

ANGULAR REGULARITY AND STRICHARTZ ESTIMATES FOR THE WAVE EQUATION

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ABSTRACT. We prove here essentially sharp $L^q(L^r)$ linear and bilinear estimates for the wave equations on Minkowski space where we assume the initial data possesses additional regularity with respect to fractional powers of the angular momentum operators $\Omega_{ij} := x^i \partial_j - x^j \partial_i$. In this setting, the range of exponents (q, r) vastly improves over what is available for the wave equations based on translation invariant derivatives of the initial data, or uniform decay of the solution.

1. INTRODUCTION: CLASSICAL STRICHARTZ ESTIMATES, IMPROVEMENTS FOR SPHERICALLY SYMMETRIC DATA, AND KNAPP COUNTEREXAMPLES

The aim of this work is to prove mixed Lebesgue space estimates for solutions to the linear wave equation on Minkowski space in a setting where the initial data is assumed to possess extra regularity with respect to weighted derivatives in the angular variable. These types of estimates arise naturally in applications to the scale invariant global existence theory of non-linear wave equations which do not possess certain “null” structures in their non-linearities. For example, in a companion to this article, we use the estimates proved here to show global existence and scattering for the (4+1) Yang–Mills equations in the Lorentz gauge for a certain class of small, scale invariant initial data.

All of the estimates we prove will be of “Strichartz type”, i.e. $L^q(L^r)$ space–time estimates for solutions of $\square\phi = 0$. Due to the presence of extra weighted angular regularity, we will get a significant gain over the usual estimates for the wave equations which are based solely on translation invariant derivatives of the initial data. What separates our estimates from the “classical” Strichartz estimates, is that they are not solely based on the uniform decay of solutions to the wave equation. Instead, we will exploit a certain “wave packet” structure these solutions exhibit in radial coordinates. This allows us to decompose our waves into a sum of pulses, each of which remains coherent for all time.¹ These pulses can then be treated on

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¹This fact makes the type of wave packets we use here more simple to handle than those used in say [18]. In fact, all of our estimates are proved by directly integrating in time, without using any complicated induction procedures or further analytical tools such as the TT^* argument. However, there is a certain lack of coherence our wave packets exhibit, which seems to limit their usefulness to situations where one can use some angular regularity. This is because, strangely enough, our wave packets become *more* coherent for large times. For relatively small times, close to the “focusing” time of our wave packets, much of this coherence seems to get lost. See Remark 4.2 after the statement of Proposition 4.1.

an essentially individual basis. This works well to prove a certain weak² endpoint estimate for the range we are considering. We then interpolate our endpoint with the endpoint from [6] to obtain the full sharp (up to an ϵ) set of estimates.

We shall also consider multilinear type estimates which involve weighted angular regularity on one or any number of the factors. For this set of estimates, we again obtain a vast improvement over the “classical” multilinear estimates for the wave equation (e.g. compared with [7]). We shall state and prove these estimates in the context of T. Tao’s dual scale machine for generating multilinear estimates (see [16]). All of the estimates we prove here are sharp, up to an arbitrarily small loss of angular regularity, when tested against Knapp counterexamples. We will explain this in more detail in the sequel.

After the proof of the estimates we describe here was discovered by the first author, a shorter proof was found by Igor Rodnianski for the case of $3 \leq n$ spatial dimensions. We have included this in an appendix to the present work and have elected to retain the discussion of our original proof in the main body of the paper because it includes the development of machinery that is interesting in its own right and is perhaps more flexible. Furthermore, our proof gives a lot of detailed and interesting information as to how wave propagation works on high spherical harmonics. In particular, we provide what seems to be a sharp localization of band limited Hankel transforms, the type of which has been studied by previous authors (see [3]). It is likely that the general procedure we employ here which uses this type phase space localization is applicable to other dispersive phenomena (e.g. wave or Schrödinger equations) on spherically symmetric backgrounds.

We now begin with a brief discussion of the usual Strichartz estimates for the wave equation. The standard reference for this material at this point is the paper [6]. Let u_1 be a unit frequency solution³ to the wave equation on Minkowski space. By this we mean a function u_1 such that:

$$(1) \quad \square u_1 = 0, \quad \text{supp} \{ \widehat{u_1}(0), \widehat{\partial_t u_1}(0) \} \subseteq \left\{ |\xi| \mid \frac{1}{2} < |\xi| < 2 \right\}.$$

Without loss of generality, we may assume that u_1 is of the form $u_1(t) = e^{it\sqrt{-\Delta}} f_1$, for some unit frequency function f_1 of the spatial variable only.⁴ Then two of the most basic quantities which determine the space–time behavior of u_1 are the following estimates:

$$(2) \quad \| e^{it\sqrt{-\Delta}} f_1 \|_{L_x^2} = \| f_1 \|_{L_x^2}, \quad (\text{Energy Estimate})$$

$$(3) \quad \| e^{it\sqrt{-\Delta}} f_1 \|_{L_x^\infty} \lesssim t^{\frac{1-n}{2}} \| f_1 \|_{L_x^1}. \quad (\text{Dispersive Estimate})$$

By interpolating between (2) and (3), and using some standard duality arguments (specifically the TT^* method), it is possible to show the following set of space–time estimates for u_1 :

²Not in the sense of real interpolation. By weak we simply mean an estimate which contains logarithmic divergences.

³For definitions of the various objects we present here, including Fourier transforms and mixed Lebesgue spaces see the next section.

⁴For more detail, see the notation list in Section 2.

Theorem 1.1 (“Classical” Strichartz estimates including endpoints (see [6], [4], and [15])). *Let $3 \leq n$ be the number of spatial dimensions, and let $\sigma = \frac{n-1}{2}$, then the following estimate holds for $2 \leq q$:*

$$(4) \quad \| e^{it\sqrt{-\Delta}} f_1 \|_{L^q(L^r)} \lesssim \| f_1 \|_{L^2} ,$$

where $\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}$, with the exception of the forbidden $L^2(L^\infty)$ endpoint on \mathbb{R}^{3+1} .

A key facet of the proof of (4), is that it does not rely on any other property of the evolution operator $e^{it\sqrt{-\Delta}}$ than the estimates (2) and (3). In other words, the proof only uses the conservation of energy and the *uniform* decay estimate (3). Furthermore, the estimate (4) is sharp in that one cannot improve the range of $L^q(L^r)$ indices stated above without replacing the L^2 norm on the right hand side of (4) with something else. This can be seen as follows: Let us consider initial data sets $\widehat{f}_1^\epsilon = \chi_{B^\epsilon}$, where χ_{B^ϵ} is the indicator function of a radially directed block of dimensions $1 \times \epsilon \times \dots \times \epsilon$, lying along the ξ_1 axis between $1/2 < \xi_1 < 2$. Then a quick calculation using the integral formula:

$$(5) \quad e^{it\sqrt{-\Delta}} f_1^\epsilon(x) = \int e^{2\pi i(t|\xi|+x \cdot \xi)} \chi_{B^\epsilon}(\xi) d\xi ,$$

shows that one has $|e^{it\sqrt{-\Delta}} f_1^\epsilon(x)| \sim \epsilon^{n-1}$ on the space-time region $S_{t,x}^\epsilon$:

$$\begin{aligned} t &= O(\epsilon^{-2}) , \\ t + x_1 &= O(1) , \\ x' &= O(\epsilon^{-1}) . \end{aligned}$$

Based on this one sees that:

$$(6) \quad \epsilon^{\frac{1}{2}(\frac{\sigma}{2} - \frac{1}{2} - \frac{\sigma}{r})} \| f_1^\epsilon \|_{L^2} \sim \epsilon^{n-1} \| \chi_{S_{t,x}^\epsilon} \|_{L^2(L^r)} \lesssim \| e^{it\sqrt{-\Delta}} f_1^\epsilon \|_{L^2(L^r)} ,$$

where $\sigma = \frac{n-1}{2}$. Therefore, the condition $\frac{1}{2} + \frac{\sigma}{r} \leq \frac{\sigma}{2}$ of Theorem 1.1 must be satisfied. We note here that initial data sets f_1^ϵ are commonly referred to as *Knapp counterexamples*.

With the above consideration in mind, it is natural to wonder if that somehow the initial data f_1 were forced to be more evenly spread out along the various radial directions in Fourier space, then one could gain some improvement on the range of indices in (4). However, any such improvement must somehow involve another mechanism than just the estimates (2) and (3). This can be seen simply from the fact that both of these estimates are sharp even for spherically symmetric data (e.g. looking at waves which focus at larger and larger times). Even so, it has been known for some time that, with the help of additional arguments based on the specific form of the integral representation for $e^{it\sqrt{-\Delta}}$, one can obtain a significant improvement over (4) for spherically symmetric initial data (see for example [8] and [9]). To understand how this can happen, consider a unit frequency radially symmetric initial data set f_1^0 , such that $\widehat{f_1^0}$ is a smooth bump function of the radial variable $|\xi|$. Then by using an integration by parts and stationary phase argument on the integral representation (5), in conjunction with the phenomena of finite speed of

propagation⁵ it is not difficult to see that one has the asymptotics (because we can let derivatives fall on $\widehat{f_1^0}$, of course this works for smooth enough Fourier data even if it is not spherically symmetric):

$$(7) \quad |e^{it\sqrt{-\Delta}} f_1^0(r)| \leq \frac{C_M}{|t|^{\frac{n-1}{2}}(1+||t|-r|)^M}.$$

That is, at time t , $e^{it\sqrt{-\Delta}} f_1^0$ is essentially the indicator function of an $O(1)$ spherical shell of radius t multiplied by the amplitude $t^{\frac{1-n}{2}}$. Based on this, one can easily compute that:

$$\|e^{it\sqrt{-\Delta}} f_1^0\|_{L^r_x} \lesssim \frac{1}{(1+|t|)^{(n-1)(\frac{1}{2}-\frac{1}{r})}}.$$

Therefore, in order for us to have that the $L^2(L^r)$ norm of $e^{it\sqrt{-\Delta}} f_1^0$ is finite, we need that $\frac{1}{2} < (n-1)(\frac{1}{2}-\frac{1}{r})$, or equivalently, that $\frac{2(n-1)}{n-2} < r$. This is a vast improvement over the requirement of $\frac{2(n-1)}{n-3} \leq r$ coming from the Knapp counterexamples. The key to this improvement is that along with the uniform decay rate (3), the waves $e^{it\sqrt{-\Delta}} f_1^0$ are highly localized in physical space along the radial variable.

Of course, an arbitrary spherically symmetric wave will not have the localization (7). However, it is possible to in a straightforward manner chop up a unit frequency spherically symmetric wave into a sum of pieces, each of which satisfy a time translated version of the asymptotic (7). This is accomplished via a suitable physical space localization of the radially symmetric Fourier transform as follows: We begin by rewriting the integral formula (5) for radially symmetric initial data f_1 as:

$$(8) \quad e^{it\sqrt{-\Delta}} f_1(r) = \frac{2\pi}{r^{\frac{n-2}{2}}} \int_0^\infty e^{2\pi i t \rho} J_{\frac{n-1}{2}}(2\pi r \rho) \widehat{f_1}(\rho) \rho^{\frac{n}{2}} d\rho,$$

where $J_{\frac{n-1}{2}}(y)$ is the Bessel function of order $\frac{n-1}{2}$. We now use the well known asymptotics for Bessel functions of relatively small order (see [17]):

$$(9) \quad J_{\frac{n-1}{2}}(y) = \begin{cases} \left(\frac{2}{\pi y}\right)^{\frac{1}{2}} \left[\cos\left(y - \frac{n-2}{4}\pi\right) \cdot m_1(y) - \sin\left(y - \frac{n-2}{4}\pi\right) \cdot m_2(y) \right], & \text{for } 1 \leq y, \\ y^{\frac{n-1}{2}} \cdot m_3(y), & \text{for } 0 \leq y \leq 1. \end{cases}$$

Here the function m_3 is C^∞ , and the remaining m_i have asymptotic expansions:

$$m_1(y) = \sum_k C_{1,k} y^{-2k},$$

$$m_2(y) = \sum_k C_{2,k} y^{-2k-1},$$

⁵The asymptotic (7) in the interior of the light-cone can be proved using integration by parts-stationary phase. In the exterior of the light cone, this asymptotic is true thanks to finite speed of propagation. Strictly speaking, the functions $\{f_1^0(0), i\sqrt{-\Delta}f(0)\}$ are not compactly supported. However, they are exponentially localized around the origin, so one may recover (7) in the exterior via weighted energy estimates. We will give an independent proof of this fact in a moment (see (14)) which does not rely on weighted energy estimates.

as $y \rightarrow \infty$. In other words, the functions $m_1(2\pi r\rho)$ and $m_2(2\pi r\rho)$ are C^∞ with derivatives in ρ uniformly bounded for all $\frac{1}{2} \leq \rho \leq 2$ and $2 \leq r$. Substituting the asymptotic (9) into the integral formula, we may assume without loss of generality that we are trying to bound integrals of the form:

$$(10) \quad I^\pm(t, r) = \frac{1}{(1+r)^{\frac{n-1}{2}}} \int_{-\infty}^{\infty} e^{2\pi i(t \pm r)\rho} m^\pm(r, \rho) \chi_{(1/4, 4)}(\rho) \widehat{f}_1(\rho) d\rho ,$$

where m^\pm is a smooth function with derivatives in ρ uniformly bounded for all $0 \leq r$, and $\chi_{(1/4, 4)}$ is a smooth bump function on the interval $(\frac{1}{4}, 4)$. It is now apparent that the integrals in (10) are essentially time translated inverse Fourier transforms of a one dimensional unit frequency function. Therefore, we can localize these integrals in physical space (i.e. the $t \pm r$ variable) on an $O(1)$ scale. This can be accomplished with the help of the so called φ -transform (see [2]), which is just a smoothed out redundant (over-sampled) version of the classical Shannon sampling. Since the function \widehat{f}_1 is compactly supported in the interval $(0, 4)$, we may take its Fourier series development:

$$(11) \quad \widehat{f}_1(\rho) = \sum_k c_k e^{i\frac{\pi}{2}k\rho} \quad , \quad \rho \in (0, 4) .$$

An important thing to notice here is that we can recover the L^2 of f_1 as a function on \mathbb{R}^n in terms of the $\{c_k\}$:

$$(12) \quad \|f_1\|_{L_x^2}^2 \sim \sum_k |c_k|^2 .$$

This can be seen from the Plancherel theorem and the fact that \widehat{f}_1 is unit frequency, so the volume part of the integral in Fourier space which comes from integrating over spheres is $O(1)$. Sticking the series (11) into the the integrals (10) yields:

$$(13) \quad I^\pm(t, r) = \sum_k \frac{c_k}{(1+r)^{\frac{n-1}{2}}} \psi_k^\pm(t, r) ,$$

where:

$$\psi_k^\pm(t, r) = \int_{-\infty}^{\infty} e^{2\pi i(t \pm r + \frac{k}{4})\rho} m^\pm(r, \rho) \chi_{(1/4, 4)}(\rho) d\rho .$$

