GLOBAL STABILITY FOR CHARGE SCALAR FIELDS ON MINKOWSKI SPACE

HANS LINDBLAD AND JACOB STERBENZ

ABSTRACT. We prove that the charge-scalar field equations are globally stable on $(3+1)$ dimensional Minkowski space for small initial data in certain gauge covariant weighted Sobolev spaces. These spaces can be chosen as to be almost scale invariant with respect to the homogeneity of the equations. This result is valid for initial data with non-zero charge that is also non-stationary at space–like infinity. The method of proof is a tensor-geometric approach which is based on a certain family of weighted bilinear L^2 space-time estimates.

1. INTRODUCTION

In this paper, we study the global in time behavior of small amplitude classical solutions to the massless Maxwell–Klein–Gordon (MKG) equations on Minkowski space. These are also sometimes referred to in the literature as Maxwell–Higgs or the Charge–Scalar–Field equations. They are constructed as follows: We let $\mathcal{M}$ denote the $(3+1)$ dimensional Minkowski space with metric $g = (-1, 1, 1, 1)$ and compatible Levi-Civita connection which we refer to here as ∇. Then let:

$$V \; = \; \mathcal{M} \times \mathbb{C} \; ,$$

denote a complex line bundle over $\mathcal{M}$ with hermitian inner product $\langle \cdot, \cdot \rangle_V$ and compatible connection D. That is, one has the formula:

$$(1) \qquad X\langle \psi, \phi \rangle_V \; = \; \langle D_X \psi, \phi \rangle_V + \langle \psi, D_X \phi \rangle_V \; ,$$

for all vector-fields X on $\mathcal{M}$. Then a section ϕ of V is said to be a *complex scalar field* if it satisfies the equation:

$$(2) \qquad \Box^{\mathbb{C}} \phi \; = \; D^\alpha D_\alpha \phi \; = \; 0 \; .$$

Since V is globally trivial, we can let $\mathbf{1}_V$ denote a unit normalized global section. That is one has $\langle \mathbf{1}_V, \mathbf{1}_V \rangle \equiv 1$. Such a choice is referred to as a *gauge*. In terms of this, any section ϕ to V can be identified with a complex valued function on $\mathcal{M}$ where we abusively write $\phi = \phi \mathbf{1}_V$. With respect to this frame for V, and for any frame $\{e_\alpha\}$ on $\mathcal{M}$, we define the connection one-form $\{A_\alpha\}$ via the relations:

$$(3) \qquad D_\alpha \mathbf{1}_V \; = \; \sqrt{-1} A_\alpha \; .$$

Notice that the compatibility condition (1) immediately implies that the $\{A_\alpha\}$ are real. Using this notation, the covariant derivative of any section ϕ with respect to a vector-field X in $\mathcal{M}$ can be written as:

$$(4) \qquad D_X \phi \; = \; X^\alpha e_\alpha(\phi) + \sqrt{-1} X^\alpha A_\alpha \cdot \phi \; .$$

Of course, the choice of $\mathbf{1}_V$, and hence representation of D_X, is somewhat arbitrary because one can always perform a local (or global) unitary transformation of it. If

we let $\mathbf{1}_V \rightsquigarrow \widetilde{\mathbf{1}}_V$ denote the transformation given by $\widetilde{\mathbf{1}}_V = e^{-i\chi}\mathbf{1}_V$, where χ is some real valued function on $\mathcal{M}$, then (4) shows that the potentials $\{A_\alpha\}$ transform as:

$$\widetilde{A}_\alpha \;=\; A_\alpha - e_\alpha(\chi) \;.$$

As it turns out, this ambiguity which is inherent to the equation (4) does *not* need to be resolved in order to proceed with a detailed analysis of the scalar field equation (2). This is because the connection (1) has a basic geometric invariant on which it is possible to base literally all analysis of (2) so long as one is content (and able) to define all analytical objects of interest in terms of geometric quantities. This is the approach we take in this paper. This basic geometric invariant is, of course, the curvature of the connection D which arises from the operation of commuting covariant differentiation. Specifically, there exists a (real) two-form F on $\mathcal{M}$, such that for any two vector-fields X, Y we have the relation:

$$(5) \qquad\qquad D_X D_Y \phi - D_Y D_X \phi - D_{[X,Y]}\phi \;=\; \sqrt{-1}F(X,Y)\cdot\phi \;.$$

In terms of the frame $\{e_\alpha\}$ we write:

$$(6) \qquad\qquad D_\alpha D_\beta \phi - D_\beta D_\alpha \phi \;=\; \sqrt{-1}F_{\alpha\beta}\cdot\phi \;.$$

Here $D_\alpha D_\beta \phi$ denotes the (α, β) component of the Hessian $D^2\phi$ and should not be confused with the repeated directional covariant differentiation $D_{e_\alpha} D_{e_\beta}\phi$. To see the difference, a short computation of (5) in the frame $\{e_\alpha\}$ shows that one has the identity:

$$D_{e_\alpha} D_{e_\beta}\phi - D_{e_\beta} D_{e_\alpha}\phi - D_{[e_\alpha, e_\beta]}\phi \;=\; \sqrt{-1}\big(e_\alpha(A_\beta) - e_\beta(A_\alpha) - c_{\alpha\beta}^\gamma A_\gamma\big)\cdot\phi \;,$$

where $[e_\alpha, e_\beta] = c_{\alpha\beta}^\gamma e_\gamma$ are the structure "constants" of the frame $\{e_\alpha\}$. Thus, one has the well known "Bianchi" identity:

$$(7) \qquad\qquad\qquad\qquad F \;=\; dA \;.$$

To define the full (MKG) system, we now couple the complex scalar field ϕ to the curvature of the connection D in such a way that one ends up with a Lagrangian field theory. For an intuitive approach to how this is done, we consider F as a solution to Maxwell's equations. Since by design F satisfies the Bianchi identity (7), we have that:

$$(8a) \qquad\qquad\qquad\qquad \nabla^\beta F_{\alpha\beta} \;=\; J_\alpha \;,$$

$$(8b) \qquad\qquad\qquad\qquad \nabla^\beta \,{}^\star F_{\alpha\beta} \;=\; 0 \;.$$

Here ${}^\star F$ denotes the Hodge dual of F and is given by ${}^\star F_{\alpha\beta} = \frac{1}{2}\in_{\alpha\beta}{}^{\gamma\delta} F_{\gamma\delta}$, where $\in_{\alpha\beta\gamma\delta}$ denotes the volume form on Minkowski space. As is the usual practice, we can use this duality operation to define the quantities:

$$(9) \qquad\qquad E_i \;=\; F_{0i} \;, \qquad\qquad\qquad H_i \;=\; {}^*F_{0i} \;,$$

in terms of which the system (8) takes the familiar form:

$$(10a) \qquad\quad -\partial_t E \;+\; \nabla_x \times H \;=\; \underline{J} \;, \qquad\qquad (\mathrm{div})E \;=\; J_0 \;,$$

$$(10b) \qquad\quad -\partial_t H \;+\; \nabla_x \times E \;=\; 0 \;, \qquad\qquad (\mathrm{div})H \;=\; 0 \;.$$

Here $\nabla_x \times \;=\in_i{}^{jk} \partial_j$ is the usual curl operation, and $\underline{J}$ denotes the spatial part of the current vector J. We note here that this latter form of the equations will be

particularly useful when dealing with questions concerning the initial data of Lie derivatives of $F_{\alpha\beta}$. In general it is the more natural language with which to discuss the initial value problem for the system (8).

For the Maxwell field F, there is an energy-momentum tensor $Q[F]$ which is given by:

$$(11) \qquad Q_{\alpha\beta}[F] \;=\; F_{\alpha\gamma}F_\beta{}^\gamma + {}^\star F_{\alpha\gamma}\,{}^\star F^\gamma{}_\beta \;=\; F_{\alpha\gamma}F_\beta{}^\gamma - \frac{1}{4}g_{\alpha\beta}\,F_{\gamma\delta}F^{\gamma\delta} \;.$$

A brief calculation using the identities (8a)–(8b) shows that one has the divergence identity:

$$(12) \qquad\qquad\qquad \nabla^\alpha Q_{\alpha\beta}[F] \;=\; -\,F_{\beta\gamma}J^\gamma \;.$$

The complex scalar field (2) also has an energy-momentum tensor which is analogous to the usual energy-momentum tensor of the (flat) D'Lambertian. This is given by:

$$(13) \qquad\qquad Q_{\alpha\beta}[\phi] \;=\; \Re(D_\alpha\phi\overline{D_\beta\phi}) - \frac{1}{2}g_{\alpha\beta}\,D^\gamma\phi\overline{D_\gamma\phi} \;.$$

A direct application of the field equation (2), the compatibility condition (1), and the commutator identity (5) shows that one has the divergence law:

$$(14) \qquad\qquad\qquad \nabla^\alpha Q_{\alpha\beta}[\phi] \;=\; F_{\beta\gamma}\Im(\phi\overline{D^\gamma\phi}) \;.$$

Notice that this formula does not contain any explicit reference to the connection coefficients (3). This is because it is a purely tensorial identity, and in particular does not depend on a choice of coordinates or frame.

The natural choice of coupling F to ϕ now comes if we simply stipulate that the current vector on the right hand side of (8a) be given by $J_\alpha = \Im(\phi\overline{D^\alpha\phi})$. With this extra condition satisfied, the total energy momentum tensor $Q = Q[F] + Q[\phi]$ becomes divergence free:

$$(15) \qquad\qquad\qquad\qquad \nabla^\alpha Q_{\alpha\beta} \;=\; 0 \;.$$

Notice that this coupling is legitimate because as an immediate consequence of the equation (2), one has that this choice of current vector J_α satisfies the continuity equation:

$$(16) \qquad\qquad\qquad\qquad \nabla^\alpha J_\alpha \;=\; 0 \;,$$

which is the only prerequisite for a vector J to show up as the right hand side of the Maxwell equation (8a). We write the resulting system of equations, which we henceforth refer to as the (MKG) equations, together as:

$$(17a) \qquad\qquad\qquad \nabla^\beta F_{\alpha\beta} \;=\; \Im(\phi\overline{D_\alpha\phi}) \;,$$

$$(17b) \qquad\qquad\qquad\qquad \Box^{\mathbb{C}}\phi \;=\; 0.$$

We note here that implicit in the system (17) is the Bianchi identity (7) or (8b). This is because we are assuming a-priori that F is the curvature of the connection which gives rise to $\Box^{\mathbb{C}}$.

In this work we study the global in time Cauchy problem for the system (17). Since we have stated the equations in such a way as to deemphasize a choice of gauge, this deserves some explanation. Since we are interested in an evolution

problem, this boils down to a discussion of the gauge covariant initial data. To specify these, we first let $\underline{D}$ denote a connection on the initial time slice bundle $\{0\} \times \mathbb{R}^3 \times \mathbb{C}$. Since this is embedded as a hyper-surface in the original bundle $\mathcal{M} \times \mathbb{C} = V$, we will also specify the initial normal derivative D_t. The initial data for the system (17) can then be written in the form:

(18a) $$ F_{0i}(0) \ = \ E_i \ , \qquad\qquad\qquad {}^*F_{0i}(0) \ = \ H_i \ , $$

(18b) $$ \phi(0) \ = \ \phi_0 \ , \qquad\qquad\qquad D_t\phi(0) \ = \ \dot{\phi}_0 \ . $$

In the above notation, we have used the labels (E, H) to denote quantities which only depend on $x \in \mathbb{R}^3$. These should not be confused with the space-time (E, H) used in the formulas (10). Now, from the form of the system (17), it is easy to see that this initial data cannot be specified freely. It must also satisfy the compatibility conditions:

(19) $$ \nabla^i E_i \ = \ \Im(\phi_0 \, \overline{\dot{\phi}_0}) \ , \qquad\qquad\qquad \nabla^i H_i \ = \ 0. $$

We will call a data set $(E, H, \phi_0, \dot{\phi}_0)$ which satisfies (19) *admissible.* The question we are now concerned with here is to describe in as detailed a way as possible the global in time behavior of solutions to the (MKG) equations whose initial data satisfies certain natural smallness assumptions. As usual, these will be stated as regularity assumptions. In general, we define the covariant and gauge covariant weighted integral Sobolev spaces:

(20) $$ \| \, T \, \|^2_{H^{k,s_0}(\mathbb{R}^3)} \ = \ \sum_{|I| \leqslant k} \int_{\mathbb{R}^3} (1 + r^2)^{s_0 + |I|} \, |\nabla^I_x T|^2 \, dx \ , $$

(21) $$ \| \, \psi \, \|^2_{H^{k,s_0}(\mathbb{R}^3)} \ = \ \sum_{|I| \leqslant k} \int_{\mathbb{R}^3} (1 + r^2)^{s_0 + |I|} \, |\underline{D}^I \psi|^2 dx \ . $$

Here I denotes the usual multiindex notation, while ∇_x denotes the restriction of the Levi-Civita connection ∇ to the time slice $\{0\} \times \mathbb{R}^3$. T denotes an arbitrary tensor. We will consider the Cauchy problem for admissible initial data sets which are in these function spaces for various values of s_0 and k.

Now, a basic regularity theorem for the system (17) says that if the norms (20)–(21) applied to (admissible) $(E, H, \phi_0, \dot{\phi}_0)$ are finite for certain values of s_0 and k, then a global solution to this system exists. In fact, there is no need to impose any of the weights $(1 + r^2)$, and almost no smoothness is needed to make the argument work. This is the content of the fundamental regularity result of Eardly-Moncrif [5]–[6], and its later significant refinement due to Klainerman-Machedon [9]. While these results in some sense give the strongest possible global existence result one could hope to ask for, they contain surprising little information as to the nature of the global solution they obtain. In fact, the only information they provide is that the unweighted Sobolev H^s norm of the solution remains bounded for all time. No practical bounds for this quantity are obtained for $1 < s$. Furthermore, these results provide no information on the profile of the solution, even assuming that the initial data is localized at time $t = 0$ in a way that is consistent with the norms (20)–(21). Finally, these results do not lend themselves to a discussion of the asymptotic behavior of the scalar field ϕ in the sense of L^2 scattering. For example, there is no

doubt that this involves long range corrections[1] due to the asymptotic behavior of the electro-magnetic field $F_{\alpha\beta}$, but from the point of view of these works it is not entirely clear what this correction should be.

The result we present here is a first attempt to fill in some of the gap in our understanding of the asymptotic behavior of solutions to the system (17), at least for that portion of initial data with small (20)–(21) norms for specific values of s_0 and k. The method we employ is a new variant of the tensorial-geometric approach of Christodoulou-Klainerman [3]–[4], which also uses in a crucial way the space-time energy estimates idea from the recent work of Lindblad-Rodnianski [12] applied to both the electro-magnetic field (8), and to the scalar field (2) through a novel geometric adaptation of a set of fundamental estimates which goes back to work of Morawetz [13]. The approach we follow also borrows some ideas from previous works on the low regularity behavior of systems of semilinear wave equations, notably [9], in that we make crucial use of a certain family of sharp weighted bilinear L^2 space-time estimates for tensorial contractions to control error terms which come up in our analysis. Furthermore, some of these estimates are proved through the use of a special weighted $L^2(L^\infty)$ "Strichartz" type estimate which holds for certain components in a tensorial setting.

There has been only one previous investigation in general[2] on the global asymptotic behavior of the equations (17). This is the paper of Shu [15]. Unfortunately, that work is highly incomplete and furthermore only covers the case compactly supported initial data. In fact, there is good reason to believe that what must have been the original method of that paper[3] would not have been sufficient to close the entire problem. Specifically, the idea of using pure L^∞ estimates combined with fixed time and characteristic energy inequalities does not seem to work for this problem. In order to avoid certain logarithmic divergences which occur when using this latter approach, we were forced to employ the full power of our space-time estimates, especially the weighted mixed Lebesgue space estimate mentioned previously. Furthermore, as is usually the case with scalar wave equations, where once one can prove global existence using weighted energies for compactly supported initial data one can move to non-compactly supported initial data by simply adding enough weights (at space-like infinity), for tensorial equations of the type (17) it is *not* possible to directly pass from the assumption of compactly supported initial data to the case of initial data with infinite support, even if one assumes that initial data decays at a very fast polynomial weight. That this should be the case is not entirely obvious at first, and is due to subtle interaction of the scaling and tensorial

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[1]We will not prove this here. However, we believe the analysis we present goes a long way towards dealing with this problem. That there should be long range corrections is consistent with what is known for the Maxwell-Dirac equations (see [7]) which has many similarities to the (MKG) equations.

[2]Meaning the system (17) with or without charge. There was a previous work of Christodoulou–Choquet-Bruhat ([2]) which involved conformal compactification. However, this type of procedure requires a lot of decay of the initial data and does not allow for either charge or dipole moments.

[3]It is not possible to say exactly what the author had in mind. Not only were the basic L^∞ estimates for the scalar field left unproven, but almost all of the error terms in the analysis are left undealt with; to directly quote the author - "I will not carry out any more of the rest of the calculation, apparently there are still a lot of works, but they are very boring, they are all estimated in the same fashion as those we gave above".

properties of the equations (17). We will explain this is more detail in the next paragraph. This being said, we think that it is fair to mention here that the work [15] was a step in the right direction, and that some of the more interesting features of the nonlinearity were first discovered there, notably the beautiful cancellation which we use in (a slightly different form) line (280) below. Finally, we should also mention here the work of Psarrelli [14] on the asymptotic behavior for the equations (17) with a non-zero mass. This work again assumed compact support of the initial data, and is ultimately very different form what we do here because the version of (2) with non-zero mass has quite different decay properties than its massless counterpart, essentially because it does not enjoy the same conformal properties as its zero mass variant.

Before moving on the statement of our main result, it is worth mentioning two of the most difficult features of attempting a physical-space weighted-geometric analysis of the equations (17). The first has to do with trying to make use of the so called "null-condition" property of this system. Unlike other semilinear geometric wave equations, for example the wave-maps [17], there is not a direct correspondence between the semilinear model equations studied in [8] and [1], which satisfy the null condition directly, and the equations of gauge field theory. There are basically two separate ways the null condition for the equations (17) can be uncovered. The first is through the use of an elliptic gauge, as in the works on the low regularity properties of this and other related systems [9]–[10]. This type of procedure leads to non-local versions of the null-forms studied in [8] and [1], which also have entirely different scaling properties from their non-local versions. To analyze the type of equation which arises in this way by using the inhomogeneous algebra (53) seems to be quite delicate, and it is not clear that this can even be done correctly. For example, the commutator of the weighted derivatives in (53) with Riesz potentials seems to cause enough of a problem that it effectively negates the savings one gets from the standard null-forms. Furthermore, any such analysis would have to somehow take into account the long range corrections the system (17) must undergo due to the lack of decay of the electro-magnetic field (8).

The second way the null condition for the MKG system can uncovered is directly through its tensorial structure. That is, the null condition makes itself evident through the contraction structure of error terms which arise after commuting the equations with geometrically defined operations of differentiation. Specifically Lie differentiation of the Maxwell field, and gauge covariant differentiation of the scalar field. Since these contractions involve all possible indices, they can be expanded in a frame which takes into account the null geometry of Minkowski space. In this way, one immediately sees that two "bad components" can never interact with each other. However, since the system (17) involves the interaction of two different types of quantities, namely a two form and a scalar field, it is not entirely clear at first that this type of tensorial structure will cause enough "good interactions" to take place, or even what the "components" of the scalar field should be. Fortunately for us, the MKG equations do contain a deep underlying structure which allows one to treat *both* quantities, $F_{\alpha\beta}$ and ϕ, as a single instance of a master "tensorial" object which we call Ψ. Whats more, there is a single (family) of bilinear estimates for this master quantity which covers all possible error terms which can come from

differentiating the system (17). This type of underlying unity goes well beyond the scalar null structure of [8] and [1], and is more akin to what happens for the pure Yang-Mills and the Einstein equations (see [16]–[4]).

The second main difficulty we need to overcome in our analysis is a consequence of an interesting interplay between the scaling and decay properties of solutions to the system (17). While in $(3 + 1)$ dimensions these equation are *subcritical* with respect to translation invariant derivatives and the conserved energy[4], which leads to the large data regularity results mentioned previously, these equations are in general *critical* with respect to decay at space-like infinity. To understand what is meant by this last statement, consider a linear wave equation of the form:

$$(22) \qquad \qquad \Box \phi \;=\; -2\sqrt{-1}\, A^\alpha \partial_\alpha \phi \;,$$

where $\Box = \partial^\alpha \partial_\alpha$ denotes the usual D'Lambertian on Minkowski space. This can be seen as a model for the equation (2) where we have fixed a gauge (which gauge does not matter for the sake of this discussion). Now, assume that the initial data for (22) decays like r^{-m} as $r \to \infty$. Then how much decay should one expect from the potentials A_α in order to guarantee that the solution keeps this decay in the far exterior region $2t < r$ (which should be the simplest to control!). By homogeneity, one expects that if A_α decays like r^{-l} for $2t < r$, then the solution ϕ will decay like r^{-m-l+1} (integrating the second order equation twice). For smooth A_α, this heuristic can be made precise through an easy use of weighted exterior energy estimates. One immediately sees from this simple analysis that to control things, A_α must decay at least as well as r^{-1} in the far exterior, and that for this critical rate of decay some work is required to avoid logarithmic divergences in the "error" term on the right hand side of (22). The decay rate r^{-1} is not a coincidence because the equation (22) scales like $\dot{H}^{\frac{1}{2}}$, for which one sees that this is precisely the critical rate of decay. Whats more, the entire system (17) scales like $\dot{H}^{\frac{1}{2}}$ at the level of the potentials A_α, and because of the tensorial nature of the MKG equations, at least some of the potentials *must* in general decay no better that r^{-1}. This is a simple effect of the elliptic constraint equation:

$$(23) \qquad \qquad \nabla^i E_i \;=\; \Im(\phi\,\overline{D_t\phi}) \;.$$

Introducing a potential function φ for the curl-free part E^{cf}, we see that unless one has:

$$\int_{\mathbb{R}^3} \Im(\phi\,\overline{D_t\phi}) \;\equiv\; 0 \;,$$

it will be the case that $\varphi \sim \frac{1}{r}$. Since this potential function must show up somewhere in field potentials A_α regardless of the gauge, we see that in general an equation of the form (22), and hence (2), will be critical for decay at space-like infinity. This means that in order to have an effective strategy, one must prove sharp space-time estimates. There will in general be no extra "convergence factors" with which one can use to integrate over time. In practice, this means that all of our space-time estimates need to be proved in a-priori (divergence) form. This should be understood in contrast to other field equations, such as the Einstein equations. In $(3+1)$ dimensions, the long range effect of the mass for this latter system leaves

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[4]The conserved energy is a simple consequence of the tensorial conservation law (15) and the fact that ∂_t is a Killing field on Minkowski space.

plenty of room with respect to the scaling properties of the equations (see for example [12]). Therefore, from the point of view of decay, the Einstein equations are in fact easier to control than the MKG equations.

We are now ready to state our main result:

Theorem 1.1 (Global Stability of MKG Equations). *Let $2 \leqslant k$ with $k \in \mathbb{N}$, and let $s_0 = s + \gamma$ be given such that $s_0 < \frac{3}{2}$, $0 < \gamma$, and $\frac{1}{2} < s$. Let $(E, H, \phi_0, \dot{\phi}_0)$ be an admissible initial data set, and define the* charge *to be the value:*

$$(24) \qquad q \;=\; \int_{\mathbb{R}^3} \Im(\phi_0 \, \overline{\dot{\phi}_0}) \;.$$

Then there exists a universal constant $\mathcal{E}_{k,s,\gamma}$, which depends only on the parameters k, s, γ, such that if $(E, H, \phi_0, \dot{\phi}_0)$ is an admissible initial data set which satisfies the smallness condition:

$$(25) \quad \| \, E^{df} \, \|_{H^{k,s_0}(\mathbb{R}^3)} + \| \, H \, \|_{H^{k,s_0}(\mathbb{R}^3)} + \| \, \underline{D}\phi_0 \, \|_{H^{k,s_0}(\mathbb{R}^3)} + \| \, \phi_0 \, \|_{H^{k,s_0}(\mathbb{R}^3)} \;\leqslant\; \mathcal{E}_{k,s,\gamma} \;,$$

where $E = E^{df} + E^{cf}$ is the Hodge decomposition of E into its divergence free and curl free components (resp.), then there exists a (unique) global solution to the system of equations (17) with this initial data set such that if $\{L, \underline{L}, e_A\}$ denotes a standard spherical null frame (see (44) and (45) below), then the following pointwise properties of this solution holds:

$$(26a) \qquad |\alpha| \;\lesssim\; \mathcal{E}_{k,s,\gamma} \cdot \tau_+^{s+\frac{3}{2}} \cdot (w)_\gamma^{\frac{1}{2}} \;, \qquad\qquad |\underline{\alpha}| \;\lesssim\; \mathcal{E}_{k,s,\gamma} \cdot \tau_+ \tau_-^{s+\frac{1}{2}} \cdot (w)_\gamma^{\frac{1}{2}} \;,$$

$$(26b) \qquad |\rho| \;\lesssim\; q \cdot r^{-2} \, \chi_{1<t<r} \;+\; \mathcal{E}_{k,s,\gamma} \cdot \tau_+^{1+s} \tau_-^{\frac{1}{2}} \cdot (w)_\gamma^{\frac{1}{2}} \;,$$

$$(26c) \qquad |\sigma| \;\lesssim\; \mathcal{E}_{k,s,\gamma} \cdot \tau_+^{1+s} \tau_-^{\frac{1}{2}} \cdot (w)_\gamma^{\frac{1}{2}} \;,$$

and:

$$(27a) \qquad |\widetilde{D}_L \phi| \;\lesssim\; \mathcal{E}_{k,s,\gamma} \cdot \tau_+^{s+\frac{3}{2}} \cdot (w)_\gamma^{\frac{1}{2}} \;, \qquad |D_{\underline{L}} \phi| \;\lesssim\; \mathcal{E}_{k,s,\gamma} \cdot \tau_+ \tau_-^{s+\frac{1}{2}} \cdot (w)_\gamma^{\frac{1}{2}} \;,$$

$$(27b) \qquad |\slashed{D}\phi| \;\lesssim\; \mathcal{E}_{k,s,\gamma} \cdot \tau_+^{1+s} \tau_-^{\frac{1}{2}} \cdot (w)_\gamma^{\frac{1}{2}} \;, \qquad |\phi| \;\lesssim\; \mathcal{E}_{k,s,\gamma} \cdot \tau_+ \tau_-^{s-\frac{1}{2}} \cdot (w)_\gamma^{\frac{1}{2}} \;.$$

Here we have set:

$$|\widetilde{D}_L \phi|^2 \;=\; |\frac{1}{r} D_L(r\phi)| \, \chi_{1<t<2r} + |D_L\phi|^2 \, \chi_{r<\frac{1}{2}t} \;, \quad |\slashed{D}\phi|^2 \;=\; \delta^{AB} D_A \phi \overline{D_B \phi} \;.$$

Also, $(\alpha, \underline{\alpha}, \rho, \sigma)$ denotes the components of the null decomposition (48) of $F_{\alpha\beta}$. Finally, the weight functions $\tau_\pm$ and w_γ are defined via the formulas:

$$\tau_+^2 \;=\; 1 + (t+r)^2 \;, \qquad \tau_-^2 \;=\; 1 + (t-r)^2 \;, \qquad w_\gamma \;=\; \tau_-^{2\gamma} \, \chi_{t<r} \;.$$

Remark 1.2. In the course of our proof of Theorem 1.1, we will also show decay (peeling) properties similar to (26)–(27) for the higher derivatives of (F, ϕ) assuming that $2 < k$. We have not stated this formally for the sake of brevity.

Remark 1.3. Note the extra term in the decay asymptotic for ρ on line (26b) above. As $r \to \infty$ on any given time slice $t = const.$ this portion of the electro-magnetic field decays only like r^{-2}. This is the long range effect of the electro-static equation (23). What is somewhat surprising, and does not seem to have been fully

realized before is that this long range effect *only* propagates in the exterior of the forward cone $t = r$. That is, after rescaling at infinity, the effect of the charge is discontinuous across this cone. This fact goes a long way towards explaining why it is so much easier to deal with the case of compactly supported initial data. Also, notice that in the case where the value (24) vanishes, this long range effect is eliminated. This, of course, is why the assumption of zero charge is amenable to other techniques, e.g. conformal compactification.

Remark 1.4. Note that we can allow any value s_0 of the weight factor in the norms (20)–(21) applied to the initial data so long as it is bigger than the scale invariant factor of $\frac{1}{2}$. At this value, the norms become homogeneous (except for the extra term of 1 in the physical space weight) with respect to the scaling properties of the equations (17). It is also at this point that essentially all of our estimates break down due to logarithmic divergences. Also, for values of $s_0 < 1$, notice that our condition on the initial data (25) requires much less decay than previous works based on weighted energy, including the classical results (e.g. [8]) for semilinear equations (notice that one get to take the gradient $\underline{D}\phi$ *first* before applying the weighted L^2 norm). Our main theorem falls just short of a scale invariant result, and can be considered a "low regularity" theorem with respect to physical space decay. Finally, we should mention that below the level of the scaling (i.e. $s_0 < \frac{1}{2}$), there is no reason for the type of analysis we do here to make sense. In fact, even the simple notion of the charge (24) cannot be defined here because this level of decay is consistent with the asymptotics $\phi_0 \sim r^{-1+\delta}$ and $\dot{\phi}_0 \sim r^{-2+\delta}$ for some $0 < \delta$.

1.1. **A brief outline of the work.** We give here a very quick overview of the paper. In the next section, we make a list of some more or less standard geometric formulas relating to tensors and complex line bundles, as well as to the geometric structure of Minkowski space. This section is probably best left as a reference, and can be avoided by the reader with a passing acquaintance of this material.

In the third section, we build a series of weighted L^2 estimates for electro-magnetic fields. This material is for the most part a straight-forward adaptation of the standard material found in [3], with some new technical devices which are crucial for the approach we take in this paper. These are: A fractionally weighted version of the usual Morawetz type estimate for null-decomposed electro-magnetic fields, the decomposition of an electro-magnetic field into its pure charge and charge free components, and fixed time, characteristic, and space-time estimates which respect this decomposition. These estimates are proved with the help of a certain weighted elliptic estimate which we demonstrate in the appendix.

In the fourth section, we set about proving analogs of the estimates of the third section for complex scalar fields. This involves an interesting new proof of the classical Morawetz energy decay estimates for scalar fields which is directly based on the conformal geometry of Minkowski space. With some work, this leads to a set of estimates which are virtually identical with what is available for electro-magnetic fields. This device is important for what we do here because it allows us to treat

the curvature and the (conjugated gradient) scalar field on the same footing.

In the fifth and sixth sections, we prove L^∞ type estimates for the electro-magnetic and complex scalar fields. Interestingly enough, these involve in a crucial way the space-time, fixed time, as well as characteristic energy estimates developed in the preceding two sections.

In the seventh section, we recast all of the estimates we prove in this paper in a uniform form. First, we introduce an auxiliary "tensor", which has the behavior of the electro-magnetic-complex-scalar field which we are studying. We then prove a series of weighted L^2 bilinear estimates for interactions of various components of this "tensor". It turns out that all of the estimates we need to control error terms can be put in a single form, which we call the abstract parity estimate. This is proved at the end of this section through a simple case analysis.

In the final two sections of the paper, we prove error estimates for the commutators of the field equations (17) with certain Lie derivatives. With the work developed in the previous sections, this turns out to be quite easy because everything is just a special case of the abstract parity estimate of section seven.

Finally, in an appendix, we prove certain Sobolev type estimates which are used in the main work. This material is included for the sake of completeness, and can simply be referred to by the reader who wants more detailed account of certain (standard) calculations we use.

2. Some Geometric Preliminaries

In this section, we introduce the basic geometric concepts and identities which we will use in our analysis of the field equations (17). All of this material is completely standard, and we review it here in detail solely for the convenience of the reader. Experienced readers will find it worthwhile to skip this section and simply refer to it at places in the sequel where a specific calculation requires one of the identities we list here.

2.1. **Lie Derivatives.** We begin here with some well known formulas involving Lie derivatives and divergences of vector-fields and differential forms. For any sections X, Y to $T\mathcal{M}$ we have the usual formula $\mathcal{L}_X Y = [X, Y]$. We also denote by $\mathcal{L}_X \omega$ and $\mathcal{L}_X F$ the Lie derivative of a one-form and two-form respectively. In the frame $\{e_\alpha\}$ these can be computed as:

$$(28a) \qquad (\mathcal{L}_X \omega)_\alpha \;=\; X(\omega_\alpha) - \omega([X, e_\alpha]) \,,$$

$$(28b) \qquad (\mathcal{L}_X F)_{\alpha\beta} \;=\; X(F_{\alpha\beta}) - F([X, e_\alpha], e_\beta) - F(e_\alpha, [X, e_\beta]) \,.$$

For vector-fields X, we form the Lorentzian divergence:

$$(29) \qquad\qquad (\mathrm{div})X \;=\; \nabla_\alpha X^\alpha \;=\; (\mathrm{trace})\nabla X \,.$$

Also, for each vector-field X on $\mathcal{M}$, we measure its effect on the Minkowski metric by forming its Lorentzian deformation tensor:

$$(30) \qquad \mathcal{L}_X g \;=\; {}^{(X)}\pi \;.$$

Using this formula, one can easily compute the commutator of the Lie derivative $\mathcal{L}_X$ and the duality operator * applied to a two-form:

$$(31) \qquad [\mathcal{L}_X, {}^*] F_{\alpha\beta} \;=\; \frac{1}{2} (\mathcal{L}_X \in)_{\alpha\beta}{}^{\gamma\delta} F_{\gamma\delta} \;=\; \big((\mathrm{div})X \big) {}^* F_{\alpha\beta} \;.$$

Now, a well known calculation shows that we have:

$$(32) \qquad {}^{(X)}\pi \;=\; 2(\mathrm{symm})\nabla X \;,$$

where (symm) denotes the symmetric part of the tensor ∇X. Since the trace of an antisymmetric tensor is always zero, using (29) this gives the following formula:

$$(33) \qquad (\mathrm{div})X \;=\; \frac{1}{2}(\mathrm{trace})\,{}^{(X)}\pi \;.$$

A direct use of this gives the calculation:

$$
\begin{aligned}
X\big((\mathrm{div})Y\big) - Y\big((\mathrm{div})X\big) \;&=\; \frac{1}{2}\mathcal{L}_X\big(g^{\alpha\beta}(\mathcal{L}_Y g)_{\alpha\beta}\big) - \frac{1}{2}\mathcal{L}_Y\big(g^{\alpha\beta}(\mathcal{L}_X g)_{\alpha\beta}\big) \;, \\
&=\; \frac{1}{2}g^{\alpha\beta}(\mathcal{L}_{[X,Y]}g)_{\alpha\beta} \;, \\
(34) \qquad &=\; (\mathrm{div})(\mathcal{L}_X Y) \;.
\end{aligned}
$$

In particular, for any vector-field X such that $(\mathrm{div})X = const.$, we have the following useful formula:

$$X\big((\mathrm{div})Y\big) \;=\; (\mathrm{div})(\mathcal{L}_X Y) \;.$$

This discussion goes through almost verbatim for one-forms ω. In this case we define:

$$(\mathrm{div})\omega \;=\; \nabla_\alpha \omega^\alpha \;.$$

Then using the identification $\omega^\alpha = g^{\alpha\beta}\omega_\beta$, we have from (34) in any frame $\{e_\alpha\}$ that:

$$(35) \qquad X\big((\mathrm{div})\omega\big) - \omega^\alpha e_\alpha\big((\mathrm{div})X\big) \;=\; -\nabla_\alpha {}^{(X)}\pi^{\alpha\beta}\omega_\beta + (\mathrm{div})(\mathcal{L}_X \omega) \;,$$

where we have used the identity:

$$(\mathcal{L}_X g^\dagger)^{\alpha\beta} \;=\; -\,{}^{(X)}\pi^{\alpha\beta} \;,$$

with $g^\dagger$ equal to the two vector $g^{\alpha\beta}$. Notice that this last line follows from multiply contracting the identity (30). Thus, again assuming that $(\mathrm{div})X = const.$ one can drop the second term on the left hand side of (35) above. This formula immediately generalizes to two-forms (or arbitrary tensors). One computes that if ${}^{(X)}\pi$ is a constant multiple of the metric $g_{\alpha\beta}$, and hence $(\mathrm{div})X = const.$ by formula (33), the following commutator formula holds:

$$(36) \qquad \mathcal{L}_X \nabla^\beta F_{\alpha\beta} \;=\; -\,{}^{(X)}\pi^{\beta\gamma}\nabla_\gamma F_{\alpha\beta} + \nabla^\beta(\mathcal{L}_X F)_{\alpha\beta} \;.$$

This formula will be used in to differentiate the Maxwell equation (8) with respect to various vector-fields which are Killing and conformal Killing, and will be the basis of our analysis for that portion of the system (17).

We would now like to set up analogs of some of the above formulas for vector-fields and one-forms which are naturally associated with the complex line bundle V. We consider complexified versions of the tangent and cotangent bundles:

$$(37) \qquad T^{\mathbb{C}}\mathcal{M} \;=\; V \otimes T\mathcal{M} \;, \qquad\qquad T^{*\mathbb{C}}\mathcal{M} \;=\; V \otimes T^*\mathcal{M} \;.$$

In local coordinates, sections to these bundles can simply be identified with complex valued vector-fields and one-forms respectively. Of special significance is the full covariant derivative of a section to V, which we denote by $D\phi$. In accordance with the above notation, this can also be seen as the complex exterior derivative of the scalar ϕ which is defined via the natural formula:

$$D\phi(X) \;=\; D_X\phi \;.$$

This should be understood in analogy with the natural formula for the exterior derivative of a real valued scalar f on $\mathcal{M}$ which is given by $df(X) = X(f)$. For real scalars, one has the Lie derivative formula: $\mathcal{L}_X df = dX(f)$, and we seek an analog of this relation in the complex setting. Therefore, we define a covariant Lie derivative, called $\mathcal{L}^{\mathbb{C}}$, on the bundles (37) implicitly via the formulas:

$$(38) \qquad \mathcal{L}^{\mathbb{C}}_Y(\phi \otimes X) \;=\; D_Y(\phi) \otimes X + \phi \otimes [Y,X] \;,$$

$$(39) \qquad \mathcal{L}^{\mathbb{C}}_Y(\phi \otimes \omega) \;=\; D_Y(\phi) \otimes \omega + \phi \otimes \mathcal{L}_Y\omega \;.$$

Using the commutator formula (5), it is seen that this Lie derivative satisfies the following natural formula with respect to D:

$$(40) \qquad \mathcal{L}^{\mathbb{C}}_Y D\phi \;=\; D(D_Y\phi) \;+\; \sqrt{-1}\, i_X F \cdot \phi \;.$$

Here $(i_X F)_\alpha \;=\; X^\beta F_{\alpha\beta}$ is the interior product of X and F.

Now, using formula (39), we have the following useful expression involving complex scalars ϕ and complex valued one-forms η:

$$(41) \qquad \mathcal{L}_X\, \Im(\phi \cdot \overline{\eta}) \;=\; \Im(D_X\phi \cdot \overline{\eta}) + \Im\left(\phi \cdot \overline{\mathcal{L}^{\mathbb{C}}_X \eta}\right) \;.$$

Also, notice that from the formulas (38)–(39), that the operation of complex Lie differentiation is well behaved with respect to contractions. For example, for a real valued two-from and real and complex vector-fields X, Y and Z (resp.) we have the formula:

$$(42) \qquad D_Y F(X,Z) \;=\; \mathcal{L}_Y F(X,Z) + F([Y,X],Z) + F(X, \mathcal{L}^{\mathbb{C}}_Y Z) \;.$$

We will use this formula *instead* of the corresponding formula involving the covariant derivative ∇_Y. This is necessary because when we differentiate F it will always be with respect to $\mathcal{L}$ and *not* ∇.

