UNIVERSITY OF CALIFORNIA, SAN DIEGO

Differentials of Measure-Preserving Flows on Path Space

A dissertation submitted in partial satisfaction
of the requirements for the degree
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by

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The dissertation of Carolyn M. Cross is approved, and it is acceptable in quality and form for publication on microfilm:

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VITA

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ABSTRACT OF THE DISSERTATION

Differentials of Measure-Preserving
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Carolyn M. Cross

Doctor of Philosophy in Mathematics

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Professor Bruce K. Driver, Chair

Let $W = \{\omega : [0,1] \to \mathbf{R}^n | \omega \text{ is continuous}\}$, equipped with Wiener measure. The classical Cameron-Martin theorem states that the mapping $(\omega \to \omega + h)$ of W to itself (for $h \in W$) preserves the measure up to a density if and only if $h \in H = \{h \in W | h(0) = 0, \int_0^1 |h'(s)|^2 ds < \infty\}$.

Bruce Driver has proved an analogous result for the space $W_o(M)$ of continuous paths on a compact manifold M with a fixed base point $o \in M$. Let $C^1 \equiv \{h \in C^1([0,1], T_oM) | h(0) = 0\}$, the space of once-continuously differentiable paths in T_oM, starting at the origin. Driver constructed a "natural" vector field X^h corresponding to each $h \in C^1$, and showed that the induced flow $t \to \sigma^h(t, \omega)$ starting at a "generic" path $\omega \in W_o(M)$ exists, and that the map $\sigma^h(t, \cdot) : W_o(M) \to W_o(M)$ preserves Wiener measure up to a density.

In my thesis I first generalize Driver's construction of measure-preserving flows to a slightly larger class V of vector fields on W. These are functions $Y : W \to W$ of the form

$$Y(\omega)(s) = \int_0^s C(\omega)(\bar{s}) d\omega(\bar{s}) + \int_0^s R(\omega)(\bar{s}) d\bar{s}$$

where, roughly speaking, C takes values in the skew-symmetric matrices and $R(\omega)$ is bounded by a "nice" function of ω.

I then show that members of V generate flows which are "smooth" in their starting path, i.e., differentiable via any vector field in V.

The proof uses a modified Picard iterates method to solve a differential equation including a term with an unbounded linear operator.

The second half of my thesis is devoted to the "geometric" result that Driver's flows are differentiable in their starting paths. This result is proved for both the "transferred" flow in W and the original flow in $W_o(M)$.

The W case is proved by showing that the class V above contains Driver's "transferred" vector fields on W, i.e., $Y^h \in V$ for all $h \in C^1$. Thus the result in Part I implies that the "transferred" flow $w^h(t, \omega)$ generated by Y^h on W is "differentiable" in its starting path ω via any of the vector fields Y^k, for $k \in C^1$. I then use certain smoothness properties of the stochastic development map to transfer this result to $W_o(M)$.

Chapter 1

Introduction

Let W be an n-dimensional, C^∞ manifold and assume X is a complete smooth vector field on W. Define the flow of X starting at $w \in W$ as the solution to the differential equation: $\dot{\sigma}(t,w) = X(\sigma(t,w))$ with $\sigma(0,w) = w$. Then by classical theorems ([34] Section 4.1, Theorem 1, p. 80), $\sigma(t,\cdot) : W \to W$ is a C^∞ function for all $t \in \mathbf{R}$. That is, the flow of a vector field on a finite-dimensional manifold depends smoothly on its starting point.

The purpose of this thesis is to prove a similar result for a class of flows on the infinite-dimensional manifold of paths on a compact Riemannian manifold.

Measure-Preserving Flows on Wiener Space

Let $W(\mathbf{R}^n) = \{\omega : [0,1] \to \mathbf{R}^n | \omega \text{ is continuous}\}$, equipped with Wiener measure μ. The classical Cameron-Martin theorem states that the mapping $(\omega \to \omega + h)$ of W to itself (for $h \in W(\mathbf{R}^n)$) preserves the measure up to a density if and only if h is in the Cameron-Martin Hilbert space H of absolutely continuous functions with one derivative in L^2.

In [8], Bruce Driver proved an analogous result for the path space of a compact manifold M with a fixed base point $o \in M$. This result states that the flows generated by certain natural vector fields on the path space $W_o(M) \equiv \{\omega \in C([0,1], M) | \omega(0) = o\}$ preserve Wiener measure, ν, up to a density. Recently, generalizations of this work have been obtained by Hsu [21], Norris [39], and Enchev-Stroock [15].

In Driver's setting, each vector field is uniquely determined by a path $h \in C^1 \equiv C^1([0,1], \mathbf{R}^n)$ according to the following construction. Since $\mathbf{R}^n$ is isomorphic to T_oM, the tangent space to M at o, we may view h as a path in T_oM. Now construct a vector field along a "generic" path ω in M by "parallel translating" the vector $h(s)$ along ω from $\omega(0) = o$ to $\omega(s)$. By doing this for a "generic" path ω in M, we create a vector field $\tilde{X}^h$ on the path space $W_o(M)$ itself.

Note that ordinary parallel translation requires the path ω to be differentiable, but the set of such paths has ν-measure zero in $W_o(M)$. Instead, we use stochastic parallel translation, which is constructed using the Ito stochastic calculus, and therefore is defined only up to ν-equivalence. In this context the notion of smoothness must involve stochastic calculus. For example, stochastic parallel translation is not in general even continuous in the sup-norm topology.

For each $h \in C^1$, the flow $\tilde{\sigma}^h : \mathbf{R} \to Maps(W_o(M), W_o(M))$ associated to the vector field $\tilde{X}^h$ is the unique solution to the equation $\dot{\tilde{\sigma}}^h(t, \omega) = \tilde{X}^h(\tilde{\sigma}^h(t, \omega))$ with $\tilde{\sigma}^h(0, \omega) = \omega$ in the space of paths $\{\sigma : \mathbf{R} \to Maps(W_o(M) \to W_o(M))\}$.

Driver showed that the flow $t \to \tilde{\sigma}^h(t, \omega)$ of the vector field $\tilde{X}^h$ starting at a "generic" path $\omega \in W_o(M)$ exists, and the map $\tilde{\sigma}^h(t, \cdot) : W_o(M) \to W_o(M)$ preserves Wiener measure up to a density.

In this paper we will show that this flow is "differentiable" in its starting path ω. This is analogous to the classical result that solutions to ordinary differential equations vary smoothly in their initial conditions.

In proving his result on $W_o(M)$, Driver uses the stochastic development map of Eells and Elworthy [12], and P. Malliavin to transfer the problem to one on $W(\mathbf{R}^n)$. The inverse of the stochastic development map may be viewed as transferring a path (thought of as wet ink) on M to one on $\mathbf{R}^n$ by "rolling" the manifold on $\mathbf{R}^n$ along the path, without slipping or twisting the manifold.

This development map can roughly be considered as a diffeomorphism betweem $W(\mathbf{R}^n)$ and $W_o(M)$, thus the "natural" vector fields $\tilde{X}^h$ on $W_o(M)$ may be transferred to vector fields $\tilde{Y}^h$ on $W(\mathbf{R}^n)$. Driver proved the existence of a measure-preserving flow on $W(\mathbf{R}^n)$ corresponding to these vector fields $\tilde{Y}^h$. He then used the development map to transfer the flows on $W(\mathbf{R}^n)$ to flows on $W_o(M)$.

In more detail, the vector field $\tilde{X}^h$ on $W_o(M)$ pulls back under the development map to a vector field $\tilde{Y}^h$ on $W(\mathbf{R}^n)$ of the form

$$\tilde{Y}^h(\omega)(s) = \int_0^s C^h(\omega)(\bar{s})d\omega(\bar{s}) + \int_0^s R^h(\omega)(\bar{s})d\bar{s}$$

where the process $s \to (C^h(\omega)(s), R^h(\omega)(s))$ is μ-a.s. $\mathrm{End}(\mathbf{R}^n) \times \mathbf{R}^n$-valued, continuous and adapted, with formulas for C^h and R^h determined by geometric properties of M. Here $\mathrm{End}(\mathbf{R}^n)$ denotes the $n \times n$ real matrices. (We will usually write this using the shorthand notation $\tilde{Y}^h(\omega) = \int C^h(\omega)d\omega + \int R^h(\omega)ds$.)

Driver proved the existence of the flow $t \to \tilde{w}^h(t,\omega)$ of the vector field $\tilde{Y}^h$, starting at a "generic" path $\omega \in W(\mathbf{R}^n)$, i.e., $\tilde{w}^h(t,\omega)$ solves

$$\frac{\partial \tilde{w}^h(t,\omega)(s)}{\partial t} = \tilde{Y}^h(\tilde{w}^h(t,\omega))(s) \tag{1.0.1}$$

with $\tilde{w}^h(0,\omega)(s) = \omega(s)$. Moreover, it was shown that the map $\tilde{w}^h(t,\cdot) : W(\mathbf{R}^n) \to W(\mathbf{R}^n)$ preserves Wiener measure up to a density.

Many others have worked in the area of nonlinear transformations on Wiener space. In particular, A. B. Cruzeiro's results [7] could be used to prove the existence and measure-preserving properties of transformations induced by the vector fields above with $C^h \equiv 0$. (See also G. Peters' results [40].) Other contributors include L. Gross [19], R. Ramer [42], S. Kusuoka [29], I. Shigekawa [47] [48], and M.-P. and P. Malliavin [36].

Smoothness of Flows on Classical Wiener Space

The purpose of the first part of this thesis is to obtain smoothness results in the $W(\mathbf{R}^n)$ setting (Chapters 3, 4 and 5). In Chapter 3 we will generalize Driver's construction of measure-preserving flows to a larger class $\tilde{V}$ of vector fields on $W(\mathbf{R}^n)$ which are independent of the geometry of M. These are functions $\tilde{Y} : W(\mathbf{R}^n) \to W(\mathbf{R}^n)$ of the form

$$\tilde{Y}(\omega) = \int C(\omega)d\omega + \int R(\omega)ds$$

where C takes values in the skew-symmetric matrices and $R(\omega)$ is bounded by a "nice" function of ω. We will also require certain smoothness conditions on the kernels C and R.

As in (1.0.1), the flow $\tilde{w} : \mathbf{R} \to Maps(W(\mathbf{R}^n), W(\mathbf{R}^n))$ associated to the vector field $\tilde{Y}$ is, roughly speaking, the unique solution to the equation $\dot{\tilde{w}}(t,\omega) = \tilde{Y}(\tilde{w}(t,\omega))$ with $\tilde{w}(0,\omega) = \omega$ in the space of paths $\{w : \mathbf{R} \to Maps(W(\mathbf{R}^n) \to W(\mathbf{R}^n))\}$.

In the main theorem of Chapter 3 (Theorem 3.2.8), we will show that the flow $t \to \tilde{w}(t,\omega)$ of such a vector field $\tilde{Y} \in \tilde{V}$ exists starting at a "generic" path $\omega \in W(\mathbf{R}^n)$, and the map $\tilde{w}(t,\cdot) : W(\mathbf{R}^n) \to W(\mathbf{R}^n)$ preserves Wiener measure up to a density. (We will sometimes use the term "quasi-invariance" for this measure preservation property.)

We will first obtain "flows" on a space of "Brownian semimartingales" rather than $W(\mathbf{R}^n)$ itself (Section 3.1). By doing this we avoid a certain technical problem in Wiener space of interdependence between existence and quasi-invariance of the flow. Then in section 3.2 this result is used to prove the existence of measure-preserving flows in the path space of $\mathbf{R}^n$.

Let $\tilde{w}_1(t,\omega)$ be the flow corresponding to the vector field $\tilde{Y}_1 \in \tilde{V}$. In Chapters 4 and 5 we will show that for each fixed t the map $\omega \to \tilde{w}_1(t,\omega)$ is "differentiable" via any vector field $\tilde{Y}_2 \in \tilde{V}$ (Theorem 5.4.2), that is,

$$[\tilde{Y}_2\tilde{w}_1](t,\cdot) \equiv \lim_{\epsilon\to 0} \frac{\tilde{w}_1(t,\tilde{w}_2(\epsilon,\cdot)) - \tilde{w}_1(t,\cdot)}{\epsilon} \tag{1.0.2}$$

where the limit is taken in L^p for all $p \in [2,\infty)$. This is analogous to the classical result that solutions to ordinary differential equations vary smoothly in their initial conditions.

As in the proof of existence of the flows, to avoid technical difficulties we will prove a "smoothness" result first in the space of Brownian semimartingales, denoted by $B^\infty\mathbf{R}^n$ (Theorem 4.1.3).

Let V be the "semimartingale version" of the vector field space $\tilde{V}$. V is to be thought of as a space of vector fields on $B^\infty\mathbf{R}^n$. Fix $Y_i \in V$, for $i = 1, 2$ and let w_i be the corresponding flow on $B^\infty\mathbf{R}^n$.

In Chapters 4 and 5 we prove that Y_2w_1 exists, where Y_2w_1 is defined as in (1.0.2) without the tildes. This is accomplished in two steps.

In Chapter 4, the equation which must be satisfied by Y_2w_1, if it exists, is obtained by formally differentiating (via Y_2) the flow equation which defines w_1. Then the existence of a unique solution Z to this equation is proved.

In Chapter 5 we prove that Y_2w_1 exists and is equal to the solution Z found in Chapter 4, by showing that the discrepancy between Z and a difference quotient approximating the derivative Y_2w_1 approaches zero in an L^p-type norm for all $p \in [2, \infty)$. In section 5.4 this result is used to prove the differentiability of the flow on $W(\mathbf{R}^n)$ (see Theorem 5.4.2).

Differentials of Flows on Path Space of a Compact Manifold

The second part of this thesis is devoted to the manifold version of Theorem 5.4.2: that Driver's flows are differentiable in their starting paths. This result is proved for both the "transferred" flow in $W(\mathbf{R}^n)$ and the original flow in $W_o(M)$.

The $W(\mathbf{R}^n)$ case is proved by showing that the class V above contains Driver's "transferred" vector fields on $W(\mathbf{R}^n)$, i.e., $\tilde{Y}^h \in V$ for all $h \in C^1$. Thus Theorem 5.4.2 implies that the "transferred" flow $w^h(t, \omega)$ generated by $\tilde{Y}^h$ on $W(\mathbf{R}^n)$ is "differentiable" in its starting path ω via any of the vector fields $\tilde{Y}^k$, for $k \in C^1$ (Theorem 7.1.3). Then we use smoothness properties of the stochastic development map to extend this differentiability result to Driver's flows on $W_o(M)$ (Theorem 7.4.3).

Chapter 2

Norms, Differentiation and Integration for Semimartingales

2.1 Definitions and Notation

This section introduces the norms and related spaces of semimartingales in which we will work throughout this paper. We also define various notions of differentiation and integration for these spaces.

Suggested references for this section include [1, 2, 13, 14, 22, 35, 38, 41, 43, 44, 45, 46]. Especially see Protter [41] for stochastic integration theory, and Emery [14] for stochastic calculus on manifolds.

Notation 2.1.1 *Throughout this paper we will use an underlying filtered probability space* $(\Omega, \{\mathcal{F}_s\}, \mathcal{F}, P)$, *satisfying the* usual hypothesis, *i.e. the* σ*-algebra* $\mathcal{F}$ *on* Ω *is complete with respect to the probability measure* P*, the filtration* $\{\mathcal{F}_s\}$ *is right continuous, and* $\mathcal{F}_0$ *contains all* P*-null sets. We assume that this space supports an* $\mathbf{R}^n$*-valued Brownian motion* b*. Two examples follow.*

1. *If* $\Omega = W(\mathbf{R}^n)$ *with* P *being Wiener measure, define*

   (a) $b_s : \Omega \to \mathbf{R}^n$ *by* $b_s(\omega) = \omega(s)$.

(b) $\mathcal{F}$ *as the completion with respect to* P *of the* σ*-algebra generated by the maps* $\{b(r) : 0 \leq r \leq 1\}$.

(c) $\mathcal{F}_s$ *as the* σ*-algebra generated by the maps* $\{b(r) : 0 \leq r \leq s\}$ *and all the* P*-null sets of* $\mathcal{F}$.

2. *If* $\Omega = W_o(M)$, *with* ν *being Wiener measure, define*

   (a) $\mathcal{F}$ *as the completion with respect to* ν *of the* σ*-algebra generated by the maps* $\sigma_o(r)(\omega) \equiv \omega(r)$ *for* $0 \leq r \leq 1$ *and* $\omega \in W_o(M)$,

   (b) $\mathcal{F}_s$ *as the* σ*-algebra generated by the maps* $\{\sigma_o(r) : 0 \leq r \leq s\}$ *and all the* ν*-null sets of* $\mathcal{F}$.

   (c) $b \equiv I^{-1} \circ H(\sigma_o)$.

Standing Conventions. In this paper a *process* on $(\Omega, \{\mathcal{F}_s\}, \mathcal{F}, P)$ means an $\{\mathcal{F}_s\}$-adapted process, and a *semimartingale* means a continuous semimartingale.

Definition 2.1.2 *An* $\mathbf{R}^n$*-valued process* w *is a* **Brownian semimartingale** *if* w *is a continuous* $\{\mathcal{F}_s\}$*-adapted process such that*

$$w(s) = w(0) + \int_0^s O(\tau)db(\tau) + \int_0^s \alpha(\tau)d\tau \qquad (2.1.1)$$

for some continuous adapted $\mathrm{End}(\mathbf{R}^n) \times \mathbf{R}^n$*-valued process* (O, α). *(Here* $\mathrm{End}(\mathbf{R}^n)$ *denotes the space of* $n \times n$ *real matrices.)*

We will usually write (2.1.1) as

$$w = w(0) + \int Odb + \int \alpha ds.$$

Notation 2.1.3 *The following norms will be used in this paper:*

1. *Let* $O(n)$ *and* $so(n)$ *denote the set of* $n \times n$ *real-valued, orthogonal, skew-symmetric matrices, respectively.*

2. *For an* $n \times m$ *matrix* A, *define* $|A| \equiv tr(A^*A)^{1/2}$ *(the Hilbert-Schmidt norm).*

3. *For* $a \in \mathbf{R}^n$, *define* $|a|$ *to be the Euclidean length of* a.

4. *If* V *is a normed space, and* $f(s)$ *is a continuous adapted* V*-valued stochastic process, define* $\|\cdot\|_{S^p(s)}$ *for* $p \in [2,\infty]$ *by*

$$\|f\|_{S^p(s)} \equiv \|\sup_{0 \le r \le s} |f(r)|_V\|_{L^p(P)}$$
$$\text{and set } \|f\|_{S^p} \equiv \|f\|_{S^p(1)}.$$

5. *If* $w = \int O db + \int \alpha ds$ *is a* V*-valued Brownian semimartingale, then for* $p \in [2,\infty]$, *let*

$$\begin{aligned} \|w\|_{B^p(s)} &\equiv \|O\|_{S^p(s)} + \|\alpha\|_{S^p(s)} \\ \text{and set } \|w\|_{B^p} &\equiv \|w\|_{B^p(1)}. \end{aligned}$$

Notation 2.1.4 *For* $p \in [2,\infty]$ *we have the following normed spaces.*

Denote by S^pV *(or just* S^p*) the space of continuous adapted* V*-valued processes* $f(s)$ *such that* $\|f\|_{S^p} < \infty$. *Let* $S^{\infty-} \equiv \cap_{p \in [2,\infty)} S^p$.

Denote by B^pV, *or just* B^p, *the space of* V*-valued Brownian semimartingales* (w) *such that* $w(0) = 0$ *and* $\|w\|_{B^p} < \infty$, *and let* $B^{\infty-} \equiv \cap_{p \in [2,\infty)} B^p$.

Note that if V *is a finite-dimensional vector space then* S^pV *and* B^pV *are Banach spaces for all* $p \in [2,\infty)$. *(Lemma 4.2, p. 304 in [8]).*

In particular, we will often use the spaces $B^p\mathbf{R}^n$, $B^p\mathrm{End}(\mathbf{R}^n)$ *and* $B^pso(n)$. *We will denote by* $B^pO(n) \subset B^p\mathrm{End}(\mathbf{R}^n)$ *the subset of processes taking values in* $O(n)$.

Definition 2.1.5 *Let* V *be a finite-dimensional vector space. Let* $J = [-\kappa,\kappa] \subset \mathbf{R}$ *be a compact interval and let* $p \in [2,\infty]$. *A function* $G : J \to S^pV$ *is called* S^p-differentiable *at* $t \in J$ *if* $\lim_{\epsilon \to 0} \|\frac{1}{\epsilon}[G(t+\epsilon) - G(t)] - G'(t)\|_{S^pV} = 0$ *for some* $G'(t) \in S^pV$.

Similarly, define a B^p-differentiable *function by replacing* S^p *with* B^p *in the above definition.*

Definition 2.1.6 *Let V be a finite-dimensional vector space, and let $S^{\infty-}V \equiv \bigcap_{p\in[2,\infty)} S^pV$ as above.*

*i) We say that a map $t \to f(t) \in S^{\infty-}V$ is $S^{\infty-}$-*continuous *if $t \to f(t)$ is S^p-continuous for all $p \in [2,\infty)$.*

*ii) Similarly, if $\|f(t) - f(\tau)\|_{S^p} \leq K_p|t-\tau|$, with K_p a function of p for all $p \in [2,\infty)$, we say f is $S^{\infty-}$-*Lipschitz.

*iii) Finally, f is $S^{\infty-}$-*differentiable *if $t \to f(t)$ is S^p-differentiable for all $p \in [2,\infty)$.*

Similarly we define the analogous $B^{\infty-}$ terms.

Definition 2.1.7 *An* **admissible curve** *is a $B^{\infty-}$-differentiable curve $\gamma : J \to B^\infty\mathbf{R}^n$ such that $\limsup_{t\to 0} \|\gamma(t)\|_{B^\infty} \leq \bar{K}_{\gamma(0)} < \infty$, and the map $t \to \dot{\gamma}(t)$ is $B^{\infty-}$-Lipschitz.*

Notation 2.1.8 *Throughout this paper the brackets $\langle\cdot\rangle$ will be used to indicate linear arguments of a function.*

Definition 2.1.9 *A function $F : B^\infty \to S^{\infty-}V$ is called $S^{\infty-}$-differentiable at $w \in B^\infty$ if there exists a linear mapping $F'(w) : B^2 \to S^2V$ and if for all admissible curves γ with $\gamma(0) = w$, $F(\gamma(t))$ is $S^{\infty-}$-differentiable at $t = 0$, with $\frac{d}{dt}|_0 F(\gamma(t)) = F'(w)\langle\dot{\gamma}(0)\rangle$. We will often use the notation $v_w F \equiv F'(w)\langle v\rangle$, where $v = \dot{\gamma}(0)$.*

*Similarly, a function $F : B^\infty\mathbf{R}^n \to B^{\infty-}V$ is $B^{\infty-}$-*differentiable *at $w \in B^\infty$ if the above holds with $S^{\infty-}$ and S^2 replaced by $B^{\infty-}$ and B^2, respectively.*

Notation 2.1.10 *Suppose $f(t,\omega)(s)$ is a process. We distinguish two types of integrals of $f(t)$:*

1. *The* pointwise integral *defined by*

$$\left(p.w.\int_a^b f(t)dt\right)(\omega)(s) \equiv \int_a^b f(t,\omega)(s)dt$$

2. The S^p-integral *satisfying*

$$\lim_{|\pi|\to 0}\left\| S^p\int_a^b f(t)dt - \sum_{\pi=\{a=t_0<t_1<\cdots<t_n=b\}} f(t_i)(t_{i+1}-t_i)\right\|_{S^p} = 0$$

where $|\pi| = \max_i |t_{i+1}-t_i|$. *We will usually write* $S^p\int$ *simply as* $\int$.

Lemma 2.1.11 *Suppose that* $g : J \to B^p\mathbf{R}^n$ *such that* $(r,s) \to g(r,\omega)(s)$ *is jointly continuous for all* $\omega \in \Omega_0$ *where* $\Omega_0 \subset \Omega$ *with* $P(\Omega_0) = 1$. *Then for* $\tau < t$, $\|p.w. \int_\tau^t g(r)dr\|_{S^p} \le \int_\tau^t \|g(r)\|_{S^p}dr$ *for all* $p \in [2,\infty)$.

Proof. Fix $p \in [2,\infty)$ and let $f(r,\omega) \equiv \sup_{0\le s\le 1} |g(r,\omega)(s)|$. The map $r \to f(r,\omega)$ is measurable since by continuity of g we may take the supremum over a countable dense subset of [0,1].

Claim:

$$\|p.w. \int_\tau^t f(r)dr\|_{L^p(P)} \le \int_\tau^t \|f(r)\|_{L^p(P)}dr.$$

Proof of claim: Let $G \ge 0$, $G \in L^{p'}(P)$, where $\frac{1}{p}+\frac{1}{p'} = 1$, and let $F(\omega) \equiv \int_\tau^t f(r,\omega)dr$, i.e., $F = p.w. \int_\tau^t f(r)dr$. Then

$$\begin{aligned}
&\|FG\|_{L^1(P)} \\
&= \int_\Omega \left[\int_\tau^t f(r,\omega)dr\right] G(\omega)P(d\omega) \\
&= \int_\tau^t dr \int_\Omega P(d\omega) f(r,\omega)G(\omega) \text{ by Tonelli} \\
&\le \int_\tau^t dr \|f(r)\|_{L^p(P)}\|G\|_{L^{p'}(P)} \text{ by Hölder's inequality}
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|F\|_{L^p(P)} &= \sup\{\|FG\|_{L^1(P)} \|G\|_{L^{p'}(P)} = 1\} \\
&\le \int_\tau^t \|f(r)\|_{L^p(P)}dr.
\end{aligned}$$

which proves the claim.

Finally, we have

$$\begin{aligned}
\|p.w.\int_\tau^t g(r)dr\|_{S^p} &= \|\sup_{0\le s\le 1}|p.w.\int_\tau^t g(r,\cdot)(s)dr|\|_{L^p(P)} \\
&\le \|p.w.\int_\tau^t \sup_{0\le s\le 1}|g(r,\cdot)(s)|dr\|_{L^p(P)} \\
&= \|p.w.\int_\tau^t f(r)dr\|_{L^p(P)} \\
&\le \int_\tau^t \|f(r)\|_{L^p(P)}dr \\
&= \int_\tau^t \|g(r)\|_{S^p}dr.
\end{aligned}$$

Q.E.D.

Lemma 2.1.12 *Let V be a finite-dimensional vector space, and suppose $f:\mathbf{R}^+\to S^{\infty-}V$ is $S^{\infty-}$-continuous. Then for all $t\in\mathbf{R}^+$ and $p,q\in[2,\infty)$, $S^p\int_0^t f(\tau)d\tau = S^q\int_0^t f(\tau)d\tau$.*

Proof. Since f is $S^{\infty-}$-continuous, for each $p\in[2,\infty)$ the Riemann sums in Notation 2.1.10 converge, so the S^p integral exists. Fix $p,q\in[2,\infty)$ with $p<q$. Let $\pi=\{0=t_0<t_1<\cdots<t_n=t\}$ and $|\pi|=\max_i|t_{i+1}-t_i|$. Then

$$\begin{aligned}
& \lim_{|\pi|\to 0}\left\|S^q\int_0^t f(\tau)d\tau-\sum_{i=0}^{n-1}f(t_i)(t_{i+1}-t_i)\right\|_{S^p} \\
\le & \lim_{|\pi|\to 0}\left\|S^q\int_0^t f(\tau)d\tau-\sum_{i=0}^{n-1}f(t_i)(t_{i+1}-t_i)\right\|_{S^q} \\
= & \ 0 \text{ by definition of the } S^q\text{-integral.}
\end{aligned}$$

Q.E.D.

Notation 2.1.13 *Let $f:\mathbf{R}^+\to S^{\infty-}$ be $S^{\infty-}$-continuous, and $t\in\mathbf{R}^+$. Let $S^{\infty-}\int_0^t f(\tau)d\tau$ denote the common integral $S^p\int_0^t f(\tau)d\tau$ for $p\in[2,\infty)$ given by Lemma 2.1.12.*

Lemma 2.1.14 *Fundamental Theorem of Calculus for S^p-integrals.*
Let $J \subset \mathbf{R}$ be a compact interval, and $f : J \to S^{\infty-}V$ be $S^{\infty-}$-continuous, where V is a finite-dimensional vector space. Then for all $p \in [2,\infty)$, the map $t \to \int_0^t f(\tau)d\tau$ is $S^{\infty-}$-differentiable, and $f(t) = \frac{d}{dt}\int_0^t f(\tau)d\tau$ for all $t \in J$.

Proof. Let $\epsilon > 0$ (the case for $\epsilon < 0$ is similar) and fix $p \in [2,\infty)$. We have

$$\begin{aligned} & \left\| \frac{1}{\epsilon}\left\{ \int_0^{t+\epsilon} f(\tau)d\tau - \int_0^t f(\tau)d\tau \right\} - f(t) \right\|_{S^p} \\ = & \left\| \frac{1}{\epsilon}\int_t^{t+\epsilon} f(\tau)d\tau - f(t) \right\|_{S^p} \\ \leq & \frac{1}{\epsilon}\int_t^{t+\epsilon} \|f(\tau) - f(t)\|_{S^p} d\tau \\ \leq & \sup_{t \leq \tau \leq t+\epsilon} \|f(\tau) - f(t)\|_{S^p} \end{aligned}$$

This converges to 0 as $\epsilon \to 0$ by the S^p-continuity of f. Q.E.D.

Chapter 3

Driver's Measure-Preserving Flows on Path Spaces

3.1 Flows in a Space of Semimartingales

In [8], Driver proved the existence of quasi-invariant flows on the path space of a compact Riemannian manifold M. He did this by first transferring the vector fields generating his flows to Wiener space via the stochastic development map of Eells and Ellworthy, and proving that these transferred vector fields generate measure-preserving flows.

Driver's proof on Wiener space will be presented in this chapter, but we will obtain a more general statement of his result by eliminating dependence on the geometry of the manifold M.

We will first obtain "flows" on a space of "Brownian semimartingales" in order to avoid a problem in Wiener space of interdependence between existence and quasi-invariance of the flow. In section 3.2 we will use this result to prove the existence of measure-preserving flows on path space itself.

The following notation and assumptions will be used throughout this paper.

Notation 3.1.1 *Let* $(C, R) : B^\infty\mathbf{R}^n \to S^{\infty-}\mathrm{End}(\mathbf{R}^n) \times S^\infty\mathbf{R}^n$ *with the following* *properties. Let* $p, p_1, p_2 \in [2, \infty)$ *be such that* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

1. *There exists* $(A, T) : B^\infty \mathbf{R}^n \to S^{\infty -}\mathrm{so}(n) \times S^\infty \mathrm{End}(\mathbf{R}^n)$ *with* $C(w) = A(w) + T(w)$ *and* $\|T(w)\|_{S^\infty} \le T_\infty < \infty$ *for all* $w \in B^\infty \mathbf{R}^n$.

2. A, T *and* R *are* $S^{\infty -}$*-differentiable (see Definition 2.1.9), and for all* $w, \tilde{w}, v \in B^\infty \mathbf{R}^n$,

$$\begin{aligned} \|v_w A\|_{S^p} &\le K\|v\|_{B^p} \\ \|v_w A - v_{\tilde{w}} A\|_{S^p} &\le K\|w - \tilde{w}\|_{B^{p_1}} \|v\|_{B^{p_2}}, \end{aligned} \tag{3.1.1}$$

   where $K = K(n, p, \bar{K}_w, \bar{K}_{\tilde{w}})$. *These conditions also hold with* A *replaced both by* T *and by* R.

3. *There exist constants* c_1 *and* c_2 *such that*

$$\|R(w)\|_{S^\infty} \le c_1 \|O\|^2_{S^\infty} + c_2 \text{ for all } w = \int O db + \int \alpha ds \in B^\infty \mathbf{R}^n.$$

4. $\|C(0)\|_{S^\infty} < \infty$.

Define

$$X(w) \equiv \int C(w) dw + \int R(w) ds \text{ for } w \in B^\infty \mathbf{R}^n$$

Let $w_o = \int O_o db + \int \alpha_o ds \in B^\infty \mathbf{R}^n$.

We explicitly state the following conditions, which are implied by 2. above, since they will often be used in this form.

$$\begin{aligned} \|C(w) - C(\tilde{w})\|_{S^p} &\le K\|w - \tilde{w}\|_{B^p} \\ \|R(w) - R(\tilde{w})\|_{S^p} &\le K\|w - \tilde{w}\|_{B^p} \end{aligned} \tag{3.1.2}$$

for all $w, \tilde{w} \in B^\infty \mathbf{R}^n$ *and* $p \in [2, \infty)$, *where* $K = K(n, p, \bar{K}_w, \bar{K}_{\tilde{w}})$. *These conditions also hold with* C *replaced both by* A *and by* T.

In theorem 3.1.5 below, we will prove existence and uniqueness (in a certain space) of a solution w to the equation

$$\dot{w}(t) = X(w(t)) \tag{3.1.3}$$

with $w(0) = w_o \in B^\infty\mathbf{R}^n$. Equivalently, we will find a solution (O, α) to the pair of equations

$$\begin{aligned} \dot{O}(t) &= C(w(t))O(t) \\ \text{and} \qquad \dot{\alpha}(t) &= C(w(t))\alpha(t) + R(w(t)) \end{aligned} \tag{3.1.4}$$

with $O(0) = O_o \in S^\infty\mathrm{End}(\mathbf{R}^n)$ and $\alpha(0) = \alpha_o \in S^\infty\mathbf{R}^n$.

Each of these solutions w and (O, α) will exist in two senses: as solutions in a space of semimartingales, and in the "pointwise" sense defined below.

Definition 3.1.2 *([8] Def. 6.3, p. 327). A 1-parameter family of* $\mathrm{End}(\mathbf{R}^n) \times \mathbf{R}^n$*-valued adapted processes* $(O(t), \alpha(t))$ *solves (3.1.4)* pointwise *if the following conditions hold:*

1. *P-a.s. the function $(t, s) \to (O(t)(s), \alpha(t)(s))$ is $C^{1,0}$.*

2. *There exist versions $A(t)$, $T(t)$ and $R(t)$ of $A(w(t))$, $T(w(t))$, and $R(w(t))$ respectively, such that P-a.s. the map $(t, s) \to (A(t)(s), T(t)(s), R(t)(s))$ is $C^{1,0}$, where $w(t) = \int O(t)db + \int \alpha(t)ds$.*

3. *There is a fixed set $\Omega_0 \subset \Omega$ of full measure such that $(O(t)(s), \alpha(t)(s))$ satisfies (3.1.4) pointwise on Ω_0 with $C(w(t))$ replaced by $C(t) \equiv A(t) + T(t)$ and $R(w(t))$ replaced by $R(t)$.*

Definition 3.1.3 *A 1-parameter family of $\mathbf{R}^n$-valued adapted processes $w(t) = \int O(t)db + \int \alpha(t)ds$ solves (3.1.3)* pointwise *if $(O(t), \alpha(t))$ solves (3.1.4) in the pointwise sense defined above.*

The following lemma gives a sufficient condition on $w(t)$ so that $C^{1,0}$ versions of $A(w(t))$, $T(w(t))$, and $R(w(t))$ exist, as required in Definition 3.1.2.