Integrating by parts as many times as necessary in the above formula, we see that we have the asymptotic:

$$(14) \quad |\psi_k^\pm(t, r)| \leq \frac{C_M}{(1 + |t \pm r + \frac{k}{4}|)^M} .$$

Using the expansion (13) and the asymptotic (14) we can directly compute that for $2 \leq p$:

$$\begin{aligned} \|I^\pm(t, \cdot)\|_{L_x^p}^p &\lesssim \int_{-\infty}^{\infty} \left(\sum_k c_k \psi_k^\pm(t, r) \right)^p (1+r)^{(n-1)(1-\frac{p}{2})} dr , \\ &\lesssim \int_{-\infty}^{\infty} \left(\sum_k \frac{c_k}{(1 + |t \pm r + \frac{k}{4}|)^3} \right)^p (1+r)^{(n-1)(1-\frac{p}{2})} dr , \\ &\lesssim \sum_k \int_{-\infty}^{\infty} \frac{|c_k|^p}{(1 + ||t + \frac{k}{4}| - r|)^2} (1+r)^{(n-1)(1-\frac{p}{2})} dr , \end{aligned}$$

The manipulation to get the last line above follows from Hölders inequality and the fact that $2 \leq p$. By integrating each expression in this line term by term and using the inclusion $\ell^2 \subseteq \ell^p$ for $2 \leq p$ we arrive at the bound:

$$\begin{aligned} \|I^\pm(t, \cdot)\|_{L_x^p} &\lesssim \left(\sum_k \frac{|c_k|^p}{(1 + |t + \frac{k}{4}|)^{p(n-1)(\frac{1}{2} - \frac{1}{p})}} \right)^{\frac{1}{p}}, \\ &\lesssim \left(\sum_k \frac{|c_k|^2}{(1 + |t + \frac{k}{4}|)^{2(n-1)(\frac{1}{2} - \frac{1}{p})}} \right)^{\frac{1}{2}}. \end{aligned}$$

Testing this last expression for L^2 in time we see that:

$$\|I^\pm\|_{L_t^2(L_x^p)}^2 \lesssim \sum_k |c_k|^2 \int_{-\infty}^{\infty} (1 + |t + \frac{k}{4}|)^{-2(n-1)(\frac{1}{2} - \frac{1}{p})} dt.$$

Now as long as $1 < 2(n-1)(\frac{1}{2} - \frac{1}{p})$, or equivalently $\frac{2(n-1)}{n-2} < p$, we have the result:

$$\|I^\pm\|_{L_t^2(L_x^p)}^2 \lesssim \sum_k |c_k|^2.$$

Using the characterization (12), we have shown that:

Proposition 1.2 (Unit frequency Strichartz estimates for spherically symmetric data). *Let u_1 be a unit frequency, spherically symmetric solution to the equation $\square u_1 = 0$ in $3 \leq n$ (spatial) dimensions, then the following space-time estimates hold:*

$$(15) \quad \|u_1\|_{L^2(L^r)} \lesssim \|u_1(0)\|_{L^2},$$

where r satisfies the bound $\frac{2(n-1)}{n-2} < r$.

Interpolating (15) with the energy estimate (2), and by various rescalings and using Littlewood–Paley theory (as in the work [6], we will discuss this in more detail in the sequel), the estimate (15) can be extended to general spherical initial data and other $L^q(L^r)$ spaces in a straight forward manner. We record this as:

Theorem 1.3 (Strichartz estimates for spherically symmetric initial data). *Let u be a spherically symmetric function on \mathbb{R}^{n+1} such that $\square u = 0$, and set $\sigma_\Omega = n-1$, then the following estimates hold:*

$$(16) \quad \|u\|_{L^q(L^r)} \lesssim (\|u(0)\|_{\dot{H}^\gamma} + \|\partial_t u(0)\|_{\dot{H}^{\gamma-1}}),$$

where $\frac{1}{q} + \frac{\sigma_\Omega}{r} < \frac{\sigma_\Omega}{2}$, and $\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma$.

Remark 1.4. In the language of [6], Theorem 1.3 says that the range of indices for Strichartz estimates for spherically symmetric initial data are non-sharp $(n-1)$ admissible. From the point of view of uniform decay and the general machinery of [6], this is like saying that these solutions morally decay like t^{1-n} . Of course, solutions to wave equation in general (if you look everywhere in Minkowski space and take the supremum) only decay like $t^{\frac{1-n}{2}}$. The point here is that if they are spherically symmetric, they only do so on a relatively thin set.

We would now like to prove a result like (15) for non-spherical initial data. By looking at the Knapp counterexample calculation (6), we see that in order for us to avoid a contradiction, we need to replace the term $\|f_1^\epsilon\|_{L^2}$ on the left hand side of that equation by something that is of the order $\epsilon^{-2(\frac{\sigma}{2}-\frac{1}{2}-\frac{\sigma}{r})}\|f_1^\epsilon\|_{L^2}$. Now, based on our experience with spherical data, it is natural to conjecture that we only need to replace the norm $\|f_1^\epsilon\|_{L^2}$ with something that incorporates angular regularity.⁶ What we will do then, is test for smoothness of f_1^ϵ in the angular direction in Fourier space. This is done by using the infinitesimal generators of the rotations on Euclidean space:

$$(17) \quad \Omega_{i,j} := x_i \partial_j - x_j \partial_i .$$

One sees immediately that:

$$(18) \quad \|f_1^\epsilon\|_{|\Omega|^{-1}L^2}^2 := \sum_{i<j} \|\Omega_{ij} f_1^\epsilon\|_{L^2}^2 = \sum_{i<j} \|\Omega_{ij} \widehat{f_1^\epsilon}\|_{L^2}^2 \sim \epsilon^{-2} \|f_1^\epsilon\|_{L^2}^2 .$$

Interpolating this with the identity, we see that:

$$(19) \quad \|f_1^\epsilon\|_{|\Omega|^{-s}L^2} := \|\Omega|^s f_1^\epsilon\|_{L^2} \sim \epsilon^{-s} ,$$

where $|\Omega|^s = (-\Delta_{sph})^{\frac{s}{2}}$, and:

$$(20) \quad \Delta_{sph} := \sum_{i<j} \Omega_{ij}^2 ,$$

is the Laplacian on the sphere of radius r .⁷ Therefore, it is natural to expect that if we add $s = 2(\frac{\sigma}{2} - \frac{1}{2} - \frac{\sigma}{r})$ angular derivatives to the left hand side of (6), we may get a true estimate in the range $\frac{2(n-1)}{n-2} < r$. Furthermore, an estimate of this kind would be sharp. This is precisely what we will prove in dimensions $4 \leq n$, with an ϵ loss of angular regularity:

Theorem 1.5 (Strichartz estimates for angularly regular data). *Let $4 \leq n$ be the number of spatial dimensions, and let u be a solution to the homogeneous wave equation $\square u = 0$. Let $\sigma_\Omega = n-1$ be the angular wave admissible Strichartz exponent, and let $\sigma = \frac{n-1}{2}$ be the classical wave admissible Strichartz exponent. Then for every $0 < \epsilon$, there is a C_ϵ depending only on ϵ such that the following set of estimates hold:*

$$(21) \quad \|u\|_{L^q(L^r)} \lesssim C_\epsilon (\|\langle \Omega \rangle^s u(0)\|_{\dot{H}^\gamma} + \|\langle \Omega \rangle^s \partial_t u(0)\|_{\dot{H}^{\gamma-1}}) ,$$

where we have that $r \neq \infty$, $s = (1+\epsilon)(\frac{n-1}{r} + \frac{2}{q} - \frac{n-1}{2})$, $\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma$, $\frac{1}{q} + \frac{\sigma}{r} \geq \frac{\sigma}{2}$, and $\frac{1}{q} + \frac{\sigma_\Omega}{r} < \frac{\sigma_\Omega}{2}$. All of the implicit constants in the above inequality depend

⁶The regularity in the radial directions does not effect the optimal estimates one gets unless you ask for $L^q(L^r)$ estimates with $q < 2$. In order to get these type of estimates, it is necessary to incorporate some decay of the initial data, even if it is spherically symmetric. This can be easily seen by taking a spherically symmetric wave ψ with the asymptotic (7), and then forming the wave $\Psi = \sum_{k=0}^N \psi(t-k)$. Then at time zero one has $\|\Psi(0)\|_{L^2} \sim N^{\frac{1}{2}}$, but one gets $\|\Psi\|_{L^q(L^r)} \sim N^{\frac{1}{q}}$ for $\frac{2(n-1)}{n-2} < r < \infty$. Therefore it is not possible to take $q < 2$.

⁷Fractional powers of the operator $-\Delta_{sph}$ can be defined in the usual way via spectral resolution. Furthermore, the interpolation identity: $(L^2, |\Omega|^{-1}L^2)_t = |\Omega|^{-t}L^2$ can easily be shown using spectral resolution and interpolation of weighted ℓ^2 sequence spaces. We'll discuss this in more detail in just a bit.

on n , q , and r . Here:

$$\| \langle \Omega \rangle^s u \|_{\dot{H}^\gamma} = \| u(0) \|_{\dot{H}^\gamma} + \| |\Omega|^s u(0) \|_{\dot{H}^\gamma} ,$$

with the analogous norm defined for $\| \langle \Omega \rangle^s \partial_t u(0) \|_{\dot{H}^{\gamma-1}}$.

Remark 1.6. A short calculation like the one done above shows that in fact (modulo ϵ angular derivatives), all of the estimates (21) are sharp when tested on Knapp counterexamples. Therefore, in this sense, they are all endpoint estimates. It would be interesting to try and remove the extra ϵ in these. We will not pursue this issue here, although we will do some extra work in the sequel to recover a sharp L^2 dispersive estimate which could be a start in this direction (see Proposition 5.1).

Remark 1.7. For the case of $n = 2, 3$ spatial dimensions, we will also prove an estimate of the type $L^{2+}(L^\infty)$ and $L^2(L^{4+})$ respectively which involves $\frac{1}{2}$ an angular derivative. However, to obtain the full range of (21) in the case of $n = 3$ spatial dimensions would require one to prove an $L^2(L^\infty)$ Strichartz estimate that involves ϵ angular derivatives. While it seems that this type of estimate is out of the reach of methods we use here, it should be attainable using the recent method wave packet of Wolff [18]. In fact, all of the estimates (21) should be able to be proved directly using that method, based on the fact that they correspond to estimates that should be true for angularly separated initial data with no extra angular regularity.

The remainder of this paper is laid out as follows. In the next section, we list briefly some of the basic notations we use here.

In the third section, we list some standard facts about analysis on the sphere that will be useful in the sequel, including formulas for Hankel transforms and Littlewood–Paley–Stein theory on the sphere. We then use this machinery to reduce the proof of Theorem 1.5 to a suitable set of “endpoint” estimates.

In the fourth section, we discuss the main tool to be used in this paper: a φ -type transform for the Hankel transform. This leads us to consider the localization properties in physical space of the bandwidth limited Hankel transform. In particular, we provide detailed asymptotics for our Hankel– φ transform which will form the backbone the Strichartz estimates to be proved here.

In the fifth, we prove an L^2 dispersive estimate for the wave equation based on angular regularity with respect to the momentum operators (17). This can be interpolated with the energy estimate to prove our “endpoints” directly via integration in time, avoiding to use of any other analytic machinery such as usual TT^* process or induction on scales.

In the sixth, we give a brief description of how our dispersive estimate can be modified in a straight forward manner to accommodate multilinear phenomena. This shows one strength of the method used here in that one gets the expected range of improved multilinear estimates virtually for free out of the machinery developed. We will also discuss why these multilinear estimates are sharp, by testing

them on a multilinear analog of the Knapp counterexamples introduced above.

In an appendix to this paper, we provide a proof of the angularly regular endpoints in the $3 \leq n$ regime based on an idea of Igor Rodnianski. The proof there is essentially independent of the machinery we develop here (modulo a somewhat similar setup in terms of multilinear estimates) and instead relies on a direct calculation involving the energy-momentum tensor for the wave equation.

2. BASIC NOTATION

We list here some of the basic notation used throughout this paper. For quantities A and B , we denote by $A \lesssim B$ to mean that $A \leq C \cdot B$ for some large constant C . The constant C may change from line to line, but will always remain fixed for any given instance where this notation appears. We will also use the notation $A \sim B$ if there exists a constant C such that $\frac{1}{C} \cdot A \leq B \leq C \cdot A$.

For a given function of two variables, say $u(t, x)$, we denote the mixed Lebesgue spaces norms $L^q(L^r)$ of u via the formulas:

$$\|u\|_{L^q(L^r)}^q := \int \|u(t)\|_{L_x^r}^q dt .$$

For a given function of the spatial variable only, we denote its Fourier transform as:

$$\widehat{f}(\xi) := \int e^{-2\pi i \xi \cdot x} f(x) dx .$$

With this normalization of the Fourier variables, the Plancherel theorem becomes $\|f\|_{L^2} = \|\widehat{f}\|_{L^2}$, and one has the Fourier inversion formula:

$$f(x) = \int e^{2\pi i \xi \cdot x} \widehat{f}(\xi) d\xi .$$

Using the Fourier transform, we define the homogeneous Sobolev space of order γ via the identity $\|f\|_{\dot{H}^\gamma} := \||\xi|^\gamma \widehat{f}\|_{L^2}$.