Using the formula (39), we compute the covariant divergence implicitly via the formula:

$$(\text{div})^{\mathbb{C}}(\phi \otimes \omega) \;=\; \omega^\alpha D_\alpha\phi + (\text{div})\omega \cdot \phi \;,$$

for any scalar field ϕ and real one-form ω. Using this last line in conjunction with the identity (35) above, we have the following commutator formula for the complex divergence and complex Lie derivative for complex valued one-forms η:

$$D_X((\text{div})^{\mathbb{C}}\eta) - \eta^\alpha e_\alpha((\text{div})X) \;=\; -D_\alpha{}^{(X)}\pi^{\alpha\beta}\eta_\beta + (\text{div})^{\mathbb{C}}(\mathcal{L}^{\mathbb{C}}_X\eta) + \sqrt{-1}X^\alpha F_{\alpha\beta}\eta^\beta \;.$$

In particular, if $^{(X)}\pi$ is a constant multiple of the metric $g_{\alpha\beta}$, we have the following formula for the commutator of the covariant wave equation and the covariant derivative (or Lie derivative if you like) D_X:

$$
(43) \quad D^\alpha D_\alpha D_X \phi \ ,
$$

$$
= (\mathrm{div})^{\mathbb{C}} D(D_X \phi) \ ,
$$

$$
= (\mathrm{div})^{\mathbb{C}} \left(\mathcal{L}_X^{\mathbb{C}} D\phi - \sqrt{-1}\, i_X F \cdot \phi \right) \ ,
$$

$$
= D_X D^\alpha D_\alpha \phi + {}^{(X)}\pi^{\alpha\beta} D_\alpha D_\beta \phi + \sqrt{-1} \left(2X^\alpha F_{\alpha\beta} D^\beta \phi - \nabla^\alpha (X^\beta F_{\alpha\beta}) \cdot \phi \right) \ .
$$

2.2. The Null Frame, Lorentz Algebra, and Associated Covariant Derivatives.
Of central importance to the analysis we do here is the freedom to perform calculations on the equations (17) in an arbitrary frame. In particular, the ability to use frames which do not arise from any system of coordinates. Since we are assuming that our initial data is both smooth and well localized around the origin in $\mathbb{R}^3$, there is a canonical choice of frame for our problem. This is the so called *standard spherical null frame.* The first two members of this are the null generators of forward and backward light cones which we define respectively as:

$$
(44) \qquad\qquad L \ = \ \partial_t + \partial_r \ , \qquad\qquad \underline{L} \ = \ \partial_t - \partial_r \ .
$$

To complete things, we need only define derivatives in the angular variables. This can be done in an identical fashion on each time slice $\{t = const.\}$ so we only need to define things on $\mathbb{R}^3$. If we let $\{e_A^0\}_{a=1,2}$ denote a local orthonormal frame for the unit sphere in $\mathbb{R}^3$, then for each value of the radial variable we can by extension define:

$$
(45) \qquad\qquad\qquad\qquad e_A \ = \ \frac{1}{r}\, e_A^0 \ .
$$

Thus the collection $\{e_A\}_{A=1,2}$ forms an orthonormal basis on each sphere $\{r = const.\}$ on each fixed time slice. We can write the usual translation invariant frame $\{\partial_i\}$ in terms of ∂_r and this basis as follows:

$$
\partial_i \ = \ \omega_i \partial_r + \omega_i^A e_A \ ,
$$

where $\omega_i^A = e^A(x_i)$ which follows from the formula:

$$
\delta_i{}^j \ = \ \omega_i \omega^j + \omega_i^A e_A(x_i) \ .
$$

Using the fact that the $\{\omega_i^A\}$ are part of a rotation matrix, we have the useful formulas:

$$
(46) \qquad \omega^i \omega_i^A \ = \ 0 \ , \qquad \omega_A^i \omega_i^B \ = \ \delta_A^B \ , \qquad \omega_i^A \omega_A^j \ = \ \delta_i{}^j - \omega_i \omega^j \ .
$$

With respect to the full frame $\{\underline{L}, L, e_A\}$, the Minkowski metric reads:

$$
g \ = \ -2\theta^{\underline{L}} \otimes \theta^L - 2\theta^L \otimes \theta^{\underline{L}} + \delta_{AB} \theta^A \otimes \theta^B \ ,
$$

where $\{\theta^{\underline{L}}, \theta^L, \theta^A\}$ is the corresponding dual frame. Associated with this null frame are the standard optical functions:

$$
(47) \qquad\qquad \underline{u} \ = \ t + r \ , \qquad\qquad u \ = \ t - r \ .
$$

Note that one has the identities:

$$L(\underline{u}) = 2 , \qquad\qquad \underline{L}(\underline{u}) = 0 ,$$
$$L(u) = 2 , \qquad\qquad \underline{L}(u) = 0 .$$

In particular both u and $\underline{u}$ solve the eikonal equation $\nabla^\alpha h \nabla_\alpha h = 0$. In terms of the frame (44)–(45) we have the standard null decomposition of the electro-magnetic field tensor F, as well as the gradient $D\phi$. As usual we define:

(48a) $\qquad\qquad \alpha_A = F_{LA} , \qquad\qquad\qquad \underline{\alpha}_A = F_{\underline{L}A} ,$

(48b) $\qquad\qquad \rho = \dfrac{1}{2} F_{\underline{L}L} , \qquad\qquad\qquad \sigma = \dfrac{1}{2} \in^{AB} F_{AB} .$

Using the duality operator *, we have the useful formulas:

(49a) $\qquad\qquad {}^*\alpha_A = \in_A^B \alpha_B , \qquad\qquad\qquad {}^*\underline{\alpha}_A = \in_A^B \underline{\alpha}_B ,$

(49b) $\qquad\qquad {}^*\rho = 2\sigma , \qquad\qquad\qquad {}^*\sigma = -\dfrac{1}{2}\rho .$

One can expand out the energy-momentum tensors (13) and (11) in the null directions (44). Using the identifications (48), for the electro-magnetic field these read:

(50a) $\qquad\qquad\qquad Q(L,L)[F] = \delta^{AB}\alpha_A\alpha_B = |\alpha|^2 ,$

(50b) $\qquad\qquad\qquad Q(\underline{L},\underline{L})[F] = \delta^{AB}\underline{\alpha}_A\underline{\alpha}_B = |\underline{\alpha}|^2 ,$

(50c) $\qquad\qquad\qquad Q(\underline{L},L)[F] = \rho^2 + \sigma^2 .$

Furthermore, for the energy-momentum tensor of the scalar field (13), one has that:

(51a) $\qquad\qquad\qquad Q(L,L)[\phi] = |D_L\phi|^2 ,$

(51b) $\qquad\qquad\qquad Q(\underline{L},\underline{L})[\phi] = |D_{\underline{L}}\phi|^2 ,$

(51c) $\qquad\qquad\qquad Q(\underline{L},L)[\phi] = \delta^{AB} D_A\phi\overline{D_B\phi} = |\slashed{D}\phi|^2 .$

Next, we record here the following standard formulas for the Levi-Civita connection ∇ with respect to this frame. This will be used many times in the sequel:

(52a) $\qquad\qquad \nabla_L L = 0 , \qquad\qquad\qquad\qquad \nabla_{\underline{L}} L = 0 ,$

(52b) $\qquad\qquad \nabla_{\underline{L}}\underline{L} = 0 , \qquad\qquad\qquad\qquad \nabla_L \underline{L} = 0 ,$

(52c) $\qquad\qquad \nabla_L e_A = 0 , \qquad\qquad\qquad\qquad \nabla_{\underline{L}} e_A = 0 ,$

(52d) $\qquad\qquad \nabla_{e_A} L = -\dfrac{1}{r} e_A , \qquad\qquad\qquad \nabla_{e_A}\underline{L} = \dfrac{1}{r} e_A ,$

(52e) $\qquad\qquad \nabla_{e_A} e_B = \overline{\nabla}_{e_A} e_B + \dfrac{1}{2r}\delta_{AB}(\underline{L} - L) ,$

(52f) $\qquad\qquad\qquad\quad = \overline{\Gamma}^D_{AB} e_D + \dfrac{1}{2r}\delta_{AB}(\underline{L} - L) .$

Here, $\overline{\nabla}$ denotes the intrinsic covariant differentiation on spheres, and $\overline{\Gamma}$ its associated frame-Christoffel symbols.

Our result is based in part on the usual process of obtaining weighted energy inequalities, which in turn give L^∞ estimates of the type (26)–(27). The weights in these kind of estimates are closely related to the Killing and conformal-Killing structure of Minkowski space, as was first realized to full effect in the seminal work [8]. Accordingly, we introduce the inhomogeneous Lorentz algebra:

$$\mathbb{L} \;=\; \{\partial_\alpha, \Omega_{\alpha\beta}, S\} \; , \tag{53}$$

where:

$$\Omega_{\alpha\beta} \;=\; x_\alpha \partial_\beta - x_\beta \partial_\alpha \; , \tag{54}$$
$$S \;=\; x^\alpha \partial_\alpha \; . \tag{55}$$

This collection of vector-fields satisfy the well known algebraic relations:

$$[\partial_\alpha, \Omega_{\beta\gamma}] \;=\; \delta_{\alpha[\beta}\partial_{\gamma]} \; , \tag{56a}$$
$$[\Omega_{\alpha\beta}, \Omega_{\gamma\sigma}] \;=\; \delta_{(\alpha\gamma}\Omega_{\beta\sigma)} \; , \tag{56b}$$
$$[\partial_\alpha, S] \;=\; \partial_\alpha \; , \tag{56c}$$
$$[\Omega_{\alpha\beta}, S] \;=0 \; . \tag{56d}$$

Expanding the formulas (54)–(55) out in the frame (44)–(45), and setting $\omega_i^A = e^A(x_i)$ we have the null decompositions:

$$\partial_0 \;=\; \frac{1}{2}(L + \underline{L}) \; , \tag{57a}$$
$$\partial_i \;=\; \frac{\omega_i}{2}(\underline{L} - L) + \omega_i^A e_A \; , \tag{57b}$$
$$\Omega_{ij} \;=\; \Omega_{ij}^A e_A \;=\; \big(x_i \omega_i^A - x_j \omega_i^A\big) e_A \; , \tag{57c}$$
$$\Omega_{i0} \;=\; \frac{\omega_i}{2}(\underline{u}L - u\underline{L}) + t\omega_i^A e_A \; , \tag{57d}$$
$$S \;=\; \underline{u}L + u\underline{L} \; . \tag{57e}$$

It will also be necessary for us to have a precise account of the algebraic relations between the frame (44)–(45) and the fields (53). These can easily be computed

using the formulas (57). We record this computation here as:

$$(58a) \qquad [L, \partial_0] \;=\; [\underline{L}, \partial_0] \;=\; [e_A, \partial_0] \;=\; 0 \;,$$

$$(58b) \qquad [L, \partial_i] \;=\; \omega_i^A[L, e_A] \;=\; -\frac{\omega_i^A}{r}e_A \;,$$

$$(58c) \qquad [\underline{L}, \partial_i] \;=\; \omega_i^A[\underline{L}, e_A] \;=\; \frac{\omega_i^A}{r}e_A \;,$$

$$(58d) \qquad [L, \Omega_{ij}] \;=\; -[\underline{L}, \Omega_{ij}] \;=\; 0 \;,$$

$$(58e) \qquad [e_B, \Omega_{ij}] \;=\; e_B(\Omega_{ij}^A)e_A + \Omega_{ij}^A\mathbf{c}_{BA}^D e_D \;,$$

$$(58f) \qquad [L, \Omega_{i0}] \;=\; \omega_i L + \frac{u}{r}\omega_i^A e_A \;,$$

$$(58g) \qquad [\underline{L}, \Omega_{i0}] \;=\; \omega_i\underline{L} + \frac{\underline{u}}{r}\omega_i^A e_A \;,$$

$$(58h) \qquad [e_B, \Omega_{i0}] \;=\; \frac{\omega_i^B}{2r}(\underline{u}L - u\underline{L}) + t\big(e_B(\omega_i^D) + \omega_i^A\mathbf{c}_{BA}^D\big)e_D \;,$$

$$(58i) \qquad [L, S] \;=\; 2L \;,$$

$$(58j) \qquad [\underline{L}, S] \;=\; 2\underline{L} \;,$$

$$(58k) \qquad [e_A, S] \;=\; \frac{1}{r}e_A.$$

Finally, to end this subsection, we list here the various formula for the action of the Levi-Civita connection ∇ on the algebra (53). All of these formulas are computed in a straightforward way using the identities (52) and (57):

$$(59a) \qquad \nabla_L\partial_\alpha \;=\; \nabla_{\underline{L}}\partial_\alpha \;=\; \nabla_{e_A}\partial_\alpha \;=\; 0 \;,$$

$$(59b) \qquad \nabla_L\Omega_{ij} \;=\; -\nabla_{\underline{L}}\Omega_{ij} \;=\; \frac{1}{r}\Omega_{ij} \;,$$

$$(59c) \qquad \nabla_{e_B}\Omega_{ij} \;=\; e_B(\Omega_{ij}^A)e_A + \Omega_{ij}^A\overline{\Gamma}_{BA}^D e_D + \frac{1}{2}\Omega_{ij}^B(\underline{L} - L) \;,$$

$$(59d) \qquad \nabla_L\Omega_{i0} \;=\; \omega_i L + \omega_i^A e_A \;,$$

$$(59e) \qquad \nabla_{\underline{L}}\Omega_{i0} \;=\; -\omega_i\underline{L} - \omega_i^A e_A \;,$$

$$(59f) \qquad \nabla_{e_B}\Omega_{i0} \;=\; \frac{\omega_i^B}{2}(L + \underline{L}) + te_B(\omega_i^A)e_A + t\omega_i^A\overline{\Gamma}_{BA}^D e_D \;,$$

$$(59g) \qquad \nabla_L S \;=\; 2L \;,$$

$$(59h) \qquad \nabla_{\underline{L}}S \;=\; 2\underline{L} \;,$$

$$(59i) \qquad \nabla_{e_A}S \;=\; -2e_A \;.$$

3. Fixed Time and Space-time Energy Estimates for the Curvature

In this section, we begin to build the estimates which lie at the center of our approach. All of these will be produced in some way through the tensorial conservation laws (12) and (14). In this section we will deal with weighted L^2 type estimates which involve the curvature tensor $F_{\alpha\beta}$. We assume that this satisfies the Maxwell equation (8) for an unspecified current density J_α.

On Minkowski space, the two most basic energy estimates (really identities) for $F_{\alpha\beta}$ are based respectively on time translation invariance and the conformal structure of the Minkowski metric. This is utilized by contracting the energy-momentum tensor (11) with the following two vector fields (respectively):

$$(60) \qquad T \;=\; \partial_0 \;=\; \frac{1}{2}(L + \underline{L}) \;,$$

$$(61) \qquad K_0 \;=\; (t^2 + |x|^2)\partial_0 + 2tx^i\partial_i \;=\; \frac{1}{2}(\underline{u}^2 L + u^2 \underline{L}) \;.$$

One readily computes their deformation tensors to be:

$$^{(T)}\pi \;=\; 0 \;, \qquad\qquad\qquad ^{(K_0)}\pi \;=\; 4t\,g \;.$$

Due to the trace-free nature of the energy-momentum tensor (11), the one-form resulting from such a contraction is seen to be divergence free. Therefore, defining the weights:

$$(62) \qquad \tau_+^2 \;=\; 1 + \underline{u}^2 \;, \qquad\qquad\qquad \tau_-^2 \;=\; 1 + u^2$$

and the hybrid vector-field:

$$\overline{K}_0 \;=\; T + K_0 \;,$$

we arrive at the following fundamental estimate through the application of the geometric Stokes theorem[5] over domains of the type $\{u \leqslant u_0\} \cap \{0 \leqslant t \leqslant t_0\}$ and expanding out the resulting quantities using the identities (50):

$$(63) \quad \int_{\mathbb{R}^3 \cap \{t=t_0\}} \tau_+^2 |\alpha|^2 + \tau_-^2 |\underline{\alpha}|^2 + (\tau_+^2 + \tau_-^2)(\rho^2 + \sigma^2) \; dx$$

$$+ \sup_u \int_{C(u) \cap \{0 \leqslant t \leqslant t_0\}} \tau_+^2 |\alpha|^2 + \tau_-^2 (\rho^2 + \sigma^2) \; dV_{C(u)}$$

$$\leqslant \int_0^{t_0} \int_{\mathbb{R}^3} |(\overline{K}_0)^\alpha F_{\alpha\beta} J^\beta| \; dxdt \;+\; \int_{\mathbb{R}^3 \cap \{t=0\}} (1 + r^2)(|E|^2 + |H|^2) \; dx \;.$$

Here $dV_{C(u)}$ is the Euclidean volume element on the cone $\{t - r = u\}$.

As it stands, estimate (63) is not terribly useful except in certain restricted situations. This is because, in general, it is not possible to assume that the right hand side of (63) above is finite, even if one assumes that the "dynamic" portion of the initial data is compactly supported. This is the effect of the constraint equation on line (10a). Indeed, taking a Hodge decomposition $E = E^{df} + E^{cf}$ of E into its divergence-free and curl-free components (respectively), and introducing the potential function:

$$E^{cf} \;=\; \nabla\varphi \;,$$

we see that this is equivalent to:

$$(64) \qquad\qquad \Delta\varphi \;=\; J_0 \;.$$

In particular, it is not possible in general for E to decay better that $E \sim \frac{1}{r^2}$ unless the following quantity, known as the *charge*, vanishes:

$$(65) \qquad\qquad q(F)(t) \;=\; \int_{\mathbb{R}^3} J_0(t) \; dx \;.$$

---

[5]We shall discuss this type of procedure in more detail in the sequel. See also [3] for a more thorough discussion and a derivation of this particular estimate.

Notice that this quantity is a constant of motion thanks to the continuity equation (16). Therefore, we shall henceforth refer to it as $q(F)$ and drop the dependence on t. Without the extra assumption $q(F) = 0$, there is no hope of obtaining a finite value for the energy (63).

In order to circumvent this problem it is necessary to either modify the energy (63) so that less decay of the initial data is required, or to modify the field equations (8) themselves so that the behavior resulting from the charge is eliminated. A naive approach to the first tactic would be to simply decrease the amount of decay required on the right hand side of (63). For example, this can be done by eliminating the use of the conformal field (61). As we shall see in a moment, it does not seem possible to do this and still retain the distinct weights (peeling) on the different components on the left hand side of (63). Furthermore, recall that the simple scaling argument used in the introduction implies that it should not be possible to close a global existence proof for the system (17) under the assumption of a total decay rate for the curvature less than $F \sim \frac{1}{r^2}$ at space-like infinity. Therefore, we use a different approach here.

A more sophisticated approach to the first tactic, which has been used by other authors in similar contexts [4], would be to employ some Lie derivatives to the field strength $F_{\alpha\beta}$ before putting it in weighted L^2, in order to kill off the effect of the charge. For instance, the Lie derivatives with respect to the rotation fields Ω_{ij} work well to accomplish this because the leading order behavior of the charge is spherically symmetric. However, this strategy runs into technical complications when it comes to dealing with the boost vector fields Ω_{i0}, which we will make essential use of in our proof to supply the sharp weighted $L^2(L^\infty)$ estimates needed to eliminate logarithmic divergences. Therefore, we shall follow a tactic which is both conceptually and technically much simpler. This is as follows: Solving the charge equation (64) explicitly, we see that:

$$\varphi(t,x) \ = \ - \ \frac{1}{4\pi} \ \int_{\mathbb{R}^3} \frac{1}{|x-y|} J_0(t,y) \ dy \ .$$

Therefore, at fixed time, we have the following asymptotic behavior for E^{cf} as $r \to \infty$:

$$(66) \qquad\qquad E_i^{cf} \ \sim \ q \cdot \frac{\omega_i}{4\pi r^2} \ .$$

Since this behavior is sufficiently simple we shall subtract it off, proving estimates for the remaining field strength. However, to do this correctly it is necessary to take into account the behavior of the charge in the wave zone $t \sim r$. Although it is not obvious at first, there is a jump type behavior of the charged component of E across the cone $u = 0$, as long as the current vector J_α satisfies certain weighted estimates which are consistent with the right hand side of (63) above. Keeping (66) in mind, the approximation we make to the charged component E_i^{cf} is the following. We first define the *charge two-form*, denoted by $\overline{F}_{\alpha\beta}$, via the exterior derivative (for $0 \leqslant t$):

$$(67) \qquad\qquad \overline{F} \ = \ - \ \frac{q(F)}{4\pi} \ d\left([\int_0^r \frac{1}{s^2} \chi^+(s-t-2) \ ds] \, dt \right) \ .$$

In the above formula, χ^+ denotes a smoothed out version of the Heaviside function; i.e. a non-decreasing C^∞ function equal to 0 on $(-\infty, 0]$ and 1 on $[1, \infty)$. Notice that by design, the tensor $\overline{F}_{\alpha\beta}$ satisfies the Bianchi identity (8b) because it is exact. Also, one has the explicit formulas for the electric-magnetic decomposition (9) of $\overline{F}$:

$$(68) \qquad \overline{E}_i \;=\; q \cdot \frac{\omega_i}{4\pi r^2} \, \chi^+(r - t - 2) \;, \qquad\qquad \overline{H}_i \;=\; 0 \;.$$

This last expression shows that $\overline{F}$ indeed represents a charge with asymptotic (66) which propagates in the exterior of the light-cone $u = 0$. In terms of the null decomposition (48), the formula (68) is even simpler and becomes:

$$(69) \qquad \overline{\rho} \;=\; \rho(\overline{F}) \;=\; \omega^i \overline{E}_i \;=\; q \cdot \frac{1}{4\pi r^2} \, \chi^+(r - t - 2) \;, \qquad \overline{\alpha} = \underline{\overline{\alpha}} = \overline{\sigma} = 0 \;,$$

and the associated current vector[6] is easily computed to be:

$$\overline{J}_{\underline{L}} \;=\; q \cdot \frac{1}{2\pi r^2} \, (\chi^+)'(r - t - 2) \;, \qquad\qquad \overline{J}_L = \overline{J}_A = 0 \;,$$

In particular, we have that:

$$(70) \qquad\qquad \overline{J}_0(t, r) \;=\; q \cdot \frac{1}{4\pi r^2} \, (\chi^+)'(r - t - 2) \;.$$

Notice that the current vector $\overline{J}_\alpha$ is only supported in the region $u \sim 0$ and has only a $\underline{L}$ component. This will allow us to obtain good estimates when it appears as an error on the right hand side of expressions such as (63).

The main estimates we need to prove now are for the remaining field strength:

$$(71) \qquad\qquad\qquad \widetilde{F} \;=\; F - \overline{F} \;.$$

This satisfies the field equations (8) with current vector:

$$(72) \qquad\qquad\qquad \widetilde{J}_\alpha \;=\; J_\alpha - \overline{J}_\alpha \;.$$

A simple calculation using the formula (70) shows that the charge for the modified field strength $\widetilde{F}$ vanishes:

$$q(\widetilde{F}) \;=\; \int_{\mathbb{R}^3} \left(J_0 - \overline{J}_0 \right) \, dx \;=\; 0 \;.$$

This formula, used in conjunction with certain weighted elliptic estimates for gradients (see appendix 9), allows control of the right hand side of (63) above with F replaced by $\widetilde{F}$. We shall return to this in a moment, after we have first discussed the previously mentioned idea of decreasing the powers occurring in the weights in estimate (63).

In the sequel we shall consider Maxwell fields $F_{\alpha\beta}$ such that the remainder $\widetilde{F}_{\alpha\beta}$ satisfies various rates of decay; not necessarily the amount which comes from the usual Morawetz estimate (63). This can be understood as making various assumptions as to how *asymptotically flat* the initial data for $F_{\alpha\beta}$ is after one has subtracted off the contribution of the charge (68) at time $t = 0$. The least amount of decay which is consistent with a finite value for the charge (65) is essentially $\widetilde{E} \sim r^{-(2+\gamma)}$ for some $0 < \gamma$. Large values for γ are available depending on the weighted decay

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[6]That is, plugging $\overline{F}$ into the field equation (8).

of J_0 as well as the vanishing of higher moments for this quantity. In order to be able to take these various rates of decay into account for the range $0 < \gamma \leqslant 1$, we would like a version of estimate (63) with weights of the form $(1 + r^2)^s$ on the right hand side for $\frac{1}{2} \leqslant s \leqslant 1$. It will be essential for us to be able to do this in such a way that distinct decay rates, also known as peeling, are still available for the various components in the null decomposition (48). This will be accomplished below through a suitable modification of the Morawetz field (61).

Finally, before stating and proving the main result of this section, let us mention in words one further adjustment to the classical Morawetz estimate (63) which will be of great use. This involves adding some space-time energies to the left hand side of (63). These turn out to be more than just a mere technical convenience and are in fact essential in order to avoid certain logarithmic divergences which come from the usual procedure of using only fixed time energy and L^∞ estimates. For us, these space-time estimates will form the backbone of our bootstrapping argument which is based on weighted bilinear $L^2(L^2)$ estimates. We now state the basic general energy estimate we will use for the curvature $F_{\alpha\beta}$ which is:

Proposition 3.1 (Generalized Morawetz estimate for the electro-magnetic field with non-vanishing charge)**.** *Let $F_{\alpha\beta}$ be a two-form which satisfies the system (8) with current vector J_α, and let $q(F)$ and $\widetilde{F}$ be the associated charge and remainder field strength defined by formulas (65) and (71) respectively. Furthermore, let $0 < \gamma, \epsilon$ and $\frac{1}{2} \leqslant s \leqslant 1$ be given parameters such that $s + \gamma < \frac{3}{2}$, and define the weights:*

$$(73) \qquad \tau_0 \;=\; \tau_-/\tau_+ \;,$$

$$(74) \qquad w_\gamma(t,r) \;=\; \chi^+(r-t) \cdot \tau_-^{2\gamma} \;+\; (1 - \chi^+(r-t)) \;,$$

$$(75) \qquad w_{\gamma,\epsilon}(t,r) \;=\; \chi^+(r-t) \cdot \tau_-^{2\gamma} \;+\; (1 - \chi^+(r-t)) \cdot \tau_-^{2\epsilon} \;,$$

$$(76) \qquad w'_{\gamma,\epsilon}(t,r) \;=\; \chi^+(r-t) \cdot \tau_-^{2\gamma-1} + (1 - \chi^+(r-t)) \cdot \tau_-^{-2\epsilon-1} \;.$$

Define the remainder null decomposition:

$$(77a) \quad \alpha_A \;=\; F_{LA} \;=\; \widetilde{F}_{LA} \;, \qquad\qquad \underline{\alpha}_A \;=\; F_{\underline{L}A} \;=\; \widetilde{F}_{\underline{L}A} \;,$$

$$(77b) \quad \widetilde{\rho} \;=\; \frac{1}{2}(F_{\underline{L}L} - \overline{F}_{\underline{L}L}) \;=\; \frac{1}{2}\widetilde{F}_{\underline{L}L} \;, \qquad \sigma \;=\; \frac{1}{2}\,\epsilon^{AB}\, F_{AB} \;=\; \frac{1}{2}\,\epsilon^{AB}\,\widetilde{F}_{AB} \;.$$

Now define the charge modified generalized Morawetz type energy content of F in the time slab $\{0 \leqslant t \leqslant t_0\} \cap \mathbb{R}^3$ to be:

$$
\begin{aligned}
(78) \quad E_0^{(s,\gamma,\epsilon)}(0,t_0)[F] \;=\;& |q(F)|^2 \\
&+ \sup_{0 \leqslant t \leqslant t_0} \int_{\mathbb{R}^3 \cap \{t\}} \left(\tau_+^{2s}|\alpha|^2 + \tau_-^{2s}|\underline{\alpha}|^2 + \tau_+^{2s}(\widetilde{\rho}^2 + \sigma^2) \right) w_\gamma \, dx \\
&+ \sup_u \int_{\{C(u)\} \cap \{0 \leqslant t \leqslant t_0\}} \left(\tau_+^{2s}|\alpha|^2 + \tau_-^{2s}(\widetilde{\rho}^2 + \sigma^2) \right) w_\gamma \, dV_{C(u)} \\
&+ \int_0^{t_0} \int_{\mathbb{R}^3} \left(\tau_+^{2s}|\alpha|^2 + \tau_0^{1+2\epsilon}\left(\tau_-^{2s}|\underline{\alpha}|^2 + \tau_+^{2s}\widetilde{\rho}^2 + \tau_+^{2s}\sigma^2 \right) \right) w'_{\gamma,\epsilon} \, dxdt \;.
\end{aligned}
$$

Then one has the following general weighted energy type estimate:

$$(79) \quad E_0^{(s,\gamma,\epsilon)}(0,t_0)[F] \ \leqslant$$

$$C_{\gamma,\epsilon} \left[\int_0^{t_0} \int_{\mathbb{R}^3} \left(\tau_+^{2s+1+2\epsilon} \tau_-^{-2\epsilon} |J_L|^2 + \tau_+^{1+2\epsilon-2s} \tau_-^{4s-2\epsilon} |J_{\underline{L}}|^2 + \tau_+^{2s} \tau_- |\slashed{J}|^2 \right) w_{\gamma,\epsilon} \, dx dt \right.$$

$$\left. + \int_{\mathbb{R}^3 \cap \{t=0\}} (1+r^2)^{s+\gamma} \left(|E^{df}|^2 + |H|^2 \right) \, dx \ + \| (1+r)^{s+\gamma} J_0(0) \|^2_{L^{\frac{6}{5}}_x} \right] .$$

Here, $C(u)$ denotes the forward facing light-cone $t - r = u$ for fixed values of u. Furthermore, $|\slashed{J}|^2 = \delta^{AB} J_A J_B$ denotes the angular portion of $|J|^2$.

Proof of estimate (79). The proof closely follows the general strategy for proving estimates of the type (63) with a few additions. We first introduce a warped generalization of the conformal killing field (61). This is:

$$(80) \qquad\qquad K_0^s \ = \ \frac{1}{2} \underline{u}^{2s} L + \frac{1}{2} |u|^{2s} \underline{L} .$$

Here we will allow s to range as $\frac{1}{2} \leqslant s \leqslant 1$. In order to gain some intuition as to the nature of K_0^s, notice that this vector-field essentially interpolates between the conformal Killing fields K_0 and S, which are its endpoint values in the interior region $0 \leqslant u$ for $s = 1$ and $s = \frac{1}{2}$ respectively. In order for this analogy to have value, it must be seen that K_0^s enjoys some positivity property when contracted with a trace-free 2-tensor which satisfies the positive energy condition. This is indeed seen to be the case through the following calculations. We let:

$$\underline{v}(\underline{u}) \ = \ \underline{u}^{2s} , \qquad\qquad v(u) \ = \ |u|^{2s} ,$$

be short hand for the fractional weights. Then, with respect to the Minkowski metric $g_{\alpha\beta}$, the deformation tensor of K_0^s is easily calculated using the identities (32) and (52):

$$(81) \qquad {}^{(K_0^s)}\pi \ = \ \frac{\underline{v} - v}{r} g + 2 \left(\frac{\underline{v} - v}{r} - (\dot{\underline{v}} + \dot{v}) \right) (\theta^{\underline{L}} \otimes \theta^L + \theta^L \otimes \theta^{\underline{L}}) .$$

In the above expression, the dot notation is used to indicate the derivative of v and $\underline{v}$ as functions of a single variable. For example, we have that $\dot{\underline{v}} = 2s |u|^{2s-1} \cdot \mathrm{sgn}(u)$. The positivity claim now follows once we have shown that the factor on the second term in (81) is non-negative:

Lemma 3.2. *Setting let $\frac{1}{2} \leqslant s \leqslant 1$ and $\underline{v} = \underline{u}^{2s}$ and $v = u^{2s}$. Then one has that:*

$$(82) \qquad\qquad 0 \ \leqslant \ \left(\frac{\underline{v} - v}{r} - (\dot{\underline{v}} + \dot{v}) \right) .$$

Proof of (82). This follows immediately from freshman calculus. For fixed t, we define the function of r:

$$f(r) \ = \ \underline{v}(t+r) - v(t-r) .$$

Notice that we have $f(0) = 0$. We also have that $f' = \underline{\dot{v}} + \dot{v}$. Therefore, from the mean value theorem there exists some $r_0 \in [0, r]$ such that:

$$\frac{f(r)}{r} \;=\; f'(r_0) \;=\; \underline{\dot{v}}(t + r_0) + \dot{v}(t - r_0) \;.$$

The claim (82) then follows if we know that f' is a non-increasing function of r. This function is computed to be:

$$(83) \qquad\qquad f'(r) \;=\; 2s(r + t)^{2s-1} + 2s|t - r|^{2s-1} \cdot \mathrm{sgn}(t - r) \;.$$

For $s = \frac{1}{2}$, this function is easily seen to be decreasing by direct inspection. For the range $\frac{1}{2} < s \leqslant 1$, (83) is continuous, so we only need to show that $f'' \leqslant 0$ when $0 < r$. Calculating the second derivative we see that:

$$f''(r) \;=\; (2s - 2)2s \left(\underline{u}^{2s-2} - |u|^{2s-2} \right) \;.$$

The desired result now follows from the fact that $0 < (2s - 2)2s$ while $(2s - 2) \leqslant 0$ and $|u| \leqslant \underline{u}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We now form a momentum density associated with the charge free portion $\widetilde{F}$ of F, as according to equations (71) and (67), and the extra γ, ϵ weights in Proposition 3.1 above:

$$(84) \qquad\qquad P_\alpha^{(s,\gamma,\epsilon)}[F] \;=\; Q_{\alpha\beta}[\widetilde{F}](\overline{K}_0^s)^\beta \cdot \widetilde{w}_{\gamma,\epsilon},$$

where $\overline{K}_s^0 = T + K_0^s$ and the weight function $\widetilde{w}_{\gamma,\epsilon}$ is defined as:

$$(85) \quad \widetilde{w}_{\gamma,\epsilon} \;=\; \left(\left(1 + (2 - u)^{2\gamma}\right) \chi^+(-u) \;+\; \left(1 + (2 + u)^{-2\epsilon}\right) (1 - \chi^+(-u)) \right) \;+$$
$$\left(1 + \underline{u}\right)^{-2\epsilon} \cdot \left((2 - u)^{2\gamma + 2\epsilon} \chi^+(-u) \;+\; 1 - \chi^+(-u) \right) \;.$$

Here χ^+ can be taken to be the same smoothed out Heaviside function used in line (67) above. In particular, $(\chi^+)'$ is positive and supported on the interval $[0, 1]$. This last assumption assures that the term $(2 + u)^{-2\epsilon}(1 - \chi^+(-u))$ on the right hand side above is never singular. The weight function $\widetilde{w}_{\gamma,\epsilon}$ and its derivatives satisfy simple bounds with respect to the weight functions w_γ and $w'_{\gamma,\epsilon}$ defined on lines (74)–(76) above. First, notice that one has the bounds:

$$C^{-1} w_\gamma \;\leqslant\; \widetilde{w}_{\gamma,\epsilon} \;\leqslant\; C w_\gamma \;,$$

for a suitable positive constant C. We will restate this estimate as follows:

$$(86) \qquad\qquad\qquad\qquad \widetilde{w}_{\gamma,\epsilon} \;\sim\; w_\gamma \;.$$

Furthermore, a brief calculation shows that:

$$-\frac{1}{2} L(\widetilde{w}_{\gamma,\epsilon}) \;=\; -\partial_{\underline{u}}(\widetilde{w}_{\gamma,\epsilon}) \;,$$
$$=\; 2\epsilon \left(1 + \underline{u}\right)^{-2\epsilon - 1} \cdot \left((2 - u)^{2\gamma + 2\epsilon} \chi^+(-u) \;+\; 1 - \chi^+(-u) \right) \;,$$
$$(87) \qquad\qquad \sim\; C_{\gamma,\epsilon}^{-1} \, \tau_0^{1 + 2\epsilon} w'_{\gamma,\epsilon} \;.$$

A similar calculation shows that:

$$-\frac{1}{2}\underline{L}(\widetilde{w}_{\gamma,\epsilon}) \;=\; -\partial_u(\widetilde{w}_{\gamma,\epsilon}) \;,$$

$$= \; \left((2-u)^{2\gamma} - (2+u)^{-2\epsilon}\right)(\chi^+)'(-u) \;+$$

$$(1+\underline{u})^{-2\epsilon}\left((2-u)^{2\gamma+2\epsilon}-1\right)(\chi^+)'(-u) \;+$$

$$2(\gamma+\epsilon)(1+\underline{u})^{-2\epsilon}(2-u)^{2\gamma+2\epsilon-1}\chi^+(-u) \;+$$

$$2\gamma\,(2-u)^{2\gamma-1}\chi^+(-u) \;+\; 2\epsilon\,(2+u)^{-2\epsilon-1}(1-\chi^+(-u)) \;,$$

$$(88) \qquad\qquad \sim\; C_{\gamma,\epsilon}^{-1}\, w'_{\gamma,\epsilon}\;.$$

The bound on the last line follows because one has:

$$(2+u)^{\delta_1} \;\leqslant\; (2-u)^{\delta_2}\;,$$

whenever $\delta_1 \leqslant \delta_2$ and $u \in [-1,0]$. Note again that $(\chi^+)'$ is a positive function. Also, it is clear that the constant implicit in the $\sim$ notation on lines (86)–(88) above can be taken to be the same value uniform in ϵ and γ because we kept the effect of these constants in our bounds with the $C_{\gamma,\epsilon}$ notation. We will continue to use the $\sim$ notation in the remainder of this proof with the understanding that the same implicit constant is being used throughout.

Now, calculating the space-time divergence of the quantity (84) we have that:

$$(89) \quad \nabla^\alpha P_\alpha^{(s,\gamma,\epsilon)}[F] \;=\; \widetilde{F}_{\alpha\beta}\widetilde{J}^\alpha(\overline{K}_0^s)^\beta \cdot \widetilde{w}_{\gamma,\epsilon} \;+\; \frac{1}{2}Q_{\alpha\beta}[\widetilde{F}]\,^{(\overline{K}_0^s)}\pi^{\alpha\beta}\cdot\widetilde{w}_{\gamma,\epsilon}$$

$$-\; \frac{1}{2}Q_{\underline{L}\beta}[\widetilde{F}](\overline{K}_0^s)^\beta \cdot L(\widetilde{w}_{\gamma,\epsilon}) \;-\; \frac{1}{2}Q_{L\beta}[\widetilde{F}](\overline{K}_0^s)^\beta \cdot \underline{L}(\widetilde{w}_{\gamma,\epsilon}) \;.$$

Integrating both sides of this last line over time slabs of the form:

$$\mathcal{R}(t_0,u_0) \;=\; \{0 \leqslant t \leqslant t_0\} \cap \{u \leqslant u_0\}\;,$$

and applying the geometric version of the Stokes theorem (Gauss theorem) we arrive at the following general integral identity:

$$(90) \quad \int_{\{t=0\}\cap\{-u_0\leqslant r\}} P_0^{(s,\gamma,\epsilon)}[F]\,dx \;=\; \int_{\{t=t_0\}\cap\{t_0-u_0\leqslant r\}} P_0^{(s,\gamma,\epsilon)}[F]\,dx$$

$$+\; \int_{C(u_0)\cap\{0\leqslant t\leqslant t_0\}} P_L^{(s,\gamma,\epsilon)}[F]\,dV_{C(u_0)} \;+\; \int\!\!\int_{\mathcal{R}(t_0,u_0)} (\text{R.H.S.})(89)\;.$$

Notice that the above identity reduces to the usual energy type estimate on the time slab $\{0 \leqslant t \leqslant t_0\} \cap \mathbb{R}^3$ when $t_0 \leqslant u_0$. In order to proceed, we now calculate each of the terms in (90) individually.

When $t = 0$, we compute that:

$$(91) \qquad\qquad P_0^{(s,\gamma,\epsilon)}[F] \;\sim\; (1+r^2)^{s+\gamma}(|\widetilde{E}|^2 + |H|^2)\;,$$

where again the $\sim$ notation here means that the ratio of the two terms above is bounded by on the left and right by C^{-1} and C respectively for a suitable *positive* constant C. In particular, if the right hand side of such an expression is positive, then so is the left. Also, $\widetilde{E}$ denotes the electric part of the tensor $\widetilde{F}$ while H is the magnetic part of F. Note that:

$$\widetilde{E} \;=\; E - \overline{E}\;,$$

where $\overline{E}$ is defined as on line (68) above. Likewise, with the help of the identities (50), we compute that at time $t = t_0$:

$$(92) \qquad P_0^{(s,\gamma,\epsilon)}[F] \;\sim\; \left(\tau_+^{2s}|\alpha|^2 + \tau_-^{2s}|\underline{\alpha}|^2 + \tau_+^{2s}(\widetilde{\rho}^2 + \sigma^2)\right) \cdot w_\gamma .$$

We also compute the characteristic energy term:

$$(93) \qquad P_L^{(s,\gamma,\epsilon)}[F] \;\sim\; \left(\tau_+^{2s}|\alpha|^2 + \tau_-^{2s}(\widetilde{\rho}^2 + \sigma^2)\right) \cdot w_\gamma .$$

It remains to calculate the terms on the right hand side of (89) above. Since the fist such term does not have a sign, we simply expand it using the null decomposition (77a)–(77b) and the definition of $\widetilde{J}$ given by (72) above as well as the estimate (86) to bound:

$$(94) \quad |\widetilde{F}_{\alpha\beta}\widetilde{J}^\alpha(\overline{K}_0^s)^\beta \cdot \widetilde{w}_{\gamma,\epsilon}| \;\lesssim\;$$
$$\left(\tau_+^{2s}(|\alpha|\cdot|\slashed{J}| + |\widetilde{\rho}|\cdot|J_L|) \;+\; \tau_-^{2s}(|\underline{\alpha}|\cdot|\slashed{J}| + |\widetilde{\rho}|\cdot|J_{\underline{L}}|)\right) \cdot w_\gamma \;+\; |\widetilde{\rho}|\cdot|q|\cdot\tau_+^{-2}\tau_-^{-10}w_\gamma .$$

Notice that the extra τ_- weight on the last term of the right hand side above comes because $\overline{J}$ is supported in the region where $\tau_- \sim 1$.

