}_{s_i}\tilde{T}_{s_i}^{-1}\right) = I. \end{cases}$$

So

$$\tilde{f}_{\mathcal{P},i}(s) - \tilde{T}_s\tilde{T}_{s_i}^{-1} = \frac{1}{2}\int_{s_i}^{s}\left(Ric_{u_{\mathcal{P}}(r)} - Ric_{\tilde{u}_r}\right)\left(\tilde{f}_{\mathcal{P},i}(r) - \tilde{T}_r\tilde{T}_{s_i}^{-1}\right) dr.$$

By Gronwall's inequality again we have:

$$\left|\tilde{f}_{\mathcal{P},i}(s)-\tilde{T}_s\tilde{T}_{s_i}^{-1}\right|\leq CK_\gamma\left|\mathcal{P}\right|^\gamma e^{\frac{1}{2}N}$$

so

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\},s\geq s_i}\left|\tilde{f}_{\mathcal{P},i}(s)-\tilde{T}_s\tilde{T}_{s_i}^{-1}\right|^q\right]\leq C_q\left|\mathcal{P}\right|^{\gamma q} \tag{6.28}$$

Combine Lemma 6.14 and (6.25), (6.26), (6.27) and (6.28) we prove this theorem.
■

6.2.3 Convergence of $\mathbf{K}_{\mathcal{P}}(s)$ to $\tilde{\mathbf{K}}_s$

Recall from Definition 5.4 that $\mathbf{K}_{\mathcal{P}}(s)$ satisfies the piecewise Jacobi equation:

$$\begin{cases}\mathbf{K}''_{\mathcal{P}}(s)=R_{u_{\mathcal{P}}(s)}\left(\beta'_{\mathcal{P}}(s_{i-1}+),\mathbf{K}_{\mathcal{P}}(s)\right)\beta'_{\mathcal{P}}(s_{i-1}+)\ \text{ for } s\in[s_{i-1},s_i) \\ \mathbf{K}'_{\mathcal{P}}(s_{i-1}+)=f^*_{\mathcal{P},i}(1)\ \text{ and } \mathbf{K}_{\mathcal{P}}(0)=0\end{cases}\quad \text{for } i=1,...,n \tag{6.29}$$

Before we state the main theorem in this section , first we need some supplementary lemmas.

Lemma 6.15 *For all $q\geq 1$, there exists a constant C_q such that*

$$\mathbb{E}\left[\sup_{r\in\mathcal{P}}\left|\mathbf{K}_{\mathcal{P}}(r)\right|^q\right]\leq C_q$$

Proof. For all $i\in\{1,\cdots,n\}$, recall from (5.4) that:

$$\mathbf{K}_{\mathcal{P}}(s_i)=\frac{1}{n}\sum_{j=0}^{i-1}f_{\mathcal{P},j+1}(s)\,f^*_{\mathcal{P},j+1}(1)$$

So for all $q \geq 1$, we have:

$$|\mathbf{K}_{\mathcal{P}}(s_i)|^q \leq i^{q-1}\frac{1}{n^q}\sum_{j=0}^{i-1}|f_{\mathcal{P},j+1}(s_i)|^q\,|f_{\mathcal{P},j+1}(1)|^q$$

apply (6.8), we have:

$$|\mathbf{K}_{\mathcal{P}}(s_i)|^q \leq e^{2qN^2\sum_{k=1}^{n}|\Delta_k\beta|^2} \tag{6.30}$$

then take expectation and we are done. ∎

Lemma 6.16 *For all $q \geq 1$, there exists a constant $C_q > 0$ such that:*

$$\mathbb{E}\left[\sup_{i\in\{1,\cdots,n\},r\in[0,1]}|\mathbf{K}_{\mathcal{P}}(r)-\mathbf{K}_{\mathcal{P}}(\underline{r})|^q\right] \leq C_q\,|\mathcal{P}|^{2q\gamma}$$

For $s \in [s_{i-1}, s_i]$,

$$\mathbf{K}_{\mathcal{P}}(s) = \mathbf{K}_{\mathcal{P}}(s_{i-1}) \tag{6.31}$$

$$+ f^{*}_{\mathcal{P},i}(1)(s-s_{i-1}) + \int_{s_{i-1}}^{s} R_{u_{\mathcal{P}}(s)}\left(\beta'_{\mathcal{P}}(s_{i-1}+),\mathbf{K}_{\mathcal{P}}(r)\right)\beta'_{\mathcal{P}}(s_{i-1}+)(s-r)\,dr. \tag{6.32}$$

Therefore

$$|\mathbf{K}_{\mathcal{P}}(s)-\mathbf{K}_{\mathcal{P}}(s_{i-1})| \tag{6.33}$$

$$\leq |f_{\mathcal{P},i}(1)|(s-s_{i-1}) \tag{6.34}$$

$$+\left|\int_{s_{i-1}}^{s} R_{u_{\mathcal{P}}(s)}\left(\beta'_{\mathcal{P}}(s_{i-1}+),\mathbf{K}_{\mathcal{P}}(r)-\mathbf{K}_{\mathcal{P}}(s_{i-1})+\mathbf{K}_{\mathcal{P}}(s_{i-1})\right)\beta'_{\mathcal{P}}(s_{i-1}+)(s-r)\,dr\right|$$

$$\leq |f_{\mathcal{P},i}(1)|(s-s_{i-1}) \tag{6.35}$$

$$+N\frac{|\Delta_i\beta|^2}{\Delta_i^2}\int_{s_{i-1}}^{s}|\mathbf{K}_{\mathcal{P}}(r)-\mathbf{K}_{\mathcal{P}}(s_{i-1})|(s-r)\,dr+\frac{1}{2}N|\Delta_i\beta|^2|\mathbf{K}_{\mathcal{P}}(s_{i-1})| := f(s),$$

where

$$f'(s) = |f_{\mathcal{P},i}(1)| + N\frac{|\Delta_i\beta|^2}{\Delta_i^2}\int_{s_{i-1}}^{s}|\mathbf{K}_{\mathcal{P}}(r) - \mathbf{K}_{\mathcal{P}}(s_{i-1})|\,dr$$

and

$$f''(s) = N\frac{|\Delta_i\beta|^2}{\Delta_i^2}|\mathbf{K}_{\mathcal{P}}(s) - \mathbf{K}_{\mathcal{P}}(s_{i-1})| \le N\frac{|\Delta_i\beta|^2}{\Delta_i^2}f(s).$$

Then $f(s)$ satisfies the following ODE

$$\begin{cases} f''(s) = N\frac{|\Delta_i\beta|^2}{\Delta_i^2}f(s) + \delta(s) \\ f'(s_{i-1}) = |f_{\mathcal{P},i}(1)| \\ f(s_{i-1}) = \frac{1}{2}N|\Delta_i\beta|^2|\mathbf{K}_{\mathcal{P}}(s_{i-1})| \end{cases} \tag{6.36}$$

where

$$\delta(s) = f''(s) - N\frac{|\Delta_i\beta|^2}{\Delta_i^2}f(s) \le 0.$$

Solving (6.36) , we have:

$$\begin{aligned} f(s) &= \mathcal{C}_i(s)\frac{1}{2}N|\Delta_i\beta|^2|\mathbf{K}_{\mathcal{P}}(s_{i-1})| + \mathcal{S}_i(s)|f_{\mathcal{P},i}(1)| \\ &\quad + \int_{s_{i-1}}^{s}\frac{\sinh\left(\sqrt{N}|\beta'_{\mathcal{P}}(s_{i-1}+)|(s-r)\right)}{\sqrt{N}|\beta'_{\mathcal{P}}(s_{i-1}+)|}\delta(r)\,dr \\ &\le \mathcal{C}_i(s)\frac{1}{2}N|\Delta_i\beta|^2|\mathbf{K}_{\mathcal{P}}(s_{i-1})| + \mathcal{S}_i(s)|f_{\mathcal{P},i}(1)| \end{aligned}$$

where

$$\mathcal{C}_i(s) := \cosh\left(\sqrt{N}|\beta'_{\mathcal{P}}(s_{i-1}+)|(s-s_{i-1})\right)$$

and

$$\mathcal{S}_i(s) := \frac{\sinh\left(\sqrt{N}|\beta'_{\mathcal{P}}(s_{i-1}+)|(s-s_{i-1})\right)}{\sqrt{N}|\beta'_{\mathcal{P}}(s_{i-1}+)|}.$$

Using the following estimate

$$\frac{\sinh\left(\sqrt{N}\left|\beta_{\mathcal{P}}'\left(s_{i-1}+\right)\right|\left(s-s_{i-1}\right)\right)}{\sqrt{N}\left|\beta_{\mathcal{P}}'\left(s_{i-1}+\right)\right|\Delta_i}$$
$$\leq \cosh\left(\sqrt{N}\left|\beta_{\mathcal{P}}'\left(s_{i-1}+\right)\right|\left(s-s_{i-1}\right)\right)\frac{\left(s-s_{i-1}\right)}{\Delta_i} \leq e^{N\left|\Delta_i\beta\right|^2}, \tag{6.37}$$

we obtain

$$f\left(s\right) \leq e^{N\left|\Delta_i\beta\right|^2}\left(\frac{1}{2}N\left|\Delta_i\beta\right|^2\left|\mathbf{K}_{\mathcal{P}}\left(s_{i-1}\right)\right| + \left|\mathcal{P}\right|\left|f_{\mathcal{P},i}\left(1\right)\right|\right) \tag{6.38}$$
$$\leq e^{NK_\gamma^2\left|\mathcal{P}\right|^{2\gamma}}\left(\frac{1}{2}NK_\gamma^2\left|\mathcal{P}\right|^{2\gamma}\sup_{i\in\{1,\cdots,n\}}\left|\mathbf{K}_{\mathcal{P}}\left(s_{i-1}\right)\right| + \left|\mathcal{P}\right|\sup_{i\in\{1,\cdots,n\},s\in[0,1]}\left|f_{\mathcal{P},i}\left(s\right)\right|\right).$$

Then apply (6.8)and (6.30), we get

$$f\left(s\right) \leq U_q\left|P\right|^{2\gamma},$$

where

$$U_q = e^{NK_\gamma^2\left|\mathcal{P}\right|^{2\gamma}}\left(\frac{1}{2}NK_\gamma^2 + \left|\mathcal{P}\right|^{1-2\gamma}\right)e^{N^2\sum_{k=1}^{n}\left|\Delta_k\beta\right|^2}$$

is a random variable with finite moments of all orders. Therefore,

$$\mathbb{E}\left[\sup_{i\in\{1,\cdots,n\},r\in[0,1]}\left|\mathbf{K}_{\mathcal{P}}\left(r\right) - \mathbf{K}_{\mathcal{P}}\left(\underline{r}\right)\right|^q\right] \leq C_q\left|\mathcal{P}\right|^{2q\gamma}$$

Remark 6.17 *Gronwall's inequality gives the control of same order.*

Lemma 6.18 *For all $q \geq 1, \gamma \in \left(0, \frac{1}{2}\right)$, there exists a constant $C_{q,\gamma}$ such that*

$$\mathbb{E}\left[\sup_{s\in\mathcal{P}}\left|\mathbf{K}_{\mathcal{P}}\left(s\right) - \mathbf{K}_s\right|^q\right] \leq C_{q,\gamma}\left|\mathcal{P}\right|^q \tag{6.39}$$

Proof. Rewrite, for all $i \in \{1, \cdots, n\}$:

$$\mathbf{K}_{\mathcal{P}}(s_i) = f_{\mathcal{P},i-1}(s_i)\, f_{\mathcal{P},i-1}(1)^{-1} \left(\sum_{j=0}^{i-1} f_{\mathcal{P},j+1}(1)\, f^*_{\mathcal{P},j+1}(1) \right) |\mathcal{P}| \tag{6.40}$$

and

$$\mathbf{K}_{s_i} = \tilde{T}_{s_i} \tilde{T}_1^{-1} \int_0^{s_i} \left(\tilde{T}_1 \tilde{T}_r^{-1} \right) \left(\tilde{T}_1 \tilde{T}_r^{-1} \right)^* dr$$

First define

$$\tilde{\mathbf{K}}_{s_i} := \tilde{T}_{s_i} \tilde{T}_1 \tilde{T}_1^{-1} \int_0^{s_i} \left(\tilde{T}_1 T_{\overline{r}}^{-1} \right) \left(\tilde{T}_1 T_{\overline{r}}^{-1} \right)^* dr,$$

we will show, for all $q \geq 1$,

$$\sup_{s \in \mathcal{P}} \left| \tilde{\mathbf{K}}_s - \mathbf{K}_s \right|^q \leq C_q |\mathcal{P}|^q \tag{6.41}$$

Recall from (4.7) that $\tilde{T}_1 \tilde{T}_r^{-1}$ satisfies the following ODE,

$$\frac{d}{dr} \left(\tilde{T}_1 \tilde{T}_r^{-1} \right) = \frac{1}{2} \left(\tilde{T}_1 \tilde{T}_r^{-1} \right) Ric_{\tilde{u}_r}.$$

So by Lemma 4.8,

$$\left| \frac{d}{dr} \left(\tilde{T}_1 \tilde{T}_r^{-1} \right) \right| \leq N \left| \tilde{T}_1 \tilde{T}_r^{-1} \right| \leq N$$

Therefore

$$\begin{aligned} \left| \left(\tilde{T}_1 \tilde{T}_r^{-1} \right) \left(\tilde{T}_1 \tilde{T}_r^{-1} \right)^* - \left(\tilde{T}_1 \tilde{T}_{\overline{r}}^{-1} \right) \left(\tilde{T}_1 \tilde{T}_{\overline{r}}^{-1} \right)^* \right| &\leq \int_r^{\overline{r}} \left| \frac{d}{ds} \left[\left(\tilde{T}_1 \tilde{T}_s^{-1} \right) \left(\tilde{T}_1 \tilde{T}_s^{-1} \right)^* \right] \right| ds \\ &\leq 2 \int_r^{\overline{r}} \left| \frac{d}{ds} \left(\tilde{T}_1 \tilde{T}_s^{-1} \right) \right| \left| \left(\tilde{T}_1 \tilde{T}_s^{-1} \right)^* \right| ds \\ &\leq C \left(\overline{r} - r \right) \\ &\leq C |\mathcal{P}| . \end{aligned}$$

So

$$\left|\tilde{\mathbf{K}}_{s_i} - \mathbf{K}_{s_i}\right| \leq \left|\tilde{T}_{s_i}\tilde{T}_1^{-1}\right| \int_0^{s_i} \left|\left(\tilde{T}_1\tilde{T}_r^{-1}\right)\left(\tilde{T}_1\tilde{T}_r^{-1}\right)^* - \left(\tilde{T}_1\tilde{T}_{\overline{r}}^{-1}\right)\left(\tilde{T}_1\tilde{T}_{\overline{r}}^{-1}\right)^*\right| dr$$
$$\leq C\left|\mathcal{P}\right|.$$

Since the right hand side is independent of i, we proved (6.41). Secondly, define:

$$\hat{\mathbf{K}}_{s_i} := \tilde{T}_{s_i}\tilde{T}_1^{-1}\left(\sum_{j=0}^{i-1} f_{\mathcal{P},j+1}(1)\, f_{\mathcal{P},j+1}^*(1)\right)\left|\mathcal{P}\right|$$

and we are about to show, for all $q \geq 1, \gamma \in \left(0, \frac{1}{2}\right)$, there exists a constant $C_{q,\gamma} > 0$ such that:

$$\mathbb{E}\left[\sup_{s\in\mathcal{P}}\left|\hat{\mathbf{K}}_s - \tilde{\mathbf{K}}_s\right|^q\right] \leq C_{q,\gamma}\left|\mathcal{P}\right|^{q\gamma} \tag{6.42}$$

for all $j \in \{1, \cdots, n\}$,

$$\begin{aligned}
&\left|f_{\mathcal{P},j+1}(1)\, f_{\mathcal{P},j+1}^*(1) - \left(\tilde{T}_1\tilde{T}_{s_{j+1}}^{-1}\right)\left(\tilde{T}_1\tilde{T}_{s_{j+1}}^{-1}\right)^*\right| \\
&\leq \left|f_{\mathcal{P},j+1}(1)\, f_{\mathcal{P},j+1}^*(1) - f_{\mathcal{P},j+1}(1)\left(\tilde{T}_1\tilde{T}_{s_{j+1}}^{-1}\right)^*\right| \\
&+ \left|f_{\mathcal{P},j+1}(1)\left(\tilde{T}_1\tilde{T}_{s_{j+1}}^{-1}\right)^* - \left(\tilde{T}_1\tilde{T}_{s_{j+1}}^{-1}\right)\left(\tilde{T}_1\tilde{T}_{s_{j+1}}^{-1}\right)^*\right| \\
&\leq \left(\left|f_{\mathcal{P},j+1}(1)\right| + \left|\tilde{T}_1\tilde{T}_{s_{j+1}}^{-1}\right|\right)\left|f_{\mathcal{P},j+1}(1) - \tilde{T}_1\tilde{T}_{s_{j+1}}^{-1}\right|
\end{aligned}$$

by (6.8),

$$\left|f_{\mathcal{P},j+1}(1)\right| \leq e^{N^2\sum_{k=1}^n\left|\Delta_k\beta\right|^2}$$

also $\left|\tilde{T}_1\tilde{T}_{s_{j+1}}^{-1}\right| \leq 1$, so

$$\begin{aligned}
&\left|f_{\mathcal{P},j+1}(1)\, f_{\mathcal{P},j+1}^*(1) - \left(\tilde{T}_1\tilde{T}_{s_{j+1}}^{-1}\right)\left(\tilde{T}_1\tilde{T}_{s_{j+1}}^{-1}\right)^*\right| \\
&\leq \left(e^{N^2\sum_{k=1}^n\left|\Delta_k\beta\right|^2} + 1\right)\sup_{j\in\{1,\cdots,n\}}\left|f_{\mathcal{P},j+1}(1) - \tilde{T}_1\tilde{T}_{s_{j+1}}^{-1}\right|
\end{aligned}$$

and for all $i \in \{1, \cdots, n\}$

$$\begin{aligned}\left|\hat{\mathbf{K}}_{s_i} - \tilde{\mathbf{K}}_{s_i}\right| &\leq |\mathcal{P}| \sum_{j=0}^{i-1} \left| f_{\mathcal{P},j+1}(1) f_{\mathcal{P},j+1}^*(1) - \left(\tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1}\right)\left(\tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1}\right)^*\right| \\ &\leq \left(e^{N^2 \sum_{k=1}^n |\Delta_k \beta|^2} + 1\right)^q \sup_{j \in \{1,\cdots,n\}} \left| f_{\mathcal{P},j+1}(1) - \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1}\right|^q .\end{aligned}$$