For every integer k we define the spatial Littlewood–Paley cutoff operator by the formula:

$$(22) \quad \widehat{P_k f} := p_k \widehat{f} ,$$

where $p_k(\xi) = p_0(2^{-k}\xi)$ and p_0 is a positive smooth bump function, $p_0 \equiv 1$ on the interval $(1, 2)$ and zero off the interval $(\frac{1}{2}, 4)$. With this notation we have the following consequence of the Littlewood–Paley theorem for $2 \leq q, r$ and $r < \infty$:

$$(23) \quad \|u\|_{L^q(L^r)}^2 \lesssim \sum_k \|P_k u\|_{L^q(L^r)}^2 .$$

For a given function of the spatial variable only, we denote its forward and backward wave propagation via the formulas:

$$W^\pm f(t, x) = e^{\pm it\sqrt{-\Delta}} f(x) = \int e^{\pm 2\pi i t|\xi|} e^{2\pi i x \cdot \xi} \widehat{f}(\xi) d\xi .$$

If u is an arbitrary solution to the homogeneous wave equation, $\square u = 0$, with initial data $u(0) = f$ and $\partial_t u(0) = g$, we may decompose it into forward and backward wave propagation in the following way:

$$(24) \quad u(t, x) = \frac{1}{2i\sqrt{-\Delta}} (W^+ g - W^- g) + \frac{1}{2} (W^+ f + W^- f) .$$

Notice that in the case where $u = u_1$ is unit frequency, the functions:

$$h_1^+ = \frac{1}{2i\sqrt{-\Delta}} g_1 + \frac{1}{2} f_1 , \quad h_1^- = -\frac{1}{2i\sqrt{-\Delta}} g_1 + \frac{1}{2} f_1 ,$$

have L^2 norm comparable to the $\dot{H}^\gamma \times \dot{H}^{\gamma-1}$ norm of (f_1, g_1) . Therefore, in the sequel, we will always assume that our unit frequency waves are of the form $e^{\pm it\sqrt{-\Delta}} h_1^\pm$, and we replace:

$$\|u_1(0)\|_{L^2} \sim (\|u_1(0)\|_{\dot{H}^\gamma} + \|\partial_t u_1(0)\|_{\dot{H}^{\gamma-1}}) .$$

3. SOME RESULTS FROM ANALYSIS ON THE SPHERE.

We list here some basic results from Fourier analysis in spherical coordinates which will be used in our proof of Theorem 3.4. We have already introduced the two basic differential elements of analysis on the sphere, the rotation vector fields $\{\Omega_{ij}\}$ and the spherical Laplacian Δ_{sph} . For every integer $0 \leq l$, there exists a finite dimensional set of functions \mathcal{Y}_l on the sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ which satisfy the equation:

$$-\Delta_{sph} Y^l = l(n+l-2) Y^l , \quad Y^l \in \mathcal{Y}_l .$$

These sets of functions exhaust the set of eigenfunctions of $-\Delta_{sph}$ and can in fact be identified with the homogeneous polynomials P^l on \mathbb{R}^n of degree l which satisfy:

$$\left(\partial_r^2 + \frac{n-1}{r} \partial_r + \frac{1}{r^2} \Delta_{sph}\right) P^l = 0 .$$

Setting $\omega_{n-1} := |\mathbb{S}^{n-1}|$, and introducing the natural inner product on \mathbb{S}^{n-2} :

$$\langle F, G \rangle := \omega_{n-1}^{-1} \int_{\mathbb{S}^{n-1}} F(\omega) \overline{G(\omega)} d\omega ,$$

we have the following basic properties of the Y^l :

Lemma 3.1 (Basic properties of spherical harmonics (see e.g. [10])).

- (1) *The dimension of the space \mathcal{Y}_l is $|\mathcal{Y}_l| = \frac{1}{l}(n+2l-2) \binom{n+l-3}{l-1}$.*
- (2) *The spaces \mathcal{Y}_l are mutually orthogonal. That is $\langle Y^l, Y^k \rangle = 0$ for $l \neq k$.*
- (3) *Let $\{Y_i^l\}_{i=1}^{|\mathcal{Y}_l|}$ be any orthonormal basis of \mathcal{Y}_l , then one has the following identity for all $\omega \in \mathbb{S}^{n-1}$: $\sum_i |Y_i^l(\omega)|^2 = |\mathcal{Y}_l|$.*
- (4) *For each $Y^l \in \mathcal{Y}_l$, we have the identity:*

$$\sum_{i < j} \|\Omega_{ij} Y^l\|_{L^2(\mathbb{S}^{n-1})}^2 = l(n+l-2) \|Y^l\|_{L^2(\mathbb{S}^{n-1})}^2$$

From now on, we fix an orthonormal basis $\{Y_i^l\}$ for each \mathcal{Y}_l . For a given function $F \in L^2(\mathbb{S}^{n-1})$, we may expand it in the L^2 sense along this basis as follows:

$$(25) \quad F = \sum_{l,i} c_i^l Y_i^l .$$

Using (25), we can define the action of $|\Omega|^s$ on this F as follows:

$$(26) \quad |\Omega|^s F = \sum_{l,i} [l(n+l-2)]^{\frac{s}{2}} c_i^l Y_i^l .$$

Then using item (2) and (4) of Lemma (3.1), and the fact that $\Omega_{ij} Y^l \in \mathcal{Y}_l$ ($\Omega_{ij} Y^l$ is a homogeneous harmonic polynomial of degree l on \mathbb{R}^n), we see that we have the identity:

$$\sum_{i < j} \|\Omega_{ij} F\|_{L^2(\mathbb{S}^{n-1})}^2 = \sum_{l,i} l(n+l-2) |c_i^l|^2 = \|\Omega F\|_{L^2(\mathbb{S}^{n-1})}^2 .$$

Using this, we see that there is, for every test function f on \mathbb{R}^n an equivalence of norms:

$$(27) \quad \|f\|_{|\Omega|^{-1}L^2} = \|\Omega f\|_{L^2} ,$$

where $|\Omega|^{-1}L^2$ is the norm from line (18). We also make the definition:

$$\begin{aligned} \|f\|_{H_\Omega^s}^2 &:= \|\langle \Omega \rangle^s f\|_{L^2}^2 , \\ &:= \|f\|_{L^2}^2 + \|\Omega f\|_{L^2}^2 . \end{aligned}$$

Now, using the fact that H_Ω^s is of the form $L^2(\ell_s^2)$, we have the following standard interpolation result (see [1]):

Proposition 3.2 (Interpolation of the angular Sobolev spaces H_Ω^s). *For any set of real numbers s_1 and s_2 , we have the following interpolation spaces for $0 < t < 1$:*

$$(28) \quad (H_\Omega^{s_1}, H_\Omega^{s_2})_t = H_\Omega^s ,$$

where $s = (1-t)s_1 + ts_2$.

We will also use here some Littlewood–Paley theory in the angular variable⁸. We proceed in analogy with (22) and let θ_0 be a smooth bump function such that $\theta_0 \equiv 1$ on the interval $(1, 2)$ and vanishing off the interval $(\frac{1}{2}, 4)$. For each $j \in \mathbb{Z}$ we denote its dyadic rescaling as $\theta_j(l) := \theta_0(2^{-j}l)$. Defining $N = 2^j$ and using the decomposition formula (25), we define the angular frequency dyadic projections of a function F on the sphere as:

$$(29) \quad F_N := \sum_{l,i} c_i^l \theta_j(l) Y_i^l .$$

We define F_0 to be the constant c^0 , which is the average of F over the sphere. A formula similar to (29) can be used to define f_N for a function on the whole of \mathbb{R}^n .

For a given function F on the sphere, we use line (1) from 3.1, as well as the fact that $\frac{1}{l}(n+2l-2) \binom{n+l-3}{l-1} \sim N^{\frac{n-2}{2}}$ when $l \sim N$ to prove *Bernstein's inequality* for

⁸Notice that strictly speaking it will not be necessary for us to use the Littlewood–Paley theorem in the angular variable due to our allowed loss of angular regularity. However, the use of Littlewood–Paley *cutoffs* in the angular variable will be essential for us.

the sphere:

$$\begin{aligned}
|F_N| &\lesssim \sum_{l,i : l \sim N} |c_i^l| \cdot |Y_i^l|, \\
&\lesssim \sum_{l \sim N} \left(\sum_i |c_i^l|^2 \right)^{\frac{1}{2}} \cdot N^{\frac{n-2}{2}}, \\
&\lesssim \left(\sum_{l,i : l \sim N} |c_i^l|^2 \right)^{\frac{1}{2}} \cdot N^{\frac{n-1}{2}}, \\
(30) \quad &= N^{\frac{n-1}{2}} \|F\|_{L^2(\mathbb{S}^{n-1})}.
\end{aligned}$$

By rescaling (30) to spheres of various radii, we have the following result on all of \mathbb{R}^n (for $N = 0$, we of course replace the $N^{\frac{n-1}{2}}$ on the right hand side by 1):

$$(31) \quad \|f_N\|_{L_r^2(L^\infty(\mathbb{S}_r^{n-1}))} \lesssim N^{\frac{n-1}{2}} \|r^{-\frac{n-1}{2}} f\|_{L^2(\mathbb{R}^n)}.$$

Here \mathbb{S}_r^{n-1} denotes the sphere of radius r centered at the origin.

Next, we record the basic result which allows us to generate certain square function expressions in terms of the f_N for L^r spaces when $2 \leq r$:

Proposition 3.3 (Littlewood–Paley–Stein theorem for the sphere (see [11], [12], and [14])). *Let $\{\theta_j\}_{j=0}^\infty$ be any set of smooth functions such that there exists a $0 < \delta$ and $\delta < \theta_j(l) < \frac{1}{\delta}$ for $l \in (2^j, 2^{j+1})$, and $\theta_j(l) = 0$ for $l \notin (2^{j-1}, 2^{j+1})$. Furthermore, let θ_j satisfy the bounds:*

$$\left| \frac{d^M}{dl^M} \theta_j \right| \leq C_M \cdot l^{-M}.$$

Then one has that for any test function F on \mathbb{S}^{n-1} , the following bound on the ratio of norms holds for $1 < p < \infty$:

$$(32) \quad \frac{1}{C_{p,\theta}} < \|F\|_{L^p(\mathbb{S}^{n-1})} / \left\| \left(\sum_{j=0}^\infty \left| \sum_{l,i} \theta_j(l) c_i^l Y_i^l \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{S}^{n-1})} < C_{p,\theta}.$$

Here the constant $C_{p,\delta}$ depends only on p and the $\{\theta_j\}$.

Using Proposition 3.3 along with the decomposition (23), we have the following estimates for functions on space–time, for $2 \leq q, r$ and $r < \infty$:

$$(33) \quad \|u\|_{L^q(L^r)}^2 \lesssim \sum_{\substack{k \in \mathbb{Z} \\ N \in 2^{\mathbb{N}} \cup \{0\}}} \|P_k u_N\|_{L^q(L^r)}^2.$$

3.1. Reduction of Theorem 3.4 to a frequency localized endpoint. We now use the setup we have introduced above to reduce the proof of Theorem 3.4 to the following frequency localized “endpoint” estimate:

Proposition 3.4 (Endpoint unit frequency Strichartz estimate for angularly regular data). *Let $3 \leq n$ be the number of spatial dimensions, and let $u_{1,N}$ be a*

unit frequency, angular frequency localized solution to the homogeneous wave equation $\square u_{1,N} = 0$. Then for every $0 < \eta$, there exists an $\frac{2(n-1)}{n-2} < r_\eta$, such that $r_\eta \rightarrow \frac{2(n-1)}{n-2}$ as $\eta \rightarrow 0$ and such that the following estimate holds:

$$(34) \quad \|u_{1,N}\|_{L^2(L^{r_\eta})} \lesssim C_\eta N^{\frac{1}{2}+\eta} \|u_{1,N}(0)\|_{L^2},$$

where the implicit constants in the above inequality depend only on n and r .

In the case of $n = 2$ spatial dimensions, we will also prove the following:

Proposition 3.5 (Endpoint $(2+1)$ dimensional unit frequency Strichartz estimate for angularly regular data). *Let $n = 2$ be the number of spatial dimensions, and let $u_{1,N}$ be a unit frequency, angular frequency localized solution to the homogeneous wave equation $\square u_{1,N} = 0$. Then for every $0 < \eta$ there is a $2 < q_\eta$ such that $q_\eta \rightarrow 2$ as $\eta \rightarrow 0$ and such that the following estimate holds:*

$$(35) \quad \|u_{1,N}\|_{L^{q_\eta}(L^\infty)} \lesssim C_\eta N^{\frac{1}{2}+\eta} \|u_{1,N}(0)\|_{L^2}.$$

Assuming now the validity of Proposition 3.4, we prove Theorem 1.5 as follows: Given exponents (q, r) such that $\frac{1}{q} + \frac{\sigma}{r} \geq \frac{\sigma}{2}$ and $\frac{1}{q} + \frac{\sigma\Omega}{r} < \frac{\sigma\Omega}{2}$, we first reduce things to the case where $q = 2$. We define t and r_1 via the formulas:

$$\begin{aligned} \frac{1}{q} &= \frac{t}{2}, \\ \frac{1}{r} &= \frac{t}{r_1} + \frac{t-1}{2}. \end{aligned}$$

Notice that we can find such a $0 \leq t \leq 1$ and $\frac{2(n-1)}{n-2} < r_1$ due to the range of (q, r) . Therefore, interpolating with the energy estimate (using Proposition 3.2), it suffices to prove (21) for indices $(2, r_1)$ with $\frac{2(n-1)}{n-2} < r_1$.

Next, using the decomposition (33) and rescaling the spatial frequency of each term in the resulting sum, it suffices to show that:

$$(36) \quad \|u_{1,N}\|_{L^2(L^{r_1})} \lesssim C_\epsilon N^{(1+\epsilon)(\frac{n-1}{r_1} - \frac{n-3}{2})} \|u_{1,N}(0)\|_{L^2}.$$

To do this, we choose:

$$(37) \quad 0 \leq \frac{2(n-1)}{r_1} - (n-3) < t_\epsilon < 1,$$

such that there exists an $0 < \eta$ with the property that:

$$(38) \quad \left(\frac{1}{2} + \eta\right)t_\epsilon \leq (1+\epsilon)\left(\frac{n-1}{r_1} - \frac{n-3}{2}\right).$$

That such choices are possible follows from our assumptions on the range of r_1 and the identity:

$$\left(\frac{1}{2} + \frac{\epsilon}{2}\right)t_0 = (1+\epsilon)\left(\frac{n-1}{r_1} - \frac{n-3}{2}\right),$$

where we have set $t_0 = \frac{2(n-1)}{r_1} - \frac{n-3}{2}$. Because of the range (37), we see that it is also possible to choose an $\frac{2(n-1)}{n-2} < r_{\eta_*} \leq \frac{2(n-1)}{n-3}$ with the property that:

$$\frac{t_\epsilon}{r_{\eta_*}} + \frac{1-t_\epsilon}{2(n-1)/(n-3)} = \frac{1}{r_1}.$$

Furthermore, using Proposition 3.4 and possibly a Sobolev embedding, we see that we have the estimate:

$$\|u_{1,N}\|_{L^2(L^{r_{n^*}})} \lesssim C_{\eta,\eta^*} N^{\frac{1}{2}+\eta} \|u_{1,N}(0)\|_{L^2}.$$

Interpolating this last line with the $L^2(L^{\frac{2(n-1)}{n-3}})$ endpoint of (4) we have achieved (36). Therefore, in the sequel, we will concentrate on the proof of (34).