We now calculate the second term on the right hand side of (89) above. Because $Q_{\alpha\beta}[F]$ is trace-free and T is Killing with respect to $g_{\alpha\beta}$, the contraction need only be taken with the second term on the right hand side of line (81). This yields:

$$(95) \qquad Q_{\alpha\beta}[\widetilde{F}]\,{}^{(\overline{K}_0^s)}\pi^{\alpha\beta} \cdot \widetilde{w}_{\gamma,\epsilon} \;\sim\; \left(\frac{\underline{v} - v}{r} - (\dot{\underline{v}} + \dot{v})\right) \cdot (\widetilde{\rho}^2 + \sigma^2) \cdot w_\gamma .$$

In particular, using Lemma 3.2 and the range restriction $\frac{1}{2} \leqslant s \leqslant 1$ we have that this last expression is non-negative.

Moving on, we calculate the third term on the right hand side of (89). Using the bound (87) and the decomposition (50) we conclude that:

$$(96) \quad -\frac{1}{2}Q_{\underline{L}\beta}[\widetilde{F}](\overline{K}_0^s)^\beta \cdot L(w_{\gamma,\epsilon}) \;\sim\; C_{\gamma,\epsilon}^{-1}\,\tau_0^{1+2\epsilon}\left(\tau_-^{2s}|\underline{\alpha}|^2 + \tau_+^{2s}(\widetilde{\rho}^2 + \sigma^2)\right) \cdot w'_{\gamma,\epsilon} .$$

Similarly, using the bound (88) we estimate:

$$(97) \qquad -\frac{1}{2}Q_{L\beta}[\widetilde{F}](\overline{K}_0^s)^\beta \cdot \underline{L}(w_{\gamma,\epsilon}) \;\sim\; C_{\gamma,\epsilon}^{-1}\left(\tau_+^{2s}|\alpha|^2 + \tau_-^{2s}(\widetilde{\rho}^2 + \sigma^2)\right) \cdot w'_{\gamma,\epsilon} .$$

Therefore, collecting the positive terms (92)–(93) and (95)–(97) onto the right hand side of (90) above, and collecting (94) with (91) on the left hand side and

using the shorthand notation(105) we can estimate:

(98)
$$\int_{\{t=0\}\cap\mathbb{R}^3} (1+r^2)^{s+\gamma}(|\widetilde{E}|^2 + |H|^2)\, dx \; + \; \int_0^{t_0}\int_{\mathbb{R}^3} |\widetilde{\rho}|\cdot|q|\cdot\tau_+^{-2}\tau_-^{-10}w_\gamma\, dxdt$$

$$+ \int_0^{t_0}\int_{\mathbb{R}^3} \left(\tau_+^{2s}(|\alpha|\cdot|\slashed{J}| + |\widetilde{\rho}|\cdot|J_L|) \; + \; \tau_-^{2s}(|\underline{\alpha}|\cdot|\slashed{J}| + |\widetilde{\rho}|\cdot|J_{\underline{L}}|))\right)\cdot w_\gamma\, dxdt$$

$$\geqslant C_{\gamma,\epsilon}^{-1}\left[\; E^{(s,\gamma,\epsilon)}(0,t_0)[\widetilde{F}] \; + \; \int_0^{t_0}\int_{\mathbb{R}^3}\left(\frac{\underline{v}-v}{r} - (\underline{\dot{v}}+\dot{v})\right)\cdot(\widetilde{\rho}^2+\sigma^2)\cdot w_\gamma\, dxdt\;\right].$$

We now make a preliminary reduction on estimate (98) above by first discarding the second positive term on the right hand side, and then using a Cauchy–Schwartz to peel off a factor of $(E^{\frac{1}{2}})^{(s,\gamma,\epsilon)}(0,t_0)[\widetilde{F}]$ from the latter two terms on the left hand side of the above expression. Setting:

(99) $\quad |||J|||_{L^2[0,t_0](L^2)(s,\gamma,\epsilon)}^2 \;=\;$
$$\int_0^{t_0}\int_{\mathbb{R}^3}\left(\tau_+^{2s}\tau_0^{-1-2\epsilon}\tau_-^{1+2\gamma}|J_L|^2 + \tau_0^{2s-1-2\epsilon}\tau_-^{2s+1+2\gamma}|J_{\underline{L}}|^2 + \tau_+^{2s}\tau_-^{1+2\gamma}|\slashed{J}|^2\right)\, w_{\gamma,\epsilon}\, dxdt\,,$$

we may replace (98) with the bound:

$$E^{(s,\gamma,\epsilon)}(0,t_0)[\widetilde{F}] \;\leqslant\; C_{\gamma,\epsilon}\Big[(E^{\frac{1}{2}})^{(s,\gamma,\epsilon)}(0,t_0)[\widetilde{F}]\cdot\big(|||J|||_{L^2[0,t_0](L^2)(s,\gamma,\epsilon)} + |q|\big)$$

$$+ \int_{\{t=0\}\cap\mathbb{R}^3}(1+r^2)^{s+\gamma}(|\widetilde{E}|^2 + |H|^2)\, dx\;\Big]\,,$$

from which easily follows the bound:
(100)
$$E^{(s,\gamma,\epsilon)}(0,t_0)[\widetilde{F}] \;\leqslant\; C_{\gamma,\epsilon}^2\Big[|||J|||_{L^2[0,t_0](L^2)(s,\gamma,\epsilon)}^2 + |q|^2 + \int_{\{t=0\}\cap\mathbb{R}^3}(1+r^2)^{s+\gamma}(|\widetilde{E}|^2+|H|^2)\, dx\;\Big]\,,$$

In order to deduce (79) from (100), we need to prove the following bounds for the range $\frac{1}{2}\leqslant s\leqslant 1$ and $0<\gamma$, where $s+\gamma<\frac{3}{2}$:

(101) $$|q|^2 \;\leqslant\; C_\gamma\|(1+r)^{s+\gamma}J_0(0)\|_{L_x^{\frac{6}{5}}}^2\,,$$

and:

(102) $$\int_{\{t=0\}\cap\mathbb{R}^3}(1+r^2)^{s+\gamma}|\widetilde{E}|^2\, dx$$

$$\leqslant\; C_\gamma\left[\int_{\{t=0\}\cap\mathbb{R}^3}(1+r^2)^{s+\gamma}|E^{df}|^2\, dx \; + \; \|(1+r)^{s+\gamma}J_0(0)\|_{L_x^{\frac{6}{5}}}^2\right]\,.$$

The first of the above estimates, (101), follows from a simple application of Hölders inequality with the weight $(1+r)^{-s-\gamma}$ in the integral (65) at time $t=0$. Note that $\frac{1}{2}<s+\gamma$ because of our range restrictions so the resulting factor integral converges with a bound depending only on γ.

The second of the above estimates results from a Hodge decomposition and the weighted elliptic estimate (282) from Appendix 9. Expanding out $\widetilde{E}$ on the left hand side we can bound:

$$\int_{\{t=0\}\cap\mathbb{R}^3} (1+r^2)^{s+\gamma}|\widetilde{E}|^2 \; dx \;\leqslant\; 2\int_{\{t=0\}\cap\mathbb{R}^3} (1+r^2)^{s+\gamma}\left(|E^{df}|^2 + |E^{cf}-\overline{E}|^2\right) \; dx \;.$$

Introducing a potential function φ for E^{cf} as in line (64) above and recalling the definition (68) of $\overline{E}$, and using line (101) to estimate:

$$\int_{\{t=0\}\cap\mathbb{R}^3} (1+r^2)^{s+\gamma}\Big|\frac{q(F)\omega^i}{r^2}\nabla_i\chi^+(r-t)\Big|^2 \; dx \;\leqslant\; C_\gamma\,|q(F)|^2 \;\leqslant\; C_\gamma\,\|\,(1+r)^{s+\gamma}J_0(0)\,\|^2_{L^{\frac{6}{5}}_x}\;,$$

we are reduced to proving the estimate:

$$(103)\quad \int_{\{t=0\}\cap\mathbb{R}^3} (1+r^2)^{s+\gamma}\Big|\nabla\big(\frac{1}{\Delta}J_0(0) + \frac{q(F)}{4\pi r}\chi^+(r-2)\big)\Big|^2 \; dx$$
$$\lesssim\; C_\gamma\,\|\,(1+r)^{s+\gamma}J_0(0)\,\|^2_{L^{\frac{6}{5}}_x}\;.$$

By using the usual $L^{\frac{6}{5}}\hookrightarrow L^2$ Sobolev embedding, as well as the bounds:

$$\int_{\{t=0\}\cap\mathbb{R}^3} \Big|\nabla\frac{q(F)}{4\pi r}\chi^+(r-2))\Big|^2 \; dx \;\lesssim\; |q(F)|^2 \;\leqslant\; C_\gamma\,\|\,(1+r)^{s+\gamma}J_0(0)\,\|^2_{L^{\frac{6}{5}}_x}\;,$$

$$\int_{\{t=0\}\cap\mathbb{R}^3} r^{2(s+\gamma)}\Big|\nabla\frac{q(F)}{4\pi r}(1-\chi^+(r-2))\Big|^2 \; dx \;\leqslant\; C_\gamma\,|q(F)|^2 \;\leqslant\; C_\gamma\,\|\,(1+r)^{s+\gamma}J_0(0)\,\|^2_{L^{\frac{6}{5}}_x}\;,$$

the second of which follows from the condition $\frac{1}{2} < s+\gamma$, we are reduced to showing that:

$$(104)\quad \int_{\{t=0\}\cap\mathbb{R}^3} r^{2(s+\gamma)}\Big|\nabla\big(\frac{1}{\Delta}J_0(0) + \frac{q(F)}{4\pi r}\big)\Big|^2 \; dx \;\lesssim\; C_\gamma\,\|\,r^{s+\gamma}J_0(0)\,\|^2_{L^{\frac{6}{5}}_x}\;.$$

Using the definition (65) of $q(F)$, this is precisely the statement of (282) in Appendix 9. Notice that the condition $\frac{1}{2} < s + \gamma < \frac{3}{2}$ is enforced by assumption. This ends the proof of estimate (79). $\qquad\qquad\square$

We conclude this section with an important generalization of Proposition 3.1. This involves estimates for derivatives of the field strength $F_{\alpha\beta}$. As it turns out, the simple decomposition (71) is remarkably robust with respect to the operation of Lie differentiation. One can show that $\mathcal{L}^I_X\widetilde{F}_{\alpha\beta}$ satisfies bounds similar to (79) with the appropriate right hand side. This will be used in the sequel to show that the various components of the null decomposition of $\widetilde{F}_{\alpha\beta}$ satisfy the expected L^∞ estimates. Together with the fact that $\overline{F}_{\alpha\beta}$ is given explicitly, this fully determines the pointwise behavior of the original field strength $F_{\alpha\beta}$. The estimate we will prove is:

Proposition 3.3 (Generalized Morawetz estimate for derivatives of the electro-magnetic field)**.** *Let $F_{\alpha\beta}$ be a two-form which satisfies the system (8) with current vector J_α, and let $q(F)$ and $\widetilde{F}$ be the associated charge and remainder field strength defined by formulas (65) and (71) respectively. Furthermore, let $0 < \gamma, \epsilon$ and $\frac{1}{2} \leqslant s \leqslant 1$ be given parameters such that $s + \gamma < \frac{3}{2}$. Define the weights $\tau_0, w_\gamma, w_{\gamma,\epsilon}, w'_{\gamma,\epsilon}$ as on*

lines (73)–(76) *above. Next, define the generalized (not charge modified) Morawetz type energy content of a two-from F in the time slab $\{0 \leqslant t \leqslant t_0\} \cap \mathbb{R}^3$ to be:*

(105)

$$
\begin{aligned}
E^{(s,\gamma,\epsilon)}(0,t_0)[F] \;=\; & \sup_{0 \leqslant t \leqslant t_0} \int_{\mathbb{R}^3 \cap \{t\}} \left(\tau_+^{2s}|\alpha|^2 + \tau_-^{2s}|\underline{\alpha}|^2 + \tau_+^{2s}(\rho^2 + \sigma^2) \right) w_\gamma \; dx \\
& + \sup_u \int_{\{C(u)\} \cap \{0 \leqslant t \leqslant t_0\}} \left(\tau_+^{2s}|\alpha|^2 + \tau_-^{2s}(\rho^2 + \sigma^2) \right) w_\gamma \; dV_{C(u)} \\
& + \int_0^{t_0} \int_{\mathbb{R}^3} \left(\tau_+^{2s}|\alpha|^2 + \tau_0^{1+2\epsilon} \left(\tau_-^{2s}|\underline{\alpha}|^2 + \tau_+^{2s}\rho^2 + \tau_+^{2s}\sigma^2 \right) \right) w'_{\gamma,\epsilon} \; dxdt \; .
\end{aligned}
$$

Now, define the kth weighted charge modified Morawetz energy:

$$
(106) \qquad E_k^{(s,\gamma,\epsilon)}(0,t_0)[F] \;=\; q^2(F) \;+\; \sum_{\substack{|I| \leqslant k \\ X \in \mathbb{L}}} E^{(s,\gamma,\epsilon)}(0,t_0)[\mathcal{L}_X^I \widetilde{F}] \; .
$$

Then, recalling the vector norm $||| \cdot |||_{L^2[0,t_0](L^2)(s,\gamma,\epsilon)}$ from line (99) above, we have the following general energy estimate:

$$
(107) \quad E_k^{(s,\gamma,\epsilon)}(0,t_0)[F] \;\leqslant\; C_{\gamma,\epsilon}^2 \left[\sum_{\substack{|I| \leqslant k \\ X \in \mathbb{L}}} \left(\; ||| \mathcal{L}_X^I J |||_{L^2[0,t_0](L^2)(s,\gamma,\epsilon)}^2 \right. \right.
$$

$$
\left. + \int_{\mathbb{R}^3 \cap \{t=0\}} (1+r^2)^{s+\gamma+|I|} \left(|\nabla_x^I E^{df}|^2 + |\nabla_x^I H|^2 \right) \; dx \;+\; \| (1+r)^{s+\gamma+|I|} \nabla_x^I J_0(0) \|_{L_x^{\frac{6}{5}}}^2 \right)
$$

$$
\left. + \sum_{|I| \leqslant k-1} \| (1+r)^{s+\gamma+|I|+1} \nabla_{t,x}^I J(0) \|_{L_x^2}^2 \right] \; .
$$

Here ∇_x and $\nabla_{t,x}$ denote the Lie derivatives of the covector J and the two-form F with respect to the spatial translation invariant fields $\{\partial_i\}$ and the full set of translation invariant fields $\{\partial_\alpha\}$ respectively.

Proof of estimate (107). *The proof is essentially that of estimate* (79) *applied to the field $\mathcal{L}_X^I \widetilde{F}_{\alpha\beta}$ with some additional calculations at the end to wrap things up to achieve the desired right hand side. First notice that by* (36) *and the fact that for any Lie derivative one has $[d, \mathcal{L}_X] = 0$, we have the following formula for $\mathcal{L}_X^I \widetilde{F}$ whenever $X \in \mathbb{L} \setminus \{S\}$:*

$$
\begin{aligned}
\nabla^\beta (\mathcal{L}_X^I \widetilde{F})_{\alpha\beta} &= (\mathcal{L}_X^I J)_\alpha - (\mathcal{L}_X^I \overline{J})_\alpha \; , \\
\nabla^{\beta\,*}(\mathcal{L}_X^I \widetilde{F}) &= 0 \; .
\end{aligned}
$$

The effect of the Lie derivative $\mathcal{L}_S$ is equally easy to account for. In this case formula (36) gives:

$$
\begin{aligned}
\nabla^\beta (\mathcal{L}_S \widetilde{F})_{\alpha\beta} &= (J - \overline{J})_\alpha \;+\; (\mathcal{L}_S J)_\alpha - (\mathcal{L}_S \overline{J})_\alpha \; , \\
\nabla^{\beta\,*}(\mathcal{L}_S \widetilde{F}) &= 0 \; .
\end{aligned}
$$

Using the above formulas, and following the proof of (79) until line (98) yields the inequality:

$$E_k^{(s,\gamma,\epsilon)}(0,t_0)[F] \leqslant C_{\gamma,\epsilon} \sum_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} \Big[|q(F)|^2 \ + \ ||| \mathcal{L}_X^I (J-\overline{J}) |||^2_{L^2[0,t_0](L^2)(s,\gamma,\epsilon)}$$

$$+ \int_{\mathbb{R}^3\cap\{t=0\}} (1+r^2)^{s+\gamma} \Big(|E(\mathcal{L}_X^I \widetilde{F})|^2 + |H(\mathcal{L}_X^I \widetilde{F})|^2 \Big) \ dx \Big] \ .$$

Using the same steps as at the end of the proof of (79) to bound the quantity $|q(F)|$, we are done once we have shown that:

$$(108) \qquad\qquad ||| \mathcal{L}_X^I \overline{J} |||^2_{L^2[0,t_0](L^2)(s,\gamma,\epsilon)} \ \lesssim \ |q(F)|^2 \ ,$$

as well as the bound:

$$(109) \quad \sum_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} \int_{\mathbb{R}^3\cap\{t=0\}} (1+r^2)^{s+\gamma} \Big(|E(\mathcal{L}_X^I \widetilde{F})|^2 + |H(\mathcal{L}_X^I \widetilde{F})|^2 \Big) \ dx$$

$$\lesssim \quad \sum_{|I|\leqslant k} \Big[\int_{\mathbb{R}^3\cap\{t=0\}} (1+r^2)^{s+\gamma+|I|} \big(|\nabla_x^I E^{df}|^2 + |\nabla_x^I H|^2 \big) \ dx$$

$$+ \ \| (1+r)^{s+\gamma+|I|} \nabla_x^I J_0(0) \|^2_{L_x^{\frac{6}{5}}} \Big]$$

$$+ \sum_{|I|\leqslant k-1} \| (1+r)^{s+\gamma+|I|+1} \nabla_{t,x}^I J(0) \|^2_{L_x^2} \ .$$

We begin with (108). This will follow from direct computation using pointwise bounds on the quantity $\mathcal{L}_X^I \overline{J}$ for $X\in\mathbb{L}$. These pointwise bounds will be provided through induction on the value $|I|$. Notice that when $|I|=0$ we may write:

(110)

$$J_{\underline{L}} \ = \ q\cdot\frac{\Omega_{\underline{L}}(\omega)}{r^2}\cdot\chi(u) \ , \quad J_L \ = \ q\cdot\frac{\Omega_L(\omega)}{r^4}\cdot\chi(u) \ , \quad J_A \ = \ q\cdot\frac{\Omega_A(\omega)}{r^3}\cdot\chi(u) \ ,$$

where χ is a C^∞ and $O(1)$ bump function adapted to the origin $u=0$ and the $\Omega_\alpha(\omega)$ are smooth function of the angular variable only. In particular, these satisfy inductive identities:

$$L(\Omega_\alpha) \ = \ \underline{L}(\Omega_\alpha) \ = \ 0 \ , \qquad\qquad e_A(\Omega_\alpha) \ = \ \frac{1}{r}\widetilde{\Omega}_\alpha \ .$$

where the $\widetilde{\Omega}_\alpha$ have the same properties of the Ω_α. Also, notice that by direct computation one can substitute the decay rates (110) into the norm $|||\cdot|||^2_{L^2[0,t_0](L^2)(s,\gamma,\epsilon)}$ to achieve a bound in terms of $|q(F)|$. We now inductively prove the identities:

$$(111a) \qquad\qquad (\mathcal{L}_X^I J)_{\underline{L}} \ = \ q\cdot\frac{\Omega_{\underline{L}}^I(\omega)}{r^2}\cdot\chi_{\underline{L}}^I(u) \ ,$$

$$(111b) \qquad\qquad (\mathcal{L}_X^I J)_L \ = \ q\cdot\frac{\Omega_L^I(\omega)}{r^4}\cdot\chi_L^I(u) \ ,$$

$$(111c) \qquad\qquad (\mathcal{L}_X^I J)_A \ = \ q\cdot\frac{\Omega_A^I(\omega)}{r^3}\cdot\chi_A^I(u) \ ,$$

where the Ω_α^I and χ_α^I depend on the specific combination of vector-fields but have the same properties as the Ω_α and χ in line (110) above. Also, the product notation is symbolic and is used to denote a finite sum of products of functions with these properties. Assuming that (111) is true for $|I| = l - 1$ it can be shown that (111) holds for $|I| = l$ through direct use of the identities (57)–(58) as well as the formula:

$$(\mathcal{L}_X J)_\alpha \ = \ X(J_\alpha) - J_{[X, e_\alpha]} \ .$$

We leave this as a straightforward, although rather tedious, exercise for the reader. Note that everything can be put in terms of the variables u, r, ω so the only thing to verify is the homogeneity claim with respect to the power of r.

It remains to prove the second bound (109) above. This is done in two separate steps. First of all, a direct computation involving the formula:

$$(\mathcal{L}_X \widetilde{F})_{\alpha\beta} \ = \ X(\widetilde{F}_{\alpha\beta}) - \widetilde{F}_{[X, e_\alpha]\, \beta} - \widetilde{F}_{\alpha\, [X, e_\beta]} \ ,$$

as well as the bracket identities (56a) and (56c) shows that at time $t = 0$ one has the pointwise bound:

$$\sum_{\substack{|I| \leqslant k \\ X \in \mathbb{L}}} \left(|E(\mathcal{L}_X^I \widetilde{F})|^2 + |H(\mathcal{L}_X^I \widetilde{F})|^2 \right) \ \lesssim \ \sum_{|I| \leqslant k} (1 + r^2)^{|I|} \left(|\nabla_{t,x}^I \widetilde{E}|^2 + |\nabla_{t,x}^I H|^2 \right) \ ,$$

Using the field equations (10), this last line can be further reduced to the estimate:

$$\sum_{\substack{|I| \leqslant k \\ X \in \mathbb{L}}} \left(|E(\mathcal{L}_X^I \widetilde{F})|^2 + |H(\mathcal{L}_X^I \widetilde{F})|^2 \right) \ \lesssim \ \sum_{|I| \leqslant k} (1 + r^2)^{|I|} \left(|\nabla_x^I \widetilde{E}|^2 + |\nabla_x^I E^{df}|^2 + |\nabla_x^I H|^2 \right)$$

$$+ \sum_{|I| \leqslant k-1} (1 + r^2)^{|I|+1} \left(|\nabla_{t,x}^I J|^2 + |q(F)|^2 \cdot |\nabla_{t,x}^I (\chi^+)'|^2 \right) \ .$$

Multiplying through by $(1 + r^2)^{s+\gamma}$ and integrating this last line over $\mathbb{R}^3$, we have achieved the bound (109) modulo the estimate:

$$\sum_{|I| \leqslant k} \int_{\mathbb{R}^3 \cap \{t=0\}} (1+r^2)^{s+\gamma+|I|} |\nabla_x^I \widetilde{E}|^2 \, dx \ \lesssim \ \sum_{|I| \leqslant k} \left(\int_{\mathbb{R}^3 \cap \{t=0\}} (1+r^2)^{s+\gamma+|I|} |\nabla_x^I E^{df}|^2 \, dx \right.$$

$$\left. + \ \| (1 + r)^{s+\gamma+|I|} \nabla_x^I J_0(0) \|_{L^{\frac{6}{5}}}^2 \right) \ .$$

Using essentially the same steps which were employed to reduce estimate (102) above to (104), this last line is a consequence of the following generalization of estimate (282) in the Appendix:

$$(112) \qquad \sum_{|I| \leqslant k} \int_{\mathbb{R}^3} r^{2(s+\gamma+|I|)} \left| \nabla \, \nabla_x^I \left(\frac{1}{\Delta} J_0(0) + \frac{q}{4\pi r} \right) \right|^2 \, dx$$

$$\leqslant \ C_\gamma \sum_{|I| \leqslant k} \| \, r^{s+\gamma+|I|} \nabla_x^I J_0(0) \, \|_{L^{\frac{6}{5}}}^2 \ .$$

Estimate (112) can be reduced to estimate (104) through a process of induction. To see this, assume that $0 < |I|$ and integrate by parts on the left hand side of

(112). This yields:

$$(\text{L.H.S.})(112) \;=\; \sum_{|I|\leqslant k} \left((s+\gamma+|I|+1)(s+\gamma+|I|) \int_{\mathbb{R}^3} r^{2(s+\gamma+|I|-1)} \left| \nabla_x^I \left(\frac{1}{\Delta} J_0(0) + \frac{q}{4\pi r} \right) \right|^2 dx \right.$$

$$\left. - \int_{\mathbb{R}^3} r^{2(s+\gamma+|I|)} \left(\nabla^I J_0(0) \right) \cdot \nabla_x^I \left(\frac{1}{\Delta} J_0(0) + \frac{q}{4\pi r} \right) dx \right) .$$

Applying Cauchy-Schwartz and Hölders inequality to the right hand side of the above expression in conjunction with the $L^2 \hookrightarrow L^6$ Sobolev embedding and setting:

$$A(k) \;=\; \sum_{|I|\leqslant k} \| \, r^{s+\gamma+I} \nabla \, \nabla_x^J \left(\frac{1}{\Delta} J_0(0) + \frac{q}{4\pi r} \right) \|_{L^2} \;,$$

$$B(k) \;=\; \sum_{|I|\leqslant k} \| \, r^{s+\gamma+|I|} \nabla_x^I J_0(0) \, \|_{L^{\frac{6}{5}}} \;,$$

we have for $1 \leqslant k$ the estimate:

(113) $\qquad |(\text{L.H.S.})(112)| \;\sim\; A(k) \;\lesssim\; A^2(k-1) + B(k) \cdot A(k) \;.$

Notice that with this notation estimate (282) reads $A(0) \lesssim B(k)$. This assumption, together with the inductive estimate (113) shows that $A(k) \lesssim B(k)$ for all $0 \leqslant k$. This ends the proof of (112) and therefore the proof of (107). $\qquad\square$

4. FIXED TIME AND SPACE-TIME ENERGY ESTIMATES FOR COMPLEX SCALAR FIELDS: CONFORMAL MODIFICATIONS AND THE MORAWETZ THEORY

In this section, we prove estimates of the type (79) for the complex scalar field (2). This is necessarily more involved than in Section 3 because the tensor (13) is not trace-free as is (11), which is a reflection of the fact that, unlike Maxwell's equations, the scalar wave equation is not conformally invariant. However, it is well known that there is an analog of the estimate (63) for the wave equation which is also known as the Morawetz estimate, and was first used in [13] to prove local energy decay for solutions to (13) in the case $F \equiv 0$. We record this estimate here as:

$$(114) \quad \int_{\{t=t_0\}\cap\mathbb{R}^3} \tau_+^2 |L\phi|^2 + \tau_-^2 |\underline{L}\phi|^2 + (\tau_+^2 + \tau_-^2)(|\nabla\!\!\!/\,\phi|^2 + |\frac{\phi}{r}|^2) \; dx$$

$$\lesssim \int_{\{t=0\}\cap\mathbb{R}^3} (1+r^2)|\nabla\phi|^2 \; dx \;.$$

Notice that we have used the usual derivatives above because we are assuming that the connection is flat. The usual procedure for proving (114) involves first coming up with a certain weighted L^2 type identity, again known as the Morawetz identity, and then performing several integration by parts in order to ultimately arrive at the estimate (114). In the context we are working, where we wish to energy estimates of the type (79) involving fractional weights as well as characteristic and space-time energy estimates, such a procedure would become unduly tedious. Therefore, we provide here a new approach to Morawetz type estimates, which provides (114) directly in divergence form. Having done this, it will be straight forward to modify the divergence identity to include various weights of the type (74)–(76) as well as

the fractional Morawetz field (80). Furthermore, our procedure leads to a more direct geometric insight as to the nature of (114) which is obscured by the usual integration by parts proof. The construction we use is directly based on the conformal geometry of Minkowski space, and yields a set of two quantities which are extremely flexible and easy to manipulate, leading to Morawetz type energy densities of the form (84) for scalar fields.

Our starting point is to conformally modify the Minkowski metric $g_{\alpha\beta}$ in such a way that the scalar field equation (2) is preserved. As is well known, if one performs the conformal change of metric:

$$\widetilde{g} = \frac{1}{\Omega^2} g ,$$

for some weight function Ω on $\mathcal{M}$, then any solution ϕ to the inhomogeneous equation:

(115) $$\Box^{\mathbb{C}} \phi = G ,$$

will transform to $\psi = \Omega\phi$, where ψ is a solution to the inhomogeneous conformal scalar field equation:

(116) $$\widetilde{\Box^{\mathbb{C}}}\psi - \frac{1}{6}\widetilde{R}\psi = \Omega^3 G .$$

Here $\widetilde{\Box^{\mathbb{C}}}$ is the covariant wave equation on the space $\widetilde{M}\times\mathbb{C}$ with connections $(\widetilde{\nabla}, \widetilde{D})$ and $\widetilde{R}$ the corresponding scalar curvature. One readily calculates that:

(117) $$\widetilde{R} = -\frac{1}{6}\Omega^2 \, \nabla^\alpha\nabla_\alpha\left(\frac{1}{\Omega}\right) .$$

In order that (116) match up with (115), we require Ω to be such that the scalar curvature vanishes, $\widetilde{R} \equiv 0$. In light of the calculation (117), this will be guaranteed if one has $\Box(\frac{1}{\Omega}) = 0$ where $\Box$ is the usual D'Lambertian on Minkowski space. There are two interesting and useful choices of Ω which give a singular solution to this problem and which are ultimately responsible for the estimate (114). These come from the fundamental solutions to the Laplace and wave equation respectively:

$$^{I}\Omega = r, \qquad\qquad ^{II}\Omega = u\underline{u} .$$

We label the resulting conformal metrics by $^{I}\widetilde{g}$ and $^{II}\widetilde{g}$ respectively. Notice that both of these metrics are singular along the varieties $r = 0$ and $|t| = r$ respectively. However, these singularities will not effect what we do here because there will always be extra factors involving positive powers of Ω in all the identities we use which will cancel the singularities off. A striking property of the metrics $^{I}\widetilde{g}$ and $^{II}\widetilde{g}$ is that the vector-field K_0 becomes Killing with respect to *both* of them (away from the singular set of course). This is seen simply from the identities:

$$K_0\left(\frac{1}{r^2}\right) = -\frac{4t}{r^2} , \qquad\qquad K_0\left(\frac{1}{(u\underline{u})^2}\right) = -\frac{4t}{(u\underline{u})^2} .$$

Thus, for example one has that:

(118) $$\mathcal{L}_{K_0}{}^{I}\widetilde{g} = K_0(\frac{1}{r^2})\,g + \frac{1}{r^2}\mathcal{L}_{K_0}g = 0,$$

with a similar calculation shows that:

(119) $$\mathcal{L}_{K_0}{}^{II}\widetilde{g} = 0 .$$

Note that the vector-field T is Killing with respect to ${}^{I}\widetilde{g}$, but is only conformal Killing with respect to ${}^{II}\widetilde{g}$. The identities (118)–(119) show that if one is to use the vector-field K_0 to produce energy estimates, it is best done with respect to the metrics ${}^{I}\widetilde{g}$ and ${}^{II}\widetilde{g}$ instead of the usual Minkowski. Accordingly, we define the *conformal energy-momentum tensors of the first and second kind* associated to (115) to simply be the usual energy-momentum tensors of the equation (116) with respect to the metrics ${}^{I}\widetilde{g}$ and ${}^{II}\widetilde{g}$:

$$(120) \qquad {}^{I}\widetilde{Q}_{\alpha\beta}[\phi] \;=\; \Re(D_\alpha(r\phi)\overline{D_\beta(r\phi)}) - \frac{1}{2}{}^{I}\widetilde{g}_{\alpha\beta}\,\widetilde{D}^\gamma(r\phi)\overline{D_\gamma(r\phi)} \;,$$

$$(121) \qquad {}^{II}\widetilde{Q}_{\alpha\beta}[\phi] \;=\; \Re(D_\alpha(u\underline{u}\phi)\overline{D_\beta(u\underline{u}\phi)}) - \frac{1}{2}{}^{II}\widetilde{g}_{\alpha\beta}\,\widetilde{D}^\gamma(u\underline{u}\phi)\overline{D_\gamma(u\underline{u}\phi)} \;.$$

Here we have used the notation $\widetilde{D}^\gamma = \widetilde{g}^{\alpha\gamma}D_\alpha$ for $\widetilde{g} = {}^{I}\widetilde{g}, {}^{II}\widetilde{g}$ on lines (120) and (121) respectively. As a direct consequence of the equation (116) and the divergence-fee property of the energy-momentum tensor for scalar fields one has the divergence laws:

$$ {}^{I}\widetilde{\nabla}^\alpha{}^{I}\widetilde{Q}_{\alpha\beta}[\phi] \;=\; r^4\left(\Re(G\cdot\overline{\frac{1}{r}D_\beta(r\phi)}) + F_{\beta\gamma}\Im(\phi\overline{\frac{1}{r}D^\gamma(r\phi)}) \right) \;,$$

$$ {}^{II}\widetilde{\nabla}^\alpha{}^{II}\widetilde{Q}_{\alpha\beta}[\phi] \;=\; (u\underline{u})^4\left(\Re(G\cdot\overline{\frac{1}{u\underline{u}}D_\beta(u\underline{u}\phi)}) + F_{\beta\gamma}\Im(\phi\overline{\frac{1}{u\underline{u}}D^\gamma(u\underline{u}\phi)}) \right) \;,$$

where ${}^{I}\widetilde{\nabla}$ and ${}^{II}\widetilde{\nabla}$ are the Levi-Civita connections of ${}^{I}\widetilde{g}$ and ${}^{II}\widetilde{g}$ respectively. Since the vector-field K_0 is Killing with respect to both of these metrics we may contract it with the tensors (120)–(121) to obtain momentum densities which satisfy the divergence laws:

$$(122)$$
$$ {}^{I}\widetilde{\nabla}^\alpha\left({}^{I}\widetilde{Q}_{\alpha\beta}[\phi](K_0)^\beta \right) \;=\; r^4\left(\Re(G\cdot\overline{\frac{1}{r}D_{K_0}(r\phi)}) + (K_0)^\beta F_{\beta\gamma}\Im(\phi\overline{D^\gamma\phi}) \right) \;,$$

$$(123)$$
$$ {}^{II}\widetilde{\nabla}^\alpha\left({}^{II}\widetilde{Q}_{\alpha\beta}[\phi](K_0)^\beta \right) \;=\; (u\underline{u})^4\left(\Re(G\cdot\overline{\frac{1}{u\underline{u}}D_{K_0}(u\underline{u}\phi)}) + (K_0)^\beta F_{\beta\gamma}\Im(\phi\overline{\frac{1}{u\underline{u}}D^\gamma(u\underline{u}\phi)}) \right) \;.$$

These last two identities can now be integrated over various space-time regions to obtain positive quantities for the scalar field ϕ at the cost of estimating the source terms on the right hand side of (122)–(123). To calculate these, notice that the volume forms of the metrics ${}^{I}\widetilde{g}$ and ${}^{II}\widetilde{g}$ are:

$$(124) \qquad dV_I \;=\; \frac{1}{r^4}dV_{\mathcal{M}}, \qquad\qquad dV_{II} \;=\; \frac{1}{(u\underline{u})^4}dV_{\mathcal{M}} \;,$$

while rT and $u\underline{u}T$ are the respective Lorentzian unit normals to the time slices $t = const$. Furthermore, notice that the vector-fields rL and $u\underline{u}L$ are the respective Lorentzian unit normal to the cones $u = const$. Therefore, applying the geometric Stokes theorem to with respect to these quantities, we arrive at the basic Morawetz type energy estimates for the complex scalar field (115):

Proposition 4.1 (First and Second Morawetz Estimates for Complex Scalar Fields)**.**
Let $\Omega = r, u\underline{u}$. Then one has the following estimates for solutions to the inhomogeneous equation (115):

$$(125) \quad \int_{\{t=t_0\}\cap\mathbb{R}^3} \frac{1}{4}\left(\underline{u}^2|\frac{1}{\Omega}D_L(\Omega\phi)|^2 + u^2|\frac{1}{\Omega}D_{\underline{L}}(\Omega\phi)|^2 + (\underline{u}^2+u^2)|\not{D}\phi|^2\right)dx$$

$$+ \sup_u \int_{C(u)\cap\{0<t<t_0\}} \frac{1}{4}\left(\underline{u}^2|\frac{1}{\Omega}D_L(\Omega\phi)|^2 + u^2|\not{D}\phi|^2\right)dV_{C(u)}$$

$$\leqslant \int_0^{t_0}\int_{\mathbb{R}^3} |G\cdot\frac{1}{\Omega}D_{K_0}(\Omega\phi)| + |(K_0)^\beta F_{\beta\gamma}\Im(\phi\overline{\frac{1}{\Omega}D^\gamma(\Omega\phi)})|\ dxdt$$

$$+ \int_{\{t=0\}\cap\mathbb{R}^3}\frac{r^2}{2}|\frac{1}{\Omega}D(\Omega\phi)|^2\ dx\ .$$

To derive (114) from estimate (125) above is a simple matter. Assume now that both $G = F \equiv 0$. Notice first that it suffices to prove (114) with the τ_+, τ_- weights replaced by $\underline{u}, u$ and $(1+r^2)$ on the right hand side replaced by r^2. This follows from the fact that, as we have already mentioned, T is Killing with respect to the metric ${}^I\widetilde{g}$. Therefore, the same procedure used to produce (125) in the case $\Omega = r$ yields the estimate:

$$\int_{\{t=t_0\}\cap\mathbb{R}^3}\frac{1}{4}\left(|\frac{1}{r}L(r\phi)|^2 + |\frac{1}{r}\underline{L}(r\phi)|^2 + |\not{\nabla}\phi|^2\right)\ dx \leqslant \frac{1}{2}\int_{\{t=0\}\cap\mathbb{R}^3}|\frac{1}{r}\nabla(r\phi)|^2\ dx\ ,$$

$$\lesssim \int_{\{t=0\}\cap\mathbb{R}^3}|\nabla\phi|^2\ dx\ .$$

The last line above follows from the Poincare inequality:

$$(126) \qquad \int_{\{t=0\}\cap\mathbb{R}^3}|\frac{\phi}{r}|^2\ dx \lesssim \int_{\{t=0\}\cap\mathbb{R}^3}|\partial_r\phi|^2\ dx\ .$$

To finish things off, we simply use the identities:

$$\underline{u}^2|\frac{1}{r}L(r\phi)|^2 = |\underline{u}L\phi + \frac{\underline{u}\phi}{r}|^2\ , \qquad u^2|\frac{1}{r}\underline{L}(r\phi)|^2 = |u\underline{L}\phi - \frac{u\phi}{r}|^2\ ,$$

$$\underline{u}^2|\frac{1}{u\underline{u}}L(u\underline{u}\phi)|^2 = |\underline{u}L\phi + 2\phi|^2\ , \qquad u^2|\frac{1}{u\underline{u}}\underline{L}(u\underline{u}\phi)|^2 = |u\underline{L}\phi + 2\phi|^2\ .$$

together with:

$$(127) \qquad\qquad \frac{-u\phi}{r} = \left(\underline{u}L\phi + 2\phi\right) - \left(\underline{u}L\phi + \frac{\underline{u}\phi}{r}\right)\ ,$$

$$(128) \qquad\qquad \frac{\underline{u}\phi}{r} = \left(u\underline{L}\phi + 2\phi\right) - \left(u\underline{L}\phi - \frac{u\phi}{r}\right)\ ,$$

which collectively imply the pointwise estimate:

$$\underline{u}^2|L\phi|^2 + u^2|\underline{L}\phi|^2 + (\underline{u}^2+u^2)|\frac{\phi}{r}|^2 \leqslant 2\sum_{\Omega=r,u\underline{u}}\underline{u}^2|\frac{1}{\Omega}L(\Omega\phi)|^2 + |\frac{1}{\Omega}\underline{L}(\Omega\phi)|^2\ .$$

This completes our demonstration of (114).