Then use Holder's inequality and Lemma 6.12, we get:

$$\mathbb{E}\left[\sup_{s \in \mathcal{P}} \left|\hat{\mathbf{K}}_s - \tilde{\mathbf{K}}_s\right|^q\right] \leq C_{q,\gamma} |\mathcal{P}|^{q\gamma}$$

Lastly, consider $\hat{\mathbf{K}}_{s_i} - \mathbf{K}_{\mathcal{P}}(s_i)$. Use (6.40) we have

$$\begin{aligned}&\left|\hat{\mathbf{K}}_{s_i} - \mathbf{K}_{\mathcal{P}}(s_i)\right| \\ &\leq \left| f_{\mathcal{P},i-1}(s_i) f_{\mathcal{P},i-1}(1)^{-1} - \tilde{T}_{s_i} \tilde{T}_1^{-1}\right| \left|\left(\sum_{j=0}^{i-1} f_{\mathcal{P},j+1}(1) f_{\mathcal{P},j+1}^*(1)\right)|\mathcal{P}|\right| \\ &\leq \left| f_{\mathcal{P},i-1}(s_i) f_{\mathcal{P},i-1}(1)^{-1} - \tilde{T}_{s_i} \tilde{T}_1^{-1}\right| \sup_{j \in \{1,\cdots,n\}} |f_{\mathcal{P},j+1}(1)|^2\end{aligned}$$

Notice that:

$$\begin{aligned}&\left| f_{\mathcal{P},i-1}(s_i) f_{\mathcal{P},i-1}(1)^{-1} - \tilde{T}_{s_i} \tilde{T}_1^{-1}\right| \\ &\quad = \left| f_{\mathcal{P},i-1}(s_i) - \tilde{T}_{s_i} \tilde{T}_{s_{i-1}}^{-1}\right| \left| f_{\mathcal{P},i-1}(1)^{-1}\right| \\ &\qquad + \left|\tilde{T}_{s_i} \tilde{T}_{s_{i-1}}^{-1}\right| \left|\left(\tilde{T}_1 \tilde{T}_{s_{i-1}}^{-1}\right)^{-1} - f_{\mathcal{P},i-1}(1)^{-1}\right| .\end{aligned}$$

From Lemma 5.5, we know $\left|f_{\mathcal{P},i-1}(1)^{-1}\right| \leq 1$, and

$$\begin{aligned}
&\left|\left(\tilde{T}_1\tilde{T}_{s_{i-1}}^{-1}\right)^{-1} - f_{\mathcal{P},i-1}(1)^{-1}\right| \\
&\leq \left|\left(\tilde{T}_1\tilde{T}_{s_{i-1}}^{-1}\right)^{-1}\right| \left|\tilde{T}_1\tilde{T}_{s_{i-1}}^{-1} - f_{\mathcal{P},i-1}(1)\right| \left|f_{\mathcal{P},i-1}(1)^{-1}\right| \\
&\leq \left|\tilde{T}_1\tilde{T}_{s_{i-1}}^{-1} - f_{\mathcal{P},i-1}(1)\right|.
\end{aligned}$$

So

$$\left|f_{\mathcal{P},i-1}(s_i) f_{\mathcal{P},i-1}(1)^{-1} - \tilde{T}_{s_i}\tilde{T}_1^{-1}\right| \leq 2 \sup_{1\leq i,j\leq n} \left|\tilde{T}_{s_j}\tilde{T}_{s_i}^{-1} - f_{\mathcal{P},i}(s_j)\right|.$$

Then apply Lemma 6.13 and 6.10 and use Holder's inequality, we get

$$\mathbb{E}\left[\sup_{s\in\mathcal{P}}\left|\hat{\mathbf{K}}_s - \mathbf{K}_{\mathcal{P}}(s)\right|^q\right] \leq C_{q,\gamma}|\mathcal{P}|^{q\gamma} \tag{6.43}$$

Combine (6.41),(6.42) and (6.43) we prove Lemma 6.18. ■

Lemma 6.19 *For all $q \geq 1$, there exists a constant $C_q > 0$ such that*

$$\sup_{s\in[0,1]} |\mathbf{K}_{\underline{s}} - \mathbf{K}_s|^q \leq C_q |\mathcal{P}|^q$$

Proof. By the fundamental theorem of calculus, we have:

$$\mathbf{K}_s = -\frac{1}{2}\int_0^s Ric_{\tilde{u}_r}\mathbf{K}_r dr + \int_0^s \left(\tilde{T}_1\tilde{T}_r^{-1}\right)^* dr$$

use Lemma 4.8 and boundness of Ric, we have

$$|\mathbf{K}_s| \leq N\int_0^s |\mathbf{K}_r|\, dr + C$$

where C and N are two constants independent of s. Then apply the Gronwall's inequality, we get:

$$|\mathbf{K}_s| \leq Ce^{Ns} \leq Ce^N \tag{6.44}$$

so $\sup_{s\in[0,1]} |\mathbf{K}_s|$ is bounded. Then use the fundamental theorem of calculus again from $\underline{s}$ to s, we have:

$$\begin{aligned}\mathbf{K}_s - \mathbf{K}_{\underline{s}} &= -\frac{1}{2}\int_{\underline{s}}^{s} Ric_{\tilde{u}_r}\mathbf{K}_r dr + \int_{\underline{s}}^{s}\left(\tilde{T}_1\tilde{T}_r^{-1}\right)^* dr \\ &= -\frac{1}{2}\int_{\underline{s}}^{s} Ric_{\tilde{u}_r}\left(\mathbf{K}_r - \mathbf{K}_{\underline{r}}\right) dr + \int_{\underline{s}}^{s}\left(\tilde{T}_1\tilde{T}_r^{-1}\right)^* dr \\ &\quad + \frac{1}{2}\int_{\underline{s}}^{s} Ric_{\tilde{u}_r}\mathbf{K}_{\underline{r}} dr\end{aligned}$$

so

$$|\mathbf{K}_s - \mathbf{K}_{\underline{s}}| \leq \frac{N}{2}\int_{\underline{s}}^{s} |\mathbf{K}_r - \mathbf{K}_{\underline{r}}|\, dr + C\,|\mathcal{P}|$$

By Gronwall's inequality, we have

$$|\mathbf{K}_s - \mathbf{K}_{\underline{s}}| \leq C\,|\mathcal{P}|\, e^{\frac{N}{2}}$$

Therefore:

$$\sup_{s\in[0,1]} |\mathbf{K}_{\underline{s}} - \mathbf{K}_s|^q \leq C_q\,|\mathcal{P}|^q$$

■

Theorem 6.20 *For all* $q \geq 1, \gamma \in \left(0, \frac{1}{2}\right)$ *there* $\exists$ *constant* $C_{q,\gamma}$ *(independent of* i*), such that:*

$$\mathbb{E}\left[\sup_{s\in[0,1]} |\mathbf{K}_s - \mathbf{K}_{\mathcal{P}}(s)|^q\right] \leq C_{q,\gamma}\,|\mathcal{P}|^{\gamma q} \tag{6.45}$$

Proof. Notice that for all $s \in [0,1]$, $s \in [s_{i-1}, s_i]$ for some $i \in \{1, \cdots, n\}$, so:

$$\begin{aligned}|\mathbf{K}_{\mathcal{P}}(s) - \mathbf{K}_s| &\leq |\mathbf{K}_{\mathcal{P}}(s) - \mathbf{K}_{\mathcal{P}}(s_{i-1})| \\ &\quad + \left|\mathbf{K}_{\mathcal{P}}(s_{i-1}) - \mathbf{K}_{s_{i-1}}\right| + \left|\mathbf{K}_{s_{i-1}} - \mathbf{K}_s\right|\end{aligned}$$

then apply Lemma 6.16, 6.18 and 6.19 we prove Theorem 6.20 ■

6.2.4 Convergence of $J_{\mathcal{P}}(s)$ to $\tilde{J}_s$

Recall that

$$J_{\mathcal{P}}(s) := \mathbf{K}_{\mathcal{P}}(s)\,\mathbf{K}_{\mathcal{P}}(1)^{-1}\,u_{\mathcal{P}}(1)^{-1}\,X(\pi\circ u_{\mathcal{P}}(1)) = \mathbf{K}_{\mathcal{P}}(s)\,\mathbf{K}_{\mathcal{P}}(1)^{-1}\,H_{\mathcal{P}} \quad (6.46)$$

where $H_{\mathcal{P}}: W_o(M)\to\mathbb{R}^d$ is given by

$$H_{\mathcal{P}}(\sigma) = u_{\mathcal{P}}(\sigma,1)^{-1}\,X(\pi\circ u_{\mathcal{P}}(\sigma,1))$$

and everything is interpreted following the Notation 3.14.

Proposition 6.21 *Let $\tilde{J}_s$ be as in Definition 4.17, then*

$$\sup_{s\in[0,1]}\left|J_{\mathcal{P}}(s)-\tilde{J}_s\right| \to 0 \text{ in } L^{\infty-}(W_o(M)).$$

Proof. We have,

$$\left|J_{\mathcal{P}}(s)-\tilde{J}_s\right| \le I+II+III,$$

where

$$\begin{aligned} I &= |\mathbf{K}_s-\mathbf{K}_{\mathcal{P}}(s)|\left|\mathbf{K}_{\mathcal{P}}(1)^{-1}\right||H_{\mathcal{P}}| \\ II &= |\mathbf{K}_s|\left|\mathbf{K}_{\mathcal{P}}(1)^{-1}-\mathbf{K}_1^{-1}\right||H_{\mathcal{P}}| \\ III &= |\mathbf{K}_s|\left|\mathbf{K}_1^{-1}\right|\left|H_{\mathcal{P}}-\tilde{H}\right| \end{aligned}$$

For I: Since X has compact support, $|H_{\mathcal{P}}(\sigma)|$ is bounded. From Lemma 5.5, $\left|\mathbf{K}_{\mathcal{P}}(1)^{-1}\right|\le 1$. Then combine Theorem 6.20, we have:

$$E[I^q] \le C_{q,\gamma}\,|\mathcal{P}|^{q\gamma} \quad (6.47)$$

For II : Notice that

$$\mathbf{K}_{\mathcal{P}}(1)^{-1} - \mathbf{K}_1^{-1} = \mathbf{K}_{\mathcal{P}}(1)^{-1}(\mathbf{K}_1 - \mathbf{K}_{\mathcal{P}}(1))\mathbf{K}_1^{-1},$$

so:

$$\begin{aligned} II &\leq |\mathbf{K}_s| \left|\mathbf{K}_{\mathcal{P}}(1)^{-1}\right| |\mathbf{K}_1 - \mathbf{K}_{\mathcal{P}}(1)| \left|\mathbf{K}_1^{-1}\right| |H_{\mathcal{P}}| \\ &\leq C \sup_{s\in[0,1]} |\mathbf{K}_s| |\mathbf{K}_1 - \mathbf{K}_{\mathcal{P}}(1)|. \end{aligned}$$

Recall that (6.44) gives the boundness of $\sup_{s\in[0,1]} |\mathbf{K}_s|$ and use Theorem 6.20 again we have:

$$\mathbb{E}[II^q] \leq C_{q,\gamma} |\mathcal{P}|^{q\gamma} \tag{6.48}$$

For III: Since $\sigma_{\mathcal{P}} \to \sigma$(Wrong) pointwise and H is bounded, so

$$III \to 0 \text{ in } L^q(W) \tag{6.49}$$

Combine (6.47),(6.48) and (6.49) we prove this proposition. ■

6.3 Convergence of $\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1}$ to $\left(\tilde{X}\right)^{tr,\nu}$

Recall

$$\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} = -\tilde{X}_{\mathcal{P}} + \int_0^1 \langle J'_{\mathcal{P}}(s+), d\beta_{\mathcal{P},s}\rangle + div\tilde{X}_{\mathcal{P}}$$

and

$$\left(\tilde{X}\right)^{tr,\nu} = -\tilde{X} + \sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1}\right)^* e_\alpha, d\beta_s \right\rangle - \sum_{\alpha=1}^{d} \left\langle X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right), e_\alpha \right\rangle.$$

Notation 6.22 $\mathcal{FC}_b^\infty := \{f \in \mathcal{FC}^\infty : f$ *is bounded* $\}$.

Theorem 6.23 *For any* $f \in \mathcal{FC}_b^\infty$, $\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f_{\mathcal{P}} - \tilde{X}^{tr,\nu} f \to 0$ *in* $L^{\infty-}(W_o(M))$.

Proof. Because of Proposition 6.21, $J_{\mathcal{P}} - Z_{\Phi} \to 0$ uniformly in $L^{\infty-}(W_o(M))$, so $\tilde{X}_{\mathcal{P}} f_{\mathcal{P}} - \tilde{X} f \to 0$ in $L^{\infty-}(W_o(M))$. So it suffices to prove the following Proposition. ■

Proposition 6.24 *Keeping the notation above,*

$$\int_0^1 \langle J'_{\mathcal{P}}(s+), d\beta_{\mathcal{P},s}\rangle - \sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1}\right)^* e_\alpha, d\beta_s \right\rangle \to 0 \tag{6.50}$$

in $L^{\infty-}(W_o(M))$.

Definition 6.25 $f : H_{\mathcal{P}}(M) \mapsto \mathbb{R}$ *is called a smooth cylinder function on* $H_{\mathcal{P}}(M)$ *if there exists a partition*

$$\mathcal{P} := \{0 < s_1 < \cdots < s_n \leq 1\}$$

of $[0,1]$ *and a smooth function* $F : \mathcal{O}(M)^n \to \mathbb{R}$ *such that:*

$$f(\sigma) = F(u_{s_1}(\sigma), u_{s_2}(\sigma), \ldots, u_{s_n}(\sigma))$$

Denote this space by $\mathcal{FC}_{\mathcal{P}}^{\infty}$.

Proposition 6.26 *Continuing the notation above,*

$$div\tilde{X}_{\mathcal{P}} - \sum_{\alpha=1}^{d} \left\langle -X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right), e_\alpha \right\rangle \to 0 \text{ in } L^{\infty-}(W_o(M)).$$

Proof of lemma 6.24.

$$\begin{aligned}\int_0^1 \langle J'_{\mathcal{P}}(s+), d\beta_{\mathcal{P},s}\rangle &= \sum_{i=1}^n \left\langle \frac{J_{\mathcal{P}}(s_i) - J_{\mathcal{P}}(s_{i-1})}{\Delta_i}, \Delta_i\beta \right\rangle \\ &= \sum_{i=1}^n \langle J'_{\mathcal{P}}(s_{i-1}), \Delta_i\beta\rangle + \sum_{i=1}^n \left\langle \int_{s_{i-1}}^{s_i} J''_{\mathcal{P}}(s)(s - s_{i-1})\, ds, \Delta_i\beta \right\rangle \\ &= I + II\end{aligned}$$

where

$$I = \sum_{i=1}^n \langle J'_{\mathcal{P}}(s_{i-1}), \Delta_i\beta\rangle$$

$$\begin{aligned} II &= \sum_{i=1}^n \left\langle \int_{s_{i-1}}^{s_i} J''_{\mathcal{P}}(s)(s - s_{i-1})\, ds, \Delta_i\beta \right\rangle \\ &= \sum_{i=1}^n \left\langle \frac{1}{\Delta_i^2} \int_{s_{i-1}}^{s_i} R_{u_{\mathcal{P}}}(\Delta_i\beta, J_{\mathcal{P}}(s))\, \Delta_i\beta\, (s - s_{i-1})\, ds, \Delta_i\beta \right\rangle. \end{aligned} \tag{6.51}$$

Since the curvature tensor is anti-symmertric, $II = 0$.

$$\begin{aligned} I &= \sum_{i=1}^n \left\langle f^*_{\mathcal{P},i}(1)\, \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}}, \Delta_i\beta \right\rangle \\ &= \sum_{i=1}^n \left\langle \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}}, f_{\mathcal{P},i}(1)\, \Delta_i\beta \right\rangle \\ &= \left\langle \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}}, \sum_{i=1}^n f_{\mathcal{P},i}(1)\, \Delta_i\beta \right\rangle \end{aligned}$$