3.2. The Hankel transform. Finally, to wrap things up for this section, we record here the following formula for the action of the inverse Fourier transform on the decomposition:

$$\widehat{f} = \sum_{l,i} \widehat{c}_i^l Y_i^l.$$

As is well known, this is given by a series of Hankel transforms. The formula is (see [10]):

$$(39) \quad f(r\omega) = \sum_{l,i} 2\pi(\sqrt{-1})^l r^{\frac{2-n}{2}} \int_0^\infty J_{\frac{n-2}{2}+l}(2\pi r\rho) \widehat{c}_i^l(\rho) \rho^{\frac{n}{2}} d\rho \cdot Y_i^l(\omega).$$

Here $J_s(y)$ is the Bessel function of order s . For $-\frac{1}{2} < s$, this is given by the integral formula:

$$(40) \quad J_s(y) = \frac{(y/2)^s}{\Gamma[(2s+1)/2]\Gamma(1/2)} \int_{-1}^1 e^{ity} (1-t^2)^{\frac{2s-1}{2}} dt.$$

4. THE HANKEL- φ TRANSFORM

As we see from the formula (39) of the last subsection, it is possible to expand the expression⁹ $e^{-it\sqrt{-\Delta}}f$ in terms of spherical harmonics as:

$$(41) \quad e^{-it\sqrt{-\Delta}}f(r\omega) = \sum_{l,i} c_i^l(t,r) Y_i^l(\omega),$$

where the coefficients c_i^l are given by the Hankel transform formula:

$$(42) \quad c_i^l(t,r) = 2\pi(\sqrt{-1})^l r^{\frac{2-n}{2}} \int_0^\infty J_{\frac{n-2}{2}+l}(2\pi r\rho) e^{-2\pi i t\rho} \widehat{c}_i^l(\rho) \rho^{\frac{n}{2}} d\rho.$$

Here, as in the previous subsection, the $\widehat{c}_i^l(\rho)$ are the coefficients in the spherical harmonic expansion of $\widehat{f}(\rho)$. Also, the coefficients $c_i^l(t,r)$ should not be confused with the inverse Fourier transform of the $\widehat{c}_i^l(\rho)$.

We would now like to be able to localize the expressions (42) in a manner analogous to the localization of the integral (8). This would be a relatively simple matter, if we could show that the asymptotic (9) held uniformly in n . That is, if there was an asymptotic of the form (9) for $J_{\frac{n-2}{2}+l}(y)$ which held uniformly as $l \rightarrow \infty$. Unfortunately, it is well known that this is only the case for the region

⁹Throughout this subsection, we will work with the operator $e^{-it\sqrt{-\Delta}}$ instead of $e^{it\sqrt{-\Delta}}$. Of course this is just a matter of notational convenience, as can be seen for instance by time reversal.

$l \ll \sqrt{y}$ (see [17]). In the transition regions, that is when $\sqrt{y} \lesssim l \lesssim y$, the asymptotic for $J_{\frac{n-2}{2}+l}(y)$ becomes quite complicated. Roughly speaking, it begins to lose oscillations in y while it gains decay in the parameter l . Because of this, it does not seem feasible to try and compute an approximate formula for $J_{\frac{n-2}{2}+l}$ and then substitute it into the integrals (42). Instead we will use a more straightforward approach, by first localizing the \widehat{c}_i^l in frequency as a Fourier series, just as we had done for \widehat{f}_1 in the integral (8), and then computing the integral (42) directly by using appropriate integral representations for the $J_{\frac{n-2}{2}+l}$.

Since we are assuming that the initial data in Theorem 3.4 is unit frequency, we will assume that all of the coefficient functions $\widehat{c}_i^l(\rho)$ in the integrals (42) are supported on the interval $(\frac{1}{2}, 2)$. We may take their Fourier series developments on the interval $(0, 4)$, and we record these as:

$$(43) \quad \widehat{c}_i^l(\rho) = \sum_k c_{i,k}^l e^{i\frac{\pi}{2}k\rho} .$$

Expanding the integral (42) in terms of the above formula, we see that:

$$(44) \quad c_i^l(t, r) = 2\pi(\sqrt{-1})^l \sum_k r^{\frac{2-n}{2}} c_{i,k}^l \psi_{t-\frac{k}{4}}^l(r) ,$$

where:

$$(45) \quad \psi_{t-\frac{k}{4}}^l(r) = \int_0^\infty J_{\frac{n-2}{2}+l}(2\pi r\rho) e^{-2\pi i(t-\frac{k}{4})\rho} \chi_{(\frac{1}{4}, 4)}(\rho) d\rho .$$

In the above formula $\chi_{(\frac{1}{4}, 4)}$ is a smooth bump function on the interval $(\frac{1}{4}, 4)$. Notice that this is not necessarily equal to 1 on any interval because we have absorbed the volume element into our definition of $\chi_{(\frac{1}{4}, 4)}$. We call the right hand side of (44) the *Hankel- φ* transform of the function $c_i^l(t, r)$. We would now like to be able to give a precise bound on how well localized the functions ψ_m^l are in physical space for the various values of the half-integer parameter l and the real variable m . This brings us to the main result of this subsection:

Proposition 4.1 (Asymptotics of the functions ψ_m^l). *Let ψ_m^l be the function given by the formula (45) for $m = t - \frac{k}{4}$. Then for every set integers $0 \leq N_1, N_2$, there exists a constant C_{N_1, N_2} depending only on the N_i (and the dimension n) such that the following asymptotics hold uniform in the parameters l, m , and r :*

$$(46) \quad |\psi_m^l(r)| \leq r^{\frac{n-2}{2}} \cdot \frac{C_{N_1, N_2}}{(1+|m|)^{N_1}} \cdot \left(\frac{1}{1+l}\right)^{N_2} , \quad 0 \leq r < 1 ,$$

$$(47) \quad |\psi_m^l(r)| \leq \frac{C_{N_1, N_2}}{(1+r+|m|)^{\frac{1}{2}} \cdot (1+|r-|m||)^{N_1}} \cdot \left(\frac{r^{\frac{1}{2}}}{1+l}\right)^{N_2} , \quad 1 \leq r \leq |m|+1 .$$

$$(48) \quad |\psi_m^l(r)| \leq \frac{1}{(r^2 - m^2)^{\frac{1}{2}}} \cdot R(l, m, r) , \quad |m|+1 < r .$$

The extra term $R(m, r)$ in line (48) above is a positive function with the bound:

$$(49) \quad R(l, m, r) \leq C_{N_1, N_2} \left(\frac{1}{(1 + ||m| - r|)^{N_1}} + \min_{\pm} \left\{ \left(\frac{l}{\sqrt{r^2 - m^2}} \right)^{\pm N_2} \right\} \right) .$$

Remark 4.2. The downside of the above asymptotic is of course the region governed by (48). When $l^2 \lesssim r$, one can see that the extra factor (49) will allow this asymptotic to look like (47). Notice that this is consistent with the fact that one has the asymptotic (9) for $J_s(y)$ in this region, and is what is responsible for the good localization (14) for spherically symmetric waves. Unfortunately, it does not seem like one can do much to improve (48) in the region where $r \ll l^2$ (except for the extra factor of $R(l, m, r)$). In fact, if one assumes there is an asymptotic for $\psi_m^l(r)$ in this region which is of the form (47), and one sets the $c_{i,k}^l \equiv 1$ in the sum (44) for a fixed l, i , by putting absolute values around the sum (44) one would get an asymptotic that looks like: $|c_i^l(0, r)| \lesssim \frac{1}{r^{\frac{1}{2}}}$. But in this case, $c_i^l(t, r)$ corresponds to a delta function along the radial variable in Fourier space, say supported at the point $\frac{1}{2\pi}$. Therefore we would have shown a bound like $|J_s(r)| \lesssim \frac{1}{r^{\frac{1}{2}}}$ uniform in s . This violates the well known asymptotic for Bessel functions: $|J_s(s)| \sim s^{-\frac{1}{3}}$ (see [17]). It would be interesting to know if there is a more coherent decomposition of the Hankel transform that could eliminate this problem.

proof of Proposition 4.1. The asymptotics (46)–(47) follow more or less directly from appropriate integral formulas for the $J_s(y)$. We will need to split the proof into the two cases: $\{r \leq 1 \text{ or } 1 < r \leq |m| + 1\}$ and $\{|m| + 1 < r\}$.

Case 1: $r \leq 1$ or $1 < r \leq |m| + 1$.

Here we use a standard integral representation for Bessel functions which differs from (40). For $s \in \mathbb{N}$, one has the following formula:

$$(50) \quad J_s(y) = \frac{(-i)^s}{2\pi} \int_0^{2\pi} e^{iy \cos \theta} e^{-is\theta} d\theta .$$

This can be proved by a simple recursive argument (see [10], Chapter 4, Lemma 3.1). All one has to do is to show that for both the integral formulas (50) and (40), the function $J_s(y)$ satisfies the recursive relation:

$$(51) \quad \frac{d}{dt} [t^{-s} J_s(y)] = t^{-s} J_{s+1}(y) , \quad 0 < t ,$$

for $s \in \mathbb{N}$. In light of (51), the equality of (50) and (40) is reduced to showing that it is true when $s = 0$. This can be achieved directly through a change of variables.

Now using periodicity, integrating over an adjacent interval of length 2π , and averaging, we see that for $s \in \mathbb{N}$ the following integral representation also holds:

$$(52) \quad J_s(y) = \frac{(-i)^s}{4\pi} \int_{-2\pi}^{2\pi} e^{iy \cos \theta} e^{-is\theta} d\theta .$$

Moreover, by a direct calculation, it's not hard to see that the recursive relation (51) is satisfied by both the integrals (52) and (40) whenever $s \in \frac{1}{2} \cdot \mathbb{N}$. Therefore,

throughout the sequel, we may assume that (52) is our definition of the Bessel function that appears in the integral formula (45) for the function ψ_m^l in dimension n . Making this substitution yields:

$$(53) \quad \begin{aligned} \psi_m^l(r) &= \frac{(-i)^s}{4\pi} \int_{-\infty}^{\infty} \int_{-2\pi}^{2\pi} e^{2\pi i(r \cos \theta - m)\rho} e^{-i(\frac{n-2}{2}+l)\theta} \chi_{(\frac{1}{4},4)}(\rho) d\theta d\rho, \\ &= \frac{(-i)^s}{4\pi} \int_{-2\pi}^{2\pi} \check{\chi}_{(\frac{1}{4},4)}(r \cos \theta - m) e^{-i(\frac{n-2}{2}+l)\theta} d\theta. \end{aligned}$$

We begin by proving the asymptotic (46). In fact, we will prove a bit more. We will show that the asymptotic (46) holds for $r \leq 30$. Our first step will be to pick up the decay in the $(1+l)$ parameter by integrating by parts the expression (53) N_2 times. The resulting expression looks like:

$$(54) \quad \psi_m^l(r) = \left(\frac{i}{\frac{n-2}{2}+l} \right)^{N_2} \sum_{k=1}^{N_2} \int_{-2\pi}^{2\pi} r^k p_k(\theta) \check{\chi}_{(\frac{1}{4},4)}^{(k)}(r \cos \theta - m) e^{-i(\frac{n-2}{2}+l)\theta} d\theta,$$

where the $p_k(\theta)$ in the above formula denote some specific trigonometric polynomials of degree k whose exact form is not important for our analysis. The next step is to gain the decay in r in conjunction with the damping in terms of inverse powers of $(1+|m|)$. To get this, we Taylor expand each $\check{\chi}_{(\frac{1}{4},4)}^{(k)}(h(\theta) - m)$, where

$$(55) \quad h(\theta) := r \cos \theta,$$

around the point $h = 0$. Notice that this is consistent with the fact that we are investigating the region where r is bounded. We now define the dimensional constant $M = \lceil \frac{n-2}{2} \rceil$, we record this Taylor expansion as:

$$(56) \quad \check{\chi}_{(\frac{1}{4},4)}^{(k)}(h - m) = \sum_{j=0}^{M-1} \frac{1}{j!} \check{\chi}_{(\frac{1}{4},4)}^{(k+j)}(-m) \cdot h^j + \frac{1}{M!} \check{\chi}_{(\frac{1}{4},4)}^{(k+M)}(u(h) - m) \cdot h^M,$$

where $u(h)$ is some smooth function such that $|u(h)| \leq 30$. Substituting the Taylor expansion (56) into the integral (54), we see that we may write:

$$\psi_m^l(r) = A + B,$$

where:

$$A = \left(\frac{i}{\frac{n-2}{2}+l} \right)^{N_2} \sum_{k+j < M} \frac{1}{j!} \check{\chi}_{(\frac{1}{4},4)}^{(k+j)}(-m) \int_{-2\pi}^{2\pi} r^{j+k} p_k(\theta) \cos^j(\theta) e^{-i(\frac{n-2}{2}+l)\theta} d\theta.$$

and:

$$|B| \leq C_{N_2} \cdot r^M \cdot \left(\frac{1}{\frac{n-2}{2}+l} \right)^{N_2} \cdot \sum_{k=M}^{N_2+M} \sup_{m-30 \leq x \leq m+30} |\check{\chi}_{(\frac{1}{4},4)}^{(k)}(x)|.$$

We can further estimate the term B above by using the fact that $\check{\chi}$ and all of its derivatives have rapid decay away from the origin. This follows immediately from the fact that $\check{\chi}$ is the inverse Fourier transform of a smooth $O(1)$ bump function. We record this observation as:

$$(57) \quad |\check{\chi}_{(\frac{1}{4},4)}^{(k)}(y)| \lesssim \frac{C_{k,N_1}}{(1+|y|)^{N_1}}.$$

By adding things up and introducing a large enough constant, this allows us to write:

$$|B| \leq C_{N_1, N_2} \cdot r^M \cdot \left(\frac{1}{\frac{n-2}{2} + l} \right)^{N_2} \cdot \frac{1}{(1 + |m|)^{N_1}} .$$

Recalling now that we have set $M = \lceil \frac{n-2}{2} \rceil$ and that we also have $3 \leq n$, we see that in order to achieve the bound (46), all we need to do is to control the expression for A . This is easy to do because a moments inspection shows that in fact one has $A \equiv 0$. This can be readily seen for even dimensions, that is when n is even, because in this case the trigonometric polynomials under the integral sign in the expression for A are of degree strictly less than $\frac{n-2}{2}$. By orthogonality, the whole expression then integrates to zero. In the case of odd dimension, each integral is still zero thanks to the fact that $\frac{n-2}{2}$ is a half integer expression, where as the term:

$$p_k(\theta) \cos^j(\theta) e^{-il} ,$$

is a trigonometric polynomial of integer degree. Since we are integrating over the double torus, $[-2\pi, 2\pi]$, a rescaling turns the expression under the integral sign in A into a product of even degree trigonometric polynomials and odd degree trigonometric polynomials. Therefore one has the needed orthogonality. This completes the proof of (46).