We now prove a fractionally weighted and space-time generalization of (125) which will be our analog of estimate (79) for the complex scalar field (115). This

is:

Proposition 4.2 (Generalized Morawetz Estimate for Complex Scalar Fields)*.
Let ϕ be a complex scalar field which satisfies the equation* (115)*, and let F be the
curvature of the corresponding connection. Let $0 < \epsilon, \gamma$ be given parameters, and
let s be chosen so that $\frac{1}{2} \leqslant s \leqslant 1$. Define the weights $\tau_0, w_\gamma, w_{\gamma,\epsilon}$, and w'_γ as on
lines* (73)–(76)*. Now, define the generalized Morawetz type energy content of ϕ in
the time slab $\{0 \leqslant t \leqslant t_0\} \cap \mathbb{R}^3$:*

(129)

$$
E^{(\epsilon,\gamma,s)}(0,t_0)[\phi] \ ,
$$

$$
= \sup_{0 \leqslant t \leqslant t_0} \int_{\{t\} \cap \mathbb{R}^3} \left(\tau_+^{2s} |\tfrac{1}{r} D_L(r\phi)|^2 + \tau_-^{2s} |D_{\underline{L}}\phi|^2 + \tau_+^{2s} \left(|\slashed{D}\phi|^2 + |\tfrac{\phi}{r}|^2 \right) \right) w_\gamma \ dx \ ,
$$

$$
+ \sup_u \int_{C(u) \cap \{0 \leqslant t \leqslant t_0\}} \left(\tau_+^{2s} |\tfrac{1}{r} D_L(r\phi)|^2 + \tau_-^{2s} |\slashed{D}\phi|^2 + \tau_+^{2s} |\tfrac{u\phi}{\underline{u}r}|^2 \right) w_\gamma \ dV_{C(u)} \ ,
$$

$$
+ \int_0^{t_0} \int_{\mathbb{R}^3} \left(\tau_+^{2s} |\tfrac{1}{r} D_L(r\phi)|^2 + \tau_0^{1+2\epsilon} \left(\tau_-^{2s} |D_{\underline{L}}\phi|^2 + \tau_+^{2s} |\slashed{D}\phi|^2 + \tau_+^{2s} |\tfrac{\phi}{r}|^2 \right) \right) w'_{\gamma,\epsilon} \ dxdt \ .
$$

Then one has the following general weighted energy type inequality:

(130) $\quad E^{(\epsilon,\gamma,s)}(0,t_0)[\phi] \ \leqslant \ C_{\epsilon,\gamma}^2 \Big[\||\, F \,\||_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2 \cdot E^{(\epsilon,\gamma,s)}(0,t_0)[\phi]$

$$
+ \int_0^{t_0} \int_{\mathbb{R}^3} \tau_+^{2s} \tau_- \ |G|^2 \ w_{\gamma,\epsilon} \ dxdt \ + \ \int_{\{t=0\} \cap \mathbb{R}^3} (1+r^2)^{s+\gamma} |D(\phi)|^2 \ dx \Big] \ ,
$$

where we have set $\||\, F \,\||_{L^\infty[0,t_0](s,\gamma,\epsilon)}$ equal to the time-slab L^∞ type norm:

(131)

$$
\||\, F \,\||_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2 \ = \ \sup_{\substack{0 \leqslant \ t \ \leqslant t_0 \\ x \in \{t\} \times \mathbb{R}^3}} \left(\tau_+^{2s+3} |\alpha|^2 + \tau_+^2 \tau_-^{2s+1} |\underline{\alpha}|^2 + \tau_+^{2s+2} \tau_- (\widetilde{\rho}^2 + \sigma^2) \right) \cdot w_\gamma
$$

$$
+ \ |q(F)|^2 \ + \ \| \tau_+^{s+1} (w')_{\gamma,\epsilon}^{\frac{1}{2}} \alpha \|_{L^2(L^\infty)[0,t_0]}^2 \ .
$$

Here the quantity $q(F)$ is defined as on line (65)*, and we have used the null decom-
position* (48)*.*

Remark 4.3. Comparing estimate (79) with (130) we see that there is a close anal-
ogy, at least with respect to energy estimates, between the various null components
of F and the gradient $D\phi$. Schematically, these are:

(132) $\qquad\qquad\qquad \alpha \ \sim \ \tfrac{1}{r} D_L(r\phi) \ , \qquad\qquad\qquad \underline{\alpha} \ \sim \ D_{\underline{L}}\phi \ ,$

(133) $\qquad\qquad\qquad \sigma \ \sim \ \slashed{D}\phi \ , \qquad\qquad\qquad\qquad \widetilde{\rho} \ \sim \ \tfrac{\phi}{r} \ .$

As we shall see in the following two sections, this analogy persists in the discussion
of L^∞ type estimates. We will make solid of this in the sequel, where we will use
the analogy (132)–(133) to reduce the field equations (17) and its commutators to
a equation for a single abstract vector quantity Ψ, which has weighted energy and
peeling properties equal to the null decompositions of F and $D\phi$.

Remark 4.4. Notice that the L^∞ norm (131) contains an extra $L^2(L^\infty)$ estimate for the component α. While this is not needed to prove (130) above, it will be crucial for controlling the error estimates which come up when estimating the non-linear problem (17). Since we will use this notation in the sequel, we have included this extra term here.

Proof of estimate (130). The proof is nearly identical to that of (79) in the previous section. Our first step is to come up with a fractionally weighted momentum density associated with the first conformal metric ${}^I\widetilde{g}$. We define this to be:

$$(134) \qquad {}^I\widetilde{P}_\alpha^{(s,\gamma,\epsilon)}[\phi] \;=\; {}^I\widetilde{Q}_{\alpha\beta}[\phi](\overline{K}_0^s)^\beta \cdot \widetilde{w}_{\gamma,\epsilon} \;,$$

where $\widetilde{w}_{\gamma,\epsilon}$ is the weight function from line (85). This satisfies the divergence law:

$$(135) \quad {}^I\widetilde{\nabla}^{\alpha\,I}\widetilde{P}_\alpha^{(s,\gamma,\epsilon)}[\phi] \;=\; r^4\left(\Re(G \cdot \overline{\frac{1}{r}D_{\overline{K}_0^s}(r\phi)}) + (\overline{K}_0^s)^\beta F_{\beta\gamma}\Im(\phi\overline{D^\gamma\phi}) \right) \cdot \widetilde{w}_{\gamma,\epsilon}$$

$$+ \; \frac{1}{2}\,{}^I\widetilde{Q}_{\alpha\beta}[\phi]\,{}^{(\overline{K}_0^s)}\widetilde{\pi}^{\alpha\beta} \cdot \widetilde{w}_{\gamma,\epsilon}$$

$$- \; \frac{r^2}{2}\,{}^I\widetilde{Q}_{\underline{L}\beta}[\phi](\overline{K}_0^s)^\beta \cdot L(\widetilde{w}_{\gamma,\epsilon}) \;-\; \frac{r^2}{2}\,{}^I\widetilde{Q}_{L\beta}[\phi](\overline{K}_0^s)^\beta \cdot \underline{L}(\widetilde{w}_{\gamma,\epsilon}) \;.$$

Integrating both sides of the above expression with respect to the volume $dV_I = r^{-4}dV_{\mathcal{M}}$ over regions of the form:

$$\mathcal{R}(t_0, u_0) \;=\; \{0 \leqslant t \leqslant t_0\} \cap \{u \leqslant u_0\} \;,$$

and noting that rT and rL are the respective Lorentzian unit normals to the time slices $t = const.$ and cones $u = const.$, we arrive at the following integral identity:

(136)
$$\int_{\{t=0\}\cap\{-u_0\leqslant r\}} r^{-2}\,{}^I\widetilde{P}_0^{(s,\gamma,\epsilon)}[\phi]\; dx \;=\; \int_{\{t=t_0\}\cap\{t_0-u_0\leqslant r\}} r^{-2}\,{}^I\widetilde{P}_0^{(s,\gamma,\epsilon)}[\phi]\; dx$$

$$+ \int_{C(u_0)\cap\{0\leqslant t\leqslant t_0\}} r^{-2}\,{}^I\widetilde{P}_L^{s,\gamma,\epsilon}[\phi]\; dV_{C(u_0)} + \int\int_{\mathcal{R}(t_0,u_0)} r^{-4}\,(\text{R.H.S.})(135)\; dxdt \;.$$

We now calculate each term in (135) individually. We will use the same $\sim$ notation as in the proof of (79). At time $t = 0$, we compute that:

$$(137) \qquad r^{-2}\,{}^I\widetilde{P}_0^{(s,\gamma,\epsilon)}[\phi] \;\sim\; (1+r^2)^{s+\gamma}|\frac{1}{r}D(r\phi)|^2 \;.$$

Likewise, at time $t = t_0$ we have that:

$$(138) \quad r^{-2}\,{}^I\widetilde{P}_0^{(s,\gamma,\epsilon)}[\phi] \;\sim\; \left(\tau_+^{2s}|\frac{1}{r}D_L(r\phi)|^2 + \tau_-^{2s}|\frac{1}{r}D_{\underline{L}}(r\phi)|^2 + \tau_+^2|\slashed{D}\phi|^2 \right) \cdot w_\gamma \;,$$

and on the cone $u_0 = const.$ we have:

$$(139) \qquad r^{-2}\,{}^I\widetilde{P}_L^{(s,\gamma,\epsilon)}[\phi] \;\sim\; \left(\tau_+^{2s}|\frac{1}{r}D_L(r\phi)|^2 + \tau_-^2|\slashed{D}\phi|^2 \right) \cdot w_\gamma \;.$$

It remains to calculate the expression r^{-4}(R.H.S.)(135). We do this for each term separately. Since the first such term does not have a sign, we put absolute value signs around it and estimate:

$$(140) \qquad \left|\Re(G \cdot \overline{\frac{1}{r}D_{\overline{K}_0^s}(r\phi)})\right| \lesssim |G| \cdot \left(\tau_+^{2s}|\frac{1}{r}D_L(r\phi)| + \tau_-^{2s}|D_{\underline{L}}\phi| + \tau_-^{2s}|\frac{\phi}{r}|\right) ,$$

$$(141) \quad \left|(\overline{K}_0^s)^\beta F_{\beta\gamma}\Im(\phi\overline{D^\gamma\phi})\right| \lesssim \tau_+^{2s+1}\left(|\alpha| \cdot |\frac{\phi}{r}| \cdot |\not{D}\phi| + |\rho| \cdot |\frac{\phi}{r}| \cdot |\frac{1}{r}D_L(r\phi)|\right)$$
$$+ \ \tau_+\tau_-^{2s}\left(|\underline{\alpha}| \cdot |\frac{\phi}{r}| \cdot |\not{D}\phi| + |\rho| \cdot |\frac{\phi}{r}| \cdot |D_{\underline{L}}\phi|\right) .$$

To calculate the second term on the right hand side of expression (135) we need to compute the deformation tensor $^{(\overline{K}_0^s)}\widetilde{\pi}_{\alpha\beta}$. Using the identities (30) and (81) this is:

$$^{(\overline{K}_0^s)}\widetilde{\pi} \ = \ \overline{K}_0^s(\frac{1}{r^2})\,g + \frac{1}{r^2}\,^{(\overline{K}_0^s)}\pi ,$$
$$= \ \frac{2}{r^2}\left(\frac{\underline{v}-v}{r} - (\underline{\dot{v}}+\dot{v})\right)(\theta^{\underline{L}} \otimes \theta^L + \theta^L \otimes \theta^{\underline{L}}) .$$

Contracting this last line with $\frac{1}{2}r^{-4\,I}\widetilde{Q}[\phi] \cdot \widetilde{w}_{\gamma,\epsilon}$ yields:

$$(142) \qquad \frac{1}{2}r^{-4\,I}\widetilde{Q}_{\alpha\beta}[\phi]\,^{(\overline{K}_0^s)}\widetilde{\pi} \cdot \widetilde{w}_{\gamma,\epsilon} \ \sim \ \left(\frac{\underline{v}-v}{r} - (\underline{\dot{v}}+\dot{v})\right) \cdot |\not{D}\phi|^2 \cdot w_\gamma .$$

In particular, we see that this term only contributes a positive addition to the right hand side of (135). To compute the last two terms of r^{-4}(R.H.S.)(135), we simply use the calculations (87)–(88) and the expansions (51) to conclude that:

(143)
$$-\frac{1}{2}r^{-2\,I}\widetilde{Q}_{\underline{L}\beta}[\phi](\overline{K}_0^s)^\beta \cdot L(\widetilde{w}_{\gamma,\epsilon}) \ \sim \ C_{\gamma,\epsilon}^{-1}\,\tau_0^{1+\epsilon}\left(\tau_-^{2s}|\frac{1}{r}D_{\underline{L}}(r\phi)|^2 + \tau_+^{2s}|\not{D}\phi|^2\right) \cdot w_{\gamma,\epsilon}' ,$$
(144)
$$-\frac{1}{2}r^{-2\,I}\widetilde{Q}_{L\beta}[\phi](\overline{K}_0^s)^\beta \cdot \underline{L}(\widetilde{w}_{\gamma,\epsilon}) \ \sim \ C_{\gamma,\epsilon}^{-1}\left(\tau_+^{2s}|\frac{1}{r}D_L(r\phi)|^2 + \tau_-^{2s}|\not{D}\phi|^2\right) \cdot w_{\gamma,\epsilon}' .$$

Now, inserting the terms (137)–(144) in the identity (136) and removing the positive contribution (142) form the right hand side of the resulting estimate, we

arrive at our first preliminary estimate to (130):

(145)
$$\int_{\{t=0\}\cap\mathbb{R}^3} (1+r^2)^{s+\gamma} |\frac{1}{r}D(r\phi)|^2 \, dx + \int_0^{t_0}\int_{\mathbb{R}^3} |G|\cdot\left(\tau_+^{2s}|\frac{1}{r}D_L(r\phi)| + \tau_-^{2s}|D_{\underline{L}}\phi| + \tau_-^{2s}|\frac{\phi}{r}|\right)\cdot w_\gamma \, dxdt$$

$$\int_0^{t_0}\int_{\mathbb{R}^3} \tau_+^{2s+1}\left(|\alpha|\cdot|\frac{\phi}{r}|\cdot|\slashed{D}\phi| + |\rho|\cdot|\frac{\phi}{r}|\cdot|\frac{1}{r}D_L(r\phi)|\right)\cdot w_\gamma \, dxdt$$

$$+ \int_0^{t_0}\int_{\mathbb{R}^3} \tau_+\tau_-^{2s}\left(|\underline{\alpha}|\cdot|\frac{\phi}{r}|\cdot|\slashed{D}\phi| + |\rho|\cdot|\frac{\phi}{r}|\cdot|D_{\underline{L}}\phi|\right)\cdot w_\gamma \, dxdt$$

$$\geqslant C_{\gamma,\epsilon}^{-1}\left[\sup_{0\leqslant t\leqslant t_0}\int_{\{t\}\cap\mathbb{R}^3}\left(\tau_+^{2s}|\frac{1}{r}D_L(r\phi)|^2 + \tau_-^{2s}|\frac{1}{r}D_{\underline{L}}(r\phi)|^2 + \tau_+^2|\slashed{D}\phi|^2\right)\cdot w_\gamma \, dx\right.$$

$$+ \sup_u\int_{C(u)\cap\{0\leqslant t\leqslant t_0\}}\left(\tau_+^{2s}|\frac{1}{r}D_L(r\phi)|^2 + \tau_-^2|\slashed{D}\phi|^2\right)\cdot w_\gamma \, dV_{C(u)}$$

$$+ \int_0^{t_0}\int_{\mathbb{R}^3}\left(\tau_+^{2s}|\frac{1}{r}D_L(r\phi)|^2 + \tau_0^{1+\epsilon}\left(\tau_-^{2s}|\frac{1}{r}D_{\underline{L}}(r\phi)|^2 + \tau_+^{2s}|\slashed{D}\phi|^2\right)\right)\cdot w'_{\gamma,\epsilon} \, dxdt\left.\right].$$

By repeating the above steps with the density (134) replaced by replaced with the following weighted version of the usual energy density for ϕ:

$$P_\alpha^{(\gamma,\epsilon)}[\phi] = Q_{\alpha\beta}[\phi](T)^\beta\cdot\widetilde{w}_{\gamma,\epsilon} ,$$

and using the bounds:

$$\left|\Re(G\cdot\overline{D_T\phi})\right| \lesssim (\text{L.H.S})(140) ,$$

(146)
$$\left|\Im(\phi\overline{D^\gamma\phi})\right| \lesssim (\text{L.H.S})(141)$$

we also have the estimate:

(147) $\quad$ (R.H.S.)(145) $\geqslant C_{\gamma,\epsilon}^{-1}\left[\sup_{0\leqslant t\leqslant t_0}\int_{\{t\}\cap\mathbb{R}^3} |D_{\underline{L}}\phi|^2\cdot w_\gamma \, dx\right.$

$$+ \sup_u\int_{C(u)\cap\{0\leqslant t\leqslant t_0\}} |D_L\phi|^2\cdot w_\gamma \, dV_{C(u)} + \int_0^{t_0}\int_{\mathbb{R}^3}\tau_0^{1+\epsilon}|D_{\underline{L}}\phi|^2\cdot w'_{\gamma,\epsilon} \, dxdt\left.\right].$$

By adding together estimates (145)–(147), we have almost achieved the statement of estimate (130). Whats missing from the right hand side are terms involving the weighted quantity $\underline{u}^{2s}|\frac{\phi}{r}|$. As with estimate (125), bounds on this quantity will come from integrating a divergence identity involving the tensor (121) and then combining the resulting estimate with (145) above. To do this, we form the second weighted momentum density:

(148) $$^{II}\widetilde{P}_\alpha^{(s,\gamma,\epsilon)}[\phi] = {}^{II}\widetilde{Q}_{\alpha\beta}[\phi](K_0)^\beta\cdot\widetilde{w}_{s,\gamma,\epsilon} ,$$

where the weight function $w_{s,\gamma,\epsilon}$ is given by:

(149) $\quad \widetilde{w}_{s,\gamma,\epsilon} = (1+\underline{u})^{2s-2}\cdot\left((2-u)^{2\gamma}\chi^+(-u) + 1-\chi^+(-u)\right)$

$$+ (1+\underline{u})^{2s-2-2\epsilon}\cdot\left((2-u)^{2\gamma+2\epsilon}\chi^+(-u) + 1-\chi^+(-u)\right)$$

We now follow through the same steps used to prove (145) above. Computing the divergence of (148) we have that:

(150)

$$
{}^{II}\widetilde{\nabla}^{\alpha}\,{}^{II}\widetilde{P}^{(s,\gamma,\epsilon)}_{\alpha}[\phi] \;=\; (u\underline{u})^4\left(\Re(G\cdot\overline{\frac{1}{u\underline{u}}D_{K_0}(u\underline{u}\phi)}) + (K_0)^\beta F_{\beta\gamma}\Im(\phi\overline{D^\gamma\phi})\right)\cdot\widetilde{w}_{s,\gamma,\epsilon}
$$
$$
-\;\frac{(u\underline{u})^2}{2}\,{}^{II}\widetilde{Q}_{\underline{L}\beta}[\phi](K_0)^\beta\cdot L(\widetilde{w}_{s,\gamma,\epsilon}) \;-\; \frac{(u\underline{u})^2}{2}\,{}^{II}\widetilde{Q}_{L\beta}[\phi](K_0)^\beta\cdot \underline{L}(\widetilde{w}_{s,\gamma,\epsilon}) \;.
$$

Integrating this expression with respect to the volume $dV_{II} = (u\underline{u})^{-4}dV_{\mathcal{M}}$ and using the Stokes theorem we arrive at the integral identity:

(151)
$$
\int_{\{t=0\}\cap\{-u_0\leqslant r\}}(u\underline{u})^{-2}\,{}^{II}\widetilde{P}^{(s,\gamma,\epsilon)}_0[\phi]\,dx \;=\; \int_{\{t=t_0\}\cap\{t_0-u_0\leqslant r\}}(u\underline{u})^{-2}\,{}^{II}\widetilde{P}^{(s,\gamma,\epsilon)}_0[\phi]\,dx
$$
$$
+\int_{C(u_0)\cap\{0\leqslant t\leqslant t_0\}}(u\underline{u})^{-2}\,{}^{II}\widetilde{P}^{s,\gamma,\epsilon}_L[\phi]\,dV_{C(u_0)} + \int\int_{\mathcal{R}(t_0,u_0)}(u\underline{u})^{-4}\,(\text{R.H.S.})(150)\,dxdt\;.
$$

We now compute in turn each term in (151) above. When $t=0$ we have that:

(152)
$$
(u\underline{u})^{-2}\,{}^{II}\widetilde{P}^{(s,\gamma,\epsilon)}_0[\phi] \;\sim\; r^{2(s+\gamma)}|\frac{1}{u\underline{u}}D(u\underline{u}\phi)|^2\;.
$$

Likewise, at time $t=t_0$ and along the cone $u=const.$ we compute that:

(153)
$$
(u\underline{u})^{-2}\,{}^{II}\widetilde{P}^{(s,\gamma,\epsilon)}_0[\phi] \;\sim\; \left(\tau_+^{2s-2}\underline{u}^2\,|\frac{1}{u\underline{u}}D_L(u\underline{u}\phi)|^2 + \tau_+^{2s-2}u^2\,|\frac{1}{u\underline{u}}D_{\underline{L}}(u\underline{u}\phi)|^2 + \tau_+^{2s-2}\underline{u}^2\,|\slashed{D}\phi|^2\right)\cdot w_\gamma\;,
$$

(154)
$$
(u\underline{u})^{-2}\,{}^{II}\widetilde{P}^{(s,\gamma,\epsilon)}_L[\phi] \;\sim\; \left(\tau_+^{2s-2}\underline{u}^2\,|\frac{1}{u\underline{u}}D_L(u\underline{u}\phi)|^2 + \tau_+^{2s-2}u^2\,|\slashed{D}\phi|^2\right)\cdot w_\gamma\;.
$$

We now turn our attention to the terms in the expression $(u\underline{u})^{-4}$ (R.H.S.)(150). First, using the identities (127)–(128) we conclude that:

$$
|\underline{u}L\phi + 2\phi| \;\leqslant\; \underline{u}|\frac{1}{r}D_L(r\phi)| + u|\frac{\phi}{r}|\;,
$$
$$
|u\underline{L}\phi + 2\phi| \;\leqslant\; u|D_{\underline{L}}\phi| + 2\underline{u}|\frac{\phi}{r}|\;.
$$

This then implies that one may estimate:

$$
\left|\Re(G\cdot\overline{\frac{1}{u\underline{u}}D_{K_0}(u\underline{u}\phi)})\right|\cdot\widetilde{w}_{s,\gamma,\epsilon} \;\lesssim\; |G|\cdot\left(\tau_+^{2s-1}|\underline{u}L\phi+2\phi| + \tau_+^{2s-2}\tau_-^2|u\underline{L}\phi+2\phi|\right)\cdot w_\gamma\;,
$$

(155)
$$
\lesssim\; |G|\cdot\left(\tau_+^{2s}|\frac{1}{r}D_L(r\phi)| + \tau_-^{2s}|D_{\underline{L}}\phi| + \tau_+^{2s}|\frac{u\phi}{\underline{u}r}|\right)\cdot w_\gamma\;.
$$

To bound the second term on the right hand side of (150) above, we simply use the estimate (141) noting that one may trade $\tau_+^{2s-2}\tau_-^2\lesssim\tau_-^{2s}$:
(156)
$$
\left|(K_0)^\beta F_{\beta\gamma}\Im(\phi\overline{D^\gamma\phi})\right|\cdot\widetilde{w}_{s,\gamma,\epsilon} \;\lesssim\; \tau_+^{2s+1}\left(|\alpha|\cdot|\frac{\phi}{r}|\cdot|\slashed{D}\phi| + |\rho|\cdot|\frac{\phi}{r}|\cdot|\frac{1}{r}D_L(r\phi)|\right)\cdot w_\gamma
$$

To compute the second two terms on the right hand side of (150), first notice that a simple calculation similar to that used in lines (87)–(88) above which we omit shows that:

$$-\frac{1}{2}L(\widetilde{w}_{\gamma,\epsilon}) \;\geqslant\; C_{\gamma,\epsilon}^{-1}\,\tau_+^{2s-2}\tau_0^{1+2\epsilon}w_{\gamma,\epsilon}' \;,$$

$$-\frac{1}{2}\underline{L}(\widetilde{w}_{\gamma,\epsilon}) \;\geqslant\; 0 \;.$$

Notice that we only care that the second term above is non-negative as this will be its only use in our proof of (130). In particular, the above two lines allow us to conclude that:

(157)
$$-\frac{1}{2(u\underline{u})^2}\,{}^{II}\widetilde{Q}_{\underline{L}\beta}[\phi](K_0)^\beta \cdot L(\widetilde{w}_{\gamma,\epsilon}) \;\geqslant\; C_{\gamma,\epsilon}^{-1}\tau_+^{2s-2}\tau_0^{1+2\epsilon}\left(u^2\,|\frac{1}{u\underline{u}}D_{\underline{L}}(u\underline{u}\phi)|^2 + \underline{u}^2\,|\slashed{D}\phi|^2\right)\cdot w_\gamma \;,$$

(158)
$$-\frac{(u\underline{u})^2}{2}\,{}^{II}\widetilde{Q}_{L\beta}[\phi](K_0)^\beta \cdot \underline{L}(\widetilde{w}_{\gamma,\epsilon}) \;\geqslant\; 0 \;.$$

Combining the estimates (152)–(158) into the identity (151) and excluding all the terms on the right hand side of the resulting inequality except those which involve the $D_{\underline{L}}$ derivative and the characteristic term involving the D_L derivative, we arrive at our compliment to estimate (145):

$$
\begin{aligned}
(159)\quad & \int_{\{t=0\}\cap\mathbb{R}^3} r^{2(s+\gamma)}|\frac{1}{u\underline{u}}D(u\underline{u}\phi)|^2\;dx \\
& + \int_0^{t_0}\int_{\mathbb{R}^3} |G|\cdot\left(\tau_+^{2s}|\frac{1}{r}D_L(r\phi)| + \tau_-^{2s}|D_{\underline{L}}\phi| + \tau_+^{2s}|\frac{u\phi}{\underline{u}r}|\right)\cdot w_\gamma\;dxdt \\
& + \int_0^{t_0}\int_{\mathbb{R}^3} \tau_+^{2s+1}\left(|\alpha|\cdot|\frac{\phi}{r}|\cdot|\slashed{D}\phi| + |\rho|\cdot|\frac{\phi}{r}|\cdot|\frac{1}{r}D_L(r\phi)|\right)\cdot w_\gamma\;dxdt \\
& + \int_0^{t_0}\int_{\mathbb{R}^3} \tau_+\tau_-^{2s}\left(|\underline{\alpha}|\cdot|\frac{\phi}{r}|\cdot|\slashed{D}\phi| + |\rho|\cdot|\frac{\phi}{r}|\cdot|D_{\underline{L}}\phi|\right)\cdot w_\gamma\;dxdt \\
& \geqslant\; C_{\gamma,\epsilon}^{-1}\left[\sup_{0\leqslant t\leqslant t_0}\int_{\{t\}\cap\mathbb{R}^3}\tau_+^{2s-2}u^2\,|\frac{1}{u\underline{u}}D_{\underline{L}}(u\underline{u}\phi)|^2\;w_\gamma\;dx\right. \\
& \quad + \sup_u \int_{C(u)\cap\{0\leqslant t\leqslant t_0\}}\tau_+^{2s-2}\underline{u}^2\,|\frac{1}{u\underline{u}}D_L(u\underline{u}\phi)|^2\;w_\gamma\;dV_{C(u)} \\
& \quad \left. + \int_0^{t_0}\int_{\mathbb{R}^3}\tau_+^{2s-2}\tau_0^{1+2\epsilon}u^2\,|\frac{1}{u\underline{u}}D_{\underline{L}}(u\underline{u}\phi)|^2\;w_{\gamma,\epsilon}'\;dxdt\right] \;.
\end{aligned}
$$

We now form estimate (130) by adding together estimates (145), (147), and (159),

while also using some further identities to simplify the resulting right and left hand sides. First, notice that by again using the expressions (127)–(128) we can estimate:

$$\tau_-^{2s}|D_{\underline{L}}\phi|^2 + \tau_+^{2s}|\frac{\phi}{r}|^2 \;\lesssim\; \tau_+^{2s-2}u^2\,|\frac{1}{u\underline{u}}D_{\underline{L}}(u\underline{u}\phi)|^2 + |D_{\underline{L}}\phi|^2 + \tau_-^{2s}|\frac{1}{r}D_{\underline{L}}(r\phi)|^2 \;,$$

$$\tau_+^{2s}|\frac{1}{r}D_L(r\phi)|^2 + \tau_+^{2s}|\frac{u\phi}{\underline{u}r}|^2 \;\lesssim\; \tau_+^{2s-2}\underline{u}^2\,|\frac{1}{u\underline{u}}D_L(u\underline{u}\phi)|^2 + |D_L\phi|^2 + \tau_+^{2s}|\frac{1}{r}D_L(r\phi)|^2 \;.$$

Thus, using the bound:

$$\tau_+^{2s}|\frac{u\phi}{\underline{u}r}|^2 \ \lesssim \ \tau_-^{2s}|\frac{\phi}{r}|^2 \ ,$$

and the energy notation (129), and expanding out the $t = 0$ energy expressions on the left hand side of the resulting estimate we conclude that:

$$E^{(s,\gamma,\epsilon)}[\phi] \ \leqslant \ C_{\gamma,\epsilon}\Bigg[\int_0^{t_0}\int_{\mathbb{R}^3} \ |G| \cdot \Big(\tau_+^{2s}|\frac{1}{r}D_L(r\phi)| + \tau_-^{2s}|D_{\underline{L}}\phi| + \tau_-^{2s}|\frac{\phi}{r}| \Big) \cdot w_\gamma \ dxdt$$

$$\int_0^{t_0}\int_{\mathbb{R}^3} \ \tau_+^{2s+1}\Big(|\alpha| \cdot |\frac{\phi}{r}| \cdot |\not{D}\phi| + |\rho| \cdot |\frac{\phi}{r}| \cdot |\frac{1}{r}D_L(r\phi)| \Big) \cdot w_\gamma \ dxdt$$

$$+ \ \int_0^{t_0}\int_{\mathbb{R}^3} \ \tau_+\tau_-^{2s}\Big(|\underline{\alpha}| \cdot |\frac{\phi}{r}| \cdot |\not{D}\phi| + |\rho| \cdot |\frac{\phi}{r}| \cdot |D_{\underline{L}}\phi| \Big) \cdot w_\gamma \ dxdt$$

$$+ \ \int_{\{t=0\}\cap\mathbb{R}^3} (1+r^2)^{s+\gamma}\Big(|D\phi|^2 + |\frac{\phi}{r}|^2 \Big) \ dx \Bigg] \ .$$

Setting:

$$|||G|||^2_{L^2[0,t_0](L^2)(s,\gamma)} \ = \ \int_0^{t_0}\int_{\mathbb{R}^3} \ \tau_+^{2s}\tau_-|G|^2 \ w_{\gamma,\epsilon} \ dxdt \ ,$$

and using the notation (131), along with several different instances of the Cauchy-Schwartz inequality we have:

$$E^{(s,\gamma,\epsilon)}[\phi] \ \leqslant \ C_{\gamma,\epsilon}\Bigg[|||G|||_{L^2[0,t_0](L^2)(s,\gamma)}\cdot(E^{\frac{1}{2}})^{(s,\gamma,\epsilon)}[\phi] + |||\,F\,|||_{L^\infty[0,t_0](s,\gamma,\epsilon)}\cdot E^{(s,\gamma,\epsilon)}[\phi]$$

$$+ \ \int_{\{t=0\}\cap\mathbb{R}^3} (1+r^2)^{s+\gamma}\Big(|D\phi|^2 + |\frac{\phi}{r}|^2 \Big) \ dx \Bigg] \ .$$

Dividing this last expression through by $(E^{\frac{1}{2}})^{(s,\gamma,\epsilon)}[\phi]$ and squaring, we easily achieve the bound:

$$E^{(s,\gamma,\epsilon)}[\phi] \ \leqslant \ C_{\gamma,\epsilon}^2\Bigg[|||G|||^2_{L^2[0,t_0](L^2)(s,\gamma)} \ + \ |||\,F\,|||^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \cdot E^{(s,\gamma,\epsilon)}[\phi]$$

$$+ \ \int_{\{t=0\}\cap\mathbb{R}^3} (1+r^2)^{s+\gamma}\Big(|D\phi|^2 + |\frac{\phi}{r}|^2 \Big) \ dx \Bigg] \ .$$

The estimate (130) now follows from this last line and the following gauge covariant Poincare type estimate which follows from the same reasoning used to produce (190) which we shall proved in the sequel:

$$\int_{\{t=0\}\cap\mathbb{R}^3} (1+r^2)^{s+\gamma}|\frac{\phi}{r}|^2 \ dx \ \lesssim \ \int_{\{t=0\}\cap\mathbb{R}^3} (1+r^2)^{s+\gamma} \ |D\phi|^2 \ dx \ .$$

$\square$

5. L^∞ Estimates for Electro-Magnetic Fields

This section is the sequel to Section 3 above in that it contains the second main set of estimates for the field strength $F_{\alpha\beta}$. Here we will prove L^∞ type estimates for this quantity without assuming that it necessarily satisfies the field equations (17). That is, in this section we will only assume that $F_{\alpha\beta}$ is an arbitrary two-form and will prove L^∞ type estimates for it at the cost of bounds on the energies (105). This is done through the well known procedure of applying global Sobolev inequalities in conjunction with geometric identity (28b) for the component decomposition of the Lie derivative of a two-form. This procedure was first explored in [3], and our treatment adds little except a bound on the $L^2(L^\infty)$ term in the norm (131). This turns out to be a direct consequence of the space-time L^2 estimate (107) and the weighted Sobolev estimates (294)–(295) and (298). However, we choose to prove the entire L^∞ estimate from scratch here primarily for the sake of completeness and because we use the fractional weights τ_+^{2s}, τ_-^{2s}.

Before we proceed with the main estimate of this section, we introduce two notational devices which will allow us to reduce things to a more simple form where the estimates (294)–(295) and (298) can be used directly. The first of these is a simple algebraic tool which will make dealing with various commutators more straight forward. This is the introduction of the "radial boost" field:

$$(160) \qquad\qquad \Omega_{0r} \;=\; \omega^i \Omega_{0i} \;=\; -\frac{1}{2}\left(\underline{u}L - u\underline{L}\right) \; .$$

It has the advantage of commuting with the rotation and scaling fields $\{\Omega_{ij}, S\}$. Furthermore, it turns out that Ω_{0r} preserves the null decomposition (48) even after one reexpands Ω_{0r} and passes back to the usual boosts Ω_{i0}. This remarkable fact[7] as well as the rule that we only need one instance of the boosts in the exterior according to (294) will help to streamline the steps we take in the sequel.

The second tool we use here is a dyadic decomposition of the distance to the standard light-cone $u = 0$, as well as the distance along its translates $u = const.$ We will use this tool to localize all of our L^∞ estimates to shells adapted to these distances. This allows us to add extra powers of τ_+, τ_- at will because they just appear as constants in our localized estimates. The language for this is as follows:

We first isolate on each fixed time slice $t = const.$ the extended exterior region $t < 2r$. We then chop this into dyadic pieces depending on the distance from the cone $u = 0$. We decompose the extended exterior region into the set of double spherical shells:

$$\mathcal{J}_i \;=\; \left\{ x \in \{t\} \times \mathbb{R}^3 \,, t < 2r \;\big|\; 2^i \leqslant \big|t - |x|\big| + 1 \leqslant 2^{i+1} \right\} \,, \qquad i \in \mathbb{N} \; .$$

For fixed $0 < i$, each of these sets can be further decomposed into exterior and inter portions $\mathcal{J}_i = \mathcal{J}_i^+ \cup \mathcal{J}_i^-$, where $\mathcal{J}_i^\pm$ contains only the points where $0 < \pm u$. Note that these sets are connected, and that $\mathcal{J}_i^-$ empty for i sufficiently large. In general, we will write $\mathcal{J}$ for one of these connected sets. We also form a partition

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[7]It is not a-priori clear that this should happen given the fact that the boosts Ω_{i0} cause quite a bit of permutation in the null decomposition (48).

of unity adapted to the decomposition $\mathcal{J}_i$ which has the homogeneity property:

$$|\partial_r \chi_{\mathcal{J}_i}| \lesssim \tau_-^{-1}(\mathcal{J}_i) ,$$

where:

(161) $\qquad \tau_+(\mathcal{J}_i) = \min_{x \in \mathcal{J}_i} \tau_+ , \qquad\qquad\qquad \tau_-(\mathcal{J}_i) = \min_{x \in \mathcal{J}_i} \tau_- .$

Furthermore, we note that it is clear that we can also choose the largest J_i^- so its cutoff is supported where $t \leqslant 4r$. When the context is understood, we will simply substitute χ for $\chi_{\mathcal{J}_i}$ and τ_+, τ_- for $\tau_+(\mathcal{J}), \tau_-(\mathcal{J})$.

Next, we deal with the light-cones $u = const$. The notation we use here is similar to what was outlined in the preceding paragraph, and we will only consider the portion of $C(u)$ in the region where $t < 2r$. For each such cone (omitting the dependence on u), we define the dyadic conical spherical shells:

$$\mathcal{I}_i = \{x \in C(u) , \; 0 \leqslant t , t < 2r \; \big| \; 2^i \leqslant t + |x| + 1 \leqslant 2^{i+1}\} , \qquad i \in \mathbb{N} .$$

We use the notation $\tau_+(\mathcal{I}_i), \tau_-(\mathcal{I}_i)$ in analogy with line (161) above. We also introduce a smooth partition of unity $\chi_{\mathcal{I}_i}$ adapted to the $\mathcal{I}_i$ which satisfies the bounds:

$$|L(\chi_{\mathcal{I}_i})| \lesssim \tau_+^{-1}(\mathcal{I}_i) .$$

In particular, notice that we have not cut off strictly alone the truncated cone $C(u) \cap \{0 \leqslant t \leqslant t_0\}$, but we may assume that each of these χ are supported where $t < 4r$.

Using these notations, we can now glue together the estimates (294)–(295), (298), and (299)–(300) into a form which will be used in the sequel:

Lemma 5.1 (Weighted global Sobolev estimates). *Let f be a test function on $\mathbb{R} \times \mathbb{R}^3$, and let w_γ be the weight function defined on line (74). Then on each time slice $t = const.$ with $1 \leqslant t$ one has the estimates:*

(162)
$$\sup_{t \leqslant 2r} |\tau_+^{1+\delta_+} \tau_-^{\frac{1}{2}+\delta_-} f|^2 \, w_\gamma \; \lesssim \sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{\partial_r, S, \Omega_{0r}\}, Y \in \mathbb{O}}} \| \tau_+^{\delta_+} \tau_-^{\delta_-} w_\gamma^{\frac{1}{2}} X^I Y^J(f) \|_{L_x^2(t<4r)}^2 ,$$

(163) $\qquad \displaystyle\sup_{2r<t} |\tau_+^{1+\delta_+} f|^2 \; \lesssim \sum_{\substack{|I| \leqslant 2 \\ X \in \{S, \Omega_{0i}\}}} \| \tau_+^{\delta_+} X^I(f) \|_{L_x^2(r<\frac{3}{4}t)}^2 ,$

Furthermore, defining the truncated cones:

$$\overline{C}(u) = C(u) \cap \{1 \leqslant t \leqslant t_0\} \cap \{t < 2r\} ,$$
$$\widetilde{C}(u) = C(u) \cap \{1 \leqslant t \leqslant t_0\} \cap \{t < 4r\} ,$$

we have the characteristic L^∞ estimate:

(164) $\qquad \displaystyle\sup_{(t,x) \in \overline{C}(u)} |\tau_+^{\frac{3}{2}+\delta_+} f|^2 \; \lesssim \sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{\Omega_{0r}, S\}, Y \in \mathbb{O}}} \| \tau_+^{\delta_+} X^I(f) \|_{L_x^2(\widetilde{C}(u))}^2 .$

In the above estimates, $\mathbb{O} = \{\Omega_{ij}\}$ denotes the Lie algebra of the rotation group.