For all $i \geq 1, s \in [s_{i-1}, s_i]$, define $g_i(s) = S_i(s) - C_i(s) S_{i-1}$ then Taylor's expansion of g_i at s_{i-1} gives:

$$g_i(s) = -S_{i-1} + (s - s_{i-1}) I + \int_{s_{i-1}}^{s} R_{u_{\mathcal{P}(r)}}(\beta'_{\mathcal{P}}(r), g_i(r))\, \beta'_{\mathcal{P}}(r)\, (s - r)\, dr$$

so

$$|g_i(s)| \leq |S_{i-1} - (s - s_{i-1}) I| + N |\beta'_{\mathcal{P}}(s_{i-1})|^2 \int_{s_{i-1}}^{s} |g_i(r)| (s - r) \, dr.$$

By Gronwall's inequality and Lemma 6.6, we have:

$$|S_i - C_i S_{i-1}| = |g_i(s_i)| \leq \frac{N}{6} K_\gamma^2 |\mathcal{P}|^{2\gamma+1} e^{\frac{1}{2} N |\Delta_i \beta|^2}$$

Therefore, by Lemma 6.4,

$$|f_{\mathcal{P},i}(1) - f_{\mathcal{P},i-1}(1)| \leq \frac{1}{|\mathcal{P}|} |C_n| \cdot \cdots \cdot |C_{i+1}| \cdot |S_i - C_i S_{i-1}| \leq \frac{N}{6} K_\gamma^2 |\mathcal{P}|^{2\gamma} e^{\sum_{i=1}^{n} N |\Delta_i \beta|^2}$$

and

$$\begin{aligned} \left| \sum_{i=1}^{n} f_{\mathcal{P},i}(1) \Delta_i \beta - \sum_{i=1}^{n} f_{\mathcal{P},i-1}(1) \Delta_i \beta \right|^q &\leq |\mathcal{P}|^{1-q} \left[\sum_{i=1}^{n} |f_{\mathcal{P},i}(1) - f_{\mathcal{P},i-1}(1)|^q |\Delta_i \beta|^q \right] \\ &\leq C_{q,\gamma} K_\gamma^{3q} |\mathcal{P}|^{3q\gamma - q} e^{\sum_{i=1}^{n} qN |\Delta_i \beta|^2} \end{aligned}$$

pick $\frac{1}{2} > \gamma > \frac{1}{3}$, we know:

$$\mathbb{E}\left[\left| \sum_{i=1}^{n} f_{\mathcal{P},i}(1) \Delta_i \beta - \sum_{i=1}^{n} f_{\mathcal{P},i-1}(1) \Delta_i \beta \right|^q \right] \to 0 \tag{6.52}$$

Notice that $f_{\mathcal{P},i-1}(1) = f_{\mathcal{P},0}(1) f_{\mathcal{P},0}^{-1}(s_{i-1}) \frac{S_{i-1}}{\Delta_{i-1}}$, so

$$\left\langle \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}}, \sum_{i=1}^{n} T_{i-1} \Delta_i \beta \right\rangle \tag{6.53}$$

$$= \left\langle f_{\mathcal{P},0}^*(1) \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}}, \sum_{i=1}^{n} f_{\mathcal{P},0}^{-1}(s_{i-1}) \frac{S_{i-1}}{\Delta_{i-1}} \Delta_i \beta \right\rangle \tag{6.54}$$

Use Lemma 5.5, we have $\left|f_{\mathcal{P},0}^{-1}\left(s_{i-1}\right)\right| \leq 1$, then use Lemma 6.6, we get:

$$\left|f_{\mathcal{P},0}^{-1}\left(s_{i-1}\right)\frac{S_{i-1}}{\Delta_{i-1}} - f_{\mathcal{P},0}^{-1}\left(s_{i-1}\right)\right|\left|\Delta_i\beta\right| \leq \left|\frac{S_{i-1}}{\Delta_{i-1}} - I\right|\left|\Delta_i\beta\right| \leq \frac{NK_\gamma^3\left|\mathcal{P}\right|^{3\gamma+1}}{6}e^{\frac{N}{2}\left|\Delta_{i-1}\beta\right|^2}$$

Therefore for all $q \geq 1$,

$$\left|\sum_{i=1}^n f_{\mathcal{P},0}^{-1}\left(s_{i-1}\right)\frac{S_{i-1}}{\Delta_{i-1}}\Delta_i\beta - \sum_{i=1}^n f_{\mathcal{P},0}^{-1}\left(s_{i-1}\right)\Delta_i\beta\right|^q \tag{6.55}$$

$$\leq \left|\mathcal{P}\right|^{1-q}\sum_{i=1}^n \frac{N^q K_\gamma^{3q}\left|\mathcal{P}\right|^{3q\gamma+q}}{6^q}e^{\frac{Nq}{2}\left|\Delta_{i-1}\beta\right|^2} \tag{6.56}$$

$$\leq C_q\left|\mathcal{P}\right|^{3\gamma q+1}K_\gamma^{3q}e^{\sum_{i=1}^n\frac{Nq}{2}\left|\Delta_{i-1}\beta\right|^2} \tag{6.57}$$

therefore:

$$\mathbb{E}\left[\left|\sum_{i=1}^n f_{\mathcal{P},0}^{-1}\left(s_{i-1}\right)\frac{S_{i-1}}{\Delta_{i-1}}\Delta_i\beta - \sum_{i=1}^n f_{\mathcal{P},0}^{-1}\left(s_{i-1}\right)\Delta_i\beta\right|^q\right] \leq C_q\left|\mathcal{P}\right|^{3\gamma q+1} \overset{|\mathcal{P}|\to 0}{\longrightarrow} 0$$

Rewrite

$$\sum_{i=1}^n f_{\mathcal{P},0}^{-1}\left(s_{i-1}\right)\Delta_i\beta \text{ as } \int_0^1 f_{\mathcal{P}}\left(s\right)d\beta_s,$$

where $f_{\mathcal{P}}\left(s\right) := \sum_{i=1}^n f_{\mathcal{P},0}^{-1}\left(s_{i-1}\right)1_{[s_{i-1},s_i)}\left(s\right)$

Consider

$$M_r := \int_0^r f_{\mathcal{P}}\left(s\right)d\beta_s - \int_0^r \tilde{T}_s^{-1}d\beta_s$$

by Burkholder-Davis-Gundy inequality, for all $q \geq 1$,

$$\mathbb{E}\left[\sup_{r\in[0,1]}\left|M_r\right|^q\right] \leq C_q\mathbb{E}\left[\langle M\rangle_1^{\frac{q}{2}}\right]$$

Since

$$\langle M\rangle_1 \leq \int_0^1 \left|f_{\mathcal{P}}(s) - \tilde{T}_s^{-1}\right|^2 ds \leq 2\int_0^1 \left|f_{\mathcal{P}}(s) - \tilde{T}_{\underline{s}}^{-1}\right|^2 ds + 2\int_0^1 \left|\tilde{T}_{\underline{s}}^{-1} - \tilde{T}_s^{-1}\right|^2 ds, \tag{6.58}$$

we have

$$\begin{aligned}\int_0^1 \left|f_{\mathcal{P}}(s) - T_{\underline{s}}^{-1}\right|^2 ds &= \sum_{i=1}^n \left|f_{\mathcal{P},0}^{-1}(s_{i-1}) - \tilde{T}_{s_{i-1}}^{-1}\right|^2 \Delta_i \\ &\leq \sum_{i=1}^n \left|f_{\mathcal{P},0}^{-1}(s_{i-1})\right|^2 \left|f_{\mathcal{P},0}(s_{i-1}) - \tilde{T}_{s_{i-1}}\right|^2 \left|\tilde{T}_{s_{i-1}}^{-1}\right|^2 \Delta_i \\ &\leq \sup_{s\in\mathcal{P}} \left|f_{\mathcal{P},0}(s) - \tilde{T}_s\right|^2 \end{aligned} \tag{6.59}$$

and

$$\int_0^1 \left|T_{\underline{s}}^{-1} - \tilde{T}_s^{-1}\right|^2 ds = \int_0^1 \left|\int_s^{\underline{s}} \left(\tilde{T}_r^{-1}\right)' dr\right|^2 ds \leq \int_0^1 N\left|s - \underline{s}\right|^2 ds \leq N\left|\mathcal{P}\right|^2.$$

Therefore,

$$\begin{aligned}\langle M\rangle_1^{\frac{q}{2}} &\leq C_q\left(\int_0^1 \left|f_{\mathcal{P}}(s) - T_{\underline{s}}^{-1}\right|^2 ds\right)^{\frac{q}{2}} + C_q\left(\int_0^1 \left|T_{\underline{s}}^{-1} - \tilde{T}_s^{-1}\right|^2 ds\right)^{\frac{q}{2}} \\ &\leq C_q\left(\sup_{s\in\mathcal{P}} \left|f_{\mathcal{P},0}(s) - \tilde{T}_s\right|^q + \left|\mathcal{P}\right|^q\right).\end{aligned}$$

Then use Theorem 6.13 we have

$$\mathbb{E}\left[\langle M\rangle_1^{\frac{q}{2}}\right] \leq C_q\left|\mathcal{P}\right|^{q\gamma}$$

from which it follows

$$\int_0^1 f_{\mathcal{P}}(s)\, d\beta_s \to \int_0^1 \tilde{T}_s^{-1} d\beta_s \text{ in } L^q\left(W_o(M)\right).$$

Then since

$$\mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}} \to \mathbf{K}_1^{-1}\tilde{H} \text{ in } L^q(W_o(M))$$

and

$$f^*_{\mathcal{P},0}(1) \to T_1^* \text{ in } L^q(W_o(M)),$$

we have

$$I \to \left\langle T_1^*\mathbf{K}_1^{-1}\tilde{H}, \int_0^1 \tilde{T}_s^{-1} d\beta_s \right\rangle \text{ in } L^q(W) \tag{6.60}$$

Lastly, notice that:

$$\mathbf{K}_1 = \tilde{T}_1 \int_0^1 \left(T_r^*\tilde{T}_r\right)^{-1} dr T_1^*.$$

So

$$\mathbf{K}_1^{-1} = T_1^{-1*}\tilde{C}$$

and

$$\left\langle T_1^*\mathbf{K}_1^{-1}\tilde{H}, \int_0^1 \tilde{T}_s^{-1} d\beta_s \right\rangle = \left\langle \tilde{C}\tilde{H}, \int_0^1 \tilde{T}_s^{-1} d\beta_s \right\rangle \tag{6.61}$$

$$= \sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1}\right)^* e_\alpha, d\beta_s \right\rangle \tag{6.62}$$

So combine (6.60) and (6.51) and we get

$$\int_0^1 \left\langle J'_{\mathcal{P}}(s+), d\beta_{\mathcal{P},s} \right\rangle - \sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1}\right)^* e_\alpha, d\beta_s \right\rangle \to 0 \text{ in } L^{\infty-}(W_o(M)).$$

Before we prove Lemma 6.26 we need some tools to allow us to differentiate with respect to paths on H^1. ■

Lemma 6.27 *Differentiating w.r.t. path:*

First we retain a definition from Chapter 0: Fix $s \in [0,1]$, consider an one parameter family of paths $\{\sigma_t\} \subset H_{\mathcal{P}}(M)$ and denote by $u_t(\cdot)$: the Horizontal lift of σ_t. For simplicity, we will denote $u_t(1)$ by u_t, σ_0 by σ, the derivative with respect *to t by $\cdot$ and the derivative with respect to s by $\prime$. For any $X \in \Gamma(TM)$, define $f_X : \mathcal{O}(M) \mapsto \mathbb{R}^d \simeq T_oM$ by*

$$f_X(u) = u^{-1}(X \circ \pi)(u)$$

Then:

$$\frac{d}{dt}|_0 f_X(u_t) = \left(\frac{d}{dt}|_0 u_t\right) f_X = u_0^{-1} \nabla_{\dot{\sigma}(1)} X \tag{6.63}$$

$$- \int_0^1 R_{u_0(r)}\left(u_0(r)^{-1} \sigma'(r+), u_0(r)^{-1} \dot{\sigma}(r)\right) dr f_X(u_0) \tag{6.64}$$

Proof. Based on the decomposition of $\mathcal{O}(M)$ as in Definition A.12, we have:

$$\dot{u}_0 = B_a(u_0) + \tilde{A}(u_0)$$

where $a = u_0^{-1} \frac{d}{dt}|_0 \sigma_t(1) = u_0^{-1} \dot{\sigma}(1) \in T_oM$ and $\tilde{A}(u_0) = \frac{d}{dt}|_0 u_0 e^{tA}$ for some $A = u_0^{-1} \frac{\nabla u_t}{dt}(0) \in \mathbf{so}(d)$ and $B_a(u_0) = \frac{d}{dt}|_0 //_t(\gamma) u_0$ where γ satisfies $\dot{\gamma}(0) = u_0 a$ and $\gamma(0) = \sigma(1)$. In this example, we can choose $\gamma(\cdot)$ to be $\sigma_{\cdot}(1)$. So

$$B_a(u_0) f_X = \frac{d}{dt}|_0 u_0^{-1} //_t^{-1}(\gamma)(X \circ \pi)(//_t(\gamma) u_0) = u_0^{-1} \nabla_{\sigma(\dot{s})} X$$

and

$$\tilde{A}(u) f_X = \frac{d}{dt}|_0 e^{-tA} u^{-1}(X \circ \pi)\left(u e^{tA}\right) = -A u_0^{-1} X(\sigma(1)) = -A f_x(u_0)$$

Following the computation in Theorem 3.3 in [3], we know that

$$A = \int_0^1 R_{u_0(r)}\left(u_0(r)^{-1} \sigma'(r+), u_0(r)^{-1} \dot{\sigma}(r)\right) dr.$$

■

Proof of Proposition 6.26. Because of Lemma 4.43, it suffices to prove:

$$div\tilde{X}_{\mathcal{P}} \to divX \circ E_1(\sigma) - \sum_{\alpha=1}^{d} \left\langle CA_1 \langle Z_\alpha \rangle \tilde{H}, e_\alpha \right\rangle.$$

Recall that

$$J_{\mathcal{P}}(s) = \mathbf{K}_{\mathcal{P}}(s) \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}}$$

So

$$J'_{\mathcal{P}}(s_{j-1}+) = \mathbf{K}'_{\mathcal{P}}(s_{j-1}+) \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}} = f^*_{\mathcal{P},j}(1) \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}}$$

and

$$X^{h_{\alpha,j}} J'_{\mathcal{P}}(s_{j-1}+) = I + II + III$$

where

$$\begin{aligned} I &= \left(X^{h_{\alpha,j}} f^*_{\mathcal{P},j}(1)\right) \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}} \\ II &= f^*_{\mathcal{P},j}(1) \left(X^{h_{\alpha,j}} \mathbf{K}_{\mathcal{P}}(1)^{-1}\right) H_{\mathcal{P}} \\ III &= f^*_{\mathcal{P},j}(1) \mathbf{K}_{\mathcal{P}}(1)^{-1} \left(X^{h_{\alpha,j}} H_{\mathcal{P}}\right) \end{aligned} \tag{6.65}$$

and this proposition is an easy corollary of the following three lemmas. ■

Lemma 6.28 *If M has constant sectional curvature, then*

$$\sum_{\alpha=1}^{d} \sum_{j=1}^{n} \langle III, e_\alpha \rangle \sqrt{\Delta_j} - divX \circ E_1 + \sum_{i=1}^{d} \left\langle \int_0^1 R_{\tilde{u}_s}(\delta\beta_s, Z_{e_i}(s)) \tilde{H}, \tilde{C}^* e_i \right\rangle \to 0$$

in $L^{\infty-}(W_o(M))$.