Lemma 3.1.4 *Let $w : J \to B^\infty\mathbf{R}^n$ be an admissible curve. Then there exist versions $A(t)$, $T(t)$ and $R(t)$ of $A(w(t))$, $T(w(t))$, and $R(w(t))$ respectively, such that P-a.s. the map $(t, s) \to (A(t)(s), T(t)(s), R(t)(s))$ is $C^{1,0}$.*

Proof. Let $t_1, t_2 \in J$, and $w_i \equiv w(t_i)$, $\dot{w}_i \equiv \frac{d}{dt}|_{t_i} w(t)$ for $i = 1, 2$. Then for all $p \in [2, \infty)$, $\frac{d}{dt}|_{t=t_i} A(w(t)) = A'(w_i)\langle \dot{w}_i \rangle$ where the derivative is taken in the S^p-topology. We have

$$\begin{aligned}
&\|A'(w_1)\langle \dot{w}_1 \rangle - A'(w_2)\langle \dot{w}_2 \rangle\|_{S^p} \\
&\qquad \leq \|A'(w_1)\langle \dot{w}_1 \rangle - A'(w_2)\langle \dot{w}_1 \rangle\|_{S^p} \\
&+\|A'(w_2)\langle \dot{w}_1 - \dot{w}_2 \rangle\|_{S^p} \\
&\qquad \leq K\|w_1 - w_2\|_{B^{p_1}} \|\dot{w}_1\|_{B^{p_2}} \\
&+K\|\dot{w}_1 - \dot{w}_2\|_{B^p} \text{ by } (3.1.1) \\
&\qquad \leq K|t_1 - t_2|.
\end{aligned}$$

Using the same argument with A replaced by T and by R, the map $t \to (A(w(t)), T(w(t)), R(w(t)))$ is $S^{\infty-}$-Lipschitz. The result follows by Lemma 8.1.4 of the appendix. Q.E.D.

Theorem 3.1.5 *Let* $(C, R) : B^\infty \mathbf{R}^n \to S^{\infty-}\mathrm{End}(\mathbf{R}^n) \times S^\infty \mathbf{R}^n$, $X : B^\infty \mathbf{R}^n \to \mathbf{R}^n$*-valued processes and* $w_o \in B^\infty \mathbf{R}^n$ *be given as in Notation 3.1.1. Then* $X : B^\infty \mathbf{R}^n \to B^{\infty-}\mathbf{R}^n$ *and the following results hold:*

(i) *There exists a unique* $B^{\infty-}$*-differentiable solution to equation (3.1.3):*

$$\dot{w}(t) = X(w(t))$$

with $w(0) = w_o \in B^\infty \mathbf{R}^n$ *in the space of paths*

$$\{w : \mathbf{R} \to B^\infty \mathbf{R}^n | \sup_{|t|<T} \|w(t)\|_{B^\infty} < \infty \ \forall T > 0\}.$$

This solution w *is in fact admissible, so* $\dot{w}(t) = \frac{d}{dt} w(t)$ *is* $B^{\infty-}$*-Lipschitz. Furthermore, there is a version of* w *which solves (3.1.3) pointwise, in the sense of Definition 3.1.3.*

(ii) *Equivalently, there exists a unique $S^{\infty-}$-differentiable solution (O, α) to the pair of equations (3.1.4):*

$$\begin{aligned} \dot{O}(t) &= C(w(t))\dot{O}(t) \\ \alpha(t) &= C(w(t))\alpha(t) + R(w(t)) \end{aligned}$$

with $O(0) = O_o \in S^\infty \mathrm{End}(\mathbf{R}^n)$ and $\alpha(0) = \alpha_o \in S^\infty \mathbf{R}^n$, where $w(t) = \int O(t)db + \int \alpha(t)ds$, in the space of paths

$$\{(O, \alpha) : \mathbf{R} \to S^\infty \mathrm{End}(\mathbf{R}^n) \times S^\infty \mathbf{R}^n | \sup_{|t|<T} [\|O(t)\|_{S^\infty} + \|\alpha(t)\|_{S^\infty}] < \infty \ \forall T > 0\}.$$

The solution (O, α) is $S^{\infty-}$-differentiable, with $(\dot{O}, \dot{\alpha})$ $S^{\infty-}$-Lipschitz. Also, there is a version of (O, α) which solves (3.1.4) pointwise, in the sense of Definition 3.1.2.

(iii) *Moreover, if we write the solution to (3.1.3) as $w(t, w_o)$, a function of its starting point, then $w(t, \cdot) : B^\infty \mathbf{R}^n \to B^\infty \mathbf{R}^n$ is a flow on $B^\infty \mathbf{R}^n$ in the sense that $w(t, w(\tau, w_o))$ and $w(t + \tau, w_o)$ are indistinguishable.*

Proof. First note that for $w = \int O db + \int \alpha ds \in B^\infty \mathbf{R}^n$,

$$\begin{aligned} \|X(w)\|_{B^p} &= \|C(w)O\|_{S^p} + \|C(w)\alpha + R(w)\|_{S^p} \\ &\leq \|C(w)\|_{S^p}[\|O\|_{S^\infty} + \|\alpha\|_{S^\infty}] + \|R(w)\|_{S^p} \text{ by Lemma 8.1.5.} \end{aligned}$$

Since this expression is finite, we have $X : B^\infty \mathbf{R}^n \to B^{\infty-}\mathbf{R}^n$.

The theorem will be proved with $\mathbf{R}$ replaced by a compact interval $J = [-\kappa, \kappa]$. This result can then be extended by existence and uniqueness to all of $\mathbf{R}$.

The equivalence between solutions of (3.1.3) and (3.1.4) follows from substituting the formula $w(t) = \int O(t)db + \int \alpha(t)ds$ into (3.1.3):

$$\begin{aligned} \int \dot{O}(t)db + \int \dot{\alpha}(t)ds &= \int C(w(t))[O(t)db + \alpha(t)ds] + \int R(w)ds \\ &= \int C(w(t))O(t)db + \int [C(w(t))\alpha(t) + R(w)]ds \end{aligned}$$

Thus $w(t)$ satisfies (3.1.3) in the B^p-norm if and only if its kernels $(O(t), \alpha(t))$ satisfy (3.1.4) in the S^p-norm.

Uniqueness. Let $\mathcal{S}$ denote the set of functions $w : J \to B^{\infty}\mathbf{R}^n$ with the following properties:

1. $w(t)$ is $B^{\infty-}$-differentiable, with $\dot{w}(t)$ $B^{\infty-}$-Lipschitz,

2. $w(0) = w_o$, and there exists a constant $C_o = C_o(c_1, c_2, T_\infty, \|w_o\|_{B^\infty})$ such that $\|w(t)\|_{B^\infty} \le C_o$ for all $t \in J$, Recall that c_1, c_2 and T_∞ are defined in Notation 3.1.1.

3. $\|O(t)\|_{S^\infty} \le \|O_o\|_{S^\infty} e^{T_\infty}$, where $w(t) = \int O(t)db + \int \alpha(t)ds$.

Notice that any solution $w(t) = \int O(t)db + \int \alpha(t)ds$ to (3.1.3) in the S^p-topologies will automatically be in $\mathcal{S}$. Indeed, by an argument similar to [8] Prop. 6.3, p. 331, $w(t)$ is $B^{\infty-}$-differentiable, with $\frac{d}{dt}w(t)$ being $B^{\infty-}$-Lipschitz, and there is a version of $w(t)$ which satisfies (3.1.3) in the pointwise sense of Definition 3.1.3. Thus by Cor. 3.1.8 below, we also have $\|O(t)\|_{S^\infty} \le \|O_o\|_{S^\infty} e^{T_\infty}$ and $\|w(t)\|_{B^\infty} \le C_o$ for all $t \in J$. Therefore $w \in \mathcal{S}$.

Define $L(w)(t) \equiv \bar{w}(t) = \int \bar{O}(t)db + \int \bar{\alpha}(t)ds$ for all $w \in \mathcal{S}$, where $\bar{O}$ and $\bar{\alpha}$ are the unique pointwise solutions to the ordinary differential equations:

$$\begin{aligned} \dot{\bar{O}}(t) &= C(w(t))\bar{O}(t) \text{ with } \bar{O}(0) = O_o \\ \text{and } \dot{\bar{\alpha}}(t) &= C(w(t))\bar{\alpha}(t) + R(w(t)) \text{ with } \bar{\alpha}(0) = \alpha_o \end{aligned} \tag{3.1.5}$$

given by fixing versions $C(t)$ and $R(t)$ of $C(w(t))$ and $R(w(t))$ which are P-a.s. $C^{1,0}$ in (t, s), via Lemma 3.1.4. Here we have solved the equations for each fixed $\omega \in \Omega$ and $s \in [0, 1]$. Thus the map $(t, s) \to (\bar{O}(t, s), \bar{\alpha}(t, s))$ is also $C^{1,0}$ P-a.s.

By Lemma 3.1.7 below, $\|\bar{O}(t)\|_{S^\infty} \le \|O_o\|_{S^\infty} e^{T_\infty}$ and $\|\bar{w}(t)\|_{B^\infty} \le C_o$ for all $t \in J$. Also, by an argument similar to Cor. 6.2, p. 330 [8], $\bar{w}(t)$ is $B^{\infty-}$-differentiable, with $\frac{d}{dt}\bar{w}(t)$ being $B^{\infty-}$-Lipschitz. Hence L preserves $\mathcal{S}$.

Claim: If $w : J \to B^{\infty}\mathbf{R}^n$ is a solution to (3.1.3) with $w(0) = w_o$, then w is a fixed point for L in the sense that

$$P(\{L(w)(t) = w(t) \text{ for all } t \in J\}) = 1.$$

(We will use this property of L in the uniqueness argument below.)

Proof of claim. By an argument similar to the proof of Proposition 6.3 p. 331 in [8], if $O(t)$ and $\alpha(t)$ are S^p solutions to (3.1.4) then we may choose versions which are solutions in the pointwise sense. Thus solving (3.1.5) with $w(t) = \int O(t)db + \int \alpha(t)ds$ yields $(\bar{O}(t), \bar{\alpha}(t)) = (O(t), \alpha(t))$ P-a.s.

Indeed, if $\bar{C}(t)$ is the version of $C(w(t))$ chosen in solving (3.1.5), and $C(t) = A(t) + T(t)$ as in Definition 3.1.2, then $\bar{C}(t) = C(t)$ for all $s \in [0,1]$ on a set $\Omega_1 \subset \Omega$ of full measure since both are P-a.s. jointly continuous versions of $C(w(t))$. Thus if $\Omega_0 \subset \Omega$ as in Def. 3.1.2, then

$$\bar{O}(t)(\omega, s) = O(t)(\omega, s)$$

for all $\omega \in \Omega_0 \cap \Omega_1$ (a set of full measure) and $s \in [0,1]$.

Claim: There is a constant $K = K(p, C_o, \|C(0)\|_{S^\infty}, \|R(0)\|_{S^\infty})$ such that

$$\|L(w)(t) - L(w)(\tau)\|_{B^p} \leq K|t - \tau| \tag{3.1.6}$$

for all $t, \tau \in J$ and $w \in \mathcal{S}$.

Proof of Claim**.** First note that by the "Lipschitz" conditions (3.1.2) on C, for any $\tilde{w} \in B^\infty \mathbf{R}^n$ with $\|\tilde{w}\|_{B^\infty} \leq C_o$ there exists a constant $K = K(p, C_o, \|C(0)\|_{S^\infty})$ (which will vary from place to place) such that

$$\begin{aligned} \|C(\tilde{w})\|_{S^p} &\leq \|C(\tilde{w}) - C(0)\|_{S^p} + \|C(0)\|_{S^p} \\ &\leq K_p \|\tilde{w}\|_{B^p} + \|C(0)\|_{S^p} \\ &\leq K \end{aligned} \tag{3.1.7}$$

Let $C(t)$ be a version of $C(w(t))$ which is P-a.s. $C^{1,0}$ in (t,s) via Lemma 3.1.4, and let $0 < \tau < t$. Then P-a.s.

$$\begin{aligned} |\bar{O}(t) - \bar{O}(\tau)| &\leq p.w. \int_\tau^t |C(r)||\bar{O}(r)|dr \\ &\leq C_o p.w. \int_\tau^t |C(r)|dr. \end{aligned}$$

It follows that

$$\begin{aligned}\|\bar{O}(t)-\bar{O}(\tau)\|_{S^p} &\leq C_o\|p.w.\int_\tau^t |C(r)|dr\|_{S^p}\\ &\leq C_o\int_\tau^t \|C(r)\|_{S^p}dr \text{ by Lemma 2.1.11}\\ &\leq K|t-\tau|.\end{aligned}$$

Similarly, using the "Lipschitz" conditions on C and R (3.1.2) there exists $K = K(p, C_o, \|C(0)\|_{S^\infty}, \|R(0)\|_{S^\infty})$ such that

$$\|\bar{\alpha}(t)-\bar{\alpha}(\tau)\|_{S^p} \leq K|t-\tau|.$$

Claim: There is a constant K_p independent of w_1 and w_2 in $\mathcal{S}$ such that

$$\|L(w_1)(t)-L(w_2)(t)\|_{B^p} \leq K_p|\int_0^t \|w_1(\tau)-w_2(\tau)\|_{B^p}d\tau| \tag{3.1.8}$$

for all $t \in J$.

Proof of Claim.

Let $\bar{w}_i = L(w_i)$ and write $\bar{w}_i(t) = \int \bar{O}_i(t)db + \int \bar{\alpha}_i(t)ds$ for $i = 1, 2$. Assume $t > 0$. (The case for $t < 0$ is similar.) By [8] Lemma 6.1(ii) there exists a constant $K = K_p(w_o, T_\infty)$ such that P-a.s.

$$|\bar{O}_1(t)(s)-\bar{O}_2(t)(s)| \leq K\int_0^t |C(w_1(\tau))(s)-C(w_2(\tau))(s)|d\tau.$$

Thus by the "Lipschitz" conditions on C (3.1.2), there exists $K = K_p(w_o, T_\infty, c_1, c_2)$ such that

$$\begin{aligned}&\|\bar{O}_1(t)(s)-\bar{O}_2(t)(s)\|_{S^p}\\ &\qquad\leq K\int_0^t \|C(w_1(\tau))-C(w_2(\tau))\|_{S^p}d\tau \text{ by Lemma 2.1.11}\\ &\qquad\leq K\int_0^t \|w_1(\tau)-w_2(\tau)\|_{B^p}d\tau.\end{aligned}$$

Similarly, using the "Lipschitz" bounds on C and R (3.1.2), there exists $K = K_p(w_o, T_\infty, c_1, c_2)$ such that

$$\|\bar{\alpha}_1(t)(s)-\bar{\alpha}_2(t)(s)\|_{S^p} \leq K\int_0^t \|w_1(\tau)-w_2(\tau)\|_{B^p}d\tau.$$

This proves the claim.

Let $L^{(n)}$ denote "L composed with itself n times". Iterating (3.1.8), we obtain

$$\|L^{(n)}(w_1)(t) - L^{(n)}(w_2)(t)\|_{B^p} \leq 2C_o K^n \frac{|t|^n}{n!} \tag{3.1.9}$$

since $\sup_{t\in J} \|w_1(t) - w_2(t)\|_{B^p} \leq 2C_o$.

This gives uniqueness: if w_1 and w_2 are both solutions to (3.1.3) then for all n, $L^{(n)}(w_i) = w_i$, so (3.1.9) shows $\|w_1(t) - w_2(t)\|_{B^p} \leq 2C_o K^n \frac{t^n}{n!}$, which tends to zero as $n \to \infty$.

Note that **(iii)** follows from the uniqueness of the pointwise solution.

Existence.

Let $w_0 : J \to B^\infty \mathbf{R}^n$ be defined by $w_0(t) = w_o$ for all $t \in J$, and define $w_n = L^{(n)}(w_0)$ for all n. Then by (3.1.9),

$$\begin{aligned} \|w_{n+1}(t) - w_n(t)\|_{B^p} &= \|L^{(n)}(w_1)(t) - L^{(n)}(w_0)(t)\|_{B^p} \\ &\leq 2C_o K^n \frac{t^n}{n!} \end{aligned}$$

Since $\sum_{n=0}^\infty 2C_o K^n \frac{t^n}{n!} < \infty$, this shows that $\{w_n\}$ is B^p-Cauchy uniformly in t, so $w(t) \equiv B^p\text{-}\lim_{n\to\infty} w_n(t)$ exists uniformly in t, and is B^p-continuous.

For each $t \in J$, $w_n(t)(s) \to w(t)(s)$ uniformly in s P-a.s., by the following argument.

$$\begin{aligned} & E(\sum_{n=0}^\infty \sup_{0\leq s\leq 1} |w_{n+1}(t)(s) - w_n(t)(s)|) \\ &= \sum_{n=0}^\infty E(\sup_{0\leq s\leq 1} |w_{n+1}(t)(s) - w_n(t)(s)|) \\ &\leq \sum_{n=0}^\infty \|\sup_{0\leq s\leq 1} |w_{n+1}(t)(s) - w_n(t)(s)|\|_{L^p(\mathcal{P})} \\ &= \sum_{n=0}^\infty \|w_{n+1}(t) - w_n(t)\|_{S^p} \\ &\leq \sum_{n=0}^\infty \|w_{n+1}(t) - w_n(t)\|_{B^p} < \infty. \end{aligned}$$

Let $\Omega_0 = \{\omega \in \Omega | \sum_{n=0}^{\infty} \sup_{0 \le s \le 1} |w_{n+1}(t)(s) - w_n(t)(s)| < \infty\}$, then $P(\Omega_0) = 1$ and $w_n(t)(s) \to w(t)(s)$ uniformly in s on Ω_0. Thus for each $t \in J$, the map $s \to w(t)(s)$ is P-a.s. continuous.

Since $\|w_n(t)\|_{B^\infty} \le C_o$ for all n, we have $\|w(t)\|_{B^\infty} \le C_o$ for all $t \in J$. Also, since w_n is $B^{\infty-}$-Lipschitz with Lipschitz constant independent of n, w is $B^{\infty-}$-Lipschitz as well. By Kolmogorov's Lemma (Lemma 8.1.3), there exists a version of w such that the function $(s,t) \to w(t)(s)$ is jointly continuous P-a.s.

Existence of a Solution in $B^{\infty-}$.

Now we will show that w is $B^{\infty-}$-differentiable, and satisfies (3.1.3).

Fix $p \in [2,\infty)$. Write the above iterates as $w_n(t) = \int O_n(t)db + \int \alpha_n(t)ds$. By (3.1.6) and the "Lipschitz" property of C (3.1.2) the function $\tau \to C(w_n(\tau))O_{n+1}(\tau)$ is $S^{\infty-}$-Lipschitz, and hence $S^{\infty-}$-integrable.

Claim:

$$O_{n+1}(t) - O_o = \int_0^t C(w_n(\tau))O_{n+1}(\tau)d\tau \; P\text{-a.s.} \tag{3.1.10}$$

Proof of claim. Let $C_n(t)$ be a P-a.s. $C^{1,0}$ version of $C(w_n(t))$ for all n. Then by applying the Fundamental Theorem of Calculus pointwise,

$$O_{n+1}(t) - O_o = p.w. \int_0^t C_n(\tau)O_{n+1}(\tau)d\tau \; P\text{-a.s.}$$

We need to show that this pointwise integral is indistinguishable from the $S^{\infty-}$-integral in (3.1.10).

To do this, fix $p \in [2,\infty)$ and let $q(t) = \int_0^t C(w_n(\tau))O_{n+1}(\tau)d\tau$ (where the integral is taken in the S^p-topology). Then $\dot{q}(t) = \frac{d}{dt}q(t)$ is $S^{\infty-}$-Lipschitz, so by Lemma 8.1.4 there exists a version $\tilde{q}(t)$ of $q(t)$ such that P-a.s. the function $(t,s) \to \tilde{q}(t)(s)$ is $C^{1,0}$. Thus P-a.s. $\dot{\tilde{q}}(t) \equiv \frac{d}{dt}\tilde{q}(t)$ exists and

$$\dot{\tilde{q}}(t) \doteq C(w_n(t))O_{n+1}(t) \doteq C_n(t)O_{n+1}(t)$$

where $\doteq$ denotes equality up to P-equivalence. This implies

$$\int_0^t C(w_n(\tau))O_{n+1}(\tau)d\tau \;\; \doteq \;\; \tilde{q}(t)$$

$$
\begin{aligned}
&\doteq \tilde{q}(0) + p.w. \int_0^t \dot{\tilde{q}}(\tau) d\tau \\
&\doteq p.w. \int_0^t \dot{\tilde{q}}(\tau) d\tau \\
&\doteq p.w. \int_0^t C_n(\tau) O_{n+1}(\tau) d\tau \\
&\doteq O_{n+1}(t) - O_o
\end{aligned}
$$

which proves the claim.

Now the right-hand side of (3.1.10) converges to $\int_0^t C(w(\tau))O(\tau)d\tau$ in the S^p-norm since

$$
\begin{aligned}
&\| \int_0^t [C(w_n(\tau))O_{n+1}(\tau) - C(w(\tau))O(\tau)] d\tau \|_{S^p} \\
&\leq \int_0^t \|C(w_n(\tau))O_{n+1}(\tau) - C(w(\tau))O(\tau)\|_{S^p} d\tau \\
&\leq \int_0^t \|C(w_n(\tau)) - C(w(\tau))\|_{S^p} \|O_{n+1}(\tau)\|_{S^\infty} d\tau \\
&\qquad + \int_0^t \|C(w(\tau))\|_{S^r} \|O_{n+1}(\tau) - O(\tau)\|_{S^{r'}} d\tau \text{ by Lemma 8.1.5} \\
&\leq K \int_0^t \|w_{n+1}(\tau) - w(\tau)\|_{B^{r'}} d\tau \text{ by (3.1.7) and (3.1.2)}
\end{aligned}
$$

and $\|w_{n+1}(\tau) - w(\tau)\|_{B^{r'}} \to 0$ uniformly in τ as $n \to \infty$. Here we are using $\frac{1}{p} = \frac{1}{r} + \frac{1}{r'}$ and $K = K(p, C_o, \|C(0)\|_{S^\infty})$.

Since the left-hand side of (3.1.10) converges to $O(t) - O_o$ in the S^p-norm, we have

$$
O(t) = O_o + \int_0^t C(w(\tau))O(\tau) d\tau.
$$

So by the Fundamental Theorem of Calculus for S^p-integrals (Lemma 2.1.14 in the Appendix), $O(t)$ is S^p-differentiable and $\dot{O}(t) = C(w(t))O(t)$. A similar argument shows that α is S^p-differentiable and satisfies (3.1.4).

Using an argument similar to [8] Proposition 6.3, p. 331, we have (1) $w : J \to B^\infty \mathbf{R}^n$ is $B^{\infty-}$-differentiable, with $\dot{w}$ being $B^{\infty-}$-Lipschitz, and (2) there exists a version of w which satisfies (3.1.3) in the pointwise sense of Definition 3.1.3. Q.E.D.

Lemma 3.1.6 *Suppose that* (T, d) *is a separable metric space and* $\{R(t)\}_{t\in T}$ *is a* $\mathbf{R}^n$*-valued* P*-a.s. continuous process. Set* $R^* \equiv \sup_{t\in T} |R(t)|$*. Then*

$$\|R^*\|_{L^\infty(P)} = \sup_{t\in T} \|R(t)\|_{L^\infty(P)}. \tag{3.1.11}$$

Proof. First notice that $|R(t)| \le R^*$ so that $\|R(t)\|_{L^\infty(P)} \le \|R^*\|_{L^\infty(P)}$ and hence

$$\sup_{t\in T} \|R(t)\|_{L^\infty(P)} \le \|R^*\|_{L^\infty(P)}. \tag{3.1.12}$$

For the opposite inequality, let $D \subset T$ be a countable dense subset of T. Without loss of generality we may assume that $\sup_{t\in T} \|R(t)\|_{L^\infty(P)} < \infty$. Let $\lambda = \sup_{t\in T} \|R(t)\|_{L^\infty(P)}$, so $\lambda \ge \|R(t)\|_{L^\infty(P)}$ for each $t \in T$. Hence for each $t \in T$, $P(|R(t)| \le \lambda) = 1$. Therefore

$$P(|R(t)| \le \lambda \text{ for all } t \in D) = P(\cap_{t\in D}\{|R(t)| \le \lambda\}) = 1.$$

Now using the continuity of $R(t)$, it follows that

$$\{|R(t)| \le \lambda \text{ for all } t \in D\} = \{|R(t)| \le \lambda \text{ for all } t \in T\}.$$

Combining the two equations above shows that

$$P(R^* \le \lambda) = P(|R(t)| \le \lambda \text{ for all } t \in T) = 1,$$

that is

$$\|R^*\|_{L^\infty(P)} \le \sup_{t\in T} \|R(t)\|_{L^\infty(P)}. \tag{3.1.13}$$

Clearly Eqs. (3.1.12) and (3.1.13) imply the lemma. Q.E.D.

Lemma 3.1.7 *Let the function* $(C, R) : B^\infty\mathbf{R}^n \to S^{\infty-}\mathrm{End}(\mathbf{R}^n) \times S^{\infty-}\mathbf{R}^n$ *and the constants* c_1*,* c_2*, and* T_∞ *be as given in Notation 3.1.1. Let* $w : J \to B^\infty\mathbf{R}^n$ *be* $B^{\infty-}$*-differentiable, with* $\dot{w}(t) = \frac{d}{dt}w(t)$ $B^{\infty-}$*-Lipschitz. Suppose that* $\|O(t)\|_{S^\infty} \le \|O(0)\|_{S^\infty} e^{T_\infty}$ *for all* $t \in J$*, where* $w(t) = \int O(t)db + \int \alpha(t)ds$*. Set* $w(0) = w_o = \int O_o db + \int \alpha_o ds \in B^\infty\mathbf{R}^n$*.*

Let $\bar{O}$ and $\bar{\alpha}$ be the unique pointwise solutions to the ordinary differential equations (3.1.5):

$$\dot{\bar{O}}(t) = C(w(t))\bar{O}(t) \text{ with } \bar{O}(0) = O_o$$

$$\text{and } \dot{\bar{\alpha}}(t) = C(w(t))\bar{\alpha}(t) + R(w(t)) \text{ with } \bar{\alpha}(0) = \alpha_o$$

given by fixing versions (via Lemma 3.1.4) of $C(w(t))$ and $R(w(t))$ which are P-a.s. $C^{1,0}$ in (t,s). (Here we have solved the ordinary differential equations for each sample point $\omega \in \Omega$ and each $s \in [0,1]$.)

Then $\|\bar{O}(t)\|_{S^\infty} \le \|O_o\|_{S^\infty} e^{T_\infty}$ for all $t \in J$, and there exists a constant $C_o = C_o(c_1, c_2, T_\infty, \|w_o\|_{B^\infty})$ such that $\|\bar{O}(t)\|_{S^\infty} + \|\bar{\alpha}(t)\|_{S^\infty} \le C_o$ for all $t \in J$.

Proof. We will follow the proof of [8] Corollary 6.1, p. 327. By [8] Lemma 6.1, p. 325 we have for P-a.e. ω and all $s \in [0,1]$ and $t \in J$,

$$\begin{aligned} |\bar{O}(t)(\omega)(s)| &\le \sup_{t\in J} |\bar{O}(t)(\omega)(s)| \\ &\le |\bar{O}(0)(\omega)(s)| e^{T_\infty}. \end{aligned}$$

It follows that

$$\|\bar{O}(t)\|_{S^\infty} \le \|\bar{O}(0)\|_{S^\infty} e^{T_\infty} \text{ for all } t \in J. \tag{3.1.14}$$

Also, by [8] Lemma 6.1, p. 325:

$$\sup_{t\in J} |\bar{\alpha}(t)(\omega)(s)| \le [|\bar{\alpha}(0)(\omega)(s)| + \sup_{t\in J} |R(t)(\omega)(s)|] e^{T_\infty}.$$

Thus

$$\begin{aligned} \sup_{t\in J} \|\bar{\alpha}(t)\|_{S^\infty} &\le \|\sup_{t\in J} \bar{\alpha}(t)\|_{S^\infty} \\ &\le [\|\bar{\alpha}(0)\|_{S^\infty} + \|\sup_{t\in J} |R(t)|\|_{S^\infty}] e^{T_\infty} \\ &\le [\|\bar{\alpha}(0)\|_{S^\infty} + ess \sup_{\omega\in\Omega} \sup_{s\in[0,1]} \sup_{t\in J} |R(t)(\omega)(s)|] e^{T_\infty} \\ &\le [\|\bar{\alpha}(0)\|_{S^\infty} + \sup_{\substack{s\in[0,1] \\ t\in J}} ess \sup_{\omega\in\Omega} |R(t)(\omega)(s)|] e^{T_\infty} \\ &\le [\|\bar{\alpha}(0)\|_{S^\infty} + \sup_{t\in J} ess \sup_{\omega\in\Omega} \sup_{s\in[0,1]} |R(t)(\omega)(s)|] e^{T_\infty} \end{aligned}$$

$$\leq \quad [\|\bar{\alpha}(0)\|_{S^\infty} + c_1 \sup_{t\in J} \|O(t)\|^2_{S^\infty} + c_2]e^{T_\infty}$$
$$\leq \quad [\|\alpha_o\|_{S^\infty} + c_1 \sup_{t\in J} \|O_o\|^2_{S^\infty} e^{2T_\infty} + c_2]e^{T_\infty}$$

where we have used Lemma 3.1.6 in the fourth and fifth lines on the P-a.s. continuous functions $(t,s) \to R(t)(s)$ and $s \to R(t)(s)$ for fixed $t \in J$.

Combining these results, we have

$$\sup_{t\in J} \|\bar{O}(t)\|_{S^\infty} + \sup_{t\in J} \|\bar{\alpha}(t)\|_{S^\infty} \leq C_o$$

where

$$C_o = [\|w_o\|_{B^\infty} + c_1 \sup_{t\in J} \|O_o\|^2_{S^\infty} e^{2T_\infty} + c_2]e^{T_\infty}.$$

Q.E.D.

Corollary 3.1.8 *Let the function* $(C,R) : B^\infty\mathbf{R}^n \to S^{\infty-}\mathrm{End}(\mathbf{R}^n) \times S^{\infty-}\mathbf{R}^n$ *and the constants* c_1, c_2, *and* T_∞ *be as given in Notation 3.1.1. Let* O *and* α *be solutions to (3.1.4) in the pointwise sense of Definition 3.1.2.*

Then $\|O(t)\|_{S^\infty} \leq \|O_o\|_{S^\infty} e^{T_\infty}$ *for all* $t \in J$, *and there exists a constant* $C_o = C_o(c_1, c_2, T_\infty, \|w_o\|_{B^\infty})$ *such that* $\|O(t)\|_{S^\infty} + \|\alpha(t)\|_{S^\infty} \leq C_o$ *for all* $t \in J$.

Proof. The proof is essentially the same as for Lemma 3.1.7 above. The difference is that in the last line of the estimate on $\sup_{t\in J} \|\bar{\alpha}(t)\|_{S^\infty}$ we should now use (3.1.14). Q.E.D.

3.2 Driver's Flows in Wiener Space

Here we will see that the "flows" on semimartingales from Section 3.1 induce flows in path space which preserve Wiener measure. The additional assumption required on the vector fields in Theorem 3.1.5 to obtain these flows in $W(\mathbf{R}^n)$ is that the function T take values in $S^{\infty-}\mathrm{so}(n)$, not $S^\infty\mathrm{End}(\mathbf{R}^n)$. This condition corresponds to the "Torsion Skew-Symmetry" assumption of Driver. We will outline Driver's proof, generalizing his result to our "nongeometric" case (in which the vector fields do not depend on the manifold M).

Notation 3.2.1 *Given a finite-dimensional vector space V,*

1. *Let $\mathcal{H}_s(V) = \sigma\{\xi_r : 0 \leq r \leq s\}$ where $\xi_r : W(V) \to V$ is the coordinate function defined by $\xi_r(\omega) = \omega(r)$ for all $\omega \in W(V)$.*

2. *Let $\bar{\mathcal{H}}^Q$ be the completion of $\mathcal{H}_1(V)$ with respect to some measure Q. (The extension of Q to $\bar{\mathcal{H}}^Q$ will still be called Q).*

3. *Denote by $\{\bar{\mathcal{H}}_s^Q\}$ the completion of the filtration $\{\mathcal{H}_s(V)\}$ with respect to Q. So $\bar{\mathcal{H}}_s^Q = \sigma(\mathcal{H}_s(V) \cup \mathcal{N}(Q))$ where $\mathcal{N}(Q) \equiv \{N \in \bar{\mathcal{H}}^Q, Q(N) = 0\}$, the set of Q-null sets.*

4. *Let $\bar{\mathcal{H}}_{s+}^Q \equiv \bigcap_{\epsilon > 0} \bar{\mathcal{H}}_{s+\epsilon}^Q$.*

With this definition, $\{\bar{\mathcal{H}}_{s+}^Q\}_{s \geq 0}$ is a right-continuous, complete filtration with respect to Q on $W(V)$, thus $(W(V), \{\bar{\mathcal{H}}_{s+}^Q\}, \bar{\mathcal{H}}_1^Q, Q)$ is a probability space satisfying the usual hypothesis (Notation 2.1.1).

Let μ denote standard Wiener measure on $W(\mathbf{R}^n)$, and equip the probability space $(W(\mathbf{R}^n), \{\bar{\mathcal{H}}_{s+}^\mu\}, \mu)$ with reference Brownian motion $\bar{b}$ defined by $\bar{b}_s(\omega) = \omega(s)$ for all $\omega \in W(\mathbf{R}^n)$ and $s \in [0, 1]$.

Notation 3.2.2 *Let $T : B^\infty \mathbf{R}^n \to S^{\infty-}\mathrm{so}(n)$ in Theorem 3.1.5, and let $\bar{w} : \mathbf{R} \to B^\infty \mathbf{R}^n$ be the solution to (3.1.3) with $\bar{w}(0) = \bar{b}$ (see Notation 3.2.1). Here the underlying probability space is $(W(\mathbf{R}^n), \{\bar{\mathcal{H}}_{s+}^\mu\}, \{\bar{b}(s)\}, \mu)$.*

Define $(\bar{O}, \bar{\alpha})$ by $\bar{w}(t) = \int \bar{O}(t) d\bar{b} + \int \bar{\alpha}(t) ds$. Since $C(\bar{w}(t))$ is μ-a.s. $\mathrm{so}(n)$-valued, $\bar{O}(t)$ is μ-a.s. $O(n)$-valued by Lemma 8.1.1.

We may choose a version such that the function $(t, s) \to (\bar{w}(t)(s), \bar{O}(t)(s), \bar{\alpha}(t)(s))$ is μ-a.s. $C^{1,0}$ by Lemma 8.1.4, since $\frac{d}{dt}\bar{w}(t)$ is $B^{\infty-}$-Lipschitz ([8] Proposition 6.3).

For each $t \in \mathbf{R}$, define $\tilde{w}(t) : W(\mathbf{R}^n) \to W(\mathbf{R}^n)$, $\tilde{O}(t) : W(\mathbf{R}^n) \to W(O(n))$ and $\tilde{\alpha}(t) : W(\mathbf{R}^n) \to W(\mathbf{R}^n)$, by $\tilde{w}(t) = \bar{w}(t)$, $\tilde{O}(t) = \bar{O}(t)$ and $\tilde{\alpha}(t) = \bar{\alpha}(t)$.

Remark 3.2.3 *Notice that $\tilde{w}(t)$ solves (3.1.3) with $\tilde{w}(0) \equiv id \in Maps(W(\mathbf{R}^n), W(\mathbf{R}^n))$. Equivalently, $(\tilde{O}(t), \tilde{\alpha}(t))$ solve (3.1.4) with $\tilde{O}(0) \equiv id \in O(n)$ and $\tilde{\alpha}(0) \equiv 0 \in \mathbf{R}^n$.*

Theorem 3.2.4 *Let $w_o = \int O_o db + \int \alpha_o ds \in B^\infty \mathbf{R}^n$ with O_o an $O(n)$-valued process and $\|\alpha_o\|_{S^\infty} < \infty$. Assume $T : B^\infty \mathbf{R}^n \to S^{\infty-}\mathrm{so}(n)$ in Theorem 3.1.5, and let $w : \mathbf{R} \to B^\infty \mathbf{R}^n$ be the solution to (3.1.3) with $w(0) = w_o$. Then for each $t \in \mathbf{R}$,*

1. *$w(t) = \int O(t)db + \int \alpha(t)ds$ where $O(t)$ is an $O(n)$-valued process and there exists a constant $C_o = C_o(c_1, c_2, \|\alpha_o\|_{S^\infty})$ such that $\|\alpha(t)\|_{S^\infty} \le C_o < \infty$,*
2. *the law of $w(t)$ is equivalent to μ, Wiener measure on $W(\mathbf{R}^n)$, and*
3. *if ρ is the Radon-Nikodym derivative $\rho \equiv \frac{d(w(t)_* P)}{d\mu}$, then ρ^r is μ-integrable for all $r \in \mathbf{R}$.*

Proof. We will restrict t to a compact interval J which for definiteness will be taken as $[-1, 1]$. Write $w(t) = \int O(t)db + \int \alpha(t)ds$, then O and α satisfy equations (3.1.4):

$$\dot{O}(t) = C(w(t))O(t)$$

$$\text{and } \dot{\alpha}(t) = C(w(t))\alpha(t) + R(w(t))$$

with $O(0) = O_o$ and $\alpha(0) = \alpha_o$. Since $A(w(t))$ and $T(w(t))$ are $\mathrm{so}(n)$-valued processes, so is $C(w(t)) = A(w(t)) + T(w(t))$.

