Commun. Math. Phys. 110, 479–501 (1987)

Communications in Mathematical Physics

Convergence of the $U(1)_4$ Lattice Gauge Theory to Its Continuum Limit

Bruce K. Driver

Institute for Advanced Study, School of Mathematics, E.C.P. Building, Princeton, NJ 08540, USA

Abstract. It is shown that in four space-time dimensions the compact $U(1)$ lattice gauge theory with general energy function converges to a renormalized free electromagnetic field on the current sector as the lattice spacing approaches zero, provided the coupling constant is sufficiently large. For the Wilson energy function, it is possible, by judicious choice of the Gibbs state, to get convergence for arbitrary coupling strengths. Furthermore, for all but a countable number of values of the coupling constant, the limit exists and is independent of the particular state chosen to define the lattice model.

1. Introduction

The problem of mathematical existence of quantized Yang–Mills' fields is (on an informal level) equivalent to defining a certain probability measure on a space of connection forms. The informal description of this (Yang–Mills') measure is

$$d\mu(A) = Z^{-1} \exp\left[\frac{1}{2g^2} \int_{\mathbb{R}^d} \sum_{i<j} \operatorname{trace}(F_{ij}^A(x)^2) dx\right] DA, \tag{1.1}$$

where A runs over a space of connection forms $(\underline{A})$ on the trivial unitary vector bundle $\mathbb{C}^N \times \mathbb{R}^d$, $F^A = dA + A \wedge A$ is the curvature of A, $DA = \prod_{i=1}^{d} \prod_{x \in \mathbb{R}^d} d(A_1(x))$ is "infinite dimensional Lebesgue measure" on $(\underline{A})$, g^2 is a positive "coupling" constant, and Z is a normalization constant which makes μ a probability measure. See Gross [5] for a discussion of (1.1) and its ailments.

A standard approach for trying to make sense of the informal expression (1.1) is to "approximate" the measure by a compact lattice gauge model introduced by K. Wilson [1], see Sect. 3 below. The problem is then to show that the lattice measures have a limit as the lattice spacing tends to zero (the continuum limit).

The procedure for removing the lattice cutoff has still not been carried out for space-time dimension larger than two with a non-abelian gauge (structure) group.

This research was supported in part by an Alfred P. Sloan Foundation Doctoral Dissertation Fellowship and in part by N.S.F. Grant Nos. DMS 84–01997 and DMS-8610730(1)

However, for partial results (involving renormalization group ideas related to ideas in Wilson [2]) in the non-abelian case, see T. Balaban [1–14] and P. Federbush [1–7].

This paper is concerned with the case where the gauge group (G) is $U(1)$. This is a considerable simplification since the measure in (1.1) may now be defined directly as an infinite-dimensional Gaussian measure, see for example Gross [5]. For our purposes, the measure (μ) of (1.1) (called the free Euclidean electromagnetic field for $G = U(1)$) is characterized by

$$\int \exp(F^A(\rho))d\mu(A) = \exp[-g^2(\Delta^{-1}d^*\rho, d^*\rho)/2], \tag{1.2}$$

where $A = \sum A_i dx^i$ with A_i being a real Schwartz distribution on $\mathbb{R}^d$, $F^A = dA$ (in distribution sense), ρ is any compactly supported C^∞-differential 2-form on $\mathbb{R}^d$, d^* is the L^2-adjoint of d, and $\Delta = -(d^*d + dd^*)$. Given that μ in (1.1) is already defined by (1.2) for $G = U(1)$ proving convergence of the lattice approximating measures is to be considered a test case for the more difficult non-abelian case.

Gross [3] showed, for $d = 3$ and $G = U(1)$ and under some natural assumptions on the lattice action, that the "lattice current" converges (in the sense of Fourier transforms) to the current in the free Euclidean electromagnetic field. Furthermore, if the lattice action is the "Villain" action, then Gross essentially shows that the lattice field strength tensor converges to the corresponding field strength tensor of the free Euclidean electromagnetic field.

This paper shows that similar results hold for $d = 4$ and $G = U(1)$, despite the fact that the heuristics motivating the definition of the lattice measure are not very convincing when $d = 4$ (see Gross [5]). In particular, for various hypothesis and lattice models, the lattice measures converge in the sense of Fourier (Laplace) transforms in the "lattice current" to the current of a renormalized free Euclidean electromagnetic field. As in Gross [3], a main technique used is the Schwinger-Dyson equations (an infinite-dimensional integration by parts). However, unlike in Gross [3], the mechanism for convergence is based on choosing a particularly nice extreme Gibbs state to represent the lattice model. These extreme Gibbs states are needed to provide the necessary decoupling of distantly separated "plaquettes."

In this paper, attention has been restricted to studying the lattice current ($J = d^*F$) rather than the field strengths (F). This allows us to avoid the Dirac monopoles (breakdown of the Bianchi identity) which are inadvertently introduced into the lattice theory. Avoidance of the monopoles seems resonable in the abelian theory because a similar mechanism for avoiding them in the non-abelian theory is now available, Gross [4].