Proof. By Lemma 6.27,

$$\sum_{\alpha=1}^{d} \sum_{j=1}^{n} \langle III, e_\alpha \rangle \sqrt{\Delta_j} = IV + V$$

where

$$IV = \sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle f_{\mathcal{P},j}^{*}\left(1\right)\mathbf{K}_{\mathcal{P}}\left(1\right)^{-1}u^{-1}\nabla_{u\sqrt{\Delta_j}f_{\mathcal{P},j}(1)e_\alpha}X, e_\alpha\right\rangle\sqrt{\Delta_j}$$

and

$$V = -\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle f_{\mathcal{P},j}^{*}\left(1\right)\mathbf{K}_{\mathcal{P}}\left(1\right)^{-1}\int_0^1 R_{u_{\mathcal{P}}(r)}\left(\beta_{\mathcal{P}}'\left(r+\right), h_{\alpha,j}\left(r\right)\right)drH_{\mathcal{P}}, e_\alpha\right\rangle\sqrt{\Delta_j}$$

Let's compute IV first. View $L\left(\cdot\right) = u^{-1}\nabla_{u\cdot}X$ as a linear functional on $\mathbb{R}^d$,then

$$\begin{aligned}
IV &= \sum_{j=1}^{n}\sum_{\alpha=1}^{d}\left\langle f_{\mathcal{P},j}^{*}\left(1\right)\mathbf{K}_{\mathcal{P}}\left(1\right)^{-1}u^{-1}L\left(T_j e_\alpha\right), e_\alpha\right\rangle\Delta_j \\
&= \sum_{j=1}^{n}Trace\left(f_{\mathcal{P},j}^{*}\left(1\right)\mathbf{K}_{\mathcal{P}}\left(1\right)^{-1}u^{-1}L\left(T_j\right)\right)\Delta_j \\
&= \sum_{j=1}^{n}Trace\left(\Delta_j f_{\mathcal{P},j}\left(1\right)f_{\mathcal{P},j}^{*}\left(1\right)\mathbf{K}_{\mathcal{P}}\left(1\right)^{-1}u^{-1}L\right) \\
&= Trace\left(\left(\sum_{j=1}^{n}\Delta_j f_{\mathcal{P},j}\left(1\right)f_{\mathcal{P},j}^{*}\left(1\right)\right)\mathbf{K}_{\mathcal{P}}\left(1\right)^{-1}u^{-1}L\right) \\
&= Trace\left(u^{-1}L\right) \\
&= divX\circ E_1.
\end{aligned}$$

Then we claim

Claim 6.29

$$V - \sum_{\alpha=1}^{d}\left\langle CA_1\left\langle Z_\alpha\right\rangle\tilde{H}, e_\alpha\right\rangle \to 0 \ in \ L^{\infty-}\left(W_o\left(M\right)\right)$$

Proof of claim. Recall that

$$V=-\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle f_{\mathcal{P},j}^{*}\left(1\right)\mathbf{K}_{\mathcal{P}}\left(1\right)^{-1}\int_{0}^{1}R_{u_{\mathcal{P}}(r)}\left(\beta_{\mathcal{P}}'\left(r+\right),h_{\alpha,j}\left(r\right)\right)drH_{\mathcal{P}},e_{\alpha}\right\rangle\sqrt{\Delta_{j}}$$

$$\begin{aligned}\int_{0}^{1}R_{u_{\mathcal{P}}(r)}\left(\beta_{\mathcal{P}}'\left(r+\right),\frac{1}{\sqrt{\Delta_{j}}}h_{\alpha,j}\left(r\right)\right)dr&=\int_{0}^{1}R_{u_{\mathcal{P}}(r)}\left(\beta_{\mathcal{P}}'\left(r+\right),f_{\mathcal{P},j}\left(r\right)e_{\alpha}\right)dr\\&=\int_{0}^{1}R_{u_{\mathcal{P}}(\underline{r})}\left(\beta_{\mathcal{P}}'\left(r+\right),f_{\mathcal{P},j}\left(\underline{r}\right)e_{\alpha}\right)dr+e_{0}\end{aligned}$$

where $e_0 := e_{0,1} + e_{0,2}$

$$e_{0,1}=\int_{0}^{1}R_{u_{\mathcal{P}}(r)}\left(\beta_{\mathcal{P}}'\left(r+\right),f_{\mathcal{P},j}\left(r\right)e_{\alpha}\right)dr-\int_{0}^{1}R_{u_{\mathcal{P}}(\underline{r})}\left(\beta_{\mathcal{P}}'\left(r+\right),f_{\mathcal{P},j}\left(r\right)e_{\alpha}\right)dr$$

and

$$e_{0,2}=\int_{0}^{1}R_{u_{\mathcal{P}}(\underline{r})}\left(\beta_{\mathcal{P}}'\left(r+\right),f_{\mathcal{P},j}\left(r\right)e_{\alpha}\right)dr-\int_{0}^{1}R_{u_{\mathcal{P}}(\underline{r})}\left(\beta_{\mathcal{P}}'\left(r+\right),f_{\mathcal{P},j}\left(\underline{r}\right)e_{\alpha}\right)dr$$

Since M has constant sectional curvature, R_u is independent of u, and therefore

$$e_{0,1}=0$$

For $e_{0,2}$, we have the following estimate:

$$\left|e_{0,2}\right|^{q}\leq N\sup_{r\in[0,1]}\left|\beta_{\mathcal{P}}'\left(r+\right)\right|^{q}\sup_{r\in[0,1],j\in\{1,\cdots,n\}}\left|f_{\mathcal{P},j}\left(r\right)-f_{\mathcal{P},j}\left(\underline{r}\right)\right|^{q}$$

use (6.14) we have

$$\left|e_{0,2}\right|^{q}\leq C_{q,\gamma}K_{\gamma}^{q}\left|\mathcal{P}\right|^{q\gamma-1}\left|\mathcal{P}\right|^{2q\gamma}e^{qN\sum_{k=1}^{n}\left|\Delta_{k}\beta\right|^{2}}K_{\gamma}^{2q}\left(1+\frac{NK_{\gamma}\left|\mathcal{P}\right|^{\gamma}}{6}\right)^{q}$$

And from which it follows:

$$\mathbb{E}\left[|e_{0,2}|^q\right] \le C_{q,\gamma} |\mathcal{P}|^{3q\gamma-1} \tag{6.66}$$

Picking $\gamma > \frac{1}{3}$, so $3q\gamma - 1 > 0$ for any $q \ge 1$ and $\mathbb{E}\left[|e_{0,2}|^q\right] \to 0$ as $|\mathcal{P}| \to 0$.

So it remains to consider:

$$\begin{aligned}\int_0^1 R_{u_{\mathcal{P}}(\underline{r})}\left(\beta'_{\mathcal{P}}(r+), f_{\mathcal{P},j}(\underline{r}) e_\alpha\right) dr &= \sum_{k=1}^{n} R_{u_{\mathcal{P}}(s_{k-1})}\left(\Delta_k \beta, f_{\mathcal{P},j}(s_{k-1}) e_\alpha\right) \\ &= \int_0^1 g_1(s)\, d\beta_s\end{aligned}$$

where

$$g_1(s) = \sum_{k=1}^{n} R_{u_{\mathcal{P}}(s_{k-1})}\left(\cdot, f_{\mathcal{P},j}(s_{k-1}) e_\alpha\right) 1_{[s_{k-1},s_k)}(s).$$

Define

$$\begin{aligned} g_2(s) &= \sum_{k=1}^{n} R_{\tilde{u}(s_{k-1})}\left(\cdot, f_{\mathcal{P},j}(s_{k-1}) e_\alpha\right) 1_{[s_{k-1},s_k)}(s) \\ g_3(s) &= \sum_{k=j+1}^{n} R_{\tilde{u}(s_{k-1})}\left(\cdot, T(s_{k-1}) \tilde{T}_{s_j}^{-1} e_\alpha\right) 1_{[s_{k-1},s_k)}(s) \\ g_4(s) &= R_{\tilde{u}(\underline{s})}\left(\cdot, \tilde{T}_s \tilde{T}_{s_j}^{-1} e_\alpha\right) 1_{[s_j,1]}(s) \\ g_5(s) &= R_{\tilde{u}(s)}\left(\cdot, \tilde{T}_s \tilde{T}_{s_j}^{-1} e_\alpha\right) 1_{[s_j,1]}(s). \end{aligned}$$

Consider

$$e_1(r) = \int_0^r g_1(s)\, d\beta_s - \int_0^r g_2(s)\, d\beta_s$$

then

$$\langle e_1 \rangle (r) \le \int_0^r |g_1(s) - g_2(s)|^2\, ds$$

so for all $q \geq 1$,

$$\begin{aligned}\mathbb{E}\left[\langle e_1\rangle^{\frac{q}{2}}(1)\right] &\leq \mathbb{E}\left[\int_0^1 |g_1(s) - g_2(s)|^q\, ds\right] \\ &\leq \mathbb{E}\left[\sum_{k=1}^{n} \left|R_{u_{\mathcal{P}}(s_{k-1})} - R_{u(s_{k-1})}\right|^q |f_{\mathcal{P},j}(s_{k-1})|^q \Delta_k\right] \\ &\leq \mathbb{E}\left[\mathcal{A}_{\mathcal{P}}^q \sup_{j\in\{1,\cdots,n\}, s\in[0,1]} |f_{\mathcal{P},j}(s)|^q\right]\end{aligned}$$

where

$$A_{\mathcal{P}} := \sup_{s\in[0,1]} \left|R_{u_{\mathcal{P}}(s)} - R_{\tilde{u}(s)}\right|$$

Using Theorem 6.2 we know

$$\mathbb{E}\left[A_{\mathcal{P}}^q\right] \leq C_{q,\gamma} |\mathcal{P}|^{q\gamma} \, \forall \gamma \in \left(0, \frac{1}{2}\right), q \geq 1$$

Then by Holder's inequality,

$$\mathbb{E}\left[\mathcal{A}_{\mathcal{P}}^q \sup_{j\in\{1,\cdots,n\}, s\in[0,1]} |f_{\mathcal{P},j}(s)|^q\right] \leq C_{q,\gamma} |\mathcal{P}|^{q\gamma}$$

then apply Burkholder-Davies-Gundy inequality, we get:

$$\mathbb{E}\left[|e_1(1)|^q\right] \leq C_q \mathbb{E}\left[\langle e_1\rangle^{\frac{q}{2}}(1)\right] \leq C_{q,\gamma} |\mathcal{P}|^{q\gamma} \tag{6.67}$$

Then consider

$$e_2(r) = \int_0^r g_2(s)\, d\beta_s - \int_0^r g_3(s)\, d\beta_s$$

then

$$\langle e_2\rangle(r) \leq \int_0^r |g_2(s) - g_3(s)|^2\, ds$$

so for all $q \geq 1$,

$$\begin{aligned}
\mathbb{E}\left[\langle e_2\rangle^{\frac{q}{2}}(1)\right] &\leq \mathbb{E}\left[\int_0^1 |g_2(s) - g_3(s)|^q \, ds\right] \\
&\leq \mathbb{E}\left[\sum_{k=1}^{n} \left|R_{\tilde{u}(s_{k-1})}\right|^q \left|f_{\mathcal{P},j}(s_{k-1}) - \tilde{T}_{s_{k-1}}\tilde{T}_{s_j}^{-1}\right|^q \Delta_k\right] \\
&\leq \mathbb{E}\left[N\sup_{j,s}\left|f_{\mathcal{P},j}(s) - \tilde{T}_s\tilde{T}_{s_j}^{-1}\right|^q\right]
\end{aligned}$$

By Holder's inequality and Theorem 6.13,

$$\mathbb{E}\left[N\sup_{j,s}\left|f_{\mathcal{P},j}(s) - \tilde{T}_s\tilde{T}_{s_j}^{-1}\right|^q\right] \leq C_{q,\gamma}|\mathcal{P}|^{q\gamma}$$

then apply Burkholder-Davies-Gundy inequality, we get:

$$\mathbb{E}\left[|e_2(1)|^q\right] \leq C_q\mathbb{E}\left[\langle e_2\rangle^{\frac{q}{2}}(1)\right] \leq C_{q,\gamma}|\mathcal{P}|^{q\gamma} \tag{6.68}$$

Then consider

$$e_3(r) = \int_0^r g_3(s)\, d\beta_s - \int_0^r g_4(s)\, d\beta_s$$

then

$$\langle e_3\rangle(r) \leq \int_0^r |g_3(s) - g_4(s)|^2 \, ds$$

so for all $q \geq 1$,

$$\begin{aligned}
\mathbb{E}\left[\langle e_3\rangle^{\frac{q}{2}}(1)\right] &\leq \mathbb{E}\left[\int_0^1 |g_3(s) - g_4(s)|^q \, ds\right] \\
&\leq \mathbb{E}\left[\sum_{k=1}^{n} \left|R_{\tilde{u}(s_{k-1})}\right|^q \int_{s_{k-1}}^{s_k} \left|\tilde{T}_s\tilde{T}_{s_j}^{-1} - \tilde{T}_{s_{k-1}}\tilde{T}_{s_j}^{-1}\right|^q ds\right] \\
&\leq C_q|\mathcal{P}|^q
\end{aligned}$$

then apply Burkholder-Davies-Gundy inequality, we get:

$$\mathbb{E}\left[\left|e_3\left(1\right)\right|^q\right] \leq C_q \mathbb{E}\left[\left\langle e_3\right\rangle^{\frac{q}{2}}\left(1\right)\right] \leq C_{q,\gamma}\left|\mathcal{P}\right|^q$$

Then consider

$$g\left(s\right) = R_{\tilde{u}_s}\left(\cdot, \tilde{T}_s \tilde{T}_{s_j}^{-1} e_\alpha\right) 1_{\left[s_j,1\right]}\left(s\right)$$

$$e_4\left(r\right) = \int_0^r g_4\left(s\right) d\beta_s - \int_0^r g_5\left(s\right) d\beta_s$$

then

$$\left\langle e_4\right\rangle\left(r\right) \leq \int_0^r \left|g_5\left(s\right) - g_4\left(s\right)\right|^2 ds$$

so for all $q \geq 1$,

$$\begin{aligned}
\mathbb{E}\left[\left\langle e_4\right\rangle^{\frac{q}{2}}\left(1\right)\right] &\leq \mathbb{E}\left[\int_0^1 \left|g_5\left(s\right) - g_4\left(s\right)\right|^q ds\right] \\
&\leq \mathbb{E}\left[\sum_{k=1}^n \left|\tilde{T}_s \tilde{T}_{s_j}^{-1}\right|^q \int_{s_{k-1}}^{s_k} \left|R_{\tilde{u}_s} - R_{\tilde{u}_{s_{k-1}}}\right|^q ds\right] \\
&\leq \mathbb{E}\left[K_\gamma^q\right]\left|\mathcal{P}\right|^{q\gamma}
\end{aligned}$$

then apply Burkholder-Davies-Gundy inequality, we get:

$$\mathbb{E}\left[\left|e_4\left(1\right)\right|^q\right] \leq C_q \mathbb{E}\left[\left\langle e_4\right\rangle^{\frac{q}{2}}\left(1\right)\right] \leq C_{q,\gamma}\left|\mathcal{P}\right|^{q\gamma} \tag{6.69}$$

Combine (6.66), (6.67), (6.68), (6.69), we get:

$$\left|V + \sum_{\alpha=1}^d \sum_{j=1}^n \left\langle \left(\tilde{T}_{s_j}^{-1}\right)^* \tilde{T}_1^* K_1^{-1} \int_{s_j}^1 R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r \tilde{T}_{s_j}^{-1} e_\alpha\right) \tilde{H}, e_\alpha \right\rangle \Delta_j\right| \to 0$$

in $L^{\infty-}(W_o(M))$. Now change the pair (e_α, e_α) to $\left(\tilde{T}_{s_j}e_\alpha, \left(\tilde{T}_{s_j}^{-1}\right)^* e_\alpha\right)$, we have

$$\begin{aligned}
&\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle \left(\tilde{T}_{s_j}^{-1}\right)^* \tilde{T}_1^* K_1^{-1} \int_{s_j}^1 R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r \tilde{T}_{s_j}^{-1} e_\alpha\right)\tilde{H}, e_\alpha \right\rangle \Delta_j \\
&= \sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle \tilde{T}_1^* K_1^{-1} \int_{s_j}^1 R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right)\tilde{H}, \tilde{T}_{s_j}^{-1}\left(\tilde{T}_{s_j}^{-1}\right)^* e_\alpha \right\rangle \Delta_j \\
&= \sum_{\alpha=1}^{d}\int_0^1 \left\langle \tilde{T}_1^* K_1^{-1} \int_{\underline{s}}^1 R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right)\tilde{H}, T_{\underline{s}}^{-1}\left(T_{\underline{s}}^{-1}\right)^* e_\alpha \right\rangle ds.
\end{aligned}$$

Then we consider

$$\begin{aligned}
I = &\sum_{\alpha=1}^{d}\int_0^1 \left\langle \tilde{T}_1^* K_1^{-1} \int_{\underline{s}}^1 R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right)\tilde{H}, \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* e_\alpha \right\rangle ds \\
&- \sum_{\alpha=1}^{d}\int_0^1 \left\langle \tilde{T}_1^* K_1^{-1} \int_{s}^1 R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right)\tilde{H}, \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* e_\alpha \right\rangle ds
\end{aligned}$$

Notice that

$$\begin{aligned}
&\left\langle \tilde{T}_1^* K_1^{-1} \int_{\underline{s}}^1 R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right)\tilde{H}, \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* e_\alpha \right\rangle \\
&- \left\langle \tilde{T}_1^* K_1^{-1} \int_{s}^1 R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right)\tilde{H}, \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* e_\alpha \right\rangle \\
&\leq II + III
\end{aligned}$$

where

$$II = \left\langle \tilde{T}_1^* K_1^{-1} \int_{\underline{s}}^s R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right)\tilde{H}, \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* e_\alpha \right\rangle$$

and

$$III = \tag{6.70}$$

$$\left\langle \tilde{T}_1^* K_1^{-1} \int_{s}^1 R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right)\tilde{H}, \left(\tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* - \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^*\right) e_\alpha \right\rangle \tag{6.71}$$

$$|II|^q \leq C_q \left| \int_{\underline{s}}^{s} R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right) \right|^q$$

By Burkholder-Davies-Gundy inequality,

$$\mathbb{E}\left[|II|^q\right] \leq C_q \left|\mathcal{P}\right|^{\frac{q}{2}}$$

Also notice that:

$$\mathbb{E}\left[|III|^q\right] \leq C_q \left|\mathcal{P}\right|^q \mathbb{E}\left[\left| \int_{\underline{s}}^{s} R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right) \right|^q\right] \leq C_q \left|\mathcal{P}\right|^q$$

So use Holder's inequality, we have:

$$\begin{aligned}
\mathbb{E}\left[|I|^q\right] &\leq C_q \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} \mathbb{E}\left[|II|^q + |III|^q\right] \\
&\leq C_q \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} \left(\left|\mathcal{P}\right|^{\frac{q}{2}} + \left|\mathcal{P}\right|^q\right) \\
&= C_q \left|\mathcal{P}\right|^{\frac{q}{2}}
\end{aligned}$$

from which it follows:

$$V + \sum_{\alpha=1}^{d} \int_0^1 \left\langle \tilde{T}_1^* K_1^{-1} \int_s^1 R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right) \tilde{H}, \tilde{T}_s^{-1} \left(\tilde{T}_s^{-1}\right)^* e_\alpha \right\rangle ds \to 0$$
$$\text{in } L^{\infty-}\left(W_o\left(M\right)\right).$$