Recall that $(O(t), \alpha(t))$ is a solution to (3.1.4) in the pointwise sense of Definition 3.1.2, so $O(t)$ is $O(n)$-valued P-a.s. since for fixed s and ω, $\dot{O}(t)(\omega)(s) = C(t)(\omega)(s)O(t)(\omega)(s)$ is a finite-dimensional linear ordinary differential equation, with $C(t)(\omega)(s) \in \mathrm{so}(n)$. ($C(t)$ is the version of $C(w(t))$ given in Definition 3.1.2).

Also, by Lemma 3.1.7 we may choose a version of $\alpha(t)$ such that $|\alpha(t)|_\infty = \sup_{0\le s\le 1}|\alpha(t)(s)| \le C_o < \infty$ (independent of $t \in J$). Thus $P(\int_0^1 |\alpha(s)|^2 ds \le C_o^2) = 1$, so we may apply Girsanov's Theorem (Lemma 8.1.2) to conclude that the law of $w(t)$ is equivalent to μ for all $t \in J$.

The fact that ρ^r is μ-integrable for all $r \in \mathbf{R}$ follows from [8] Corollary 8.1, p. 349. Q.E.D.

Theorem 3.2.5 *Let $(\Omega, \{\mathcal{F}_s\}, P)$ be a filtered probability space satisfying the usual hypothesis, equipped with an $\mathbf{R}^n$-valued Brownian motion $b(s)$.*

Let $\tilde{w}(t) : W(\mathbf{R}^n) \to W(\mathbf{R}^n)$ be as defined in Notation 3.2.2. Let $w_o = \int O_o db + \int \alpha_o ds \in B^\infty \mathbf{R}^n$ with O_o an $O(n)$-valued process and $\|\alpha_o\|_{S^\infty} < \infty$, as in Theorem 3.2.4. Let $w : \mathbf{R} \to B^\infty \mathbf{R}^n$ be the solution to (3.1.3) with $w(0) = w_o$ as given by Theorem 3.1.5, where $T : B^\infty \mathbf{R}^n \to S^{\infty-}\mathrm{so}(n)$.

Then for each $t \in R$, $w(t)$ is P-indistinguishable from $\tilde{w}(t) \circ w_o$.

Proof. By Theorem 3.2.4(i), we have the hypotheses of Girsanov's Theorem satsified by $\tilde{w}(t)$ on $(W(\mathbf{R}^n), \{\bar{\mathcal{H}}^\mu_{s+}\}, \{\bar{b}(s)\}, \mu)$, and by $w(t)$ on $(\Omega, \{\mathcal{F}_s\}, \{b(s)\}, P)$, since the sup-norm bound implies an L^2 bound on $\tilde{\alpha}(t)$ and $\alpha(t)$ respectively. Thus we have both $\tilde{w}(t)_*\mu \sim \bar{b}_*\mu = \mu$ and $w(t)_*P \sim b_*P = \mu$, where $\sim$ means equivalence.

Since $w_o : \Omega \to W(\mathbf{R}^n)$ is an adapted process, we have for all $s \in [0,1]$, $w_0(s)$ is $\mathcal{F}_s/\mathcal{B}(\mathbf{R}^n)$-measurable, i.e., $w_o(s) \in \mathcal{F}_s/\mathcal{B}(\mathbf{R}^n)$, where $\mathcal{B}(\mathbf{R}^n)$ denotes the Borel sets on $\mathbf{R}^n$. But $w_o(s) = \bar{b}_s \circ w_o$, thus

$$\begin{aligned} & w_o(s) \in \mathcal{F}_s/\mathcal{B}(\mathbf{R}^n) \\ \Leftrightarrow \quad & \bar{b}_s \circ w_o \in \mathcal{F}_s/\mathcal{B}(\mathbf{R}^n) \\ \Leftrightarrow \quad & w_o \in \mathcal{F}_s/\mathcal{H}_s \text{ since } \{\bar{b}_r\}_{0\le r\le s} \text{ generate } \mathcal{H}_s. \end{aligned}$$

Claim: w_o is $\mathcal{F}_s/\bar{\mathcal{H}}^\mu_{s+}$-measurable for all $s \in [0,1]$.

First note that if $N \in \mathcal{N}(\mu)$ then $\mu(N) = 0$, and hence $P \circ w_o^{-1}(N) = 0$ since $w_{o*}P \sim \mu$. Thus $w_o^{-1}(N) \in \mathcal{F}_r$ for all $r \geq 0$ since $\mathcal{F}_r$ contains all P-null sets.

Thus for each $\epsilon > 0$, $w_o \in \mathcal{F}_{s+\epsilon}/\mathcal{N}(\mu)$, and since $w_o \in \mathcal{F}_{s+\epsilon}/\mathcal{H}_{s+\epsilon}$ we have w_o is $\mathcal{F}_{s+\epsilon}/\sigma(\mathcal{H}_{s+\epsilon} \cup \mathcal{N}(\mu))$-measurable. Let $B \in \bar{\mathcal{H}}_{s+}^{\mu} = \bigcap_{\epsilon>0} \sigma(\mathcal{H}_{s+\epsilon} \cup \mathcal{N}(\mu))$. Then $w_o^{-1}(B) \in \mathcal{F}_{s+\epsilon}$ for all $\epsilon > 0$, hence $w_o^{-1}(B) \in \bigcap_{\epsilon>0} \mathcal{F}_{s+\epsilon} = \mathcal{F}_s$ by right continuity. This proves the claim.

Thus since $\tilde{w}(t) : W(\mathbf{R}^n) \to W(\mathbf{R}^n)$ is $\bar{\mathcal{H}}_{s+}^{\mu}/\bar{\mathcal{H}}_{s+}^{\mu}$-measurable, the process $\tilde{w}(t) \circ w_o$ is $\mathcal{F}_s/\bar{\mathcal{H}}_{s+}^{\mu}$-measurable for all $s \in [0,1]$. By [8] Proposition 8.2, p. 352, $\tilde{w}(t) \circ w_o$ is P-indistinguishable from

$$\begin{aligned}
&\int \tilde{O}(t) \circ w_o dw_o + \int \tilde{\alpha}(t) \circ w_o ds \\
&= \int \tilde{O}(t) \circ w_o \cdot O_o db + \int [\tilde{O}(t) \circ w_o \cdot \alpha_o + \tilde{\alpha}(t) \circ w_o] ds
\end{aligned}$$

Set $O(t) = \tilde{O}(t) \circ w_o \cdot O_o$ and $\alpha(t) = \tilde{O}(t) \circ w_o \cdot \alpha_o + \tilde{\alpha}(t) \circ w_o$

Claim: $O(t)$ and $\alpha(t)$ are $S^{\infty-}$-continuously differentiable and solve (3.1.4) with $O(0) = O_o$ and $\alpha(0) = \alpha_o$.

With this claim we have

$$w(t) = \int O(t) db + \int \alpha(t) ds \doteq \tilde{w}(t) \circ w_o$$

by the uniqueness assertion in Theorem 3.1.5, thus proving the theorem.

To prove the claim we will need the following lemma:

Lemma 3.2.6 *Keeping the same notation as in Theorem 3.2.5, let $Z : W(\mathbf{R}^n) \to W(\mathbf{R}^N)$ be a $\bar{\mathcal{H}}_1^{\mu}/\mathcal{H}_1(\mathbf{R}^N)$-measurable process. Then for each $p \in [2,\infty)$ and each $r \in (2,\infty)$ there is a constant $C = C(r,p)$ independent of Z such that*

$$\|Z \circ w_o\|_{S^p(P)} \leq C \|Z\|_{S^r(\mu)}$$

Proof. Let $\rho \equiv \frac{d(w_{o*}P)}{d\mu}$, then

$$\|Z \circ w_o\|_{S^p(P)} = \|Z\|_{S^p(w_{o*}P)}$$

$$
\begin{aligned}
&= \|Z\|_{S^p(\rho\mu)} \\
&= [\int (Z^* \rho^{\frac{1}{p}})^p d\mu]^{\frac{1}{p}} \\
&= \|Z^* \rho^{\frac{1}{p}}\|_{L^p(\mu)} \\
&\leq \|\rho\|_{L^{r'}(\mu)} \|Z\|_{S^r(\mu)}
\end{aligned}
$$

where $\frac{1}{r'} = \frac{1}{p} - \frac{1}{r}$. Set $C \equiv \|\rho^{\frac{1}{p}}\|_{L^{r'}(\mu)}$ which is finite by [8] Corollary 8.1, p. 349. Q.E.D. (Lemma)

Now since the functions $t \to \tilde{O}(t)$ and $t \to \tilde{\alpha}(t)$ are $S^{\infty -}(\mu)$-continuously differentiable, it is clear from the above lemma that $t \to \tilde{O}(t) \circ w_o$ and $t \to \tilde{\alpha}(t) \circ w_o$ are $S^{\infty -}(P)$-continuously differentiable with derivatives $\dot{\tilde{O}}(t) \circ w_o$ and $\dot{\tilde{\alpha}}(t) \circ w_o$ respectively. This implies that $t \to O(t)$ and $t \to \alpha(t)$ are also $S^{\infty -}(P)$-continuously differentiable with

$$
\begin{aligned}
\dot{O}(t) &= \dot{\tilde{O}}(t) \circ w_o \cdot O_o \text{ and} \\
\dot{\alpha}(t) &= \dot{\tilde{O}}(t) \circ w_o \cdot \alpha_o + \dot{\tilde{\alpha}}(t) \circ w_o.
\end{aligned}
$$

To finish the proof we will show that $O(t)$ and $\alpha(t)$ solve (3.1.4), that is,

$$
\begin{aligned}
\dot{O}(t) &= C(\hat{w}(t))O(t) \\
\text{and} \qquad \dot{\alpha}(t) &= C(\hat{w}(t))\alpha(t) + R(\hat{w}(t))
\end{aligned}
\tag{3.2.1}
$$

with $O(0) = O_o$ and $\alpha(0) = \alpha_o$ where $\hat{w}(t) \equiv \int O(t)db + \int \alpha(t)ds$.

First note that since $\tilde{O}(0) = id \in O(n)$ and $\tilde{\alpha}(0) \equiv 0 \in \mathbf{R}^n$ (see Remark 3.2.3) we have $O(0) = O_o$ and $\alpha(0) = \alpha_o$.

Let $\bar{C}$ and $\bar{R}$ be continuous versions of $C(b)$ and $R(b)$ respectively, so we may view $\bar{C} : W(\mathbf{R}^n) \to W(\mathrm{so}(n))$ and $\bar{R} : W(\mathbf{R}^n) \to W(\mathbf{R}^n)$. These are functions which "implement" $C : B^\infty \mathbf{R}^n \to S^{\infty -}\mathrm{so}(n)$ and $R : B^\infty \mathbf{R}^n \to S^\infty \mathbf{R}^n$ in the following sense. Let $Z = \int O db + \int \alpha ds \in B^\infty \mathbf{R}^n$ with O being $O(n)$-valued and $\|\alpha\|_{S^\infty} < \infty$ $\bar{C} \circ Z$ and $\bar{R} \circ Z$ are indistinguishable from $C(Z)$ and $R(Z)$ by Girsanov's Theorem (Lemma 8.1.2) and [8] Proposition 8.2, p. 352.

We can now easily verify (3.2.1). By definition we have $\dot{\tilde{O}}(t) = [\bar{C} \circ \tilde{w}(t)]\tilde{O}(t)$ and hence

$$\begin{aligned}
\dot{O}(t) &= \dot{\tilde{O}}(t) \circ w_o \cdot O_o \\
&= \bar{C} \circ \underbrace{\tilde{w}(t) \circ w_o}_{\hat{w}(t)} \cdot \underbrace{\tilde{O}(t) \circ w_o \cdot O_o}_{O(t)} \\
&= C(\hat{w}(t))O(t).
\end{aligned}$$

Similarly, by definition, $\dot{\tilde{\alpha}}(t) = \bar{C} \circ \tilde{w}(t)\tilde{\alpha}(t) + \bar{R} \circ \tilde{w}(t)$ and thus

$$\begin{aligned}
\dot{\alpha}(t) &= \dot{\tilde{O}}(t) \circ w_o \cdot \alpha_o + \dot{\tilde{\alpha}}(t) \circ w_o \\
&= \bar{C} \circ \underbrace{\tilde{w}(t) \circ w_o}_{\hat{w}(t)} \cdot \underbrace{\tilde{O}(t) \circ w_o \cdot \alpha_o + \tilde{\alpha}(t) \circ w_o}_{\alpha(t)} \\
&\quad + \bar{R} \circ \underbrace{\tilde{w}(t) \circ w_o}_{\hat{w}(t)} \\
&= C(\hat{w}(t))\alpha(t) + R(\hat{w}(t))
\end{aligned}$$

This proves the claim and the theorem. Q.E.D.

Remark 3.2.7 *Notice that the notion of a solution to (3.1.3) (with derivative taken in $B^{\infty-}$) is independent of the particular choice of a reference Brownian motion. This follows from the fact that the S^p-norm of a process is invariant under multiplication by an $O(n)$-valued process (Lemma 4.2.3). Any $\mathbf{R}^n$-valued Brownian motion (B) on $(W(\mathbf{R}^n), \{\bar{\mathcal{H}}^{\mu}_{s+}\}, \mu)$ must be of the form $B = \int O d\bar{b}$, where O is a predictable $O(n)$-valued process. (See [41] Theorem 42, p. 155.)*

Theorem 3.2.8 *([8] Theorem 8.4). Suppose $\tilde{w}$, $\tilde{O}$, and $\tilde{\alpha}$ are as in Notation 3.2.2. Then $\tilde{w}$ is a flow on $W(\mathbf{R}^n)$ which leaves Wiener measure μ quasi-invariant. More explicitly,*

1. *for all $t, \tau \in \mathbf{R}$, $\tilde{w}(t+\tau) = \tilde{w}(t) \circ \tilde{w}(\tau)$ μ-a.s., and*

2. *$\frac{d(\tilde{w}(t)_* \mu)}{d\mu} = Z(t)$ where*

$$Z(t) = \exp\{-\int_0^1 \tilde{\alpha}(-t) \cdot \tilde{O}(-t) d\bar{b} - \frac{1}{2}\int_0^1 |\tilde{\alpha}(-t)(s)|^2 ds\}.$$

Proof.

1. By Theorem 3.2.4, we may take $(\bar{O}, \bar{\alpha})$ such that $\bar{O}(t)$ is an $O(n)$-valued process and $\|\bar{\alpha}(t)\|_{S^\infty} \leq C_o < \infty$. Thus fixing $\tau \in \mathbf{R}$ and setting $w_o = \tilde{w}(\tau)$ $(= \bar{w}(\tau))$ and $(\Omega, \{\mathcal{F}_s\}, P) = (W(\mathbf{R}^n), \{\bar{\mathcal{H}}^\mu_{s+}\}, \mu)$ in Theorem 3.2.5 we have $w(t) \equiv \tilde{w}(t) \circ \tilde{w}(\tau)$ solves (3.1.3) with $w(0) = \tilde{w}(\tau)$. But the function $t \to \tilde{w}(t+\tau)$ also solves (3.1.3) with the same initial condition. Thus by uniqueness of solutions (Theorem 3.1.5), we have $\tilde{w}(t+\tau) = \tilde{w}(t) \circ \tilde{w}(\tau)$ μ-a.s.

2. Note that $\|\bar{\alpha}(t)\|_{S^\infty} \leq C_o$ implies $P(\int_0^1 |\bar{\alpha}(t)(s)|^2 ds \leq C_o^2) = 1$. By Girsanov's Theorem (Lemma 8.1.2), with $P = \mu$, $b = \bar{b}$, and $w = \tilde{w}(t)$, (and hence $Z_1 = Z(-t)$), we have

$$\mu = \bar{b}_* \mu = \tilde{w}(t)_*[Z(-t) \cdot \mu], \text{ thus}$$

$$\mu(f) = \mu(f \circ \bar{b}) = \mu(Z(-t) f \circ \tilde{w}(t))$$

for all bounded measurable functions $f : W(\mathbf{R}^n) \to \mathbf{R}$. Letting $f \to f \circ \tilde{w}(-t)$ we find

$$\mu(f \circ \tilde{w}(-t)) = \mu(Z(-t) f \circ \tilde{w}(-t) \circ \tilde{w}(t)) = \mu(Z(-t) f) \tag{3.2.2}$$

since by part (i), $f \circ \tilde{w}(-t) \circ \tilde{w}(t) = f$ μ-a.s. Replacing t by $-t$ in (3.2.2),

$$\mu(f \circ \tilde{w}(t)) = \mu(Z(t) f)$$

which implies $\frac{d(\tilde{w}(t)_* \mu)}{d\mu} = Z(t)$.

Q.E.D.

Chapter 4

Existence and Uniqueness for the "Formal" Equation

4.1 Derivation of the Formal Equation for the Differential of the Flow

In this chapter and the next we will prove the existence of differentials of Driver's flows on semimartingales (Section 3.1). This "machine" will then be used in Section 5.4 to obtain differentials of the quasi-invariant flows in the path space $W(\mathbf{R}^n)$ (from Section 3.2).

The following notation will be used throughout this chapter and the next. These are the same as in 3.1.5 except that now C will take values in $S^{\infty-}\mathrm{so}(n)$, not $S^{\infty-}\mathrm{End}(\mathbf{R}^n)$. This assumption facilitates many of the calculations in this section and the next, but the results in these sections hold in the more general case as well.

Notation 4.1.1 *In this chapter we return to the abstract underlying filtered probability space* $(\Omega, \{\mathcal{F}_s\}, \mathcal{F}, P)$*, satisfying the usual conditions, with reference Brownian motion* b*.*

Fix $C : B^\infty\mathbf{R}^n \to S^{\infty-}\mathrm{so}(n)$*,* $R : B^\infty\mathbf{R}^n \to S^\infty\mathbf{R}^n$*,* $X : B^\infty\mathbf{R}^n \to$

$B^{\infty-}\mathbf{R}^n$, $w : \mathbf{R} \to B^\infty\mathbf{R}^n$ $O : \mathbf{R} \to S^\infty O(n)$ and $\alpha : \mathbf{R} \to S^\infty\mathbf{R}^n$ satisfying the following conditions as in Theorem 3.1.5. (Let $p, p_1, p_2 \in [2, \infty)$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.)

1. *C and R are $S^{\infty-}$-differentiable (see Definition 2.1.9), and for all $w, \tilde{w} \in B^\infty\mathbf{R}^n$ and $v \in B^{\infty-}\mathbf{R}^n$,*

$$\begin{aligned} \|v_w C\|_{S^p} &\leq K\|v\|_{B^p} \text{ and} \\ \|v_w C - v_{\tilde{w}} C\|_{S^p} &\leq K\|w - \tilde{w}\|_{B^{p_1}}\|v\|_{B^{p_2}} \end{aligned} \tag{4.1.1}$$

similarly

$$\begin{aligned} \|v_w R\|_{S^p} &\leq K\|v\|_{B^p} \text{ and} \\ \|v_w R - v_{\tilde{w}} R\|_{S^p} &\leq K\|w - \tilde{w}\|_{B^{p_1}}\|v\|_{B^{p_2}} \end{aligned} \tag{4.1.2}$$

where $K = K(n, p_1, p_2, \bar{K}_w, \bar{K}_{\tilde{w}})$.

2. *There exist constants c_1 and c_2 such that*

$$\|R(w)\|_{S^\infty} \leq c_1\|O\|_{S^\infty}^2 + c_2 \text{ for all } w = \int O db + \int \alpha ds \in B^\infty\mathbf{R}^n.$$

3. *$\|C(0)\|_{S^\infty} < \infty$.*

4. *$X : B^\infty\mathbf{R}^n \to B^{\infty-}\mathbf{R}^n$ is defined by*

$$X(w) \equiv \int C(w)dw + \int R(w)ds \text{ for } w \in B^\infty\mathbf{R}^n.$$

5. *$w : \mathbf{R} \to B^\infty\mathbf{R}^n$ is the $B^{\infty-}$-differentiable solution to the equation (3.1.3):*

$$\dot{w}(t) = X(w(t))$$

given by Theorem 3.1.5 with $w(0) = w_o = \int O_o db + \int \alpha_o ds$ *in the space of paths*

$$\{w : \mathbf{R} \to B^\infty\mathbf{R}^n | \sup_{|t|<T} \|w(t)\|_{B^\infty} < \infty \; \forall T > 0\}.$$

6. $(O,\alpha):\mathbf{R}\to S^\infty\mathrm{End}(\mathbf{R}^n)\times S^\infty\mathbf{R}^n$ *is defined by* $w(t)=\int O(t)db+\int\alpha(t)ds$. *That is,* (O,α) *is the unique* $S^{\infty-}$*-differentiable solution to the equations (3.1.4):*

$$\begin{aligned}\dot{O}(t) &= C(w(t))O(t)\\ \text{and}\qquad \dot{\alpha}(t) &= C(w(t))\alpha(t)+R(w(t))\end{aligned}$$

with $O(0)=O_o$ *and* $\alpha(0)=\alpha_o$*, in the space of paths*

$$\{(O,\alpha):\mathbf{R}\to S^\infty\mathrm{End}(\mathbf{R}^n)\times S^\infty\mathbf{R}^n\mid \sup_{|t|<T}[\|O(t)\|_{S^\infty}+\|\alpha(t)\|_{S^\infty}]<\infty\ \forall T>0\}.$$

Notation 4.1.2 *For the following we write the solution to the equation* $\dot{w}(t)=X(w(t))$ *as a function of its starting point, so* $w(t,w_o)$ *denotes the solution to equation (3.1.3):*

$$\dot{w}(t,w_o)=X(w(t,w_o))\text{ with }w(0,w_o)=w_o\in B^\infty\mathbf{R}^n.$$

We also extend this notation to O *and* α*, so that* $O(t,w_o)$ *and* $\alpha(t,w_o)$ *are defined by*

$$w(t,w_o)=\int O(t,w_o)db+\int\alpha(t,w_o)ds.$$

The following theorem gives the main smoothness result in the semi-martingale setting.

Theorem 4.1.3 **i)** *The functions* $O(t,\cdot):B^\infty\mathbf{R}^n\to S^\infty\mathbf{End}(\mathbf{R}^n)$ *and* $\alpha(t,\cdot):B^\infty\mathbf{R}^n\to S^\infty\mathbf{R}^n$ *are* $S^{\infty-}$*-differentiable at* w_o*. (See Definition 2.1.9.)*

In particular, let $\gamma:J\to B^\infty\mathbf{R}^n$ *be an admissible curve (Def. 2.1.7) with* $\gamma_0=w_o$*, and let* $Y=\frac{d}{dt}|_0\gamma_\epsilon$*. Then*

$$\begin{aligned}[Y_{w_o}O](t) &\equiv \lim_{\epsilon\to0}\tfrac{O(t,\gamma_\epsilon)-O(t,w_o)}{\epsilon}\\ \text{and }[Y_{w_o}\alpha](t) &\equiv \lim_{\epsilon\to0}\tfrac{\alpha(t,\gamma_\epsilon)-\alpha(t,w_o)}{\epsilon}\end{aligned}\tag{4.1.3}$$

exist where the limits are taken in the $S^p\mathrm{End}(\mathbf{R}^n)$*- and* $S^p\mathbf{R}^n$*-topologies for* $p\in[2,\infty)$*. Furthermore,* $Z(t)\equiv\begin{bmatrix}[Y_{w_o}O](t)\\ [Y_{w_o}\alpha](t)\end{bmatrix}\equiv\begin{bmatrix}Z_1(t)\\ Z_2(t)\end{bmatrix}$ *satisfies the equation*

$$\dot{Z}(t)=C(w(t))Z(t)+A_t(Z(t))+K(t)\text{ with }Z(0)=0\tag{4.1.4}$$

where A_t and $K(t)$ are defined below.

ii) *The function $w(t,\cdot) : B^\infty\mathbf{R}^n \to B^\infty\mathbf{R}^n$ is $B^{\infty-}$-differentiable at w_o (see Definition 2.1.9.)*

In particular, let $\gamma : J \to B^\infty\mathbf{R}^n$ and Y be as in **i)***, then*

$$[Y_{w_o}w](t) \equiv \lim_{\epsilon\to 0}\frac{w(t,\gamma_\epsilon) - w(t,w_o)}{\epsilon}$$

exists where the limit is taken in the $B^p\mathbf{R}^n$-topologies for $p \in [2,\infty)$. Furthermore, $[Y_{w_o}w](t) = \tilde{Z}(t)$ where

$$\tilde{Z}(t) \equiv \int_0^s Z_{1,\bar{s}}(t)db(\bar{s}) + \int_0^s Z_{2,\bar{s}}(t)d\bar{s} + \int_0^s O_{\bar{s}}(t)dY(\bar{s}). \qquad (4.1.5)$$

Notation 4.1.4 *For $D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$ an $\{\mathcal{F}_s\}$-adapted* $\mathrm{End}(\mathbf{R}^n) \times \mathbf{R}^n$*-valued continuous process, define*

$$\begin{aligned} A_t(D) \quad &\equiv \quad \begin{bmatrix} C'(w(t))\langle \int D_1 db + \int D_2 ds\rangle O(t) \\ C'(w(t))\langle \int D_1 db + \int D_2 ds\rangle \alpha(t) \end{bmatrix} \\ &+ \quad \begin{bmatrix} 0 \\ R'(w(t))\langle \int D_1 db + \int D_2 ds\rangle \end{bmatrix}, \end{aligned}$$

and $K(t) \equiv \begin{bmatrix} C'(w(t))\langle \int O(t)dY\rangle O(t) \\ C'(w(t))\langle \int O(t)dY\rangle\alpha(t) + R'(w(t))\langle \int O(t)dY\rangle \end{bmatrix}.$

In this chapter we prove there exists a unique solution to the "formal" equation (4.1.4). This equation must be satisfied by the differential (if it exists) of the flow, since (4.1.4) may be obtained by formally "differentiating" (3.1.4) via Y. We will now demonstrate this procedure.

Remark 4.1.5 *By formally differentiating (3.1.4) via Y, we obtain (4.1.4). That is, if $Z_1 =$ "$Y_{w_o}O$" and $Z_2 =$ "$Y_{w_o}\alpha$" then $Z \equiv \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ satisfies Equation (4.1.4):*

$$\dot{Z}(t) = C(w(t))Z(t) + A_t(Z(t)) + K(t) \text{ with } Z(0) = 0.$$

Equivalently,

$$\dot{Z}_1(t) = C(w(t))Z_1(t) + C'(w(t))\langle\tilde{Z}(t)\rangle O(t) \text{ with } Z_1(0) = 0 \text{ and} \tag{4.1.6}$$

$$\dot{Z}_2(t) = C(w(t))Z_2(t) + C'(w(t))\langle\tilde{Z}(t)\rangle\alpha(t) + R'(w(t))\langle\tilde{Z}(t)\rangle \text{ with } Z_2(0) = 0 \tag{4.1.7}$$

where $\tilde{Z}(t)$ is defined by (4.1.5).

We first "differentiate" the equation $\dot{O}(t) = C(w(t))O(t)$ by Y :

$$\begin{aligned} \frac{d}{dt}[YO] &= C(w)YO + [YC(w)]O \\ &= C(w)YO + C'(w)\langle Yw\rangle O \\ &= C(w)YO + C'(w)\langle\int YOdb + \int OdY + \int Y\alpha ds\rangle O. \end{aligned}$$

For $Z_2 \equiv$ "$Y\alpha$" we start with the equation $\dot{\alpha} = C(w)\alpha + R(w)$ and use the same procedure to obtain

$$\begin{aligned} \frac{d}{dt}(Y\alpha) &= C(w)Y\alpha + [YC(w)]\alpha + Y[R(w)] \\ &= C(w)Y\alpha + C'(w)\langle Yw\rangle\alpha + R'(w)\langle Yw\rangle \\ &= C(w)Y\alpha + C'(w)\langle\int YOdb + \int OdY + \int Y\alpha ds\rangle\alpha \\ &\quad + R'(w)\langle\int YOdb + \int OdY + \int Y\alpha ds\rangle. \end{aligned}$$

Outline of Proof. To prove existence and uniqueness of solutions to equation (4.1.4), we will give proofs for three intermediate equations, each (essentially) adding in one more term in the formal equation.

The first equation ((4.2.8) below)

$$\dot{T}(t) = C(w(t))T(t) \text{ with } T(0) = I$$

is necessary in order to handle the unbounded operator $C(w(t))$. This result is give in Proposition 4.2.7.

Secondly, in Proposition 4.3.4 we have a solution to the equation (4.3.10):

$$Q(t)(\cdot) = I + \int_0^t \mathcal{L}_\tau(Q(\tau)(\cdot))d\tau$$

where $\mathcal{L}_t(D) \equiv T(t)^{-1}A_t(T(t)D)$ and I is the identity map on $S^{\infty-}(\mathrm{End}(\mathbf{R}^n) \times \mathbf{R}^n)$.

Thirdly, we show in Lemma 4.4.3 that $V(t) \equiv \int_0^t Q(t)Q(r)^{-1}T(r)^{-1}K(r)dr$ solves (4.4.13):

$$\dot{V}(t) = \mathcal{L}_t(V(t)) + T(t)^{-1}K(t) \text{ with } V(0) = 0.$$

Finally, we have (Theorem 4.4.4) that $Z(t) \equiv T(t)V(t)$ solves equation (4.1.4):

$$\dot{Z}(t) = C(w(t))Z(t) + A_t(Z(t)) + K(t) \text{ with } Z(0) = 0.$$

4.2 Step 1: The "T-equation" - with an Unbounded Skew-Symmetric Operator

In this section we isolate the term involving the skew-symmetric function C. We will first obtain the $O(n)$-valued pointwise solution for the equation (4.2.8)

$$\dot{T}(t) = C(w(t))T(t) \text{ with } T(0) = I.$$

Then we will show that this solution also satisfies the above equation with the derivative taken in the S^p-norm, for all $p \in [2, \infty)$.

Remark 4.2.1 *The results in this section could be proved by a general argument similar to the proof of Theorem 3.1.5, but in the current case the flow $w(t)$ is already known to exist, and $C(w)$ is $\mathrm{so}(n)$-valued for all $w \in B^\infty\mathbf{R}^n$. These conditions make it possible to give a shorter proof of a different style.*

Notation 4.2.2 *Let $\mathcal{E} \equiv \mathrm{End}(\mathbf{R}^n) \times \mathbf{R}^n$ with norm defined by $|(B, b)| = |B| + |b|$. Define multiplication by $A \in \mathrm{End}(\mathbf{R}^n)$: $A(B, b) \equiv (AB, Ab)$ for all $(B, b) \in \mathcal{E}$.*

Lemma 4.2.3 *Let $p \in [2, \infty)$, then for $D = (D_1, D_2) \in S^p\mathcal{E}$ and G an $O(n)$-valued continuous adapted process, $\|D\|_{S^p\mathcal{E}} = \|GD\|_{S^p\mathcal{E}}$. This result also holds for $D \in S^p\mathrm{End}(\mathbf{R}^n)$ and $D \in S^p\mathbf{R}^n$.*

Proof. For fixed $\omega \in \Omega$ and $s \in [0, 1]$ we have

$$|D_i(s)(\omega)| = |G(s)D_i(s)(\omega)| \text{ for } i = 1, 2 \text{ (since } G(s) \in O(n) \text{ pointwise).}$$

$$\begin{aligned} \text{Thus } \|D\|_{S^p} &= \|\sup\nolimits_{0 \le s \le 1} |D(s)|\|_{L^p(P)} \\ &= \|\sup\nolimits_{0 \le s \le 1} |G(s)D(s)|\|_{L^p(P)} \\ &= \|GD\|_{S^p}. \end{aligned}$$

Q.E.D.

Notation 4.2.4 *Let $C(t)$ be a version of $C(w(t))$ such that the mapping $(t, s) \to C(t, \omega)(s)$ is jointly continuous for all $\omega \in \Omega_0$ where $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$. Such a version exists by Kolmogorov's Lemma (Lemma 8.1.3) since $C \circ w$ is $S^{\infty -}$-Lipschitz (3.1.2).*

Lemma 4.2.5 *Let $C : \mathbf{R} \to S^{\infty -}\mathrm{so}(n)$ and $\Omega_0 \subset \Omega$ be defined as in Notation 4.2.4. Then there exists a unique pointwise solution to*

$$\dot{T}(t, \omega)(s) = C(t, \omega)(s)T(t, \omega)(s) \text{ with } T(0, \omega)(s) = I \tag{4.2.8}$$

for all $\omega \in \Omega_0$. Moreover, the process $s \to T_s(t)$ is $O(n)$-valued for all $t \in J$.

Proof. For fixed $s \in [0, 1]$ and ω, (4.2.8) is a linear ordinary differential equation in finite dimensions, so it has a unique solution. By Lemma 8.1.1, $s \to T_s(t)$ is an $O(n)$-valued process. Q.E.D.

Lemma 4.2.6 *Let $T(t)$ be the pointwise solution to (4.2.8) given in Lemma 4.2.5. Then $\|T(\tau) - T(t)\|_{S^p} \le K|t - \tau|$ for all $\tau, t \in J$ and $p \in [2, \infty)$. In addition, $T(t)^{-1}$ is S^p-Lipschitz in t. Here we are taking $S^p = S^p\mathrm{End}(\mathbf{R}^n)$.*

Proof. Fix $p \in [2, \infty)$, and let $K = K(p, C_o, \|C(0)\|_{S^\infty})$ be a constant which varies from line to line (here and throughout this paper),

$$\begin{aligned}
\|T(\tau) - T(t)\|_{S^p} &= \|p.w. \int_t^\tau \frac{d}{dr}[T(r)]dr\|_{S^p} \\
&= \|p.w. \int_t^\tau C(r)T(r)dr\|_{S^p} \\
&\leq K \int_t^\tau \|C(r)T(r)\|_{S^p} dr \text{ by Lemma 2.1.11} \\
&= K \int_t^\tau \|C(r)\|_{S^p} dr \text{ by Lemma 4.2.3} \\
&\leq K|\tau - t| \text{ by (3.1.7)}
\end{aligned}$$

Also,

$$\begin{aligned}
\left\|T(t)^{-1} - T(\tau)^{-1}\right\|_{S^p} &= \left\|T(\tau)^{-1}[T(\tau) - T(t)]T(t)^{-1}\right\|_{S^p} \\
&= \|T(\tau) - T(t)\|_{S^p} \text{ by Lemma 4.2.3} \\
&\leq K|\tau - t|.
\end{aligned}$$

Q.E.D.