Closely related results to Gross [3] and this paper have been obtained by C. King [1–2]. King has used the results and idea's of Balaban [1–3] to prove the existence of the continuum $U(1)$–Higgs (abelian) model in two and three space-time dimensions. However, the approximating measures that King uses are non-compact versions of the Wilson lattice approximation. The generalization of these non-compact approximations to the non-abelian setting is as yet unknown.

This paper has been divided into eight sections with one appendix. Sections two and three contain the basic notation and the definition of the lattice model. Section four gives the statements of the main results. The remaining sections are devoted to the proofs of the main theorems. The appendix describes the necessary notations and facts about pressure which are needed in Sect. 8.

Finally, let me introduce some notation which will be used throughout the paper. The notation $\mu(f)$ will be used to denote $\int f\,d\mu$, and the notation $A \subset\subset B$ will be used to denote $A \subset B$ and $|A| < \infty$ ($|A|$ is the number elements of the set A).

2. Lattice Complexes and Their Basic Properties

Definition 2.1. Let $\{e_i\}_{i=1}^d$ be the standard unit vectors in $\mathbb{R}^d$, and k be a non-negative integer less than d. The positively (/negatively) oriented k-cells based at $x \in Z^d$, are the formal symbols: $\pm(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k})_x$, where $1 \leqq i_1 < i_2 < \cdots < i_k \leqq id$, provided $k \geqq 1$. The positive (/negative) oriented 0-cells are the symbols $(\pm)_x$, where $x \in Z^d$. Let $(Z^d)^{(k)}$ denote the set of k-cells of both orientations.

A k-chain is defined in the usual way as the formal sum of a finite number of k-cells with integer coefficients with $-1c$ identified with the k-cell $(-c)$ of opposite orientation. Similarly, the boundary operator (∂) is defined on chains by

$$\partial c = \sum_{\varepsilon=0}^{1} \sum_{j=1}^{k} (-1)^{\varepsilon+j} (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k})_{x+\varepsilon e_{i_j}} \tag{2.1}$$

(where the basis vector under the circumflex is to be omitted) for $c = (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k})_x$, and then extended linearly.

Definition 2.2. A (lattice) k-form is a homomorphism on the Z module of k-chains to the complex numbers. So if ψ is a k-form and c, and c' are the k-chains and a is an integer, then $\psi(c + ac') = \psi(c) + a\psi(c')$. A k-form is said to have compact support if it is identically zero on all k-cells based sufficiently far from 0.

The differential $(d\psi)$ of k-form (ψ) is the $k+1$ form determined on $(k+1)$-cells (c) by

$$d\psi(c) = \psi(\partial c) = \sum_{p \in \partial c} \psi(p). \tag{2.3}$$

If φ and ψ are two k-forms with compact support, set

$$(\varphi, \psi) = \tfrac{1}{2} \sum_{c \in (Z^d)^{(k)}} \varphi(c)\psi(c). \tag{2.4}$$

The co-differential $(d^*\psi)$ of $(k+1)$-form (ψ) is the k form determined on k-cells (p) by

$$d^*\psi(p) = \sum_{c.p \in \partial c} \psi(c). \tag{2.5}$$

The co-differential (d^*) is the adjoint of (d) with respect to (2.4).

The rest of this section deals with approximating differential (C^∞) forms on $\mathbb{R}^d$ by lattice forms on Z^d. If ψ is a differential k-form on $\mathbb{R}^d$, and $a > 0$ (thought of as the lattice spacing), then define the lattice approximation (ψ_a) by

$$\psi_a(c) = \int_{ac} \psi \equiv \int_{[0,a]^k} \psi_{i_1 i_2 \ldots i_k}(ax + s_1 e_{i_1} + \cdots + s_k e_{i_k}) ds_1 \ldots ds_k, \tag{2.6}$$

where $c = (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k})_x$ and $\psi_{i_1 i_2 \ldots i_k} = \psi(e_{i_1}, e_{i_2}, \ldots, e_{i_k})$ the components of ψ. (Note: $\psi_a(c) \simeq a^k \psi_{i_1 i_2 \ldots i_k}(ax)$ for a close to zero.)

We now have two meanings for the symbol (d). However, the next lemma shows that for lattice approximations the two differentials may be interchanged.

Lemma 2.1. *Let* ψ *be a differential k-form on* $\mathbb{R}^d$ *and* ψ_a *be a lattice approximation. Then* $(d\psi)_a$ *and* $d(\psi_a)$ *are the same k-forms on* Z^d, *where the first d is the exterior differential and the second d is the lattice differential.*

Proof.
$$(d\psi)_a(c) = \int_{ac} d\psi = \int_{\partial(ac)} \psi = \int_{a(\partial c)} \psi = \psi_a(\partial c) = (d\psi_a)(c).$$

The second equality is Stokes' theorem, see Spivak [1]. The rest is only a matter of unwinding definitions. Q.E.D.