We now turn our attention to proving the asymptotic (47) for the regime where $1 < r \leq |m| + 1$. For the remainder of this section we will assume that m is positive, as the other case can be dealt with by a similar argument. Furthermore, using the bound that we proved in the previous discussion, we can without loss of generality assume that $20 \leq m$.

Our first step is to split the integral on the right hand side of (53) smoothly into the regions where $(1 - \cos \theta) \ll 1$ and otherwise. To realize this split, we restrict the integral (53) to the regions:

$$\begin{aligned} R_1 &= \{ \theta \mid |\theta| < 1 \} , \\ R_2 &= \{ \theta \mid |\theta \pm 2\pi| < 1 \} , \\ R_3 &= [-2\pi, 2\pi] \setminus (\{ \theta \mid |\theta| < \frac{1}{2} \} \cup \{ \theta \mid |\theta \pm 2\pi| < \frac{1}{2} \}) . \end{aligned}$$

Notice that by symmetry, we only need to consider the regions R_1 and R_3 . On these regions, a bit of explicit computation using Taylor expansions shows that:

$$(58) \quad h(\theta) - m = -\frac{1}{2}r[u(\theta)]^2 - (m - r) , \quad \theta \in R_1 ,$$

$$(59) \quad |h(\theta) - m| \geq \frac{1}{100}m , \quad \theta \in R_2 \text{ and } ,$$

where the function $u(\theta)$ in the first line above satisfies the bound:

$$(60) \quad \frac{1}{2} < u(\theta)/\theta^2 < 2 .$$

Notice that to get (59) we have used the condition $20 \leq m$. Next, using another simple calculation involving the Taylor series of trigonometric functions, as well as

the estimates (57), (58) and (60), we see that one may write for $|\theta| < 1$:

$$(61) \quad \left| \frac{d^{N_2}}{d\theta^{N_2}} [\check{\chi}(h(\theta) - m)] \right| \leq C_{N_2} \sup_{0 \leq j \leq N_2} |\check{\chi}^{(j)}(h(\theta) - m)| \cdot \sum_{k=0}^{\lfloor \frac{N_2}{2} \rfloor} r^{N_2-k} |\theta|^{N_2-2k},$$

$$\leq \frac{C_{N_1, N_2}}{(2 + m - r + r\theta^2)^{N_1+N_2+1}} \cdot \sum_{k=0}^{\lfloor \frac{N_2}{2} \rfloor} r^{N_2-k} |\theta|^{N_2-2k}.$$

We are now in a position to bound the integral (53) on the region R_1 . To do this, we will employ a smooth cutoff function $\chi_{(-1,1)}$ such that:

$$\chi_{(-1,1)}(\theta) = \begin{cases} 1, & \theta \in (-\frac{1}{2}, \frac{1}{2}), \\ 0, & \theta \notin (-1, 1). \end{cases}$$

We are trying to prove the estimate (47) for the expression:

$$\int \chi_{(-1,1)}(\theta) \cdot \check{\chi}_{(\frac{1}{4},4)}(h(\theta) - m) e^{-i(\frac{n-2}{2}+l)\theta} d\theta.$$

Integrating by parts N_2 times in the above integral and using the bound (61) (along with the fact that we are in dimension $3 \leq n$) we compute:

$$\begin{aligned} & \left| \int \chi_{(-1,1)}(\theta) \cdot \check{\chi}_{(\frac{1}{4},4)}(h(\theta) - m) e^{-i(\frac{n-2}{2}+l)\theta} d\theta \right|, \\ & \leq \left(\frac{1}{l+1} \right)^{N_2} \cdot \left| \int \frac{d^{N_2}}{d\theta^{N_2}} [\chi_{(-1,1)}(\theta) \cdot \check{\chi}_{(\frac{1}{4},4)}(h(\theta) - m)] e^{-i(\frac{n-2}{2}+l)\theta} d\theta \right|, \\ & \leq \left(\frac{1}{l+1} \right)^{N_2} \cdot \sum_{0 \leq k \leq \frac{N_2}{2}} \int \frac{C_{N_1, N_2}}{(2 + m - r + r\theta^2)^{N_1+N_2+2}} \cdot r^{N_2-k} |\theta|^{N_2-2k} d\theta, \\ & \leq C_{N_1, N_2} \frac{1}{r^{\frac{1}{2}}} \cdot \left(\frac{r^{\frac{1}{2}}}{l+1} \right)^{N_2} \int \frac{1 + |\theta|^{N_2}}{(2 + m - r + \theta^2)^{N_1+N_2+2}} d\theta, \\ & \leq \frac{C_{N_1, N_2}}{r^{\frac{1}{2}}(2 + m - r)^{N_1+1}} \cdot \left(\frac{r^{\frac{1}{2}}}{l+1} \right)^{N_2} \int \frac{1 + |\theta|^{N_2}}{(1 + \theta^2)^{N_2+1}} d\theta, \\ & \leq \frac{C_{N_1, N_2}}{(1 + m + r)^{\frac{1}{2}}(2 + m - r)^{N_1}} \cdot \left(\frac{r^{\frac{1}{2}}}{l+1} \right)^{N_2}. \end{aligned}$$

Thus, we have proved (47) for this portion of things.

It remains to prove the estimate (47) for the region R_3 . We suggestively (c.f. (59)) denote the cutoff here by $\chi_{m \leq |h(\theta)|} = (1 - \chi_{(-1,1)})$. The calculation is essentially the same as what was done above, except that here we can afford to be more careless about the powers of r which come up through integration by parts. Integrating by parts N_2 times and using the bound (59) in conjunction with the

estimate (57), we see that:

$$\begin{aligned}
& \left| \int \chi_{m \lesssim |h(\theta)|}(\theta) \cdot \check{\chi}_{(\frac{1}{4}, 4)}(h(\theta) - m) e^{-i(\frac{n-2}{2}+l)\theta} d\theta \right|, \\
& \leq C_{N_2} \left(\frac{1}{l+1} \right)^{N_2} \cdot \sum_{0 \leq k \leq N_2} r^{N_2} \cdot \sup_{\frac{1}{100}m \leq |h-m|} |\check{\chi}_{(\frac{1}{4}, 4)}^{(k)}(h-m)|, \\
& \leq C_{N_1, N_2} \left(\frac{1}{l+1} \right)^{N_2} \cdot \frac{r^{N_2}}{(1+m)^{N_1 + \frac{N_2+1}{2}}}, \\
& \leq \frac{C_{N_1, N_2}}{(1+m+r)^{\frac{1}{2}}(1+m-r)^{N_1}} \cdot \left(\frac{r^{\frac{1}{2}}}{l+1} \right)^{N_2}.
\end{aligned}$$

This completes the proof of (47) and ends the demonstration of **case 1**.

Case 2: $|m| + 1 < r$.

In this section, the main difficulty will be for us to incorporate the remainder term (49) into our asymptotic. In order to motivate the steps we will take here, we argue heuristically as follows: We first define the auxiliary function:

$$g(\theta) := r \cos \theta - m.$$

It is clear that the main contribution to the integral (53) integrating around points where the function $g(\theta)$ defined above vanishes. By symmetry, we need only consider such points which are positive. We'll call the one closest to zero θ_0 . Now, expanding $g(\theta)$ around this zero gives:

$$(62) \quad h(\theta) = -\sqrt{r^2 - m^2}(\theta - \theta_0) - \frac{m}{2}(\theta - \theta_0)^2 + O((\theta - \theta_0)^3),$$

where again $0 < \theta_0 = \cos^{-1}(\frac{m}{r})$. Therefore, it is natural to expect that:

$$\begin{aligned}
\text{(R.H.S.)}(53) & \approx \frac{(-i)^s}{4\pi} \int_{-2\pi}^{2\pi} \check{\chi}_{(\frac{1}{4}, 4)}(-\sqrt{r^2 - m^2}(\theta - \theta_0)) e^{-i(\frac{n-2}{2}+l)\theta} d\theta, \\
& \approx \frac{e^{-i(\frac{n-2}{2}+l)\theta_0}}{(r^2 - m^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \check{\chi}_{(\frac{1}{4}, 4)}(\theta) e^{i\frac{(n-2)/2+l}{\sqrt{r^2 - m^2}}\theta} d\theta, \\
(63) \quad & = \frac{e^{-i(\frac{n-2}{2}+l)\theta_0}}{(r^2 - m^2)^{\frac{1}{2}}} \cdot \left(i \frac{(n-2)/2+l}{\sqrt{r^2 - m^2}} \right)^{N_2} \int_{-\infty}^{\infty} \check{\chi}_{(\frac{1}{4}, 4)}^{(-N_2)}(\theta) e^{i\frac{(n-2)/2+l}{\sqrt{r^2 - m^2}}\theta} d\theta,
\end{aligned}$$

where the $\check{\chi}_{(\frac{1}{4}, 4)}^{(-N_2)}$, $N_2 = 0, \pm 1, \pm 2, \dots$, are the (unique) set of derivatives and derivatives of $\check{\chi}_{(\frac{1}{4}, 4)}$ which vanish at infinity. These satisfy the bounds similar to (57), even if N_2 is positive:

$$(64) \quad \check{\chi}_{(\frac{1}{4}, 4)}^{(-N_2)}(y) \leq \frac{C_{N_2}}{(1+|y|)^2}.$$

This, of course, comes from the fact that $\check{\chi}_{(\frac{1}{4}, 4)}$ is the inverse Fourier transform of a unit frequency function, and is therefore supported away from the origin in Fourier space. Using (64) we easily get the following bound for arbitrary non-negative N_2 :

$$|(\text{R.H.S.})(63)| \leq \frac{C_{N_2}}{(r^2 - m^2)^{\frac{1}{2}}} \cdot \min_{\pm} \left\{ \left(\frac{l}{\sqrt{r^2 - m^2}} \right)^{\pm N_2} \right\}.$$

In order to make the previous argument rigorous, we need to justify the two approximations used in the lines directly above (63). We will not be able to do this completely, which will be responsible for the extra term in the formula (49) for $R(l, m, r)$. As in the previous two subsections, we argue by isolating the interval of integration. By symmetry, and the decay bound (57), we can without loss of generality assume that we are integrating (53) over the interval $[0, \frac{3\pi}{4}]$. We will first need to go a little further and chop some more off of the left hand side of this interval. What we'll do is take θ_1 (we're still keeping $\theta_0 = \cos^{-1}(\frac{m}{r})$) to be such that:

$$(65) \quad \frac{r}{\sqrt{r^2 - m^2}} \sin(\theta_1) = \frac{1}{2}.$$

Taylor expanding $\sin \theta$ gives us the bound:

$$(66) \quad \theta_1 \lesssim \frac{(r - m)^{\frac{1}{2}}}{r^{\frac{1}{2}}}.$$

Furthermore, we also have the bound:

$$(67) \quad \frac{3}{4}(r - m) \leq \frac{\sqrt{3r^2 + m^2}}{2} - m \leq r \cos \theta - m = g(\theta), \quad \theta \in [0, \theta_1].$$

Now, using (66) and (67) above, in conjunction with the asymptotic (57), we immediately see that:

$$\int_0^{\theta_1} |\check{\chi}_{(\frac{1}{4}, 4)}(g(\theta))| d\theta \lesssim \frac{(r - m)^{\frac{1}{2}}}{r^{\frac{1}{2}}} \cdot \frac{C_{N_1}}{(r - m)^{N_1}}.$$

This is enough to give (48) for this portion of things. Therefore, it remains to compute the integral:

$$I_1 = \int_{\theta_1}^{\frac{3\pi}{4}} \check{\chi}_{(\frac{1}{4}, 4)}(g(\theta)) e^{-i(\frac{n-2}{2}+l)\theta} d\theta.$$

Keeping the Taylor expansion (62) in mind, we now make the following change of variable for I_1 :

$$\varphi(\theta) = g\left(\frac{1}{\sqrt{r^2 - m^2}}\theta + \theta_0\right).$$

Using this, we can write:

$$(68) \quad |I_1| = \frac{1}{\sqrt{r^2 - m^2}} \left| \int_R \check{\chi}_{(\frac{1}{4}, 4)}(\varphi(\theta)) e^{-i\frac{(n-2)/2+l}{\sqrt{r^2 - m^2}}\theta} d\theta \right|,$$

Where R is the interval:

$$R = \left[\sqrt{r^2 - m^2}(\theta_1 - \theta_0), \sqrt{r^2 - m^2}\left(\frac{3\pi}{4} - \theta_0\right) \right].$$

The desired result will now follow by integrating by parts as many times as necessary the integral (68). However, some care needs to be taken in order to control terms involving φ' , which can be as big as $\frac{r}{\sqrt{r^2 - m^2}}$. Also, one needs to know that the higher derivatives of φ possess some decay in order to deal with terms where the

derivatives fall on $\frac{1}{\varphi'}$ instead of the exponential factor. We address these issues now. The first observation we make is that we may integrate the bound:

$$(69) \quad \frac{1}{2} \leq \frac{r}{\sqrt{r^2 - m^2}} \sin \theta, \quad \theta \in [\theta_1, \frac{3\pi}{4}],$$

using the fact that $\varphi(0) = 0$ to get that:

$$(70) \quad |\theta| \lesssim |\varphi(\theta)|.$$

over the range of integration. However, this is not enough to address the issue of a product of the form $(\varphi')^k \cdot \check{\chi}_{(\frac{1}{4}, 4)}^{(j)}$. To handle this, notice that one has the bound:

$$(71) \quad \frac{r}{\sqrt{r^2 - m^2}} \sin \theta \leq m - r \cos \theta + 1, \quad \theta \in [\theta_1, \frac{3\pi}{4}],$$

as can be seen from the fact that this bound is true for $\theta = \theta_1$, and that upon differentiating both sides of this expression one is reduced to showing (the inequality for the derivatives is trivial for $\theta \in [\frac{\pi}{2}, \frac{3\pi}{4}]$):

$$\frac{1}{\sqrt{r^2 - m^2}} \leq \tan \theta, \quad \theta \in [\theta_1, \frac{\pi}{2}].$$

This then follows immediately from the increasing nature of $\tan \theta$ on this interval and from the identity $\tan(\theta_1) = \frac{\sqrt{r^2 - m^2}}{m}$. Now, combining the bounds (71), (70) and (57), we see that for any positive integers k, j , one has the estimates:

$$(72) \quad (\varphi')^k(\theta) \cdot \check{\chi}_{(\frac{1}{4}, 4)}^{(j)}(\varphi(\theta)) \lesssim \frac{C_{j,k}}{(1 + |\theta|)^2}.$$