Proposition 4.2.7 *Let $T(t)$ be the pointwise solution to (4.2.8) given in Lemma 4.2.5. Then $T(t)$ satisfies (4.2.8) with the derivative taken in the S^p-topologies for all $p \in [2, \infty)$.*

Proof. Fix $p \in [2, \infty)$ and let p_1 and p_2 be such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Then

$$\begin{aligned}
&\overline{\lim}_{\epsilon \to 0} \|\frac{1}{\epsilon}[T(t+\epsilon) - T(t)] - C(t)T(t)\|_{S^p} \\
&\quad = \overline{\lim}_{\epsilon \to 0} \|\frac{1}{\epsilon} p.w. \int_t^{t+\epsilon} C(\tau)T(\tau)d\tau - C(t)T(t)\|_{S^p} \\
&\quad \text{(by definition of the pointwise solution)} \\
&\quad = \overline{\lim}_{\epsilon \to 0} \|p.w. \int_t^{t+\epsilon} [C(\tau)T(\tau) - C(t)T(t)]\frac{d\tau}{\epsilon}\|_{S^p} \\
&\quad \leq \overline{\lim}_{\epsilon \to 0} \epsilon^{-1} |\int_t^{t+\epsilon} \|C(\tau)T(\tau) - C(t)T(t)\|_{S^p} d\tau| \text{ by Lemma 2.1.11}
\end{aligned}$$

Now

$$\begin{aligned}
\|C(\tau)T(\tau) - C(t)T(t)\|_{S^p} &\leq \|C(\tau)[T(\tau) - T(t)]\|_{S^p} \\
&\quad + \|[C(\tau) - C(t)]T(t)\|_{S^p}
\end{aligned}$$

Applying Hölder's inequality to the first term we have

$$\begin{aligned}
\|C(\tau)[T(\tau)-T(t)]\|_{S^p} &\leq \|C(\tau)\|_{S^{p_1}}\|T(\tau)-T(t)\|_{S^{p_2}} \\
&\leq K\|C(\tau)\|_{S^{p_1}}|\tau - t| \text{ by Lemma 4.2.6} \\
&\leq K|\tau - t| \text{ by (3.1.7)}
\end{aligned}$$

For the second term we have

$$\begin{aligned}
\|[C(\tau)-C(t)]T(t)\|_{S^p} &= \|C(\tau)-C(t)\|_{S^p} \text{ (by Lemma 4.2.3)} \\
&\leq K\|w(\tau)-w(t)\|_{B^p} \text{ (by (3.1.2))} \\
&\leq K|\tau - t| \text{ since } w \text{ is } B^p\text{-Lipschitz}
\end{aligned}$$

Using these estimates we have

$$\begin{aligned}
\overline{\lim}_{\epsilon\to 0}\|\frac{1}{\epsilon}[T(t+\epsilon)-T(t)] - C(t)T(t)\|_{S^p} &\leq K\overline{\lim}_{\epsilon\to 0}\int_t^{t+\epsilon}|\tau - t|\frac{d\tau}{\epsilon} \\
&= 0.
\end{aligned}$$

Q.E.D.

4.3 Step 2: The "Q-equation"

In this section the method of Piccard iterates will be used to prove there exists a unique curve $Q : J \to Maps(S^{\infty-}\mathcal{E}, S^{\infty-}\mathcal{E})$ which is strongly differentiable in the following sense: for all $D \in S^{\infty-}\mathcal{E}$, $\dot{Q}(t)(D) \equiv \frac{d}{dt}[Q(t)(D)]$ exists in $S^{\infty-}$, and which satisfies

$$\dot{Q}(t)(D) = \mathcal{L}_t(Q(t)(D)) \text{ with } Q(0)(D) = D \tag{4.3.9}$$

for all $D \in S^{\infty-}\mathcal{E}$ where $\mathcal{L}_t(D) \equiv T(t)^{-1}A_t(T(t)D)$, $T(t)$ is the solution to (4.2.8), and I is the identity map on $S^{\infty-}\mathcal{E}$. In addition, for each $p \in [2,\infty)$ there exists a unique extension $\bar{Q}(t)$ of $Q(t)$ such that $\bar{Q}(t) : S^p\mathcal{E} \to S^p\mathcal{E}$.

Remark 4.3.1 *In the proof below (Lemma 4.3.3) we will show that the map $t \to \mathcal{L}_t$ is continuous when $\mathcal{L}_t$ is viewed as an operator from S^p to S^q, where $q > p$. If we had the stronger condition that $t \to \mathcal{L}_t \in \mathbf{End}(S^p)$ were continuous, we could easily conclude that there is a unique solution to (4.3.9). The fact that we do not have this condition necessitates a more delicate proof. In particular, our conditions on $\mathcal{L}_t$ arise from the fact that $p_1 > p$ in (4.1.1). This is dictated by the application of this result in geometrical settings.*

Lemma 4.3.2 *Let A_t be defined as in Notation 4.1.4. For all $p, q \in [2, \infty)$ with $q > p$ and $t, \tau \in J$,*
i) $A_t : S^p\mathcal{E} \to S^p\mathcal{E}$ and $\|A_t\|_{S^p\mathcal{E} \to S^p\mathcal{E}} \leq K$ for some $K > 0$ independent of t.
ii) For all $D \in S^{\infty -}\mathcal{E}$ we have

$$\|A_t(D) - A_\tau(D)\|_{S^p\mathcal{E}} \leq \quad K_p |t - \tau| \|D\|_{S^q\mathcal{E}}.$$

Proof of i**).** Let $D = (D_1, D_2) \in S^p\mathcal{E}$ and define the Brownian semimartingale $M_s \equiv \int_0^s D_1(\bar{s}) db(\bar{s}) + \int_0^s D_2(\bar{s}) d\bar{s}$. Then

$$\begin{aligned} \|M\|_{B^p} &= \|D_1\|_{S^p} + \|D_2\|_{S^p} \\ &\leq K \|D\|_{S^p\mathcal{E}}. \end{aligned}$$

Here we are using the fact that

$$\sup_{0 \leq r \leq s} |D_1| + \sup_{0 \leq r \leq s} |D_2| \leq 2 \sup_{0 \leq r \leq s} \{|D_1| + |D_2|\}.$$

Now

$$\begin{aligned} \|A_t(D)\|_{S^p\mathcal{E}} &\leq \|C'(w(t))\langle M \rangle O(t)\|_{S^p \mathrm{End}(\mathbf{R}^n)} \\ &\quad + \|C'(w(t))\langle M \rangle \alpha(t)\|_{S^p\mathbf{R}^n} \\ &\quad + \|R'(w(t))\langle M \rangle\|_{S^p\mathbf{R}^n} \\ &\leq \|C'(w(t))\langle M \rangle\|_{S^p\mathbf{R}^n} \\ &\quad + \|C'(w(t))\langle M \rangle\|_{S^p\mathbf{R}^n} \|\alpha(t)\|_{S^\infty\mathbf{R}^n} \\ &\quad + \|R'(w(t))\langle M \rangle\|_{S^p\mathbf{R}^n} \end{aligned}$$

$$\begin{aligned} &\le K\|M\|_{B^p\mathbf{R}^n} \\ &\le K\|D\|_{S^p\mathcal{E}}. \end{aligned}$$

where the third inequality follows from (4.1.1) and (4.1.2). **Proof of** ii)**.** Let D and M be as in the proof of i). Then

$$\begin{aligned} \|A_t(D) - A_\tau(D)\|_{S^p\mathcal{E}} \le\ & \|C'(w(t))\langle M\rangle O(t) - C'(w(\tau))\langle M\rangle O(\tau)\|_{S^p} \\ & +\|C'(w(t))\langle M\rangle \alpha(t) - C'(w(\tau))\langle M\rangle \alpha(\tau)\|_{S^p} \\ & +\|R'(w(t))\langle M\rangle - R'(w(\tau))\langle M\rangle\|_{S^p} \end{aligned}$$

Let $p_2 = q$ and let p_1 be such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Then

$$\begin{aligned} &\|C'(w(t))\langle M\rangle O(t) - C'(w(\tau))\langle M\rangle O(\tau)\|_{S^p} \\ &\qquad \le \|[C'(w(t))\langle M\rangle - C'(w(\tau))\langle M\rangle]O(t)\|_{S^p} \\ &\qquad +\|C'(w(\tau))\langle M\rangle[O(t) - O(\tau)]\|_{S^p} \\ &\qquad \le \|C'(w(t))\langle M\rangle - C'(w(\tau))\langle M\rangle\|_{S^p} \text{ by Lemma 4.2.3} \\ &\qquad +\|C'(w(\tau))\langle M\rangle\|_{S^{p_2}}\|O(t) - O(\tau)\|_{S^{p_1}} \text{ by Lemma 8.1.5} \\ &\qquad \le K\|w(t) - w(\tau)\|_{B^{p_1}}\|M\|_{B^{p_2}} \text{ by (4.1.1)} \\ &\qquad \le K|t-\tau|\|D\|_{S^{p_2}\mathcal{E}}. \end{aligned}$$

The second and third terms are similar, using $\sup_{t\in J}\|\alpha(t)\|_{S^\infty} < \infty$ (Notation 4.1.1). Q.E.D.

Lemma 4.3.3 *Let $T(t)$ be the solution to (4.2.8). For $p \in [2,\infty)$ and $D \in S^p \equiv S^p\mathcal{E}$ define $\mathcal{L}_t$ by $\mathcal{L}_t(D) \equiv T(t)^{-1}A_t(T(t)D)$. Then*

i) *$\mathcal{L}_t \in \mathrm{End}(S^p)$, the space of bounded linear functions from S^p to S^p, and $\|\mathcal{L}_t\|_{S^p\to S^p} \le K_p$, a constant independent of t.*

ii) *If $t \to D(t)$ is $S^{\infty-}$-continuous (Lipschitz) then so is $t \to \mathcal{L}_t(D(t))$.*

iii) *The map $t \to \mathcal{L}_t \in \mathbf{Hom}(S^p, S^q)$ is Lipschitz for $q > p$.*

Proof of i). Clearly $\mathcal{L}_t$ is linear. Given $D \in S^p$ we have

$$\begin{aligned}
\|\mathcal{L}_t(D)\|_{S^p} &= \|T(t)^{-1}A_t(T(t)D)\|_{S^p} \\
&= \|A_t(T(t)D)\|_{S^p} \text{ by Lemma 4.2.3} \\
&\leq K\|T(t)D\|_{S^p} \text{ by Lemma 4.3.2} \\
&\leq K\|D\|_{S^p} \text{ by Lemma 4.2.3}
\end{aligned}$$

Thus $\mathcal{L}_t : S^p \to S^p$, with $\|\mathcal{L}_t\|_{S^p \to S^p} = \sup_{D\neq 0} \frac{\|\mathcal{L}_t(D)\|_{S^p}}{\|D\|_{S^p}} \leq K$. **Proof of ii).** Let $p \in [2, \infty)$, and let p_1 and p_2 be such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. We have

$$\begin{aligned}
\|\mathcal{L}_t(D(t)) - \mathcal{L}_\tau(D(\tau))\|_{S^p} &\leq \|T(t)^{-1}A_t[T(t)\{D(t) - D(\tau)\}]\|_{S^p} \\
&\quad + \|T(t)^{-1}A_t[\{T(t) - T(\tau)\}D(\tau)]\|_{S^p} \\
&\quad + \|T(t)^{-1}(A_t - A_\tau)[T(\tau)D(\tau)]\|_{S^p} \\
&\quad + \|[T(t)^{-1} - T(\tau)^{-1}]A_\tau[T(\tau)D(\tau)]\|_{S^p} \\
&\leq \|A_t[T(t)\{D(t) - D(\tau)\}]\|_{S^p} \text{ by Lemma 4.2.3} \\
&\quad + \|A_t[\{T(t) - T(\tau)\}D(\tau)]\|_{S^p} \\
&\quad + \|(A_t - A_\tau)[T(\tau)D(\tau)]\|_{S^p} \\
&\quad + \|T(t)^{-1} - T(\tau)^{-1}\|_{S^{p_2}}\|A_\tau[T(\tau)D(\tau)]\|_{S^{p_1}} \\
&\qquad \text{by Lemma 8.1.5} \\
&\leq K\|T(t)\{D(t) - D(\tau)\}\|_{S^p} \text{ by Lemma 4.3.2} \\
&\quad + K\|\{T(t) - T(\tau)\}D(\tau)\|_{S^p} \\
&\quad + K|t - \tau|\|T(\tau)D(\tau)\|_{S^{p_1}} \text{ by Lemma 4.3.2} \\
&\quad + K|t - \tau|\|T(\tau)D(\tau)\|_{S^{p_1}} \text{ by Lemma 4.2.6} \\
&\leq K\|D(t) - D(\tau)\|_{S^p} \text{ by Lemma 4.2.3} \\
&\quad + K|t - \tau|\|D(\tau)\|_{S^p} \text{ by Lemma 4.2.6} \\
&\quad + K|t - \tau|\|D(\tau)\|_{S^{p_1}}
\end{aligned}$$

This expression tends to 0 as $\tau \to t$, since $\tau \to \|D(\tau)\|_{S^p}$ and $\tau \to \|D(\tau)\|_{S^{p_1}}$ are both continuous and therefore bounded for τ near t.

iii) This is proved by taking $p_1 = q$ and $D(t) \equiv D \in S^q$ in the estimate above. Q.E.D.

Proposition 4.3.4 *Let $\mathcal{L}_t \in \mathrm{End}(S^p)$ for all $p \in [2, \infty)$ be defined as in Lemma 4.3.3. For all $p \in [2, \infty)$ there exists a unique solution $Q : J \to Maps(S^{\infty -}\mathcal{E}, S^{\infty -}\mathcal{E})$ to the integral equation*

$$Q(t)(\cdot) = I + \int_0^t \mathcal{L}_\tau(Q(\tau)(\cdot))d\tau \tag{4.3.10}$$

where I is the identity map on $S^{\infty -}\mathcal{E}$.

That is, for each $D \in S^{\infty -}\mathcal{E}$, Q solves

$$Q(t)(D) = D + \int_0^t \mathcal{L}_\tau(Q(\tau)(D))d\tau$$

where the integral is an $S^{\infty -}\mathcal{E}$-integral (see Notation 2.1.13).

Also, for each $p \in [2, \infty)$, $t, \tau \in J$ and $D \in S^{\infty -}\mathcal{E}$ we have

1. $\|Q(t)(D)\|_{S^p} \le K_p \|D\|_{S^p}$
2. $\|Q(t)(D) - Q(\tau)(D)\|_{S^p} \le K_p \|D\|_{S^p} |t - \tau|$
3. *There exists a unique extension $\bar{Q}(t)$ of $Q(t)$ such that $\bar{Q}(t) : S^p\mathcal{E} \to S^p\mathcal{E}$.*

Proof. Existence. For all $D \in S^{\infty -}\mathcal{E}$ define $Q_0(t)(D) = D \in S^{\infty -}\mathcal{E}$, and recursively define

$$Q_{m+1}(t)(D) \equiv D + \int_0^t \mathcal{L}_\tau(Q_m(\tau)(D))d\tau \text{ for } m = 0, 1, 2, ... \tag{4.3.11}$$

where the integral is taken in $S^{\infty -}\mathcal{E}$.

Note that by induction on m, the map $t \to Q_m(t)(D)$ is $S^{\infty -}$-continuous for each $D \in S^{\infty -}\mathcal{E}$. Indeed, if $t \to Q_m(t)(D)$ is $S^{\infty -}$-continuous then by Lemma 4.3.3, the map $\tau \to \mathcal{L}_\tau(Q_m(\tau)(D))$ is also, so the $S^{\infty -}$-integral exists by

Lemma 2.1.12. By the Fundamental Theorem of Calculus for S^p- integrals (Lemma 2.1.14), the map $t \to \int_0^t \mathcal{L}_\tau(Q_m(\tau)(D))d\tau$ is $S^{\infty-}$-differentiable, and therefore $S^{\infty-}$-continuous.

We have for each $D \in S^p$,

$$\begin{aligned}
\|Q_{m+1}(t)(D) - Q_m(t)(D)\|_{S^p} &\leq \int_0^t \|\mathcal{L}_\tau(Q_m(\tau)(D)) - \mathcal{L}_\tau(Q_{m-1}(\tau)(D))\|_{S^p} d\tau \\
&= \int_0^t \|\mathcal{L}_\tau[Q_m(\tau)(D) - Q_{m-1}(\tau)(D)]\|_{S^p} d\tau \\
&\leq K \int_0^t \|Q_m(\tau)(D) - Q_{m-1}(\tau)(D)\|_{S^p} d\tau \\
&\qquad \text{by Lemma 4.3.3.}
\end{aligned}$$

Iterating, we have

$$\begin{aligned}
&\|Q_{m+1}(t)(D) - Q_m(t)(D)\|_{S^p} \\
&\qquad \leq K^m \textstyle\int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{m-1}} \|Q_1(\tau_m)(D) - Q_0(\tau_m)(D)\|_{S^p} d\tau_m \cdots d\tau_1 \\
&\qquad \leq K^m \textstyle\int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{m-1}} [\|Q_1(\tau_m)(D)\|_{S^p} + \|Q_0(\tau_m)(D)\|_{S^p}] d\tau_m \cdots d\tau_1 \\
&\qquad \leq K^m \textstyle\int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{m-1}} \|D\|_{S^p}[2 + tK] d\tau_m \cdots d\tau_1 \\
= \quad & K^m \tfrac{t^m}{m!} \|D\|_{S^p}[2 + tK].
\end{aligned}$$

The estimate in the third step comes from the following:

$$\begin{aligned}
\|Q_1(\tau_m)(D)\|_{S^p} + \|Q_0(\tau_m)(D)\|_{S^p} &= \|D + \int_0^{\tau_m} \mathcal{L}_\tau(D)d\tau\|_{S^p} + \|D\|_{S^p} \\
&\leq 2\|D\|_{S^p} + \tau_m \sup_{0 \leq \tau \leq \tau_m} \|\mathcal{L}_\tau(D)\|_{S^p} \\
&\leq 2\|D\|_{S^p} + \tau_m K \|D\|_{S^p} \text{ (by Lemma 4.3.3)} \\
&\leq \|D\|_{S^p}(2 + tK).
\end{aligned}$$

Thus

$$\|Q_{m+1}(t)(D) - Q_m(t)(D)\|_{S^p} \leq (2 + tK)K^m \frac{t^m}{m!} \|D\|_{S^p} \tag{4.3.12}$$

Summing, we have

$$\begin{aligned}
\sum_{m=0}^{\infty} \|Q_{m+1}(t)(D) - Q_m(t)(D)\|_{S^p} &\leq \sum_{m=0}^{\infty} K^m \frac{t^m}{m!}[2 + tK]\|D\|_{S^p} \\
&= [2 + tK]e^{tK}\|D\|_{S^p}.
\end{aligned}$$

So $\{Q_n(t)(D)\}_{n=0}^{\infty}$ is S^p-Cauchy uniformly in t, thus $Q(t)(D) \equiv \lim_{n\to\infty} Q_n(t)(D)$ exists where the limit is taken in the S^p-topology, uniformly in t. Also $t \to Q(t)(D)$ is S^p-continuous (as a uniform limit of continuous functions).

Now let $p, q \in [2, \infty)$ with $q < p$, and let $Q(t)(D) \equiv \lim_{n\to\infty} Q_n(t)(D)$ in S^p as above, then

$$\begin{aligned} & \lim_{m,n\to\infty} \|Q_m(t)(D) - Q_n(t)(D)\|_{S^q} \\ \leq \;& \lim_{m,n\to\infty} \|Q_m(t)(D) - Q_n(t)(D)\|_{S^p} \\ = \;& 0 \end{aligned}$$

So $\{Q_n(t)(D)\}_{n=0}^{\infty}$ is also S^q-Cauchy uniformly in t, so $Q_n(t)(D) \to_{n\to\infty} Q(t)(D)$ in S^q uniformly in t, and $t \to Q(t)(D)$ is S^q-continuous. Since this holds for all $p, q \in [2, \infty)$ with $q < p$, we have $t \to Q(t)(D)$ is $S^{\infty-}$-continuous.

Now by (4.3.12) we have for all $p \in [2, \infty)$,

$$\begin{aligned} & \|Q(t)(D)\|_{S^p} \\ \leq \;& \|Q_o(D)\|_{S^p} + \sum_{k=0}^{\infty} \|Q_{k+1}(t)(D) - Q_k(t)(D)\|_{S^p} \\ \leq \;& [(2 + tK)e^{tK} + 1]\|D\|_{S^p}. \end{aligned}$$

Also, since $L^\infty(P)$ is dense in $L^p(P)$, S^∞ is dense in S^p, and hence $S^{\infty-} \supset S^\infty$ is also dense in S^p. Thus the linear operator $Q : S^{\infty-} \to S^p$ has a unique extension to a linear operator $\bar{Q} : S^p \to S^p$.

Now we show that $Q(t)$ satisfies (4.3.10). We have for all $D \in S^{\infty-}\mathcal{E}$,

$$\begin{aligned} Q(t)(D) &= \lim_{n\to\infty} Q_n(t)(D) \\ &= \lim_{n\to\infty} [D + \int_0^t \mathcal{L}_\tau(Q_{n-1}(\tau)(D))d\tau] \\ &= D + \int_0^t [\mathcal{L}_\tau(\lim_{n\to\infty} Q_{n-1}(\tau)(D))]d\tau \text{ (by uniform convergence)} \\ &= D + \int_0^t [\mathcal{L}_\tau(Q(\tau)(D))]d\tau. \end{aligned}$$

Finally, we have for all $D \in S^{\infty-}\mathcal{E}$,

$$\begin{aligned}\|Q(t)(D)-Q(\tau)(D)\|_{S^p} &\leq \int_\tau^t \|\mathcal{L}_r(Q(r)(D))\|_{S^p}dr\\ &\leq K_p\|D\|_{S^p}|t-\tau| \text{ by Prop. 4.3.3.}\end{aligned}$$

Uniqueness. Suppose $Q(t)$ and $S(t)$ are both solutions to (4.3.10). Fix $D \in S^{\infty-}\mathcal{E}$ and $p \in [2,\infty)$. Let $f(t) \equiv \|Q(t)(D) - S(t)(D)\|_{S^p}$ for all $t \in J$. Then

$$\begin{aligned} f(t) &= \|\int_0^t \{\mathcal{L}_\tau[Q(\tau)(D)] - \mathcal{L}_t[S(\tau)(D)]\}d\tau\|_{S^p}\\ &= \|\int_0^t \mathcal{L}_\tau[Q(\tau)(D) - S(\tau)(D)]d\tau\|_{S^p} \text{ (since } \mathcal{L}_\tau \text{ is linear)}\\ &\leq \int_0^t[\|\mathcal{L}_\tau[Q(\tau)(D) - S(\tau)(D)]\|_{S^p}]d\tau\\ &\leq K\int_0^t \|Q(\tau)(D) - S(\tau)(D)\|_{S^p}d\tau \text{ (by Lemma 4.3.3)}\\ &= K\int_0^t f(\tau)d\tau.\end{aligned}$$

By Gronwall's inequality (Lemma 8.1.7) with $\epsilon = 0$ we have $f \equiv 0$. Since $p \in [2,\infty)$ is arbitrary we have $Q(t)(D) = S(t)(D)$ in S^p for all $t \in J$ and $p \in [2,\infty)$, thus $Q(t)(D) = S(t)(D)$ in $S^{\infty-}$ for all $t \in J$. Q.E.D.

Lemma 4.3.5 *Let $Q(t) : S^{\infty-}\mathcal{E} \to S^{\infty-}\mathcal{E}$ be the solution to the integral equation (4.3.10) as given by Proposition 4.3.4.* **i)** *For all $D \in S^{\infty-}\mathcal{E}$, $\dot{Q}(t)(D) \equiv \frac{d}{dt}[Q(t)(D)]$ exists in $S^{\infty-}$ and $Q(t)$ satisfies (4.3.9):*

$$\dot{Q}(t)(D) = \mathcal{L}_t(Q(t)(D)) \text{ with } Q(0)(D) = D.$$

ii) *If the map $t \to D(t) \in S^{\infty-}\mathcal{E}$ is $S^{\infty-}\mathcal{E}$-differentiable, then so is the map $t \to Q(t)D(t)$, and*

$$\frac{d}{dt}[Q(t)D(t)] = \mathcal{L}_t(Q(t)D(t)) + Q(t)\dot{D}(t).$$

Proof. **i)** By Proposition 4.3.4, for all $D \in S^{\infty-}\mathcal{E}$ the map $t \to Q(t)(D)$ is $S^{\infty-}$-continuous, thus by Lemma 4.3.3 the map $t \to \mathcal{L}_t(Q(t)(D))$ is also. Hence by the Fundamental Theorem of Calculus for S^p-integrals (Lemma 2.1.14), $t \to \int_0^t \mathcal{L}_\tau(Q(\tau)(D))d\tau$ is $S^{\infty-}$-differentiable, and

$$\frac{d}{dt}\left[D + \int_0^t \mathcal{L}_\tau(Q(\tau)(D))d\tau\right] \quad = \quad \mathcal{L}_t(Q(t)(D)).$$

Thus for all $D \in S^{\infty-}\mathcal{E}$, $\dot{Q}(t)(D) \equiv \frac{d}{dt}[Q(t)(D)]$ exists in $S^{\infty-}$ and

$$\dot{Q}(t)(D) = \mathcal{L}_t(Q(t)(D)) \text{ with } Q(0)(D) = D.$$

ii) Let $t \in J$ and $p \in [2, \infty)$. We have

$$\begin{aligned}
&\overline{\lim}_{\epsilon\to 0}\|\frac{Q(t+\epsilon)D(t+\epsilon) - Q(t)D(t)}{\epsilon} - [\mathcal{L}_t(Q(t)D(t)) + Q(t)\dot{D}(t)]\|_{S^p} \\
&\quad \le \overline{\lim}_{\epsilon\to 0}\|\frac{[Q(t+\epsilon) - Q(t)]D(t)}{\epsilon} - \mathcal{L}_t(Q(t)D(t))\|_{S^p} \\
&\quad +\overline{\lim}_{\epsilon\to 0}\|Q(t)[\frac{D(t+\epsilon) - D(t)}{\epsilon} - \dot{D}(t)]\|_{S^p} \\
&\quad +\overline{\lim}_{\epsilon\to 0}\frac{1}{\epsilon}\|[Q(t+\epsilon) - Q(t)][D(t+\epsilon) - D(t)]\|_{S^p}.
\end{aligned}$$

The first term tends to zero by part **i)**, and the second and third by the boundedness and Lipschitz properties of Q (see Prop. 4.3.4 and the S^p-continuity of the map $t \to D(t)$. Q.E.D.

4.4 Step 3: Existence and Uniqueness for the Formal Equation

In this section we use the solutions $T(t)$ to (4.2.8) and $Q(t)$ to (4.3.10), and Gronwall's inequality to obtain a unique solution to the equation (4.4.13)

$$\dot{V}(t) = \mathcal{L}_t(V(t)) + T(t)^{-1}K(t) \text{ with } V(0) = 0.$$

Then by variation of parameters, we show existence of a unique solution to the formal equation

$$\dot{Z}(t) = A_t(Z(t)) + C(w(t))Z(t) + K(t) \text{ with } Z(0) = 0.$$

Lemma 4.4.1 *Let $K(t)$ be defined as in Notation 4.1.4. Then for all $p \in [2,\infty)$, and $T < \infty$, $\sup_{0\le t<T} \|K(t)\|_{S^p\mathcal{E}} < \infty$.*

Proof. Fix $p \in [2,\infty)$ and let p_1 and p_2 be such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Let r_1 and r_2 be such that $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{p_1}$.

$$
\begin{aligned}
\|K(t)\|_{S^p\mathcal{E}} &\le \|C'(w(t))\langle \int O(t)dY\rangle O(t)\|_{S^p\mathrm{End}(\mathbf{R}^n)} \\
&\quad + \|C'(w(t))\langle \int O(t)dY\rangle \alpha(t)\|_{S^p\mathbf{R}^n} \\
&\quad + \|R'(w(t))\langle \int O(t)dY\rangle\|_{S^p\mathbf{R}^n} \\
&\le \|C'(w(t))\langle \int O(t)dY\rangle\|_{S^p\mathrm{End}(\mathbf{R}^n)} \text{ by Lemma 4.2.3} \\
&\quad + \|C'(w(t))\langle \int O(t)dY\rangle\|_{S^{p_1}\mathrm{End}(\mathbf{R}^n)}\|\alpha(t)\|_{S^{p_2}\mathbf{R}^n} \text{ and Hölder} \\
&\quad + \|R'(w(t))\langle \int O(t)dY\rangle\|_{S^p\mathbf{R}^n} \\
&\le K\| \int O(t)dY\|_{B^{p_1}\mathbf{R}^n} \text{ by (4.1.1) and (4.1.2)} \\
&\le K\|O(t)\|_{S^{r_1}\mathrm{End}(\mathbf{R}^n)}\|Y\|_{B^{r_2}\mathbf{R}^n} \text{ by [8] Lemma 4.1(v), p. 302}
\end{aligned}
$$

The last term is finite since $\|O(t)\|_{S^{r_1}} \le \|w(t)\|_{B^{r_1}} < \infty$ and $\|Y\|_{B^{r_2}\mathbf{R}^n} < \infty$ (see Notation 4.1.1). Q.E.D.

Lemma 4.4.2 *Define $K(t)$ as in Notation 4.1.4 For all $p \in [2,\infty)$, the map $t \to K(t)$ is S^p-Lipschitz.*

Proof. Fix $p \in [2,\infty)$ and let p_1 and p_2 be such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Let $M(t) \equiv \int O(t)dY$. Then

$$
\begin{aligned}
K(t) &= C'(w(t))\langle M(t)\rangle O(t) \\
&\quad + C'(w(t))\langle M(t)\rangle \alpha(t) + R'(w(t))\langle M(t)\rangle
\end{aligned}
$$

We have for all $q, q_1, q_2 \in [2,\infty)$ with $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$,

$$
\begin{aligned}
\|M(t) - M(\tau)\|_{S^q} &= \|\int [O(t) - O(\tau)]dY\|_{S^q} \\
&\le \|O(t) - O(\tau)\|_{S^{q_1}}\|Y\|_{B^{q_2}} \text{ by [8] Lemma 4.1, p. 302} \\
&\le K|t-\tau| \text{ since } w \text{ is } B^{q_1}\text{-Lipschitz and } \|Y\|_{B^{q_2}} < \infty.
\end{aligned}
$$

Thus $M(t)$ is $S^{\infty-}$-Lipschitz. Also

$$\begin{aligned}\sup_{t\in J}\|M(t)\|_{S^q} &= \sup_{t\in J}\|\int O(t)dY\|_{S^q}\\ &\le \sup_{t\in J}\|O(t)\|_{S^{q_1}}\|Y\|_{B^{q_2}}\end{aligned}$$

which is finite since $\sup_{t\in J}\|O(t)\|_{S^{q_1}} \le \sup_{t\in J}\|w(t)\|_{B^\infty} < \infty$. So $M(t)$ is $S^{\infty-}$-bounded for $t \in J$.

Now

$$\begin{aligned}&\|C'(w(t))\langle M(t)\rangle - C'(w(\tau))\langle M(\tau)\rangle\|_{S^p}\\ &\le \|[C'(w(t)) - C'(w(\tau))]\langle M(t)\rangle\|_{S^p}\\ &+\|C'(w(\tau))\langle M(t) - M(\tau)\rangle\|_{S^p}\\ &\le K\|w(t) - w(\tau)\|_{B^{p_1}}\|M(t)\|_{B^{p_2}} \text{ by (4.1.1)}\\ &+K\|M(t) - M(\tau)\|_{B^p}\\ &\le K|t-\tau|.\end{aligned}$$

Thus the map $t \to C'(w(t))\langle M(t)\rangle$ is $S^{\infty-}$-Lipschitz. A similar result holds for C replaced by R, using the Lipschitz assumption on R' (4.1.2).

Finally, since $O(t)$ and $\alpha(t)$ are also $S^{\infty-}$-Lipschitz, the result follows from [8] Lemma 4.6(ii). Q.E.D.

Lemma 4.4.3 *Let $T(t)$ be the solution to (4.2.8). For $p \in [2,\infty)$ the following equation has a unique solution:*

$$\dot{V}(t) = \mathcal{L}_t(V(t)) + T(t)^{-1}K(t) \text{ with } V(0) = 0 \tag{4.4.13}$$

where the derivative is taken in the S^p-topology.

Proof. Existence. Let $Q(t) : S^{\infty-}\mathcal{E} \to S^{\infty-}\mathcal{E}$ be the solution to (4.3.9) as given in Lemma 4.3.5. In the following we will write $Q(t)(D)$ as $Q(t)D$ for $D \in S^{\infty-}\mathcal{E}$ since $Q(t)$ is linear. Notice that by linearity and since $t \to Q(t)D$ is $S^{\infty-}$-continuous for all $D \in S^{\infty-}\mathcal{E}$, $Q(t)^{-1}$ exists for t near 0.

Now $r \to T(r)^{-1}$ and $r \to K(r)$ are $S^{\infty-}$-continuous, by Lemmas 4.2.6 and 4.4.2, so by [8] Lemma 4.6, p. 314 we also have $S^{\infty-}$-continuity of the map $r \to Q(t)Q(r)^{-1}T(r)^{-1}K(r)$. Thus $V(t) \equiv \int_0^t Q(t)Q(r)^{-1}T(r)^{-1}K(r)dr$ is $S^{\infty-}$-differentiable by the Fundamental Theorem of Calculus for S^p-integrals (Lemma 2.1.14) and we have by [8] Lemma 4.6(iii), p. 314,

$$\begin{aligned}
\dot{V}(t) &\equiv \int_0^t \dot{Q}(t)Q(r)^{-1}T(r)^{-1}K(r)dr + [Q(t)Q(r)^{-1}T(r)^{-1}K(r)]|_{r=t} \\
&= \int_0^t \mathcal{L}_t(Q(t))Q(r)^{-1}T(r)^{-1}K(r)dr + T(t)^{-1}K(t) \\
&= \mathcal{L}_t[\int_0^t Q(t)Q(r)^{-1}T(r)^{-1}K(r)dr] + T(t)^{-1}K(t) \\
&= \mathcal{L}_t[V(t)] + T(t)^{-1}K(t).
\end{aligned}$$

Uniqueness. Fix $p \in [2,\infty)$. Let $V(t)$ and $W(t)$ be two solutions to (4.4.13). Integrating (4.4.13) in S^p we have $V(t) = \int_0^t[\mathcal{L}_r(V(r)) + T(r)^{-1}K(r)]dr$. Let $f(t) \equiv \|V(t) - W(t)\|_{S^p}$. Then

$$\begin{aligned}
f(t) &= \|V(t) - W(t)\|_{S^p} \\
&= \|\int_0^t [\mathcal{L}_r(V(r)) - \mathcal{L}_r(W(r))]dr\|_{S^p} \\
&\leq K\int_0^t \|V(r) - W(r)\|_{S^p}dr \text{ by Lemma 4.3.3} \\
&= K\int_0^t f(r)dr.
\end{aligned}$$

So by Gronwall's inequality (Lemma 8.1.7) with $\epsilon = 0$ we have $f \equiv 0$. Q.E.D.

Theorem 4.4.4 *Let $T(t)$ be the solution to the equation $\dot{T}(t) = C(w(t))T(t)$ with $T(0) = I$ as in Proposition 4.2.7. Let $V(t)$ be the solution to $\dot{V}(t) = \mathcal{L}_t(V(t)) + T(t)^{-1}K(t)$ with $V(0) = 0$ as in Lemma 4.4.3.*

Then $Z(t) \equiv T(t)V(t)$ solves the following equation (4.1.4) in $S^{\infty-}$:

$$\dot{Z}(t) = A_t(Z(t)) + C(w(t))Z(t) + K(t) \text{ with } Z(0) = 0.$$

Proof. By $S^{\infty-}$-differentiation of $Z(t)$ (via [8] Lemma 4.6, p. 314) we have

$$\dot{Z}(t) = \dot{T}(t)V(t) + T(t)\dot{V}(t)$$

$$\begin{aligned} &= C(w(t))T(t)V(t) + T(t)T(t)^{-1}[A_t(T(t)V(t)) + K(t)] \\ &= C(w(t))Z(t) + A_t(Z(t)) + K(t). \end{aligned}$$

Also $Z(0) = T(0)V(0) = 0$. Q.E.D.

Chapter 5

Existence and Uniqueness for Differentials of the Flow

5.1 Equation for the Difference Quotient

In this section we derive the equation satisfied by the the difference between the solution $Z(t) = (Z_1(t), Z_2(t))$ to the formal equation (4.1.4) and a difference quotient which will converge to $(YO, Y\alpha)$ in the S^p-norms for all $p \in [2, \infty)$.

To facilitate the proof, we multiply this difference on the left by $O(t)^{-1}$ which is $O(n)$-valued, thus the value of the S^p-norm is unchanged. The benefit is that the equation satisfied by this "modified difference" is free of unbounded terms; see Lemma 5.1.3.