The last step is to show a change of integration order:

$$\begin{aligned}
&\sum_{\alpha=1}^{d} \int_0^1 \left\langle \tilde{T}_1^* K_1^{-1} \int_s^1 R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right) \tilde{H}, \tilde{T}_s^{-1} \left(\tilde{T}_s^{-1}\right)^* e_\alpha \right\rangle ds \qquad (6.72) \\
&= \sum_{\alpha=1}^{d} \int_0^r \left\langle \tilde{T}_1^* K_1^{-1} \int_0^1 R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right) \tilde{H}, \tilde{T}_s^{-1} \left(\tilde{T}_s^{-1}\right)^* e_\alpha \right\rangle ds
\end{aligned}$$

Then the claim is easily seen following changing the pair (e_α, e_α) to

$$\left(\int_0^r \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* ds e_\alpha, \left[\left(\left(\int_0^r \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* ds\right)^{-1}\right]^* e_\alpha\right)$$

and recognizing

$$\tilde{T}_r \int_0^r \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* ds e_\alpha = Z_\alpha(r).$$

now we prove (6.72). Define:

$$f(s) = \sum_{\alpha=1}^d \int_0^t \left\langle \tilde{T}_1^* K_1^{-1} \int_s^t R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right)\tilde{H}, \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* e_\alpha \right\rangle ds$$

$$g(s) = \sum_{\alpha=1}^d \int_0^r \left\langle \tilde{T}_1^* K_1^{-1} \int_0^t R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right)\tilde{H}, \int_0^r \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* ds e_\alpha \right\rangle$$

then

$$df = \sum_{\alpha=1}^d \left\langle \tilde{T}_1^* K_1^{-1} R_{\tilde{u}_t}\left(d\beta_t, \tilde{T}_t e_\alpha\right)\tilde{H}, \int_0^t \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* ds e_\alpha \right\rangle$$

and $f(0) \equiv 0$ notice that

$$dg = \sum_{\alpha=1}^d \left\langle \tilde{T}_1^* K_1^{-1} R_{\tilde{u}_t}\left(d\beta_t, \tilde{T}_t e_\alpha\right)\tilde{H}, \int_0^t \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* ds e_\alpha \right\rangle = df$$

and $g(0) = 0$. Therefore, (6.72) is obtained by observing that left hand side= $f_1 = g_1 =$ right hand side. ■ ■

Lemma 6.30 *If M has constant sectional curvature, $\sum_{\alpha,j=1,1}^{d,n} \langle I, e_\alpha\rangle \sqrt{\Delta_j} \to 0$ as $|\mathcal{P}| \to 0$ in $L^{\infty-}(W_o(M))$.*

Proof. Define $\tilde{g}_j(s) := X^{h_{\alpha,j}} f_{\mathcal{P},j}(s)$ and $g_j(s) := \tilde{g}_j(s) - \tilde{g}_j(\underline{s})$. Then we know that $g_j(s)$ satisfies the following ODE: for $k = j, \cdots, n$

$$\begin{cases} g_j''(s) = A_{\mathcal{P},k}(s)\, g_j(s) + \dot{A_{\mathcal{P},k}}(s)\left(f_{\mathcal{P},j}(s) - f_{\mathcal{P},j}(\underline{s})\right) & s \in [s_{k-1}, s_k] \\ g_j(\underline{s}) = 0 \\ g_j'(\underline{s}) = 0 \end{cases}$$

where

$$\dot{A_{\mathcal{P},k}}(s) = \frac{d}{dt}|_0 \left(R_{u_{\mathcal{P}}(t,s)}\left(\beta_{\mathcal{P}}'(t,s), \cdot\right)\beta_{\mathcal{P}}'(t,s)\right).$$

For $s \in [s_{k-1}, s_k]$, we know:

$$g_j(s) = \int_{s_{k-1}}^{s} S_k(s-r)\, \dot{A_{\mathcal{P}}^k}(r)\left(f_{\mathcal{P},j}(r) - f_{\mathcal{P},j}(s_{k-1})\right) dr$$

use Lemma 6.6 and 6.14, we have:

$$\begin{aligned} \left|f_{\mathcal{P},i}(s) - f_{\mathcal{P},i}(\underline{s})\right|^q &\leq \frac{N^q}{2^q}\left|\Delta_k\beta\right|^{2q} e^{N|\Delta_k\beta|^2} e^{\frac{1}{2}qN\sum_{k=1}^n|\Delta_k\beta|^2}\left(1 + \frac{NK_\gamma\left|\mathcal{P}\right|^\gamma}{6}\right)^q \\ &\leq C_q\left|\mathcal{P}\right|^{2q\gamma} e^{qN\sum_{k=1}^n|\Delta_k\beta|^2} K_\gamma^{2q}\left(1 + \frac{NK_\gamma\left|\mathcal{P}\right|^\gamma}{6}\right)^q \end{aligned}$$

and

$$\left|S_k(s-r)\right| \leq (s-r)\left(1 + \frac{N}{6}K_\gamma^2\left|\mathcal{P}\right|^{2\gamma} e^{\frac{1}{2}N\sum_{i=1}^n|\Delta_i\beta|^2}\right)$$

therefore:

$$\begin{aligned} &\left|g_j(s)\right| \\ &\leq \int_{s_{k-1}}^{s}\left|S_k(s-r)\right|\left|\dot{A_{\mathcal{P}}^k}(r)\right|\left|f_{\mathcal{P},j}(r) - f_{\mathcal{P},j}(s_{k-1})\right| dr \\ &\leq C\sup_{k\in\{1,\ldots,n\},r\in[0,1]}\left|\dot{A_{\mathcal{P},k}}(r)\right|\left|\mathcal{P}\right|^{2\gamma} K_\gamma^2\left(1 + \frac{NK_\gamma\left|\mathcal{P}\right|^\gamma}{6}\right) e^{N\sum_{i=1}^n|\Delta_i\beta|^2}\int_{s_{k-1}}^{s}(s-r)\, dr \\ &= C\sup_{k\in\{1,\ldots,n\},r\in[0,1]}\left|\dot{A_{\mathcal{P}}^k}(r)\right|\left|\mathcal{P}\right|^{2\gamma+2} K_\gamma^2\left(1 + \frac{NK_\gamma\left|\mathcal{P}\right|^\gamma}{6}\right) e^{N\sum_{i=1}^n|\Delta_i\beta|^2} \end{aligned}$$

so

$$|\tilde{g}_j(1)| \leq \sum_{k=j}^{n} |g_j(s_k)| \tag{6.73}$$

$$\leq C \sup_{k\in\{1,\ldots,n\},r\in[0,1]} \left|\dot{A}_{\mathcal{P}}^{k}(r)\right| |\mathcal{P}|^{2\gamma+1} K_\gamma^2 \left(1+\frac{NK_\gamma |\mathcal{P}|^\gamma}{6}\right) e^{N\sum_{i=1}^{n}|\Delta_i\beta|^2} \tag{6.74}$$

lastly we analyze $\sup_{k\in\{1,\ldots,n\},r\in[0,1]} \left|\dot{A}_{\mathcal{P}}^{k}(r)\right|$:

$$\dot{A}_{\mathcal{P}}^{k}(r) = \left(\frac{d}{dt}|_0 R_{u_{\mathcal{P}}(t,s)}\right)\left(\beta_{\mathcal{P}}'(s),\cdot\right)\beta_{\mathcal{P}}'(s) + R_{u_{\mathcal{P}}(s)}\left(\frac{d}{dt}|_0\beta_{\mathcal{P}}'(t,s),\cdot\right)\beta_{\mathcal{P}}'(s)$$
$$+ R_{u_{\mathcal{P}}(s)}\left(\beta_{\mathcal{P}}'(s),\cdot\right)\frac{d}{dt}|_0\beta_{\mathcal{P}}'(t,s)$$

since M has the constant sectioanl curvature, then

$$\left(\frac{d}{dt}|_0 R_{u_{\mathcal{P}}(t,s)}\right)\left(\beta_{\mathcal{P}}'(s),\cdot\right)\beta_{\mathcal{P}}'(s) = 0.$$

notice that

$$\beta_{\mathcal{P}}'(t,s) = //_s(\sigma_t)^{-1}\sigma_{\mathcal{P}}'(t,s),$$

apply Lemma 6.27 and we have:

$$X^{h_{\alpha,j}}\beta_{\mathcal{P}}'(s_{k-1}+) = \frac{\delta_k^j e_\alpha}{\sqrt{\Delta_j}} - \int_0^{s_{k-1}} R_{u_{\mathcal{P}}(\tau)}\left(\beta_{\mathcal{P}}'(\tau+), h_{\alpha,j}(\tau)\right)d\tau\,\beta_{\mathcal{P}}'(s_{k-1}+) \tag{6.75}$$

Therefore,

$$\begin{aligned}\left|\dot{A}_{\mathcal{P}}^{k}(r)\right| &\leq N\left|X^{h_{\alpha,j}}\beta_{\mathcal{P}}'(s_{k-1}+)\right|\left|\beta_{\mathcal{P}}'(s_{k-1})\right| \\ &\leq N\left(\frac{1}{\sqrt{|\mathcal{P}|}}+N\sup_{j,s}\left|h_{\alpha,j}(s)\right|\sup_{s\in[0,1]}\left|\beta_{\mathcal{P}}'(s)\right|^{2}\right)\left|\beta_{\mathcal{P}}'(s_{k-1})\right| \\ &\leq N\left(\frac{1}{\sqrt{|\mathcal{P}|}}+Nf(K_{\gamma})\sqrt{|\mathcal{P}|}\,|\mathcal{P}|^{2(\gamma-1)}\right)K_{\gamma}\,|\mathcal{P}|^{\gamma-1} \\ &\leq f(K_{\gamma})\,|\mathcal{P}|^{3\gamma-\frac{5}{2}}\end{aligned}$$

where $f(K_{\gamma})$ is some random variable in $L^{1}(W_{o}(M))$, so

$$\left|\tilde{g}_{j}(1)\right| \leq Cf(K_{\gamma})\,|\mathcal{P}|^{5\gamma-\frac{3}{2}} \tag{6.76}$$

From there we can see:

$$\begin{aligned}\sum_{\alpha,j=1,1}^{d,n}\langle I,e_{\alpha}\rangle\sqrt{\Delta_{j}} &= \sum_{\alpha,j=1,1}^{d,n}\left\langle\left(X^{h_{\alpha,j}}T_{j}^{*}\right)\mathbf{K}_{\mathcal{P}}^{-1}(1)\,H_{\mathcal{P}},e_{\alpha}\right\rangle\sqrt{\Delta_{j}} \\ &= \sum_{\alpha=1}^{d}\left\langle\sum_{j=1}^{n}\left(\tilde{g}_{j}^{*}(1)\sqrt{|\mathcal{P}|}\right)\mathbf{K}_{\mathcal{P}}(1)^{-1}H_{\mathcal{P}},e_{\alpha}\right\rangle\end{aligned}$$

From (6.76) we know that $\sum_{j=1}^{n}\left(\tilde{g}_{j}^{*}(1)\sqrt{|\mathcal{P}|}\right)\to 0$ in $L^{\infty-}(W)$, also notice that $\mathbf{K}_{\mathcal{P}}(1)^{-1}H_{\mathcal{P}}\to\mathbf{K}(1)^{-1}\tilde{H}$ in $L^{\infty-}(W_{o}(M))$, so:

$$\sum_{\alpha=1}^{d}\left\langle\sum_{j=1}^{n}\left(\tilde{g}_{j}^{*}(1)\sqrt{|\mathcal{P}|}\right)\mathbf{K}_{\mathcal{P}}(1)^{-1}H_{\mathcal{P}},e_{\alpha}\right\rangle\to 0 \text{ in } L^{\infty-}(W_{o}(M)).$$

■

Lemma 6.31 *If M has constant sectional curvature, then*

$$\sum_{\alpha,j=1,1}^{d,n}\langle II,e_{\alpha}\rangle\sqrt{\Delta_{j}}\to 0$$

as $|\mathcal{P}| \to 0$ *in* $L^{\infty -}(W_o(M))$.

Proof. Notice that

$$X^{h_{\alpha,j}}\left(\mathbf{K}_{\mathcal{P}}(1)^{-1}\right) = -\mathbf{K}_{\mathcal{P}}(1)^{-1} X^{h_{\alpha,j}}\left(\mathbf{K}_{\mathcal{P}}(1)\right)\mathbf{K}_{\mathcal{P}}(1)^{-1}$$

so

$$\left|X^{h_{\alpha,j}}\left(\mathbf{K}_{\mathcal{P}}(1)^{-1}\right)\right| \leq \left|X^{h_{\alpha,j}}\left(\mathbf{K}_{\mathcal{P}}(1)\right)\right|$$

then use $\tilde{g}_j(s) := X^{h_{\alpha,j}}(\mathbf{K}_{\mathcal{P}}(s))$ and the Lemma follows from a Lemma 6.30-type argument. ■

Chapter 7

Proof of Main Theorem

Before proving Theorem 1.11, first we need some supplementary results. Recall that the manifold considered in Theorem 1.11 is a Hadamard manifold with constant sectional curvature.

Proposition 7.1 *For any $f \in \mathcal{FC}^{\infty}$, $X \in \Gamma(TM)$ with compact support,*

$$\tilde{X}^{tr,\nu} f \in L^{\infty-}(W_o(M), \nu).$$

Lemma 7.2

$$\sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1}\right)^* e_\alpha, d\beta_s \right\rangle \cdot f \in L^{\infty-}(W_o(M), \nu).$$

Proof. For any $v \in \mathbb{C}^d$,

$$\left\langle \left(\int_0^1 \tilde{T}_r^{-1} T^*(r)^{-1} dr \right) v, v \right\rangle = \int_0^1 \left\| T^*(r)^{-1} v \right\|^2 dr \geq C \left\| v \right\|^2$$

So

$$\left\| \left(\int_0^1 \tilde{T}_r^{-1} T^*(r)^{-1} dr \right)^{-1} \right\| \leq \frac{1}{C}$$

Since X has compact support and is smooth, $\|X(\cdot)\| \in C_0(M)$ and

$$\left\|\tilde{H}(\sigma)\right\| = \|X \circ E_1(\sigma)\| \le \sup \|X\| < C$$

also notice that $C(\sigma)$ is independent of (σ), so we have

$$\left\|\left\langle C(\sigma)\,\tilde{H}(\sigma), e_\alpha\right\rangle\right\| \le \|C(\sigma)\| \left\|\tilde{H}(\sigma)\right\| \le C \left\|\tilde{H}(\sigma)\right\| \le C.$$

Since

$$\left[\tilde{T}(s)^{-1}\right] \text{ is bounded}$$

$$\left[\tilde{T}(s)^{-1}\right] \in L^\infty([0,1])$$

Using Burkholder's inequality, we get:

$$\int_0^1 \left\langle \left[\tilde{T}(s)^{-1}\right]^* e_\alpha, d\beta_s \right\rangle \in L^{\infty-}(W_o(M))$$

Therefore,

$$\sum_{\alpha=1}^{d} \left\langle C(\sigma)\,\tilde{H}(\sigma), e_\alpha\right\rangle \int_0^1 \left\langle \left[\tilde{T}(s)^{-1}\right]^* e_\alpha, d\beta_s \right\rangle \cdot f \in L^{\infty-}(W_0(M))$$

■

Lemma 7.3

$$\sum_{\alpha=1}^{d} \left\langle C(\sigma)\, X^{Z_{\int_0^{\cdot}\left[\tilde{T}_r^{-1}\right]^* e_\alpha dr}} \tilde{H}, e_\alpha \right\rangle \cdot f \in L^{\infty-}(W_o(M), \nu).$$

Proof. From Lemma 4.43 we know:

$$-\sum_{\alpha=1}^{d}\left\langle X^{Z_{\int_0^\cdot[\tilde{T}(r)^{-1}]^*e_\alpha dr}}\left(C(\sigma)\tilde{H}(\sigma)\right),e_\alpha\right\rangle$$
$$=divX\circ E_1(\sigma)-\sum_{\alpha=1}^{d}\left\langle CA\left\langle Z_{\int_0^\cdot\left[\tilde{T}(r)^{-1}\right]^*e_\alpha dr}\right\rangle(1)\,\tilde{H}(\sigma),e_\alpha\right\rangle. \tag{7.1}$$

where

$$A\left\langle Z_{\int_0^\cdot\left[\tilde{T}(r)^{-1}\right]^*e_\alpha dr}\right\rangle(1)=\int_0^1 R_{\tilde{u}(s)}\left(Z_{\int_0^\cdot\left[\tilde{T}(r)^{-1}\right]^*e_\alpha dr}(s),\delta\beta_s\right) \tag{7.2}$$

Since $\int_0^\cdot\left[\tilde{T}(r)^{-1}\right]^* e_\alpha dr$ is bounded, by Gronwall's inequality one can see that $Z_{\int_0^\cdot\left[\tilde{T}(r)^{-1}\right]^*e_\alpha dr}$ is bounded and thus using Burkholder's inequality, we have:

$$A\left\langle Z_{\int_0^\cdot\left[\tilde{T}(r)^{-1}\right]^*e_\alpha dr}\right\rangle(1)\in L^{\infty-}(W_o(M)). \tag{7.3}$$

It is easy to see $divX\circ E_1(\sigma)$ is bounded because $X\in\Gamma(TM)$ with compact support. Therefore:

$$\sum_{\alpha=1}^{d}\left\langle C(\sigma)\,X^{Z_{\int_0^\cdot\left[\tilde{T}_r^{-1}\right]^*e_\alpha dr}}\tilde{H},e_\alpha\right\rangle\cdot f\in L^{\infty-}(W_o(M),\nu).$$

■

Proof of Proposition 7.1. Recall that from Lemma 4.43 and 4.42, we have:

$$\tilde{X}^{tr,\nu}f=-X^{Z_\Phi}f+\sum_{\alpha=1}^{d}\left\langle\tilde{C}\tilde{H},e_\alpha\right\rangle\int_0^1\left\langle\left(\tilde{T}_s^{-1}\right)^*e_\alpha,d\beta_s\right\rangle\cdot f$$
$$-\sum_{\alpha=1}^{d}\left\langle C(\sigma)\,X^{Z_{\int_0^\cdot\left[\tilde{T}_r^{-1}\right]^*e_\alpha dr}}\tilde{H},e_\alpha\right\rangle\cdot f$$

A similar argument as in Lemma 4.43 can show that $\tilde{X}f\in L^{\infty-}(W_o(M))$, then combine Lemma 7.2 and 7.3 and we can prove Proposition 7.1. ■

Lemma 7.4 *For any* $f\in\mathcal{FC}_{\mathcal{P}}^{\infty}$, $\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1}f\in L^{\infty-}(H_{\mathcal{P}}(M),\nu_{\mathcal{P}}^1)$.