Finally, we record the fact that the higher derivatives of φ satisfy the following simple bounds, which can be verified through direct calculation:

$$(73) \quad |\varphi^{(k)}| \lesssim \frac{1}{(r - m)^{k-1}}, \quad M = 2, 3, \dots$$

We are now ready to bound (68). First, we integrate as many times as necessary, letting the derivatives fall on the term $\check{\chi}_{(\frac{1}{4}, 4)}(\varphi(\theta))$. The resulting expression looks like:

$$\begin{aligned} & \left| \int_R \check{\chi}_{(\frac{1}{4}, 4)}(\varphi(\theta)) e^{-i \frac{(n-2)/2+l}{\sqrt{r^2 - m^2}} \theta} d\theta \right|, \\ & \lesssim \left(\frac{\sqrt{r^2 - m^2}}{1 + l} \right)^{N_2} \int_R \left| \frac{d^{N_2}}{d\theta^{N_2}} \check{\chi}_{(\frac{1}{4}, 4)}(\varphi(\theta)) \right| d\theta + \sum_{k=0}^{N_2-1} \left| \frac{d^k}{d\theta^k} \check{\chi}_{(\frac{1}{4}, 4)}(\varphi(\theta)) \right| \Big|_{\partial R}. \end{aligned}$$

Expanding out the derivatives in the first term of the above expression, and using the bounds (71) and (70), we see that we can bound:

$$\left| \frac{d^{N_2}}{d\theta^{N_2}} \check{\chi}_{(\frac{1}{4}, 4)}(\varphi(\theta)) \right| \lesssim \frac{C_{N_2}}{(1 + |\theta|)^2}.$$

Therefore, the integral in the first term above does not cause us any trouble. As far as the boundary values are concerned, we can use the bounds (65), (67), (71),

and (57) to show that:

$$\begin{aligned}
& \sum_{k=0}^{N_2-1} \left| \frac{d^k}{d\theta^k} \check{\chi}_{(\frac{1}{4},4)}(\varphi(\theta)) \right| \Big|_{\partial R}, \\
& \lesssim C_{N_2} \sup_{k_i \leq N_2} \left(|\varphi^{(k_1)}(\theta)|^{k_2} \cdot |\check{\chi}_{(\frac{1}{4},4)}^{(k_3)}(\varphi(\theta))| \right) \Big|_{\frac{\sqrt{r^2-m^2}(\frac{3r}{4}-\theta_0)}{\sqrt{r^2-m^2}(\theta_1-\theta_0)}}, \\
& \lesssim C_{N_1, N_2} \left(\frac{1}{(r-m)^{N_1}} + \frac{1}{(r+m)^{N_1}} \right), \\
& \lesssim \frac{C_{N_1, N_2}}{(r-m)^{N_1}}.
\end{aligned}$$

It remains to bound (68) where we integrate by parts and let the derivatives fall on the exponential factor. Doing this N_2 times yields:

$$\begin{aligned}
& \left| \int_R \check{\chi}_{(\frac{1}{4},4)}(\varphi(\theta)) e^{-i\frac{(n-2)/2+l}{\sqrt{r^2-m^2}}\theta} d\theta \right|, \\
& = \left| \int_R \left(\frac{1}{\varphi'(\theta)} \frac{d}{d\theta} \right)^{N_2} \left(\check{\chi}_{(\frac{1}{4},4)}^{(-N_2)}(\varphi(\theta)) \right) \cdot e^{-i\frac{(n-2)/2+l}{\sqrt{r^2-m^2}}\theta} d\theta \right|, \\
& \lesssim \int_R |\check{\chi}_{(\frac{1}{4},4)}^{(-N_2)}(\varphi(\theta))| \cdot \left| \left(\frac{1}{\varphi'(\theta)} \frac{d}{d\theta} \right)^{N_2} \left(e^{-i\frac{(n-2)/2+l}{\sqrt{r^2-m^2}}\theta} \right) \right| d\theta \\
(74) \quad & + \sum_{k=1}^{N_2} |\check{\chi}_{(\frac{1}{4},4)}^{(-k)}(\varphi(\theta))| \cdot \left| \frac{1}{\varphi'(\theta)} \left(\frac{1}{\varphi'(\theta)} \frac{d}{d\theta} \right)^{k-1} \left(e^{-i\frac{(n-2)/2+l}{\sqrt{r^2-m^2}}\theta} \right) \right| \Big|_{\partial R}
\end{aligned}$$

To control the first term above, we bound:

$$\begin{aligned}
(75) \quad & \left| \left(\frac{1}{\varphi'(\theta)} \frac{d}{d\theta} \right)^{N_2} \left(e^{-i\frac{(n-2)/2+l}{\sqrt{r^2-m^2}}\theta} \right) \right|, \\
& \lesssim \sup_{0 \leq k \leq 2N_2-1} \left| \frac{1}{\varphi'(\theta)} \right|^k \cdot \sum_{j=0}^{N_2} \left(\frac{1+l}{\sqrt{r^2-m^2}} \right)^j |P_{N_2-j}(\varphi^{(1+\bullet)}(\theta))|,
\end{aligned}$$

where P_0 is a constant, and the other P_k denote a homogeneous expression of weight k in the variables $(\varphi^{(2)}, \dots, \varphi^{(i)}, \dots)$, where each $\varphi^{(i)}$ is given weight $i-1$. Therefore, using the bound (69) (which in particular implies that $|1/\varphi'| \leq 2$) as well as the bounds (73), we see that we can estimate:

$$\begin{aligned}
(75) \quad & \lesssim C_{N_2} \sum_{j=0}^{N_2} \left(\frac{1+l}{\sqrt{r^2-m^2}} \right)^j \cdot \frac{1}{(r-m)^{N_2-j}}, \\
& \lesssim C_{N_2} \left(\left(\frac{1+l}{\sqrt{r^2-m^2}} \right)^{N_2} + \frac{1}{(r-m)^{N_2}} \right).
\end{aligned}$$

Finally, to deal with the boundary terms on the right hand side of (74) we use the bounds (67) and (64) and simply estimate:

$$\begin{aligned} & \sum_{k=1}^{N_2} |\check{\chi}_{(\frac{1}{4},4)}^{(-k)}(\varphi(\theta))| \cdot \left| \frac{1}{\varphi'(\theta)} \left(\frac{1}{\varphi'(\theta)} \frac{d}{d\theta} \right)^{k-1} \left(e^{-i \frac{(n-2)/2+l}{\sqrt{r^2-m^2}} \theta} \right) \right| \Big|_{\partial R}, \\ & \lesssim C_{N_2} \sup_{0 \leq k \leq N_2} |\check{\chi}_{(\frac{1}{4},4)}^{(-k)}(\varphi(\theta))| \Big|_{\partial R}, \\ & \lesssim \frac{C_{N_1, N_2}}{(r-m)^{N_1}}. \end{aligned}$$

Combining the above estimates together, we have shown that:

$$|I_1| \lesssim \frac{C_{N_1, N_2}}{\sqrt{r^2-m^2}} \left(\frac{1}{(r-m)^{N_1}} + \min_{\pm} \left\{ \left(\frac{l}{\sqrt{r^2-m^2}} \right)^{\pm N_2} \right\} \right).$$

This completes the proof of the asymptotic (48), and the demonstration of Proposition 4.1. \square

5. SOME L^2 DISPERSIVE ESTIMATES FOR THE WAVE EQUATION; LINEAR AND BILINEAR ESTIMATES

We are now ready to directly proceed with the proof of estimate (34). As we have mentioned previously, we may assume that $u_{1,N}$ is of the form $e^{-it\sqrt{-\Delta}} f_{1,N}$, for some unit frequency and dyadic angular frequency function $f_{1,N}$. For this $u_{1,N}$, we use the formulas from lines (41), (42), (43), (44), and (45) to expand its wave propagation into harmonics and Hankel- φ transforms as follows:

$$(76) \quad e^{-it\sqrt{-\Delta}} f_{1,N}(r) = \sum_{\substack{l \sim N \\ i,k}} r^{\frac{2-n}{2}} c_{i,k}^l \psi_{t-\frac{k}{4}}^l(r) \cdot Y_i^l$$

In the above expression, we have absorbed the constants $2\pi(\sqrt{-1})^l$ into each $c_{i,k}^l$. By orthogonality of everything in sight, we have that:

$$(77) \quad \|f_{1,N}\|_{L^2}^2 \sim \sum_{\substack{i,k \\ l \sim N}} |c_{i,k}^l|^2.$$

We now state our main result as follows:

Proposition 5.1 (L^2 dispersive estimate for the wave equation). *For dimensions $2 \leq n$, let $e^{-it\sqrt{-\Delta}} f_{1,N}$ be given by the formula (76). Then one has the following estimate uniform in t and r :*

$$(78) \quad |e^{-it\sqrt{-\Delta}} f_{1,N}(r)| \lesssim \left(\sum_{\substack{l \sim N \\ i,k}} \frac{|c_{i,k}^l|^2}{(1+|t-\frac{k}{4}|)^{n-1}} \right)^{\frac{1}{2}} \cdot N^{\frac{n-1}{2}}.$$

Before proceeding with the proof, let us first show briefly how the above estimate may be used to show (34)–(35). We begin with (34). What we need to do is to show that for every $0 < \eta$, there exists an $\frac{2(n-1)}{n-2} < r_\eta$, such that the following estimate holds:

$$(79) \quad \| e^{-it\sqrt{-\Delta}} f_{1,N} \|_{L^2(L^{r_\eta})} \lesssim N^{\frac{1}{2}+\eta} \cdot \| f_{1,N} \|_{L^2} ,$$

where the implicit constants depend on both η and r . We also need r_η to approach $\frac{2(n-1)}{n-2}$ as $\eta \rightarrow 0$. Now, interpolating¹⁰ (78) with the energy estimate (2) gives the following result for $2 \leq r_\eta$:

$$(80) \quad \| e^{-it\sqrt{-\Delta}} f_{1,N} \|_{L_x^{r_\eta}} \lesssim \left(\sum_{\substack{l \sim N \\ i,k}} \frac{|c_{i,k}^l|^2}{(1 + |t - \frac{4}{k}|)^{(\frac{1}{2} - \frac{1}{r_\eta})2(n-1)}} \right)^{\frac{1}{2}} \cdot N^{(\frac{1}{2} - \frac{1}{r_\eta}) \frac{n-1}{2}} .$$

Choosing r_η according to the formula $\frac{1}{2} + \eta = (\frac{1}{2} - \frac{1}{r_\eta}) \frac{n-1}{2}$, we see that we have the following identity holds: $(\frac{1}{2} - \frac{1}{r_\eta})2(n-1) = 1 + 2\eta$. Therefore, for this choice of r_η , we may square (80) and integrate directly in time to achieve (79). Also note that $r_\eta \rightarrow \frac{2(n-1)}{n-2}$ as $\eta \rightarrow 0$.

In the case of estimate (35), that is $n = 2$ spatial dimensions, we use Hölders inequality to see that for every $0 < \eta$ the following estimate holds:

$$\left(\sum_{\substack{l \sim N \\ i,k}} \frac{|c_{i,k}^l|^2}{(1 + |t - \frac{4}{k}|)} \right)^{\frac{1}{2}} \lesssim \left(\sum_{\substack{l \sim N \\ i,k}} \frac{|c_{i,k}^l|^2}{(1 + |t - \frac{4}{k}|)^{1+\eta}} \right)^{\frac{1}{2(1+\eta)}} \cdot \left(\sum_{\substack{l \sim N \\ i,k}} |c_{i,k}^l|^2 \right)^{\frac{\eta}{2(1+\eta)}} .$$

Therefore, if we choose $q_\eta = 2(1 + \eta)$ a direct integration and the energy bound (77) shows that we have:

$$\| e^{-it\sqrt{-\Delta}} f_{1,N} \|_{L^{q_\eta}(L^\infty)} \lesssim N^{\frac{1}{2}} \cdot \| f_{1,N} \|_{L^2} .$$

We are now reduced to proving the estimate (78).

proof of Proposition 5.1. Writing everything out, and using a Cauchy–Schwartz in the sum over i for each fixed l we have that:

(81)

$$\begin{aligned} |e^{-it\sqrt{-\Delta}} f_{1,N}(r)| &\lesssim \left| \sum_{\substack{l \sim N \\ i,k}} r^{\frac{2-n}{2}} c_{i,k}^l \psi_{t-\frac{k}{4}}^l(r) \cdot Y_i^l \right| , \\ &\lesssim \left(\sum_i \left| \sum_{\substack{l \sim N \\ k}} r^{\frac{2-n}{2}} c_{i,k}^l \psi_{t-\frac{k}{4}}^l(r) \right|^2 \right)^{\frac{1}{2}} \cdot \sup_{l \sim N} \left(\sum_i |Y_{i,k}^l|^2 \right)^{\frac{1}{2}} . \end{aligned}$$

¹⁰This can be achieved by simply interpolating in weighted ℓ^2 and the usual Lebesgue spaces using the map $\{c_{i,k}^l\} \rightarrow e^{-it\sqrt{-\Delta}} f_{1,N}$ for fixed time (see [1]).