Finally, if ψ, and φ are compactly supported C^∞ differential k-forms on $\mathbb{R}^d$, put

$$(\psi, \varphi) = \sum \int_{\mathbb{R}^d} \psi_{i_1 i_2 \ldots i_k}(x) \varphi_{i_1 i_2 \ldots i_k}(x) dx, \tag{2.7}$$

where the sum is over increasing subsequences of length k of $\{1, 2, 3, \ldots, d\}$, and $\psi_{i_1 i_2 \ldots i_k}(x)$ and $\varphi_{i_1 i_2 \ldots i_k}(x)$ are the components of the differential k-forms ψ and φ respectively. The following lemma connecting the two bilinear forms (2.4) and (2.7) is easily proved using standard Riemann integral techniques.

Lemma 2.2. *Let* ψ *and* φ *be compactly supported differential k-forms on* $\mathbb{R}^d$, *then*

$$(\psi, \varphi) = \lim_{a \downarrow 0} a^{(d-2k)}(\psi_a, \varphi_a). \tag{2.8}$$

3. Definition of the Lattice Model

Let $h: \mathbb{R} \to \mathbb{R}$ be a real twice continuously differentiable even periodic function with period 2π. Any such function (h) will be called an energy function. The main examples of interest are given below.

Example 3.1. Wilson energy: $h(x) = 1 - \cos(x)$.

Example 3.2. Generalized Wilson energy: $h(x) = 1 - \cos(mx)$, where m is any integer.

Example 3.3. Wilson-like energy: $h(x) = -\sum_{k=1}^{N} b_k \cos(kx)$, where $b_k \geqq 0$ and N is an positive integer.

Example 3.4. Villain energy: For each $\beta > 0$, define h_β by

$$\exp(-\beta h_\beta(x)) = c_\beta \sum_{n=-\infty}^{\infty} \exp[-\beta(x - 2\pi n)^2/2], \tag{3.1}$$

where c_β is a constant chosen such that the right-hand side is one at $x = 0$.

To each energy function (h) we will associate a lattice statistical mechanical model. See Israel [1] Preston [1] for the general notation and facts about lattice statistical mechanics.

Let $(\mathscr{B}_+)\mathscr{B}$ be the set of (positively oriented) one cells on Z^d. Let $(\mathscr{P}_+)\mathscr{P}$ be the set of (positively oriented) two cells. The one cells will also be referred to as bonds and the two cells as plaquettes. The lattice for the model is the set of positively oriented bonds $\mathscr{B}_+$. The state space for the model is S^1, the unit circle. The unit circle will be identified with the interval $[-\pi,\pi]$ with the end points identified. The "apriori" measure on S^1 is taken to be normalized Lebesgue measure (λ) on $[-\pi,\pi]$. The configuration space (Ω) is then $(S^1)^{\mathscr{B}_+} \equiv \{\omega:\mathscr{B}_+ \to [-\pi,\pi]\}$.

It is convenient to embed configuration space (Ω) in $(S^1)^{\mathscr{B}}$, by defining $\omega(-b) \equiv -\omega(b)$, where $b \in \mathscr{B}_+$ and $\omega \in \Omega$. (Remember that the unit circle has been identified with $[-\pi,\pi]$.) With this convention each configuration may be extended to a 1-form on Z^d.

Definition 3.1. Let h be a given energy function, then the associated interaction potential $(\varphi^h = \{\varphi^h_B\}_{B\subset\subset\mathscr{B}_+})$ is given by

$$\varphi^h_B(\omega) = \begin{cases} h(d\omega(p)) & \text{if } \bar{B} = \overline{\partial p} \quad \text{for some } p\in\mathscr{P}_+ \\ 0 & \text{otherwise} \end{cases}, \tag{3.2}$$

where $\bar{B}$ denotes the set of bonds in B disregarding orientation. Hence, the energy $(H^h_B(\omega|\omega'))$ of a configuration (ω) given the boundary conditions (ω') over $B \subset\subset \mathscr{B}_+$ is

$$H^h_B(\omega|\omega') = \sum_{p\in\mathscr{P}_+:\bar{p}\cap\bar{B}\neq\phi} h(d[\omega_B \times \omega'_{\mathscr{B}_+\setminus B}](p)), \tag{3.3}$$

where

$$\omega_B \times \omega'_{\mathscr{B}_+\setminus B}(b) \equiv \begin{cases} \omega(b) & \text{if } b\in B \\ \omega'(b) & \text{otherwise} \end{cases}. \tag{3.4}$$

The corresponding specification $(\Pi^h \equiv \Pi^{\varphi^h} \equiv \{\Pi^h_B\}_{B\subset\subset\mathscr{B}_+})$ is defined in the standard way as follows. If (f) is a continuous function on Ω (a compact space with the product topology) then,

$$\Pi^h_B(\omega, f) = Z^h_B(\omega)^{-1} \int_{\Omega(B)} e^{-H^h_B(\omega'|\omega)} f(\omega'_B \times \omega_{\mathscr{B}_+\setminus B})\, d\lambda^B(\omega'_B), \tag{3.5}$$

where $Z^h_B(\omega)$ is the normalization constant such that $\Pi^h_B(\omega,1)=1, \Omega(B)\equiv(S^1)^B$, $\omega'_B = \omega'|_B \in \Omega(B)$, and $d\lambda^B(\omega'_B)$ is the product of the normalized Lebesgue measures on $\Omega(B)$. The set of (extreme) Gibbs states associated to Π^h will be denoted by $(G_e(h))\,G(h)$.