In this chapter we will use the underlying filtered probability space $(\Omega, \{\mathcal{F}_s\}, \mathcal{F}, P)$, satisfying the usual conditions, with reference Brownian motion b.

Recall that we have fixed the functions $C : B^\infty\mathbf{R}^n \to S^{\infty-}\mathrm{so}(n)$, $R : B^\infty\mathbf{R}^n \to S^\infty\mathbf{R}^n$ as in Notation 4.1.1, and have defined $X : B^\infty\mathbf{R}^n \to B^{\infty-}\mathbf{R}^n$ by

$$X(w) \equiv \int C(w)dw + \int R(w)ds \text{ for } w \in B^\infty\mathbf{R}^n.$$

and $w : \mathbf{R} \to B^\infty\mathbf{R}^n$ as the $S^{\infty-}$-differentiable solution to

$$\dot{w}(t) = X(w(t)) \text{ with } w(0) = w_o,$$

We have also defined $O : \mathbf{R} \to S^\infty O(n)$ and $\alpha : \mathbf{R} \to S^\infty\mathbf{R}^n$ by $w(t) = \int O(t)db + \int \alpha(t)ds$, and have seen in Theorem 3.1.5 that $O(t)$ and $\alpha(t)$ satisfy the following equations:

$$\dot{O}(t) = C(w(t))O(t) \text{ with } O(0) = O_o$$

$$\text{and } \dot{\alpha}(t) = C(w(t))\alpha(t) + R(w(t)) \text{ with } \alpha(0) = \alpha_o.$$

Notation 5.1.1 *Let $\gamma_\epsilon = \int Q_\epsilon db + \int \beta_\epsilon ds$ be an admissible curve with $\gamma_0 = w_o$, and let $w_\epsilon(t) \in B^\infty\mathbf{R}^n$ be the solution to the equation*

$$\frac{d}{dt}w_\epsilon(t) \equiv \dot{w}_\epsilon(t) = X(w_\epsilon(t)) \text{ with } w_\epsilon(0) = \gamma_\epsilon.$$

given by Theorem 3.1.5. (That is, $w_\epsilon(t) = e^{tX}(\gamma_\epsilon)$.) Note that $w_\epsilon(t)|_{\epsilon=0} = w(t)$.

Define O_ϵ and α_ϵ by

$$w_\epsilon(t)(s) = \int_0^s O_\epsilon(t)(\bar{s})d\gamma_\epsilon(\bar{s}) + \int_0^s \alpha_\epsilon(t)(\bar{s})d\bar{s}.$$

Lemma 5.1.2 *O_ϵ and α_ϵ satisfy*

$$\dot{O}_\epsilon(t) = C(w_\epsilon(t))O_\epsilon(t) \text{ with } O_\epsilon(0) = I$$

$$\text{and } \dot{\alpha}_\epsilon(t) = C(w_\epsilon(t))\alpha_\epsilon(t) + R(w_\epsilon(t)) \text{ with } \alpha_\epsilon(0) = 0.$$

Proof. We have

$$\begin{aligned} w_\epsilon(t) &= \int [O_\epsilon(t)]d\gamma_\epsilon + \int [\alpha_\epsilon(t)]ds \\ &= \int [O_\epsilon(t)][Q_\epsilon db + \beta_\epsilon ds] + \int [\alpha_\epsilon(t)]ds \\ &= \int [O_\epsilon(t)Q_\epsilon]db + \int [\alpha_\epsilon(t) + O_\epsilon(t)\beta_\epsilon]ds \end{aligned}$$

Let $\tilde{O}(t) \equiv O_\epsilon(t)Q_\epsilon$ and $\tilde{\alpha}(t) \equiv \alpha_\epsilon(t) + O_\epsilon(t)\beta_\epsilon$. Then by Theorem 3.1.5 we have

$$\begin{aligned} \dot{O}_\epsilon(t) &= \dot{\tilde{O}}(t)Q_\epsilon^{-1} \\ &= C(w_\epsilon(t))\tilde{O}(t)Q_\epsilon^{-1} \\ &= C(w_\epsilon(t))O_\epsilon(t) \end{aligned}$$

and

$$\begin{aligned}
\dot{\alpha}_\epsilon(t) &= \dot{\tilde{\alpha}}(t) - \dot{O}_\epsilon(t)\beta_\epsilon \\
&= C(w_\epsilon(t))\tilde{\alpha}(t) + R(w_\epsilon(t)) - C(w_\epsilon(t))O_\epsilon(t)\beta_\epsilon \\
&= C(w_\epsilon(t))\alpha_\epsilon(t) + R(w_\epsilon(t))
\end{aligned}$$

Q.E.D.

In the following lemma we will supress the t parameter from the notation.

Lemma 5.1.3 *Let Z_1 and Z_2 be the solutions to the "formal" equations (4.1.6) and (4.1.7) as given in Theorem 4.4.4, with $\tilde{Z}$ given by (4.1.5).*
Let $\epsilon > 0$ and define

$$\begin{aligned}
E_1 &\equiv \frac{1}{\epsilon}(O^{-1}O_\epsilon) - I) - O^{-1}Z_1, \\
E_2 &\equiv \frac{1}{\epsilon}(O^{-1}\alpha_\epsilon) - O^{-1}\alpha) - O^{-1}Z_2, \\
\text{and } \tilde{E} &\equiv \frac{1}{\epsilon}(w_\epsilon - w) - \tilde{Z}.
\end{aligned}$$

Then E_1 and E_2 satisfy the following equations where the derivatives are taken in the S^p-norm for all $p \in [2, \infty)$:

$$\begin{aligned}
\dot{E}_1 &= O^{-1}[\frac{1}{\epsilon}(C(w_\epsilon) - C(w))(O_\epsilon - O)] + O^{-1}[\frac{1}{\epsilon}(C(w_\epsilon) - C(w)) - C'(w)\langle\tilde{Z}\rangle]O, \\
\dot{E}_2 &= O^{-1}[\frac{1}{\epsilon}(C(w_\epsilon) - C(w))(\alpha_\epsilon - \alpha)] + O^{-1}[\frac{1}{\epsilon}(C(w_\epsilon) - C(w)) - C'(w)\langle\tilde{Z}\rangle]\alpha \\
&\quad + O^{-1}[\frac{1}{\epsilon}(R(w_\epsilon) - R(w)) - R'(w)\langle\tilde{Z}\rangle].
\end{aligned}$$

Proof. Recall that Z_1 and Z_2 solve

$$\begin{aligned}
\dot{Z}_1 &= C(w)Z_1 + C'(w)\langle\tilde{Z}\rangle O \text{ with } Z_1(0) = 0 \\
\text{and } \dot{Z}_2 &= C(w)Z_2 + C'(w)\langle\tilde{Z}\rangle\alpha + R'(w)\langle\tilde{Z}\rangle \text{ with } Z_2(0) = 0 \\
\text{where } \tilde{Z} &\equiv \int_0^s Z_{1,\bar{s}}db(\bar{s}) + \int_0^s Z_{2,\bar{s}}d\bar{s} + \int_0^s O_{\bar{s}}dY(\bar{s}).
\end{aligned}$$

Since $O(t)$ is orthogonal and $C(w(t))$ is skew symmetric, we have $\frac{d}{dt}(O(t)^{-1}) = (\frac{d}{dt}O(t))^{tr} = O(t)^{tr}C(w(t))^{tr} = -O(t)^{-1}C(w(t))$.

By $S^{\infty-}$-differentiation E_1 solves:

$$\begin{aligned}\dot{E}_1 &= \frac{1}{\epsilon}[\dot{\widehat{O^{-1}}}O_\epsilon + O^{-1}\dot{O}_\epsilon] - \dot{\widehat{O^{-1}}}Z_1 \\ &\qquad -O^{-1}[C(w)Z_1 + C'(w)\langle\tilde{Z}\rangle O] \\ &= O^{-1}[\frac{1}{\epsilon}(C(w_\epsilon)O_\epsilon - C(w)O_\epsilon) - C'(w)\langle\tilde{Z}\rangle O] \\ &= O^{-1}[\frac{1}{\epsilon}(C(w_\epsilon) - C(w))(O_\epsilon - O)] \\ &\qquad +O^{-1}[\frac{1}{\epsilon}(C(w_\epsilon) - C(w)) - C'(w)\langle\tilde{Z}\rangle]O.\end{aligned}$$

Notice that the unbounded terms $-O^{-1}C(w)Z_1$ and $-\dot{\widehat{O^{-1}}}Z_1$ in the first line cancel. A similar calculation gives the E_2 equation. Q.E.D.

5.2 Estimates

Lemma 5.2.1 *Let $w_\epsilon(t) = \int O_\epsilon(t)d\gamma_\epsilon + \int \alpha_\epsilon ds \in B^\infty\mathbf{R}^n$ be given as in Notation 5.1.1, and let E_1 and E_2 be as defined in Lemma 5.1.3. For all $p \in [2,\infty)$ and $t \in J$ we have:*

(i) $\|w_\epsilon(t) - w(t)\|_{B^p}$ is $O(\epsilon)$ *(recall that $w_\epsilon(t)|_{\epsilon=0} = w(t)$) ;*

(ii) $\|O_\epsilon(t) - O(t)\|_{S^p}$ is $O(\epsilon)$

(iii) $\|\alpha_\epsilon(t) - \alpha(t)\|_{S^p}$ is $O(\epsilon)$

(iv) $\|O(t)^{-1}\frac{1}{\epsilon}[C(w_\epsilon(t)) - C(w(t))][O_\epsilon(t) - O(t)]\|_{S^p}$ is $O(\epsilon)$ for all $t \in J$;

(v) $\|O(t)^{-1}\frac{1}{\epsilon}[C(w_\epsilon(t)) - C(w(t))][\alpha_\epsilon(t) - \alpha(t)]\|_{S^p}$ *is $O(\epsilon)$ for all $t \in J$;*

(vi) $\|\tilde{E}(t)\|_{B^p} \le \|E_1(t)\|_{S^p} + \|E_2(t)\|_{S^p} + O(\epsilon)$;

(vii) $$\|O(t)^{-1}[\frac{1}{\epsilon}(C(w_\epsilon(t)) - C(w(t))) - C'(w(t))\langle\tilde{Z}(t)\rangle]O(t)\|_{S^p} \le K_p[\|E_1(t)\|_{S^p} + \|E_2(t)\|_{S^p} + O(\epsilon)];$$

(viii) $$\|O(t)^{-1}[\frac{1}{\epsilon}(C(w_\epsilon(t)) - C(w(t))) - C'(w(t))\langle\tilde{Z}(t)\rangle]\alpha(t)\|_{S^p} \le K_p[\|E_1(t)\|_{S^p} + \|E_2(t)\|_{S^p} + O(\epsilon)];$$

(ix) $\|O(t)^{-1}[\frac{1}{\epsilon}(R(w_\epsilon(t)) - R(w(t))) - R'(w(t))\langle\tilde{Z}(t)\rangle]\|_{S^p}$
$$\leq K_p[\|E_1(t)\|_{S^p} + \|E_2(t)\|_{S^p} + O(\epsilon)].$$

Proof. Fix $p \in [2,\infty)$. **(i)** We have

$$\begin{aligned}
w_\epsilon(t) - w(t) &= \int O_\epsilon(t)d\gamma_\epsilon + \int \alpha_\epsilon(t)ds - \int O(t)db - \int \alpha(t)ds \\
&= \int O_\epsilon(t)[Q_\epsilon db + \beta_\epsilon ds] + \int \alpha_\epsilon(t)ds - \int O(t)db - \int \alpha(t)ds \\
&= \int [O_\epsilon(t)Q_\epsilon - O(t)]db + \int [\alpha_\epsilon(t) - \alpha(t) + O_\epsilon(t)\beta_\epsilon]ds
\end{aligned}$$

Let $F_1(t) \equiv O(t)^{-1}O_\epsilon(t)Q_\epsilon - I$ and $F_2(t) \equiv O(t)^{-1}[\alpha_\epsilon(t) - \alpha(t) + O_\epsilon(t)\beta_\epsilon]$, then (dropping the t parameter) we have

$$\begin{aligned}
\|\dot{F}_1\|_{S^p} &= \|[\dot{\widehat{O^{-1}O_\epsilon}}]Q_\epsilon\|_{S^p} \\
&= \|\dot{\widehat{O^{-1}O_\epsilon}}\|_{S^p} \text{ by Lemma 4.2.3} \\
&= \|\dot{\widehat{O^{-1}}}O_\epsilon + O^{-1}\dot{O}_\epsilon\|_{S^p} \\
&= \|-O^{-1}C(w)O_\epsilon + O^{-1}C(w_\epsilon)O_\epsilon\|_{S^p} \\
&= \|O^{-1}(C(w_\epsilon) - C(w))O_\epsilon\|_{S^p} \\
&= \|C(w_\epsilon) - C(w)\|_{S^p} \text{ by Lemma 4.2.3} \\
&\leq K_p\|w_\epsilon - w\|_{B^p} \text{ by (3.1.2).}
\end{aligned}$$

$$\begin{aligned}
\text{Also, } \|\dot{F}_2\|_{S^p} &\leq \|\dot{\widehat{O^{-1}}}\alpha_\epsilon + O^{-1}\dot{\alpha}_\epsilon - \dot{\widehat{O^{-1}}}\alpha - O^{-1}\dot{\alpha}\|_{S^p} \\
&\qquad + \|\dot{\widehat{O^{-1}O_\epsilon}}\beta_\epsilon\|_{S^p} \\
&= \|-O^{-1}C(w)\alpha_\epsilon + O^{-1}[C(w_\epsilon)\alpha_\epsilon + R(w_\epsilon)] \\
&\qquad + O^{-1}C(w)\alpha - O^{-1}[C(w)\alpha + R(w)]\|_{S^p} \\
&\qquad + \|\beta_\epsilon\|_{S^\infty}\|\dot{\widehat{O^{-1}O_\epsilon}}\|_{S^p} \\
&\leq \|O^{-1}(C(w_\epsilon) - C(w))\alpha_\epsilon\|_{S^p} + \|O^{-1}(R(w_\epsilon) - R(w))\|_{S^p} \\
&\qquad + \|\beta_\epsilon\|_{S^\infty}\|\dot{\widehat{O^{-1}O_\epsilon}}\|_{S^p} \\
&\leq \|C(w_\epsilon) - C(w)\|_{S^p}\|\alpha_\epsilon\|_{S^\infty} \\
&\qquad + \|R(w_\epsilon) - R(w)\|_{S^p} \text{ by Lemma 4.2.3}
\end{aligned}$$

$$
\begin{aligned}
&\quad +\|\beta_\epsilon\|_{S^\infty}\|w_\epsilon - w\|_{B^p} \text{ by the proof for } \dot{F}_1 \text{ above} \\
&\leq \; K_p\|w_\epsilon - w\|_{B^p} + O(\epsilon).
\end{aligned}
$$

The last step follows from the Lipschitz assumptions on C and R (3.1.2), and since $\|a\|_{S^\infty} < \infty$ and $\sup_{t\in J}\|\alpha_\epsilon(t)\|_{S^\infty}$ is bounded (Theorem 3.1.5).

Using these estimates we have by Lemma 4.2.3:

$$
\begin{aligned}
\|F_1(t)\|_{S^p} &\leq \|F_1(0)\|_{S^p} + \textstyle\int_0^t \|\dot{F}_1(\tau)\|_{S^p} d\tau \\
&\leq \|O_o^{-1}Q_\epsilon - I\|_{S^p} + K_p \textstyle\int_0^t \|w_\epsilon(\tau) - w(\tau)\|_{B^p} d\tau
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\|F_2(t)\|_{S^p} &\leq \|\alpha_\epsilon(0) - \alpha(0) + \beta_\epsilon\|_{S^p} + K_p \int_0^t \|w_\epsilon(\tau) - w(\tau)\|_{B^p} d\tau \\
&= \|\beta_\epsilon - \alpha_o\|_{S^p} + K_p \int_0^t \|w_\epsilon(\tau) - w(\tau)\|_{B^p} d\tau.
\end{aligned}
$$

Using these two estimates we have

$$
\begin{aligned}
\|w_\epsilon - w\|_{B^p} &= \|O_\epsilon Q_\epsilon - O\|_{S^p} + \|\alpha_\epsilon - \alpha + O_\epsilon\beta_\epsilon\|_{S^p} \\
&= \|OF_1\|_{S^p} + \|OF_2\|_{S^p} \\
&= \|F_1\|_{S^p} + \|F_2\|_{S^p} \\
&\leq \|\gamma_\epsilon - w_o\|_{B^p} + K_p \int_0^t \|w_\epsilon(\tau) - w(\tau)\|_{B^p} d\tau
\end{aligned}
$$

Finally, we have $\|\gamma_\epsilon - w_o\|_{B^p} \leq K_p\epsilon$ since $\epsilon \to \gamma_\epsilon$ is $B^{\infty -}$-Lipschitz. So by Gronwall's inequality (8.1.7) $\|w_\epsilon(t) - w(t)\|_{B^p}$ is $O(\epsilon)$ for all $t \in J$. Also note that $\|w_\epsilon(t) - w(t)\|_{B^p}$
$= \|F_1\|_{S^p} + \|F_2\|_{S^p}$, so $\|F_1\|_{S^p}$ and $\|F_2\|_{S^p}$ are also $O(\epsilon)$.

(ii) Define $D_1(t) \equiv O_\epsilon(t) - O(t)$.

$$
\begin{aligned}
\|D_1\|_{S^p} &\leq \|O_\epsilon - O_\epsilon Q_\epsilon\|_{S^p} + \|O_\epsilon Q_\epsilon - O\|_{S^p} \\
&= \|I - Q_\epsilon\|_{S^p} + \|F_1\|_{S^p} \text{ by Lemma 4.2.3} \\
&\leq O(\epsilon) + \|w_\epsilon - w\|_{B^p} \\
&\leq O(\epsilon) \text{ by } \textbf{(i)}.
\end{aligned}
$$

where the third line follows by the continuity of $\epsilon \to Q_\epsilon$ and the definition of the B^p-norm.

(iii) Define $D_2(t) \equiv \alpha_\epsilon(t) - \alpha(t)$. Then

$$\begin{aligned}\|D_2\|_{S^p} &\leq \|\alpha_\epsilon - \alpha + O_\epsilon\beta_\epsilon\|_{S^p} + \|O_\epsilon\beta_\epsilon\|_{S^p} \\ &= \|F_2\|_{S^p} + \|\beta_\epsilon\|_{S^p} \text{ by Lemma 4.2.3} \\ &\leq O(\epsilon)\end{aligned}$$

by **(i)** and the S^p-continuity of the map $\epsilon \to \beta_\epsilon$.

(iv) Fix $p \in [2, \infty)$ and let p_1 and p_2 be such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Then

$$\begin{aligned}&\|O^{-1}[\frac{1}{\epsilon}(C(w_\epsilon) - C(w))(O_\epsilon - O)]\|_{S^p} \\ &\quad = \|\frac{1}{\epsilon}(C(w_\epsilon) - C(w))(O_\epsilon - O)\|_{S^p} \\ &\quad \leq \frac{1}{\epsilon}\|C(w_\epsilon) - C(w)\|_{S^{p_1}}\|O_\epsilon - O\|_{S^{p_2}} \text{ (by Hölder's inequality)} \\ &\quad \leq \frac{K_p}{\epsilon}\|w_\epsilon - w\|_{B^{p_1}}\|O_\epsilon - O\|_{S^{p_2}} \text{ (by [8] Prop 6.2 (vii))}\end{aligned}$$

But $\|w_\epsilon - w\|_{B^{p_1}}$ and $\|O_\epsilon - O\|_{S^{p_2}}$ are both $O(\epsilon)$ by **(i)** and **(ii)**, so the last line above is $O(\epsilon)$.

(v) The proof is similar to that for **(iv)**, using **(iii)** above.

(vi) Recall that $w_\epsilon = \int O_\epsilon d\gamma_\epsilon + \int \alpha_\epsilon ds$ We have

$$\begin{aligned}\tilde{E} &\equiv \frac{w_\epsilon - w}{\epsilon} - \tilde{Z} \\ &= \int [\frac{O_\epsilon - O}{\epsilon} - Z_1]db \\ &\quad + \int [\frac{\alpha_\epsilon - \alpha}{\epsilon} - Z_2]ds \\ &\quad + \int O_\epsilon d\Delta_\epsilon - \int OdY\end{aligned}$$

where $\Delta_\epsilon = \frac{\gamma_\epsilon - w_o}{\epsilon}$. So

$$\begin{aligned}\|\tilde{E}\|_{B^p} &\leq \|\int OE_1db + \int OE_2ds\|_{B^p} \\ &\quad + \|\int O_\epsilon d(\Delta_\epsilon - Y)\|_{B^p} + \|\int [O_\epsilon - O]dY\|_{B^p}.\end{aligned}$$

Let $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Then

$$\begin{aligned} \|\int O_\epsilon d(\Delta_\epsilon - Y)\|_{B^p} &\leq \|O_\epsilon\|_{S^{p_1}}\|\Delta_\epsilon - Y\|_{B^{p_2}} \text{ by [8] Lemma 4.1(v)} \\ &\leq K_{p_1}\|\frac{\gamma_\epsilon - \gamma_0}{\epsilon} - Y\|_{B^{p_2}} \\ &= O(\epsilon) \end{aligned}$$

by the definition of the $B^{\infty-}$-derivative (see Notation 4.1.1). Also,

$$\begin{aligned} \|\int [O_\epsilon - O]dY\|_{B^p} &\leq \|O_\epsilon - O\|_{S^{p_1}}\|Y\|_{B^{p_2}} \text{by [8] Lemma 4.1(v)} \\ &= O(\epsilon) \text{ by } \textbf{(ii)}. \end{aligned}$$

$$\begin{aligned} \text{So } \|\tilde{E}\|_{B^p} &\leq \|OE_1\|_{S^p} + \|OE_2\|_{S^p} + O(\epsilon) \\ &= \|E_1\|_{S^p} + \|E_2\|_{S^p} + O(\epsilon). \end{aligned}$$

(vii) By the definition of the S^p-derivative (Definition 2.1.5) we have

$$\begin{aligned} C(w_\epsilon) - C(w) &= C'(w)\langle w_\epsilon - w\rangle + \mathcal{E}_1(w_\epsilon, w) \\ \text{and } R(w_\epsilon) - R(w) &= R'(w)\langle w_\epsilon - w\rangle + \mathcal{E}_2(w_\epsilon, w) \end{aligned}$$

where $\|\mathcal{E}_i(w_\epsilon, w)\|_{S^p}$is $O(\|w_\epsilon - w\|_{S^p}^2)$ for $i = 1, 2$. Thus

$$\begin{aligned} &\|O^{-1}[\tfrac{1}{\epsilon}(C(w_\epsilon) - C(w)) - C'(w)\langle\tilde{Z}\rangle]O\|_{S^p} \\ &\quad = \|\tfrac{1}{\epsilon}(C(w_\epsilon) - C(w)) - C'(w)\langle\tilde{Z}\rangle\|_{S^p} \\ &\quad = \|C'(w)\langle\tfrac{w_\epsilon - w}{\epsilon} - \tilde{Z}\rangle + \tfrac{1}{\epsilon}\mathcal{E}_1(w_\epsilon, w)\|_{S^p} \\ &\quad \leq \|C'(w)\langle\tilde{E}\rangle\|_{S^p} + \tfrac{1}{\epsilon}\|\mathcal{E}_1(w_\epsilon, w)\|_{S^p} \\ &\quad \leq K_p\|\tilde{E}\|_{B^p} + O(\epsilon) \text{ by (4.1.1) and } \textbf{(i)} \text{ above} \\ &\quad \leq K_p[\|E_1\|_{S^p} + \|E_2\|_{S^p} + O(\epsilon)] \text{ by } \textbf{(vi).} \end{aligned}$$

(viii) Similar to the proof of **(vii)**, using Lemma 8.1.5 and the fact that $\sup_{t\in J}\|\alpha(t)\|_{S^\infty} < \infty$.

(ix) Similar to the proof of **(vii)**, using (4.1.2) the bound on R'. Q.E.D.

5.3 Existence of the Differential

We now prove one of the main results of this paper, Theorem 4.1.3: $[YO](t,b)$ and $[Y\alpha](t,b)$ exist in $S^{\infty-}$ and satisfy the formal equation (4.1.4), and $[Yw](t,b)$ is given by (4.1.5).

Proof. Fix $p \in [2,\infty)$ and let $E(t) \equiv \int E_1(t)db + \int E_2(t)ds$. We will use the modified Gronwall's inequality (Lemma 8.1.8) on $\|E(t)\|_{B^p} \equiv \|E_1(t)\|_{S^p} + \|E_2(t)\|_{S^p}$. We have $\|E(0)\|_{B^p} \equiv \|E_1(0)\|_{S^p} + \|E_2(0)\|_{S^p} = 0$.

$$\begin{aligned}
\|\dot{E}_1\|_{S^p} &\leq \|O^{-1}[\frac{1}{\epsilon}(C(w_\epsilon) - C(w))(O_\epsilon - O)]\|_{S^p} \\
&\quad + \|O^{-1}[\frac{1}{\epsilon}(C(w_\epsilon) - C(w)) - C'(w)\langle\tilde{Z}\rangle]O\|_{S^p} \\
&\leq K_p\|E(t)\|_{B^p} + O(\epsilon) \text{ by Lemma 5.2.1 } \textbf{(iv)} \text{ and } \textbf{(vii)}, \text{ and} \\
\|\dot{E}_2\|_{S^p} &= \|O^{-1}[\frac{1}{\epsilon}(C(w_\epsilon) - C(w))(\alpha_\epsilon - \alpha)]\|_{S^p} \\
&\quad + \|O^{-1}[\frac{1}{\epsilon}(C(w_\epsilon) - C(w)) - C'(w)\langle\tilde{Z}\rangle]\alpha\|_{S^p} \\
&\quad + \|O^{-1}[\frac{1}{\epsilon}(R(w_\epsilon) - R(w)) - R'(w)\langle\tilde{Z}\rangle]\|_{S^p} \\
&\leq K_p\|E(t)\|_{B^p} + O(\epsilon) \text{ by Lemma 5.2.1 } \textbf{(v)}, \textbf{(viii)} \text{ and } \textbf{(ix)}.
\end{aligned}$$

Thus $\|\dot{E}(t)\|_{B^p} \equiv \|\dot{E}_1(t)\|_{S^p} + \|\dot{E}_2(t)\|_{S^p} \leq K_p\|E(t)\|_{B^p} + O(\epsilon)$, so by Lemma 8.1.8 , $\|E(t)\|_{B^p} = \|E_1(t)\|_{S^p} + \|E_2(t)\|_{S^p}$ is $O(\epsilon)$ for all $t \in J$. Thus

$$\begin{aligned}
0 &= \lim_{\epsilon\to 0} \|E_1\|_{S^p} \\
&= \lim_{\epsilon\to 0} \|O^{-1}[\frac{1}{\epsilon}(O_\epsilon - O) - Z_1]\|_{S^p} \\
&= \lim_{\epsilon\to 0} \|\frac{1}{\epsilon}(O_\epsilon - O) - Z_1\|_{S^p}
\end{aligned}$$

and similarly, $\lim_{\epsilon\to 0}\|\frac{1}{\epsilon}(\alpha_\epsilon - \alpha) - Z_2\|_{S^p} = 0$. Thus the S^p-derivatives defined in (4.1.3) exist, with $YO = Z_1$, and $Y\alpha = Z_2$. Finally, by **(vi)** the B^p-derivative Yw exists and is given by $\tilde{Z}$. Q.E.D.

5.4 Differentials of the Flows in Wiener Space

In this section we will use our semimartingale "machine" Theorem 4.1.3 to get differentials of the quasi-invariant flows on $W(\mathbf{R}^n)$ obtained in Section 3.2.

Notation 5.4.1

(i) *Let $T : B^\infty \mathbf{R}^n \to S^{\infty-}\mathrm{so}(n)$ in Theorem 3.1.5, and take the underlying probability space to be $(W(\mathbf{R}^n), \{\bar{\mathcal{H}}^\mu_{s+}\}, \{\bar{b}(s)\}, \mu)$ (see Notation 3.2.1). We will write the solution to equation (3.1.3) as a function of its starting point as in Notation 4.1.2, so $w(t, w_o)$ denotes the solution to equation:*

$$\dot{w}(t, w_o) = X(w(t, w_o)) \text{ with } w(0, w_o) = w_o \in B^\infty \mathbf{R}^n.$$

Also define $O(t, w_o)$ and $\alpha(t, w_o)$ by

$$w(t, w_o) = \int O(t, w_o) db + \int \alpha(t, w_o) ds.$$

(ii) *Let $\tilde{w}(t) : W(\mathbf{R}^n) \to W(\mathbf{R}^n)$, be the solution to (3.1.3) with $\tilde{w}(0) \equiv id \in Maps(W(\mathbf{R}^n), W(\mathbf{R}^n))$ and let $\tilde{O}(t) : W(\mathbf{R}^n) \to W(O(n))$ and $\tilde{\alpha}(t) : W(\mathbf{R}^n) \to W(\mathbf{R}^n)$, solve (3.1.4) with $\tilde{O}(0) \equiv I \in O(n)$ and $\tilde{\alpha}(0) \equiv 0 \in \mathbf{R}^n$ as given in Notation 3.2.2. That is, $\tilde{w}(t)$, $\tilde{O}(t)$ and $\tilde{\alpha}(t)$ are jointly continuous versions of $w(t, \bar{b})$, $O(t, \bar{b})$ and $\alpha(t, \bar{b})$, respectively.*

Theorem 5.4.2 *Let $\tilde{\gamma} : J \to Maps(W(\mathbf{R}^n), W(\mathbf{R}^n))$ be an admissible curve (Def. 2.1.7) such that $\tilde{\gamma}_\epsilon$ has law equivalent to μ for all $\epsilon \in J$ with $\tilde{\gamma}_0 = \bar{b}$, and let $\tilde{Y} = \frac{d}{d\epsilon}|_0 \tilde{\gamma}_\epsilon$. For all $t \in J$,*

$$\begin{aligned} [\tilde{Y}.\tilde{O}](t) &\equiv \lim_{\epsilon \to 0} \tfrac{\tilde{O}(t)(\tilde{\gamma}_\epsilon(\cdot)) - \tilde{O}(t)(\cdot)}{\epsilon} \\ \text{and } [\tilde{Y}.\tilde{\alpha}](t) &\equiv \lim_{\epsilon \to 0} \tfrac{\tilde{\alpha}(t)(\tilde{\gamma}_\epsilon(\cdot)) - \tilde{\alpha}(t)(\cdot)}{\epsilon} \end{aligned} \tag{5.4.1}$$

exist where the limits are taken in the S^p-topologies for $p \in [2, \infty)$.

Equivalently,

$$[\tilde{Y}.\tilde{w}](t) \equiv \lim_{\epsilon \to 0} \frac{\tilde{w}(t)(\tilde{\gamma}_\epsilon(\cdot)) - \tilde{w}(t)(\cdot)}{\epsilon}$$

exists where the limit is taken in the B^p-topologies for $p \in [2, \infty)$.

Note that we do not define these derivatives for each $\omega \in W(\mathbf{R}^n)$ since the limits exist only P-a.s.

Proof. Let $t \in J$. We know $[\tilde{Y}_{\bar{b}} O](t) \equiv \lim_{\epsilon \to 0} \frac{O(t,\tilde{\gamma}_\epsilon) - O(t,\bar{b})}{\epsilon}$ exists in $S^{\infty-}$ by Theorem 4.1.3 (here we are viewing $\tilde{Y} \in B^{\infty-}\mathbf{R}^n$). Since $\bar{b} : W(\mathbf{R}^n) \to W(\mathbf{R}^n)$ is the identity map, and $\tilde{\gamma}_\epsilon$ has law equivalent to μ, we have (supressing the parameter t) $O(\tilde{\gamma}_\epsilon) \doteq O(\bar{b} \circ \tilde{\gamma}_\epsilon) \doteq \tilde{O} \circ \tilde{\gamma}_\epsilon$.

Thus $\tilde{O}(t) \circ \tilde{\gamma}_\epsilon - \tilde{O}(t)$ is a version of $O(t,\tilde{\gamma}_\epsilon) - O(t,\bar{b})$, so $[\tilde{Y}\tilde{O}](t) : W(O(n)) \to W(O(n))$ as defined above exists and is a version of $[\tilde{Y}_{\bar{b}} O](t)$. Similarly, $[\tilde{Y}\tilde{\alpha}](t)$ and $[\tilde{Y}\tilde{w}](t)$ exist and are versions of $[\tilde{Y}_{\bar{b}}\alpha](t)$ and $[\tilde{Y}_{\bar{b}} w](t)$ respectively. Q.E.D.

Chapter 6

Geometric Definitions

6.1 Geometric Definitions

Notation 6.1.1 *Throughout this paper (M, ∇, g) will be a smooth compact n-dimensional Riemannian manifold with metric g and g-compatible covariant derivative ∇. We will also assume that the torsion tensor of ∇ satisfies the skew symmetry property $g\langle T\langle X, Y\rangle, Y\rangle \equiv 0$ for all $X, Y \in \Gamma(TM)$ (see Definition 6.1.11), i.e. that ∇ is Torsion Skew Symmetric (TSS).*

Notation 6.1.2 *Given a manifold M and a fixed point $o \in M$, let*
(i) $W(M) \equiv C([0,1], M)$,
(ii) $W_o(M) \equiv \{\omega \in W(M) : \omega(0) = o \in M\}$,
(iii) $W_o^\infty(M)$ *denote the set of smooth paths in* $W_o(M)$.

Definition 6.1.3 *Denote the orthonormal frame bundle of M by $\pi : O(M) \to M$, or just $O(M)$. Recall that if $v_m \in O(M)$ then $v : \mathbf{R}^n \to T_mM$ is a linear isometry. Throughout this paper we will refer to a fixed frame $u_o \in O(M)$ with $\pi u_o = o$.*

Definition 6.1.4 *Given a vector bundle $\pi : E \to M$ and $\sigma \in W_o^\infty(M)$, denote the set of* smooth sections of E along σ *by* $\Gamma_\sigma^\infty(E)$.

Definition 6.1.5 *Given $\sigma \in W_o^\infty(M)$ and a smooth vector field along σ, $Z \in \Gamma_\sigma^\infty(TM)$, let $\frac{\nabla Z}{ds} \in \Gamma_\sigma^\infty(TM)$ denote the* covariant derivative of Z along σ.

Definition 6.1.6 *Let $E = \mathrm{Hom}(\mathbf{R}^n, TM)$, the vector bundle over M with fiber* $E_m = \mathrm{Hom}(\mathbf{R}^n, T_m M)$ for all $m \in M$. *For $u \in \Gamma_\sigma^\infty(O(M))$, define $\frac{\nabla u}{ds} \in \Gamma_\sigma^\infty(E)$ by $\frac{\nabla u}{ds}(s)(\zeta) = (\frac{\nabla}{ds})(u(s)\zeta)$ for all $\zeta \in \mathbf{R}^n$. (Note that $u(s)\zeta \in T_{\sigma(s)}M$ for all $s \in [0,1]$, i.e., $u(\cdot)\zeta \in \Gamma_\sigma(TM)$.)*

Notation 6.1.7 *Let $\omega \equiv \omega^\nabla$ be the connection 1-form on $O(M)$ with values in* $\mathrm{so}(n)$ *defined by $\omega\langle u'(s)\rangle = u(s)^{-1}\frac{\nabla u}{ds}(s)$ for any smooth path u in $O(M)$.*

Definition 6.1.8

1. *A path $u \in W^\infty(O(M))$ is said to be* **horizontal** *if $\frac{\nabla u}{ds}(s) = 0$ or equivalently if $\omega\langle u'(s)\rangle = 0$ for all $s \in [0,1]$. Denote by $HW_{u_o}^\infty(O(M))$ the set of smooth horizontal paths in $O(M)$ based at u_o.*

2. *Given a curve $\sigma \in W_o^\infty(M)$, define the* **horizontal lift** *of σ, $H(\sigma)$, to be the unique curve $u \in HW_{u_o}^\infty(O(M))$ with $\pi \circ u(s) = \sigma(s)$ for all $s \in [0,1]$. The function $H : W_o^\infty(M) \to HW_{u_o}^\infty(O(M))$ defined above will be called the* **horizontal lift map.**

Definition 6.1.9 *Denote the standard horizontal vector fields by $B\langle a\rangle(\cdot) \in \Gamma(TO(M))$ for all $a \in \mathbf{R}^n$, where $B\langle a\rangle(u) \in T_uO(M)$ is the unique vector such that $\pi_* B\langle a\rangle(u) = ua$ and $\omega\langle B\langle a\rangle(u)\rangle = 0$.*

Definition 6.1.10 *Denote by θ the canonical ($\mathbf{R}^n$-valued) 1-form on $O(M)$, defined by $\theta\langle\xi\rangle = u^{-1}\pi_*\xi$ for all $\xi \in T_uO(M)$ and $u \in O(M)$.*

Definition 6.1.11 *Let $X, Y, Z \in \Gamma(TM)$. Define*
(i) the curvature tensor of ∇ by $R\langle X, Y\rangle Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$,
(ii) the torsion tensor of ∇ by $T\langle X, Y\rangle = \nabla_X Y - \nabla_Y X - [X, Y]$.