Proof. From Theorem 6.23 we know that

$$\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f\left(\phi\left(b_{\mathcal{P}}\right)\right) - \tilde{X}\tilde{f} \to 0 \text{ in } L^{\infty-}\left(W_o\left(M\right)\right). \tag{7.4}$$

where $\tilde{f}\left(\sigma\right) = f\left(\tilde{u}\right) \in \mathcal{FC}^{\infty}$.

From Proposition 7.1 we know $\tilde{X}\tilde{f} \in L^{\infty-}\left(W_o\left(M\right)\right)$, so $\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f\left(\phi\left(b_{\mathcal{P}}\right)\right) \in L^{\infty-}\left(W_o\left(M\right)\right)$.

Since the law of $\phi\left(b_{\mathcal{P}}\right)$ under ν is $\nu_{\mathcal{P}}^1$, so

$$\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f \in L^{\infty-}\left(H_{\mathcal{P}}\left(M\right), \nu_{\mathcal{P}}^1\right) \iff \tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f\left(\phi\left(b_{\mathcal{P}}\right)\right) \in L^{\infty-}\left(W_o\left(M\right)\right).$$

■

Notation 7.5 *Denote by g any one of $\{g_i\}_{i=0}^d$ defined in Theorem 3.3 and $\left\{g^{(m)}\right\}_m$ be the approximating sequence in $L^{\frac{d}{d-1}}\left(M\right)$ as defined in Remark 3.5.*

Lemma 7.6 *Define $\tilde{g}\left(\sigma\right) = g\left(\sigma\left(1\right)\right)$ and $\tilde{g}^{(m)}\left(\sigma\right) = g^{(m)}\left(\sigma\left(1\right)\right)$, then for any $f \in \mathcal{FC}^{\infty}$,*

$$\int_{W(M)} \tilde{g}\left(\sigma\right) \tilde{X}^{tr,\nu} f\left(\sigma\right) d\nu\left(\sigma\right) \text{ makes sense}$$

and

$$\lim_{m\to\infty} \int_{W(M)} \tilde{g}^{(m)}\left(\sigma\right) \tilde{X}^{tr,\nu} f\left(\sigma\right) d\nu\left(\sigma\right) = \int_{W(M)} \tilde{g}\left(\sigma\right) \tilde{X}^{tr,\nu} f\left(\sigma\right) d\nu\left(\sigma\right)$$

Proof. Since $\nu\left\{\sigma : \sigma\left(1\right) = e\right\} = 0$, so $\tilde{g}$ is $\nu - a.s.$ well-defined. In particular, for any $p > 0$,

$$\int_{W(M)} \left|\tilde{g}\left(\sigma\right)\right|^p d\nu\left(\sigma\right) = \int_M \left|g\left(x\right)\right|^p p_1\left(0, x\right) d\lambda\left(x\right) \tag{7.5}$$

Since g has compact support and $p_1(0,\cdot) \in C^\infty(M)$,

$$\int_M |g(x)|^p p_1(0,x)\, d\lambda(x) \le C \|g\|^p_{L^p(M)} \tag{7.6}$$

Since $g \in L^{1+\frac{1}{d-1}}(M)$, we have

$$\tilde{g} \in L^{1+\frac{1}{d-1}}(W_o(M)).$$

Notice that from Proposition 7.1, we have $\tilde{X}^{tr,\nu} f \in L^{\infty-}(W_o(M))$, so by Holder's inequality, we get:

$$\int_{W(M)} \left| \tilde{g}(\sigma) \tilde{X}^{tr,\nu} f(\sigma) \right| d\nu(\sigma) < \infty.$$

To prove (7.6), just notice that $\cup_m suppg^{(m)}$ is compact, so we have, following the same argument as before:

$$\int_{W(M)} \left| \tilde{g}^{(m)} - \tilde{g} \right|^p (\sigma)\, d\nu(\sigma) = \int_M \left| g^{(m)}(x) - g(x) \right|^p p_1(0,x)\, d\lambda(x) \tag{7.7}$$

$$\le C \left\| g^{(m)} - g \right\|^p_{L^p(M)} \tag{7.8}$$

Using Holder's inequality again we can get (7.6). ∎

Lemma 7.7 *Define* $\tilde{g} : H_{\mathcal{P}}(M) \to \mathbb{R}$ *to be* $\tilde{g}(\sigma) = g(\sigma(1))$*, then*

$$\tilde{g} \in L^{\frac{d}{d-1}}\left(H_{\mathcal{P}}(M), \nu^1_{\mathcal{P}}\right).$$

Proof. Apply the Co-area formula to $|\tilde{g}|^{\frac{d}{d-1}}$, we have:

$$\int_{H_{\mathcal{P}}(M)} |\tilde{g}(\sigma)|^{\frac{d}{d-1}}\, d\nu^1_{\mathcal{P}}(\sigma) = \int_M |g(x)|^{\frac{d}{d-1}} h_{\mathcal{P}}(x)\, dx$$

where $h_{\mathcal{P}}(x) \in C(M)$ is defined in Theorem 3.33. Since g has compact support, we know:

$$\int_M |g(x)|^{\frac{d}{d-1}} h_{\mathcal{P}}(x)\,dx \le C\int_M |g(x)|^{\frac{d}{d-1}}\,dx \tag{7.9}$$

Therefore $\tilde{g} \in L^{\frac{d}{d-1}}\left(H_{\mathcal{P}}(M), \nu_{\mathcal{P}}^1\right)$. ■

Lemma 7.8 *Define $\tilde{g}(\sigma) = g(\sigma(1))$ and $\tilde{g}^{(m)}(\sigma) = g^{(m)}(\sigma(1))$, then for any $f \in \mathcal{FC}_{\mathcal{P}}^{\infty}$,*

$$\int_{H_{\mathcal{P}}(M)} \tilde{g}(\sigma)\,\tilde{X}^{tr,\nu_{\mathcal{P}}^1} f(\sigma)\,d\nu_{\mathcal{P}}^1(\sigma) \ \textit{ makes sense}$$

and

$$\lim_{m\to\infty}\int_{H_{\mathcal{P}}(M)} \tilde{g}^{(m)}(\sigma)\,\tilde{X}^{tr,\nu_{\mathcal{P}}^1} f(\sigma)\,d\nu_{\mathcal{P}}^1(\sigma) = \int_{H_{\mathcal{P}}(M)} \tilde{g}(\sigma)\,\tilde{X}^{tr,\nu_{\mathcal{P}}^1} f(\sigma)\,d\nu_{\mathcal{P}}^1(\sigma)$$

Proof. Using Lemma 7.4, Lemma 7.7 and Holder's inequality, we can easily see

$$\int_{H_{\mathcal{P}}(M)} \tilde{g}(\sigma)\,\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f(\sigma)\,d\nu_{\mathcal{P}}^1(\sigma) \ \text{ makes sense}$$

Then apply the co-area formula, we have:

$$\int_{H_{\mathcal{P}}(M)} \left|\left(\tilde{g}^{(m)} - \tilde{g}\right)(\sigma)\right|^{\frac{d}{d-1}} d\nu_{\mathcal{P}}^1(\sigma) = \int_M |(g^m - g)(x)|^{\frac{d}{d-1}} h_{\mathcal{P}}(x)\,dx$$

Since $h_{\mathcal{P}}(x) \in C(M)$ and $\cup_m supp\,(g^m - g)$ is compact, so

$$\int_M |(g^m - g)(x)|^{\frac{d}{d-1}} h_{\mathcal{P}}(x)\,dx \to 0 \text{ as } m \to 0$$

and

$$\tilde{g}^{(m)} - \tilde{g} \to 0 \text{ in } L^{\frac{d}{d-1}}\left(d\nu_{\mathcal{P}}{}^1\right)$$

Using Holder's inequality again we have:

$$\left|\int_{H_{\mathcal{P}}(M)} \left(\tilde{g}^{(m)}(\sigma) - \tilde{g}(\sigma)\right) \tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}} f(\sigma)\, d\nu_{\mathcal{P}}^{1}(\sigma)\right| \tag{7.10}$$

$$\leq \left\|\tilde{g}^{(m)} - \tilde{g}\right\|_{L^{\frac{d}{d-1}}\left(\nu_{\mathcal{P}}^{1}\right)} \left\|\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}} f\right\|_{L^{d}\left(\nu_{\mathcal{P}}^{1}\right)}. \tag{7.11}$$

Therefore

$$\lim_{m\to\infty} \int_{H_{\mathcal{P}}(M)} \tilde{g}^{(m)}(\sigma)\, \tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}} f(\sigma)\, d\nu_{\mathcal{P}}^{1}(\sigma) = \int_{H_{\mathcal{P}}(M)} \tilde{g}(\sigma)\, \tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}} f(\sigma)\, d\nu_{\mathcal{P}}^{1}(\sigma)$$

■

Theorem 7.9 *For any $f \in \mathcal{FC}_{\mathcal{P}}^{\infty}$:*

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} \tilde{g}(\sigma)\, \tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}} f(\sigma)\, d\nu_{\mathcal{P}}^{1}(\sigma) = \int_{W(M)} \tilde{g}(\sigma)\, \tilde{X}^{tr,\nu} \tilde{f}(\sigma)\, d\nu(\sigma)$$

Proof. Since the law of $\phi \circ b_{\mathcal{P}}$ under ν is ${\nu_{\mathcal{P}}}^{1}$, we have:

$$\int_{H_{\mathcal{P}}(M)} \tilde{g}(\sigma)\, \tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}} f(\sigma)\, d\nu_{\mathcal{P}}^{1}(\sigma) = \mathbb{E}_{\nu}\left[\tilde{g} \cdot \tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}} f(\phi \circ b_{\mathcal{P}})\right] \tag{7.12}$$

Also

$$\int_{W(M)} \tilde{g}(\sigma)\, \tilde{X}^{tr,\nu} \tilde{f}(\sigma)\, d\nu(\sigma) = \mathbb{E}_{\nu}\left[\tilde{g} \cdot \tilde{X}^{tr,\nu} \tilde{f}\right] \tag{7.13}$$

So

$$\left|\int_{H_{\mathcal{P}}(M)} \tilde{g}(\sigma)\, \tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}} f(\sigma)\, d\nu_{\mathcal{P}}^{1}(\sigma) - \int_{W(M)} \tilde{g}(\sigma)\, \tilde{X}^{tr,\nu} \tilde{f}(\sigma)\, d\nu(\sigma)\right| \tag{7.14}$$

$$\leq \mathbb{E}\left[\left|\tilde{g} \cdot \tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}} f(\phi \circ b_{\mathcal{P}}) - \tilde{g} \cdot \tilde{X}^{tr,\nu} \tilde{f}\right|\right] \tag{7.15}$$

$$\leq \mathbb{E}\left[\left|\tilde{g}(\phi \circ \beta_{\mathcal{P}})\right| \cdot \left|\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}} f(\phi \circ b_{\mathcal{P}}) - \tilde{X}^{tr,\nu} \tilde{f}\right|\right] + \mathbb{E}\left[\left|\tilde{g}(\phi \circ b_{\mathcal{P}}) - \tilde{g}\right| \cdot \left|\tilde{X}^{tr,\nu} \tilde{f}\right|\right] \tag{7.16}$$

From Lemma 7.7, we have

$$\tilde{g}\left(\phi \circ \beta_{\mathcal{P}}\right) \in L^{\frac{d}{d-1}}\left(W_o\left(M\right)\right)$$

and from Theorem we have

$$\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f\left(\phi \circ b_{\mathcal{P}}\right) - \tilde{X}^{tr,\nu}\tilde{f} \to 0 \text{ in } L^{\infty}\left(W_o\left(M\right)\right).$$

So by Holder's inequality,

$$\mathbb{E}\left[\left|\tilde{g}\left(\phi \circ \beta_{\mathcal{P}}\right)\right| \cdot \left|\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f\left(\phi \circ b_{\mathcal{P}}\right) - \tilde{X}^{tr,\nu}\tilde{f}\right|\right] \to 0 \text{ as } \left|\mathcal{P}\right| \to 0. \tag{7.17}$$

Then we consider

$$\mathbb{E}\left[\left|\tilde{g}\left(\phi \circ b_{\mathcal{P}}\right) - \tilde{g}\right| \cdot \left|\tilde{X}^{tr,\nu}\tilde{f}\right|\right]$$

By Holder's inequality,

$$\mathbb{E}\left[\left|\tilde{g}\left(\phi \circ b_{\mathcal{P}}\right) - \tilde{g}\right| \cdot \left|\tilde{X}^{tr,\nu}\tilde{f}\right|\right] \leq \mathbb{E}\left[\left|\tilde{g}\left(\phi \circ b_{\mathcal{P}}\right) - \tilde{g}\right|^p\right]^{\frac{1}{p}} \cdot \mathbb{E}\left[\left|\tilde{X}^{tr,\nu}\tilde{f}\right|^q\right]^{\frac{1}{q}} \tag{7.18}$$

where $p > 0$ and $q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

From Proposition 7.1 we know $\tilde{X}^{tr,\nu}\tilde{f} \in L^{\infty-}\left(W_o\left(M\right)\right)$, therefore in order to show

$$\left|\int_{H_{\mathcal{P}}(M)} \tilde{g}\left(\sigma\right) \tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f\left(\sigma\right) d\nu_{\mathcal{P}}^1\left(\sigma\right) - \int_{W(M)} \tilde{g}\left(\sigma\right) \tilde{X}^{tr,\nu}\tilde{f}\left(\sigma\right) d\nu\left(\sigma\right)\right| \to 0 \text{ as } \left|\mathcal{P}\right| \to 0, \tag{7.19}$$

It suffices to show there exists $p > 1$ such that

$$\mathbb{E}_{\nu}\left[\left|\tilde{g}\left(\phi \circ b_{\mathcal{P}}\right) - \tilde{g}\right|^p\right] \to 0. \tag{7.20}$$

Notice that for some $\epsilon > 0$ to be determined,

$$\left|\tilde{g}\left(\phi \circ b_{\mathcal{P}}\right) - \tilde{g}\right|^{p(1+\epsilon)} \leq C_{p,\epsilon}\left(\left|\tilde{g}\left(\phi \circ b_{\mathcal{P}}\right)\right|^{p(1+\epsilon)} + \left|\tilde{g}\right|^{p(1+\epsilon)}\right)$$

It is easy to see that as long as $p\left(1+\epsilon\right) < \frac{d}{d-1}$, $\mathbb{E}\left[\left|\tilde{g}\right|^{p(1+\epsilon)}\right] < \infty$.

Lemma 7.10 *For any* $p \leq \frac{d}{d-1}$,

$$\sup_{\mathcal{P}} \mathbb{E}\left[\left|\tilde{g}\left(\phi \circ b_{\mathcal{P}}\right)\right|^{p}\right] < \infty. \tag{7.21}$$

Proof. Since the law of $\phi \circ b_{\mathcal{P}}$ under ν is $\nu_{\mathcal{P}}^{1}$, we have:

$$\mathbb{E}\left[\left|\tilde{g}\left(\phi \circ b_{\mathcal{P}}\right)\right|^{p}\right] = \int_{H_{\mathcal{P}}(M)} \left|\tilde{g}\right|^{p}\left(\sigma\right) d\nu_{\mathcal{P}}^{1}\left(\sigma\right). \tag{7.22}$$

Then apply co-area formula, we get:

$$\int_{H_{\mathcal{P}}(M)} \left|\tilde{g}\right|^{p}\left(\sigma\right) d\nu_{\mathcal{P}}^{1}\left(\sigma\right) = \int_{M} \left|g\left(x\right)\right|^{p} h_{\mathcal{P}}\left(x\right) dx \tag{7.23}$$

where $h_{\mathcal{P}}\left(x\right)$ is defined as in Theorem 3.12.