Up to this point it has been convenient to absorb certain numeric factors associated with the lattice spacing parameter a and the coupling strength g into the energy function h. These factors are easily made explicit when necessary.

Definition 3.2. A Gibbs state $\mu \in G(g^{-2}a^{(d-4)}h)$ is said to be a lattice approximation (for the lattice aZ^d) to the free Euclidean electromagnetic field with coupling strength g. (The factor $a^{(d-4)}$ is motivated by (2.8).)

Definition 3.3. The lattice field strength tensor is

$$F(p)(\omega) = h'(d\omega(p)), \tag{3.6}$$

where $p\in\mathscr{P}$, and $\omega\in\Omega$. (Remember that each ω is extended to be a 1-form on Z^d, so $d\omega(p)$ is defined.)

Remark 3.1. Since the energy function (h) is assumed to be even, h' is then odd. So for each configuration (ω), $F(\cdot)(\omega)$ may be considered as a 2-form on Z^d.

Definition 3.4. The "lattice current" associated to F is

$$J(b)(\omega) = d^*F(b)(\omega) = \sum_{p\in\mathscr{P}:b\in\partial p} F(p)(\omega), \tag{3.7}$$

for all $b\in\mathscr{B}$, and $\omega\in\Omega$.

4. Statement of the Main Results

Theorem 4.1. *Let $d>4$, and φ be a closed $(d\varphi=0)$ complex valued test 2-form on $\mathbb{R}^d$. For each $a>0$, let $\mu_a \in G(g^{-2}a^{(d-4)}h)$, then*

$$\lim_{a\downarrow 0} \mu_a(\exp(a^{(d-4)})(F,\varphi_a)) = 1, \tag{4.1}$$

where $(F,\varphi_a)(\omega) \equiv \frac{1}{2}\sum_{p\in\mathscr{P}} F(p)(\omega)\varphi_a(p)$ as in (2.5).

Remark 4.1. The fact that φ is closed is equivalent to the existence of a test 1-form (j) such that $\varphi = dj$ (provided $d>2$), as follows from standard compact De Rham cohomology theory (Bott and Tu [1]). Hence the theorem is a statement about the lattice current d^*F, since $(F,\varphi)=(F,dj)=(d^*F,j)=(J,j)$.

Theorem 4.1 asserts that the continuum limit on the current sector of the lattice models for $d>4$ is "trivial." The limiting measure is a δ-function at the zero configuration.

In the following, we will restrict the space-time dimension to the case of most interest; $d=4$. This is an exceptional case, since the lattice approximating Gibbs state of Definition 3.2 no longer depends on the lattice spacing parameter (a). The influence of the lattice spacing is only felt in the lattice approximations φ_a to the test 2-form φ.

Theorem 4.2. $(d=4)$ *Suppose that $G(g^{-2}h)=\{\mu\}$, is a one element set. Then for any closed complex test 2-form (φ) on $\mathbb{R}^4$,*

$$\lim_{a\downarrow 0} \mu(e^{(F,\varphi_a)}) = \exp\left(\frac{\alpha}{2} g^2(\varphi,\varphi)\right), \tag{4.2}$$

where $\alpha \equiv \mu(h''(\omega(\partial p))) \geqq 0$ (independent of $p\in\mathscr{P}$). Furthermore, $\alpha=0$ if and only if h is a constant function.

Remark 4.2. The set of Gibbs states $G(g^{-2}h)$ is closed under translations and 90°-rotations of the lattice, since the specifications are invariant under these operations. Hence, the unique Gibbs state (μ) must be invariant under translations and 90°-rotations.

The theorem says that the Laplace transforms of the lattice current converges to the corresponding Laplace transforms of the free Euclidean field (compare (4.2) with (1.2)) provided the coupling constant (g^2) is renormalised by the factor α. (Note: $(\Delta^{-1}d^*\rho, d^*\rho) = -(\rho,\rho)$ if $d\rho=0$.)

An application of Dobrushin's uniqueness theorem gives an easy method for checking the hypothesis of Theorem 4.2. This is the subject of the next lemma.

Lemma 4.1. *Let h be a given energy function, and let the coupling constant* (g^2) *be chosen sufficiently large such that* $[\sup(h) - \inf(h)] < 2g^2/9$. *Then* $|G(g^{-2}h)| = 1$, *and hence the conclusion of Theorem 4.2 holds.*

Theorem 4.3. ($d = 4$) *Let φ be a real-valued closed* ($d\varphi = 0$) *test 2-form on* $\mathbb{R}^4$. *Suppose that $\mu \in G_e(g^{-2}h)$ is an extreme Gibbs state which is also translation and 90°-rotation invariant. Then*

$$\lim_{a \downarrow 0} \mu(e^{i(F, \varphi_a)}) = \exp\left(\frac{-\alpha}{2} g^2(\varphi, \varphi)\right), \tag{4.3}$$

where α is the constant defined in Theorem 4.2.