Definition 6.1.12 *For all $a, b \in \mathbf{R}^n$ and $u \in O(M)$ let*

1. $\Omega_u\langle a, b\rangle \equiv u^{-1}R\langle ua, ub\rangle u \in \mathrm{so}(n)$,

2. $\Theta_u\langle a, b\rangle \equiv u^{-1}T\langle ua, ub\rangle \in \mathbf{R}^n$.

For a proof that $\Omega_u\langle a, b\rangle \in \mathrm{so}(n)$, see [23] Section III Theorem 2.4 and Section III.5. Notice that if ∇ is Torsion Skew Symmetric, i.e. $g\langle T\langle X, Y\rangle, Y\rangle \equiv 0$ for all $X, Y \in \Gamma(TM)$, then we also have $\Theta_u\langle a, \cdot\rangle \in so(n)$ for all $a \in \mathbf{R}^n$

6.2 Stochastic Geometric Definitions

Recall that throughout this paper we are using an underlying filtered probability space $(\Omega, \{\mathcal{F}_s\}, \mathcal{F}, P)$, satisfying the *usual hypothesis* (Notation 2.1.1).

Definition 6.2.1 *Define an M-*valued semimartingale *to be a continuous M-valued $\{\mathcal{F}_s\}$-adapted stochastic process Z, with the property that for all $f \in C^\infty(M)$, $f(Z)$ is a real-valued $(\Omega, \{\mathcal{F}_s\}, \mathcal{F}, P)$ semimartingale.*

Notation 6.2.2 *Suppose Q is an imbedded submanifold of $\mathbf{R}^N$ and that $q_0 \in Q$ is a fixed base point. (If $Q = \mathbf{R}^n, M,$ or $O(M)$, take $q_0 = 0, o,$ or u_o, respectively.) Let $B^\infty Q$ denote the space of Q-valued Brownian semimartingales starting at q_0 which are in also $B^\infty \mathbf{R}^N$, as defined in Notation 2.1.4.*

Definition 6.2.3 *Given semimartingales X and Y, let $\int X\delta Y$ denote the process $(s \to \int_0^s X\delta Y)$ where the integral is the Fisk-Stratonovich stochastic integral. In terms of Itô integrals, $\int X\delta Y = \int XdY + \frac{1}{2}[X, Y]$, where $s \to [X, Y]_s$ is the mutual variation process of X and Y. We will write $dXdY$ for the differential of $[X, Y]$.*

Definition 6.2.4 *Let $\bar{\pi} : T^*M \to M$ be the dual of the tangent bundle on M. Suppose that $\alpha(s)$ is a T^*M-valued semimartingale and Z is the M-valued semimartingale defined by $Z(s) \equiv \bar{\pi}(\alpha(s))$. The* Stratonovich integral *$\int \alpha\langle\delta Z\rangle$ is defined* *to be the real-valued semimartingale* $\sum \int f_i(s)\delta(g^i(Z(s)))$*, where* $\{f_i\}$ *is a finite collection of real-valued semimartingales, and* $\{g^i\}$ *is a finite subset of* $C^\infty(M)$ *such that* $\alpha(s) = \sum f_i(s)dg^i|_{Z(s)}$*. See [8], Remark 3.2, p. 290 for a proof that such a decomposition exists, and that the integral is well defined. References for stochastic integration include [14], [22], and [25].*

Definition 6.2.5 *Note that if* β *is a smooth 1-form on* M *and* Z *is an* M*-valued semimartingale, then* $\alpha(s) \equiv \beta|_{Z(s)}$ *is a* T^*M*-valued semimartingale with* $Z(s) \equiv \bar{\pi}(\alpha(s))$*. In this case the Stratonovich integral* $\int \beta\langle\delta Z\rangle$ *is defined as the process* $\int \alpha\langle\delta Z\rangle$*.*

Definition 6.2.6 *Suppose that* Q *is a manifold and* $F : \mathbf{R}^n \to \Gamma(TQ)$ *is a linear map defined by* $(a \to F\langle a\rangle(\cdot))$*. Given an* $\mathbf{R}^n$*-valued semimartingale* (w)*, a* Q*-valued semimartingale* (q) *is said to satisfy the* Stratonovich stochastic differential equation

$$\delta q = F\langle\delta w\rangle(q) \tag{6.2.1}$$

if and only if for all $f \in C^\infty(Q)$*,* $d(f(q)) = df\langle F\langle\delta w\rangle(q)\rangle$*. That is,*

$$f(q(s)) - f(q(0)) = \sum_{i=1}^n \int_0^s (F\langle e_i\rangle(q(\tau))f)\delta w^i(\tau)$$

where $\{e_i\}_{i=1}^n$ *is the standard basis for* $\mathbf{R}^n$*.*

See Theorem 3.1 in [8], p. 292 for a proof that (6.2.1) has a unique solution if F has compact support.

Recall that $\omega \equiv \omega^\nabla$ is the connection 1-form on $O(M)$ with values in so(n) (Notation 6.1.7).

Definition 6.2.7 *An* $O(M)$*-valued semimartingale* u *is said to be* ω-horizontal *(or just* horizontal*) if* $\int\omega\langle\delta u\rangle \equiv 0$*.*

Definition 6.2.8 *Given an* M*-valued semimartingale* σ*, a* horizontal lift of σ *is an* $O(M)$*-valued semimartingale* u *satisfying (i)* $\pi \circ u = \sigma$*, and (ii)* u *is horizontal.*

See Theorem 3.2 in [8], p. 294 for a proof that a unique horizontal lift exists when $u(0) = u_o \in O(M)$ is fixed.

Notation 6.2.9 *Denote by*

1. $\mathcal{S}M$ *the space of based M-valued semimartingales starting at $o \in M$,*
2. $H\mathcal{S}O(M)$ *the horizontal $O(M)$-valued semimartingales starting at $u_o \in O(M)$.*
3. $\mathcal{S}\mathbf{R}^n$ *the space of $\mathbf{R}^n$-valued semimartingales starting at $0 \in \mathbf{R}^n$.*

Notation 6.2.10 *Define the maps*

1. *Stochastic Horizontal Lift $H : \mathcal{S}M \to H\mathcal{S}O(M)$ such that $H(\sigma)$ is the horizontal lift of $\sigma \in \mathcal{S}M$ to $O(M)$ starting at u_o,*
2. $\pi : H\mathcal{S}O(M) \to \mathcal{S}M$ *by $\pi(u) = \pi \circ u$,*
3. *Stochastic Development (Eells and Elworthy [12]) $I : \mathcal{S}\mathbf{R}^n \to H\mathcal{S}O(M)$ such that for all $w \in \mathcal{S}\mathbf{R}^n$, $I(w) \equiv u$ where u is the solution to the Stratonovich differential equation $\delta u = B\langle \delta w \rangle(u)$ with $u(0) = u_o$,*
4. $I^{-1} : H\mathcal{S}O(M) \to \mathcal{S}\mathbf{R}^n$ *by $I^{-1}(u) = \int \theta\langle \delta u \rangle$.*

Theorem 6.2.11
The functions H and π are inverses of each other as are I and I^{-1}.

For a proof, see for example [8] Theorem 3.3 p. 297.

Definition 6.2.12 *Let Q be a manifold. A Q-valued semimartingale (Z) is said to be a Brownian semimartingale iff $f \circ Z$ is an $\mathbf{R}$-valued Brownian semimartingale for all $f \in C^\infty(Q)$.*

Definition 6.2.13 *Given a smooth function $f : O(M) \to V$, where V is a vector space, define the* horizontal derivative of f, $f^H : O(M) \to \mathrm{End}(\mathbf{R}^n, V)$ *by $f^H(u)\langle a \rangle = df\langle B\langle a \rangle(u)\rangle$ for all $u \in O(M)$ and $a \in \mathbf{R}^n$.*

Notation 6.2.14 *We will use the horizontal derivatives of two functions, $u \to \Omega_u\langle\cdot,\cdot\rangle$ and $u \to \Theta_u\langle\cdot,\cdot\rangle$, given in definition 6.1.12. For $f(u) \equiv \Omega_u\langle\cdot,\cdot\rangle$ we denote the horizontal derivative $f^H(u)\langle a\rangle$ by $\Omega_u^H\langle a,\cdot,\cdot\rangle$, and define $\Theta_u^H\langle a,\cdot,\cdot\rangle$ similarly.*

Also, for $A, B \in \mathrm{End}(\mathbf{R}^n)$ and $a \in \mathbf{R}^n$, define

$$\bar{\Omega}_u\langle A,a,B\rangle \equiv \sum_{i=1}^n \Omega_u^H\langle Ae_i, a, Be_i\rangle$$

$$\text{and } \bar{\Theta}_u\langle A,a,B\rangle \equiv \sum_{i=1}^n \Theta_u^H\langle Ae_i, a, Be_i\rangle.$$

6.3 Existence of a Measure-Preserving Flow on $W_o(M)$

Let (M,∇,g) be as in Notation 6.1.1 (Recall that the torsion tensor of ∇ satisfies the skew symmetry property $g\langle T\langle X,Y\rangle,Y\rangle \equiv 0$ for all $X,Y \in \Gamma(TM)$.)

Theorem 6.3.1 *(Driver [8] Theorem 8.5, p. 361) Fix $h \in C^1 \equiv \{h \in C^1([0,1],\mathbf{R}^n) : h(0) = 0\}$. Let $\bar{H}$ be a fixed version of the horizonotal lift map H (see Notation 6.2.10). Define the vector field $\tilde{X}^h$ on $W_o(M)$ by $\tilde{X}^h(\tilde{\sigma}) \equiv \bar{H}(\tilde{\sigma})h$. Then there exists a unique solution to the equation*

$$\dot{\tilde{\sigma}}^h = \tilde{X}^h(\tilde{\sigma}^h) \text{ with } \tilde{\sigma}^h(0) = id \tag{6.3.2}$$

in the space of paths $\{\tilde{\sigma} : \mathbf{R} \to Maps(W_o(M) \to W_o(M))\}$. Furthermore, $\tilde{\sigma}$ is a flow on $W_o(M)$, and $\tilde{\sigma}(t)_*\nu$ is equivalent to $\nu \equiv$ Wiener measure on $W_o(M)$.

We will also use the "semimartingale version" of this theorem, which is stated below.

Theorem 6.3.2 *(Driver [8] Cor. 6.3, p. 336) Suppose M is an imbedded submanifold of $\mathbf{R}^N$ for some N. Define the vector field X^h on $B^\infty M$ by $X^h(\sigma) \equiv$*

$H(\sigma)h$. Let $\sigma_o \in B^\infty M$ with $\sigma_o(0) \equiv o \in M$. Then there exists a unique solution to the equation

$$\dot{\sigma}^h = X^h(\sigma^h) \text{ with } \sigma^h(0) = \sigma_o \tag{6.3.3}$$

in the space of paths $\{\sigma : \mathbf{R} \to B^\infty M\}$.

Note that for $\sigma \in B^\infty M$, $H(\sigma)$ is well-defined (see Definition 6.2.7).

Notation 6.3.3 *For $w = \int Odb + \int \alpha ds \in B^\infty \mathbf{R}^n$, and $u \equiv I(w)$ define C^h : $B^\infty \mathbf{R}^n \to \mathrm{so}(n)$-valued processes, and $R^h : B^\infty \mathbf{R}^n \to \mathbf{R}^n$-valued processes by*

$$C^h(w) = \int \Omega_u \langle h, \delta w \rangle + \Theta_u \langle h, \cdot \rangle$$

$$\text{and } R^h(w) \equiv \frac{1}{2}\{Ric_u\langle h, O, O\rangle + \bar{\Theta}_u \langle O, h, O\rangle\} + h'$$

where Ω_u and Θ_u are given in Definition 6.1.12, $\bar{\Theta}_u$ is given in Notation 6.2.14, and

$$Ric_u\langle h, O, O\rangle \equiv \sum_{i=1}^{n} \Omega_u \langle h, Oe_i\rangle Oe_i.$$

Note: We have $C^h : B^\infty \mathbf{R}^n \to S^{\infty-}\mathrm{so}(n)$ by the proof of [8] Cor. 6.2, p. 330, and $R^h : B^\infty \mathbf{R}^n \to S^\infty \mathbf{R}^n$ by the proof of [8] Cor. 6.1, p. 328.

Define $Y^h : B^\infty \mathbf{R}^n \to B^{\infty-}\mathbf{R}^n$ by

$$Y^h(w) \equiv \int C^h(w) dw + \int R^h(w) ds \text{ for } w \in B^\infty \mathbf{R}^n.$$

Theorem 6.3.4 *(Driver [8] Thm 5.1, p. 320 and Thm. 6.1, p. 332) Let σ^h : $\mathbf{R} \to B^\infty M$ be as defined in 6.3.2. Let $w^h(t) \equiv I^{-1} \circ H(\sigma^h(t)) \in B^\infty \mathbf{R}^n$ for all $t \in \mathbf{R}$.*

Then w^h is the unique solution to the equation

$$\dot{w}^h(t) = Y^h(w^h(t)) \text{ with } w^h(0) = b \tag{6.3.4}$$

in the space of paths $\{w : \mathbf{R} \to B^\infty \mathbf{R}^n\}$.

Theorem 6.3.5 *(Driver [8] Prop. 6.1, p. 323) Let O^h and α^h be defined by $w^h(t) = \int O^h(t)db + \int \alpha^h(t)ds$. Then $O^h : \mathbf{R} \to O(n)$-valued processes, and α^h : $\mathbf{R} \to \mathbf{R}^n$-valued processes, and these satisfy the following equations:*
$\dot{O}^h(t) = C^h(w^h(t))O^h(t)$ with $O^h(0) = I$ and
$\dot{\alpha}^h(t) = C^h(w^h(t))\alpha^h(t) + R^h(w^h(t))$ with $\alpha^h(0) = 0$.

Chapter 7

Existence and Uniqueness of the Derivative of the Geometric Flow

7.1 Pulled-Back Flow Equation to $W(\mathbf{R}^n)$ via the Itô Map

The following will be fixed throughout the rest of this paper.

Notation 7.1.1 *Fix $h, k \in C^1$, and let $C \equiv C^h : B^\infty\mathbf{R}^n \to S^{\infty-}\mathrm{so}(n)$, $R \equiv R^h : B^\infty\mathbf{R}^n \to S^\infty\mathbf{R}^n$, and $Y^k : B^\infty\mathbf{R}^n \to B^{\infty-}\mathbf{R}^n$ be as given in Notation 6.3.3. Let $w^k, w^h : \mathbf{R} \to B^\infty\mathbf{R}^n$, $O^h : \mathbf{R} \to S^\infty O(n)$ and $\alpha^h : \mathbf{R} \to S^\infty\mathbf{R}^n$ be defined as in Theorems 6.3.4 and 6.3.5.*

Let $C_o = C_o(|h'|_\infty, \|w(0)\|_{B^\infty}) \in (0,\infty)$ such that $\sup_{t\in J}\|w^h(t)\|_{B^\infty} \le C_o$. Existence of such a C_o is given in [8], Cor. 6.1, p. 327.

Notation 7.1.2 *We write the solution to the flow equation $\dot{w}^h(t) = Y^h(w^h(t))$ as a function of its starting point, so $w^h(t, w_o)$ denotes the solution to the equation*

$$w^h(t, w_o) = Y^h(w^h(t, w_o)) \text{ with } w^h(0, w_o) = w_o \in B^\infty\mathbf{R}^n. \tag{7.1.1}$$

Thus the solution to (6.3.4) is given by $w^h(t,b)$. *Existence of a unique solution to (7.1.1) is given by Driver [8], Theorem 6.1, p. 332. We also extend this notation to* O^h *and* α^h, *so that* $O^h(t,w_o)$ *and* $\alpha^h(t,w_o)$ *are defined by:*

$$w^h(t,w_o) = \int O^h(t,w_o)dw_o + \int \alpha^h(t,w_o)ds.$$

The following is one of the main results in this paper for the geometric case.

Theorem 7.1.3 **i)** *The functions* $O^h(t,\cdot) : B^\infty\mathbf{R}^n \to S^\infty\mathbf{End}(\mathbf{R}^n)$ *and* $\alpha^h(t,\cdot) : B^\infty\mathbf{R}^n \to S^\infty\mathbf{R}^n$ *are* $S^{\infty-}$*-differentiable at* w_o. *Let* γ *be and admissible curve with* $\gamma_0 = w_o$, *and let* $Y = \frac{d}{d\epsilon}|_0\gamma_\epsilon$. *(In particular, we could take* $\gamma_\epsilon = w^k(\epsilon,w_o)$ *and* $Y = Y^k(w_o)$.*) Then*

$$\begin{aligned} [Y_{w_o}O^h](t,w_o) &\equiv \lim_{\epsilon\to 0}\tfrac{O^h(t,\gamma_\epsilon)-O^h(t,w_o)}{\epsilon} \\ \text{and } [Y_{w_o}\alpha^h](t,w_o) &\equiv \lim_{\epsilon\to 0}\tfrac{\alpha^h(t,\gamma_\epsilon))-\alpha^h(t,w_o)}{\epsilon} \end{aligned} \tag{7.1.2}$$

exist where the limits are taken in the $S^p\mathrm{End}(\mathbf{R}^n)$*- and* $S^p\mathbf{R}^n$*-topologies for* $p \in [2,\infty)$. *Furthermore,* $Z(t) \equiv \begin{bmatrix} Y_{w_o}O^h(t) \\ Y_{w_o}\alpha^h(t) \end{bmatrix} \equiv \begin{bmatrix} Z_1(t) \\ Z_2(t) \end{bmatrix}$ *satisfies the equation:*

$$\dot{Z}(t) = C(w^h(t))Z(t) + A_t(Z(t)) + K(t) \text{ with } Z(0) = 0 \tag{7.1.3}$$

where A_t *and* $K(t)$ *are defined in Notation 7.1.4 below.*

ii) *The function* $w^h(t,\cdot) : B^\infty\mathbf{R}^n \to B^\infty\mathbf{R}^n$ *is* $B^{\infty-}$*-differentiable at* w_o. *(See Definition 2.1.9.) That is, for any admissible curve* γ *with* $\gamma_0 = w_o$ *and* $Y \equiv \frac{d}{d\epsilon}|_0\gamma_\epsilon$,

$$[Y_{w_o}w^h](t,w_o) \equiv \lim_{\epsilon\to 0}\frac{w^h(t,\gamma_\epsilon) - w^h(t,w_o)}{\epsilon}$$

exists where the limit is taken in the $B^p\mathbf{R}^n$*-topologies for* $p \in [2,\infty)$. *Furthermore,* $[Y_{w_o}w^h](t,w_o) = \tilde{Z}(t)$ *where*

$$\tilde{Z}(t) \equiv \int_0^s Z_{1,\bar{s}}(t)db(\bar{s}) + \int_0^s Z_{2,\bar{s}}(t)d\bar{s} + \int_0^s O^h_{\bar{s}}(t)dY(\bar{s}). \tag{7.1.4}$$

Notation 7.1.4 *Let $O = O^h$, $\alpha = \alpha^h$, $w = w^h$, $C = C^h$, $R = R^h$, and let γ be a fixed admissible curve with $\gamma_0 = w_o$ and let $Y \equiv \frac{d}{d\epsilon}|_0 \gamma_\epsilon$.*

Define

$$\begin{aligned} A_t(Q) &\equiv \begin{bmatrix} C'(w(t))\langle \int Q_1 db + \int Q_2 ds \rangle O(t) \\ C'(w(t))\langle \int Q_1 db + \int Q_2 ds \rangle \alpha(t) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ R'(w(t))\langle \int Q_1 db + \int Q_2 ds \rangle \end{bmatrix}, \end{aligned}$$

and $K(t) \equiv \begin{bmatrix} C'(w(t))\langle \int O(t) dY \rangle O(t) \\ C'(w(t))\langle \int O(t) dY \rangle \alpha(t) + R'(w(t))\langle \int O(t) dY \rangle \end{bmatrix},$

where $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$ *is an $\{\mathcal{F}_s\}$-adapted* $\mathrm{End}(\mathbf{R}^n) \times \mathbf{R}^n$*-valued continuous process.*

The definition of A_t involves derivatives of C and R, so we will prove their existence first.

7.2 Properties of the Pulled-Back Kernels

In this section we show that C and R are differentiable (Theorems 7.2.3 and 7.2.5), and satisfy the appropriate Lipschitz properties (Theorem 7.2.19).

Notation 7.2.1 *For $w = \int O db + \int \alpha ds \in B^\infty \mathbf{R}^n$, let $u \equiv I(w)$ then $C(w) = \mathcal{A}(w) + \mathcal{T}(w)$ where $\mathcal{A}(w) \equiv \int \Omega_u \langle h, \delta w \rangle$ and $\mathcal{T}(w) \equiv \Theta_u \langle h, \cdot \rangle$, (see Definition 6.1.12). Notice that $u \to \Omega_u \langle \cdot, \cdot \rangle$ and $u \to \Theta_u \langle \cdot, \cdot \rangle$ are smooth fuctions.*

Remark 7.2.2 *By the Whitney imbedding theorem (see Ref. [Au] of [49]) we may view $O(M)$ as a compact imbedded submanifold of $\mathbf{R}^N$ for some $N < \infty$, and thus we may extend smooth functions on $O(M)$, in particular, the maps $u \to \Omega_u \langle \cdot, \cdot \rangle$ and $u \to \Theta_u \langle \cdot, \cdot \rangle$, to smooth functions with compact support in $\mathbf{R}^N$. This will allow us to apply several nonintrinsic results from [8].*

Theorem 7.2.3 *The map $C : B^\infty\mathbf{R}^n \to S^{\infty-}\mathrm{so}(n)$ (Notation 7.1.1) is $S^{\infty-}$-differentiable (see Def. 2.1.9).*

Furthermore, if γ is an admissible curve with $w = \gamma_0$, $v = \frac{d}{d\epsilon}|_0\gamma_\epsilon$ and $u = I(w)$, then

$$v_w C = \sum_{i=1}^{n}\{\int f_i'(u)\langle v_w I\rangle\delta w^i + \int f_i(u)\delta v^i\} + g'(u)\langle v_w I\rangle \tag{7.2.5}$$

where $f_i(\tilde{u}) \equiv \Omega_{\tilde{u}}\langle h, e_i\rangle$ and $g(\tilde{u}) \equiv \Theta_{\tilde{u}}\langle h, \cdot\rangle$ for $\tilde{u} \in O(M)$ and $i = 1, ..., n$. (See Definition 6.1.12).

This theorem will be proved by Theorems 7.2.4 and 7.2.19 below.

Theorem 7.2.4 *The map $t \to C(\gamma(t))$ is $S^{\infty-}$-differentiable (Def. 2.1.5) for any admissible curve γ. Furthermore, if $w = \gamma_0$, $v = \frac{d}{d\epsilon}|_0\gamma_\epsilon$ and $u = I(w)$, then (7.2.5) holds.*

Proof. Fix γ, w, v and u as above. Let $u_\epsilon = I(\gamma_\epsilon)$ for each ϵ. We have

$$\begin{aligned}
\mathcal{A}(\gamma_\epsilon) &= \int \Omega_{u_\epsilon}\langle h, \delta\gamma_\epsilon\rangle \\
&= \sum_{i=1}^{n}\int \Omega_{u_\epsilon}\langle h, e_i\rangle\delta\gamma_\epsilon^i \\
&= \sum_{i=1}^{n}\int G_i(\epsilon)\delta H_i(\epsilon)
\end{aligned}$$

where $G_i(\epsilon) \equiv \Omega_{u_\epsilon}\langle h, e_i\rangle$, and $H_i(\epsilon) \equiv \gamma_\epsilon^i$. Note that $G_i(\epsilon)$ and $H_i(\epsilon)$ are Brownian semimartingales for each ϵ, and $\epsilon \to \gamma_\epsilon$ is $B^{\infty-}$-differentiable, with $\epsilon \to \dot{\gamma}_\epsilon$ being $B^{\infty-}$-Lipschitz. By [8] Corollary 4.2, p. 313, $\epsilon \to u_\epsilon$ is $B^{\infty-}$-differentiable, and $\epsilon \to \dot{u}_\epsilon$ is $B^{\infty-}$-Lipschitz. Therefore G_i is $B^{\infty-}$-differentiable and the map $\epsilon \to G_i(\epsilon)$ is $B^{\infty-}$-Lipschitz by [8] Lemma 4.8, p. 317, and Remark 7.2.2. Moreover,

$$\dot{G}_i(\epsilon) = df_i\langle\dot{u}_\epsilon\rangle = f_i(u_\epsilon)\langle\dot{u}_\epsilon\rangle$$

where $f_i(\tilde{u}) \equiv \Omega_{\tilde{u}}\langle h, e_i\rangle$ for $\tilde{u} \in O(M)$. Using this we have by [8] Lemma 4.7, p. 316, that the $B^{\infty -}$-derivative of $\epsilon \to \mathcal{A}(\gamma_\epsilon)$ exists, the map $\epsilon \to \frac{d}{d\epsilon}[\mathcal{A}(\gamma_\epsilon)]$ is $B^{\infty -}$-Lipschitz, and

$$\begin{aligned}
&\tfrac{d}{d\epsilon}[\mathcal{A}(\gamma_\epsilon)] \\
&= \textstyle\sum_{i=1}^n \{\int \dot{G}_i(\epsilon)\delta H_i(\epsilon) + \int G_i(\epsilon)\delta \dot{H}_i(\epsilon)\} \\
&= \textstyle\sum_{i=1}^n \{\int f_i'(u_\epsilon)\langle \dot{u}_\epsilon\rangle \delta\gamma_\epsilon^i + \int f_i(u_\epsilon)\delta\dot{\gamma}_\epsilon^i\}.
\end{aligned}$$

Similarly, since $\tilde{u} \to g(\tilde{u}) \equiv \Theta_{\tilde{u}}\langle h, \cdot\rangle$ is a smooth function, and $\epsilon \to u_\epsilon$ is $B^{\infty -}$-differentiable, and $\epsilon \to \dot{u}_\epsilon$ is $B^{\infty -}$-Lipschitz, we have by [8] Lemma 4.8 and Remark 7.2.2 above, that $\epsilon \to \mathcal{T}(\gamma_\epsilon) = g(u_\epsilon)$ is $B^{\infty -}$-differentiable for all $p \in [2, \infty)$, with

$$\frac{d}{d\epsilon}[\mathcal{T}(\gamma_\epsilon)] \quad = \quad g'(u_\epsilon)\langle \dot{u}_\epsilon\rangle$$

and $\epsilon \to \frac{d}{d\epsilon}[\mathcal{T}(\gamma_\epsilon)]$ $B^{\infty -}$-Lipschitz.

Thus $\epsilon \to \frac{d}{d\epsilon}[\mathcal{A}(\gamma_\epsilon) + \mathcal{T}(\gamma_\epsilon)] = \frac{d}{d\epsilon}[C(\gamma_\epsilon)] = C'(\gamma_\epsilon)\langle\dot{\gamma}_\epsilon\rangle$ is $B^{\infty -}$-Lipschitz, with formula at $\epsilon = 0$ given by (7.2.5). Q.E.D.

Theorem 7.2.5 *The map $R : B^\infty\mathbf{R}^n \to S^\infty\mathbf{R}^n$ (Notation 7.1.1) is $S^{\infty -}$-differentiable (see Def. 2.1.9). Furthermore, if $\gamma_\epsilon = \int Q_\epsilon db + \int \beta_\epsilon ds$ is an admissible curve with $w = \int O db + \int \alpha ds = \gamma_0$, $v = \frac{d}{d\epsilon}|_0\gamma_\epsilon$ and $u = I(w)$, then*

$$\begin{aligned}
v_w R \quad = \quad & \tfrac{1}{2}[Ric_u'\langle v_w I, h, O, O\rangle + Ric_u\langle h, \dot{Q}_0, O\rangle + Ric_u\langle h, O, \dot{Q}_0\rangle] \\
& + \textstyle\sum_{i=1}^n [\tilde{g}'(u)\langle v_w I\rangle\langle Oe_i, Oe_i\rangle + \tilde{g}(u)\langle \dot{Q}_0 e_i, Oe_i\rangle \\
& + \tilde{g}(u)\langle Oe_i, \dot{Q}_0 e_i\rangle].
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\dot{\gamma}_0 R \quad \equiv \quad & \tfrac{d}{d\epsilon}\,|_0\; R(\gamma_\epsilon) \\
= \quad & \textstyle\sum_{i=1}^n [\tilde{f}'(u)\langle v_w I, Oe_i, Oe_i\rangle + \tilde{f}(u)\langle \dot{Q}_0 e_i\rangle Oe_i \\
& + \tilde{f}(u)\langle Oe_i\rangle \dot{Q}_0 e_i \\
& + \tilde{g}'(u)\langle v_w I\rangle\langle Oe_i, Oe_i\rangle + \tilde{g}(u)\langle \dot{Q}_0 e_i, Oe_i\rangle \\
& + \tilde{g}(u)\langle Oe_i, \dot{Q}_0 e_i\rangle]
\end{aligned} \tag{7.2.6}$$

where $\tilde{f}(\tilde{u}) \equiv \Omega_{\tilde{u}}\langle h, \cdot\rangle$, $\tilde{g}(\tilde{u}) \equiv \Theta^H_{\tilde{u}}\langle \cdot, h, \cdot\rangle$. *for all* $\tilde{u} \in O(M)$. *(See Notation 6.2.14). Also,* $\tilde{f}'(u)\langle \xi, a, b\rangle \equiv \xi(u \to \tilde{f}(u)\langle a\rangle b)$ *and* $\tilde{g}'(u)\langle \xi, a, b\rangle \equiv \xi(u \to \tilde{g}(u)\langle a, b\rangle)$ *for all* $\xi \in T_uO(M) \cong \mathbf{R}^N$ *and* $a, b \in \mathbf{R}^n$.

This theorem will be proved by Theorems 7.2.6 and 7.2.19 below.

Theorem 7.2.6 *The map* $t \to R(\gamma(t))$ *is* $S^{\infty-}$*-differentiable (Def. 2.1.5) for any admissible curve* γ.

Furthermore, if $\gamma_\epsilon = \int Q_\epsilon db + \int \beta_\epsilon ds$ *is an admissible curve with* $w = \int Odb + \int \alpha ds = \gamma_0$, $v = \frac{d}{d\epsilon}|_0 \gamma_\epsilon$ *and* $u = I(w)$, *then (7.2.6) holds.*

Proof. Let $u_\epsilon = I(\gamma_\epsilon)$ for each ϵ. We have

$$R(\gamma_\epsilon) \equiv \frac{1}{2}\{Ric_{u_\epsilon}\langle h, Q_\epsilon, Q_\epsilon\rangle + \bar{\Theta}_{u_\epsilon}\langle Q_\epsilon, h, Q_\epsilon\rangle\} + h'$$

$$\begin{aligned} \text{where } Ric_u\langle h, O, O\rangle &\equiv \sum_{i=1}^n \Omega_u\langle h, Oe_i\rangle Oe_i, \\ \bar{\Theta}_u\langle O, h, O\rangle &\equiv \sum_{i=1}^n \Theta^H_u\langle Oe_i, h, Oe_i\rangle, \end{aligned}$$

and Θ^H_u is the horizontal derivative of Θ_u as defined in Notation 6.2.14.
Let $F : \mathbf{R} \to \mathrm{Hom}(\mathbf{R}^n, \mathrm{End}(\mathbf{R}^n))$ by $F(\epsilon) \equiv \Omega_{u_\epsilon}\langle h, \cdot\rangle$. Now $\tilde{u} \to \Omega_{\tilde{u}}\langle h, \cdot\rangle \equiv \tilde{f}(\tilde{u})$ is a smooth function with compact support in $\mathbf{R}^N$, and by [8] Corollary 4.2, p. 313, $\epsilon \to u_\epsilon$ is $B^{\infty-}$-differentiable with $\epsilon \to \dot{u}_\epsilon$ being $B^{\infty-}$-Lipschitz. Thus by [8] Lemma 4.8, p. 317, and Remark 7.2.2, $F(\epsilon)$ is $B^{\infty-}$-differentiable, and hence is $S^{\infty-}\mathrm{End}(\mathbf{R}^n)$-differentiable, with $\dot{F}(\epsilon) = \tilde{f}'(u_\epsilon)\langle \dot{u}_\epsilon\rangle$ being $S^{\infty-}$-Lipschitz.

Now by the definition of the B^p-norm, $\epsilon \to \gamma_\epsilon$ being $B^{\infty-}$-differentiable with $\epsilon \to \dot{\gamma}_\epsilon$ being $B^{\infty-}$-Lipschitz implies $\epsilon \to Q_\epsilon$ is $S^{\infty-}$-differentiable $\epsilon \to \dot{Q}_\epsilon$ being $S^{\infty-}$-Lipschitz. Thus by repeated applications of [8] Lemma 4.6 (Product Rule), p. 314, $\epsilon \to F(\epsilon)\langle Q_\epsilon e_i\rangle Q_\epsilon e_i = \Omega_{u_\epsilon}\langle h, Q_\epsilon e_i\rangle Q_\epsilon e_i$ is $S^{\infty-}$-differentiable, with $\epsilon \to \frac{d}{d\epsilon}[F(\epsilon)\langle Q_\epsilon e_i\rangle Q_\epsilon e_i]$ being $S^{\infty-}$-Lipschitz.

Also $\frac{d}{d\epsilon}[F(\epsilon)\langle Q_\epsilon e_i\rangle] = \dot{F}(\epsilon)\langle Q_\epsilon e_i\rangle + F(\epsilon)\langle \dot{Q}_\epsilon e_i\rangle$ and

$$\begin{aligned}\frac{d}{d\epsilon}[F(\epsilon)\langle Q_\epsilon e_i\rangle Q_\epsilon e_i] &= [\dot{F}(\epsilon)\langle Q_\epsilon e_i\rangle + F(\epsilon)\langle \dot{Q}_\epsilon e_i\rangle]Q_\epsilon e_i \\ &\quad + F(\epsilon)\langle Q_\epsilon e_i\rangle \dot{Q}_\epsilon e_i \\ &= [\tilde{f}'(u_\epsilon)\langle \dot{u}_\epsilon\rangle\langle Q_\epsilon e_i\rangle + \tilde{f}(u_\epsilon)\langle \dot{Q}_\epsilon e_i\rangle]Q_\epsilon e_i \\ &\quad + \tilde{f}(u_\epsilon)\langle Q_\epsilon e_i\rangle \dot{Q}_\epsilon e_i\end{aligned}$$

Define $G(\epsilon) \equiv \Theta^H_{u_\epsilon}\langle \cdot, h, \cdot\rangle$. Using [8] Corollary 4.2 and Lemma 4.8 as above, (with $\tilde{g}(\tilde{u}) \equiv \Theta^H_{\tilde{u}}\langle \cdot, h, \cdot\rangle$ and $G(\epsilon)$ replacing $\tilde{f}(\tilde{u})$ and $F(\epsilon)$ respectively), $\epsilon \to G(u_\epsilon)$ is $S^{\infty-}$-differentiable, with derivative given by $\dot{G}(\epsilon) = \tilde{g}'(u_\epsilon)\langle \dot{u}_\epsilon\rangle$, and $\epsilon \to \dot{G}(\epsilon)$ being $S^{\infty-}$-Lipschitz.