Proof of (162)–(164). These follow almost directly from their local versions (294)–(295), (298), and (299)–(300). We'll just deal with (162) here as the other two estimates follow from similar reasoning and are left to the reader. For a dyadic shell J, by combining estimates (294)–(295) in tandem and using the estimate:

$$|\tau_- \partial_r \chi f| \; \lesssim \; |\widetilde{\chi} f| + |\widetilde{\chi} S(f)| + |\widetilde{\chi} \Omega_{0r}(f)| + |\widetilde{\chi} \partial_r f| \, ,$$

where $\widetilde{\chi}$ is a cutoff on $\mathrm{supp}\{\chi\}$, we arrive at:

$$\sup_x \; \tau_+^2 \tau_-^1 |\chi f|^2 \; \lesssim \sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I| + |J| \leqslant 2 \\ X \in \{\partial_r, S, \Omega_{0r}\}, Y \in \mathbb{O}}} \| \, \widetilde{\chi} X^I Y^J(f) \, \|_{L_x^2}^2 \, .$$

Since both sides of this last estimate are restricted to a doubling of the shell $\mathcal{J}$, we can multiply both sides by the constant $\tau_+^{2\delta_+}(\mathcal{J}) \tau_-^{2\delta_- + 2\gamma}(\mathcal{J})$ if $t < r$, and by the constant $\tau_+^{2\delta_+}(\mathcal{J}) \tau_-^{2\delta_-}(\mathcal{J})$ if $r < t$. Doing this and then summing the result over all shells $\mathcal{J}$ in the extended exterior $t \leqslant 2r$ yields the desired result. $\qquad\square$

We now state and prove the main result of this section:

Proposition 5.2 (General L^∞ estimate for electro-magnetic fields)**.** *Let $F_{\alpha\beta}$ be an arbitrary two-form on $\mathbb{R} \times \mathbb{R}^3$, and let w_γ be the weight function defined on line* (74)*. Then in terms of null decomposition* (48) *and the general energy norm* (105)*, one has the following L^∞ estimate:*

$$(165) \qquad \sup_{\substack{0 \leqslant t \leqslant t_0 \\ x \in \{t\} \times \mathbb{R}^3}} \big(\tau_+^{2s+3} |\alpha|^2 + \tau_+^2 \tau_-^{2s+1} |\underline{\alpha}|^2 + \tau_+^{2s+2} \tau_- (\rho^2 + \sigma^2) \big) \cdot w_\gamma$$

$$+ \; \| \, \tau_+^{s+1} \tau_-^{\frac{1}{2}} (w')^{\frac{1}{2}}_{\gamma, \epsilon} \, \alpha \, \|_{L_t^2(L_x^\infty)[0, t_0]}^2 \; \lesssim \sum_{\substack{|I| \leqslant 2 \\ X \in \mathbb{L}}} E^{(s, \gamma, \epsilon)}(0, t_0)[\mathcal{L}_X^I F] \, .$$

In particular, using the charge modified energy norm (106)*, and the general weighted L^∞ norm* (131) *we have that:*

$$(166) \qquad\qquad ||| \, F \, |||_{L^\infty[0, t_0](s, \gamma, \epsilon)} \; \lesssim \; E_2^{(s, \gamma, \epsilon)}(0, t_0)[F] \, .$$

Proof of estimate (165). In what follows, we shall only consider the estimate (165) in the region $1 \leqslant t$. For $0 \leqslant t \leqslant 1$ the result follows from a weighted estimate on the electric-magnetic decomposition (9) in conjunction with a simple weighted version of the usual Sobolev embedding. We leave the details of this to the reader.

The proof can now be broken down into three sections. All of the components in the first term on the left hand side of (165) except α can be treated with fixed time energy estimates. The pure L^∞ estimate for α is a consequence of characteristic energies, while the mixed norm estimate follows from the space-time energy estimate contained in (107).

Bounds on $\{\underline{\alpha}, \rho, \sigma\}$. We prove the ascertation separately for the region $t < 2r$ and its compliment, starting with the former. We begin with the worst decaying component $\underline{\alpha}$. To estimate this in the extended exterior region $t < 2r$, we treat it as a scalar (really two separate scalars) and use (162) with $\delta_+ = 0$ and $\delta_- = s$. This yields for fixed t:

$$\sup_{t<2r} \tau_+^2 \tau_-^{2s+1} w_\gamma |\underline{\alpha}|^2 \lesssim \sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{\partial_r, S, \Omega_{0r}\}, Y \in \mathbb{O}}} \| \tau_-^s w_\gamma^{\frac{1}{2}} X^I Y^J(\underline{\alpha}) \|_{L_x^2(t \leqslant 4r)}^2 \ .$$

Therefore, all we need to do to bound the right hand side of the above expression in terms of the energy on the right hand side of (165) is to prove the following pointwise bound in the region where $t < 2r$:

$$(167) \qquad \sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{\partial_r, S, \Omega_{0r}\}, Y \in \mathbb{O}}} |X^I Y^J(\underline{\alpha})|^2 \lesssim \sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{\partial_i, S, \Omega_{0i}\}, Y \in \mathbb{O}}} |\underline{\alpha}(\mathcal{L}_X^I \mathcal{L}_Y^J F)|^2 \ .$$

This is in turn an immediate consequence of the formula (28b) and the expansion identities of the Lie derivatives (58), with some help from the identities (46). Notice that to expand vector-fields of the form $X_r = \omega^i X_i$, the formula (28b) implies that:

$$(168) \qquad (\mathcal{L}_{X_r} F_{\alpha\beta}) \ = \ \omega^i (\mathcal{L}_{X_i})_{\alpha\beta} + e_\alpha(\omega^i) X^\gamma F_{\gamma\beta} + e_\beta(\omega^i) X_i^\gamma F_{\alpha\gamma} \ .$$

Therefore, after some computations we have:

$$(169a) \qquad\qquad \partial_r(\underline{\alpha}_A) \ = \ \omega^i \underline{\alpha}_A(\mathcal{L}_{\partial_i} F) \ ,$$

$$(169b) \qquad\qquad \Omega_{ij}(\underline{\alpha}_A) \ = \ \underline{\alpha}_A(\mathcal{L}_{\Omega_{ij}} F) + [\Omega_{ij}, e_A]^B \underline{\alpha}_B \ ,$$

$$(169c) \qquad\qquad S(\underline{\alpha}_A) \ = \ \underline{\alpha}_A(\mathcal{L}_S F) - 2\underline{\alpha}_A \ ,$$

$$(169d) \qquad\qquad \Omega_{0r}(\underline{\alpha}_A) \ = \ \omega^i \underline{\alpha}_A(\mathcal{L}_{\Omega_{0i}} F) - \underline{\alpha}_A \ .$$

The component formulas (169) work directly to produce bounds on the zeroth and first derivative terms on the left hand side of (168). Therefore, to complete the bound (168), we only need to deal with the terms on the left hand side of this expression which contain two derivatives. Since the terms containing two derivatives must contain an angular derivative, it suffices to apply an angular derivative to both sides the formulas (169) and express the resulting right hand side in terms of the collection $\{\underline{\alpha}, \underline{\alpha}(\mathcal{L}_X F), \underline{\alpha}(\mathcal{L}_X \mathcal{L}_Y F)\}$ for $X \in \mathbb{L}$ and $Y \in \mathbb{O}$. First of all, if the extra angular derivative lands on a $\underline{\alpha}$ component on the right hand side of (169), we can use these formulas to reexpand the result using the above remarks to bound the resulting coefficients. If on the other has the extra angular derivative lands on a coefficient on the right hand side of (169) we are done because of the bounds:

$$(170) \qquad\qquad |\Omega_{jk}(\omega_i)| \ , \ |\Omega_{jk}([\Omega_{lm}, e_A]^B)| \quad \lesssim \quad 1 \ ,$$

which easily follow from the homogeneity property (45) of the fields $\{e_A\}$ and the fact that $\Omega_{ij}^A = r \widetilde{\Omega}_{ij}^A(\omega)$ where $\widetilde{\Omega}_{ij}^A$ is a function of the angular variable only.

To bound the terms ρ and σ in the extended exterior $t < 2r$, we proceed exactly as above. First using (162) with $\delta_+ = s$ and $\delta_- = 0$ we have that:

$$\sup_{t<2r} \tau_+^{2s+2}\tau_- w_\gamma(\rho^2 + \sigma^2) \lesssim \sum_{\substack{|I|\leqslant 1, |J|\leqslant 2 \\ |I|+|J|\leqslant 2 \\ X\in\{\partial_r, S, \Omega_{0r}\}, Y\in\mathbb{O}}} \| \tau_+^s w_\gamma^{\frac{1}{2}} X^I Y^J (\rho, \sigma) \|_{L_x^2(t\leqslant 4r)}^2 .$$

Thus, to bound things in terms of energy, we need to only provide the pointwise bound for $t < 2r$:
$$(171) \qquad \sum_{\substack{|I|\leqslant 1, |J|\leqslant 2 \\ |I|+|J|\leqslant 2 \\ X\in\{\partial_r, S, \Omega_{0r}\}, Y\in\mathbb{O}}} |X^I Y^J(\rho, \sigma)|^2 \lesssim \sum_{\substack{|I|\leqslant 1, |J|\leqslant 2 \\ |I|+|J|\leqslant 2 \\ X\in\{\partial_i, S, \Omega_{0i}\}, Y\in\mathbb{O}}} \rho^2(\mathcal{L}_X^I \mathcal{L}_Y^J F) + \sigma^2(\mathcal{L}_X^I \mathcal{L}_Y^J F) .$$

This in turn is a consequence the formulas:

$$(172a) \qquad \partial_r(\rho) = \omega^i \rho(\mathcal{L}_{\partial_i} F) , \qquad\qquad \Omega_{ij}(\rho) = \rho(\mathcal{L}_{\Omega_{ij}} F) ,$$

$$(172b) \qquad S(\rho) = \rho(\mathcal{L}_S F) - 2\rho , \qquad\qquad \Omega_{0r}(\rho) = \omega^i \rho(\mathcal{L}_{\Omega_{0i}} F) .$$

and:

$$(173a) \qquad \partial_r(\sigma) = \omega^i \sigma(\mathcal{L}_{\partial_r} F) , \qquad\qquad \Omega_{ij}(\sigma) = \sigma(\mathcal{L}_{\Omega_{ij}} F) ,$$

$$(173b) \qquad S(\sigma) = \rho(\mathcal{L}_S F) - 2\sigma , \qquad\qquad \Omega_{0r}(\sigma) = \omega^i \sigma(\mathcal{L}_{\Omega_{0i}} F) .$$

Notice that the set (172) is easily proved using the identities (28b), (168), (46), and (58), while the second set (173) follows from (172) and the duality formulas (31) and (49). As above, these formulas directly provide bounds for the terms on the right hand side of (171) which contain at most one derivative. The second derivative terms can be bounded as for the $\underline{\alpha}$ component above using (170). This completes the proof of the bound (165) for the terms $\{\underline{\alpha}, \rho, \sigma\}$ in the extended exterior region $t < 2r$.

To complete this subsection, we must prove the bound (165) for the terms $\{\underline{\alpha}, \rho, \sigma\}$ in the deep interior region $r < \frac{1}{2}t$. In fact, this can be done for all the components of F including α. To do this, we use the following bound on the electric-magnetic decomposition (9) in the region $r < \frac{1}{2}t$:

$$(174) \qquad \tau_+^{2s}(|E|^2 + |H|^2) \lesssim \left(\tau_+^{2s}|\alpha|^2 + \tau_-^{2s}|\underline{\alpha}|^2 + \tau_+^{2s}(\rho^2 + \sigma^2)\right) .$$

Therefore, to prove (165) in this region, after an application of the scalar estimate (163) we only need to prove the pointwise bound:

$$(175) \qquad \sum_{\substack{|I|\leqslant 2 \\ X\in\{S, \Omega_{0i}\}}} |X^I(E, H)|^2 \lesssim \sum_{\substack{|I|\leqslant 2 \\ X\in\{S, \Omega_{0i}\}}} |E(\mathcal{L}_X^I F)|^2 + |H(\mathcal{L}_X^I F)|^2 .$$

This in turn follows directly from the Lie derivative rule (28b) and the commutator identities (56).

The $L^2(L^\infty)$ estimate for α. We now prove the mixed norm estimate:

$$(176) \qquad \| \tau_+^{s+1} \tau_-^{\frac{1}{2}} (w')_{\gamma,\epsilon}^{\frac{1}{2}} \alpha \|_{L^2(L^\infty)}^2 \lesssim \sum_{\substack{|I| \leqslant 2 \\ X \in \mathbb{L}}} E^{(s,\gamma,\epsilon)}(0,t_0)[\mathcal{L}_X^I F] \ .$$

Due to the space-time energy estimate for α contained in the right hand side of (176), it suffices to prove the two fixed time estimates:

$$\sup_{t<2r} \tau_+^{2s+2} \tau_- |\alpha|^2 \, w'_{\gamma,\epsilon} \ \lesssim \sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{\partial_i, S, \Omega_{0i}\}, Y \in \mathbb{O}}} \| \tau_+^s \tau_-^{-\frac{1}{2}} (w')_{\gamma,\epsilon}^{\frac{1}{2}} \alpha(\mathcal{L}_X^I \mathcal{L}_Y^J F) \|_{L^2(t<4r)}^2 \ ,$$

$$\sup_{r<\frac{1}{2}t} \tau_+^{2s+2-2\epsilon} |\alpha|^2 \ \lesssim \sum_{\substack{|I| \leqslant 2 \\ X \in \{S, \Omega_{0i}\}}} \| \tau_+^{s-1-\epsilon} \Big(E(\mathcal{L}_X^I F), H(\mathcal{L}_X^I F) \Big) \|_{L^2(r<\frac{3}{4}t)} \ .$$

These in turn follow from the scalar estimates[8] (162)–(163) and the same procedure used above to prove the bound (167) and (175). Notice that by the same type of calculations used to produce (169) we have that:

$$(177a) \qquad\qquad \partial_r(\alpha_A) \ = \ \omega^i \alpha_A(\mathcal{L}_{\partial_i} F) \ ,$$

$$(177b) \qquad\qquad \Omega_{ij}(\alpha_A) \ = \ \alpha_A(\mathcal{L}_{\Omega_{ij}} F) + [\Omega_{ij}, e_A]^B \alpha_B \ ,$$

$$(177c) \qquad\qquad S(\alpha_A) \ = \ \alpha_A(\mathcal{L}_S F) - 2\alpha_A \ ,$$

$$(177d) \qquad\qquad \Omega_{0r}(\alpha_A) \ = \ \omega^i \alpha_A(\mathcal{L}_{\Omega_{0i}} F) + \alpha_A \ .$$

The L^∞ bound for α. We need only consider the case $t < 2r$, as the desired estimate in the compliment was proved on lines (174) and (175) above. That is, we only need to prove this bound along the cones $\overline{C}(u)$ which were introduced just before estimate (164) above. Using that estimate with $\delta_+ = s$, we have the scalar bound:

$$\sup_{(t,x) \in \overline{C}(u)} \tau_+^{2s+3} w_\gamma |\alpha|^2 \ \lesssim \sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{S, \Omega_{0r}\}, Y \in \mathbb{O}}} \| \tau_+^s w_\gamma^{\frac{1}{2}} X^I Y^J(\alpha) \|_{L_x^2(\tilde{C}(u))}^2 \ .$$

Notice that the w_γ factor can be added in at will because it is a constant on the cones $C(u)$. Using this last line, we see that is suffices to prove the pointwise bound:

$$\sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{S, \Omega_{0r}\}, Y \in \mathbb{O}}} |X^I Y^J(\alpha)|^2 \ \lesssim \sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{S, \Omega_{0i}\}, Y \in \mathbb{O}}} |\alpha(\mathcal{L}_X^I \mathcal{L}_Y^J F)|^2 \ .$$

This in turn follows from the decomposition formulas (177) and the coefficient bounds (170). This completes the proof of estimate (165). $\square$

We end this section with a discussion of the peeling properties of Lie derivatives of the charge portion $\overline{F}$ of the curvature F. This will be used in the following sections wherever the quantities $\mathcal{L}_X^I \overline{F}_{\alpha\beta}$ occur, because while there are nice explicit

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[8]It is clear that one can replace w_γ in estimate (162) with $w'_{\gamma,\epsilon}$.

formulas for these objects, they obey no useful space-time estimates.

Proposition 5.3 (Peeling properties of the pure charge fields $\overline{F}_{\alpha\beta}$). *Let $F_{\alpha\beta}$ be an electro-magnetic field, and let $\overline{F}_{\alpha\beta}$ be its pure charge component as defined on the line* (69). *Then for each multiindex I we have the following L^∞ estimates:*

$$(178) \qquad\qquad |\alpha(\mathcal{L}_X^I \overline{F})| \;\leqslant\; C_I \;|q(F)| \cdot \tau_0 \tau_+^{-2} \chi_{t<r+1} \;,$$

$$(179) \qquad |\underline{\alpha}(\mathcal{L}_X^I \overline{F})| \;,\; |\rho(\mathcal{L}_X^I \overline{F}))| \;,\; |\sigma(\mathcal{L}_X^I \overline{F}))| \;,\; \leqslant\; C_I \;|q(F)| \cdot \tau_+^{-2} \chi_{t<r+1} \;,$$

where all $X \in \mathbb{L}$.

Proof of the estimates (178)–(179). The proof is a simple inductive procedure similar to what was done for $\overline{J}_\alpha$ in section 3 starting on line (110). We begin by making the inductive hypothesis that:

$$(180) \qquad\qquad \alpha(\mathcal{L}_X^I \overline{F}) \;=\; q \cdot \sum_{k=1}^{|I|} \frac{\Omega_\alpha^I(\omega)}{r^{2+k}} \cdot (\chi_k^+)_\alpha^I (r-t-2) \;,$$

$$(181) \qquad\qquad \underline{\alpha}(\mathcal{L}_X^I \overline{F}) \;=\; q \cdot \sum_{k=0}^{|I|} \frac{\Omega_{\underline{\alpha}}^I(\omega)}{r^{2+k}} \cdot (\chi_k^+)_{\underline{\alpha}}^I (r-t-2) \;,$$

$$(182) \qquad\qquad \rho(\mathcal{L}_X^I \overline{F}) \;=\; q \cdot \sum_{k=0}^{|I|} \frac{\Omega_\rho^I(\omega)}{r^{2+k}} \cdot (\chi_k^+)_\rho^I (r-t-2) \;,$$

$$(183) \qquad\qquad \sigma(\mathcal{L}_X^I \overline{F}) \;=\; q \cdot \sum_{k=0}^{|I|} \frac{\Omega_\sigma^I(\omega)}{r^{2+k}} \cdot (\chi_k^+)_\sigma^I (r-t-2) \;,$$

where the $\Omega_\bullet^I$ are smooth functions of the angular variable whose C^∞ bounds depend only on I and the specific component being considered, and the $(\chi_k^+)_\bullet^I(s)$ are smooth functions of the single variable s which vanish for $s \leqslant -1$ and satisfy the homogeneity bound:

$$|\partial_s^j (\chi_k^+)_\bullet^I| \;\leqslant\; C_I \; s^{k-j} \;,$$

for $1 \leqslant s$. Here, as in previous discussions, the product notation $\Omega \cdot \chi^+$ is symbolic for a sum of products of functions with these properties.

Now, from the formulas (69), it is clear that the assumption (180)–(183) holds for $|I| = 0$. Furthermore, simple explicit calculations using the formulas (57), and which we leave to the reader, show that differentiating as scalars the quantities on the right hand side of the formulas (180)–(183) yields quantities with these same properties. This follows simply from expressing everything in the derivative in terms of the variables u, r, ω. Therefore, it remains to show that one can inductively reproduce identities of this form after the bracket terms on the right hand side of formula (28b) are taken into account.

First notice that if the Lie derivative is with respect to the fields $\{\Omega_{ij}, S\}$ then the null decomposition (48) is preserved, so the claim follows from the fact that the coefficients in the expansions (58) can all be put in the form $\Omega(\omega) \cdot \varphi(\frac{u}{r})$ for some

affine function φ.

Next, notice that while the translation invariant fields $\{\partial_\alpha\}$ do not preserve the decomposition (48), their coefficients in the formulas (58) introduce a factor which is bounded by $\frac{1}{r}$. Therefore, it remains to deal with the bracket portion when dealing boosts $\{\Omega_{0i}\}$. Notice that it suffices to limit discussion to the component α because the remaining components have the same general form in (181)–(183) above and the homogeneity of the coefficients in the formulas (58) guarantee that this general form is preserved after multiplication by them. The claim now follows from the explicit formulas (58f) and (58h), because the only thing that can disturb the structure of (180) is when one of the other components $\underline{\alpha}(\mathcal{L}_X^I \overline{F}), \rho(\mathcal{L}_X^I \overline{F})$ or $\sigma(\mathcal{L}_X^I \overline{F})$ gets moved to the α position. Notice that this in turn can only happen when an extra factor of $\frac{u}{r}$ is introduced, having the effect of causing the sum in formulas (181)–(183) to start with $k = 1$. This completes the demonstration of (178)–(179). $\qquad\square$

6. L^∞ estimate for complex scalar fields

This section is the companion to Section 4. It contains L^∞ type estimates comparable to (165) for complex scalar fields in terms of bounds on energy norms of the kind (129). These will be proved as in the last section through the application of global Sobolev estimates and several geometric identities which reduce bounds on scalar derivatives to bounds on covariant derivatives. The device which enables us to do this the following simple, although far reaching, estimate of Kato:

Lemma 6.1 (Kato's "diamagnetic" inequality). *Let ϕ be a section of V over $\mathcal{M}$, and denote by $|\phi|^2 = \langle \phi, \phi \rangle_V$. Then one has the point-wise inequality:*

$$(184) \qquad\qquad \left| X(|\phi|) \right| \leqslant |D_X \phi| \ ,$$

for any vector-field X on $\mathcal{M}$.

Proof of (184). This is a straight forward consequence of the compatibility condition (1). We compute:

$$
\begin{aligned}
X(|\phi|) &= X\left(\langle \phi, \phi \rangle_V^{\frac{1}{2}} \right) \ , \\
&= \frac{X\langle \phi, \phi \rangle_V}{2\langle \phi, \phi \rangle_V^{\frac{1}{2}}} \ , \\
&= \frac{\langle D_X \phi, \phi \rangle_V + \langle \phi, D_X \phi \rangle_V}{2|\phi|} \ , \\
&= \Re\left(\langle \frac{\phi}{|\phi|}, D_X \phi \rangle_V \right).
\end{aligned}
$$

Applying absolute values to both sides of the above equation and using the Cauchy–Schwartz inequality, the desired result follows. $\qquad\square$

As a first application of estimate (184), we prove the following generalization of estimates (162)–(164) to complex scalar fields:

Lemma 6.2 (Weighted global Sobolev estimates for complex scalar fields)**.** *Let and let ϕ be a section of V over $\mathcal{M}$, and let w_γ be the weight function defined on line* (74)*. Then on each time slice $t = const.$, with $1 \leqslant t$, one has the estimates:*

(185)
$$\sup_{t \leqslant 2r} \ |\tau_+^{1+\delta_+} \tau_-^{\frac{1}{2}+\delta_-} \phi|^2 \, w_\gamma \ \lesssim \sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{\partial_r, S, \Omega_{0r}\}, Y \in \mathbb{O}}} \| \, \tau_+^{\delta_+} \tau_-^{\delta_-} w_\gamma^{\frac{1}{2}} \, D_X^I D_Y^J \phi \, \|^2_{L_x^2(t < 4r)} \ ,$$

(186)
$$\sup_{2r < t} \ |\tau_+^{1+\delta_+} \phi|^2 \ \lesssim \sum_{\substack{|I| \leqslant 2 \\ X \in \{S, \Omega_{0i}\}}} \| \, \tau_+^{\delta_+} \, D_X^I \phi \, \|^2_{L_x^2(r < \frac{3}{4}t)} \ ,$$

Furthermore, defining the truncated cones:

$$\overline{C}(u) \ = C(u) \cap \{1 \leqslant t \leqslant t_0\} \cap \{t < 2r\} \ ,$$
$$\widetilde{C}(u) \ = C(u) \cap \{1 \leqslant t \leqslant t_0\} \cap \{t < 4r\} \ ,$$

we have the characteristic L^∞ estimate:

(187)
$$\sup_{(t,x) \in \overline{C}(u)} \ |\tau_+^{\frac{3}{2}+\delta_+} \phi|^2 \ \lesssim \sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{S-\Omega_{0r}\}, Y \in \mathbb{O}}} \| \, \tau_+^{\delta_+} \, D_X^I \phi \, \|^2_{L_x^2\left(\widetilde{C}(u)\right)} \ .$$

In the above estimates, $\mathbb{O} = \{\Omega_{ij}\}$ denotes the Lie algebra of the rotation group.

Remark 6.3. Notice that we have used the vector-field $S - \Omega_{0r}$ instead of the individuals $\{S, \Omega_{0r}\}$ in the statement of estimate (187) above. This additional structure will be important in the sequel.

Proof of estimates (185)–(187)*.* The proof of these are virtually identical to the proof of (162)–(164), with the added twist that one uses (184) after each application of the scalar estimates (294)–(295), (298), and (299)–(300) to expand the scalar derivatives to covariant derivatives. The details are left to the interested reader. $\square$

It will also be useful for us to have a version of the estimate (185) which is tailored to deal with the initial data (18b) in the context of the norms which appear on the left hand side of (25).

Lemma 6.4 (Weighted Sobolev estimates for the initial data)**.** *Let and let ϕ be a section of the hyperplane bundle $\{0\} \times \mathbb{R}^3 \times \mathbb{C}$ which vanishes as $r \to \infty$, and let*

Let $\frac{1}{2} < s_0$ be a given parameter. Then the following weighted L^∞ estimates hold:

$$(188) \qquad \sup_x (1+r)^{s_0+\frac{1}{2}} |\phi| \;\lesssim\; \sum_{\substack{1 \leqslant |I| \leqslant 2 \\ X \in \{\partial_i\}}} \| (1+r)^{s_0-1+|I|} D_X^I \phi \|_{L^2(\mathbb{R}^3)} \;,$$

$$(189) \qquad \sup_x (1+r)^{s_0+\frac{3}{2}} |\phi| \;\lesssim\; \sum_{\substack{|I| \leqslant 2 \\ X \in \{\partial_i\}}} \| (1+r)^{s_0+|I|} D_X^I \phi \|_{L^2(\mathbb{R}^3)} \;.$$

Proof of the estimates (188)–(189). The proof of these is almost identical to the proof of estimate (187) above. We simply apply the estimate (294)–(295) of the appendix in order to the functions $\chi\phi$, χ now being the cutoff on a spherical shell of dyadic distance from the origin, using the Kato inequality (184) after each step. Then, using the differential bounds:

$$|\partial_i^I \chi| \;\lesssim\; (1+r)^{-|I|} \, \widetilde{\chi} \;,$$

where $\widetilde{\chi}$ is some cutoff on the support of χ, and adding together over all the (finitely overlapping) χ we arrive at the set of estimates:

$$\sup_x (1+r)^{s_0+\frac{1}{2}} |\phi| \;\lesssim\; \sum_{\substack{|I| \leqslant 2 \\ X \in \{\partial_i\}}} \| (1+r)^{s_0-1+|I|} D_X^I \phi \|_{L^2(\mathbb{R}^3)} \;,$$

$$\sup_x (1+r)^{s_0+\frac{3}{2}} |\phi| \;\lesssim\; \sum_{\substack{|I| \leqslant 2 \\ X \in \{\partial_i\}}} \| (1+r)^{s_0+|I|} D_X^I \phi \|_{L^2(\mathbb{R}^3)} \;.$$

The second of the above estimates is already of the form (189). To conclude (188) from the first estimate above we simply need to eliminate the zero order derivative term. This can be done through the use of the following covariant Poincare type estimate which holds for $\frac{1}{2} < s_0$ assuming that ϕ vanishes at infinity:

$$(190) \qquad \int_{\mathbb{R}^3} (1+r)^{2s_0-2} |\phi|^2 \; dx \;\lesssim\; \int_{\mathbb{R}^3} (1+r)^{2s_0} |D_r\phi|^2 \; dx \;.$$

By an application of the Kato estimate (184), this last estimate follows from the corresponding statement with ϕ replaced by a real valued (compactly supported) test function φ, and the covariant derivative D_r replaced by ∂_r. This in turn is achieved by integrating both sides of the following identity with respect to the measure $dr \, d\omega$ and then applying a Cauchy-Schwartz:

$$\partial_r \left((1+r)^{2s_0-1} r^2 \, \varphi^2 \right) \;=\; (2s_0 - 1)(1+r)^{2s_0-2} r^2 \, \varphi^2 \; + $$
$$2(1+r)^{2s_0-1} r \, \varphi^2 \;+\; 2(1+r)^{2s_0-1} r^2 \, \varphi \, \partial_r\varphi \;.$$

This completes the proof of (188)–(189). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Before proceeding to the main estimate of this section, we first prove another set of preliminary estimates which will be extremely useful in the sequel. These are conjugated versions of the estimate (190) above which also involve the optical weight τ_-, and which we also call Poincare estimates:

Lemma 6.5 (Covariant Poincare estimates for time slices). *Let ϕ be a section to V over $\mathcal{M}$, and let D_r denote the corresponding radial covariant derivative. Then for constants p, q such that both $|q| < p+1$ and $-1 < p$ one has the following estimate:*

$$(191) \qquad \int_{\mathbb{R}^3} \tau_-^p \tau_+^q |\phi|^2 \, dx \;\lesssim\; \int_{\mathbb{R}^3} \tau_-^{p+2} \tau_+^q |\frac{1}{r} D_r(r\phi)|^2 \, dx \;,$$

where the implicit constant depends on both p and q. More specifically, in the exterior region one has the estimate:

$$(192) \qquad \int_{t<r} \tau_-^p \tau_+^q |\phi|^2 \, dx \;\lesssim\; \int_{t<r} \tau_-^{p+2} \tau_+^q |\frac{1}{r} D_r(r\phi)|^2 \, dx \;,$$

whenever $0 < p+1+q$ and $-1 < p$. Furthermore, in the interior one has that:

$$(193) \qquad \int_{r<t} \tau_-^p \tau_+^q |\phi|^2 \, dx \;\lesssim\; \int_{r<t} \tau_-^{p+2} \tau_+^q |\frac{1}{r} D_r(r\phi)|^2 \, dx \;,$$

for values $q < p+1$ and $-1 < p$.

Proof of estimate (191). Keeping in mind the bound (184), it is clear that we only need to consider the incarnation of (191) with ϕ replaced by the real valued test function φ and D_r replaced by the usual radial derivative ∂_r. We now consider the regions $t < r$ and $r < t$ separately. In the first case, it suffices to prove (191) with τ_- replaced by $1 - \underline{u}$ and τ_+ replaced by $1 + \underline{u}$. We then integrate the identity:

$$\partial_r \left((1-u)^{p+1}(1+\underline{u})^q (r\varphi)^2\right) = (p+1)(1-u)^p(1+\underline{u})^q (r\varphi)^2$$
$$+ q(1-u)^{p+1}(1+\underline{u})^{q-1}(r\varphi)^2 + 2(1-u)^{p+1}(1+\underline{u})^q \partial_r(r\phi)(r\phi) \;.$$

with respect to the measure $drd\omega$ over the region $t < r$ to achieve:

$$(194) \quad \int_{t<r} h^{p,q}(t,r)|\varphi|^2 \, dx \;+\; \int_{\mathbb{S}^2_{t=r}} (1+2r)^q |\varphi|^2 dV_{\mathbb{S}^2_{t=r}}$$
$$= 2 \int\!\!\int (1-u)^{p+1}(1+\underline{u})^q \partial_r(r\phi)(r\phi) \; drd\omega \;,$$

where $h^{p,q}(t,r)$ is the weight function given by:

$$h^{p,q}(t,r) = (p+1)(1-u)^p(1+\underline{u})^q + q(1-u)^{p+1}(1+\underline{u})^{q-1} \;,$$
$$= (1-u)^p(1+\underline{u})^q \left((p+1) + q\frac{1-u}{1+\underline{u}}\right) \;.$$

The conditions $-1 < p$ and $-q < p+1$ now easily guarantee the existence of a constant $C_{p,q}$ such that we have the point-wise bounds:

$$C_{p,q}^{-1}(1-u)^p(1+\underline{u})^q \;\leqslant\; h^{p,q}(t,r) \;\leqslant\; C_{p,q}(1-u)^p(1+\underline{u})^q \;.$$

Applying this last bound and a Cauchy–Schwartz to the square of (194) with the angular integral on the left hand side discarded, we arrive at the estimate:

$$\left(\int_{t<r} (1-u)^p(1+\underline{u})^q |\varphi|^2 \, dx\right)^2$$
$$\lesssim \int_{t<r} (1-u)^p(1+\underline{u})^q |\varphi|^2 \, dx \;\cdot\; \int_{t<r} (1-u)^{p+2}(1+\underline{u})^q |\frac{1}{r}\partial_r(r\varphi)|^2 \, dx \;.$$

This proves the assertion (192) by moving back to the weights τ_+, τ_-.

We now prove the exterior estimate (193). This follows essentially the same steps as above. First, we replace the weights τ_+, τ_- by $(1+u)$ and $(1+\underline{u})$ respectively. The analog of (194) now reads:

$$(195) \quad \int_{r<t} h^{p,q}(t,r)|\varphi|^2 \; dx \; - \; \int_{\mathbb{S}^2_{t=r}} (1+2r)^q |\varphi|^2 dV_{\mathbb{S}^2_{t=r}}$$
$$= \; 2\int\int (1+u)^{p+1}(1+\underline{u})^q \partial_r(r\phi)(r\phi) \; dr d\omega \; ,$$

where this time $h^{p,q}(t,r)$ is given by the expression:

$$h^{p,q}(t,r) \;=\; (1+u)^p(1+\underline{u})^q \left(-(p+1) + q\frac{1+u}{1+\underline{u}} \right) \; .$$

The bounds $-1 < p$ and $q < p+1$ now imply the existence of a constant $C_{p,q}$ such that:

$$C_{p,q}^{-1}(1+u)^p(1+\underline{u})^q \;\leqslant\; -h^{p,q}(t,r) \;\leqslant\; C_{p,q}(1+u)^p(1+\underline{u})^q \; .$$

The remainder of the proof now proceeds by squaring and using Cauchy-Schwartz as above. $\qquad\square$

We are now ready to state and prove the main estimate of this section:

Proposition 6.6 (L^∞ estimates for complex scalar fields). *Let ϕ be a section to V over $\mathcal{M}$ with norm $|\cdot|^2$ and compatible connection D. Let $F_{\alpha\beta}$ be the curvature tensor of D. Define the kth weighted generalized energy content of ϕ in the time slab $0 \leqslant t \leqslant t_0$ to be:*

$$(196) \qquad E_k^{(s,\gamma,\epsilon)}(0,t_0)[\phi] \;=\; \sum_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} E^{(s,\gamma,\epsilon)}(0,t_0)[D_X^I \phi] \; ,$$

where $E^{(s,\gamma,\epsilon)}(0,t_0)[\phi]$ is defined on line (129). Now define the time-slab L^∞ type norm:

(197)

$$||| \phi |||^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \;=\; \| \tau_+^{s+1}\tau_-^{\frac{1}{2}}(w')_{\gamma,\epsilon}^{\frac{1}{2}} \left(\frac{1}{r}D_L(r\phi)\chi_{t<2r} + D_L\phi\,\chi_{r<\frac{1}{2}t} \right) \|^2_{L^2(L^\infty)[0,t_0]}$$
$$+ \; \tau_+^{2s+3}|\frac{1}{r}D_L(r\phi)\chi_{t<2r} + D_L\phi\,\chi_{r<\frac{1}{2}t}|^2 w_\gamma$$
$$\left(\tau_+^2\tau_-^{2s-1}|\phi|^2 + \tau_+^2\tau_-^{2s+1}|D_{\underline{L}}\phi|^2 + \tau_+^{2s+1}\tau_-\left(|\not{D}\phi|^2 + \sum_{\substack{|I|\leqslant 1 \\ X\in\mathbb{L}}}|\frac{D_X^I\phi}{\tau_+}|^2\right) \right) \cdot w_\gamma \; .$$

Then recalling the definition of the energy norm (106) for the curvature $F_{\alpha\beta}$, we have the following nonlinear estimate for the field ϕ under the additional assumption that $\epsilon < s - \frac{1}{2}$:

$$(198) \quad ||| \phi |||^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \;\lesssim\; E_2^{(s,\gamma,\epsilon)}(0,t_0)[\phi] \cdot \left(1 + E_2^{(s,\gamma,\epsilon)}(0,t_0)[F] \right)$$
$$+ \; E_2^{(s,\gamma,\epsilon)}(0,t_0)[F] \cdot ||| \phi |||^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \; .$$

Proof of estimate (198). To begin with, we may assume that $1 \leqslant t$, as the complimentary case can easily be dealt with by a straightforward application of radially weighted Sobolev estimates and energy bounds similar to what we will use in the deep interior region $r < \frac{1}{2}t$. We now split cases according to whether $t < 2r$ or $r < \frac{1}{2}t$. We deal with the more difficult former case first. Here we will prove estimate (198) by using an essentially separate argument for each term on the left hand side of (197). We proceed as follows:

Estimate (197) **for $\frac{\phi}{r}$ and $\not{D}\phi$ in the region** $t < 2r$. We begin with the undifferentiated estimate for $\frac{\phi}{r}$. Applying estimate (185) with $\delta_+ = s - 1$ and $\delta_- = 0$, and using the pointwise bound:

$$(199) \qquad \sum_{X \in \{\partial_r, S, \Omega_{ij}, \Omega_{0r}\}} |D_X \phi| \; \lesssim \; \tau_+ |\frac{1}{r} D_L(r\phi)| + \tau_- |D_L \phi| + \tau_+ \left(|\not{D}\phi| + |\frac{\phi}{r}| \right) ,$$

which follows easily from the expansions (57), we arrive at the L^∞ estimate:

$$(200)$$
$$\sup_{1 \leqslant t < 2r} \tau_+^{2s+2} \tau_- |\frac{\phi}{r}|^2 w_\gamma \; \lesssim \; \sum_{\substack{|I| \leqslant 1 \\ X \in \mathbb{O}}} \int_{t < 4r} \left(\tau_+^{2s} |\frac{1}{r} D_L(r D_X^I \phi)|^2 \; + \; \tau_-^{2s} |D_{\underline{L}}(D_X^I \phi)|^2 \right.$$
$$\left. + \; \tau_+^{2s} \left(|\not{D}(D_X^I \phi)|^2 + |\frac{D_X^I \phi}{r}|^2 \right) \right) w_\gamma \; dx \; .$$

Notice that one can add another derivative to ϕ on the right hand side of (200) and still remain bounded by the energy $E_2^{(s, \gamma, \epsilon)}[\phi]$. Therefore we also have the desired bound for $\frac{D_x \phi}{r}$.