Now, using the formula in item (3) of Lemma 3.1, and the fact that $\frac{1}{l}(n+2l-2)\binom{n+l-3}{l-1} \sim N^{n-2}$ for $l \sim N$, we have:

$$\begin{aligned} \text{(L.H.S.)}(81) &\lesssim \left(\sum_i \left| \sum_{\substack{l \sim N \\ k}} r^{\frac{2-n}{2}} c_{i,k}^l \psi_{t-\frac{k}{4}}^l(r) \right|^2 \right)^{\frac{1}{2}} \cdot N^{\frac{n-2}{2}}, \\ &\lesssim \left(\sum_{\substack{l \sim N \\ i}} \left| \sum_k r^{\frac{2-n}{2}} c_{i,k}^l \psi_{t-\frac{k}{4}}^l(r) \right|^2 \right)^{\frac{1}{2}} \cdot N^{\frac{n-1}{2}}, \end{aligned}$$

where the last line above follows from a Cauchy–Schwartz and the bound $l \sim N$. The proposition will now be shown if we can prove that:

$$(82) \quad \sum_k r^{\frac{2-n}{2}} |c_{i,k}^l| \cdot |\psi_{t-\frac{k}{4}}^l(r)| \lesssim \left(\sum_{\substack{l \sim N \\ i,k}} \frac{|c_{i,k}^l|^2}{(1+|t-\frac{k}{4}|)^{n-1}} \right)^{\frac{1}{2}}.$$

We now use the asymptotics of Proposition 4.1 and a little computation to split:

$$r^{\frac{2-n}{2}} |\psi_{t-\frac{k}{4}}^l(r)| \lesssim A\left(r, t - \frac{k}{4}\right) + B\left(r, t - \frac{k}{4}\right),$$

where $B\left(r, t - \frac{k}{4}\right)$ is supported where $1 < |t - \frac{k}{4}| < (r-1)$, and:

$$\begin{aligned} A\left(r, t - \frac{k}{4}\right) &= \frac{1}{(1+|t-\frac{k}{4}|)^{\frac{n-1}{2}}} \cdot \frac{1}{(1+|r-|t-\frac{k}{4}||)}, \\ B\left(r, t - \frac{k}{4}\right) &= \frac{1}{(1+|t-\frac{k}{4}|)^{\frac{n-1}{2}}} \cdot \frac{1}{(1+|r-|t-\frac{k}{4}||)^{\frac{1}{2}}} \cdot R\left(l, t - \frac{k}{4}, r\right). \end{aligned}$$

Here $R\left(l, t - \frac{k}{4}, r\right)$ is the function from line (49). Substituting $A\left(r, t - \frac{k}{4}\right)$ into the left hand side of the sum (82), and using a Cauchy–Schwartz immediately yields the desired result for this half of things. In order to finish up, then, we only need substitute to $B\left(r, t - \frac{k}{4}\right)$ into the right hand side. Doing this and using a Cauchy–Schwartz and re-indexing, we see that it is enough to show:

$$(83) \quad \sum_{\substack{k \in \frac{1}{4} \cdot \mathbb{Z} + t \\ |k| < (r-1)}} \frac{1}{(1+|r-|k||)} \cdot R(l, k, r) \lesssim 1.$$

uniform in r and l . This can be done by comparing things to the appropriate integrals and is left to the reader. Notice that the convergence factor $R(l, k, r)$ avoids any logarithmic divergences in this sum. This completes the proof of (78) and therefore the proof of Theorem 1.5. \square

6. BILINEAR ESTIMATES FOR ANGULARLY REGULAR DATA

As is well known, for applications in the lower dimensional setting, linear estimates of the form (21) are not sufficient. What is needed are multilinear versions. An extremely versatile method for building these type of estimates is based on the

“fine and coarse scale” machine of T. Tao (see [7]). The basic idea is to fix a scale, say $\frac{1}{\mu}$ for $\mu \ll 1$, and then decompose the domain of spatial variable into cubes with side lengths of this scale. Then, one replaces the usual L^r norm in the spatial variable with $\ell^r(L^2)$, where the L^2 norm is taken on the “fine” scale of each individual cube, while the ℓ^r norm represents the “coarse” scale which is summation over all cubes. One reason this method is so important, is that it allows one to use the bilinear construction process directly in an iteration procedure where resorting to the canned estimates that this method ultimately provides may be unduly burdensome. Also, the way these estimates are constructed will allow us to generate bilinear estimates where only one term in the product contains extra angular regularity. These type of estimates are extremely important for applications when one proves inductive estimates via an iteration procedure where it is necessary to consider estimates for products with an angular derivative falling on one or the other term in the product. Because of these considerations, we will content ourselves here with the dual scale estimates themselves, and not bother with listing out the various multilinear estimates which follow from them. In the $(4 + 1)$ and higher dimensions, the usual dual scale estimates read:

Theorem 6.1 (“Improved” Strichartz estimates). *Let $4 \leq n$ be the number of spatial dimensions. Let $0 < \mu \lesssim 1$ be given, and let $\{Q_\alpha\}$ be a partition of \mathbb{R}^n into cubes of side length $\sim \frac{1}{\mu}$. Then if u_1 is a unit frequency solution to the equation $\square u_1 = 0$, the following estimates hold:*

$$(84) \quad \left\| \left(\sum_{\alpha} \|u_1(t)\|_{L^2(Q_\alpha)}^r \right)^{\frac{1}{r}} \right\|_{L_t^2} \lesssim \mu^{-1} \|u_1(0)\|_{L^2},$$

where $\frac{2(n-1)}{n-3} \leq r$.

We show here that both the range of (84) and the power of μ that appears there can be significantly improved. In what follows, we will only bother proving an estimate which is the analog of (3.4). More general estimates which involve various amounts of angular regularity can then be gained by interpolating this estimate with (84). Also, it is not so easy to work out the Littlewood–Paley theory in the angular variable for localized norms like those that appear on the left hand side of (84). However, since we are already losing a small amount of angular regularity, one may simply replace the Littlewood–Paley sum (33) by an ℓ^1 sum. As we have noted before, this has no bearing on applications.

Theorem 6.2 (“Improved” frequency localized Strichartz estimates for angularly regular data; endpoint case). *Let $3 \leq n$ be the number of spatial dimensions, and let $u_{1,N}$ be a unit frequency and angular frequency localized solution to the homogeneous wave equation $\square u_{1,N} = 0$. Let $0 < \mu \lesssim 1$ be given, and let $\{Q_\alpha\}$ be a partition of \mathbb{R}^n into cubes of side length $\sim \frac{1}{\mu}$. Then for every $0 < \eta$, there is a C_η and $\frac{2(n-1)}{n-2} < r_\eta$ depending on η , such that $r_\eta \rightarrow \frac{2(n-1)}{n-2}$ as $\eta \rightarrow 0$ such that the following estimate holds:*

$$(85) \quad \left\| \left(\sum_{\alpha} \|u_{1,N}(t)\|_{L^2(Q_\alpha)}^{r_\eta} \right)^{\frac{1}{r_\eta}} \right\|_{L_t^2} \lesssim C_\eta \mu^{-\frac{1}{2}-2\eta} N^{\frac{1}{2}+\eta} \|u_{1,N}(0)\|_{L^2}.$$

Remark 6.3. Up to the small loss in $\frac{1}{\mu}$ and N , the estimate (85) is sharp when tested against the bilinear analog of the Knapp counterexamples (5). We construct these briefly as follows. We consider (frequency) initial data sets χ_{B^ϵ} which along with being highly localized in the angular variable, are also well localized in the radial variable. That is, we are now assuming χ_{B^ϵ} is supported on a small square of dimensions $\sim \epsilon \times \epsilon \times \dots \times \epsilon$, lying along the ξ_1 axis between $1/2 < \xi_1 < 2$. A quick calculation then shows that the integral:

$$e^{it\sqrt{-\Delta}} f_1^\epsilon(x) = \int e^{2\pi i(t|\xi|+x\cdot\xi)} \chi_{B^\epsilon}(\xi) d\xi ,$$

behaves like $|e^{it\sqrt{-\Delta}} f_1^\epsilon(x)| \sim \epsilon^n$ on the space–time region $S_{t,x}^\epsilon$:

$$\begin{aligned} t &= O(\epsilon^{-2}) , \\ t + x_1 &= O(\epsilon^{-1}) , \\ x' &= O(\epsilon^{-1}) . \end{aligned}$$

Choosing our cubes Q_α with side lengths $\frac{1}{\mu} \sim \frac{1}{\epsilon}$, we see that for any $\frac{2(n-1)}{n-2} < r$ (in fact for any $1 \leq r \leq \infty$) the following is true:

$$(86) \quad \mu^{-\frac{1}{2}} \epsilon^{-\frac{1}{2}} \|f_1^\epsilon\|_{L^2} \sim \epsilon^n \cdot \frac{1}{\epsilon^{\frac{n}{2}}} \cdot \frac{1}{\epsilon} \lesssim \left\| \left(\sum_\alpha \|e^{it\sqrt{-\Delta}} f_1^\epsilon\|_{L^2(Q_\alpha)}^r \right)^{\frac{1}{r}} \right\|_{L_t^2}$$

Therefore, using (19), we see that (85) is indeed sharp for this sequence of initial data. Of course the condition $\frac{2(n-1)}{n-2} < r$ cannot be improved, even for spherically symmetric initial data. In fact, one can also see for these type of waves (say with the asymptotic (7)), (85) is again sharp. We leave this simple calculation to the interested reader.

proof of estimate (85). The proof will be similar in spirit to that of (34). However, we will need to use orthogonality in a more fundamental way here. This is because we will not be able to rely solely on L^∞ as we did in the proof of (78). Ultimately, this has to do with the fact that the wave packets $\psi_m^l(r)$ are not well localized in physical space when $|m| < r$ and $r \ll l^2$ (see the remark after Proposition 4.1).

In order to prove estimate (85), we begin by writing the basic energy estimate (2) in the following way (we are still using the notation from line (76)):

$$(87) \quad \left\| \left(\sum_\alpha \|e^{-it\sqrt{-\Delta}} f_{1,N}\|_{L^2(Q_\alpha)}^2 \right)^{\frac{1}{2}} \right\|_{L_t^\infty} \lesssim \left(\sum_{\substack{l \sim N \\ i,k}} |c_{i,k}^l|^2 \right)^{\frac{1}{2}} .$$

Our next step is to prove the following spatially localized fixed time estimate:

$$(88) \quad \sup_\alpha \|e^{-it\sqrt{-\Delta}} f_{1,N}\|_{L^2(Q_\alpha)} \lesssim |\ln \mu| \mu^{-\frac{n-1}{2}} N^{\frac{n-1}{2}} \left(\sum_{\substack{l \sim N \\ i,k}} \frac{|c_{i,k}^l|^2}{(1 + |t - \frac{k}{4}|)^{n-1}} \right)^{\frac{1}{2}} .$$

Assuming for the moment the validity of (88), we may interpolate between (87) and (88) to achieve the following estimate for $2 \leq r_\eta$:

$$(89) \quad \left(\sum_{\alpha} \| e^{-it\sqrt{-\Delta}} f_{1,N} \|_{L^2(Q_\alpha)}^{r_\eta} \right)^{\frac{1}{r_\eta}} \\ \lesssim \left(\sum_{\substack{l \sim N \\ i,k}} \frac{|c_{i,k}^l|^2}{(1 + |t - \frac{4}{k}|)^{(\frac{1}{2} - \frac{1}{r_\eta})(n-1)}} \right)^{\frac{1}{2}} \cdot |\ln \mu| \mu^{-\left(\frac{1}{2} - \frac{1}{r_\eta}\right) \frac{n-1}{2}} N^{\left(\frac{1}{2} - \frac{1}{r_\eta}\right) \frac{n-1}{2}}$$

As in the previous section, choosing r_η by the identity, $\frac{1}{2} + \eta = \left(\frac{1}{2} - \frac{1}{r_\eta}\right) \frac{n-1}{2}$, we have that $\left(\frac{1}{2} - \frac{1}{r_\eta}\right) 2(n-1) = 1 + 2\eta$. Therefore, squaring (89) and integrating directly in time, we will have achieved (85). Therefore, we now concentrate on proving (88).

It suffices to show (88) for a fixed α . Therefore, we will now assume that we are on a fixed cube Q_α . Our first step is foliate Q_α with the hypersurfaces $\mathbb{S}_r^{n-1} \cap Q_\alpha$, where \mathbb{S}_r^{n-1} is the sphere of radius r centered at the origin. Notice that for each r , one has the area estimate $|\mathbb{S}_r^{n-1} \cap Q_\alpha| \lesssim \mu^{-(n-1)}$. Therefore, using a Cauchy-Schwartz, it suffices to prove the estimate:

$$(90) \quad \| e^{-it\sqrt{-\Delta}} f_{1,N} \|_{L_r^2(L^\infty(\mathbb{S}_r^{n-1}))(Q_\alpha)} \lesssim |\ln \mu| N^{\frac{n-1}{2}} \left(\sum_{\substack{l \sim N \\ i,k}} \frac{|c_{i,k}^l|^2}{(1 + |t - \frac{k}{4}|)^{n-1}} \right)^{\frac{1}{2}}.$$

Next, we chop Q_α into at most $|\ln \mu|$ dyadic pieces, which are of the form $R_j \cap Q_\alpha$, where $R_j = \{x \mid 2^j < |x| < 2^{j+1}\}$ is the radial dyadic region of size $r \sim 2^j$. We only need to do this for $0 < j$, i.e we keep the ball of bounded radius (say radius = 2) as a single region R_0 . Therefore, to show (90), it suffices to estimate:

$$(91) \quad \sup_j \| e^{-it\sqrt{-\Delta}} f_{1,N} \|_{L_r^2(L^\infty(\mathbb{S}_r^{n-1}))(R_j)} \lesssim N^{\frac{n-1}{2}} \left(\sum_{\substack{l \sim N \\ i,k}} \frac{|c_{i,k}^l|^2}{(1 + |t - \frac{k}{4}|)^{n-1}} \right)^{\frac{1}{2}}.$$

Now fix R_j . To prove estimate (91) here, we run the decomposition (76) and decompose the resulting sum into the sum of pieces:

$$e^{-it\sqrt{-\Delta}} f_{1,N} = \Sigma_0 + \Sigma_1,$$

where:

$$\Sigma_0 = \sum_{\substack{l \sim N \\ i,k \\ |t - \frac{4}{k}| > 4 \cdot 2^j}} r^{\frac{2-n}{2}} c_{i,k}^l \psi_{t - \frac{k}{4}}^l(r) \cdot Y_i^l, \\ \Sigma_1 = \sum_{\substack{l \sim N \\ i,k \\ |t - \frac{4}{k}| \leq 4 \cdot 2^j}} r^{\frac{2-n}{2}} c_{i,k}^l \psi_{t - \frac{k}{4}}^l(r) \cdot Y_i^l,$$

To estimate (91) on the sum Σ_1 , we simply keep it as a whole object and use the Bernstein inequality (31) to estimate:

$$\begin{aligned}
& \sup_j \|\Sigma_1\|_{L_r^2(L^\infty(\mathbb{S}_r^{n-1}))(R_j)}, \\
& \lesssim N^{\frac{n-1}{2}} 2^{-\frac{n-1}{2}j} \|\Sigma_1\|_{L_x^2}, \\
& \lesssim N^{\frac{n-1}{2}} 2^{-\frac{n-1}{2}j} \left(\sum_{\substack{l \sim N \\ i,k \\ |t-\frac{k}{4}| \leq 4 \cdot 2^j}} |c_{i,k}^l|^2 \right), \\
& \lesssim N^{\frac{n-1}{2}} \left(\sum_{\substack{l \sim N \\ i,k \\ |t-\frac{k}{4}| \leq 4 \cdot 2^j}} \frac{|c_{i,k}^l|^2}{(1+|t-\frac{k}{4}|)^{n-1}} \right).
\end{aligned}$$

As was to be shown. Therefore, we are reduced to bounding Σ_0 . Notice that for each fixed $r \in R_j$, and for each term in this sum, we have $|t - \frac{k}{4}| > r$. Therefore, we can use the well localized asymptotic (47) and a Cauchy-Schwartz in the k summation (in a way similar to the computation started on line (81)) to bound this term in $L^\infty(\mathbb{S}_r^{n-1})$ by:

$$|\Sigma_0(r)| \lesssim \left(\sum_{\substack{l \sim N \\ i,k \\ |t-\frac{k}{4}| > 4 \cdot 2^j}} \frac{|c_{i,k}^l|^2}{(1+|t-\frac{k}{4}|)^{n-1}} \cdot \frac{1}{(1+|r-|t-\frac{k}{4}||)^2} \right)^{\frac{1}{2}} \cdot N^{\frac{n-1}{2}}.$$

Squaring this last expression, and integrating each term in the sum with respect to dr (notice there is no extra volume element because we took the sup on \mathbb{S}_r^{n-1}), we see that we have proved (91) for this portion of things. This then completes the proof of estimate (88), and therefore the proof of (85). \square

7. APPENDIX

We present here a simplified proof of Theorem 3.4 and Theorem 6.2. First, we note that in order to prove both (34)

$$(92) \quad \|u_{1,N}\|_{L_t^2 L_x^{p_\eta}} \lesssim C_\eta N^{\frac{1}{2}+\eta} \|u_{1,N}(0)\|_{L^2}$$

and (85)

$$(93) \quad \|u_{1,N}\|_{L_t^2 \ell^{p_\eta} L_\mu^2} \lesssim C_\eta \mu^{-\frac{1}{2}-\eta} N^{\frac{1}{2}+\eta} \|u_{1,N}(0)\|_{L^2}$$

with $p_\eta \rightarrow 2(n-1)/(n-2-\eta)$ and arbitrary small $\eta > 0$, it suffices to prove the estimate (85). Here $u_{1,N}$ is a unit frequency solution of the wave equation $\square u = 0$ of angular frequency N and the norm

$$\|f\|_{\ell^p L_\mu^2} = \left(\sum_\alpha \|f\|_{L^2(Q_\alpha)}^p \right)^{\frac{1}{p}},$$

where $\{Q_\alpha\}$ is a partition of \mathbb{R}^n into cubes Q_α of side length of μ^{-1} .