Also as above $\epsilon \to Q_\epsilon$ is $S^{\infty-}$-differentiable, so by repeated applications of [8] Lemma 4.6, p. 314, $\epsilon \to G(\epsilon)\langle Q_\epsilon e_i, Q_\epsilon e_i\rangle = \Theta^H_{u_\epsilon}\langle Q_\epsilon e_i, h, Q_\epsilon e_i\rangle$ is $S^{\infty-}$-differentiable, with

$$\begin{aligned}\frac{d}{d\epsilon}[G(\epsilon)\langle Q_\epsilon e_i, Q_\epsilon e_i\rangle] &= \dot{G}(\epsilon)\langle Q_\epsilon e_i, Q_\epsilon e_i\rangle + G(\epsilon)\langle \dot{Q}_\epsilon e_i, Q_\epsilon e_i\rangle \\ &\quad + G(\epsilon)\langle Q_\epsilon e_i, \dot{Q}_\epsilon e_i\rangle. \\ &= \tilde{g}'(u_\epsilon)\langle \dot{u}_\epsilon\rangle\langle Q_\epsilon e_i, Q_\epsilon e_i\rangle + \tilde{g}(u_\epsilon)\langle \dot{Q}_\epsilon e_i, Q_\epsilon e_i\rangle \\ &\quad + \tilde{g}(u_\epsilon)\langle Q_\epsilon e_i, \dot{Q}_\epsilon e_i\rangle\end{aligned}$$

and $\epsilon \to \frac{d}{d\epsilon}[G(\epsilon)\langle Q_\epsilon e_i, Q_\epsilon e_i\rangle]$ is $S^{\infty-}$-Lipschitz.

Thus $\epsilon \to \frac{d}{d\epsilon}[R(\gamma_\epsilon)] = R'(\gamma_\epsilon)\langle \dot{\gamma}_\epsilon\rangle$ is $S^{\infty-}$-Lipschitz, with formula at $\epsilon = 0$ given by (7.2.6). Q.E.D.

Remark 7.2.7 *Let w be a Brownian semimartingale and $u \equiv I(w)$ (Notation 6.2.10), that is, u solves $\delta u = B\langle \delta w\rangle(u)$, so by the definition of the Stratonovich stochastic differential equation (Def. 6.2.6) we have $d[f(u)] = df\langle B\langle \delta w\rangle(u)\rangle \equiv f^H(u)\langle \delta w\rangle$ for all $f \in C^\infty(O(M))$.*

Theorem 7.2.8 *The map $(w, v) \to v_w I$ satisfies the following conditions for all $p, p_1, p_2 \in [2, \infty)$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and for all w,$\tilde{w}$,$v \in B^\infty \mathbf{R}^n$:*

i) $\|v_w I\|_{B^p} \le K\|v\|_{B^p}$

ii) $\|v_w I - v_{\tilde{w}} I\|_{S^p} \le K\|w-\tilde{w}\|_{B^{p_1}}\|v\|_{B^{p_2}}$

where $K = K(n, p_1, p_2, \bar{K}_w, \bar{K}_{\tilde{w}})$.

Proof of i). Let γ be an admissible curve with $w = \gamma_0$ and $v = \frac{d}{d\epsilon}|_0 \gamma_\epsilon$. (For example, we could take $\gamma_\epsilon = w + \epsilon v$ for all ϵ.)

Let $u_\epsilon = I(\gamma_\epsilon)$ for each ϵ. For all $\epsilon \in (-\delta, \delta)$ we have $K_\delta = K_\delta(n, p, \sup_{\epsilon\in(-\delta,\delta)} \|\gamma_\epsilon\|_{B^\infty})$ such that

$$\|\frac{u_\epsilon - u_0}{\epsilon}\|_{B^p} \le K_\delta \|\frac{\gamma_\epsilon - \gamma_0}{\epsilon}\|_{B^p} \text{ by [8] Cor. 4.1, p. 306.}$$

Since $\dot{u}_0$ and $\dot{\gamma}_0$ both exist, by taking the limit as $\epsilon \to 0$ we have $\|\dot{u}_0\|_{B^p} \le K_\delta \|\dot{\gamma}_0\|_{B^p}$. Now let $K = \limsup_{\delta\to 0} K_\delta$.

Proof of ii). Let $\gamma : J \to B^\infty\mathbf{R}^n$ and $\tilde{\gamma} : J \to B^\infty\mathbf{R}^n$ be admissible curves (Def. 2.1.7) with $\dot{\gamma}_0 = v_w$ and $\dot{\tilde{\gamma}}_0 = v_{\tilde{w}}$. (For example, we could take $\gamma_\epsilon = w + \epsilon v$ and $\tilde{\gamma}_\epsilon = \tilde{w} + \epsilon v$ for all ϵ.)

Let $u = I(w)$, $\tilde{u} = I(\tilde{w})$, $u_\epsilon = I(\gamma_\epsilon)$ and $\tilde{u}_\epsilon = I(\tilde{\gamma}_\epsilon)$ for each $\epsilon \in J$.

Let $\xi \equiv \dot{u}_0 = v_w I$ and $\tilde{\xi} \equiv \dot{\tilde{u}}_0 = v_{\tilde{w}} I$. By [8], Corollary 4.2, p. 313, ξ and $\tilde{\xi}$ respectively solve the stochastic differential equations:

$$\begin{aligned} d\xi &= Z(u)\delta v + Z'(u)\langle\xi\rangle\delta w \\ \text{and } d\tilde{\xi} &= Z(\tilde{u})\delta v + Z'(\tilde{u})\langle\tilde{\xi}\rangle\delta\tilde{w}, \\ \text{with } \quad & \xi(s)|_{s=0} = \tilde{\xi}(s)|_{s=0} = 0 \end{aligned}$$

where $Z : O(M) \to \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^N)$ is defined by $Z(\bar{u}) \equiv B\langle\cdot\rangle(\bar{u})$ for all $\bar{u} \in O(M)$, so Z is a smooth function (see Lemma 8.1.9 in the appendix). Here we have identified $T_{\bar{u}}O(M)$ with $\mathbf{R}^N$ for each $\bar{u} \in O(M)$. Thus by Remark 7.2.2 we may view Z as a smooth function with compact support on $\mathbf{R}^N$.

Using the Itô differential the equation for ξ becomes

$$\begin{aligned} d\xi &= Z(u)dv + \frac{1}{2}Z'(u)\langle Z(u)dw\rangle dv + Z'(u)\langle\xi\rangle dw \\ &\quad + \frac{1}{2}Z''(u)\langle\xi, Z(u)dw\rangle dw. \\ \\ &= Z(u)dv + \frac{1}{2}V_1(u)[O\otimes A]ds + Z'(u)\langle\xi\rangle dw \\ &\quad + \frac{1}{2}V_2(u)\langle\xi\rangle[O\otimes O]ds \end{aligned}$$

where $w = \int Odb + \int \alpha ds$, $v = \int Adb + \int ads$, $V_1(u)[O\otimes A] \equiv \sum_{i=1}^n Z'(u)\langle Z(u)Oe_i\rangle Ae_i$ and $V_2(u)\langle\xi\rangle[O\otimes O] \equiv \sum_{i=1}^n Z''(u)\langle\xi, Z(u)Oe_i\rangle Oe_i$. Here $\{e_i\}_{i=1}^n$ is the standard basis on $\mathbf{R}^n$. So the maps $u \to V_1(u)$ and $u \to V_2(u)$ may also be viewed as smooth functions with compact support on $\mathbf{R}^N$.

Thus we have

$$\begin{aligned} \|\xi - \tilde{\xi}\|^p_{S^p(s)} &\le K\|\int[Z(u) - Z(\tilde{u})]dv\|^p_{S^p(s)} \\ &\quad + K\|\int Z'(u)\langle\xi\rangle dw - \int Z'(\tilde{u})\langle\tilde{\xi}\rangle d\tilde{w}\|^p_{S^p(s)} \\ &\quad + K\|\int V_1(u)[O\otimes A]ds - \int V_1(\tilde{u})[\tilde{O}\otimes A]ds\|^p_{S^p(s)} \\ &\quad + K\|\int V_2(u)\langle\xi\rangle[O\otimes O]ds - \int V_2(\tilde{u})\langle\tilde{\xi}\rangle[\tilde{O}\otimes\tilde{O}]ds\|^p_{S^p(s)} \end{aligned}$$

We will consider each of these four terms separately.

Term I.

$$\begin{aligned} \|\int[Z(u) - Z(\tilde{u})]dv\|^p_{S^p(s)} &\le K\|v\|^p_{B^{p_2}(s)}\|Z(u) - Z(\tilde{u})\|^p_{S^{p_1}(s)} \text{ by Lemma 8.1.6} \\ &\le K_p\|w - \tilde{w}\|^p_{B^{p_1}(s)}\|v\|^p_{B^{p_2}(s)} \text{ by [8] Cor. 4.1, p. 306.} \end{aligned}$$

Term II.

$$\begin{aligned} &\|\int Z'(u)\langle\xi\rangle dw - \int Z'(\tilde{u})\langle\tilde{\xi}\rangle d\tilde{w}\|^p_{S^p(s)} \\ &\qquad \le K\|\int Z'(u)\langle\xi\rangle - Z'(\tilde{u})\langle\tilde{\xi}\rangle dw\|^p_{S^p(s)} \\ &\qquad\quad + K\|\int Z'(\tilde{u})\langle\tilde{\xi}\rangle d[w - \tilde{w}]\|^p_{S^p(s)} \end{aligned}$$

$$
\begin{aligned}
&\leq K\|w\|^p_{B^\infty(s)}\|\int_0^s \|Z'(u)\langle\xi\rangle - Z'(\tilde{u})\langle\tilde{\xi}\rangle\|^p_{S^p(r)}dr \text{ by [8] Lemma 4.1(iv)}\\
&\qquad +K\|Z'(\tilde{u})\langle\tilde{\xi}\rangle\|^p_{S^{p_2}(s)}\|w-\tilde{w}\|^p_{B^{p_1}(s)} \text{ by Lemma 8.1.6}\\
&\leq K\int_0^s \|[Z'(u) - Z'(\tilde{u})]\langle\xi\rangle\|^p_{S^p(r)}dr\\
&\qquad +K\int_0^s \|Z'(\tilde{u})\langle\xi-\tilde{\xi}\rangle\|^p_{S^p(r)}dr\\
&\qquad +K\|\tilde{\xi}\|^p_{S^{p_2}(s)}\|\|w-\tilde{w}\|^p_{B^{p_1}(s)} \text{ since } |Z'| \text{ is uniformly bounded}\\
&\leq K\|Z'(u) - Z'(\tilde{u})\|^p_{S^{p_1}(s)}\|\xi\|^p_{B^{p_2}(s)} \text{ by Lemma 8.1.6}\\
&\qquad +K\int_0^s \|\xi-\tilde{\xi}\|^p_{S^p(r)}dr \text{ since } |Z'| \text{ is uniformly bounded}\\
&\qquad +K\|w-\tilde{w}\|^p_{B^{p_1}(s)}\|v\|^p_{B^{p_2}(s)} \text{ by Theorem 7.2.8 (i)}\\
&\leq K_p\|w-\tilde{w}\|^p_{B^{p_1}(s)}\|v\|^p_{B^{p_2}(s)}\\
&\qquad +K\int_0^s \|\xi-\tilde{\xi}\|^p_{S^p(r)}dr
\end{aligned}
$$

The last inequality follows by the Lipschitz property of Z', [8] Cor. 4.1 p. 306, and Theorem 7.2.8 (i).

Term III.

Again using Theorem 7.2.8 (i) and the Lipschitz properties of $u \to V_1(u)$ we have

$$
\begin{aligned}
&\|\int V_1(u)[O\otimes A]ds - \int V_1(\tilde{u})[\tilde{O}\otimes A]ds\|^p_{S^p(s)}\\
&\qquad \leq K\int_0^s \|(V_1(u) - V_1(\tilde{u}))[O\otimes A]\|^p_{S^p(r)}dr\\
&\qquad\qquad +K\int_0^s \|V_1(\tilde{u})[(O-\tilde{O})\otimes A]\|^p_{S^p(r)}dr\\
&\qquad \leq K\|V_1(u) - V_1(\tilde{u})\|^p_{S^{p_1}(s)}\|w\|^p_{B^\infty}\|v\|^p_{B^{p_2}(s)}\\
&\qquad\qquad +K\|V_1(\tilde{u})[(O-\tilde{O})\otimes A]\|^p_{S^p(s)}\\
&\qquad \leq K\|w-\tilde{w}\|^p_{B^{p_1}(s)}\|v\|^p_{B^{p_2}(s)}
\end{aligned}
$$

Term IV.

Similar reasoning shows:

$$
\begin{aligned}
K\|\int V_2(u)\langle\xi\rangle[O\otimes O]ds - \int V_2(\tilde{u})\langle\tilde{\xi}\rangle[\tilde{O}\otimes\tilde{O}]ds\|^p_{S^p(s)} \\
&\leq K\|[V_2(u)-V_2(\tilde{u})]\langle\xi\rangle[O\otimes O]\|^p_{S^p(s)} \\
&\quad +K\int_0^s \|V_2(\tilde{u})\langle\xi-\tilde{\xi}\rangle[O\otimes O]\|^p_{S^p(r)}dr \\
&\quad +K\|V_2(\tilde{u})\langle\tilde{\xi}\rangle[(O-\tilde{O})\otimes O]\|^p_{S^p(s)} \\
&\quad +K\|V_2(\tilde{u})\langle\tilde{\xi}\rangle[\tilde{O}\otimes(O-\tilde{O})]\|^p_{S^p(s)} \\
&\leq K\|V_2(u)-V_2(\tilde{u})\|^p_{S^{p_1}(s)}\|\xi\|^p_{S^{p_2}(s)} \\
&\quad +K\int_0^s \|\xi-\tilde{\xi}\|^p_{S^p(r)}dr \\
&\quad +K\|\tilde{\xi}\|^p_{B^{p_2}(s)}\|O-\tilde{O}\|^p_{S^{p_1}(s)} \\
&\leq K\|w-\tilde{w}\|^p_{B^{p_1}(s)}\|v\|^p_{B^{p_2}(s)} + K\int_0^s \|\xi-\tilde{\xi}\|^p_{S^p(r)}dr
\end{aligned}
$$

Thus, combining the four terms above gives

$$
\|\xi-\tilde{\xi}\|^p_{S^p(s)} \leq K\|w-\tilde{w}\|^p_{B^{p_1}(s)}\|v\|^p_{B^{p_2}(s)} + K\int_0^s \|\xi-\tilde{\xi}\|^p_{S^p(r)}dr.
$$

Now Gronwall's inequality (Lemma 8.1.7) gives the result. Q.E.D.

Definition 7.2.9 *Given a map $f : B^\infty\mathbf{R}^n \times B^\infty\mathbf{R}^n \to S^{\infty-}V$, we say that f satisfies the B^{p+}*-condition *(or* is *B^{p+}) if for all $p \in [2,\infty)$, $w, \tilde{w}, v \in B^\infty\mathbf{R}^n$ we have*

$$
\begin{aligned}
\|f(w,v)-f(\tilde{w},v)\|_{S^pV} &\leq K\|w-\tilde{w}\|_{B^{p_1}\mathbf{R}^n}\|v\|_{B^{p_2}\mathbf{R}^n} \\
\text{and } \|f(w,v)\|_{S^pV} &\leq K\|v\|_{B^p\mathbf{R}^n}
\end{aligned}
$$

where $K = K(n,p_1,p_2,\bar{K}_w,\bar{K}_{\tilde{w}})$ and $p_1,p_2 \in [2,\infty)$ are such that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$.

Lemma 7.2.10 *Suppose $F : B^\infty\mathbf{R}^n \to S^\infty\mathrm{Hom}(\mathbf{R}^N,V)$ satisfies*

$$
\|F(w)-F(\tilde{w})\|_{S^p\mathrm{Hom}(\mathbf{R}^N,V)} \leq K\|w-\tilde{w}\|_{B^p\mathbf{R}^n}
$$

for all $p \in [2,\infty)$,

and $G : B^\infty \mathbf{R}^n \times B^\infty \mathbf{R}^n \to S^{\infty-}\mathbf{R}^N$ *is* B^{p+}, *i.e., there exists* $K = K(n, p_1, p_2, \bar{K}_w, \bar{K}_{\tilde{w}})$ *such that*

$$\begin{aligned} \|G(w,v) - G(\tilde{w},v)\|_{S^p\mathbf{R}^N} &\leq K\|w-\tilde{w}\|_{B^{p_1}\mathbf{R}^n}\|v\|_{B^{p_2}\mathbf{R}^n} \\ \text{and } \|G(w,v)\|_{S^p\mathbf{R}^N} &\leq K\|v\|_{B^p\mathbf{R}^n} \end{aligned}$$

for all $p, p_1, p_2 \in [2,\infty)$ *with* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.
Then $H : B^\infty \mathbf{R}^n \times B^\infty \mathbf{R}^n \to S^{\infty-}V$ *defined by* $H(w,v) \equiv F(w)G(w,v)$ *is also* B^{p+}, *i.e.,*

$$\begin{aligned} \|H(w,v) - H(\tilde{w},v)\|_{S^pV} &\leq K\|w-\tilde{w}\|_{B^{p_1}\mathbf{R}^n}\|v\|_{B^{p_2}\mathbf{R}^n} \\ \text{and } \|H(w,v)\|_{S^pV} &\leq K\|v\|_{B^p\mathbf{R}^n} \end{aligned}$$

for all $p, p_1, p_2 \in [2,\infty)$ *such that* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. *Here* $K = K(n, p_1, p_2, \bar{K}_w, \bar{K}_{\tilde{w}}, \|F\|_{S^\infty})$.

Proof. We have

$$\begin{aligned} \|H(w,v) - H(\tilde{w},v)\|_{S^p} &\leq \|[F(w) - F(\tilde{w})]G(w,v)\|_{S^p} \\ &\quad + \|F(\tilde{w})[G(w,v) - G(\tilde{w},v)]\|_{S^p} \\ &\leq \|F(w) - F(\tilde{w})\|_{S^{p_1}}\|G(w,v)\|_{S^{p_2}} \\ &\quad + \|F(\tilde{w})\|_{S^\infty}\|G(w,v) - G(\tilde{w},v)\|_{S^p} \text{ by Lemma 8.1.6} \\ &\leq K\|F(w) - F(\tilde{w})\|_{S^{p_1}}\|v\|_{B^{p_2}} \\ &\quad + K\|G(w,v) - G(\tilde{w},v)\|_{S^p} \\ &\leq K\|w-\tilde{w}\|_{B^{p_1}}\|v\|_{B^{p_2}}. \end{aligned}$$

Also

$$\begin{aligned} \|H(w,v)\|_{S^p} &\leq \sup_{w \in B^\infty \mathbf{R}^n} \|F(w)\|_{S^\infty}\|G(w,v)\|_{S^p} \\ &\leq K\|v\|_{B^p}. \end{aligned}$$

Q.E.D.

Definition 7.2.11 *A function $F : B^\infty \mathbf{R}^n \to B^{\infty-}V$ is called B^p-*Lipschitz *if*

$$\|F(w) - F(\tilde{w})\|_{B^p} \le K\|w - \tilde{w}\|_{B^p}$$

for all $w, \tilde{w} \in B^\infty \mathbf{R}^n$, where $p \in [2,\infty)$ and $K = K(n, p, \|w\|_{B^\infty}, \|\tilde{w}\|_{B^\infty})$.

Lemma 7.2.12 *Suppose $F : B^\infty \mathbf{R}^n \to B^\infty \mathrm{Hom}(\mathbf{R}^n, V)$ is also S^∞-bounded and is B^p-Lipschitz for all $p \in [2,\infty)$.*

Then $H : B^\infty \mathbf{R}^n \times B^\infty \mathbf{R}^n \to S^{\infty-}V$ defined by $H(w,v) \equiv \int F(w)\delta v$ is B^{p+}, i.e.,

$$\begin{aligned}
\|H(w,v) - H(\tilde{w},v)\|_{S^pV} &\le K\|w - \tilde{w}\|_{B^{p_1}\mathbf{R}^n}\|v\|_{B^{p_2}\mathbf{R}^n} \\
\text{and } \|H(w,v)\|_{S^pV} &\le K\|v\|_{S^p\mathbf{R}^n}
\end{aligned}$$

where $K = K(n, p_1, p_2, \bar{K}_w, \bar{K}_{\tilde{w}})$.

Proof. Since $F(w)$ and v are Brownian semimartingales we may write

$$\begin{aligned}
v &= \int A db + \int a ds \\
\text{and } F(w) &= \int B(w) db + \int \beta(w) ds
\end{aligned}$$

Then $d[F(w)]dw = \sum_{i=1}^n B(w)e_i A e_i ds$, and

$$\begin{aligned}
\|B(w) - B(\tilde{w})\|_{S^p} &\le \|F(w) - F(\tilde{w})\|_{B^p} \\
&\le K\|w - \tilde{w}\|_{B^p}.
\end{aligned}$$

We have

$$\begin{aligned}
\int F(w)\delta v &= \int F(w)dv + \frac{1}{2}\int d[F(w)]dv \\
&= \int F(w)dv + \frac{1}{2}\sum_{i=1}^n \int B(w)e_i A e_i ds.
\end{aligned}$$

Thus

$$\begin{aligned}
\|H(w,v) - H(\tilde{w},v)\|_{S^p} &\le \|\int F(w)dv - \int F(\tilde{w})dv\|_{B^p} \\
&\quad + \frac{1}{2}\sum_{i=1}^n \|\int [B(w) - B(\tilde{w})]e_i A e_i ds\|_{B^p}.
\end{aligned}$$

Now

$$\begin{aligned}\|\int F(w)dv - \int F(\tilde{w})dv\|_{B^p} &= \|\int [F(w) - F(\tilde{w})]dv\|_{B^p} \\ &\le \|F(w) - F(\tilde{w})\|_{S^{p_1}}\|v\|_{B^{p_2}} \text{ by Lemma 8.1.6} \\ &\le K\|w - \tilde{w}\|_{B^{p_1}}\|v\|_{B^{p_2}}.\end{aligned}$$

Also

$$\begin{aligned}&\|\textstyle\int [B(w) - B(\tilde{w})]e_i A e_i ds\|_{B^p} \\ &\quad = \|[B(w) - B(\tilde{w})e_i]Ae_i\|_{S^p} \\ &\quad \le \|[B(w) - B(\tilde{w})]e_i\|_{S^{p_1}}\|Ae_i\|_{S^{p_2}} \text{ by Lemma 8.1.5} \\ &\quad \le K\|w - \tilde{w}\|_{B^{p_1}}\|v\|_{B^{p_2}}.\end{aligned}$$

Finally we have

$$\begin{aligned}\|H(w,v)\|_{S^p} &= \|\int F(w)\delta v\|_{S^p} \\ &\le \|F(w)\|_{S^\infty}\|v\|_{B^p} \text{ by Lemma 8.1.6} \\ &\le K\|v\|_{B^p}.\end{aligned}$$

Q.E.D.

Notation 7.2.13 *Let $w, v \in B^\infty\mathbf{R}^n$, $u = I(w)$ as given in Notation 6.2.10, and $\xi = v_w I$. Then by [8], Corollary 4.2, p. 313, we may write*

$$\begin{aligned}du &= Z(u)\delta w \text{ with } u(0) = u_o \\ \text{and } d\xi &= Z(u)\delta v + Z'(u)\langle\xi\rangle\delta w \text{ with } \xi(s)|_{s=0} = 0\end{aligned}$$

where $Z : O(M) \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^N)$ is defined by $Z(\bar{u}) \equiv B\langle\cdot\rangle(\bar{u})$ for all $\bar{u} \in O(M)$, so Z may be viewed as a smooth function with compact support in $\mathbf{R}^N$ (see Lemma 8.1.9 in the appendix). Here we have identified $T_{\bar{u}}O(M)$ with $\mathbf{R}^N$ for each $\bar{u} \in O(M)$.

Lemma 7.2.14 *Let $F : \mathbf{R}^N \to V$ be a smooth function with compact support and suppose $I : B^\infty\mathbf{R}^n \to B^{\infty-}\mathbf{R}^N$ is B^p-Lipschitz for all $p \in [2, \infty)$. Then $F \circ I : B^\infty\mathbf{R}^n \to B^{\infty-}V$ is also B^p-Lipschitz for all $p \in [2, \infty)$.*

Proof. Let $p \in [2,\infty)$, $w = \int Odb + \int \alpha ds \in B^\infty \mathbf{R}^n$, $\tilde{w} = \int \tilde{O}db + \int \tilde{\alpha} ds \in B^\infty \mathbf{R}^n$, $u = I(w)$ and $\tilde{u} = I(\tilde{w})$.

By Notation 7.2.13, u solves

$$du = Z(u)\delta w$$

where $Z : O(M) \to \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^N)$ is a smooth function with compact support in $\mathbf{R}^N$.

Thus, using Itô's formula we have:

$$\begin{aligned} d[F(u)] &= F'(u)\langle du\rangle + \frac{1}{2}F''(u)\langle du, du\rangle \\ &= G(u)dw + H(u)\langle O, O\rangle ds \end{aligned}$$

where

$$\begin{aligned} G(u)dw &\equiv F'(u)\langle Z(u)dw\rangle \\ H(u)\langle O, O\rangle &\equiv \frac{1}{2}\sum_{i=1}^{n} F''(u)\langle Z(u)Oe_i, Z(u)Oe_i\rangle. \end{aligned}$$

In the following K will be a constant (which will vary from line to line), depending on p, $\|w\|_{B^\infty}$, $\|\tilde{w}\|_{B^\infty}$, and the Lipschitz constants of F, F', and F''.

$$\begin{aligned} \|F(u) - F(\tilde{u})\|_{B^p} &\leq K\|\int [G(u) - G(\tilde{u})]dw\|_{B^p} \\ &\quad + K\|\int G(\tilde{u})d[w - \tilde{w}]\|_{B^p} \\ &\quad + K\|\int [H(u) - H(\tilde{u})]\langle O, O\rangle ds\|_{B^p} \\ &\quad + K\|\int H(\tilde{u})\langle \tilde{O} - O, O\rangle ds\|_{B^p} \\ &\quad + K\|\int H(\tilde{u})\langle \tilde{O}, \tilde{O} - O\rangle ds\|_{B^p} \\ &\leq K\|G(u) - G(\tilde{u})\|_{S^p}\|w\|_{B^\infty} \\ &\quad + K\|G(\tilde{u})\|_{S^\infty}\|w - \tilde{w}]\|_{B^p} \\ &\quad + K\|H(u) - H(\tilde{u})\|_{S^p} \\ &\quad + 2K\|H(\tilde{u})\|_{S^\infty}\|\int [\tilde{O} - O]ds\|_{B^p} \end{aligned}$$

$$
\begin{aligned}
&\leq K\|w-\tilde{w}]\|_{B^p}+K\|O-\tilde{O}\|_{S^p} \\
&\leq K\|w-\tilde{w}]\|_{B^p}
\end{aligned}
$$

Q.E.D.

Theorem 7.2.15 *The function $C' : B^\infty\mathbf{R}^n \times B^\infty\mathbf{R}^n \to S^{\infty-}V$ defined in Equation (7.2.5) satisfies the B^{p+} condition (Definition 7.2.9), that is, for all $w, \tilde{w}, v \in B^\infty\mathbf{R}^n$ we have*

$$
\begin{aligned}
\|v_w C\|_{S^p} &\leq K\|v\|_{B^p} \\
\text{and } \|v_w C - v_{\tilde{w}} C\|_{S^p} &\leq K\|w-\tilde{w}\|_{B^{p_1}}\|v\|_{B^{p_2}}
\end{aligned}
$$

where $p, p_1, p_2 \in [2,\infty)$ such that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$ and $K = K(n, p_1, p_2, \bar{K}_w, \bar{K}_{\tilde{w}})$.

Proof. Let $\gamma : J \to B^\infty\mathbf{R}^n$ and $\tilde{\gamma} : J \to B^\infty\mathbf{R}^n$ be admissible curves (Def. 2.1.7) with $\dot{\gamma}_0 = v_w$ and $\dot{\tilde{\gamma}}_0 = v_{\tilde{w}}$. (For example, we could take $\gamma_\epsilon = w + \epsilon v$ and $\tilde{\gamma}_\epsilon = \tilde{w} + \epsilon v$ for all ϵ.) Let $u = I(w)$, $\tilde{u} = I(\tilde{w})$, and $u_\epsilon = I(\gamma_\epsilon)$ and $\tilde{u}_\epsilon = I(\tilde{\gamma}_\epsilon)$ for each $\epsilon \in \mathbf{R}$.

By Theorem 7.2.3 we have

$$
\begin{aligned}
v_w C &= \sum_{i=1}^{n}\{\int f_i'(u)\langle v_w I\rangle \delta w^i + \int f_i(u)\delta v^i\} + g'(u)\langle v_w I\rangle \\
&= \mathbf{F}(w)v + \mathbf{G}(w)v + \mathbf{H}(w)v
\end{aligned}
$$

where $f_i(\bar{u}) \equiv \Omega_{\bar{u}}\langle h, e_i\rangle \in so(n)$ and $g(\bar{u}) \equiv \Theta_{\bar{u}}\langle h, \cdot\rangle \in so(n)$ for $i = 1, ..., n$, and $\bar{u} \in O(M)$. (Recall we are assuming that ∇ is Torsion Skew Symmetric.)

By Remark 7.2.2 we may view $O(M)$ as a compact submanifold of $\mathbf{R}^N$ for some N, and thus extend $\{f_i\}_{i=1}^n$ and g to smooth functions with compact support on $\mathbf{R}^N$.

The proof will consist of obtaining bounds for each of the terms $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ of the form:

$$
\begin{aligned}
\|\mathbf{F}(w)v\|_{S^p} &\leq K\|v\|_{B^p} \\
\text{and } \|\mathbf{F}(w)v - \mathbf{F}(\tilde{w})v\|_{S^p} &\leq K\|w-\tilde{w}\|_{B^{p_1}}\|v\|_{B^{p_2}}
\end{aligned}
$$

The F Term.

Write $\Phi = f_i'$. We have:

$$\begin{aligned}\int \Phi(u)\langle v_w I\rangle \delta w &= \int \Phi(u)\langle v_w I\rangle dw + \frac{1}{2}\int d[\Phi(u)\langle v_w I\rangle] dw \\ &= \int \Phi(u)\langle v_w I\rangle dw \\ &\quad + \frac{1}{2}\int \Phi'(u)\langle Z(u)dw, v_w I\rangle dw \\ &\quad + \frac{1}{2}\int \Phi(u)\langle d[v_w I]\rangle dw\end{aligned}$$

We will consider each of these three terms separately.

I) By [8] Cor. 4.1, p. 306, the map $w \to u = I(w)$ is B^p-Lipschitz, so $w \to \Phi(u)$ is B^p-Lipschitz by Lemma 7.2.14. Also, since u is $O(n)$-valued, and $\Phi : \mathbf{R}^N \to \mathbf{Hom}(\mathbf{R}^N, \mathrm{so}(n))$ is bounded (as a smooth function with compact support in $\mathbf{R}^N$), $\Phi(u)$ is S^∞-bounded by a constant independent of u. Also, Theorem 7.2.8 implies that the map $(w, v) \to v_w I$ satisfies the B^{p+}-condition. Thus by Lemma 7.2.10 the function $(w, v) \to \Phi(u)\langle v_w I\rangle$ is B^{p+}.

Now

$$\begin{aligned}&\|\int \Phi(u)\langle v_w I\rangle dw - \int \Phi(u)\langle v_{\tilde{w}} I\rangle d\tilde{w}\|_{B^p} \\ &\leq \|\int [\Phi(u)\langle v_w I\rangle - \Phi(u)\langle v_{\tilde{w}} I\rangle] dw\|_{B^p} \\ &\quad + \|\int \Phi(u)\langle v_{\tilde{w}} I\rangle d[w - \tilde{w}]\|_{B^p} \\ &\leq \|\Phi(u)\langle v_w I\rangle - \Phi(u)\langle v_{\tilde{w}} I\rangle\|_{S^p}\|w\|_{B^\infty} \\ &\quad + \|\Phi(u)\langle v_{\tilde{w}} I\rangle\|_{S^{p_2}}\|w - \tilde{w}]\|_{B^{p_1}} \\ &\leq K\|w - \tilde{w}\|_{B^{p_1}}\|v\|_{B^{p_2}} \text{ by Theorem 7.2.8}\end{aligned}$$

Also,

$$\begin{aligned}\|\int \Phi(u)\langle v_w I\rangle dw\|_{S^p} &\leq \|\Phi(u)\langle v_w I\rangle\|_{S^p}\|w\|_{B^\infty} \text{ by Lemma 8.1.6} \\ &\leq K\|v\|_{B^p}.\end{aligned}$$

II) Write $w = \int O db + \int \alpha ds$ and $\tilde{w} = \int \tilde{O} db + \int \tilde{\alpha} ds$.

Let $\{e_i\}_{i=1}^n$ be the standard basis on $\mathbf{R}^n$. Then

$$\int \Phi'(u)\langle Z(u)dw, v_w I\rangle dw \quad = \quad \sum_{i=1}^n \int \Phi'(u)\langle Z(u)Oe_i, v_w I\rangle Oe_i ds.$$

Write $G(u)\langle O, v_w I, O\rangle \equiv \sum_{i=1}^n \Phi'(u)\langle Z(u)Oe_i, v_w I\rangle Oe_i$. Then

$$\begin{aligned}
&\| \int \Phi'(u)\langle Z(u)dw, v_w I\rangle dw - \int \Phi'(\tilde{u})\langle Z(\tilde{u})d\tilde{w}, v_{\tilde{w}} I\rangle d\tilde{w}\|_{S^p} \\
&\quad = \| \int G(u)\langle O, v_w I, O\rangle ds - \int G(\tilde{u})\langle \tilde{O}, v_{\tilde{w}} I, \tilde{O}\rangle ds\|_{S^p} \\
&\quad \leq K\| \int [G(u) - G(\tilde{u})]\langle O, v_w I, O\rangle ds\|_{S^p} \\
&\qquad + K\| \int G(\tilde{u})\langle O - \tilde{O}, v_w I, O\rangle ds\|_{S^p} \\
&\qquad + K\| \int G(\tilde{u})\langle \tilde{O}, v_w I - v_{\tilde{w}} I, O\rangle ds\|_{S^p} \\
&\qquad + K\| \int G(\tilde{u})\langle \tilde{O}, v_{\tilde{w}} I, O - \tilde{O}\rangle ds\|_{S^p} \\
&\quad \leq K\|G(u) - G(\tilde{u})\|_{S^{p_1}} \|v_w I\|_{S^{p_2}} \\
&\qquad + K\|G(\tilde{u})\|_{S^\infty} \|O - \tilde{O}\|_{S^{p_1}} \|v_w I\|_{S^{p_2}} \\
&\qquad + K\|G(\tilde{u})\|_{S^\infty} \|v_w I - v_{\tilde{w}} I\|_{S^p} \\
&\quad \leq K\|w - \tilde{w}\|_{B^{p_1}} \|v\|_{B^{p_2}} \text{ by Theorem 7.2.8.}
\end{aligned}$$

Also,

$$\begin{aligned}
\| \int \Phi'(u)\langle Z(u)dw, v_w I\rangle dw\|_{S^p} \quad &= \quad \| \int G(u)\langle O, v_w I, O\rangle ds\|_{S^p} \\
&\leq \quad K\|v_w I\|_{S^p} \\
&\leq \quad K\|v\|_{B^p}
\end{aligned}$$

III) Let $\xi \equiv v_w I$. Then ξ solves the stochastic differential equation:

$$\begin{aligned}
d\xi \quad &= \quad Z(u)\delta v + Z'(u)\langle \xi\rangle \delta w \\
\text{with} \qquad & \quad \xi(s)|_{s=0} = 0
\end{aligned}$$

where $Z : O(M) \to \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^N)$ is a smooth function with compact support in $\mathbf{R}^N$ (see Notation 7.2.13).