Apply Proposition 3.32 we know that:

$$\sup_{\mathcal{P}} h_{\mathcal{P}}\left(x\right) < \infty \tag{7.24}$$

Since g has compact support, $\sup_{\mathcal{P}} h_{\mathcal{P}}\left(x\right)$ is bounded on its support and the bound is independent of $\mathcal{P}$, from there it follows that (using Holder's inequality):

$$\sup_{\mathcal{P}} \int_{M} \left|g\left(x\right)\right|^{p} h_{\mathcal{P}}\left(x\right) dx < \infty \tag{7.25}$$

■ Apply Lemma 7.10 with a choice of ϵ such that $p\left(1+\epsilon\right) < \frac{d}{d-1}$, we get:

$$\sup_{\mathcal{P}} \mathbb{E}_{\nu}\left[\left|\tilde{g}\left(\phi \circ b_{\mathcal{P}}\right) - \tilde{g}\right|^{p(1+\epsilon)}\right] < \infty \tag{7.26}$$

Therefore

$$\{|\tilde{g}(\phi \circ b_{\mathcal{P}}) - \tilde{g}|^{p}\} \text{ is uniformly integrable under } \nu.$$

Then consider

$$U_{\mathcal{P}} := \left\{\sigma \in W_{o}(M) : \pi \circ \Phi^{-1} \circ \pi_{\mathcal{P}} \circ \tilde{\Phi} \circ \Sigma(\sigma) = e\right\} \tag{7.27}$$

Since the law of $\Phi^{-1} \circ \pi_{\mathcal{P}} \circ \tilde{\Phi} \circ \Sigma$ under ν is $\nu^{1}_{\mathcal{P}}$, denote

$$V_{\mathcal{P}} := \{\sigma \in H_{\mathcal{P}}(M) : E_{1}(\sigma) = e\} \tag{7.28}$$

Combine this with uniform integrability, we get

Then

$$\nu^{1}_{\mathcal{P}}(V_{\mathcal{P}}) = \nu(U_{\mathcal{P}}).$$

$$\left\{b \in W_{0}\left(\mathbb{R}^{d}\right) : (\pi \circ \phi \circ \pi_{\mathcal{P}})(b) = e\right\}$$

Apply the co-area formula with $f(x) = 1_{\{x=e\}}$, we get:

$$\nu^{1}_{\mathcal{P}}(V_{\mathcal{P}}) = \int_{H_{\mathcal{P}}(M)} f(\sigma(1))\, d\nu^{1}_{\mathcal{P}}(\sigma) = \int_{M} f(x)\, h_{\mathcal{P}}(x)\, dx = 0. \tag{7.29}$$

From there we can construct a $\nu-$Null set:

$$N := \cup_{\mathcal{P}} U_{\mathcal{P}} \cup \{\sigma \in W_{o}(M) : E_{1}(\sigma) = e\}.$$

Recall from Corollary 6.3, we have

$$\mathbb{E}_{\nu}\left[|u_{\mathcal{P}}(1) - \tilde{u}(1)|^{q}\right] \to 0 \text{ as } |\mathcal{P}| \to 0 \text{ for any } q \geq 1. \tag{7.30}$$

This implies that

$$|u_{\mathcal{P}}(1) - \tilde{u}(1)| \to 0 \text{ in probability.}$$

Notice that $g \in C^{\infty}(M/e)$ and $\pi : \mathcal{O}(M) \to M$ is smooth, so excluding N, we have

$$|\tilde{g}(\phi \circ b_{\mathcal{P}}) - \tilde{g}| = |g \circ \pi(u_{\mathcal{P}}(1)) - g \circ \pi(\tilde{u}(1))| \to 0 \text{ in probability.} \tag{7.31}$$

Combine 7.20 and 7.31 we know

$$\mathbb{E}\left[|\tilde{g}(\phi \circ b_{\mathcal{P}}) - \tilde{g}| \cdot \left|\tilde{X}^{tr,\nu}\tilde{f}\right|\right] \to 0$$

■

Notation 7.11 *Denote by $\mathcal{FC}_{1-}^{\infty}$ the subspace of $\mathcal{FC}^{\infty}$consisting of functions that are $\mathcal{F}_s$ – measurable for some $s < 1$.*

Proposition 7.12 *Let $f \in L^1(W_o(M), d\nu)$ be $\mathcal{F}_s$ – measurable for some $s < 1$, then*

$$\lim_{m\to\infty}\int_{W_o(M)} \delta_x^{(m)}(\Sigma_1) f d\nu = \int_{W_o(M)} f d\nu_x.$$

Here $\mathcal{F}_s := \sigma(\Sigma_r : 0 \le r \le s)$ where $\Sigma_r(\sigma) = \sigma(r)$ is the canonical process on $W_o(M)$.

Proof. First off recall that ν_x and ν are absolutely continuous relative to one another when restricted to $\mathcal{F}_s$ and in fact,

$$d\nu_x = p_{1-s}(\Sigma_s, x)\, d\nu, \tag{7.32}$$

where $p_t(x, y)$ is the heat kernel for (M, g). As $p_{1-s}(\Sigma_s, x)$ is bounded on M, we may conclude that $f \in L^1(W_o(M), d\nu_x)$ for all $x \in M$ so that $\int_{W_o(M)} f d\nu_x$ is well defined.

By the Markov property

$$\int_{W_o(M)} \delta_x^{(m)}(\Sigma_1) f d\nu = \int_{W_o(M)} \left(P_{1-s}\delta_x^{(m)}\right)(\Sigma_s) f d\nu, \tag{7.33}$$

where (using dz for the Riemannian volume measure),

$$\left(P_{1-s}\delta_x^{(m)}\right)(y) = \int_M p_{1-s}(y,z)\,\delta_x^{(m)}(z)\,dz \to p_{1-s}(y,x) \text{ as } m \to \infty.$$

Using this limiting result, Eq. (7.32), the fact that

$$\left|\left(P_{1-s}\delta_x^{(m)}\right)(y)\right| \leq \sup_{y,z\in M} p_{1-s}(y,z) =: K_s < \infty,$$

along with DCT, we may pass to the limit in Eq. (7.33) to find

$$\lim_{m\to\infty} \int_{W_o(M)} \delta_x^{(m)}(\Sigma_1) f d\nu = \int_{W_o(M)} p_{1-s}(\Sigma_s, x) f d\nu = \int_{W_o(M)} f d\nu_x.$$

■

Proof of Theorem 1.11. Recall from Remark 3.5 that we can approximate the delta mass δ_x on M in the following way:

$$\delta_x^{(m)} := g_0^{(m)} + \sum_{j=1}^{d} X_j g_j^{(m)} \in C_0^\infty(M)$$

and

$$\delta_x^{(m)} \to \delta_x \text{ in } \mathcal{D}'(M)$$

where $\left\{g_j^{(m)} : 0 \leq j \leq d, m \geq 1\right\} \subset C_0^\infty(M)$ and $\{X_j : 1 \leq j \leq d\} \subset \Gamma(TM)$ with compact supports. Using the Orthogonal lift, we get:

$$\delta_x^{\tilde{(m)}} := g_0^{\tilde{(m)}} + \sum_{j=1}^{d} X_{\mathcal{P},j} g_j^{\tilde{(m)}} \in C_0^\infty(M)$$

where $\tilde{g}(\sigma) = g \circ E_1(\sigma)$ for any $g \in C(M)$ and $X_{\mathcal{P},i}$ is the Orthogonal lift of X_i into $\Gamma(TH_{\mathcal{P}}(M))$.

For any $0 \leq j \leq d$ (with the convention that $X_{\mathcal{P},0} = I$), using integration by parts, we get:

$$\int_{H_{\mathcal{P}}(M)} \left(\tilde{g_0^{(m)}} + \sum_{j=1}^{d} X_{\mathcal{P},j} \tilde{g_j^{(m)}} \right) f d\nu_{\mathcal{P}}^1 = \int_{H_{\mathcal{P}}(M)} \left(\tilde{g_0^{(m)}} \cdot f + \sum_{j=1}^{d} X_{\mathcal{P},j}^{tr,\nu_{\mathcal{P}}^1} f \cdot \tilde{g_j^{(m)}} \right) d\nu_{\mathcal{P}}^1 \tag{7.34}$$

Now let $m \to \infty$, from Corollary 3.33 we have:

$$\text{LHS of 7.34} = \int_{H_{\mathcal{P},x}(M)} f d\nu_{\mathcal{P},x}^1$$

Apply Lemma 7.8 to each $\left(\tilde{g_j^{(m)}}, X_{\mathcal{P},j} \right)$, we have:

$$\text{RHS of 7.34} = \int_{H_{\mathcal{P}}(M)} \left(\tilde{g}_0 \cdot f + \sum_{j=1}^{d} X_{\mathcal{P},j}^{tr,\nu_{\mathcal{P}}^1} f \cdot \tilde{g}_j \right) d\nu_{\mathcal{P}}^1 \tag{7.35}$$

Then let $|\mathcal{P}| \to 0$, from Theorem 7.9 we have:

$$\lim_{|\mathcal{P}| \to 0} \int_{H_{\mathcal{P},x}(M)} f d\nu_{\mathcal{P},x}^1 = \int_{W_o(M)} \left(\tilde{g}_0 \cdot f + \sum_{j=1}^{d} \tilde{X}_j^{tr,\nu} f \cdot \tilde{g}_j \right) d\nu \tag{7.36}$$

According to Lemma 7.6,

$$\int_{W_o(M)} \left(\tilde{g}_0 \cdot f + \sum_{j=1}^{d} \tilde{X}_j^{tr,\nu} f \cdot \tilde{g}_j \right) d\nu \tag{7.37}$$

$$= \lim_{m \to \infty} \int_{W_o(M)} \left(\tilde{g_0^{(m)}} \cdot f + \sum_{j=1}^{d} \tilde{X}_j^{tr,\nu} f \cdot \tilde{g_j^{(m)}} \right) d\nu \tag{7.38}$$

Then use integration by parts formula developed in Lemma 4.42 we have:

$$\int_{W_o(M)} \left(g_0^{\tilde{(m)}} \cdot f + \sum_{j=1}^{d} \tilde{X}_j^{tr,\nu} f \cdot g_j^{\tilde{(m)}} \right) d\nu = \int_{W_o(M)} \left(g_0^{\tilde{(m)}} + \sum_{j=1}^{d} \tilde{X}_j g_j^{\tilde{(m)}} \right) \cdot f d\nu \tag{7.39}$$

$$= \int_{W_o(M)} \tilde{\delta_x}^{(m)} f d\nu. \tag{7.40}$$

If $f \in \mathcal{FC}_{1-}^{\infty}$, then apply Proposition 7.12 we have

$$\int_{W_o(M)} \tilde{\delta_x}^{(m)} f d\nu \to \int_{W_o(M)} f d\nu_x.$$

Therefore

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P},x}(M)} f d\nu_{\mathcal{P},x}^1 = \int_{W_o(M)} f d\nu_x. \tag{7.41}$$

■

Appendix A

Riemannian Manifolds

A.1 Hadamard Manifold

Definition A.1 (Hadamard Manifold) *A Hadamard manifold is a complete Riemannian manifold, simply connected and with non–positive sectional curvature.*

Hadamard manifolds share very nice global properties as recorded in the following theorem as the Theorem of Hadamard.

Theorem A.2 *If M is a Hadamard manifold, then M is diffeomorphic to $\mathbb{R}^d$, $d = \dim M$; more precisely for any $x \in M$, $\exp_x : T_xM \to M$ is a diffeomorphism.*

A.2 Connections on Principal Bundle

Notation A.3 *Denote by $\Gamma^\infty(TM)$ the smooth sections of the tangent bundle. You can think of this as the space of smooth vector field.*

Definition A.4 (Affine connection) *An affine connection is a map $\nabla : \Gamma(TM) \times \Gamma(TM) \mapsto \Gamma(TM)$ or $(X, Y) \mapsto \nabla_X Y$ satisfying the following conditions: for*

$X, Y, Z \in \Gamma(TM)$ *and* $f, g \in C^{\infty}(M)$:

$$\begin{aligned} \nabla_X fY &= (Xf)\, Y + f\nabla_X fY \\ \nabla_X (Y+Z) &= \nabla_X Y + \nabla_X Z \\ \nabla_{fX+gY} Z &= f\nabla_X Z + g\nabla_Y Z \end{aligned}$$

Definition A.5 *An affine connection* ∇ *is said to be* ***metric compatible*** *if the following is true for any* $X, Y, Z \in \Gamma(TM)$*:*

$$\nabla_Z g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

A metric compatible connection is also called the ***metric connection****.*

Definition A.6 *For any* $X, Y, Z \in \Gamma(TM)$, *define the* ***Riemann curvature tensor*** $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ *and* ***torsion tensor*** $T : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ *to be:*

$$\begin{aligned} R(X, Y, Z) &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ T(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y] \end{aligned}$$

A connection is said to be symmetric if $T \equiv 0$.

Theorem A.7 (Levi-Civita) *There exists a unique symmetric metric connection, which is called the Levi-Civita connection.*

Throughout this paper we stick with the Levi-Civita connection ∇.

Definition A.8 (Principal bundle) *A principal bundle* $(P, G, \pi, M, \{U_\alpha\}, \phi_\alpha)$ *consists of the following data:*

- P, M *are smooth manifolds.* $\pi : P \to M$ *smooth submersion is called the fibre projection map.*

- *A Lie group G is said to be the structure group of P: i.e. G admits a free and transitive group action on P on the right:*

$$(G, P) \ni (g, u) \to u \cdot g \in P$$

- *(Local trivialization) $\{U_\alpha\}$ is an open covering of M, then $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is a diffeomorphism.*

Example A.9 (Frame bundle $L(M)$) *Let G be the general linear group $GL(d, \mathbb{R})$ where $d = \dim M$ and for each $x \in M$, denote by $L(M)_x$ the linear frames of T_xM (Here we will identify a linear frame with a linear isomorphism from $\mathbb{R}^d \to T_xM$). Then $L(M) := \cup_{x \in M} L(M)_x$ can be made a principal bundle with structure group $GL(d, \mathbb{R})$. We will call this principal bundle the frame bundle over M, simply denoted by $L(M)$.*

Example A.10 (Orthonormal frame bundle $(\mathcal{O}(M), \pi)$) *See Definition 2.1*

Definition A.11 (Fundamental vector field) *Given a principal bundle P over M with structure group G, for any $p \in M$, dentote by $G_p := \pi^{-1}(\{p\})$ the fiber at $p = \pi(u)$. Let V_uP be the tangent space of P at u which is tangent to G_p. Since $G_p \cong G$, so*

$$\dim V_uP = \dim G = \dim \mathfrak{g}.$$

One can construct a base of V_uP in the following way: take a basis $\{A_i\}$ of $\mathfrak{g}$, consider

$$u(s) := u \exp(sA_i)$$

then $u(s)$ is a differentiable curve on V_uP with $u(0) = u$. Define:

$$A_i^\dagger := \frac{d}{ds} |_0 \, u(s)$$

This is called the fundamental vector field generated by A_i. Using substitution, one can see that the map $A \to A^\dagger$ is a real vector space isomorphism. (Actually this is a Lie algebra isomorphism.) However, there is no unique way to specify the "orthogonal compliment" of this vector bundle VP unless some more structures are involved, which is called connection on P.

Definition A.12 (Connection on principal bundle) *A (smooth) connection on a principal bundle P is a choice of (smooth) decomposition of the tangent bundle TP over P as follows, for any $u \in P$:*

$$T_uP = V_uP \oplus H_uP$$

and

$$H_{ug}P = R_{g*}H_uP$$

where $R_g : P \ni u \to ug \in P$ is the right action of G on P.

Definition A.13 (Connection one-form) *A connection one-form is a Lie-algebra-valued one form on P, i.e. $\omega \in \mathfrak{g} \otimes T^*P$ satisfying the following requirement:*

$$\begin{array}{lll} (i) & \omega\left(A^\dagger\right) = A & \text{for any } A \in \mathfrak{g} \\ (ii) & R_g^*\omega = Ad_{g^{-1}}\omega & \text{for any } g \in G \end{array} \tag{A.1}$$

here $Ad_{g^{-1}}X = g^{-1}Xg$ for any $X \in \mathfrak{g}$.

Remark A.14 *Given a smooth connection on P, we can naturally get a connection one-form ω in the following way: for each $X \in T_uP$, there exists unique $A_X \in \mathfrak{g}$ and $X^H \in H_uP$ such that $X = A_X^\dagger + X^H$. define $\omega(X) = A_X$. It is easy to see that ω satisfies A.1. Conversely, given a smooth connection one-form ω, we can define $H_uP = \ker \omega_u$ and it gives a smooth connection on P.*

Remark A.15 *It is known that a smooth connection on a principal bundle P induces a smooth connection on its associated vector bundles. In particular, it gives* *rise to a connection on* M *defined as in Definition A.4. There are usually two ways to see that. One is to use the connection on* P *to derive "horizontal lift" and further parallel translation, then use parallel translation to define covariant derivative and further a connection on* M*. Interested readers can refer to the Chapter III section 1 in the classical book [27] by Kobayashi and Nomizu for a more detailed exposition. The other way is to use local one-forms of* ω *in* P *and the push-forward of the representation of* G *to derive a compatible local one-forms on* M *from which one can construct a connection on* M*.*

Conversely, an affine connection on M *gives rise to a connection on the frame bundle* $L(M)$ *introduced on Example A.9, see Chapter III section 2 in [27] and section 2.1 in [22]. In particular, if the connection* ∇ *is a metric connection on* M*, the connection on* $L(M)$ *reduces to a connection on* $\mathcal{O}(M)$*. Throughout this paper we will fix* ∇ *to be the Levi-civita connection and consider only the connection on* $\mathcal{O}(M)$ *induced by* ∇*. We also fix a* $u_0 \in \mathcal{O}(M)_o$ *so that* $\mathcal{O}(M)$ *becomes a pointed manifold and further we use* u_0 *to identify* T_oM *with* $\mathbb{R}^d$*.*

Remark A.16 π *induces an isomorphism* $\pi_* : H_u\mathcal{O}(M) \to T_{\pi(u)}M$ *following the decomposition specified by* ∇*. This is a result of the fact that* $\pi_*\{V_u\mathcal{O}(M)\}$ *and* $\dim T_u\mathcal{O}(M) = d + \dim \mathfrak{so}(d) = d + \dim V_u\mathcal{O}(M)$*. Therefore for any* $x \in M$*,* $u \in \pi^{-1}(\{x\})$*,* $X \in T_xM$*, there exists a unique tangent vector* $X^* \in H_u\mathcal{O}(M)$ *such that* $\pi_*X^* = X$*.* X^* *is called the* ***horizontal lift*** *of* X *to* u*. ss*

Appendix B

ODE estimates

Proposition B.1 *Consider an ODE:*

$$Y''(s) = A(s)\, Y(s)$$

where $Y(s), A(s) \in M_{n\times n}(\mathbb{R})$ *are real* $n \times n$ *matrices and* $A(s)$ *is positive semi-definite.*

Denote by $\{C(s), S(s)\}$*the solutions to this ODE with initial values:*

$$C(0) = I, C'(0) = 0 \text{ and } S(0) = 0, S'(0) = I$$

Recall that in this paper we use $eig(X)$ *to denote the set of eigenvalues of matrix* X*. Then*

- *If* $\lambda \in eig(C(s))$, *then* $|\lambda| \geq 1$.
- *If* $\lambda \in eig(S(s))$, *then* $|\lambda| \geq s$.