The proofs of Theorems 4.1–4.3 are given in Sect. 6.

Remark 4.3. Theorem 4.3 is a special case of Theorem 4.2 if $|G(g^{-2}h)| = 1$.

The existence of a Gibbs state satisfying the hypothesis of Theorem 4.3 is not known for general energy functions (h). However, if h is a Wilson-like action (Example 3.3), then results of Messager et al. [1] guarantee the existence of such a Gibbs state.

Theorem 4.4. (*Messager et al.* [1]) *Assume $d = 4$, and suppose h is a Wilson-like energy function. Let $\mu_g^0 \in G(g^{-2}h)$ be the Gibbs state constructed by taking the thermodynamic limit with zero boundary conditions*:

$$\mu_g^0 \equiv \text{weak-}\lim_{B \uparrow \mathscr{B}_+} \Pi_B^{g^{-2}h}(0, \cdot), \tag{4.4}$$

where 0 denotes the zero configuration, and $B \uparrow \mathscr{B}_+$ means that B should eventually contain any finite set of bonds in $\mathscr{B}_+$. Then μ_g^0 exists and defines an extreme invariant Gibbs state, i.e. *satisfies the hypothesis of Theorem 4.3.*

The proofs of Lemma 4.1 and Theorem 4.4 are in Sect. 7.

Following ideas in Pfister [1] and Fröhlich and Pfister [1], one finds for a Wilson-like energy function and for almost all values of the coupling constant (g) that the limit in Theorem 4.3 exists and is independent of the particular choice of invariant Gibbs state.

Theorem 4.5. ($d = 4$) *Let h be a Wilson-like energy function. Then for all but at most a countable number of $g > 0$,*

$$\lim_{a \downarrow 0} \mu_g(e^{i(F, \varphi_a)}) = \exp\left(\frac{-\alpha}{2} g^2(\varphi, \varphi)\right), \tag{4.5}$$

where μ_g is any (*translation and* 90°-*rotation*) *invariant element of* $G(g^{-2}h)$, *φ is any real closed test 2-form, and* $\alpha \equiv \mu_g^0(h''(\omega(\partial p)))$ (μ_g^0 *is defined in* (4.4)) *independent of the particular choice* (μ_g).

The proof is given in Sect. 8.

It should be pointed out that Theorems 4.3–4.5 are only needed when there is a first order phase transition for the model, that is when $|G(g^{-2}h)| > 1$. Otherwise the stronger result of Theorem 4.2 would be applicable. The existence of a first order phase transition is still an open question for these $U(1)_4$-gauge models. Recall that Guth's theorem (Guth [1]) asserts that in $d=4$, the $U(1)_4$ lattice gauge theory with the Villain action does have a phase transition. However, this phase transition is characterized by the decay properties of "Wilson loop variables" as the size of the loop increases to infinity. It is not clear how a phase transition of this type is related to the more standard phase transitions which are characterized by smoothness properties of the thermodynamic potentials.

5. Preliminary Lemmas and Estimates

This section contains some basic identities which result from integration by parts. These identities will then be used to derive a number of basic estimates of quantities involving the Laplace or Fourier transforms of the lattice currents. Throughout this section it will be convenient to absorb the factor $g^{-2}a^{(d-4)}$ into the energy function (h). These factors are easily reinserted later when they are needed. Also, unless explicitly stated, the dimension of space time (d) is allowed to be arbitrary.

Lemma 5.1. (*Schwinger–Dyson Equations, see Gross* [3].) *Let h be an energy function, $\mu\in G(h)$ be an arbitrary Gibbs state. Let f be a differentiable periodic function depending on only finitely many bond variables. Then for any bond b*

$$\mu(\partial f/\partial\omega(b))=\mu(J(b)\cdot f), \tag{5.1}$$

where $J(b)$ is the lattice current in Definition 3.4.

Proof. Choose $B\subset\subset\mathscr{B}_+$ such that $f(\omega)$ depends only on ω_B and such that b and the bonds of any neighboring plaquettes are contained in B. Then by finite dimensional integration by parts, noting that the boundary terms are zero by periodicity,

$$\Pi_B^h(\omega,\partial f/\partial\omega(b))=\Pi_B^h(\omega,J(b)\cdot f), \tag{5.2}$$

for all configurations ω. See Eqs. (3.3), (3.5) and the definition of $J(b)$ (Definition 3.4) and the following computation:

$$\begin{aligned}\partial H_B^h(\omega'|\omega)/\partial\omega'(b)&=\sum_{p\in\mathscr{P}_+:\bar{p}\cap\bar{B}\neq\phi}h'(d\omega'(p))\cdot\partial d\omega'(p)/\partial\omega'(b)\\&=\sum_{p\in\mathscr{P}:b\in\partial p}h'(d\omega'(p))=J(b)(\omega').\end{aligned}$$

In the second equality we used the assumption that h' is odd, the convention of ω' being extended to a 1-chain, and the assumption that b was in the "interior" of B.