Thus

$$\begin{aligned}\int \Phi(u)\langle d[v_w I]\rangle dw &= \int \Phi(u)\langle d\xi\rangle dw \\ &= \int \Phi(u)\langle Z(u)dv + Z'(u)\langle\xi\rangle dw\rangle dw \\ &= \int G(u)\langle A, O\rangle ds + \int H(u)\langle \xi, O, O\rangle ds\end{aligned}$$

where now

$$\begin{aligned}G(u)\langle A, O\rangle &\equiv \sum_{i=1}^{n} \Phi(u)\langle Z(u)Ae_i\rangle Oe_i \\ \text{and } H(u)\langle \xi, O, O\rangle &\equiv \sum_{i=1}^{n} \Phi(u)\langle Z'(u)\langle\xi\rangle Oe_i\rangle Oe_i.\end{aligned}$$

Now

$$\begin{aligned}&\| \int \Phi(u)\langle d[v_w I]\rangle dw - \int \Phi(\tilde{u})\langle d[v_{\tilde{w}} I]\rangle d\tilde{w}\|_{S^p} \\ &\qquad \le K\| \int G(u)\langle A, O\rangle ds - \int G(\tilde{u})\langle A, \tilde{O}\rangle ds\|_{S^p} \\ &\qquad\qquad + K\| \int H(u)\langle v_w I, O, O\rangle ds - \int H(\tilde{u})\langle v_{\tilde{w}} I, \tilde{O}, \tilde{O}\rangle ds\|_{S^p} \\ &\qquad \le K\| \int [G(u) - G(\tilde{u})]\langle A, O\rangle ds\|_{S^p} \\ &\qquad\qquad + K\| \int G(\tilde{u})\langle A, O - \tilde{O}\rangle ds\|_{S^p} \\ &\qquad\qquad + K\| \int [H(u) - H(\tilde{u})]\langle v_w I, O, O\rangle ds\|_{S^p} \\ &\qquad\qquad + K\| \int H(\tilde{u})\langle v_w I - v_{\tilde{w}} I, O, O\rangle ds\|_{S^p} \\ &\qquad\qquad + K\| \int H(\tilde{u})\langle v_{\tilde{w}} I, O - \tilde{O}, \tilde{O}\rangle ds\|_{S^p} \\ &\qquad\qquad + K\| \int H(\tilde{u})\langle v_{\tilde{w}} I, \tilde{O}, O - \tilde{O}\rangle ds\|_{S^p} \\ &\qquad \le K\|w - \tilde{w}\|_{B^{p_1}} \|v\|_{B^{p_2}} \text{ by Theorem 7.2.8.}\end{aligned}$$

Also,

$$\begin{aligned}&\| \int \Phi(u)\langle d[v_w I]\rangle dw\|_{S^p} \\ &\qquad \le K\| \int G(u)\langle A, O\rangle ds\|_{S^p} \\ &\qquad\qquad + K\| \int H(u)\langle v_w I, O, O\rangle ds\|_{S^p} \\ &\qquad \le K\|G(u)\|_{S^\infty}\|A\|_{S^p} + K\|H(u)\|_{S^\infty}\|v_w I\|_{S^p} \\ &\qquad \le K\|v\|_{B^p} \text{ by Theorem 7.2.8.}\end{aligned}$$

The G Term.

The map $w \to f_i(u)$ is B^p-Lipschitz (by Lemma 7.2.14), and is S^∞-bounded since u is $O(n)$-valued, and f_i is a smooth function having compact support in $\mathbf{R}^N$. Thus the map $(w,v) \to \int f_i(u)\delta v^i$ is B^{p+} by Lemma 7.2.12.

The H Term.

The map $w \to g'(u)$ is also B^p-Lipschitz (by Lemma 7.2.14), and S^∞-bounded. Also, Theorem 7.2.8 implies that the map $(w,v) \to v_w I$ satisfies the B^{p+}-condition. Thus the map $(w,v) \to g'(u)\langle v_w I\rangle$ is B^{p+} by Lemma 7.2.10. Q.E.D.

Theorem 7.2.16 *The function $R' : B^\infty\mathbf{R}^n \times B^\infty\mathbf{R}^n \to S^{\infty-}V$ satisfies the B^{p+} condition (Definition 7.2.9), that is, for all $w, \tilde{w}, v \in B^\infty\mathbf{R}^n$,*

$$\begin{aligned} \|v_w R\|_{S^p} &\leq K\|v\|_{B^p} \\ \text{and } \|v_w R - v_{\tilde{w}} R\|_{S^p} &\leq K\|w-\tilde{w}\|_{B^{p_1}}\|v\|_{B^{p_2}} \end{aligned}$$

where $p, p_1, p_2 \in [2,\infty)$ such that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$ and $K = K(n, p_1, p_2, \bar{K}_w, \bar{K}_{\tilde{w}})$.

Proof. We will write $w = \int Odb + \int \alpha ds$ and $\tilde{w} = \int \tilde{O}db + \int \tilde{\alpha}ds$. Let $\gamma : J \to B^\infty\mathbf{R}^n$ and $\tilde{\gamma} : J \to B^\infty\mathbf{R}^n$ be admissible curves (Def. 2.1.7) with $\dot{\gamma}_0 = v_w$ and $\dot{\tilde{\gamma}}_0 = v_{\tilde{w}}$. Let $u = I(w)$, $\tilde{u} = I(\tilde{w})$, and $u_\epsilon = I(\gamma_\epsilon)$ and $\tilde{u}_\epsilon = I(\tilde{\gamma}_\epsilon)$ for each $\epsilon \in \mathbf{R}$.

By Theorem 7.2.5 we have

$$\begin{aligned} v_w R \;= \sum_{i=1}^n \; & [\tilde{f}'(u)\langle v_w I, Oe_i, Oe_i\rangle + \tilde{f}(u)\langle Ae_i\rangle Oe_i \\ & +\tilde{f}(u)\langle Oe_i\rangle Ae_i \\ & +\tilde{g}'(u)\langle v_w I, Oe_i, Oe_i\rangle + \tilde{g}(u)\langle Ae_i, Oe_i\rangle \\ & +\tilde{g}(u)\langle Oe_i, Ae_i\rangle] \end{aligned}$$

where $\tilde{f}(\bar{u}) \equiv \Omega_{\bar{u}}\langle h, \cdot\rangle$ and $\tilde{g}(\bar{u}) \equiv \Theta^H_{\bar{u}}\langle \cdot, h, \cdot\rangle$. for all $\bar{u} \in O(M)$ (Notation 6.2.14), and $\tilde{f}'(u)\langle \xi, a, b\rangle \equiv \xi(u \to \tilde{f}(u)\langle a\rangle b)$ and $\tilde{g}'(u)\langle \xi, a, b\rangle \equiv \xi(u \to \tilde{g}(u)\langle a, b\rangle)$ for all $\xi \in T_u O(M) \cong \mathbf{R}^N$ and $a, b \in \mathbf{R}^n$.

By Remark 7.2.2 we may view $O(M)$ as a compact submanifold of $\mathbf{R}^N$ for some N, and thus extend $\tilde{f}$ and $\tilde{g}$ to smooth functions with compact support on $\mathbf{R}^N$.

Our goal is to show that the map $(w, v) \to v_w R$ satisfies the B^{p+}-condition in Definition 7.2.9. This is obtained by repeated application of Lemma 7.2.10 using the following facts:

(i) The functions $\tilde{f}$, $\tilde{f}'$, $\tilde{g}$, and $\tilde{g}'$ are all smooth with compact support in $\mathbf{R}^N$, thus the maps $w \to \tilde{f}(u)$, etc. are all bounded and are B^p-Lipschitz by Lemma 7.2.14 and [8] Cor. 4.1, p. 306.

(ii) The map $(w, v) \to v_w I$ is B^{p+} (by Theorem 7.2.8).

(iii) The map $w \to O$ satisfies $\|O - \tilde{O}\|_{S^p} \leq \|w - \tilde{w}\|_{B^p}$ and $\|O\|_{S^\infty} \leq \|w\|_{B^\infty}$. Q.E.D.

The following lemma and theorem show that for all $w \in B^\infty \mathbf{R}^n$ we may extend the map $C' : B^\infty \mathbf{R}^n \to \mathbf{End}(B^\infty)$ to a map $\tilde{C}' : B^\infty \mathbf{R}^n \to \mathbf{End}(B^2)$ such that $\tilde{C}'(w) : B^p \to B^p$, and the properties in Theorem 7.2.15 still hold.

In the following we are using the fact that B^∞ is dense in B^p for each $p \in [2, \infty)$.

Lemma 7.2.17 *For each $p \in [2, \infty)$ and $w \in B^\infty \mathbf{R}^n$ there exists a unique linear operator $C'_p(w) : B^p \to S^p$ defined by $C'_p(w)\langle v\rangle \equiv S^p\text{-}\lim_{n\to\infty} C'(w)\langle v_n\rangle$ where $v \in B^p$ and $\{v_n\} \subset B^\infty$ is any sequence such that $v_n \to v$ in B^p. Moreover, $C'_p(w) = C'_2(w)|_{B^p}$. The result also holds with C' replaced by R'.*

Proof. Let $p \in [2, \infty)$, $w \in B^\infty \mathbf{R}^n$, $v \in B^p$ and $\{v_n\} \subset B^\infty$ such that $v_n \to v$ in B^p. We have

$$\begin{aligned} \|C'(w)\langle v_m\rangle - C'(w)\langle v_n\rangle\|_{S^p} &= \|C'(w)\langle v_m - v_n\rangle\|_{S^p} \\ &\leq K_p\|v_m - v_n\|_{B^p} \text{ by Theorem 7.2.15} \\ &\to 0 \text{ as } m, n \to \infty. \end{aligned}$$

Thus we may define $C'_p(w)\langle v\rangle$ as the limit of the S^p-Cauchy sequence $\{C'(w)\langle v_n\rangle\}$. The linearity of the map $v \to C'_p(w)\langle v\rangle$ follows from the linearity of $C'(w)\langle\cdot\rangle$.

Now let $\{\tilde{v}_n\} \subset B^\infty$ be another sequence with $\tilde{v}_n \to v$ in B^p. Then

$$\begin{aligned}
\|C'(w)\langle v_n\rangle - C'(w)\langle\tilde{v}_n\rangle\|_{S^p} &\leq K_p\|v_n - \tilde{v}_n\|_{B^p} \text{ by Theorem 7.2.15} \\
&\leq K_p[\|v_n - v\|_{B^p} + \|v - \tilde{v}_n\|_{B^p}] \\
&\quad \to 0 \text{ since } v_n \to v \text{ and } \tilde{v}_n \to v \text{ in } B^p
\end{aligned}$$

Thus the limit $C'_p(w)\langle v\rangle \in S^p$ defined above is unique.

Finally, we have

$$\begin{aligned}
&\|C'_p(w)\langle v\rangle - C'(w)\langle v_n\rangle\|_{S^2} \\
&\leq \|C'_p(w)\langle v\rangle - C'(w)\langle v_n\rangle\|_{S^p} \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

But $v \in B^p \subset B^2$, so this defines $C'_2(w)\langle v\rangle$, that is, $C'_p(w)\langle v\rangle = C'_2(w)\langle v\rangle$ P-a.s. The proof for R' is similar. Q.E.D.

Notation 7.2.18 *We will still denote the extensions (to B^2) by $C'(w)$ and $R'(w)$, for all $w \in B^\infty\mathbf{R}^n$.*

The following theorem completes the proofs that C and R are $S^{\infty-}$-differentiable (Theorems 7.2.3 and 7.2.5.)

Theorem 7.2.19 *Fix $w, \tilde{w} \in B^\infty\mathbf{R}^n$ and $v \in B^{\infty-}\mathbf{R}^n$ Then*

$$\begin{aligned}
\|v_w C\|_{S^p} &\leq K\|v\|_{B^p} \text{ and} \\
\|v_w C - v_{\tilde{w}} C\|_{S^p} &\leq K\|w - \tilde{w}\|_{B^{p_1}}\|v\|_{B^{p_2}}
\end{aligned} \tag{7.2.7}$$

similarly

$$\begin{aligned}
\|v_w R\|_{S^p} &\leq K\|v\|_{B^p} \text{ and} \\
\|v_w R - v_{\tilde{w}} R\|_{S^p} &\leq K\|w - \tilde{w}\|_{B^{p_1}}\|v\|_{B^{p_2}}
\end{aligned} \tag{7.2.8}$$

where $p, p_1, p_2 \in [2, \infty)$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $K = K(n, p_1, p_2, \bar{K}_w, \bar{K}_{\tilde{w}})$.

Proof. Fix $p, p_1, p_2 \in [2, \infty)$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Let $\{v_n\} \subset B^\infty$ such that $v_n \to v$ in B^{p_2} (so $v_n \to v$ in B^p as well). By Theorem 7.2.15, $\|C'(w)\langle v_n \rangle\|_{S^p} \leq K_p \|v_n\|_{B^p}$ for all n, so by taking the limit as $n \to \infty$ we have $\|C'(w)\langle v \rangle\|_{S^p} \leq K_p \|v\|_{B^p}$.

Similarly, $C'(w)\langle v_n \rangle - C'(\tilde{w})\langle v_n \rangle \to C'(w)\langle v \rangle - C'(\tilde{w})\langle v \rangle$ in S^p as $n \to \infty$, so the conditions $\|C'(w)\langle v_n \rangle - C'(\tilde{w})\langle v_n \rangle\|_{S^p} \leq K\|w - \tilde{w}\|_{B^{p_1}} \|v_n\|_{B^{p_2}}$ for all n imply $\|C'(w)\langle v \rangle - C'(\tilde{w})\langle v \rangle\|_{S^p} \leq K\|w - \tilde{w}\|_{B^{p_1}} \|v\|_{B^{p_2}}$. The proof for R' is similar. Q.E.D.

7.3 Existence of the Derivative of the Flow on $W(\mathbf{R}^n)$

We now prove Theorem 7.1.3: $[Y_{w_o} O^h](t, w_o)$ and $[Y_{w_o} \alpha^h](t, w_o)$ exist in $S^{\infty-}$ and satisfies (7.1.3), and $[Y_{w_o} w^h](t, w_o)$ exists in $B^{\infty-}$ and satisfies (7.1.4).

Proof of Theorem 7.1.3. The proof will consist of showing that Theorem 4.1.3 is applicable. For this it is necessary to verify that C^h and R^h satisfy the conditions of C and R in Notation 4.1.1. By Theorem 7.2.19, condition 1 is satisfied. In [8], conditions 2 and 3 are verified in the proofs of Corollaries 6.1 and 6.2, respectively. By the definitions of w^h, O^h and α^h, conditions 4, 5, and 6 are satisfied (taking $X = Y^h$). Note that C^h and R^h satisfy the hypotheses of Theorem 3.1.5.

Q.E.D.

Now we have the analog of Theorem 7.1.3 for flows on the space of paths $W(\mathbf{R}^n)$. We will take the underlying probability space $\Omega = W(\mathbf{R}^n)$ with reference Brownian motion $\bar{b}$ defined by $\bar{b}_s(\omega) = \omega(s)$ for all $\omega \in W(\mathbf{R}^n)$.

Notation 7.3.1 *Let $h \in C^1$, and let $\tilde{Y}^h : W(\mathbf{R}^n) \to W(\mathbf{R}^n)$ be a continuous* *version of* $Y^h(\bar{b})$. *Let* $\tilde{w}^h(t) = \int \tilde{O}^h(t)d\bar{b} + \int \tilde{\alpha}^h(t)ds$ *be the solution to the equation* $\dot{\tilde{w}}^h(t) = \tilde{Y}^h(\tilde{w}^h(t))$ *with* $\tilde{w}^h(0) = id : W(\mathbf{R}^n) \to W(\mathbf{R}^n)$. *(See Notation 3.2.2 and Remark 3.2.3).*

Theorem 7.3.2 *Let* $h, k \in C^1$. *For all* $t \in J$,

i)

$$\begin{aligned} [\tilde{Y}^k \tilde{O}^h](t) &\equiv \lim_{\epsilon \to 0} \tfrac{\tilde{O}^h(t)(\tilde{w}^k(\epsilon)(\cdot)) - \tilde{O}^h(t)(\cdot)}{\epsilon} \\ \text{and } [\tilde{Y}^k \tilde{\alpha}^h](t) &\equiv \lim_{\epsilon \to 0} \tfrac{\tilde{\alpha}^h(t)(\tilde{w}^k(\epsilon)(\cdot)) - \tilde{\alpha}^h(t)(\cdot)}{\epsilon} \end{aligned} \tag{7.3.1}$$

exist where the limits are taken in the S^p*-topologies for* $p \in [2, \infty)$.

ii)

$$[\tilde{Y}^k \tilde{w}^h](t) \equiv \lim_{\epsilon \to 0} \frac{\tilde{w}^h(t)(\tilde{w}^k(\epsilon)(\cdot)) - \tilde{w}^h(t)(\cdot)}{\epsilon}$$

exists where the limit is taken in the B^p*-topologies for* $p \in [2, \infty)$.

Note that we do not define these derivatives for each $\omega \in W(\mathbf{R}^n)$ *since the limits exist only* P*-a.s.*

Proof. In the proof of Theorem 7.1.3 above we have verified the conditions in Notation 4.1.1. Also, for all $\epsilon \in J$, $\tilde{w}^k(\epsilon)_* P$ is equivalent to standard Wiener measure μ on $W(\mathbf{R}^n)$, so this result follows directly from Theorem 5.4.2. Q.E.D.

7.4 Existence of the Derivative of the Flow on $W_o(M)$

Notation 7.4.1 *Given* $h \in C^1$, *let* X^h *be the vector field on* $B^\infty M$ *defined in Theorem 6.3.2. We will write the solution to the flow equation* $\dot{\sigma}^h(t) = X^h(\sigma^h(t))$ *on* $B^\infty M$ *as a function of its starting point, as we did for the flow on* $B^\infty \mathbf{R}^n$ *(Notation 7.1.2). We write* $\sigma^h(t, \sigma_0)$ *for the solution to the equation*

$$\sigma^h(t,\sigma_0) = X^h(\sigma^h(t,\sigma_0)) \text{ with } \sigma^h(0,\sigma_0) = \sigma_0 \in B^\infty M. \tag{7.4.1}$$

The existence of a unique solution to (7.4.1) is given by [8] Corollary 6.3, p. 336.

Theorem 7.4.2 *Suppose $M \subset \mathbf{R}^N$ is an imbedded submanifold. Let $h,k \in C^1$, and define σ^h, σ^k and X^k as in Notation 7.4.1. Let $\sigma_o \in B^\infty M$ such that $\sigma_{o*}P$ is equivalent to ν. Then for all $t \in J$,*

$$[X^k\sigma^h](t,\sigma_o) \equiv \lim_{\epsilon\to 0} \frac{\sigma^h(t,\sigma^k(\epsilon,\sigma_o)) - \sigma^h(t,\sigma_o)}{\epsilon} \tag{7.4.2}$$

exists where the limit is taken in the $B^p\mathbf{R}^N$-topology for all $p \in [2,\infty)$.

Proof. Fix $t \in J$. By [8] Theorems 5.1 and 3.3, we have a 1-1 correspondence between flows on $B^\infty M$ defined by (7.4.1) and those on $B^\infty\mathbf{R}^n$ defined by (7.1.1), given by $\sigma^h = \pi \circ I(w^h)$. In the expanded notation 7.4.1, $\sigma^h(t,\sigma_0) = \pi \circ I(w^h)(t,w_0)$ means that σ^h satisfies (7.4.1) with starting point $\sigma_0 = \pi \circ I(w_0)$.

So we have $\sigma^h(t,\sigma^k(\epsilon,\sigma_o)) = \pi \circ I[w^h(t,w^k(\epsilon,b))]$ since $\sigma^k(\epsilon,\sigma_o) = \pi \circ I(w^k(\epsilon,b))$. Let $\gamma_\epsilon \equiv w^h(t,w^k(\epsilon,b))$. Now the map $\epsilon \to \gamma_\epsilon$ is $B^{\infty-}$-differentiable by Theorem 7.1.3, and we have by [8] Cor. 4.2, p. 313, that $\epsilon \to I(\gamma_\epsilon)$ is also $B^{\infty-}$-differentiable. (Note that the $B^{\infty-}$-Lipschitz requirement on the derivative $\epsilon \to \frac{d}{d\epsilon}\gamma_\epsilon$ in [8] Cor. 4.2, p. 313 is used only to obtain a Lipschitz bound for $\epsilon \to \frac{d}{d\epsilon}I(\gamma_\epsilon)$.)

Finally, since π is smooth we have by the Chain Rule ([8] Lemma 4.6, p. 314) that $[X^k\sigma^h](t,\sigma_o) \equiv \frac{d}{d\epsilon}|_0[\pi \circ I(\gamma_\epsilon)]$ exists. Q.E.D.

Theorem 7.4.3 *Fix $h,k \in C^1$. Let $\bar{H}$ be a fixed version of the horizonotal lift map H (see Notation 6.2.10). Define the vector field $\tilde{X}^h$ on $W_o(M)$ by $\tilde{X}^h(\tilde{\sigma}) \equiv \bar{H}(\tilde{\sigma})h$. Let $\tilde{\sigma}^h : \mathbf{R} \to Maps(W_o(M) \to W_o(M))$ be the solution to equation (6.3.2):*

$$\dot{\tilde{\sigma}}^h = \tilde{X}^h(\tilde{\sigma}^h) \text{ with } \tilde{\sigma}^h(0) = id$$

given by Theorem 6.3.1 ([8] Theorem 8.5, p. 361). Similarly define $\tilde{X}^k$ *and* $\tilde{\sigma}^k$*. Then for all* $t \in J$*,*

$$[\tilde{X}^k \tilde{\sigma}^h](t)(\cdot) \equiv \lim_{\epsilon \to 0} \frac{\tilde{\sigma}^h(t, \tilde{\sigma}^k(\epsilon)(\cdot)) - \tilde{\sigma}^h(t)(\cdot)}{\epsilon} \tag{7.4.3}$$

exists where the limit is taken in the $B^p\mathbf{R}^N$*-topology for all* $p \in [2, \infty)$*.*

Proof. Let σ^h be the solution to (7.4.1) with underlying probability space $(W_o(M), \{\bar{\mathcal{H}}^\nu_{s+}\}, \nu)$ and $\sigma_o = \bar{\sigma}_o$ where $\bar{\sigma}_o(s) : W_o(M) \to M$ is defined by $\bar{\sigma}_o(s)(\omega) = \omega(s)$ for $0 \le s \le 1$ and $\omega \in W_o(M)$ (see Notation 3.2.1).

We know that for all $t \in J$, $[X^k\sigma^h](t, \bar{\sigma}_o) \equiv \lim_{\epsilon \to 0} \frac{\sigma^h(t, \sigma^k(\epsilon, \bar{\sigma}_o)) - \sigma^h(t, \bar{\sigma}_o)}{\epsilon}$ exists in $B^{\infty-}$ by Theorem 7.4.2. Since $\sigma^k(\epsilon, \bar{\sigma}_o)$ has law equivalent to ν, we have (supressing the parameters t and ϵ) $(\sigma^h \circ \sigma^k)(\bar{\sigma}_o) \doteq \sigma^h(\bar{\sigma}_o) \circ \sigma^k(\bar{\sigma}_o) \doteq \tilde{\sigma}^h \circ \tilde{\sigma}^k$.

Thus $\tilde{\sigma}^h(t)(\tilde{\sigma}^k(\epsilon)) - \tilde{\sigma}^h(t)$ is a version of $\sigma^h(t, \sigma^k(\bar{\sigma}_o)) - \sigma^h(t, \bar{\sigma}_o)$, so $[\tilde{X}^k\tilde{\sigma}^h](t)$ as defined above exists and is a version of $[X^k\sigma^h](t, \bar{\sigma}_o)$. Q.E.D.

Chapter 8

Appendix

8.1 Classical Theorems of Girsanov, Kolmogorov and Gronwall

Lemma 8.1.1 *Let $J = [-\kappa, \kappa] \subset \mathbf{R}$ and $C : J \to \mathrm{so}(n)$. Then there exists a unique solution to*

$$\dot{T}(t) = C(t)T(t) \text{ with } T(0) = I \tag{8.1.1}$$

Moreover, $T(t) \in O(n)$ for all $t \in \mathbf{R}$.

Proof. Since (8.1.1) is a linear ordinary differential equation in finite dimensions, it has a unique solution. To show that $T(t) \in O(n)$ note that

$$\begin{aligned} \frac{d}{dt}[T(t)^*T(t)] &= \frac{d}{dt}[T(t)^*]T(t) + T(t)^*\dot{T}(t) \\ &= T(t)^*C(t)^*T(t) + T(t)^*C(t)T(t) \\ &= 0 \text{ (since } C(t) \text{ is so}(n)\text{-valued).} \end{aligned}$$

Since $T(0)^*T(0) = I$, this shows that $T(t)^*T(t) = I$ for all $t \in J$. Q.E.D.

The following is a corollary of Girsanov's theorem using Novikov's criterion. For a proof see [8] Lemma 8.2, p. 347.

Lemma 8.1.2 *(Girsanov's Theorem) Let* $w = \int Odb + \int \alpha ds \in B^\infty \mathbf{R}^n$ *such that* (O, α) *is a predictable* $O(n) \times \mathbf{R}^n$*-valued process, and* $P(\int_0^1 |\alpha(s)|^2 ds \leq K) = 1$ *for some constant* $K < \infty$*. Then*

1. $\mu \equiv b_* P$ *and* $w_* P$ *are equivalent.*

2. *Let* $Z_s \equiv \exp\{-\int_0^s \alpha \cdot Odb - \frac{1}{2}\int_0^s |\alpha|^2 d\bar{s}\}$ *and define* $Q \equiv Z_1 \cdot P$*, that is,* Q *is the probability measure on* Ω *such that* $\frac{dQ}{dP} = Z_1$*. Then* $w_* Q = \mu$*.*

Lemma 8.1.3 *Kolmogorov's Lemma (See [41] Theorem 53, p. 171, and Corollary, p. 173.)*

Let $p \in [1, \infty)$ *and* V *be a finite-dimensional vector space. Suppose* $f : J \to S^p V$ *is* S^p*-Lipschitz. Then there is a version of* f *such that* P*-a.s. the function* $(t \to f(t)) : J \to W(V) \equiv C([0,1], V)$ *is continuous. In particular, there is a version of* f *such that the function* $((t, s) \to f(t)(s) : J \times [0,1] \to V$ *is* P*-a.s. continuous.*

Lemma 8.1.4 *(Driver [8] Lemma 4.5, p. 306) Let* $p > 1$ *and suppose* $q : J \to S^p \mathbf{R}^n$ *is an* S^p*-differentiable function and the derivative* $\dot{q}$ *is* S^p*-Lipschitz. Then there is a version of* q *such that* P*-a.s. the function* $(t, s) \to q(t)(s)$ *is* $C^{1,0}$*.*

Lemma 8.1.5 *Let* $p, p_1, p_2 \in [2, \infty]$ *and be such that* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$*. Suppose* V *is a finite-dimensional vector space. Let* X *and* Y *be continuous adapted processes such that* $XY \in S^p V$*. Then* $\|XY\|_{S^p} \leq \|X\|_{S^{p_1}} \|Y\|_{S^{p_2}}$*.*

Proof. For $p_2 = \infty$ we have

$$\|XY\|_{S^p} = \{\int_\Omega [\sup_{0 \leq r \leq s} |X(\omega, r) Y(\omega, r)|]^p d\omega\}^{\frac{1}{p}}$$

$$\begin{aligned}
&\leq \{\int_\Omega [\sup_{0\leq r\leq s} |X(\omega,r)| \sup_{0\leq r\leq s} |Y(\omega,r)|]^p d\omega\}^{\frac{1}{p}} \\
&\leq \{\int_\Omega [\sup_{0\leq r\leq s} |X(\omega,r)| \sup_{0\leq r\leq s} \text{ess sup}_{\sigma\in\Omega} |Y(\sigma,r)|]^p d\omega\}^{\frac{1}{p}} \\
&= \{\int_\Omega [\sup_{0\leq r\leq s} |X(\omega,r)|]^p d\omega\}^{\frac{1}{p}} \sup_{0\leq r\leq s} \text{ess sup}_{\sigma\in\Omega} |Y(\sigma,r)| \\
&= \|X\|_{S^p} \|Y\|_{S^\infty}
\end{aligned}$$

The proof for $p_1 = \infty$ is similar. For $p_1, p_2 \in (p, \infty)$ we have

$$\begin{aligned}
\|XY\|_{S^p} &= \| \sup_{0\leq r\leq 1} |X(r)Y(r)| \|_{L^p(\mathcal{P})} \\
&\leq \| \sup_{0\leq r\leq 1} |X(r)| \sup_{0\leq r\leq 1} |Y(r)| \|_{L^p(\mathcal{P})} \\
&\leq \| \sup_{0\leq r\leq 1} |X(r)| \|_{L^{p_1}(\mathcal{P})} \| \sup_{0\leq r\leq 1} |Y(r)| \|_{L^{p_2}(\mathcal{P})} \text{ by Hölder} \\
&= \|X\|_{S^{p_1}} \|Y\|_{S^{p_2}}
\end{aligned}$$

Q.E.D.

Lemma 8.1.6 *Let $p, p_1, p_2 \in [2, \infty]$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Suppose V and W are finite-dimensional vector spaces $w \equiv \int O db + \int \alpha ds$ is an V-valued Brownian semimartingale, and Z is a* $\mathrm{Hom}(V, W)$*-valued continuous adapted process. Then $\| \int Z dw\|_{B^p} \leq \|Z\|_{S^{p_1}} \|w\|_{B^{p_2}}$.*

Proof. We have

$$\begin{aligned}
\| \int Z dw\|_{B^p} &= \| \int Z O db + \int Z\alpha ds\|_{B^p} \\
&= \|ZO\|_{S^p} + \|Z\alpha\|_{S^p} \\
&\leq \|Z\|_{S^{p_1}} \|O\|_{S^{p_2}} + \|Z\|_{S^{p_1}} \|\alpha\|_{S^{p_2}} \text{ by Lemma 8.1.5} \\
&= \|Z\|_{S^{p_1}} \|w\|_{B^{p_2}}.
\end{aligned}$$

Q.E.D.

Lemma 8.1.7 Gronwall's Lemma. *Suppose $f : \mathbf{R}^+ \to \mathbf{R}^+$ and $\epsilon, K \geq 0$ are constants such that $f(t) \leq \epsilon + K \int_0^t f(\tau) d\tau$ for all $t \in \mathbf{R}^+$. Then $f(t) \leq \epsilon e^{Kt}$ for all $t \in \mathbf{R}$.*

Proof. Let $F(t) \equiv K\int_0^t f(\tau)d\tau$, then $\dot{F}(t) = Kf(t) \leq K[\epsilon + F(t)]$ by hypothesis. Thus we have

$$\begin{aligned}\frac{d}{dt}[e^{-Kt}F(t)] &= e^{-Kt}[\dot{F}(t) - KF(t)] \\ &\leq K\epsilon e^{-Kt}.\end{aligned}$$

Integrating and solving for $F(t)$, we have $F(t) \leq \epsilon e^{Kt} - \epsilon$, thus $f(t) = F(t) + \epsilon \leq \epsilon e^{Kt}$. Q.E.D.

The following lemma is a consequence of [8] Lemma 7.4, p. 339.

Lemma 8.1.8 Modified Gronwall's Lemma *Let $g : J \to S^{\infty-}$ be $S^{\infty-}$-differentiable. Suppose there exist constants $K, \epsilon \geq 0$ such that*

$$\|\dot{g}(t)\|_{S^p} \leq K\|g(t)\|_{S^p} + O(\epsilon)$$

for all $t \in J$, and $\|g(0)\|_{S^p}$ is $O(\epsilon)$. Then $\|g(t)\|_{S^p}$ is $O(\epsilon)$ for all $t \in J$. This result also holds with $S^{\infty-}$ and S^p replaced by $B^{\infty-}$ and B^p.

Lemma 8.1.9 *For all $a \in \mathbf{R}^n$, the map $B\langle a\rangle(\cdot) : O(M) \to TO(M)$ is smooth.*

Proof. Let ∇ be a given covariant derivative on TM. For $u \in O(M)$ we have $B\langle a\rangle(u) = [\pi_*|_{\mathcal{H}_u^\nabla}]^{-1}ua$ where $\mathcal{H}_u^\nabla = \mathcal{H}_u^\nabla O(M) \subset TO(M)$ is a horizontal tangent space, also called a connection on $O(M)$. The covariant derivative ∇ determines $\mathcal{H}_u^\nabla$ since $\mathcal{H}_u^\nabla O(M) = \{\dot{\alpha}(0)|\alpha(0) = u \text{ and } \frac{\nabla u}{dt}(0) = 0\}$.

To investigate the smoothness of the map $[\pi_*|_{\mathcal{H}_u^\nabla}]^{-1}$ we will represent $\mathcal{H}_u^\nabla$ in local coordinates.

First we represent ∇ (locally) as a covariant derivative $\tilde{\nabla}$ on the trivial vector bundle $M \times \mathbf{R}^n$.

We have $TM \tilde{=} M \times \mathbf{R}^n$ as an isomorphism of vector bundles by the map $\phi : M \times \mathbf{R}^n \to TM$ defined by $\phi(m, a) = U(m)a$ where $U(p) \in O_p(M)$ for all $p \in M$ (i.e. U is a fixed moving frame).

Identify $\Gamma(M \times \mathbf{R}^n)$ with $C^\infty(M, \mathbf{R}^n)$. Then for all $S \in \Gamma(M \times \mathbf{R}^n)$ we may write

$$\tilde{\nabla}_v S = dS\langle v\rangle + \Gamma^u\langle v\rangle S \tag{8.1.2}$$

where Γ^U is a smooth so(n)-valued 1-form on M defined by $\Gamma^U\langle v\rangle \equiv U^{-1}\nabla_v U$.

Now we also have $O(M) \tilde{=} M \times \mathrm{so}(n)$ via the map $\Psi : M \times \mathrm{so}(n) \to O(M)$ defined by $\Psi(m, g) = U(m)g$.

So we may represent $\mathcal{H}_u^\nabla O(M)$ locally as $\mathcal{H}^{\tilde{\nabla}}_{(m,g)}(M \times \mathrm{so}(n)) \equiv \{(\dot{m}(0), \dot{g}(0)) | m(0) = m,\ g(0) = g,\ \frac{\tilde{\nabla} g}{dt}(0) = 0\}$, where $u = U(m)g$.

Now we have via (8.1.2) a representation of the local covariant derivative along a smooth curve $\sigma(t)$ in M : $\frac{\tilde{\nabla}}{dt}S(t) = \dot{S}(t) + \Gamma^U\langle\dot{\sigma}(t)\rangle S(t)$ where $S(t) \in \Gamma_{\sigma(t)}(M \times \mathrm{so}(n))$.

Since $g(t) \in \Gamma_{m(t)}(M \times \mathrm{so}(n))$ we have $\frac{\tilde{\nabla} g}{dt}(0) = \dot{g}(0) + \Gamma^U\langle\dot{m}(0)\rangle g(0)$, so setting $v = \dot{m}(0)$, we have $\mathcal{H}^{\tilde{\nabla}}_{(m,g)}(M \times \mathrm{so}(n)) \equiv \{(v, (-\Gamma^U\langle v\rangle g)_g) | v \in T_m M\} \subset T_m M \times T_g \mathrm{so}(n)$.

Also, the map $\pi_* : \mathcal{H}^{\tilde{\nabla}}_{(m,g)} \to T_m M$ is just a projection, so we have $[\pi_*|_{\mathcal{H}^{\tilde{\nabla}}_{(m,g)}}]^{-1} v = (v, (-\Gamma^U\langle v\rangle g)_g)$.

Finally, in the local representation we have for each $a \in \mathbf{R}^n$, $\tilde{B}\langle a\rangle(\cdot) : M \times \mathrm{so}(n) \to \mathcal{H}^{\tilde{\nabla}}_{(m,g)}$ defined by $\tilde{B}\langle a\rangle((m, g)) = (U(m)ga, (-\Gamma^U\langle U(m)ga)_g)$ which is clearly smooth in (m, g). Q.E.D.

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