To see this, we choose a partition $\{Q_\alpha\}$ of size $\mu = 1$ and compute, using the Sobolev embedding on Q_α , that for any $2 \leq p$:

$$\begin{aligned} \|u_{1,N}\|_{L_t^2 L_x^p} &= \left\| \left(\sum_\alpha \|u_{1,N}\|_{L^p(Q_\alpha)}^p \right)^{\frac{1}{p}} \right\|_{L_t^2}, \\ &\lesssim \left\| \left(\sum_\alpha \|u_{1,N}\|_{L^2(Q_\alpha)}^p \right)^{\frac{1}{p}} \right\|_{L_t^2}. \end{aligned}$$

Therefore, we see that it suffices to deal with the estimate (93). First, using the Sobolev inequality $L_\omega^\infty \subset \langle \Omega \rangle^{-\frac{n-1}{2}} L_\omega^2$ on the unit sphere \mathbb{S}^{n-1} for angular frequency localized functions, we have the following estimate for any tiling of \mathbb{R}^n by cubes $\{Q_\alpha\}$ of side length $\sim \mu^{-1}$:

$$\begin{aligned} \|u_{1,N}(t)\|_{L^2(Q_\alpha)}^2 &\lesssim \int_{r\omega \in Q_\alpha} |u_{1,N}(t)|^2 dx \lesssim \mu^{-\frac{n-1}{2}} \int_0^\infty dr r^{n-1} \sup_{\omega \in \mathbb{S}^{n-1}} |u_{1,N}(r\omega, t)|^2 \\ &\lesssim \mu^{-\frac{n-1}{2}} \|(1+r)^{-\frac{n-1}{2}} \langle \Omega \rangle^{\frac{n-1}{2}} u_{1,N}(t)\|_{L_x^2}. \end{aligned}$$

Thus,

$$\|u_{1,N}(t)\|_{\ell^\infty L_\mu^2} \lesssim \mu^{-\frac{n-1}{2}} N^{\frac{n-1}{2}} \|(1+r)^{-\frac{n-1}{2}} u_{1,N}(t)\|_{L_x^2}.$$

Interpolating this with the trivial estimate

$$\|u_{1,N}(t)\|_{\ell^2 L_\mu^2} \lesssim \|u_{1,N}(t)\|_{L_x^2},$$

we arrive at the following estimate which will be our point of departure:

$$(94) \quad \|u_{1,N}\|_{L_t^2 \ell^{p_\eta} L_\mu^2} \lesssim C_\eta \mu^{-\frac{1}{2}-\eta} \|(1+r)^{-\frac{1}{2}(1+\eta)} \langle \Omega \rangle^{\frac{1}{2}(1+\eta)} u_{1,N}\|_{L_t^2 L_x^2},$$

where $p_\eta = \frac{2(n-1)}{n-2-\eta}$. Therefore, using (94) and that $\langle \Omega \rangle^{\frac{1}{2}(1+\eta)} u_{1,N}$ is another unit frequency solution of the wave equation localized at the angular frequency N , we see that in order to prove (93), it suffices to prove the following space-time Morawetz type estimate for unit frequency solutions of the homogeneous wave equation when $3 \leq n$:

$$(95) \quad \int_{-\infty}^\infty \int_{\mathbb{R}^n} \frac{1}{(1+r)^{1+\eta}} |u_1(t,x)|^2 dx dt \lesssim \|u_1(0)\|_{L_x^2}^2.$$

In the above estimate, the implicit constant depends on η and the dimension, and from now on we will keep this dependence implicit. Estimate (95), as well as its version with $\eta = 0$ and a logarithmic loss in time, was proved in the work of Keel-Smith-Sogge [5] in dimension three with the help of the sharp Huygen's principle. We will prove (95) directly using essentially nothing but an integration by part argument. Before we continue, let us make one more reduction. It turns out that (95) is more naturally proved for the spatial gradient $\nabla_x u_1$. Since u_1 is unit frequency, this reduction does not effect the validity of (95). Indeed, suppose we have the following estimate:

$$(96) \quad \int_{-\infty}^\infty \int_{\mathbb{R}^n} \frac{1}{(1+r)^{1+\eta}} |\nabla_x u_1(t,x)|^2 dx dt \lesssim \|\nabla_{t,x} u_1(0)\|_{L_x^2}^2.$$

for any unit frequency solution u_1 . Let \tilde{u}_1 be another unit frequency solution with the property that $\Delta\tilde{u}_1 = -u_1$. Then for any $\epsilon > 0$,

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{1}{(1+r)^{1+\eta}} |u_1(t,x)|^2 dxdt &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{1}{(1+r)^{1+\eta}} \Delta\tilde{u}_1 \bar{u}_1 dxdt \\ &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{1}{(1+r)^{1+\eta}} \nabla_x \tilde{u}_1 \nabla_x \bar{u}_1 dxdt + (1+\eta) \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{r \nabla_x r}{(1+r)^{3+\eta}} \nabla_x \tilde{u}_1 \bar{u}_1 dxdt \\ &\lesssim \epsilon^{-1} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{1}{(1+r)^{1+\eta}} |\nabla_x \tilde{u}_1(t,x)|^2 dxdt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{1}{(1+r)^{1+\eta}} |\nabla_x u_1(t,x)|^2 dxdt \\ &\quad + \epsilon \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{1}{(1+r)^{1+\eta}} |u_1(t,x)|^2 dxdt \end{aligned}$$

and (95) follows from (96) by choosing a sufficiently small ϵ . Therefore, we may now assume that we are trying to achieve estimate (96). The fact that (96) contains the gradient ∇_x will allow us to prove it for an arbitrary solution to the homogeneous wave equation. This will be done using some more or less standard energy-momentum tensor techniques. Let ϕ be a solution to the homogeneous wave equation:

$$(97) \quad \square\phi = 0,$$

and let $Q_{\alpha\beta}[\phi]$ be its energy-momentum tensor:

$$Q_{\alpha\beta}[\phi] = \partial_\alpha\phi \partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}\partial^\gamma\phi\partial_\gamma\phi.$$

Here the Greek indices run on the set $\alpha, \beta, \gamma = 0, \dots, n$, and $g_{\alpha\beta} = \text{diag}(-1, 1, \dots, 1)$ is the standard Minkowski metric. The key feature of $Q_{\alpha\beta}[\phi]$ is that it is space-time divergence free:

$$(98) \quad D^\alpha Q_{\alpha\beta}[\phi] = 0,$$

where ∇ denotes the Levi-Civita connection of $g_{\alpha\beta}$. We now contract $Q_{\alpha\beta}[\phi]$ a radial vector-field:

$$X = f(r)\partial_r.$$

Define the momentum density:

$$P_\alpha[\phi, X] = Q_{\alpha\beta}[\phi]X^\beta.$$

A quick calculation shows that the divergence of $P_\alpha[\phi, X]$ satisfies the identity:

$$(99) \quad D^\alpha P_\alpha[\phi, X] = \frac{1}{2}Q_{\alpha\beta}[\phi]\pi^{\alpha\beta},$$

where:

$$\pi_{\alpha\beta} = D_\alpha X_\beta + D_\beta X_\alpha,$$

is the deformation tensor of X . Introducing the orthonormal frame $\{\partial_t, \partial_r, e_A\}$, where the $\{e_A\}$ form an orthonormal frame on each $n-1$ sphere $S_{t,r} = \{r = \text{const.}\}$, one can easily calculate this quantity to be:

$$(100) \quad \pi_{\alpha\beta} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & f'(r) & \dots & 0 \\ 0 & 0 & \frac{f(r)}{r}\delta_{AB} & \end{pmatrix},$$

where δ_{AB} denotes the metric on the spheres $S_{t,r}$. In particular, we see that:

$$(101) \quad \text{tr}\pi = f'(r) + (n-1)\frac{f(r)}{r},$$

Next, using (99) and (100), a direct computation shows that:

$$2 D^\alpha P_\alpha[\phi, X] = f'(r)|\partial_r \phi|^2 + \frac{f(r)}{r} |\nabla \phi|^2 - \frac{1}{2} \text{tr} \pi \partial^\gamma \phi \partial_\gamma \phi .$$

In the above formula, $|\nabla \phi|^2 = \delta^{AB} |e_A(\phi)| \cdot |e_B(\phi)|$ denotes the angular portion of the spatial gradient $|\nabla_x \phi|^2$. Since, $\partial^\gamma \phi \partial_\gamma \phi = \frac{1}{2} \square |\phi|^2$, if we define the modified momentum density:

$$\tilde{P}_\alpha[\phi, X] = P_\alpha[\phi, X] + \frac{1}{4} \text{tr} \pi \phi \partial_\alpha \phi - \frac{1}{8} \partial_\alpha (\text{tr} \pi) |\phi|^2 ,$$

we end up with the identity:

$$(102) \quad D^\alpha \tilde{P}_\alpha[\phi, X] = \frac{1}{2} \left(f'(r)|\partial_r \phi|^2 + \frac{f(r)}{r} |\nabla \phi|^2 \right) - \frac{1}{8} \Delta (\text{tr} \pi) |\phi|^2 .$$

Integrating (102) over a time slab, we arrive at the following a-priori estimate for ϕ :

$$(103) \quad \int_{\mathbb{R}^n} \tilde{P}_0[\phi, X](0) dx = \int_{\mathbb{R}^n} \tilde{P}_0[\phi, X](T) dx + \int_0^T \int_{\mathbb{R}^n} \frac{1}{2} \left(f'(r)|\partial_r \phi|^2 + \frac{f(r)}{r} |\nabla \phi|^2 \right) - \frac{1}{8} \Delta (\text{tr} \pi) |\phi|^2 dx dt ,$$

where:

$$(104) \quad \int_{\mathbb{R}^n} \tilde{P}_0[\phi, X](0) dx = \int_{\mathbb{R}^n} \left(X^\alpha \partial_t \phi(0) \partial_\alpha \phi(0) + \frac{1}{4} \text{tr} \pi \phi(0) \partial_t \phi(0) \right) dx .$$

with an identical expression for the time = T boundary piece on the right hand side of (103) above. In particular, using (101), one has that if $|f(r)| \lesssim 1$ then:

$$(105) \quad \left| \int_{\mathbb{R}^n} \tilde{P}_0[\phi, X](0) dx \right| \lesssim \| r^{-1} \phi(0) \|_{L_x^2} \cdot \| \nabla_{t,x} \phi(0) \|_{L_x^2} , \\ \lesssim \| \nabla_{t,x} \phi(0) \|_{L_x^2}^2 ,$$

where in the last line we used the Hardy inequality. Similar estimate holds for the other boundary term $\tilde{P}_0[\phi, X](T)$. Therefore, using the usual conservation of energy, we have the following a-priori estimate for solutions to the homogeneous wave equation:

$$(106) \quad \left| \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{1}{2} \left(f'(r)|\partial_r \phi|^2 + \frac{f(r)}{r} |\nabla \phi|^2 \right) - \frac{1}{8} \Delta (\text{tr} \pi) |\phi|^2 dx dt \right| \\ \lesssim \| \nabla_{t,x} \phi(0) \|_{L_x^2}^2 .$$

We now use (106) to derive (96) by choosing the weight function $f(r)$.

$$f(r) = \frac{r}{\epsilon + r}$$

for some $\epsilon > 0$. A direct calculation shows that, with this choice of f , we have

$$\Delta (\text{tr} \pi) = \Delta \left(\frac{\epsilon}{(\epsilon + r)^2} + \frac{n-1}{\epsilon + r} \right) = -\frac{1}{r(\epsilon + r)^3} \left((n-3)r + 3(n-3) \frac{\epsilon r}{\epsilon + r} + \frac{3\epsilon^2(n-1)}{\epsilon + r} \right) < 0$$

in dimensions $n \geq 3$. Therefore this gives the a-priori estimate:

$$(107) \quad \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \left(\frac{\epsilon}{(\epsilon + r)^2} |\partial_r \phi|^2 + \frac{1}{\epsilon + r} |\nabla \phi|^2 \right) dx dt \lesssim \| \nabla_{t,x} \phi(0) \|_{L_x^2}^2$$

with an implicit constant in \lesssim independent of ϵ . In particular, choosing $\epsilon = 2^k$ with an integer $k \geq 0$, we obtain

$$\int_{-\infty}^{\infty} \int_{|x| \leq 1} |\nabla_x \phi|^2 dx dt + \int_{-\infty}^{\infty} \int_{2^{k-1} \leq |x| \leq 2^{k+1}} \frac{|\nabla_x \phi|^2}{r} dx dt \lesssim \|\nabla_{t,x} \phi(0)\|_{L_x^2}^2.$$

Dividing the above inequality by $2^{-\eta k}$ and summing over k immediately yields the desired estimate (96).

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