Proof. For all $v \in \mathbb{C}^d$, define $v(s) := C(s)\, v$, then:

$$\langle v''(s), v(s)\rangle = \langle A(s)\, v(s), v(s)\rangle \geq 0.$$

Therefore,

$$\frac{d}{ds}\left\langle v'\left(s\right),v\left(s\right)\right\rangle = \left\langle v''\left(s\right),v\left(s\right)\right\rangle + \left\|v'\left(s\right)\right\|^2 \geq 0.$$

Since $\left\langle v'\left(0\right),v\left(0\right)\right\rangle = 0$, so $\left\langle v'\left(s\right),v\left(s\right)\right\rangle \geq 0$. Therefore

$$\frac{d}{ds}\left\|v\left(s\right)\right\|^2 = 2Re\left\langle v'\left(s\right),v\left(s\right)\right\rangle \geq 0.$$

Notice that $\left\|v\left(0\right)\right\|^2 = \left\|v\right\|^2$, so

$$\left\|v\left(s\right)\right\|^2 \geq \left\|v\right\|^2.$$

Therefore if $\lambda \in eig\left(C\left(s\right)\right)$, choose $v \in \mathbb{C}^d$ to be an eigenvector associated to λ, then

$$\left\|\lambda v\right\|^2 = \left\|C\left(s\right)v\right\|^2 \geq \left\|v\right\|^2.$$

So

$$|\lambda| \geq 1.$$

Therefore $C\left(s\right)$ is invertible and

$$\left\|C\left(s\right)\right\| = \max_{\lambda \in eig(C(s))} |\lambda| \geq 1.$$

A lower bound result for $\|S(s)v\|$ can be found in [29, Appendix E]:

$$\|S(s)v\| \geq s\left\|v\right\|.$$

From there it follows

$$\text{If } \lambda \in eig\left(S\left(s\right)\right), \text{ then } |\lambda| \geq s$$

and $S(s)$ is invertible with

$$\|S(s)\| = \max_{\lambda \in eig(S(s))} |\lambda| \geq s.$$

■

Definition B.2 *Denote $R_{u(s)}(\xi, \cdot)\xi$ by $A_\xi(s)$, $(C_\xi(s), S_\xi(s))^t$ is the fundamental solution to the ODE:*

$$V'(s) = \begin{pmatrix} 0 & 1 \\ A_{\xi_x} & 0 \end{pmatrix} V(s)$$

Proposition B.3 *If R is bounded by a constant N, i.e. $|R(\xi, \cdot)\xi| \leq N|\xi|^2$, then*

$$|C_\xi(s)| \leq \cosh\left(\sqrt{N}|\xi| s\right) \leq e^{\frac{1}{2}N|\xi|^2 s^2} \tag{B.1}$$

$$\begin{aligned} |S_\xi(s)| &\leq \sqrt{N}|\xi| s \frac{\sinh\left(\sqrt{N}|\xi| s\right)}{\sqrt{N}|\xi| s} \\ &\leq \cosh\left(\sqrt{N}|\xi| s\right)\sqrt{N}|\xi| s \\ &\leq \sqrt{N}|\xi| s e^{\frac{1}{2}N|\xi|^2 s^2} \end{aligned} \tag{B.2}$$

$$|S_\xi(s) - sI| \leq \frac{N|\xi|^2 s^3}{6} e^{\frac{1}{2}N|\xi|^2 s^2} \tag{B.3}$$

and

$$|C_\xi(s) - I| \leq \frac{N|\xi|^2 s^2}{2} e^{\frac{1}{2}N|\xi|^2 s^2} \tag{B.4}$$

Proof. B.1 and B.2 are quite elementary, so here we only resent the proof of B.3 and B.4.

By Taylor's expansion,

$$S_\xi(s) = sI + \int_0^s R_{\tilde{u}_r}(\xi, S_\xi(r))\xi(s-r)\,dr.$$

$$\begin{aligned}\left|S_{\xi}(s)-sI\right| &\leq N\left|\xi\right|^{2}\int_{0}^{s}\left|S_{\xi}(r)\right|(s-r)\,dr\\ &\leq N\left|\xi\right|^{2}\int_{0}^{s}\left[\left|S_{\xi}(r)-rI\right|+r\right](s-r)\,dr\end{aligned}$$

Define $f(s):=\left|S_{\xi}(s)-sI\right|$, then we have:

$$f(s)\leq\int_{0}^{s}N\left|\xi\right|^{2}(s-r)\,f(r)\,dr+N\left|\xi\right|^{2}\frac{s^{3}}{6}$$

By Gronwall's inequality:

$$f(s)\leq N\left|\xi\right|^{2}\frac{s^{3}}{6}e^{\frac{1}{2}N|\xi|^{2}s^{2}}$$

Then we consider $C_{\xi}(s)$:

$$C_{\xi}(s)=I+\int_{0}^{s}R_{\tilde{u}_{r}}\left(\xi,C_{\xi}(r)\right)\xi\,(s-r)\,dr.$$

So

$$\begin{aligned}\left|C_{\xi}(s)-I\right| &\leq N\left|\xi\right|^{2}\int_{0}^{s}\left|C_{\xi}(r)\right|(s-r)\,dr\\ &\leq N\left|\xi\right|^{2}\int_{0}^{s}\left[\left|C_{\xi}(r)-I\right|+1\right](s-r)\,dr.\end{aligned}$$

Define $f(s):=\left|C_{\xi}(s)-I\right|$, then we have:

$$f(s)\leq\int_{0}^{s}N\left|\xi\right|^{2}(s-r)\,f(r)\,dr+N\left|\xi\right|^{2}\frac{s^{2}}{2}.$$

By Gronwall's inequality:

$$f(s)\leq N\left|\xi\right|^{2}\frac{s^{2}}{2}e^{\frac{1}{2}N|\xi|^{2}s^{2}}.$$

■

Appendix C

Calculus on Differential Forms

Theorem C.1 (change of variable formula on manifold) *If $F : M \to N$ is an orientation preserving diffeomorphism and α is a $d-$form on N with $d = \dim M$. Then $F^*\alpha$ is a $d-$form on M and the following is true:*

$$\int_M F^*\alpha = \int_N \alpha. \tag{C.1}$$

In particular, if M and N are Riemannian manifolds with volume forms vol_M and vol_N, then

$$F^* vol_N = \mathcal{J}_F vol_M. \tag{C.2}$$

where $\mathcal{J}_F = \sqrt{\det (DF)^{tr} DF}$.

Proof. Since the integral of forms are independent of the choice of open coverings, so it suffices to prove for in a chart (U, x) of N,

$$\int_{F^{-1}(U)} F^*\alpha = \int_U \alpha$$

Locally on U, $\alpha = f(x)\, dx_1 \wedge \cdots \wedge dx_d$ and $F^*\alpha = f \circ F d(x_1 \circ F) \wedge \cdots \wedge d(x_d \circ F)$.

Choose a chart map y on $F^{-1}(U) \cong \mathbb{R}^d$, then

$$F^*\alpha = f \circ F \circ y^{-1} d\left(x_1 \circ F \circ y^{-1}\right) \wedge \cdots \wedge d\left(x_d \circ F \circ y^{-1}\right) \tag{C.3}$$

$$= f \circ F \circ y^{-1} \det\left(\frac{\partial\left(x_i \circ F \circ y^{-1}\right)}{\partial y_j}\right) dy_1 \wedge \cdots \wedge dy_d \tag{C.4}$$

Notice that F is orientation preserving, so Equation C.1 is easily follows from the change of variable formula on $\mathbb{R}^d$ applied to $x \circ F \circ y^{-1} : \mathbb{R}^d \to \mathbb{R}^d$. Equation C.1 is thus easily obtained by using orthonormal frames on M and N. ■

C.1 A Structure Theorem for $\mathrm{div}_g\left(\tilde{X}\right)$

This section is devoted to a structure theorem for $\mathrm{div}_g\left(\tilde{X}\right)$ which is t

Let $\pi : (M, g) \to (N, h)$ be a submersion of two smooth Riemannian manifolds. To each $m \in M$ and $v \in T_{\pi(m)}N$, let $\hat{v} := \pi_{*m}^{\mathrm{tr}}\left(\pi_{*m}\pi_{*m}^{\mathrm{tr}}\right)^{-1} v \in T_m M$ so that $\hat{v}$ is the unique shortest vector in $T_m M$ such that $\pi_{*m}\hat{v} = v$. So if $X \in \Gamma(TN)$ is a vector field on N, then $\hat{X} \in \Gamma(TM)$ is defined by $\hat{X}(m) = \pi_{*m}^{\mathrm{tr}}\left(\pi_{*m}\pi_{*m}^{\mathrm{tr}}\right)^{-1} X(\pi(m))$ and we have $\pi_*\hat{X} = X \circ \pi$. Finally, let Vol_g and Vol_h be the volume forms on (M, g) and (N, h) respectively.

Lemma C.2 *If* $K := \dim M > k := \dim N$, *then there exists a unique* $K - k$ – *form* (γ) *on* M *such that;*

1. $\mathrm{Vol}_g = (\pi^* \mathrm{Vol}_h) \wedge \gamma$
2. $i_{\hat{v}}\gamma = 0$ *for any* $v \in T_{\pi(m)}N$ *and* $m \in M$.

Proof. Uniqueness. Assuming such a γ exists, choose an orthonormal basis

$\{e_1, \ldots, e_k\}$ for $T_{\pi(m)}N$ such that $\mathrm{Vol}_h(e_1, \ldots, e_k) = 1$. Then it follows that

$$\begin{aligned}\mathrm{Vol}_g(\hat{e}_1, \ldots, \hat{e}_k, \cdot, \ldots, \cdot) &= (\pi^* \mathrm{Vol}_h)(\hat{e}_1, \ldots, \hat{e}_k) \wedge \gamma \\ &= \mathrm{Vol}_h(\pi_* \hat{e}_1, \ldots, \pi_* \hat{e}_k) \wedge \gamma \\ &= \mathrm{Vol}_h(e_1, \ldots, e_k) \wedge \gamma = \gamma\end{aligned}$$

which shows γ is unique if it exists.

Existence. Now suppose that $\{e_1, \ldots, e_k\}$ is a local orthonormal frame on M in a neighborhood of $\pi(m)$ such that $\mathrm{Vol}_h(e_1, \ldots, e_k) = 1$. Then by above we must define

$$\gamma := \mathrm{Vol}_g(\hat{e}_1, \ldots, \hat{e}_k, \cdot, \ldots, \cdot) \text{ in a neighborhood of } m.$$

It is now straightforward to check that this γ has the desired properties and is defined independent of the choice of frame. ■

Corollary C.3 *If $X \in \Gamma(TN)$ and $\hat{X} \in \Gamma(TM)$ is its lift as described above, then*

$$\mathrm{div}_g\left(\hat{X}\right) = \mathrm{div}_h(X) \circ \pi + \rho_{\hat{X}}$$

where $\rho_{\hat{X}}(m)$ is a function on M depending only on $\hat{X}(m)$. {To compute $\rho_{\hat{X}}$ explicitly will require a better understanding of $d\gamma$.]

Proof. From Lemma C.2 we learn,

$$\begin{aligned}\mathrm{div}_g\left(\hat{X}\right) \mathrm{Vol}_g &= d\left[i_{\hat{X}} \mathrm{Vol}_g\right] = d\left[i_{\hat{X}}\left((\pi^* \mathrm{Vol}_h) \wedge \gamma\right)\right] \\ &= d\left[\left(i_{\hat{X}}(\pi^* \mathrm{Vol}_h) \wedge \gamma\right)\right] \\ &= \left[d\left(i_{\hat{X}}(\pi^* \mathrm{Vol}_h)\right)\right] \wedge \gamma + (-1)^k \left(i_{\hat{X}}(\pi^* \mathrm{Vol}_h) \wedge d\gamma\right).\end{aligned}$$

Since

$$i_{\hat{X}}\left(\pi^{*}\operatorname{Vol}_{h}\right)=\left(\pi^{*}\operatorname{Vol}_{h}\right)\left(\hat{X},--\right)=\operatorname{Vol}_{h}\left(\pi_{*}\hat{X},\pi_{*}--\right)$$
$$=\operatorname{Vol}_{h}\left(X\circ\pi,\pi_{*}--\right)=\pi^{*}\left(i_{X}\operatorname{Vol}_{h}\right)$$

it follows that

$$d\left(i_{\hat{X}}\left(\pi^{*}\operatorname{Vol}_{h}\right)\right)=d\left(\pi^{*}\left(i_{X}\operatorname{Vol}_{h}\right)\right)=\pi^{*}\left(d\left(i_{X}\operatorname{Vol}_{h}\right)\right)$$
$$=\pi^{*}\left(\operatorname{div}_{h}(X)\operatorname{Vol}_{h}\right)=\operatorname{div}_{h}(X)\circ\pi\cdot\pi^{*}\operatorname{Vol}_{h}.$$

Combining these equations then shows,

$$\operatorname{div}_{g}\left(\hat{X}\right)\operatorname{Vol}_{g}=\operatorname{div}_{h}(X)\circ\pi\cdot\left(\pi^{*}\operatorname{Vol}_{h}\right)\wedge\gamma+(-1)^{k}\left(i_{\hat{X}}\left(\pi^{*}\operatorname{Vol}_{h}\right)\wedge d\gamma\right)$$
$$=\left[\operatorname{div}_{h}(X)\circ\pi+\rho_{\hat{X}}\right]\cdot\operatorname{Vol}_{g}$$

where

$$\rho_{\hat{X}}=\frac{(-1)^{k}\left(i_{\hat{X}}\left(\pi^{*}\operatorname{Vol}_{h}\right)\wedge d\gamma\right)}{\operatorname{Vol}_{g}}.$$

■

Appendix D

Some matrix analysis

Consider

$$a := \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n \text{ and } S = \begin{bmatrix} I_{n\times n} \\ a^{\mathrm{tr}} \end{bmatrix}$$

so that

$$S^{\mathrm{tr}} = \begin{bmatrix} I_{n\times n} & a \end{bmatrix}.$$

Notice that S is a $(n+1)\times n$ and S^{tr} is $n\times(n+1)$ matrix. For $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ we have

$$S^{\mathrm{tr}} \begin{bmatrix} x \\ u \end{bmatrix} = x + ua \text{ and } Sx = \begin{bmatrix} x \\ a \cdot x \end{bmatrix}$$

$$S^{\mathrm{tr}} Sx = x + (a \cdot x)\, a = x + a\ a^{\mathrm{tr}} x = \left(I + aa^{\mathrm{tr}}\right) x.$$

Thus choosing an orthonormal basis $\{u_i\}_{i=1}^n$ for $\mathbb{R}^n$ such that $u_1 = \hat{a}$ we learn that

$$S^{\mathrm{tr}} S u_1 = \left(1 + \|a\|^2\right) u_1 \text{ and } S^{\mathrm{tr}} S u_i = u_i \text{ for } i > 1.$$

Thus it follows that $\det(S^{\text{tr}}S) = 1 + \|a\|^2$. We record the higher dimensional generalization of the result above. It is used in computing some determinants in the thesis.

Theorem D.1 *Suppose that V is a finite dimensional inner product space, $A : V^n \to V$ is a linear map, and*

$$S := \begin{bmatrix} I_{V^n \times V^n} \\ A \end{bmatrix} : V^n \to V^{n+1}.$$

Then

$$\det\left[S^{\text{tr}}S\right] = \det\left[I_V + AA^{\text{tr}}\right].$$

Proof. First observe that

$$S^{\text{tr}}S = \begin{bmatrix} I & A^{\text{tr}} \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} = I + A^{\text{tr}}A.$$

We let $\{u_j\}_{j=1}^n \subset V$ be an orthonormal basis of eigenvectors for $AA^{\text{tr}} : V \to V$ so that $AA^{\text{tr}}u_j = \lambda_j u_j$ and then let $v_j := A^{\text{tr}}u_j$. Then it follows that

$$A^{\text{tr}}Av_j = A^{\text{tr}}AA^{\text{tr}}u_j = A^{\text{tr}}\lambda_j u_j = \lambda_j A^{\text{tr}}u_j = \lambda_j v_j.$$

Now extend $\{v_j\}_{j=1}^n$ to a basis for all V^n. From this we will find that $S^{\text{tr}}S$ has eigenvalues $\{1\} \cup \{1 + \lambda_j\}_{j=1}^n$ and therefore

$$\det\left(S^{\text{tr}}S\right) = \prod_{j=1}^n (1 + \lambda_j) = \det\left(I + AA^{\text{tr}}\right).$$

■

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