To bound the term $\not{D}\phi$, we simply use (200) in conjunction with the following estimate which holds where $1 \leqslant t < 2r$:

$$\tau_+ |\not{D}\phi| \; \lesssim \; \sum_{i < j} |D_{\Omega_{ij}} \phi| \; .$$

Estimate (197) **for $D_{\underline{L}}\phi$ in the region** $t < 2r$. Our first step here is to apply the estimate (185) with $\delta_+ = 0$ and $\delta_- = s$ to the section $D_{\underline{L}}\phi$ which yields the bound:

$$(201)$$
$$\sup_{1 \leqslant t \leqslant 2r} \tau_+^2 \tau_-^{2s+1} |D_{\underline{L}}\phi|^2 w_\gamma \; \lesssim \; \sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I| + |J| \leqslant 2 \\ X \in \{\partial_r, S, \Omega_{0r}\}, Y \in \mathbb{O}}} \| \tau_-^s w_\gamma^{\frac{1}{2}} D_X^I D_Y^J D_{\underline{L}}\phi \|_{L_x^2(t < 4r)}^2 \; .$$

We are done once we can handle the commutator $[D_X^I D_Y^J, D_{\underline{L}}]$. To compute the effect of the rotations, we use formulas (5), (28b), and (56b) together with the fact that $\alpha(\overline{F}) = 0$ which yields:

$$(202\text{a}) \qquad [D_{\Omega_{ij}}, D_{\underline{L}}] \; = \; - \sqrt{-1} \Omega_{ij}^A \underline{\alpha}_A \; ,$$

$$(202\text{b}) \qquad \Omega_{kl}([D_{\Omega_{ij}}, D_{\underline{L}}]) \; = \; - \sqrt{-1} \left(\Omega_{ij}^A \underline{\alpha}_A (\mathcal{L}_{\Omega_{kl}} \widetilde{F}) + \delta_{(lk} \Omega_{ij)}^A \; \underline{\alpha}_A \right) \; .$$

To compute the effect of the fields $\{\partial_r, S, \Omega_{0r}\}$, we proceed as above with the help of the identities (169) and the fact that $\partial_r(\Omega_{ij}^A) = r^{-1}\Omega_{ij}^A$. This yields:

$$(203a) \qquad [D_r, D_{\underline{L}}] = -\sqrt{-1}\rho \,,$$

$$(203b) \qquad [D_S, D_{\underline{L}}] = -D_{\underline{L}} - \sqrt{-1}\underline{u}\rho \,,$$

$$(203c) \qquad [D_{\Omega_{0r}}, D_{\underline{L}}] = -D_{\underline{L}} + \sqrt{-1}\underline{u}\rho$$

$$(203d) \qquad \partial_r([D_{\Omega_{ij}}, D_{\underline{L}}]) = -\sqrt{-1}\Omega_{ij}^A\left(\omega^k\underline{\alpha}_A(\mathcal{L}_{\partial_k}\widetilde{F}) + \frac{1}{r}\underline{\alpha}_A\right) \,,$$

$$(203e) \qquad S([D_{\Omega_{ij}}, D_{\underline{L}}]) = -\sqrt{-1}\Omega_{ij}^A\left(\underline{\alpha}_A(\mathcal{L}_S\widetilde{F}) - \underline{\alpha}_A\right)$$

$$(203f) \qquad \Omega_{0r}([D_{\Omega_{ij}}, D_{\underline{L}}]) = -\sqrt{-1}\Omega_{ij}^A\left(\omega^k\underline{\alpha}_A(\mathcal{L}_{\Omega_{0i}}\widetilde{F}) - \frac{u}{r}\underline{\alpha}_A\right) \,.$$

Combining the estimates (202)–(203), and expanding out the fields ∂_r and Ω_{0r} using the identity $\underline{L}(\omega_i) = 0$ to move the contractions past the $D_{\underline{L}}$ derivative, we arrive at the following pointwise bounds for the commutator in the region $1 \leqslant t < 2r$:

$$(204) \qquad \sum_{\substack{|I|\leqslant 1,|J|\leqslant 2 \\ |I|+|J|\leqslant 2 \\ X\in\{\partial_r,S,\Omega_{0r}\},Y\in\mathbb{O}}} |D_X^I D_Y^J D_{\underline{L}}\phi| \lesssim \sum_{\substack{|I|\leqslant 1,|J|\leqslant 2 \\ |I|+|J|\leqslant 2 \\ X\in\{\partial_i,S,\Omega_{0i}\},Y\in\mathbb{O}}} |D_{\underline{L}}D_X^I D_Y^J \phi|$$

$$+ \sum_{\substack{|I|+|J|\leqslant 1 \\ X,Y\in\mathbb{L}}} \tau_+|\underline{\alpha}(\mathcal{L}_X^I\widetilde{F})|\cdot|D_Y^J\phi| + \sum_{\substack{|I|\leqslant 1 \\ X\in\mathbb{O}}} \tau_+|\rho|\cdot|D_X^I\phi| \,.$$

Expanding out $\rho = \widetilde{\rho} + \overline{\rho}$, and using the definitions of the energies (106) and (196) and the L^∞ estimates (131) and (197) wherever possible (which we leave as a straightforward exercise to the reader), the bounds (201) and (204) together imply that:

$$\sup_{1\leqslant t\leqslant 2r} \tau_+^2\tau_-^{2s+1}|D_{\underline{L}}\phi|^2 w_\gamma \lesssim E_2^{(s,\gamma,\epsilon)}[\phi]\cdot\left(1 + \||F\||_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2\right) +$$

$$E_1^{(s,\gamma,\epsilon)}[F]\cdot\||\phi\||_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2 + \||F\||_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2\cdot\sum_{\substack{|I|\leqslant 1 \\ X\in\mathbb{L}}} \|\tau_-^{-\frac{1}{2}}D_X^I\phi\|_{L^2(\mathbb{R}^3)}^2 \,.$$

Every term on the right hand side of this last expression is acceptable save for the last. Unfortunately, this expression cannot be directly bounded by the energy (196). This is where we need to make use of the fact that there is the additional room $0 < s - \frac{1}{2}$. Doing this, we may replace the $\tau_-^{-\frac{1}{2}}$ weight in this term by τ_-^{s-1} and assume that we are trying to estimate:

$$(205) \qquad \sum_{\substack{|I|\leqslant 1 \\ X\in\mathbb{L}}} \|\tau_-^{s-1}D_X^I\phi\|_{L^2(\mathbb{R}^3)}^2 \lesssim \sum_{\substack{|I|\leqslant 1 \\ X\in\mathbb{L}}} \left(\|\tau_+^s(w)_\gamma^{\frac{1}{2}}\frac{1}{r}D_L(rD_X^I\phi)\|_{L^2(\mathbb{R}^3)}^2\right.$$

$$\left. + \|\tau_-^s(w)_\gamma^{\frac{1}{2}}D_L D_X^I\phi\|_{L^2(\mathbb{R}^3)}^2 + \|\tau_+^s(w)_\gamma^{\frac{1}{2}}\frac{1}{r}D_X^I\phi\|_{L^2(\mathbb{R}^3)}^2\right).$$

This last line follows from Poincare estimate (191) with $p = 2s - 2$ and $q = 0$ along with the simple pointwise bound:

$$(206) \qquad \tau_-^s|\frac{1}{r}D_r(r\phi)| \lesssim \left(\tau_+^s|\frac{1}{r}D_L(r\phi)| + \tau_-^s|D_{\underline{L}}\phi| + \tau_+^s|\frac{\phi}{r}|\right)\cdot(w)_\gamma^{\frac{1}{2}} \,.$$

Improved L^∞ estimates for the term $\frac{\phi}{r}$ in the region $t < 2r$. This estimate follows from a simple combination of the global Sobolev and Poincare estimates. Applying (185) with $\delta_+ = 0$ and $\delta_- = s - 1$, and expanding out the fields ∂_r and Ω_{0r} we have that:

$$\sup_{1 \leqslant t < 2r} \tau_+^2 \tau_-^{2s-1} |\phi|^2 w_\gamma \lesssim \sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{\partial_i, S, \Omega_{0i}\}, Y \in \mathbb{O}}} \| \tau_-^{s-1}(w)_\gamma^{\frac{1}{2}} D_X^I D_Y^J \phi \|_{L^2(t<4r)} \ .$$

We are now finished by applying estimate (191) in the same way as used to prove (205) above, with the help of the pointwise bound (206) to convert the result into energy.

The $L^2(L^\infty)$ estimate for $\frac{1}{r} D_L(r\phi)$ in the region $t < 2r$. Using estimate (185) with $\delta_+ = s$, $\delta_- = 0$ and the weight w_γ replaced by $w'_{\gamma,\epsilon}$ on each fixed time slice, we directly have the bound:

$$(207) \quad \| \tau_+^{s+1} \tau_-^{\frac{1}{2}} (w')_{\gamma,\epsilon}^{\frac{1}{2}} \frac{1}{r} D_L(r\phi) \|_{L^2(L^\infty)[1,t_0] \cap \{t<2r\}} \lesssim$$

$$\sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{\partial_r, S, \Omega_{0r}\}, Y \in \mathbb{O}}} \| \tau_+^s (w')_{\gamma,\epsilon}^{\frac{1}{2}} D_X^I D_Y^J \frac{1}{r} D_L(r\phi) \|_{L^2(L^2)[1,t_0] \cap \{t<4r\}} \ .$$

Keeping in mind now the space-time energy norm contained in $E_2^{(s,\gamma,\epsilon)}[\phi]$, we are finished once we have taken into account the commutator $[D_X^I D_Y^J, \frac{1}{r} D_L(r\cdot)]$. As above, we compute the effect of the rotations to be:

$$(208a) \qquad [D_{\Omega_{ij}}, \frac{1}{r} D_{\underline{L}}(r\cdot)] = -\sqrt{-1}\Omega_{ij}^A \alpha_A \ ,$$

$$(208b) \qquad \Omega_{kl}([D_{\Omega_{ij}}, D_{\underline{L}}]) = -\sqrt{-1}\left(\Omega_{ij}^A \alpha_A (\mathcal{L}_{\Omega_{kl}} \widetilde{F}) + \delta_{(lk} \Omega_{ij)}^A \alpha_A \right) \ .$$

Similarly, using the formulas (177) and the fact that $\partial_r(\Omega_{ij}^A) = r^{-1}\Omega_{ij}^A$, we compte the effect of the fields $\{\partial_r, S, \Omega_{0r}\}$ to be:

$$(209a) \qquad [D_r, \frac{1}{r} D_L(r\cdot)] = -\sqrt{-1}\rho - \frac{1}{r^2} \ ,$$

$$(209b) \qquad [D_S, \frac{1}{r} D_L(r\cdot)] = \sqrt{-1}u\,\rho - \frac{1}{r} D_L(r\cdot) \ ,$$

$$(209c) \qquad [D_{\Omega_{0r}}, \frac{1}{r} D_L(r\cdot)] = \sqrt{-1}u\,\rho + \frac{1}{r} D_L(r\cdot) - \frac{u}{r^2} \ ,$$

$$(209d) \qquad \partial_r([D_{\Omega_{ij}}, \frac{1}{r} D_L(r\cdot)]) = -\sqrt{-1}\Omega_{ij}^A \left(\omega^k \alpha_A (\mathcal{L}_{\partial_k} \widetilde{F}) + \frac{1}{r}\alpha_A \right) \ ,$$

$$(209e) \qquad S([D_{\Omega_{ij}}, \frac{1}{r} D_L(r\cdot)]) = -\sqrt{-1}\Omega_{ij}^A \left(\alpha_A (\mathcal{L}_S \widetilde{F}) - \alpha_A \right)$$

$$(209f) \qquad \Omega_{0r}([D_{\Omega_{ij}}, \frac{1}{r} D_L(r\cdot)]) = -\sqrt{-1}\Omega_{ij}^A \left(\omega^k \alpha_A (\mathcal{L}_{\Omega_{0i}} \widetilde{F}) - \frac{u}{r}\alpha_A \right) \ .$$

Combining the identities (208)–(209), and expanding out the fields ∂_r and Ω_{0r} using the identity $\underline{L}(\omega_i) = 0$ to move the contractions past the $\frac{1}{r} D_L(r\cdot)$ derivative, we

arrive at the following pointwise bounds for the commutator in the region $1 \leqslant t < 2r$:

(210)

$$\sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{\partial_r, S, \Omega_{0r}\}, Y \in \mathbb{O}}} |D_X^I D_Y^J \frac{1}{r} D_L(r\phi)| \;\; \lesssim \;\; \sum_{\substack{|I| \leqslant 1, |J| \leqslant 2 \\ |I|+|J| \leqslant 2 \\ X \in \{\partial_i, S, \Omega_{0i}\}, Y \in \mathbb{O}}} |\frac{1}{r} D_L(r D_X^I D_Y^J \phi)|$$

$$+ \sum_{\substack{|I|+|J| \leqslant 1 \\ X, Y \in \mathbb{L}}} \tau_+ |\alpha(\mathcal{L}_X^I \widetilde{F})| \cdot |D_Y^J \phi| \;\; + \sum_{\substack{|I| \leqslant 1 \\ X \in \mathbb{O}}} \left(\tau_- |\rho| \cdot |D_X^I \phi| \;\; + \;\; \tau_0 |\frac{D_X^I \phi}{r}| \right) \;.$$

We now insert each term on the right hand side of (210) into the right hand side of estimate (207) and bound the result using the norms (106), (196), (131), and (197). Since there are some choices involved and the condition $\epsilon \leqslant s - \frac{1}{2}$ makes its appearance here we do this explicitly. Note that the first term on the right hand side of (210) is already of the right form so we move on to the second. For this term, we split cases depending on wether the derivatives are falling on α or ϕ. In the first case, we estimate:

$$\sum_{\substack{|I| \leqslant 1 \\ X \in \mathbb{L}}} \| \tau_+^{s+1} (w')^{\frac{1}{2}}_{\gamma, \epsilon} \; \alpha(\mathcal{L}_X^I \widetilde{F}) \phi \|^2_{L^2(L^2)[0, t_0]} \;,$$

$$\lesssim \sum_{\substack{|I| \leqslant 1 \\ X \in \mathbb{L}}} \| \tau_+^{s} (w')^{\frac{1}{2}}_{\gamma, \epsilon} \; \alpha(\mathcal{L}_X^I \widetilde{F}) \|^2_{L^2(L^2)} \cdot \| \tau_+ \phi \|^2_{L^\infty(L^\infty)[0, t_0]} \;,$$

$$\lesssim E_1^{(s, \gamma, \epsilon)}(0, t_0)[F] \cdot \|| \phi \||^2_{L^\infty[0, t_0](s, \gamma \epsilon)} \;.$$

In the second case, we use Hölders inequality to conclude that:

$$\sum_{\substack{|I| \leqslant 1 \\ X \in \mathbb{L}}} \| \tau_+^{s+1} (w')^{\frac{1}{2}}_{\gamma, \epsilon} \; \alpha \, D_X^I \phi \|^2_{L^2(L^2)[0, t_0]} \;,$$

$$\lesssim \| \tau_+^{s+1} (w')^{\frac{1}{2}}_{\gamma, \epsilon} \; \alpha \|^2_{L^2(L^\infty)[0, t_0]} \cdot \sum_{\substack{|I| \leqslant 1 \\ X \in \mathbb{L}}} \| \tau_-^{-\frac{1}{2}} D_X^I \phi \|^2_{L^\infty(L^2)[0, t_0]} \;.$$

Notice that the first factor on this last line is contained in the norm (131). To bound the second factor on the last line above, we again use the extra room $\frac{1}{2} < s$ and apply the estimate (205).

It remains to deal with the last term on the right had side of (210). To handle this, we expand $\rho = \widetilde{\rho} + \overline{\rho}$. Using the identity (69), we see that we can combine the term $\tau_- \overline{\rho} \, D_X^I \phi$ with the second term in this expression. Notice that this is automatically included in the energy (196). Thus, it suffices to be able to prove the

bound:

$$\sum_{\substack{|I|\leqslant 1 \\ X\in\mathbb{L}}} \| \, \tau_+^s \tau_-(w')^{\frac{1}{2}}_{\gamma,\epsilon} \, \widetilde{\rho} \, D_X^I \phi \, \|^2_{L^2(L^2)[0,t_0]} \, \lesssim$$

$$\||\, F \,|\|^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \cdot \sum_{\substack{|I|\leqslant 1 \\ X\in\mathbb{L}}} \| \, \tau_-^{\frac{1}{2}}(w')^{\frac{1}{2}}_{\gamma,\epsilon} \, \frac{1}{r} D_X^I \phi \, \|^2_{L^2(L^2)[0,t_0]} \,.$$

To bound the second factor on the right hand side of this last expression, we simply use the condition $\epsilon \leqslant s - \frac{1}{2}$ which guarantees that $\tau_- \leqslant \tau_+^{2s} \tau_0^{1+2\epsilon}$. This concludes the $t < 2r$ estimate for $\frac{1}{r} D_L(r\phi)$ in $L^2(L^\infty)$.

The pure L^∞ for $\frac{1}{r} D_L(r\phi)$ in the region $t < 2r$. We begin here by applying estimate (187) to the term $\frac{1}{r} D_L(r\phi)$ with $\delta_+ = s$. This yields the bound:

$$(211) \qquad \sup_{(t,x)\in\overline{C}(u)} \tau_+^{2s+3} |\frac{1}{r} D_L(r\phi)|^2 w_\gamma \, \lesssim$$

$$\sum_{\substack{|I|\leqslant 1,|J|\leqslant 2 \\ |I|+|J|\leqslant 2 \\ X\in\{S-\Omega_{0r}\},Y\in\mathbb{O}}} \| \, \tau_+^s (w)^{\frac{1}{2}}_{\gamma} \, D_X^I D_Y^J \frac{1}{r} D_L(r\phi) \, \|_{L^2\big(\widetilde{C}(u)\big)} \,.$$

To deal with the commutator, we apply the identities (208)–(209) which provide the following refinement of (210):

$$(212) \qquad \sum_{\substack{|I|\leqslant 1,|J|\leqslant 2 \\ |I|+|J|\leqslant 2 \\ X\in\{S-\Omega_{0r}\},Y\in\mathbb{O}}} |D_X^I D_Y^J \frac{1}{r} D_L(r\phi)| \, \lesssim \, \sum_{\substack{|I|\leqslant 1,|J|\leqslant 2 \\ |I|+|J|\leqslant 2 \\ X\in\{S,\Omega_{0i}\},Y\in\mathbb{O}}} |\frac{1}{r} D_L(r D_X^I D_Y^J \phi)|$$

$$+ \, \sum_{\substack{|I|+|J|\leqslant 1 \\ X\in\mathbb{L} \\ Y\in\{S-\Omega_{0r}\}\cup\mathbb{O}}} \tau_+ |\alpha(\mathcal{L}_X^I \widetilde{F})| \cdot |D_Y^J \phi| \, + \, \sum_{\substack{|I|\leqslant 1 \\ X\in\mathbb{O}}} \left(|u| \cdot |\rho| \cdot |D_X^I \phi| \, + \, |\frac{u D_X^I \phi}{r^2}| \right) \,.$$

Notice that the weight τ_- is replaced by $|u|$ in the last term on the right hand side of (212). This is because we do not have any term involving the radial derivative ∂_r to contend with on the left hand side. It is necessary to have this if we are to use the characteristic energy bound contained in (196). We now insert the terms on the right hand side of (212) into the right hand side of (211). Notice that the first such expression is automatically bounded by the energy (196). Therefore, it suffices to treat the last two groups of terms which we do in reverse order.

To bound the last group of terms on the right hand side of (212), we expand $\rho = \widetilde{\rho} + \overline{\rho}$, and group the term $|u| \cdot \overline{\rho} D_X^I \phi$ with the term $r^{-2} |u| D_X^I \phi$ using the identity

(69). The resulting term is contained in (196), so we are left with bounding:

$$\sum_{\substack{|I|\leqslant 1 \\ X\in\mathbb{L}}} \|\,\tau_+^s\tau_-(w)_{\widetilde{\gamma}}^{\frac{1}{2}}\,\widetilde{\rho}D_X^I\phi\,\|_{L^2\left(\widetilde{C}(u)\right)}^2\ ,$$

$$\lesssim \|\,\tau_-^s(w)_{\widetilde{\gamma}}^{\frac{1}{2}}\,\widetilde{\rho}\,\|_{L^2\left(\widetilde{C}(u)\right)}^2 \cdot \sum_{\substack{|I|\leqslant 1 \\ X\in\mathbb{L}}} \|\,\tau_+^s\tau_-^{1-s}\,D_X^I\phi\,\|_{L^\infty[0,t_0]}^2\ ,$$

$$\lesssim E^{(s,\gamma,\epsilon)}[F]\cdot|||\,\phi\,|||_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2\ .$$

To wrap things up here, we need to bound the middle set of terms on the right hand side of (212) substituted into the right hand side of (211). Dealing first with the term where the derivatives fall on α, we estimate:

$$\sum_{\substack{|I|\leqslant 1 \\ X\in\mathbb{L}}} \|\,\tau_+^{s+1}(w)_{\widetilde{\gamma}}^{\frac{1}{2}}\,\alpha(\mathcal{L}_X^I\widetilde{F})\,\phi\,\|_{L^2\left(\widetilde{C}(u)\right)}^2\ ,$$

$$\lesssim \sum_{\substack{|I|\leqslant 1 \\ X\in\mathbb{L}}} \|\,\tau_+^s(w)_{\widetilde{\gamma}}^{\frac{1}{2}}\,\alpha(\mathcal{L}_X^I\widetilde{F})\,\|_{L^2\left(\widetilde{C}(u)\right)}^2 \cdot \|\,\tau_+\phi\,\|_{L^\infty(L^\infty)[0,t_0]}^2\ ,$$

$$\lesssim E_1^{(s,\gamma,\epsilon)}(0,t_0)[F]\cdot|||\,\phi\,|||_{L^\infty[0,t_0](s,\gamma\epsilon)}^2\ .$$

Therefore, it remains to deal with the case where the derivatives fall on ϕ. Here we have to split cases depending on whether we have $D_Y\phi$ with $Y=S-\Omega_{0r}$, or $Y\in\mathbb{O}$. In the first case, we write $Y=\underline{u}L$ and we expand:

$$|\underline{u}L\phi|\ \leqslant\ \tau_+|\frac{1}{r}D_L(r\phi)|+\frac{\tau_+}{r}|\phi|\ .$$

Using the fact that $t<2r$ this allows us to estimate:

$$\|\,\tau_+^{s+1}(w)_{\widetilde{\gamma}}^{\frac{1}{2}}\,\alpha\,D_{S-\Omega_{0r}}\phi\,\|_{L^2\left(\widetilde{C}(u)\right)}^2\ ,$$

$$\lesssim \|\,\tau_+^s(w)_{\widetilde{\gamma}}^{\frac{1}{2}}\,\alpha\,\|_{L^2\left(\widetilde{C}(u)\right)}^2 \cdot \left(\|\,\tau_+^2\frac{1}{r}D_L(r\phi)\,\|_{L^\infty[0,t_0]}^2+\|\,\tau_+\phi\,\|_{L^\infty[0,t_0]}^2\right)\ ,$$

$$\lesssim E_1^{(s,\gamma,\epsilon)}(0,t_0)[F]\cdot|||\,\phi\,|||_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2\ .$$

We are now left with estimating the expression $\sum_{Y\in\mathbb{O}}|\tau_+^{s+1}(w)_{\widetilde{\gamma}}^{\frac{1}{2}}\,\alpha\,D_Y\phi|$. This turns out to require an argument which is a bit more involved than what we have been doing so far. Notice that we cannot directly use either L^∞ or L^2 bounds on the $D_{\Omega_{ij}}\phi$ term. Furthermore, a Poincare estimate similar to (191) for light-cones will not work in this situation because it introduces boundary terms which cannot be controlled. Therefore, being unable to deal with things on the light-cone $\widetilde{C}(u_0)$, we extend the integral to the truncated time slab:

$$\mathcal{R}(u_0,t_0)=\left([\max\{0,u_0\},t_0]\times\mathbb{R}^3\right)\cap\{u\leqslant u_0\}\ .$$

Doing this via Stokes theorem yields the estimate:

$$(213) \quad \sum_{Y \in \mathbb{O}} \| \tau_+^{s+1}(w)_\gamma^{\frac{1}{2}} \, \alpha \, D_Y \phi \|_{L^2(\widetilde{C}(u_0))}^2 \ \leqslant$$

$$\sum_{Y \in \mathbb{O}} \left(\left| \int \int_{\mathcal{R}(u_0, t_0)} \tau_+^{2s+2} \, \underline{L} \left(|\alpha \, D_Y \phi|^2 \right) \cdot w_\gamma \, dx dt \right| + \| \tau_+^{s+1}(w)_\gamma^{\frac{1}{2}} \, \alpha \, D_Y \phi \|_{L^2(\{t=u_0\} \times \mathbb{R}^3)}^2 \right) \, .$$

Notice that things have been arranged so that the derivative does not fall on the w_γ weight. This is merely a notational convenience and is possible due to the fact that w_γ is a constant on the cone $\widetilde{C}(u_0)$ and non-decreasing as $u \to -\infty$ in the region $\mathcal{R}(u_0, t_0)$. To bound the second term on the right hand side of (213) above, we estimate:

$$\sum_{Y \in \mathbb{O}} \| \tau_+^{s+1}(w)_\gamma^{\frac{1}{2}} \, \alpha \, D_Y \phi \|_{L^2(\{t=u_0\} \times \mathbb{R}^3)}^2 \ \lesssim$$

$$\sum_{Y \in \mathbb{O}} \||\, F \,\||_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2 \cdot \| \tau_+^{-\frac{1}{2}} \, D_Y \phi \|_{L^2(\{t=u_0\} \times \mathbb{R}^3)}^2 \, .$$

We are now in the familiar territory where estimate (205) can be applied.

Moving on to the first term on the right hand side of (213), we expand the $\underline{L}$ derivative to bound:

$$(214) \quad \sum_{Y \in \mathbb{O}} \left| \int \int_{\mathcal{R}(u_0, t_0)} \tau_+^{2s+2} \, \underline{L} \left(|\alpha \, D_Y \phi|^2 \right) \cdot w_\gamma \, dx dt \right| \ \leqslant$$

$$\sum_{\substack{Y \in \mathbb{O} \\ X \in \{\partial_r, \partial_t, S, \Omega_{0r}\}}} \int_0^{t_0} \int_{\mathbb{R}^3} \tau_+^{2s+2} \tau_-^{-1} \left| X \left(|\alpha \, D_Y \phi|^2 \right) \right| \cdot w_\gamma \, dx dt \, .$$

Using the formulas (177) and the fact that $\partial_t(\alpha_A) = \alpha_A(\mathcal{L}_{\partial_t} \widetilde{F})$, we expand the sum:

$$(215) \quad \sum_{\substack{Y \in \mathbb{O} \\ X \in \{\partial_r, \partial_t, S, \Omega_{0r}\}}} \left| X \left(|\alpha \, D_Y \phi|^2 \right) \right| \ \lesssim \ \sum_{\substack{|I| \leqslant 2 \\ X \in \mathbb{L}}} |\alpha|^2 \cdot |D_X^I \phi|^2 \ +$$

$$\sum_{\substack{|I| \leqslant 1 \\ X \in \mathbb{O}, Y \in \mathbb{L}}} |\alpha| \cdot |\alpha(\mathcal{L}_Y \widetilde{F})| \cdot |D_Y \phi|^2 \, .$$

Substituting the first term on the right hand side of (215) in the right hand integral in (214), and using the mixed norm estimate for α we bound:

$$\sum_{\substack{|I| \leqslant 2 \\ X \in \mathbb{L}}} \| \tau_+^{s+1} \tau_-^{-\frac{1}{2}}(w)_\gamma^{\frac{1}{2}} \, \alpha \, D_X^I \phi \|_{L^2([0,t_0] \times \mathbb{R}^3)}^2 \ \lesssim$$

$$\||\, F \,\||_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2 \cdot \sum_{\substack{|I| \leqslant 2 \\ X \in \mathbb{L}}} \| \tau_-^{-\frac{1}{2}+\epsilon} \, D_X^I \phi \|_{L^\infty(L^2)[0,t_0]}^2 \, .$$

Using now the condition that $\epsilon \leqslant s - \frac{1}{2}$, we are in a position to again apply (205).

We have now reduced things to estimating the integral:

$$I = \sum_{X \in \mathbb{L}, Y \in \mathbb{O}} \int_0^{t_0} \int_{\mathbb{R}^3} \tau_+^{2s+2} \tau_-^{-1} \, |\alpha| \cdot |\alpha(\mathcal{L}_X \widetilde{F})| \cdot |D_Y \phi|^2 \cdot w_\gamma \; dx dt \; .$$

Since this cannot be done directly, we use the identities:

$$|D_Y \phi|^2 = 2Y(\Re\langle \phi, D_Y \phi\rangle - 2\Re\langle \phi, D_Y^2 \phi\rangle \; ,$$
$$2\Re\langle \phi, D_Y \phi\rangle = Y(|\phi^2|) \; .$$

to integrate by parts several times with respect to the $D_{\Omega_{ij}}$ derivatives to achieve the bound:

$$I \lesssim \sum_{\substack{|I|,|J| \leqslant 2 \\ X \in \mathbb{L}, Y \in \mathbb{O}}} \int_0^{t_0} \int_{\mathbb{R}^3} \tau_+^{2s+2} \tau_-^{-1} \, |\alpha| \cdot |\alpha(\mathcal{L}_X^I \widetilde{F})| \cdot |\phi| \cdot |D_Y^J \phi| \cdot w_\gamma \; dx dt$$

$$+ \sum_{\substack{|I| \leqslant 2 \\ X \in \mathbb{L}}} \int_0^{t_0} \int_{\mathbb{R}^3} \tau_+^{2s+2} \tau_-^{-1} \, |\alpha(\mathcal{L}_X^I \widetilde{F})|^2 \cdot |\phi|^2 \cdot w_\gamma \; dx dt \; ,$$

$$= I_1 + I_2 \; .$$

To estimate the first integral on the right hand side above, we use a multiple Hölder inequality to conclude:

$$I_1 \lesssim \sum_{\substack{|I|,|J| \leqslant 2 \\ X \in \mathbb{L}, Y \in \mathbb{O}}} \| \tau_+^{s+1} \tau_-^{\frac{1}{2}} (w')_{\gamma,\epsilon}^{\frac{1}{2}} \alpha \|_{L^2(L^\infty)[0,t_0]} \cdot \| \tau_+^s (w')_{\gamma,\epsilon}^{\frac{1}{2}} \alpha(\mathcal{L}_X^I \widetilde{F}) \|_{L^2(L^2)[0,t_0]}$$

$$\cdot \| \tau_+ \tau_-^\epsilon \phi \|_{L^\infty[0,t_0]} \cdot \| \tau_-^{-\frac{1}{2}+\epsilon} D_Y^J \phi \|_{L^\infty(L^2)[0,t_0]} \; ,$$

$$\lesssim \left(\|\| F \|\|_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2 + E_2^{(s,\gamma,\epsilon)}[F] \right) \cdot \left(\|\| \phi \|\|_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2 + E_2^{(s,\gamma,\epsilon)}[\phi] \right) \; .$$

Notice that the passage to the last line above is guaranteed by the condition $\epsilon \leqslant s - \frac{1}{2}$ and the estimate (205). To estimate the second integral, I_2 above, we proceed as follows:

$$I_2 \lesssim \sum_{\substack{|I| \leqslant 2 \\ X \in \mathbb{L}}} \| \tau_+^s (w')_{\gamma,\epsilon} \, \alpha(\mathcal{L}_X^I \widetilde{F}) \|_{L^2(L^2)[0,t_0]}^2 \cdot \| \tau_+ \tau_-^\epsilon \phi \|_{L^\infty[0,t_0]}^2 \; ,$$

$$\lesssim E_2^{(s,\gamma,\epsilon)}[F] \cdot \|\| \phi \|\|_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2 \; .$$

This completes the proof of the pure L^∞ bound for $\frac{1}{r} D_L(r\phi)$ in the region $t < 2r$.

The estimate (197) **in the deep interior region** $r < \frac{1}{2}t$. We begin with the pure L^∞ estimates. Applying estimate (186) with $\delta_+ = s - 1$ and simply expanding out all of the resulting derivatives we have the bound:

$$(216) \qquad\qquad \sup_{r < \frac{1}{2}t} \tau_+^{2s+1} \, |\phi|^2 \lesssim \sum_{\substack{|I| \leqslant 2 \\ X \in \mathbb{L}}} \| \tau_+^{s-1} D_X^I \phi \|_{L^2(r < \frac{3}{4}r)} \; .$$

The desired bounds on the gradient $D\phi$ now follow from the estimates:

$$\tau_+ |D\phi| \lesssim \sum_{\substack{|I|\leqslant 1 \\ X\in\mathbb{L}}} |D_X^I \phi| \, ,$$

$$\sum_{\substack{|I|\leqslant 3 \\ X\in\mathbb{L}}} |D_X^I \phi| \lesssim \sum_{\substack{|I|\leqslant 2 \\ X\in\mathbb{L}}} \tau_+ |D(D_X^I \phi)| \, ,$$

which both hold due to the condition $r < \frac{3}{4}t$. To estimate the expression $\sum_{\substack{|I|\leqslant 1 \\ X\in\mathbb{L}}} |\tau_+^{-1} D_X^I \phi|$, we apply (186) to it with $\delta_+ = s$ and bound the resulting right hand side with the help of the pointwise estimate:

$$\sum_{\substack{|I|\leqslant 2, |J|\leqslant 1 \\ X,Y\in\mathbb{L}}} |D_X^I \ \tau_+^{-1} \ D_Y^J \phi| \lesssim \sum_{\substack{|I|\leqslant 2 \\ X\in\mathbb{L}}} \left(|D\, D_X^I \phi| + \tau_+^{-1} |D_X^I \phi| \right) \, .$$

To wrap things up for this subsection, we bound the D_ϕ term in weighted $L^2(L^\infty)$ by first applying the Sobolev embedding (186) on fixed time slices and then using the same reasoning as above to estimate the resulting space-time integral in terms of the energy (196). Notice that this does not depend on the structure of $D_L\phi$, and that in fact the same process works for any component of $D\phi$.

Combining all of the estimates proved in the various subsections above, we have proved the estimate:

$$||| \phi |||^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \lesssim E_2^{(s,\gamma,\epsilon)}[0,t_0][\phi]\cdot\left(1 + ||| F |||^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} + E_2^{(s,\gamma,\epsilon)}[0,t_0][F]\right)$$

$$+ \ \left(||| F |||^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} + E_2^{(s,\gamma,\epsilon)}[0,t_0][F]\right) \cdot ||| \phi |||^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \, .$$

Using now the L^∞ estimate (165) for the curvature, we arrive at estimate (198). $\square$

7. Abstract Weight Notation and the Bilinear Space-Time Estimates in General Form

In this section, we will organize and consolidate the L^2 and L^∞ type estimate that we have proved in the preceding sections. This will help to streamline notation for the remainder of the paper, and will ultimately reduce a lot of overlapping which occurs in the commutator estimates needed to deal with the nonlinear problem (17). This will also give a chance for the reader to review the various notations, parameters, weights, etc. which have been introduces thus far. Our first order of business here is to set up some generic markers for the quantities which arise in the estimates we consider. These are the null components of the curvature F given by (48), the corresponding null decomposition of the gradient $D\phi$, and the weighted scalar field $\frac{\phi}{\tau_+}$. As we have already mentioned in Remark 4.3 above, there is a straightforward analogy between different members of these objects. To make this

analogy precise, we form the following sets of field quantities:

$$(217a) \quad \Psi_{(1)} = \{\alpha \, , \, \frac{1}{r}D_L(r\phi)\,\chi_{t<2r} + D_L\phi\,\chi_{r<\frac{1}{2}t}\} \, , \quad \Psi_{(-1)} = \{\underline{\alpha} \, , \, D_{\underline{L}}\phi\} \, ,$$

$$(217b) \quad \Psi_{(0)} = \{(\widetilde{\rho} \, , \, \sigma \, , \, \slashed{D}\phi \, , \, \sum_{|J|\leqslant 1, Y\in\mathbb{L}} \frac{D_Y^J\phi}{\tau_+}\} \, , \qquad Q_{(0)} = \{\overline{\rho}\} \, .$$

We will also use the following designation for sets containing the component decompositions of Lie derivatives of the various field quantities:

$$(218a) \quad \mathcal{L}_{\mathbb{L}}^k\Psi_{(1)} = \bigcup_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} \{\alpha(\mathcal{L}_X^I\widetilde{F}) \, , \, \frac{1}{r}D_L(rD_X^I\phi)\,\chi_{t<2r} + D_L(D_X^I\phi)\,\chi_{r<\frac{1}{2}t}\} \, ,$$

$$(218b) \quad \mathcal{L}_{\mathbb{L}}^k\Psi_{(0)} = \bigcup_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} \{\rho(\mathcal{L}_X^I\widetilde{F}) \, , \, \sigma(\mathcal{L}_X^I\widetilde{F}) \, , \, \slashed{D}(D_X^I\phi) \, , \, \sum_{|J|\leqslant 1, Y\in\mathbb{L}} \frac{D_X^I D_Y^J\phi}{\tau_+}\} \, ,$$

$$(218c) \quad \mathcal{L}_{\mathbb{L}}^k\Psi_{(-1)} = \bigcup_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} \{\underline{\alpha}(\mathcal{L}_X^I\widetilde{F}) \, , \, D_{\underline{L}}(D_X^I\phi)\} \, ,$$

$$(218d) \quad \mathcal{L}_{\mathbb{L}}^k Q_{(1)} = \bigcup_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} \{\alpha(\mathcal{L}_X^I\overline{F})\} \, ,$$

$$(218e) \quad \mathcal{L}_{\mathbb{L}}^k Q_{(0)} = \bigcup_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} \{\rho(\mathcal{L}_X^I\overline{F}) \, , \, \sigma(\mathcal{L}_X^I\overline{F})\} \, ,$$

$$(218f) \quad \mathcal{L}_{\mathbb{L}}^k Q_{(-1)} = \bigcup_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} \{\underline{\alpha}(\mathcal{L}_X^I\overline{F})\} \, .$$

While it is true that the above objects are properly sets, they can be manipulated as if they are in fact functions by making each designator stand for the absolute sum of each element in the set, and letting products of the various sets denote the absolute sum of products of each element in the corresponding sets. As a first application of this principle, we use the above notation to recast the norms (106), (196), (131), and (197):

(219)
$$E_k^{(s,\gamma,\epsilon)}(0,t_0)[\Psi] = E_k^{(s,\gamma,\epsilon)}(0,t_0)[F] + E_k^{(s,\gamma,\epsilon)}(0,t_0)[\phi] \, ,$$
(220)
$$|||\,\Psi\,|||_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2 = |q(F)|^2 + \|\tau_+^{s+1}\tau_-^{\frac{1}{2}}(w')^{\frac{1}{2}}_{\gamma,\epsilon}\Psi_{(1)}\|_{L^2(L^\infty)[0,t_0]}^2$$
$$\sup_{0\leqslant t\leqslant t_0} (\tau_+^2\tau_-^{2s-1}|\phi| + \tau_+^{2s+3}|\Psi_{(1)}|^2 + \tau_+^2\tau_-^{2s+1}|\Psi_{(-1)}|^2 + \tau_+^{2s+2}\tau_-|\Psi_{(0)}|^2) \cdot w_\gamma \, .$$

Notice that we have included the extra L^∞ norm for ϕ on the right hand side of (220). This extra estimate turns out to be very important for us, but will be used in a somewhat auxiliary manner.

We can now write out the content of line (219) with the symbolic estimate (neglecting the characteristic energy terms which are only used to prove L^∞ estimates

and which don't quite fit into our scheme):

(221)

$$E^{(s,\gamma,\epsilon)}(0,t_0)[\Psi] \;\geqslant\; \int_{\{t_0\}\times\mathbb{R}^3} \left(\tau_+^{2s}|\Psi_{(1)}|^2 + \tau_-^{2s}|\Psi_{(-1)}|^2 + \tau_+^{2s}|\Psi_{(0)}|^2\right)\cdot w_\gamma\;dx$$

$$+\; \int_0^{t_0}\int_{\mathbb{R}^3} \left(\tau_+^{2s}|\Psi_{(1)}|^2 + \tau_0^{1+2\epsilon}\big(\tau_-^{2s}|\Psi_{(-1)}|^2 + \tau_+^{2s}|\Psi_{(0)}|^2\big)\right)\cdot w'_{\gamma,\epsilon}\;dxdt\;.$$

We can also write the L^∞ estimates (178)–(179), (165), and (198) in the following consolidated symbolic form:

Proposition 7.1 (L^∞ estimates for field quantities in abstract from)**.** *Let $0\leqslant t_0$ be a given fixed time, and let $0<\gamma,\epsilon,s$ be parameters chosen so that $\epsilon\leqslant s-\frac{1}{2}$, then one has the following abstract non-linear L^∞ estimate for the field quantities $\mathcal{L}_{\mathbb{L}}^k\Psi$ and $\mathcal{L}_{\mathbb{L}}^k Q$:*

(222) $\quad \||\,\mathcal{L}_{\mathbb{L}}^k\Psi\,||_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2 \;\lesssim\; E_{k+2}^{(s,\gamma,\epsilon)}(0,t_0)[\Psi]\cdot\left(1+E_2^{(s,\gamma,\epsilon)}(0,t_0)[\Psi]\right)$

$$+\; E_2^{(s,\gamma,\epsilon)}(0,t_0)[\Psi]\cdot\||\,\mathcal{L}_{\mathbb{L}}^k\Psi\,||_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2\;,$$

and:

(223a) $$\mathcal{L}_{\mathbb{L}}^k Q_{(1)} \;\leqslant\; C_k\,|q|\cdot\tau_0\tau_+^{-2}\cdot\chi_{t<r+1}\;,$$

(223b) $$\mathcal{L}_{\mathbb{L}}^k Q_{(0)}\;,\;\mathcal{L}_{\mathbb{L}}^k Q_{(-1)} \;\leqslant\; C_k\,|q|\cdot\tau_+^{-2}\cdot\chi_{t<r+1}\;.$$

We now conclude this section by proving all of the estimates we will encounter in the rest of the paper in a simple abstract form. It turns out that everything we need can be cast in the language of L^2 bilinear space-time estimates. These in turn follow directly from Hölders inequality and the L^2 and L^∞ type estimates contained in the right hand sides of (221) and (220)–(223b).