The theorem now follows by integrating both sides of Eq. (5.2) with respect to the Gibbs state (μ) and using the D.L.R. equations; Preston [1], Israel [1] or Ruelle [1]. Q.E.D.

Lemma 5.2. (*See Gross* [3].) *Let j be a lattice* 1*-form with finite support, $\mu\in G(h)$, and $u(s)=\mu(e^{s(J,j)})$ for s real. Then*

$$u'(s)=s\sum_{p\in\mathscr{P}_+}dj(p)^2\mu(h''(d\omega(p)\cdot e^{s(J,j)})). \tag{5.3}$$

Proof.

$$u'(s) = s\mu((J,j)e^{s(J,j)}) = (1/2)\sum j(b)\mu(J(b)e^{s(J,j)})$$
$$= (1/2)\sum j(b)\mu\left(\frac{\partial}{\partial\omega(b)}e^{s(J,j)}\right)$$
$$= (s/2)\sum j(b)\mu\left(\frac{\partial(J,j)}{\partial\omega(b)}\cdot e^{s(J,j)}\right), \tag{5.4}$$

where the sums are over all bonds. The Schwinger–Dyson equations were used in the third equality.

Set $\varphi = dj$, so by Definitions 3.3 and 3.4, $(J,j) = (F,\varphi)$. We now compute the derivative in (5.4).

$$\sum_b \frac{\partial(F,\varphi)}{\partial\omega(b)} j(b) = (1/2)\sum_b\sum_p j(b)\varphi(p)\{h''(d\omega(p))[1_{\partial p}(b) - 1_{\partial p}(-b)]\}$$
$$= (1/2)\sum_p \varphi(p)h''(d\omega(p))[dj(p) - dj(-p)]$$
$$= \sum_p \varphi(p)^2 h''(d\omega(p)), \tag{5.5}$$

where

$$1_{\partial p}(b) \equiv \begin{cases} 1 & \text{if } b\in\partial p \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

Plug (5.5) into (5.4) to finish the proof, noting the extra factor of two in (5.3) arises from the restriction of the sum to $\mathscr{P}_+$. Q.E.D.

Lemma 5.3. (*Dimension* $= d$.) *Let j be a complex lattice* 1-*form with finite support, $\mu\in G(h)$ a translation invariant Gibbs state, and $u(s) = \mu(e^{s(J,j)})$. Then*

$$|u(s)| \leqq \exp\{(s^2/2)\|dj\|^2\cdot\|h''\|_\infty\}, \tag{5.7}$$

where $\|dj\|^2 \equiv (dj_a, \overline{dj}_a)$.

Proof. Without loss of generality, it may be assumed that j is real since $|u(s)| \leqq \mu(|e^{s(J,j)}|) = \mu(e^{s(J,\mathrm{Re}\,j)})$, and $\|d\,\mathrm{Re}(j)\|^2 \leqq \|dj\|^2$. So assume j is real.

By Eq. (5.3),

$$|u'(s)| \leqq s\|dj\|^2\|h''\|_\infty u(s). \tag{5.8}$$

Hence

$$|\ln(u(s)/u(0))| \leqq \int_{t\in[0,s]} |d\ln(u(t))/dt|\,dt \leqq \|dj\|^2\|h''\|_\infty s^2/2.$$

The lemma follows by exponentiating this last inequality using $u(0) = 1$. Q.E.D.

For notational ease, let $K(\cdot,\cdot,\dots,\cdot)$ denote a generic function which is increasing in each of its variables. From lemma to lemma and even line to line there may be many such functions K, which will all be denoted by the same letter.

Lemma 5.4. (*Dimension* $= d$) *Let $\mu\in G(h)$ be an invariant Gibbs state, and j be a* *complex lattice 1-form on Z^d, with finite support. Then*

$$\mu(|e^{(J,j)}-1|^2) \leqq K(\|h''\|_\infty \cdot \|dj\|_2)\|h''\|_\infty \cdot \|dj\|_2^2. \tag{5.9}$$

Proof. Let

$$v(s)=\mu(|e^{s(J,j)}-1|^2)=\mu(e^{2s\mathrm{Re}(J,j)}+1-e^{s(J,\bar{j})}-e^{s(J,\bar{j})}). \tag{5.10}$$

Differentiating (5.10) using Eq. (5.3), one easily finds the estimate

$$|v'(s)| \leqq s\|h''\|_\infty \sum_p |dj(p)^2|[\mu(|e^{2s(J,j)}|)+\mu(|e^{s(J,j)}|)].$$

By using the estimate in Eq. (5.8) and this last equation, we conclude that

$$|v'(s)| \leqq K(\|h''\|_\infty \cdot \|dj\|_2)\|h''\|_\infty \cdot \|dj\|_2^2 \tag{5.11}$$

for $s \in [0,1]$. Since $v(0)=0$, the same estimate holds for v. Q.E.D.