Proposition 7.2 (Weighted L^2 bilinear estimates for field quantities in abstract form)**.** *Let $0\leqslant t_0$ be a given fixed time, and let $0<\gamma,\epsilon,s$ be parameters chosen so that $\epsilon\leqslant s-\frac{1}{2}$, and define the auxiliary weight:*

(224) $$\widetilde{w}_{\gamma,\epsilon}\;=\;\tau_-^{2\gamma}\chi_{t<r}+\tau_-^{-\epsilon}\chi_{r<t}\;,$$

then one has the following abstract estimates for field quantities Ψ and Φ:

(225) $\qquad \| \, \tau_+^{2s+1} \, \widetilde{w}_{\gamma,\epsilon} \; \Psi_{(1)} \cdot \Phi_{(0)} \, \|^2_{L^2[0,t_0]} \; \lesssim$
$$E^{(s,\gamma,\epsilon)}(0,t_0)[\Psi_{(1)}] \cdot \| |\, \Phi_{(0)} \, |\|^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \; ,$$

(226) $\qquad \| \, \tau_+^{2s+1} \, \widetilde{w}_{\gamma,\epsilon} \; \Psi_{(1)} \cdot \Phi_{(0)} \, \|^2_{L^2[0,t_0]} \; \lesssim$
$$E^{(s,\gamma,\epsilon)}(0,t_0)[\Phi_{(0)}] \cdot \| |\, \Psi_{(1)} \, |\|^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \; ,$$

(227) $\| \, \tau_+^{2s+\frac{1}{2}-\epsilon} \tau_-^{\frac{1}{2}+\epsilon} \widetilde{w}_{\gamma,\epsilon} \; \Psi_{(0)} \cdot \Phi_{(0)} \, \|^2_{L^2[0,t_0]} \; \lesssim$
$$E^{(s,\gamma,\epsilon)}(0,t_0)[\Psi_{(0)}] \cdot \| |\, \Phi_{(0)} \, |\|^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \; ,$$

(228) $\qquad \| \, \tau_+^{s+1} \tau_-^s \widetilde{w}_{\gamma,\epsilon} \; \Psi_{(1)} \cdot \Phi_{(-1)} \, \|^2_{L^2[0,t_0]} \; \lesssim$
$$E^{(s,\gamma,\epsilon)}(0,t_0)[\Psi_{(1)}] \cdot \| |\, \Phi_{(-1)} \, |\|^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \; ,$$

(229) $\qquad \| \, \tau_+^{s+1} \tau_-^s \widetilde{w}_{\gamma,\epsilon} \; \Psi_{(1)} \cdot \Phi_{(-1)} \, \|^2_{L^2[0,t_0]} \; \lesssim$
$$E^{(s,\gamma,\epsilon)}(0,t_0)[\Phi_{(-1)}] \cdot \| |\, \Psi_{(1)} \, |\|^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \; ,$$

(230)
$$\| \, \tau_+^{s+\frac{1}{2}-\epsilon} \tau_-^{s+\frac{1}{2}+\epsilon} \widetilde{w}_{\gamma,\epsilon} \; \Psi_{(0)} \cdot \Phi_{(-1)} \, \|^2_{L^2[0,t_0]} \; \lesssim$$
$$E^{(s,\gamma,\epsilon)}(0,t_0)[\Psi_{(0)}] \cdot \| |\, \Phi_{(-1)} \, |\|^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \; ,$$

(231)
$$\| \, \tau_+^{s+\frac{1}{2}-\epsilon} \tau_-^{s+\frac{1}{2}+\epsilon} \widetilde{w}_{\gamma,\epsilon} \; \Psi_{(0)} \cdot \Phi_{(-1)} \, \|^2_{L^2[0,t_0]} \; \lesssim$$
$$E^{(s,\gamma,\epsilon)}(0,t_0)[\Phi_{(-1)}] \cdot \| |\, \Psi_{(0)} \, |\|^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \; ,$$

Recalling the definition of the weight function w_γ from line (74), we also have the following space-time estimates for the interaction of the charge Q and the field quantity Ψ:

(232) $\qquad \| \, \tau_+^{s+\frac{5}{2}-\epsilon} \tau_-^{-1+\epsilon} (w_\gamma)^{\frac{1}{2}} Q_{(1)} \cdot \Psi_{(0)} \, \|_{L^2[0,t_0]} \; \lesssim \; |q|^2 \cdot E^{(s,\gamma,\epsilon)}[\Phi] \; ,$

(233) $\qquad \| \, \tau_+^{s+\frac{3}{2}-\epsilon} \tau_-^{\epsilon} (w_\gamma)^{\frac{1}{2}} \left(Q_{(-1)} \cdot \Psi_{(0)} + Q_{(0)} \cdot \Psi_{(0)} + Q_{(1)} \cdot \Psi_{(-1)} \right) \|_{L^2[0,t_0]}$
$$\lesssim \; |q|^2 \cdot E^{(s,\gamma,\epsilon)}[\Phi] \; ,$$

(234) $\qquad \| \, \tau_+^{\frac{3}{2}-\epsilon} \tau_-^{s+\epsilon} (w_\gamma)^{\frac{1}{2}} Q_{(0)} \cdot \Psi_{(-1)} \, \|_{L^2[0,t_0]} \; \lesssim \; |q|^2 \cdot E^{(s,\gamma,\epsilon)}[\Phi] \; .$

Remark 7.3. Since the proofs of these are a direct consequence of the estimates (221) and (220)–(223b) we simply make a few comments here. First of all, notice that the product estimate (229) must use the extra mixed space $L^2(L^\infty)$ norm on the right hand side of (220). If one were instead to use the $\Psi_{(1)}$ peeling property times the space-time L^2 estimate for $\Phi_{(-1)}$, there would be an additional need for a factor of τ_0^ϵ. As we shall see in Section 9, such an extra convergence factor is not available in our application. This is precisely the reason we have included the extra $L^2(L^\infty)$ norms. Furthermore, notice that these mixed Lebesgue space norms can also be used in the other case where we need to put $\Psi_{(1)}$ in L^∞. Thus, as far as our proof of Theorem 1.1 is concerned, we never need to make use of the pure L^∞ estimate for this quantity.

The second thing we call the readers attention to here is the fact that the estimates (232)–(234) will be used in their precise form for $2t < r$. Notice that there are no extra factors of τ_-^γ which can be put in these estimates as was the case for (225)–(231) above. From a technical point of view (aside from convenience), this is the reason it is necessary to have the exact space-time energy norms on the right hand side of (221). Without these, estimates of the form (232)–(234) would lead to logarithmic divergences which we see no other way of controlling.

Before we finish, we take one last look at the estimates (225)–(234). Our purpose is to recast these in such a way that they conform more closely to how they will be applied to estimate the system (17). In doing this, a surprisingly simple and elegant picture emerges of the underlying structure of the system (17). This reenforces our point of view that the complex scalar field can be treated as if it were a tensorial quantity with roughly the same properties as the curvature $F_{\alpha\beta}$. We will put this structure to use through the so called "parity condition". This is a numerical device used to keep track of weights and components in contractions such as the right hand side of (43), which is typical of the kind of error terms we treat in the next two sections. It turns out that all of the estimates we will need can be reduced to a single streamlined form, the motivation for which will become a bit more clear through its use in the sequel.

Proposition 7.4 (Abstract parity form of the weighted bilinear L^2 estimates.).
Let $0 \leqslant t_0$ be a given fixed time, and let $0 < \gamma, \epsilon, s$ be parameters chosen so that this time $2\epsilon \leqslant s - \frac{1}{2}$. Recall the definition of the auxiliary weight $w_{\gamma,\epsilon}$ on line (75), and define the parity weights $w_{\gamma,\epsilon}(a)$ via the formulas:

(235)
$$ w_{\gamma,\epsilon}(1) \;=\; \tau_0^{-\epsilon} \cdot (w)_{\gamma,\epsilon}^{\frac{1}{2}} \;, \quad w_{\gamma,\epsilon}(0) \;=\; \tau_0^{\frac{1}{2}} \cdot (w)_{\gamma,\epsilon}^{\frac{1}{2}} \quad w_{\gamma,\epsilon}(-1) \;=\; \tau_0^{2s-\epsilon} \cdot (w)_{\gamma,\epsilon}^{\frac{1}{2}} \;. $$

Also define the parity optical weights:

(236)
$$ \tau_{(1)} \;=\; \tau_- \;, \qquad\qquad \tau_{(0)} \;=\; \tau_{(-1)} \;=\; \tau_+ \;. $$

Then the following abstract bilinear L^2 estimate holds for the field quantities Ψ, Φ:

(237)
$$ \| \, \tau_+^{s-\frac{1}{2}} \, w_{\gamma,\epsilon}(a+b+c+d) \, \tau_{(a)}\tau_{(b)} \, \Psi_{(c)} \cdot \Phi_{(d)} \, \|_{L^2[0,t_0]}^2 \;\lesssim $$
$$ E^{(s,\gamma,\epsilon)}(0,t_0)[\Psi] \cdot ||| \, \Phi \, |||_{L^\infty[0,t_0](s,\gamma,\epsilon)}^2 \;. $$

whenever the condition $-1 \leqslant a+b+c+d \leqslant 1$ holds. One also has the analogous estimate for the interaction of Q and Φ under the same condition on a, b, c, d:

(238)
$$ \| \, \tau_+^{s-\frac{1}{2}} \, w_{\gamma,\epsilon}(a+b+c+d) \, \tau_{(a)}\tau_{(b)} \, Q_{(c)} \cdot \Phi_{(d)} \, \|_{L^2[0,t_0]}^2 \;\lesssim $$
$$ |q|^2 \cdot E^{(s,\gamma,\epsilon)}(0,t_0)[\Phi] \;. $$

Proof of the estimates (237)–(238). We concentrate on the first estimate (237), the second being similar and much easier. First of all, we claim that this can be reduced

to the more restricted special case:

$$(239) \quad \| \tau_+^{s+\frac{1}{2}} w_{\gamma,\epsilon}(b+c+d) \, \tau_{(b)} \, \Psi_{(c)} \cdot \Phi_{(d)} \|^2_{L^2[0,t_0]} \; \lesssim$$
$$E^{(s,\gamma,\epsilon)}(0,t_0)[\Psi] \cdot \| | \Phi | \|^2_{L^\infty[0,t_0](s,\gamma,\epsilon)} \; ,$$

where now we impose the two conditions $-1 \leqslant c+d \leqslant 1$ and $-1 \leqslant b+c+d \leqslant 1$. Notice that (239) implies (237) with these extra conditions enforced because in that case one simply has the bound:

$$w_{\gamma,\epsilon}(a+b+c+d) \cdot \tau_+^{-1} \cdot \tau_{(a)} \; \leqslant \; w_{\gamma,\epsilon}(b+c+d) \; ,$$

as long as we are not in the case where both $a = 1$ and $b+c+d = -1$. If this happens, we see from the previous conditions that $b = -1, 0$ so we can use the bound:

$$w_{\gamma,\epsilon}(a+b+c+d) \cdot \tau_+^{-1} \cdot \tau_{(a)}\tau_{(b)} \; \leqslant \; w_{\gamma,\epsilon}(b+c+d+1) \cdot \tau_{(b+1)} \; ,$$

which holds in this case.

To derive the estimate (237) in the case where $c+d = -1$ and $b = -1$, notice that one must then have $a = 1$, so the bound $\tau_+^{-1}\tau_{(1)} \leqslant 1$ reduces things to (239) with $c+d = -1$ and $b = 0$. The case $c+d = 1$, $b = 1$, $a = -1$ can be treated similarly.

To derive (237) from (239) in the case where either $c = d = -1$ or $c = d = 1$ we use the symbolic inequality:

$$\tau_+^{-1}\tau_{(1)}\Psi_{(-1)} \; \leqslant \; \Psi_{(0)},$$

and the fact that the product $\Psi_{(1)} \cdot \Phi_{(1)}$ so favorable that it satisfies the space-time L^2 estimate (237) for any of the weights $w_{\gamma,\epsilon}(a)$.

It now remains to prove (239). This involves a simple case by case analysis, split along the value of $c+d$:

Case: $c+d = 1$. In this case we can either have $b = -1$ or $b = 0$. In either case the $\tau_{(b)}$ weight is the same, so it suffices to consider the case which maximizes the weight $w_{\gamma,\epsilon}(b+c+d)$. In this case we are dealing with $w_{\gamma,\epsilon}(1)$. Substituting this into estimate (239), and using the bound:

$$\tau_+^{s+\frac{1}{2}} w_{\gamma,\epsilon}(1) \, \tau_{(0)} \; = \; \tau_+^{s+\frac{3}{2}+\epsilon}\tau_-^{-\epsilon}(w)^{\frac{1}{2}}_{\gamma,\epsilon} \; ,$$
$$\lesssim \; \tau_+^{2s+1}\widetilde{w}_{\gamma,\epsilon} \; ,$$

which holds due to the condition $2\epsilon \leqslant s - \frac{1}{2}$, we see that the desired result follows from estimates (225)–(226) above.

Case: $c+d = 0$. In this case, we can have all choices $-1 \leqslant b \leqslant 1$. It suffices to consider the one which maximizes the product $w_{\gamma,\epsilon}(b) \cdot \tau_{(b)}$ which is easily seen to

be $b = 0$. In this case we compute the total weight in estimate (239) to be:

$$\tau_+^{s+\frac{1}{2}} w_{\gamma,\epsilon}(0)\, \tau_{(0)} \;=\; \tau_+^{s+1} \tau_-^{\frac{1}{2}}(w)_{\gamma,\epsilon}^{\frac{1}{2}} \;,$$
$$\lesssim\; \tau_+^{s+1} \tau_-^{s}\, \widetilde{w}_{\gamma,\epsilon} \;,$$
$$\lesssim\; \tau_+^{2s+\frac{1}{2}-\epsilon} \tau_-^{\frac{1}{2}+\epsilon}\, \widetilde{w}_{\gamma,\epsilon} \;,$$

where the last two inequalities follow from the condition $2\epsilon \leqslant s - \frac{1}{2}$. We are now in a position to directly apply estimates (227)–(229).

Case: c+d=-1. In this case we can have either $b = 1$ or $b = 0$ so it suffices to consider the one which maximizes the product $w_{\gamma,\epsilon}(b-1) \cdot \tau_{(b)}$. This is $b = 0$ in which case we are dealing with the total weight:

$$\tau_+^{s+\frac{1}{2}} w_{\gamma,\epsilon}(-1)\, \tau_{(0)} \;=\; \tau_+^{\frac{3}{2}+\epsilon-s} \tau_-^{2s-\epsilon}(w)_{\gamma,\epsilon}^{\frac{1}{2}} \;,$$
$$\lesssim\; \tau_+^{s+\frac{1}{2}-\epsilon} \tau_-^{s+\frac{1}{2}+\epsilon}\, \widetilde{w}_{\gamma,\epsilon} \;,$$

where, as before, the inequality is guaranteed by the condition $2\epsilon \leqslant s - \frac{1}{2}$. Substituting this last line into estimate (239) we can then directly apply estimates (230)–(231). This completes our proof of (237). $\qquad\square$

8. Differentiating the Field Equations I: Error Estimates for the curvature $F_{\alpha\beta}$

We are now ready to begin in earnest our proof of Theorem 1.1. We assume that we have fixed some level of regularity for the problem, say k derivatives with $2 \leqslant k$. The result will be demonstrated through a bootstrapping argument on the energy (219). Recalling the definition of the norms for the initial data on the left hand side of (25), we define:

$$(240) \quad E_k^{(s,\gamma)}(0)[F,\phi] \;=\; \|\, E^{df}\,\|_{H^{k,s+\gamma}(\mathbb{R}^3)}^2 \;+\; \|\, H\,\|_{H^{k,s+\gamma}(\mathbb{R}^3)}^2 \;+$$
$$\|\, \underline{D}\phi_0\,\|_{H^{k,s+\gamma}(\mathbb{R}^3)}^2 \;+\; \|\, \phi_0\,\|_{H^{k,s+\gamma}(\mathbb{R}^3)}^2 \;.$$

We will now show that:

Theorem 8.1 (Bootstrapping form of the Theorem 1.1). *Let k be a given level of regularity, and assume that we are given parameters $0 < s, \gamma, \epsilon$ with the properties that $s \leqslant 1$, and $2\epsilon \leqslant s - \frac{1}{2}$, and $s + \gamma < \frac{3}{2}$. Let $0 < T$ be a given time parameter, and let Ψ denote the totality of components of the system (17) as defined by (218) with the associated k^{th} level energy content (219). Let $E_k^{(s,\gamma)}(0)[F,\phi]$ denote the initial k^{th} level energy content in the initial data (18). Then there exists a constant $1 \leqslant C_{k,s,\gamma,\epsilon}$ which depend only on k,s,ϵ,γ, but* not *on T, such that if one first assumes that both:*

$$(241) \qquad\qquad E_k^{(s,\gamma)}(0)[F,\phi] \;,\; E_k^{(s,\gamma,\epsilon)}(0,T)[\Psi] \;\leqslant\; C_{k,s,\gamma,\epsilon}^{-1} \;,$$

then the following nonlinear estimate also holds:

$$(242) \quad E_k^{(s,\gamma,\epsilon)}(0,T)[\Psi] \;\leqslant\; C_{k,s,\gamma,\epsilon}\left(E_k^{(s,\gamma)}(0)[F,\phi] + \sum_{l=2}^{7} \left[E_k^{(s,\gamma,\epsilon)}(0,T)[\Psi] \right]^l \right) \;.$$

In particular, the estimate (242) shows that the assumptions:

$$E_k^{(s,\gamma)}(0)[F,\phi] \;\leqslant\; \frac{1}{8}C_{k,s,\gamma,\epsilon}^{-2} \;, \qquad\qquad E_k^{(s,\gamma,\epsilon)}(0,T)[\Psi] \;\leqslant\; \frac{1}{2}C_{k,s,\gamma,\epsilon}^{-1} \;,$$

together imply that:

$$E_k^{(s,\gamma,\epsilon)}(0,T)[\Psi] \leqslant \frac{1}{4}C_{k,s,\gamma,\epsilon}^{-1} \;.$$

This, combined with the usual local existence theorem for the system (17) and with the main L^∞ estimate (222) (assuming that $C_{k,s,\gamma,\epsilon}$ is chosen so large that the condition (241) allows one to absorb the extra L^∞ terms on the right hand side of this estimate), implies the claim of Theorem 1.1. Therefore, for the remainder of this paper we will assume that the solution exists up to time T, and we will concentrate on proving the a-priori non-linear estimate (242).

In this section we concentrate on proving (242) for the portion of Ψ which contains the curvature. By the energy estimate (107), this boils down to estimating the differentiated current vector $\mathcal{L}_X^I J$ at time $t = 0$, as well as over the space–time slab $[0,T]$. We do this separately. For the latter it suffices to be able to prove the following bounds:

Proposition 8.2 (Space-time error bounds for the curvature $F_{\alpha\beta}$). *Let k be a given level of regularity, and assume that we are given parameters $0 < s, \gamma, \epsilon$ with the properties that $s \leqslant 1$, and $2\epsilon \leqslant s - \frac{1}{2}$, $s + \gamma < \frac{3}{2}$. Let $0 < T$ be a given time parameter, and let:*

$$J \;=\; \Im(\phi\overline{D\phi}) \;,$$

be the current vector for the system (17). Recall the current vector norm (99). Then there exists a constant $1 \leqslant C_{k,s,\gamma,\epsilon}$ depending only on k, s, ϵ, γ, such that if:

$$(243) \qquad\qquad E_k^{(s,\gamma,\epsilon)}(0,T)[\Psi] \;\leqslant\; C_{k,s,\gamma,\epsilon}^{-1} \;,$$

then one has the following weighted space-time estimate:

$$(244) \qquad \sum_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} \left|\left|\left| \mathcal{L}_X^I J \right|\right|\right|_{L^2[0,T](s,\gamma,\epsilon)}^2 \;\lesssim\; \sum_{l=2}^{5} \left[E_k^{(s,\gamma,\epsilon)}(0,T)[\Psi] \right]^l \;.$$

Proof of estimate (244). In light of the abstract L^∞ estimate (222), and the assumption (243) with the constant $C_{k,s,\gamma,\epsilon}^{-1}$ chosen small enough that the right hand side of (222) containing the extra L^∞ norm can be absorbed into the left hand side, it suffices to be able to show that:

$$(245) \quad \sum_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} \left|\left|\left| \mathcal{L}_X^I J \right|\right|\right|_{L^2[0,T](s,\gamma,\epsilon)}^2 \;\lesssim\;$$

$$E_k^{(s,\gamma,\epsilon)}(0,T)[\Psi] \cdot \left|\left|\left| \mathcal{L}_{\mathbb{L}}^{k-2}\Psi \right|\right|\right|_{L^\infty[0,T](s,\gamma,\epsilon)}^2 \left(1 + \left|\left|\left| \mathcal{L}_{\mathbb{L}}^{k-2}\Psi \right|\right|\right|_{L^\infty[0,T](s,\gamma,\epsilon)}^2 \right) \;.$$

Expanding out the norm on the left hand side of this estimate, and using the weight notation (235) we see that:

$$\sum_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} ||| \mathcal{L}_X^I J |||^2_{L^2[0,T](s,\gamma,\epsilon)} \;=\; \sum_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} \left(\| \tau_+^{s+\frac{1}{2}} w_{\gamma,\epsilon}(1) \; \mathcal{L}_X^I J_L \|^2_{L^2[0,T]} \right.$$
$$\left. +\; \| \tau_+^{s+\frac{1}{2}} w_{\gamma,\epsilon}(0) \; \mathcal{L}_X^I \not{J} \|^2_{L^2[0,T]} \;+\; \| \tau_+^{s+\frac{1}{2}} w_{\gamma,\epsilon}(-1) \; \mathcal{L}_X^I J_{\underline{L}} \|^2_{L^2[0,T]} \right) \;.$$

Therefore, using the abstract parity estimate (237), it suffices to prove the following symbolic bounds:

(246)
$$\sum_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} |\mathcal{L}_X^I J_L|$$
$$\lesssim \sum_{\substack{l+m=k-1 \\ l=0,m=k \\ a+b+c=1}} \tau_{(a)} \; \left(\mathcal{L}_\mathbb{L}^l \Psi_{(b)} + \mathcal{L}_\mathbb{L}^l Q_{(b)}\right) \cdot \mathcal{L}_\mathbb{L}^m \Psi_{(c)} \cdot \left(1 + ||| \mathcal{L}_\mathbb{L}^{k-2} \Psi |||_{L^\infty[0,T](s,\gamma,\epsilon)}\right) \;,$$

(247)
$$\sum_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} |\underline{\mathcal{L}}_X^I J_L|$$
$$\lesssim \sum_{\substack{l+m=k-1 \\ l=0,m=k \\ a+b+c=-1}} \tau_{(a)} \; \left(\mathcal{L}_\mathbb{L}^l \Psi_{(b)} + \mathcal{L}_\mathbb{L}^l Q_{(b)}\right) \cdot \mathcal{L}_\mathbb{L}^m \Psi_{(c)} \cdot \left(1 + ||| \mathcal{L}_\mathbb{L}^{k-2} \Psi |||_{L^\infty[0,T](s,\gamma,\epsilon)}\right) \;,$$

(248)
$$\sum_{\substack{|I|\leqslant k \\ X\in\mathbb{L}}} |\mathcal{L}_X^I \not{J}|$$
$$\lesssim \sum_{\substack{l+m=k-1 \\ l=0,m=k \\ a+b+c=0}} \tau_{(a)} \; \left(\mathcal{L}_\mathbb{L}^l \Psi_{(b)} + \mathcal{L}_\mathbb{L}^l Q_{(b)}\right) \cdot \mathcal{L}_\mathbb{L}^m \Psi_{(c)} \cdot \left(1 + ||| \mathcal{L}_\mathbb{L}^{k-2} \Psi |||_{L^\infty[0,T](s,\gamma,\epsilon)}\right) \;.$$

We will only prove the bounds (246)–(248) in the extended exterior region $t < 2r$. These bounds in the complimentary region $r < \frac{1}{2}t$ follows from similar reasoning and is much simpler because the weights $\tau_{(a)}$ are all identical there.

Our first step is the following simple inductive calculation of the Lie derivative $\mathcal{L}_X^I J$, based on repeated use of the formula (41):

(249)

$$
\begin{aligned}
\mathcal{L}_X^I J &= \sum_{\substack{K_1+K_2=I \\ X_1,X_2\in\mathbb{L}}} \left(D_{X_1}^{K_1}\phi \cdot \overline{\mathcal{L}_{X_2}^{\mathbb{C}}{}^{K_2} D\phi} \right) + \left(D_{X_2}^{K_2}\phi \cdot \overline{\mathcal{L}_{X_1}^{\mathbb{C}}{}^{K_1} D\phi} \right) , \\
&= \sum_{\substack{K_1+K_2=I \\ X,Y\in\mathbb{L}}} \left(D_{X_1}^{K_1}\phi \cdot \overline{D(D_{X_2}^{K_2}\phi)} \right) + \left(D_{X_2}^{K_2}\phi \cdot \overline{D(D_{X_1}^{K_1}\phi)} \right) \\
&\qquad + \sum_{\substack{K_1+K_2=I \\ X_1,X_2\in\mathbb{L}}} \left(D_{X_1}^{K_1}\phi \cdot \overline{[\mathcal{L}_{X_2}^{\mathbb{C}}{}^{K_2},D]\phi} \right) + \left(D_{X_2}^{K_2}\phi \cdot \overline{[\mathcal{L}_{X_1}^{\mathbb{C}}{}^{K_1},D]\phi} \right) , \\
&= A+B .
\end{aligned}
$$

We now compute each of the A and B terms on the right hand side of the above identities separately. Each of these terms can be seen as real valued two forms, and we denote their components by A_α and B_α respectively.

To compute the A term, notice that since the sum is symmetric in the K and L multiengined, we can introduce the needed extra factor of $\frac{1}{r}$ and r to the L component. Doing this, and putting absolute values around the different components of A while using the condition $|I| \leqslant k$ we have the bounds:

(250)
$$
\begin{aligned}
|A_L| \lesssim &\sum_{\substack{|K|=k \\ X\in\mathbb{L}}} \tau_+ \,|\frac{\phi}{\tau_+}| \cdot |\frac{1}{r}D_L(rD_X^K\phi)| \\
&+ \sum_{\substack{|K_1|+|K_2|\leqslant k-1 \\ X_1,X_2,Y\in\mathbb{L}}} |D_Y D_{X_1}^{K_1}\phi| \cdot |\frac{1}{r}D_L(rD_{X_2}^{K_2}\phi)| ,
\end{aligned}
$$

(251)
$$
\begin{aligned}
|A_{\underline{L}}| \lesssim &\sum_{\substack{|K|=k \\ X\in\mathbb{L}}} \tau_+ \,|\frac{\phi}{\tau_+}| \cdot |D_{\underline{L}}(D_X^K\phi)| \\
&+ \sum_{\substack{|K_1|+|K_2|\leqslant k-1 \\ X_1,X_2,Y\in\mathbb{L}}} |D_Y(D_{X_1}^{K_1}\phi)| \cdot |D_{\underline{L}}(D_{X_2}^{K_2}\phi)| ,
\end{aligned}
$$

(252)
$$
\begin{aligned}
|A\!\!\!/| \lesssim &\sum_{\substack{|K|=k \\ X\in\mathbb{L}}} \tau_+ \,|\frac{\phi}{\tau_+}| \cdot |D\!\!\!/(D_X^K\phi)| \\
&+ \sum_{\substack{|K_1|+|K_2|\leqslant k-1 \\ X_1,X_2,Y\in\mathbb{L}}} |D_Y(D_{X_1}^{K_1}\phi)| \cdot |D\!\!\!/(D_{X_2}^{K_2}\phi)| .
\end{aligned}
$$

For the first term in each of the above sums, it suffices to merely recall the designations (217)–(218) and the definition of the parity weights $\tau_{(a)}$ to achieve the bounds (246)–(248). To achieve these bounds for the second sum in (250)–(252) above, it

suffices to show the symbolic bounds:

$$\sum_{\substack{|K_1|\leqslant l \\ X_1,Y\in\mathbb{L}}} |D_Y(D_{X_1}^{K_1}\phi)| \;\lesssim\; \sum_{a+b=0} \tau_{(a)}\, \mathcal{L}_{\mathbb{L}}^l \Psi_{(b)}\;.$$

This last line follows from a straight forward application of the bounds:

$$(253) \qquad |X^L|\,,\; |X^A| \;\lesssim\; \tau_+\,, \qquad |X^{\underline{L}}| \;\lesssim\; \tau_-\,, \qquad X\in\mathbb{L}\,,$$

which follows from an inspection of the identities (57), together with the designations (217)–(218) and the weight definition (236).

We now move on to dealing with the B portion of identity (249). This boils down to computing the commutator actions $[\mathcal{L}_X^{\mathbb{C}\,K},D]\phi$. By a repeated use of the formula (40), and the fact that $\mathbb{L}$ is a Lie algebra (so all its commutators have constant coefficients), we easily have the bounds:

$$(254) \qquad |B_\alpha| \;\lesssim\; \sum_{\substack{|K_1|+|K_2|+|K_3|\leqslant|I|-1 \\ X_1,X_2,X_3,Y\in\mathbb{L}}} |(i_Y\mathcal{L}_{X_1}^{K_1}F)_\alpha|\cdot|D_{X_2}^{K_2}\phi|\cdot|D_{X_3}^{K_3}\phi|\;.$$

We now use the fact that either $|K_2|\leqslant k-2$ or $|K_3|\leqslant k-2$, which comes from the restriction given above on their sum, to employ the L^∞ estimate for $D_X^K\phi$ contained in (222) to bound the second factor above as follows:

$$|D_{X_2}^{K_2}\phi|\cdot|D_{X_3}^{K_3}\phi| \;\lesssim\; \mathcal{L}_{\mathbb{L}}^{\max\{|K_2|,|K_3|\}}\Psi_{(0)}\cdot\|\,\mathcal{L}_{\mathbb{L}}^{k-2}\Psi\,\|_{L^\infty[0,T](s,\gamma,\epsilon)}\;.$$

Therefore, to achieve the right hand side of (246)–(248) for the B term, it suffices to show the bounds:

$$(255) \qquad \sum_{\substack{|K|\leqslant l \\ X,Y\in\mathbb{L}}} |(i_Y\mathcal{L}_X^K F)_L| \;\lesssim\; \sum_{a+b=1} \tau_{(a)}\left(\mathcal{L}_{\mathbb{L}}^l\Psi_{(b)}+\mathcal{L}_{\mathbb{L}}^l Q_{(b)}\right)\;,$$

$$(256) \qquad \sum_{\substack{|K|\leqslant l \\ X,Y\in\mathbb{L}}} |(i_Y\mathcal{L}_X^K F)_{\underline{L}}| \;\lesssim\; \sum_{a+b=-1} \tau_{(a)}\left(\mathcal{L}_{\mathbb{L}}^l\Psi_{(b)}+\mathcal{L}_{\mathbb{L}}^l Q_{(b)}\right)\;,$$

$$(257) \qquad \sum_{\substack{|K|\leqslant l \\ X,Y\in\mathbb{L}}}\sum_{A} |(i_Y\mathcal{L}_X^K F)_A| \;\lesssim\; \sum_{a+b=0} \tau_{(a)}\left(\mathcal{L}_{\mathbb{L}}^l\Psi_{(b)}+\mathcal{L}_{\mathbb{L}}^l Q_{(b)}\right)\;.$$

Splitting $\mathcal{L}_X^K F = \mathcal{L}_X^K\widetilde{F}+\mathcal{L}_X^K\overline{F}$, it suffices to do this calculation for the $\widetilde{F}$ portion of things. The computation for the charge field $\overline{F}$ is identical. Since this is an abstract counting argument for an arbitrary two form, we can drop the Lie derivatives and the tilde notation. Making now the identifications:

$$(258) \qquad L \;\Leftrightarrow\; (1)\,, \qquad \underline{L} \;\Leftrightarrow\; (-1)\,, \qquad A,B \;\Leftrightarrow\; (0)\,,$$

we see that in the groupings (217)–(218), the component $F_{\alpha\beta}$ is put in the set that corresponds to the sum of α and β as they range over the null frame $\{L,\underline{L},e_A\}$. Using this observation in conjunction with the bound (253) and the weight definition (236), it is immediate that the estimates (255)–(257) are simply a translation of the contractions $X^\beta F_{\alpha\beta}$, computed in the null frame, into the more simple numerical parity sum form. This completes the proof of the estimate (244). $\square$

To complete the proof of the bootstrapping estimate (242) for the electro-magnetic field $F_{\alpha\beta}$, it suffices to be able to bound the initial data type norms for the current vector J given on the right hand side of (107) in terms of the energy (240). To do this, it is enough to show that:

Proposition 8.3 (Initial data bounds for the system (17))**.** *Let $2 \leqslant k$ be a given level of regularity, and assume that we are given parameters $0 < \gamma$, and $\frac{1}{2} < s$, and $s + \gamma < \frac{3}{2}$. Let (F, ϕ) be a solution to the system (17) with initial data $(E, H, \phi_0, \dot{\phi}_0)$. Then there exists a constant C_{k,s_0} such that if one first assumes the smallness condition:*

$$(259) \qquad\qquad E_k^{(s,\gamma)}(0)[F, \phi] \;\leqslant\; C_{k,s,\gamma,\epsilon}^{-1} \;,$$

then one has the initial bounds:

(260)
$$\sum_{|I| \leqslant k} \| (1+r)^{s+\gamma+|I|} \nabla_x^I J_0(0) \|_{L_x^{\frac{6}{5}}}^2 \;\; + \sum_{|I| \leqslant k-1} \| (1+r)^{s+\gamma+|I|+1} \nabla_{t,x}^I J(0) \|_{L_x^2}^2$$
$$\lesssim \; \left[\; E_k^{(s,\gamma)}(0)[F, \phi] \;\right]^2 \;.$$

Proof of the estimate (260). We begin by bounding the first term on the left hand side of (260). At time $t = 0$ we directly compute that in terms of the initial data:

$$(261) \qquad\qquad \sum_{|I|=l} |\nabla_x^I J_0| \;\lesssim\; \sum_{|K_1|+|K_2|=l} |\underline{D}^{K_1} \phi_0| \cdot |\underline{D}^{K_2} \dot{\phi}_0| \;.$$

This, combined with the two L^∞ estimates:

$$(262) \qquad (1+r)^{s+\gamma+\frac{1}{2}} |\phi_0| \;\lesssim\; \sum_{|I| \leqslant 1} \| (1+r)^{s+\gamma+|I|} \underline{D}^I \underline{D} \phi_0 \|_{L^2(\mathbb{R}^3)} \;,$$

$$(263) \qquad (1+r)^{s+\gamma+\frac{3}{2}} |\dot{\phi}_0| \;\lesssim\; \sum_{|I| \leqslant 2} \| (1+r)^{s+\gamma+|I|} \underline{D}^I \dot{\phi}_0 \|_{L^2(\mathbb{R}^3)} \;,$$

which in turn follow for the estimates (188)–(189), achieves the bound (260) for this term by using the L^2–L^3 Hölder inequality and directly integrating the L^3 estimate using (262)–(263) and the fact that $\frac{1}{2} < s + \gamma$.

To estimate the second term on the left hand side of (260), we first expand the derivatives $\nabla_{t,x} J$ as we did in line (261) above. Doing this, and using the L^∞ estimate:

(264)
$$\sum_{\substack{|K| \leqslant k-1 \\ X \in \{\partial_\alpha\}}} (1+r)^{1+|K|} |D_X^K \phi| \;\lesssim\; \sum_{\substack{1 \leqslant |I_1|+|I_2| \leqslant k-1 \\ Y \in \{\partial_i\}, X \in \{\partial_\alpha\}}} \| (1+r)^{s+\gamma-1+|I_1|+|I_2|} D_Y^{I_1} D_X^{I_2} \phi \|_{L^2(\mathbb{R}^3)} \;,$$

which follows from the condition $\frac{1}{2} < s + \gamma$ as well as the estimate (188) applied to functions of the form $(1 + r)^{|K|} D_X^K \phi$, we are reduced to needing to show the general estimate:

$$(265) \qquad\qquad \sum_{\substack{1 \leqslant |I| \leqslant k+1 \\ X \in \{\partial_\alpha\}}} \| (1+r)^{s+\gamma-1+|I|} D_X^I \phi \|_{L^2(\mathbb{R}^3)}^2 \;\lesssim\; E_k^{(s,\gamma)}(0)[F] \;,$$

holds, provided that one first assumes the smallness condition (259) as well as the constraint $2 \leqslant k$ are in effect. It is clear that the work in showing (265) is simply a matter of controlling the commutator of the derivatives D_t and D_i. The simplest way to do this is to fix a value for k and then to induct on the number of time derivatives on the left hand side of this estimate. If this number is zero, then the claim is obvious, so we may now fix it at some value $1 \leqslant l \leqslant k+1$. We would now like to prove the bounds:

$$(266) \qquad \| (1+r)^{s+\gamma-1+|I|} D_X^{K_1} D_t D_Y^{K_2} \phi \|_{L^2(\mathbb{R}^3)}^2 \lesssim E_k^{(s,\gamma)}(0)[F] \, ,$$

where $|K_1| + |K_2| \leqslant |I| - 1$ and $X \in \{\partial_\alpha\}$ as well as $Y \in \{\partial_i\}$, provided that the same estimate holds whenever the D_t is replaced by one of D_i. In this regard, it will be useful for us to use the following notation: If K is a multiindex, then we denote by $|K^0|$ the number of time derivatives in D_X^K with $X \in \{\partial_\alpha\}$. In terms of this, we by design have that $|K_1^0| \leqslant l-1$ in (266) above. Now, computing multiple commutators, we see have the pointwise bound:

$$(267) \quad |D_X^{K_1} D_t D_Y^{K_2} \phi| \quad \lesssim \quad |D_X^{K_1} D_Y^{K_2} D_t \phi|$$
$$+ \sum_{\substack{|I_1|+|I_2|=|K_2|-1 \\ X \in \{\partial_\alpha\} \\ Y_1, Y_2, Z \in \{\partial_i\}}} \left| D_X^{K_1} D_{Y_1}^{I_1} [D_t, D_Z] D_{Y_2}^{I_2} \phi \right| \, .$$

We estimate the second term in the above expression first. Substituting this into the left hand side of (266), and expanding out the commutator in terms of the electric field E_i, distributing the derivatives according to the covariant Leibnitz rule, using the field equations (10) to remove time derivatives from E, and splitting result along the Hodge decomposition $E = E^{df} + E^{cf}$ we have the bound:

$$(268)$$

$$\sum_{\substack{|K_1^0| \leqslant l-1 \\ |K_2|+|I_1|+|I_2| \leqslant k-1 \\ X \in \{\partial_\alpha\} \\ Y_1, Y_2, Z \in \{\partial_i\}}} \| (1+r)^{s+\gamma+1+|I_1|+|I_2|} D_X^{K_1} D_{Y_1}^{I_1} [D_t, D_Z] D_{Y_2}^{I_2} \phi \|_{L^2(\mathbb{R}^3)} \, ,$$

$$\lesssim \sum_{\substack{|I_1|+|I_2| \leqslant k-1 \\ |I_2^0| \leqslant l-1 \\ X \in \{\partial_\alpha\}}} \left(\| (1+r)^{s+\gamma+1+|I_1|+|I_2|} (\nabla_x^{I_1} E^{df}) \cdot (D_X^{I_2} \phi) \|_{L^2(\mathbb{R}^3)} \right.$$

$$+ \left. \| (1+r)^{s+\gamma+1+|I_1|+|I_2|} (\nabla_x^{I_1} E^{cf}) \cdot (D_X^{I_2} \phi) \|_{L^2(\mathbb{R}^3)} \right)$$

$$+ \sum_{\substack{|I_1|+|I_2| \leqslant k-2 \\ |I_1^0|+|I_2^0| \leqslant l-1 \\ X \in \{\partial_i\}}} \| (1+r)^{s+\gamma+2+|I_1|+|I_2|} (\nabla_{t,x}^{I_1} J) \cdot (D_X^{I_2} \phi) \|_{L^2(\mathbb{R}^3)} \, ,$$

$$= A + B + C \, .$$

We now bound each of the three terms on the right hand side above separately. For the term A, we are done by induction after a use of the L^∞ estimate (264). To bound the B term, we add and subtract off the term $\nabla \frac{1}{2\pi r^2} \cdot \chi^+(r-2)$ from the curl-free component E^{cf}. Then using the estimate (112), and the bounds we have already established for the first term on the left hand side of (260) in conjunction

with an application of the estimate (264), this task is reduced to showing that:

$$\sum_{\substack{|I_2|\leqslant k-1 \\ |I_2^0|\leqslant l-1 \\ X\in\{\partial_\alpha\}}} \| (1+r)^{s+\gamma-1+|I_2|} D_X^{I_2}\phi \|_{L^2(\mathbb{R}^3)}^2 \ \lesssim\ E_k(0)^{(s,\gamma)}[\phi,F] \ .$$

But this last estimate follows from our inductive hypothesis together with a use of Poincare estimate (190) to handle the zero derivative term.

Our next task here is to prove the (inductive) bound (260) for the C term on the right hand side of line (268) above. As before, we use the estimate (264) to deal with the factor involving $(1+r)^{1+|I_2|}D_X^{I_2}\phi$. Doing this, and using the continuity equation (16) and the smallness condition, we arrive at the bound (as long as $0 < l-1$):

$$C \ \lesssim\ \sum_{\substack{|I_1|\leqslant k-2 \\ |I_1^0|\leqslant l-2}} \| (1+r)^{s+\gamma+1+|I_1|} \nabla_{t,x}^{I_1} J \|_{L^2(\mathbb{R}^3)} \ .$$

We can now start on line (259) above, and repeat the argument up until this point to bound the right hand side of this last expression by induction.

To finish here, we only need to prove the (inductive) bound (260) for the first term on the left hand side of (267). If $|K_1^0| = 0$, then we are automatically done. If however, $1 \leqslant |K_1^0|$, we simply repeat the steps outlined above for the second term on the right hand side (267), but this time applied to the second term on the right hand side of the bound:

$$(269) \quad |D_X^{\widetilde{K}_1} D_t D_Y^{K_2} D_t\phi| \ \lesssim\ |D_X^{\widetilde{K}_1} D_Y^{K_2} D_t^2\phi|$$
$$+ \sum_{\substack{|I_1|+|I_2|=|K_2|-1 \\ X\in\{\partial_\alpha\} \\ Y_1,Y_2,Z\in\{\partial_i\}}} \left| D_X^{\widetilde{K}_1} D_{Y_1}^{I_1} [D_t, D_Z] D_{Y_2}^{I_2} D_t\phi \right| \ ,$$

where this time $|\widetilde{K}_1| = |K_1| - 1$ while $\widetilde{K}_1^0 = l - 2$. Notice that these conditions guarantee that no more than a total of $k - 1$ derivatives altogether, and no more than $l - 1$ time derivatives in particular can fall on ϕ in this second term, so we can still put it in L^∞ and use induction as we did for the second term on the right hand side of (267) above.

Finally, to deal with the first term on the right hand side of (269), we simply use the wave equation (2) which trades the derivatives D_t^2 for the covariant Laplacian $D^i D_i$. Having reduced the total number of time derivatives in this last expression, we are done by induction. This completes the proof of the estimate (260).   □

9. Differentiating the Equations II: Error Estimates for the complex scalar field ϕ

To wrap things up, we need to prove the boot-strapping estimate (242) for the ϕ portion of things. Using the energy estimate (130) for complex scalar fields, and using the assumption that $C_{k,s,\gamma,\epsilon}^{-1}$ in (243) is chosen so small that the L^∞ norm

term on the right hand side of (130) can be absorbed into the right hand side of that inequality, and using the estimates (190) and (265) above to deal with the terms involving the initial data, we see that it is enough to prove the following L^2 estimate for the commutator of covariant derivatives and the complex D'Lambertian:

Proposition 9.1 (Error bounds for the complex scalar field ϕ). *Let k be a given level of regularity, and assume that we are given parameters $0 < s, \gamma, \epsilon$ with the properties that $s \leqslant 1$, and $2\epsilon \leqslant s - \frac{1}{2}$, and $s + \gamma < \frac{3}{2}$. Let $0 < T$ be a given time parameter. Then there exists a constant $1 \leqslant C_{k,s,\gamma,\epsilon}$ depending only on k, s, ϵ, γ, such that if:*

$$(270) \qquad E_k^{(s,\gamma,\epsilon)}(0,T)[\Psi] \ \leqslant \ C_{k,s,\gamma,\epsilon}^{-1} \ ,$$

then one has the following weighted space-time estimate for commutators:

$$(271) \qquad \sum_{\substack{|I| \leqslant k \\ X \in \mathbb{L}}} \| \, \tau_+^s \tau_-^{\frac{1}{2}}(w)_{\gamma,\epsilon}^{\frac{1}{2}} \left[\Box^{\mathbb{C}}, D_X^I \right] \phi \, \|_{L^2[0,T]} \ \lesssim \ \sum_{l=2}^{7} \left[E_k^{(s,\gamma,\epsilon)}(0,T)[\Psi] \right]^l \ ,$$

where $w_{\gamma,\epsilon}$ is the weight function from line (75) above.

Proof of estimate (271). As in the previous section, we may assume that $C_{k,s,\gamma,\epsilon}^{-1}$ is chosen small enough that the right hand side of (222) containing the extra L^∞ norm can be absorbed into the left hand side. Taking this into account, it suffices to be able to show:

$$(272) \qquad \sum_{\substack{|I| \leqslant k \\ X \in \mathbb{L}}} \| \, \tau_+^s \tau_-^{\frac{1}{2}}(w)_{\gamma,\epsilon}^{\frac{1}{2}} \left[\Box^{\mathbb{C}}, D_X^I \right] \phi \, \|_{L^2[0,T]} \ \lesssim$$

$$E_k^{(s,\gamma,\epsilon)}(0,T)[\Psi] \cdot |\!|\!| \, \mathcal{L}_{\mathbb{L}}^{k-2} \Psi \, |\!|\!|_{L^\infty[0,T](s,\gamma,\epsilon)}^2 \left(1 + |\!|\!| \, \mathcal{L}_{\mathbb{L}}^{k-2} \Psi \, |\!|\!|_{L^\infty[0,T](s,\gamma,\epsilon)}^2 \right)^2 \ .$$

Our first step in this process is to expand the commutator $\left[\Box^{\mathbb{C}}, D_X^I \right]$. This can be done through the following multiindex identity:

$$(273) \qquad [\Box^{\mathbb{C}}, D_X^I]\phi \ = \ \sum_{\substack{K_1 + K_2 + K_3 = I \\ |K_2| = 1 \\ X_1, X_2, X_3 \in \mathbb{L}}} D_{X_1}^{K_1} [\Box^{\mathbb{C}}, D_{X_2}^{K_2}] D_{X_3}^{K_3} \phi \ .$$

Now, using the single derivative commutator formula (43) in conjunction with a repeated application the two identities:

$$\begin{aligned} D_X \left(Y^\alpha F_{\alpha\beta} D^\beta \phi \right) \ = \ & [X, Y]^\alpha F_{\alpha\beta} D^\beta \phi \\ & + \ Y^\alpha \mathcal{L}_X F_{\alpha\beta} D^\beta \phi \ + \ Y^\alpha F_{\alpha\beta} D^\beta (D_X \phi) \\ & + \ Y^\alpha F_{\alpha\beta} \, {}^{(X)}\pi^{\beta\gamma} D_\gamma \phi \ + \ \sqrt{-1} Y^\alpha F_{\alpha\beta} F^\beta_{\ \gamma} X^\gamma \cdot \phi \ , \end{aligned}$$

and:

$$\begin{aligned} D_X \left(\nabla^\beta (Y^\alpha F_{\alpha\beta}) \cdot \phi \right) \ = \ & \nabla^\beta ([X,Y]^\alpha F_{\alpha\beta}) \cdot \phi \ + \ \nabla^\beta (Y^\alpha \mathcal{L}_X F_{\alpha\beta}) \cdot \phi \\ & + \ \nabla^\beta (Y^\alpha F_{\alpha\beta}) \cdot D_X \phi \ + \ {}^{(X)}\pi^{\beta\gamma} \nabla_\gamma (Y^\alpha F_{\alpha\beta} \cdot \phi \ , \end{aligned}$$

which in turn follow from the geometric formulas (42), (40), and (35), we arrive at the following pointwise bound for the left hand side of (273) above:

$$(274) \quad \sum_{\substack{|I| \leqslant k \\ X \in \mathbb{L}}} |[\Box^{\mathbb{C}}, D_X^I] \phi| \ ,$$

$$\lesssim \sum_{\substack{|K_1| + |K_2| \leqslant k-1 \\ X_1, X_2, Y \in \mathbb{L}}} \left(\left| Y^\beta (\mathcal{L}_{X_1}^{K_1} J_\beta) \right| \cdot \left| D_{X_2}^{K_2} \phi \right| \right.$$

$$+ \left| 2 Y^\alpha (\mathcal{L}_{X_1}^{K_1} F_{\alpha\beta}) \cdot D^\beta (D_{X_2}^{K_2} \phi) - \nabla^\alpha (Y^\beta) \cdot (\mathcal{L}_{X_1}^{K_1} F_{\alpha\beta}) \cdot (D_{X_2}^{K_2} \phi) \right| \right)$$

$$+ \sum_{\substack{|K_1| + |K_2| + |K_3| \leqslant k-2 \\ X_1, X_2, X_3, Y_1, Y_2 \in \mathbb{L}}} \left| Y_1^\alpha (\mathcal{L}_{X_1}^{K_1} F_{\alpha\beta}) \right| \cdot \left| Y_2^\gamma (\mathcal{L}_{X_2}^{K_2} F_{\beta\gamma}) \right| \cdot \left| D_{X_3}^{K_3} \phi \right| \ ,$$

$$= A + B + C \ .$$

Our goal in now to prove the bound (272) for each of these three terms. We do this separately and in order. For the A term above, taking in to account the abstract parity estimates (237)–(238), our task is reduced to showing the symbolic bound:

$$(275) \quad A \lesssim$$
$$\sum_{\substack{l+m=k-1 \\ a+b+c+d=0}} \tau_+^{-1} \, \tau_{(a)} \tau_{(b)} \, \left(\mathcal{L}_{\mathbb{L}}^l \Psi_{(c)} + \mathcal{L}_{\mathbb{L}}^l Q_{(c)} \right) \cdot \mathcal{L}_{\mathbb{L}}^m \Psi_{(d)} \cdot \left(1 + ||| \, \mathcal{L}_{\mathbb{L}}^{k-2} \Psi \, |||_{L^\infty[0,T](s,\gamma,\epsilon)} \right)^2 \ .$$

Expanding out the contraction in β on the right hand side of (275) above, and using the identifications (258), we have the straight forward bound:

$$(276) \qquad \left| Y^\beta \mathcal{L}_{X_1}^{K_1} J_\beta \right| \lesssim \sum_{a+b=0} \tau_{(a)} \left| \mathcal{L}_{X_1}^{K_1} J_{(b)} \right| \ .$$

To bound the $\mathcal{L}_{X_1}^{K_1} J_{(b)}$ term, we can simply use lines (246)–(248) above. However, it will be necessary for us to use the following refinement of those bounds, which can easily be checked by recalling lines (250)–(254):

$$\sum_{\substack{|K_1| \leqslant l \\ X_1 \in \mathbb{L}}} \left| \mathcal{L}_{X_1}^{K_1} J_{(a)} \right| \lesssim$$

$$\sum_{\substack{b+c=a \\ l_1+l_2=l}} \tau_+^{-1} \, \tau_{(b)} \, \left(\mathcal{L}_{\mathbb{L}}^{l_1} \Psi_{(c)} + \mathcal{L}_{\mathbb{L}}^{l_1} Q_{(c)} \right) \cdot \sum_{\substack{|K_1| \leqslant l_2 \\ X_1 \in \mathbb{L}}} |D_{X_1}^{K_1} \phi| \cdot \left(1 + ||| \, \mathcal{L}_{\mathbb{L}}^{l-2} \Psi \, |||_{L^\infty[0,T](s,\gamma,\epsilon)} \right) \ .$$

Expanding this into (276) above, and tacking on the extra factor of $D_{X_2}^{K_2} \phi$ while using the bound:

$$\sum_{\substack{|K_1| \leqslant l_2, |K_2| \leqslant m \\ l_2+m \leqslant k-1 \\ X_1, X_2 \in \mathbb{L}}} |D_{X_1}^{K_1} \phi| \cdot |D_{X_2}^{K_2} \phi| \lesssim \mathcal{L}_{\mathbb{L}}^{\max\{l_2, m\}} \Psi_{(0)} \cdot ||| \, \mathcal{L}_{\mathbb{L}}^{k-2} \Psi \, |||_{L^\infty[0,T](s,\gamma,\epsilon)} \ ,$$

noting that by design we have $l_1 + \max\{l_2, m\} \leqslant k-1$, we see that we have achieved (275).

We now move on to bounding the B term in line (274) above. Here we will prove the symbolic bound:

$$(277) \qquad B \lesssim \sum_{\substack{a+b+c=0 \\ l+m \leqslant k-1}} \tau_{(a)} \left(\mathcal{L}_{\mathbb{L}}^l \Psi_{(b)} + \mathcal{L}_{\mathbb{L}}^l Q_{(b)} \right) \cdot \mathcal{L}_{\mathbb{L}}^m \Psi_{(c)} .$$

Through an application of estimate (237), this will prove the bound (272) for this portion of things. To simplify matters, it suffices to prove these bounds with $l = m = 0$. The estimate with the derivatives is just a matter of notation. With this in mind, it suffices to show that both:

$$(278) \qquad \left| Y^\alpha (F_{\alpha\beta}) \cdot \frac{1}{r} D^\beta (r\phi) \right| \lesssim \sum_{a+b+c=0} \tau_{(a)} \left(Q_{(b)} + \Psi_{(b)} \right) \cdot \Psi_{(c)} ,$$

(279)

$$\left| \left(\frac{2}{r} \nabla^\alpha(r) \cdot X^\beta - \nabla^\alpha(X^\beta) \right) \cdot F_{\alpha\beta} \right| \cdot |\phi| \lesssim \sum_{a+b+c=0} \tau_{(a)} \left(Q_{(b)} + \Psi_{(b)} \right) \cdot \Psi_{(c)} .$$

To turn (278) into the bound (277) (after adding derivatives) is a simple matter of applying the definitions (217), (236), and the weight bounds (253). Notice that since this is a full contraction, its parity weight must be zero. Also, note that the term $\frac{1}{r} D_{\underline{L}}(r\phi)$ can be treated as a $\Psi_{(-1)}$ term on account of the symbolic bounds:

$$\Psi_{(0)} \lesssim \Psi_{(-1)} .$$

Therefore, show (277), it now remains to prove the bounds (279). For the most part, this is a simple matter of noticing that from the formulas (59) (or by homogeneity!) each component of the covariant derivatives on the left hand side of (279) satisfies the bounds:

$$\left| \frac{2}{r} \nabla^\alpha X^\beta \right| + |\nabla^\alpha X^\beta| \lesssim 1 .$$

Thus, if the parity weight of $F_{\alpha\beta}$ is (0) or (1), we can pass to the left hand side after multiplication by $\tau_+ \cdot \tau_+^{-1}$. The only place this general procedure does not work is when $F_{\alpha\beta}$ has weight (-1). In this case, to pass to the right hand side we must pick up an extra factor of τ_0. This indeed turns out to be the case and is perhaps the most striking structural property of the commutator (43) as was pointed out in [15]. What we need to show is that:

$$(280) \qquad \left| \frac{2}{r} \nabla^{\underline{L}}(r) \cdot X^A - \nabla^{\underline{L}}(X^A) + \nabla^A(X^{\underline{L}}) \right| \lesssim \tau_0 ,$$

for each $X \in \mathbb{L}$ in the region $t < 2r$. This follows from a direct use of the identities (59). Notice that for the case $X \in \{\partial_\alpha, S\}$, the above sum either vanishes or is (τ_+^{-1}). Thus, the main thing to check is that (280) holds when $X \in \{\Omega_{ij}, \Omega_{i0}\}$. This in turn follows easily from the formulas:

$$\nabla^{\underline{L}}(\Omega_{ij}^A) = -\frac{1}{2r} \, \Omega_{ij}^A , \qquad\qquad \nabla^A(\Omega_{ij}^{\underline{L}}) = \frac{1}{2r} \, \Omega_{ij}^A ,$$

$$\nabla^{\underline{L}}(\Omega_{0i}^A) = -\frac{1}{2} \, \omega_i^A , \qquad\qquad \nabla^A(\Omega_{0i}^{\underline{L}}) = \frac{1}{2} \, \omega_i^A .$$

To finish our proof of estimate (272), we only need to show this bound for the C term on line (274) above. Writing $F = \widetilde{F} + \overline{F}$ and expanding out this term into parity notation, we have the symbolic bound:

$$
C \lesssim
$$
$$
\sum_{\substack{l+m=k-1 \\ a+b+c+d=0}} \tau_+^{-1}\ \tau_{(a)}\tau_{(b)}\ \left(\mathcal{L}_{\mathbb{L}}^l \Psi_{(c)} + \mathcal{L}_{\mathbb{L}}^l Q_{(c)}\right) \cdot \mathcal{L}_{\mathbb{L}}^m \Psi_{(d)} \cdot \left(1 + |||\,\mathcal{L}_{\mathbb{L}}^{k-2}\Psi\,|||_{L^\infty[0,T](s,\gamma,\epsilon)}\right)\ ,
$$
$$
+ \sum_{\substack{l+m=k-1 \\ a+b+c+d=0}} \tau_+\ \tau_{(a)}\tau_{(b)}\ \mathcal{L}_{\mathbb{L}}^{k-1} Q_{(c)} \cdot \mathcal{L}_{\mathbb{L}}^{k-1} Q_{(d)} \cdot \mathcal{L}_{\mathbb{L}}^{k-1}\Psi_{(0)}\ ,
$$
$$
=\ C_1 + C_2\ .
$$

To prove the bound (272) for the term C_1 is a simple matter of referring to the abstract parity estimate (237). Thus, we are left with bounding the term C_2. This reduces directly to a special case of the estimate (238) after an application of the L^∞ bound (223) which easily implies:

$$
\sum_{\substack{l+m=k-1 \\ a+b+c+d=0}} \tau_+\ \tau_{(a)}\tau_{(b)}\ \mathcal{L}_{\mathbb{L}}^{k-1} Q_{(c)} \cdot \mathcal{L}_{\mathbb{L}}^{k-1} Q_{(d)} \lesssim
$$
$$
\tau_-^{-1}\ \tau_{(0)}\tau_{(0)}\ \mathcal{L}_{\mathbb{L}}^{k-1} Q_{(0)} \cdot |||\,\mathcal{L}_{\mathbb{L}}^{k-2}\Psi\,|||_{L^\infty[0,T](s,\gamma,\epsilon)}\ .
$$

This completes the proof of estimate (272) and hence the proof of (271). □

Appendix

This appendix contains the proofs of several more or less standard weighted Sobolev type estimates which are used at various places in the main paper. In particular in Sections 3 and 5–6. We make no claim to originality, rather our purpose here is to have things stated in the precise form in which we find them convenient to use in our work. The first such estimate is a simple weighted version of the usual $L^2 \hookrightarrow L^6$ embedding:

Lemma 9.2. *Let φ be a real valued test function on $\mathbb{R}^3$, and set:*

$$
(281) \qquad\qquad q\ =\ \int_{\mathbb{R}^3} \varphi\ dx\ ,
$$

the average of φ. Let δ be given such that $\frac{1}{2} < \delta < \frac{2}{3}$. Then the following weighted integral inequality holds:

$$
(282) \qquad\qquad \int_{\mathbb{R}^3} r^{2\delta} \left| \nabla \left(\frac{1}{\Delta}\varphi + \frac{q}{4\pi r} \right) \right|^2 dx\ \lesssim\ \|\, r^\delta\, \varphi\, \|^2_{L^{\frac{6}{5}}}\ .
$$

In the above estimate, the implicit constants depend on δ.

Remark 9.3. In effect, what estimate (282) shows is that is possible to commute the weight r^δ past the Riesz operator $\nabla\Delta^{-1}$ so long as one first subtracts off the leading order term in the asymptotic expansion of $\Delta^{-1}\varphi$. This is:

$$(283) \qquad \nabla_i \frac{1}{\Delta}\varphi \;\sim\; \frac{q\cdot\omega_i}{4\pi r^2} \;.$$

That such a correction is necessary is clear from the range of weights we are considering; for $\frac{1}{2} < \delta$, a function with this behavior at infinity in general could not possible be in L^2 weighted by r^δ. The idea, of course, is that once the right hand side of (283) is subtracted off from $\nabla\Delta^{-1}\varphi$, the resulting function will decay enough to be in r^δ weighted L^2 as long as the appropriately weighted $L^{\frac{6}{5}}$ norm of φ is bounded. However, there is a limit to how much weight one can expect to apply this way. This is because one can further expand (283) to include higher order terms:

$$(284) \qquad \nabla_i \frac{1}{\Delta}\varphi \;\sim\; -\nabla_i\left(\frac{q}{4\pi r^2} + 2\sum_k \frac{q_k\cdot\omega_k}{4\pi r^2} + \{higher\} \right) \;,$$

where:

$$q_k \;=\; \int_{\mathbb{R}^3} y_k\,\varphi(y)\;dy \;,$$

are the higher moments of φ. Without the additional vanishing of these other quantities, it is clear that one cannot put $r^\delta\,\nabla\Delta^{-1}\varphi$ in L^2 for $\frac{3}{2} \leqslant \delta$. (284) is sometimes referred to as the *multipole* expansion of $\nabla\frac{1}{\Delta}\varphi$. Since we are only interested in the decay of the initial data (18) modulo charge of the order no greater than $O(r^{-3})$, the first term in (284) will be the only one which concerns us in this work.

Proof of (282). Our first step here is to simply integrate by parts several times on the left hand side of (282). This yields the identity:

$$
\begin{aligned}
(285) \quad \text{(L.H.S.)}(282) \;=\; &\delta(2\delta+1)\int_{\mathbb{R}^3} r^{2\delta-2}\,\left(\frac{1}{\Delta}\varphi + \frac{q}{4\pi r}\right)^2 dx \\
&- \int_{\mathbb{R}^3} r^{2\delta}\,\varphi\cdot\left(\frac{1}{\Delta}\varphi + \frac{q}{4\pi r}\right)\,dx \;.
\end{aligned}
$$

To bound the integral in the first term on the right hand side of (285) above, we use the definition (281) to compute:

$$\left|\int_{\mathbb{R}^3} r^{2\delta-2}\left(\frac{1}{\Delta}\varphi + \frac{q}{4\pi r}\right)^2 dx\right| \;\lesssim\; \int_{\mathbb{R}^3} r^{2\delta-2}\left(\int_{\mathbb{R}^3} |\frac{1}{|x-y|} - \frac{1}{|y|}|\,|\varphi(y)|\,dy\right)^2 dx\;.$$

We now split cases depending on whether $\frac{1}{2} < \delta \leqslant 1$ or $1 < \delta < \frac{3}{2}$. In the first case, we use the bound:

$$|\frac{1}{|x-y|} - \frac{1}{|y|}| \;\lesssim\; |y|^\delta\left(\frac{1}{|x|^\delta\,|x-y|} + \frac{1}{|x|\,|x-y|^\delta}\right) \;.$$

This reduces our work to estimating the two integrals:

$$A \;=\; \int_{\mathbb{R}^3} \frac{1}{|x|^2} \left(\int_{\mathbb{R}^3} \frac{1}{|x-y|} \, |y|^\delta \, |\varphi(y)| \, dy \right)^2 \, dx \,, \qquad \frac{1}{2} \;<\; s \;\leqslant\; 1 \,,$$

$$B \;=\; \int_{\mathbb{R}^3} \frac{1}{|x|^{4-2\delta}} \left(\int_{\mathbb{R}^3} \frac{1}{|x-y|^\delta} \, |y|^\delta \, |\varphi(y)| \, dy \right)^2 \, dx \,, \qquad \frac{1}{2} \;<\; s \;\leqslant\; 1 \,.$$

In the second case, we simply use the bound:

$$\left| \frac{1}{|x-y|} - \frac{1}{|y|} \right| \;\leqslant\; \frac{|y|}{|x| \, |x-y|} \,.$$

This reduces us to bounding the integral:

$$C \;=\; \int_{\mathbb{R}^3} \frac{1}{|x|^{4-2\delta}} \left(\int_{\mathbb{R}^3} \frac{1}{|x-y| \, |y|^{1-\delta}} \, |y|^\delta \, |\varphi(y)| \, dy \right)^2 \, dx \,, \qquad \frac{1}{2} \;<\; s \;\leqslant\; 1 \,.$$

In all cases, the bound we wish to prove is:

$$(286) \qquad\qquad A, B, C, \;\lesssim\; \| \, r^\delta \, \varphi \, \|_{L^{\frac{6}{5}}} \,.$$

This last estimate follows from several different instances of the generalized fractional integration Lemma 9.4 below. Notice that in the case of integral A we have (notation of Lemma 9.4) $\alpha = \beta = 1$, and $\gamma = 0$. In case of integral B we have $\alpha = 2 - \delta$, $\beta = \delta$, and $\gamma = 0$. In the case of integral C above we have $\alpha = 2 - \delta$, $\beta = 1$, and $\gamma = \delta - 1$. In all cases we have that $p = p' = 2$ and $q = \frac{6}{5}$ and $\alpha + \beta + \gamma = 2$ so the scaling condition (290) is satisfied. Also, note that in each case we have that $\alpha < \frac{3}{2}$ and $\gamma < \frac{1}{2}$ so the "gap" condition (291) is satisfied. This completes the proof of the bound:

$$(287) \qquad\qquad \left| \int_{\mathbb{R}^3} r^{2\delta-2} \left(\frac{1}{\Delta}\varphi + \frac{q}{4\pi r} \right)^2 \, dx \right| \;\lesssim\; \| \, r^\delta \, \varphi \, \|_{L^{\frac{6}{5}}} \,.$$

It remains to bound the second term on the right hand side expression (285) above. By Hölders inequality we have that:

$$\left| \int_{\mathbb{R}^3} r^{2\delta} \, \varphi \cdot \left(\frac{1}{\Delta}\varphi + \frac{q}{4\pi r} \right) \, dx \, \right| \;\leqslant\; \| \, r^\delta \, \varphi \, \|_{L^{\frac{6}{5}}} \cdot \| \, r^\delta \left(\frac{1}{\Delta}\varphi + \frac{q}{4\pi r} \right) \|_{L^6} \,.$$

By the usual $L^2 \hookrightarrow L^6$ Sobolev embedding, the Leibnitz rule, and the triangle inequality, the second factor on the right hand side above can be bounded by:

$$\| \, r^\delta \left(\frac{1}{\Delta}\varphi + \frac{q}{4\pi r} \right) \|_{L^6} \;\lesssim\; \| \, r^{\delta-1} \left(\frac{1}{\Delta}\varphi + \frac{q}{4\pi r} \right) \|_{L^2} + \| \, r^\delta \, \nabla \left(\frac{1}{\Delta}\varphi + \frac{q}{4\pi r} \right) \|_{L^2} \,.$$

Therefore, using the bound (287) above, we can estimate:

$$(288) \quad \left| \int_{\mathbb{R}^3} r^{2\delta} \, \varphi \cdot \left(\frac{1}{\Delta}\varphi + \frac{q}{4\pi r} \right) \, dx \, \right|$$

$$\lesssim\; \| \, r^\delta \, \varphi \, \|_{L^{\frac{6}{5}}} \cdot \left(\| \, r^\delta \, \varphi \, \|_{L^{\frac{6}{5}}} + \| \, r^\delta \, \nabla \big(\frac{1}{\Delta}\varphi + \frac{q}{4\pi r} \big) \|_{L^2} \right) \,.$$

Adding estimates (287) and (288) into the identity (285), dividing through by the quantity $\| \, r^\delta \, \nabla \big(\frac{1}{\Delta}\varphi + \frac{q}{4\pi r} \big) \|_{L^2}$ and resquaring, we have achieved the desired estimate (282). $\qquad\qquad \square$

To complete the proof of (282), we need to show the bound (286) for the integrals $A, B,$ and C above. This will be a consequence of the following generalization of the classical Hardy-Littlewood-Sobolev fractional integration theorem (see [11]):

Lemma 9.4 (General fractional integration lemma). *Let $1 < q < p < \infty$ and $0 \leqslant \alpha, \beta, \gamma$ be given parameters. Then the following integral estimate holds for (positive) test functions F and G on $\mathbb{R}^3$:*

$$(289) \qquad \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(x) \, \frac{1}{|x|^\alpha \, |x-y|^\beta \, |y|^\gamma} \, F(y) \, dxdy \; \lesssim \; \| \, G \, \|_{L^{p'}} \cdot \| \, F \, \|_{L^q} \; ,$$

where the following conditions hold on the various indices:

$$(290) \qquad 3 - (\alpha + \beta + \gamma) \;=\; 3(\frac{1}{q} - \frac{1}{p}) \; , \qquad\qquad (scaling) \; ,$$

$$(291) \qquad \alpha \;<\; 3(1 - \frac{1}{p}) \; , \;\; \gamma \;<\; 3(1 - \frac{1}{q}) \; , \qquad\qquad (\text{``gap''}) \; .$$

The implicit constant in the above estimate depends on $p, q, \alpha, \beta, \gamma$.

Proof of (289). By the restricted weak type version of the Marcinkiewicz interpolation theorem, it suffices to prove that:

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \chi_E(x) \, \frac{1}{|x|^\alpha \, |x-y|^\beta \, |y|^\gamma} \, \chi_F(y) \, dxdy \; \lesssim \; |E|^{\frac{1}{p'}} \cdot |F|^{\frac{1}{q}} \; ,$$

for measurable sets E and F where $1 < q < p < \infty$ and the conditions (290)–(291) hold. By the Riesz rearrangement inequality (see [11]), this is reduced to showing that:

$$(292) \quad \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\chi_E(x) \cdot |x|^{-\alpha})^* \, \frac{1}{|x-y|^\beta} \, (\chi_F(y) \cdot |y|^{-\gamma})^* \, dxdy \; \lesssim \; |E|^{\frac{1}{p'}} \cdot |F|^{\frac{1}{q}} \; ,$$

where f^* denotes the symmetric decreasing rearrangement[9] of the function f. We now compute that:

$$\begin{aligned}
(\chi_E(x) \cdot |x|^{-\alpha})^*(r) \;&=\; \int_0^\infty \chi^*_{\{\chi_E \cdot |x|^{-\alpha} > s\}}(r) \, ds \; , \\
&=\; \int_0^\infty \chi_{[0, |E \cap B(s^{-\frac{1}{\alpha}})|^{\frac{1}{3}})}(r) \, ds \; , \\
&\leqslant\; \chi_{[0, |E|^{\frac{1}{3}})}(r) \cdot \int_0^\infty \chi_{[0, 4\pi s^{-\frac{1}{\alpha}})}(r) \, ds \; , \\
&=\; (4\pi)^{-1} (\chi_E)^*(r) \cdot r^{-\frac{1}{\alpha}} \; ,
\end{aligned}$$

and similarly for $(\chi_F(x) \cdot |x|^{-\gamma})^*$. We now plug these formulas into the right hand side of (292) and apply the usual fractional integration theorem to yield:

$$(293) \quad \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\chi_E(x))^* \cdot |x|^{-\alpha} \, \frac{1}{|x-y|^\beta} \, (\chi_F(y))^* \cdot |y|^{-\gamma} \, dxdy$$
$$\lesssim \; \| \, (\chi_E(x))^* \cdot |x|^{-\alpha} \, \|_{L^{\theta_1}} \cdot \| \, (\chi_F(x))^* \cdot |x|^{-\gamma} \, \|_{L^{\theta_2}} \; ,$$

---

[9]Here we use a definition of f^* which does not include the usual normalization factor of 4π to avoid additional typesetting. This is $\chi^*_E(r) = \chi_{[0, |E|^{\frac{1}{3}})}$. Clearly this does not effect the use of rearrangements in the Riesz inequality.

where θ_1 and θ_2 are given by the expressions $\frac{1}{\theta_1} = \frac{1}{p'} + \frac{\alpha}{3}$ and $\frac{1}{\theta_2} = \frac{1}{q} + \frac{\gamma}{3}$ respectively. Notice that from the scaling condition (290) and some quick algebra we have that: $6 - \beta = 3(\frac{1}{\theta_1} + \frac{1}{\theta_2})$, which is the scaling condition for (293) to hold. Furthermore, notice that by the gap condition (291) we have that both $1 < \theta_1, \theta_2 < \infty$ which is also needed for (293) to hold in general. The proof of (292) is now accomplished by simply computing:

$$\| \, (\chi_E(x))^* \cdot |x|^{-\alpha} \, \|_{L^{\theta_1}}^{p'} \;=\; \Big(\int_0^{|E|^{\frac{1}{3}}} r^{2 - \alpha\theta_1} \; dr \Big)^{\frac{p'}{\theta_1}} \; ,$$
$$\lesssim \; |E|^{\frac{p'}{3\theta_1}(3 - \alpha\theta_1)} \; ,$$
$$= \; |E| \; .$$

Notice that the inequality follows because we automatically have that $\alpha\theta_1 < 3$. An identical computation shows that:

$$\| \, (\chi_F(x))^* \cdot |x|^{-\gamma} \, \|_{L^{\theta_2}}^{q} \;\lesssim\; |F| \; .$$

This completes the proof of (292) and therefore the proof of estimate (289). $\square$

The second main set of estimates we prove here are a localized version of the so called global Sobolev inequalities. These were first utilized in [8] to prove global existence and asymptotic behavior for general non-linear wave equations. The versions which we state here are sufficiently "atomic" to provide all of the L^∞ we will need in this paper. To discuss these, we shall employ the same notation as introduced in the beginning of Section 5. In particular, the notion of a dyadic cone distance (CD) shell and the associated $\tau_+(\mathcal{J}), \tau_-(\mathcal{J})$ notation.

Lemma 9.5 (Exterior global Sobolev estimate). *In the exterior region $1 \leqslant t < 2r$, let $\mathcal{J}$ be a given dyadic CD shell and let q be given such that $2 < q < \infty$. Then for test functions f the following estimates hold:*

$$(294) \qquad |\chi f(t, r, \omega)|^2 \;\lesssim\; \tau_+^{-\frac{4}{q}}(\mathcal{J}) \sum_{i<j} \| \, \Omega_{ij} \, \chi f(t, r) \, \|_{L^q(\mathbb{S}_r^2)}^2 \; ,$$

where χ is the cutoff on the dyadic CD shell $\mathcal{J}$. We also have that:

$$(295) \quad \| \, \chi f \, \|_{L_r^\infty(L^q(\mathbb{S}_r^2))} \;\lesssim\;$$
$$\tau_+^{-2(\frac{1}{2} - \frac{1}{q})}(\mathcal{J}) \, \tau_-^{-\frac{1}{2}}(\mathcal{J}) \Big(\| \, \chi f \, \|_{L^2} \;+\; \sum_{i<j} \| \, \Omega_{ij} \, \chi f \, \|_{L^2} \;+\; \| \, \tau_- \, \partial_r \, \chi f \, \|_{L^2} \Big) \; .$$

Proof of estimates (294)–(295). The proof of both of these is entirely standard as they are essentially just rescaled versions of the usual translation invariant Sobolev estimates. First off, (294) is just the Sobolev theorem on spheres $\mathbb{S}_r$. For the sake of completeness, we give a proof of (295). The first step is to introduce a new set of variables which are supported in the unit dyadic annular region $\mathcal{A} = [\frac{1}{8}, 8] \times \mathbb{S}^2$:

$$r\omega \;=\; (\tau_-(J) \cdot \widetilde{r} + t)\omega \; .$$

Using the dilation identity for spheres, we have that:

$$\| \, \chi f \, \|_{L_r^\infty(L^q(\mathbb{S}_r^2))} \;\sim\; \tau_+^{\frac{2}{q}}(\mathcal{J}) \cdot \| \, \chi f \, \|_{L_{\widetilde{r}}^\infty(L^q(\mathbb{S}^2))} \; .$$

Furthermore, a direct calculation using polar integrals gives:

$$\tau_+(\mathcal{J})\tau_-^{\frac{1}{2}}(\mathcal{J}) \cdot \left(\| \chi f \|_{L^2} + \sum_{i<j} \| \Omega_{ij}\, \chi f \|_{L^2} + \| \tau_-\, \partial_r\, \chi f \|_{L^2} \right) \sim$$
$$\| \chi f \|_{L^2(\mathcal{A})} + \| \partial_{\tilde{r}} \chi f \|_{L^2(\mathcal{A})} + \| \overline{\nabla} \chi f \|_{L^2(\mathcal{A})} \ .$$

Therefore, it suffices to be able to prove the estimate:

$$(296) \qquad \| \chi f \|_{L_{\tilde{r}}^\infty(L^q(\mathbb{S}^2))} \lesssim \| \chi f \|_{H^1(\mathcal{A})} \ .$$

Notice that by design, the rescaled χf does not intersect the boundary $\partial \mathcal{A}$. Using two charts on the interior of $\mathcal{A}$, estimate (296) follows from the general mixed Sobolev embedding on $\mathbb{R}^3$ for test functions φ:

$$(297) \qquad \| \varphi \|_{L_x^p(L_y^q)} \lesssim \| \varphi \|_{H^1} \ ,$$

where $(x,y) \in \mathbb{R} \times \mathbb{R}^2$ and $\frac{3}{2} - \frac{1}{p} - \frac{2}{q} < 1$. This last estimate in turn follows from running the Sobolev lemma in the x and y variables separately to achieve:

$$\| \varphi \|_{L_x^p(L_y^q)} \lesssim \| \langle D_x \rangle^{(\frac{1}{2} - \frac{1}{p})+} \langle D_y \rangle^{2(\frac{1}{2} - \frac{1}{q})} \varphi \|_{L^2} \ .$$

where the "+" notation denotes an arbitrarily small quantity which gets us around the case $p = \infty$. (297) now follows from this last estimate and the symbol bounds:

$$(1 + |\xi_x|^2)^{\frac{1}{2}(\frac{1}{2} - \frac{1}{p})+}(1 + |\xi_y|^2)^{(\frac{1}{2} - \frac{1}{q})} \lesssim (1 + |\xi_x|^2 + |\xi_y|^2)^{\frac{1}{2}} \ ,$$

whenever $\frac{3}{2} - \frac{1}{p} - \frac{2}{q} < 1$. We have now shown (295). $\qquad \square$

The second global Sobolev estimate we prove here is the analog of Lemma 9.5 for the interior region $r < \frac{1}{2}t$. This is:

Lemma 9.6 (Interior global Sobolev estimate). *Fix $1 \leqslant t$, then in the interior region $r < \frac{3}{4}t$, one has the following weighted estimate for test functions f:*

$$(298) \qquad \| f \|_{L^q} \lesssim t^{-3(\frac{1}{p} - \frac{1}{q})} \left(\| f \|_{L^p} + \sum_{X \in \{S, \Omega_{0i}\}} \| X(f) \|_{L^p} \right) \ ,$$

whenever $\frac{1}{p} - \frac{1}{q} < \frac{1}{3}$.

Proof of estimate (298). Note that this estimate is taking place in a ball of radius t. By rescaling and using the usual Sobolev embedding, it suffices to be able to show that:

$$t \| \nabla f \|_{L^p} \lesssim \sum_{X \in \{S, \Omega_{0i}\}} \| X(f) \|_{L^p} \ .$$

This follows immediately from the using the following identities in turn:

$$t\, \partial_i = \Omega_{i0} - x_i \partial_t \ ,$$
$$\partial_t = (t^2 - r^2)^{-1} \left(tS - x^i \Omega_{i0} \right) \ .$$

$\qquad \square$

Finally, we end this appendix with a characteristic version of the exterior estimates (294)–(295). This estimate will be used to prove the peeling properties of the best decaying components of F and ϕ in the main work. Because of this, unlike the previous two estimates we will need to cut this one off sharply along the time slices $1 \leqslant t \leqslant t_0$.

Lemma 9.7 (Characteristic (truncated) global Sobolev estimate). *Let $C(u)$ denote the cones $u = const.$ and let t_0 be a fixed parameter. Now define the truncated cone:*

$$\overline{C}(u) \;=\; C(u) \cap \{1 \leqslant t \leqslant t_0\} \cap \{t < 2r\} \;,$$

Let $\mathcal{I}$ be a dyadic shell along the extended exterior region $t < 2r$ of the cone $C(u)$, and χ its smooth cutoff. Then for any test function f, one has the following estimate for $2 < q < \infty$:

$$(299) \qquad |\chi\, f(t, r\omega)|^2 \;\lesssim\; \tau_+^{-\frac{4}{q}}(\mathcal{I}) \sum_{i<j} \| \Omega_{ij}(\chi\, f(t, r)) \|_{L^q(\mathbb{S}_r^2)} \;.$$

We also have that:

$$(300) \quad \| \chi\, f \|_{L_u^\infty(L^q(\mathbb{S}_r^2))(\overline{C}(u))} \;\lesssim\;$$
$$\tau_+^{-\frac{3}{2}-\frac{2}{q}}(\mathcal{I}) \left(\| \chi\, f \|_{L^2(\overline{C}(u))} \;+\; \| \underline{u} L(\chi\, f) \|_{L^2(\overline{C}(u))} \;+\; \sum_{i<j} \| \Omega_{ij}(\chi\, f) \|_{L^2(\overline{C}(u))} \right) \;.$$

Proof of estimates (299)–(300). The first estimate (299) is of course just a restatement of (294). Therefore, we concentrate on (300). This estimate is proved using a rescaling argument similar to that used to prove (295) along with the mixed norm Sobolev estimate (297). However, one need to take a bit of care to deal with the sharp cutoff along the slab $\{1 \leqslant t \leqslant t_0\}$. This can be taken care of using the extension theorem for H^1 functions (see [18]).

Specifically, rescaling all variables along the cone $\chi\overline{C}(u)$ by the factor $\tau_+(\mathcal{I})$, we are again reduced to showing the annular estimate (296). However, this time it is possible for χf to extend past the boundary $\partial \mathcal{A}$ even though we only care about its behavior inside of $\mathcal{A}$. However, this is not a problem because $\mathcal{A}$ is a C^∞ submanifold of $\mathbb{R}^3$, which allows us to extend $\chi_{\mathcal{A}}\chi f$ to a function $\widetilde{\chi f}$ with the properties:

$$\widetilde{\chi f}|_{\mathcal{A}} \;=\; \chi f|_{\mathcal{A}} \;, \qquad\qquad \| \widetilde{\chi f} \|_{H^1} \;\lesssim\; \| \chi f \|_{H^1(\mathcal{A})} \;.$$

Applying now estimate (297) to this extension yields the desired result. $\qquad\square$

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