6. Proofs of Theorems 4.1–4.3

Our first goal will be to prove Theorems 4.2 and 4.3. Theorem 4.1 will be proved at the end of this section. The proof of Theorems 4.2 and 4.3 will be broken up into a number of propositions and lemmas.

Proposition 6.1. ($d=4$) *Suppose that j is a complex valued test 1-form on $\mathbb{R}^4$ and $\mu \in G(h)$ is an invariant Gibbs state. Define*

$$c(a) \equiv \sup\{|\mathrm{cov}_\mu(h''(d\omega(p)), e^{s(J,j_a)})| : p \in \mathscr{P}_+, s \in [0,1]\}, \tag{6.1}$$

where $\mathrm{cov}_\mu(f,g) \equiv \mu(fg)-\mu(f)\mu(g)$. *Suppose that* $\lim_{a\downarrow 0} c(a)=0$, *then*

$$\lim_{a\downarrow 0} \mu(e^{(J,j_a)}) = \exp\left(\frac{\alpha}{2}(dj,dj)\right), \tag{6.2}$$

with α as in Theorem 4.2.

Proof. Let $k_a=\alpha(dj_a,dj_a)$, $u_a(s)=\mu(e^{s(J,j_a)})$, and $v_a(s)=\exp(-k_a s^2/2)\cdot u_a(s)$. Then by Lemma 5.2

$$\begin{aligned} v_a'(s) &= \exp(-k_a s^2/2)\{-k_a s u_a(s)+u_a'(s)\} = \exp(-k_a s^2/2)\{(1/2)s \\ &\quad \times \sum_p dj_a(p)^2\, \mathrm{cov}_\mu(h''(d\omega(p)), e^{s(J,j_a)})\}. \end{aligned} \tag{6.3}$$

So

$$|v_a'(s)| \leqq K \cdot c(a) \cdot \|dj_a\|^2, \tag{6.4}$$

where

$$K=\sup\{(s/2)\exp(-k_a s^2/2) | 0<a<1, 0 \leqq s \leqq 1\}, \tag{6.5}$$

and

$$\|dj_a\|^2 \equiv (dj_a, \overline{dj_a}). \tag{6.6}$$

By Lemma 2.2 $k_a \to k \equiv \alpha(dj,dj)$ as $a \to 0$, so that $K<\infty$. Since by assumption $c(a)\to 0$ as $a \to 0$, $v_a' \to 0$ as $a \to 0$ uniformly in $s \in [0,1]$. Since $v_a(0)=1$ ($u_a(0)=1$), it follows that v_a converges to one uniformly in s. Putting $s=1$, conclude that $u_a(1)\to e^{k/2}$ as $a\to 0$. Q.E.D.

In view of this proposition, the proofs of Theorems 4.2 and 4.3 are reduced to showing in each case that $c(a)\to 0$ as $a\to 0$. This will be done by using the cluster properties of unique or extreme Gibbs states. To carry this procedure out we must divide the test 1-form into "near" and "far" pieces. The cluster properties will be applied to the far pieces. The following lemma enables us to control this spliting of the test 1-form.

For the remainder of this section let g be a real infinitely differentiable function with compact support on $\mathbb{R}^d$. Furthermore assume g is radial, $0\leqq g\leqq 1$, and

$$g(x)=\begin{cases}0 & \text{if } |x|\geqq 2\\ 1 & \text{if } |x|\leqq 1\end{cases}.$$

Put $g^r(x)=g(r^{-1}x)$ for all $r>0$.

Lemma 6.1. *Let j be a complex test 1-form on $\mathbb{R}^d$ and g^r be as above. Define $j^r\equiv g^r\cdot j$ for all $r>0$. Then*

$$a^{d-4}\|(dj^r)_a\|_2^2\leqq K\cdot[\|j\|_\infty^2\cdot r^{(d-2)}+\|dj\|_\infty^2 r^d], \tag{6.7}$$

for all positive r and a. The sup-norms are supremums over all the components of the forms and over all of $\mathbb{R}^d$. K is a constant which only depends on the dimension (d), and the function (g).

Proof.

$$\begin{aligned}\|(dj^r)_a\|_2^2 &\leqq [\|(dg^r\wedge j)_a\|_2+\|(g^r\cdot dj)_a\|_2]^2\\ &\leqq 2[\|(dg^r\wedge j)_a\|_2^2+\|(g^r\cdot dj)_a\|_2^2].\end{aligned} \tag{6.8}$$

We estimate the two terms of (6.8) separately. Starting with the first term observe,

$$|(dg^r\wedge j)_a(p)|\leqq \int_{ap}|dg^r\wedge j|\leqq \|j\|_\infty\cdot\int_{ap}|dg^r|, \tag{6.9}$$

where the absolute value of a form denotes the maximum over the absolute value of the components. By the definition of g^r, there exists a dimension dependent constant (c) such that

$$\int_{ap}|dg^r|=0 \quad \text{if} \quad \operatorname{dist}(ap,0)\geqq cr. \tag{6.10}$$

Using the estimate,

$$|dg^r|=|r^{-1}\nabla g(r^{-1}x)\cdot dx|\leqq r^{-1}\|\nabla g\|_\infty, \tag{6.11}$$

and (6.10), we conclude that

$$\int_{ap}|dg^r|\leqq \|\nabla g\|_\infty r^{-1}a^2\chi_{(cr/a)}(p), \tag{6.12}$$

where

$$\chi_s(p)\equiv\begin{cases}1 & \text{if dist }(p,0)\leqq s\\ 0 & \text{otherwise}\end{cases} \tag{6.13}$$

Combine the estimates (6.9) into (6.12) to get

$$|(dg^r \wedge j)_a(p)| \leqq K\|j\|_\infty r^{-1}a^2\chi_{cr/a}(p). \tag{6.14}$$

Square and then sum this last estimate over all plaquettes to obtain

$$a^{(d-4)}\|(dg^r \wedge j)_a\|_2^2 \leqq K\cdot\|j\|_\infty^2\cdot r^{(d-2)}. \tag{6.15}$$

To obtain this last inequality I have used

$$|\{p:\operatorname{dist}(p,0) \leqq cr/a\}| \leqq K\cdot(r/a)^d, \tag{6.16}$$

where K denotes a constant only depending on d. By a similar (easier) argument it follows

$$a^{(d-4)}\|(g^r dj)_a\|_2^2 \leqq K\cdot\|dj\|_\infty^2\cdot r^d, \tag{6.17}$$

where K is a constant only depending on d.

The theorem now follows from the estimates (6.8), (6.15) and (6.17). Q.E.D.

We are now ready to apply the cluster properties of extreme or unique Gibbs states. So for convenience, let me remaind the reader of these cluster properties. A Gibbs state (μ) is said to have the (strong) cluster property, if for all $f \in C(\Omega)$ there exists $B \subset\subset \mathscr{B}_+$ such that for any $g \in C(\Omega)$ that depends only on $\omega_{\mathscr{B}_+\setminus B}$, then

$$|\operatorname{cov}_\mu(f,g)| \leqq \begin{cases} \mu(|g|) \text{ (strong)} \\ \|g\|_\infty \end{cases}, \tag{6.18}$$

where $\operatorname{cov}_\mu(f,g) \equiv \mu(fg) - \mu(f)\mu(g)$. and $\|\cdot\|_\infty$ is the sup-norm.

Theorem 6.1. (*Ruelle* [*1*] *or Preston* [*1*]) *If a Gibbs state μ is (unique) extreme, then μ has the (strong) cluster property.*

Theorem 6.2. ($d = 4$) *Let h be an energy function, $\mu \in G(h)$, and j be a complex test 1-form on $\mathbb{R}^4$. Fix a plaquette (p_0) based at $0 \in Z^4$. Then*

a) *If $|G(h)| = 1$, then*

$$|\operatorname{cov}_\mu(h''(d\omega(p_0)), e^{(J,j_a)})| \leqq K(\|j\|_\infty, \|dj\|_\infty, \|dj_a\|_2)\cdot O_{p_0}(a). \tag{6.19}$$

b) *If μ is extreme ($\mu \in G_e(h)$) and j is purely imaginary, then the estimate (6.19) still holds, and in (6.19), $O_{p_0}(a)$ denotes a function which is independent of the test 1-form (j), and tends to zero as a tends to zero.*

Proof. Let r be a positive number less than one, j^r be as above, and put $k^r \equiv j - j^r$. For notational ease put $j_a^r = (j^r)_a$, $k_a^r = (k^r)_a$, and $f(\omega) = h''(d\omega(p_0))$. The parameter ($r$) will eventually be chosen to be a function of a which converges to zero as a goes to zero.

First split $\operatorname{cov}_\mu(f, e^{(J,j_a)})$ into two parts,

$$\begin{aligned}\operatorname{cov}_\mu(f, e^{(J,j_a)}) &= \operatorname{cov}_\mu(f, e^{(J,j_a^r)}e^{(J,k_a^r)}) \\ &= \operatorname{cov}_\mu(f, [e^{(J,j_a^r)} - 1]e^{(J,k_a^r)}) + \operatorname{cov}_\mu(f, e^{(J,k_a^r)}).\end{aligned} \tag{6.20}$$

Call the first term in (6.20) A and the second term B. We now estimate $|A|$,

$$\begin{aligned}|A| &\leqq 2\|f\|_\infty \mu(|[e^{(J,j_a^r)} - 1]e^{(J,k_a^r)}|) \leqq 2\|f\|_\infty \|[e^{(J,j_a^r)} - 1]\|_{L^2(\mu)} \|e^{(J,k_a^r)}\|_{L^2(\mu)} \\ &\leqq K(\|dj_a^r\|_2, \|dk_a^r\|_2)\cdot\|dj_a^r\|_2 \leqq K(\|dj\|_\infty, \|j\|_\infty, \|dj_a\|_2)\cdot r,\end{aligned} \tag{6.21}$$

where $K(\cdot,\cdot,\cdot)$ denotes a function (depending on h) which is increasing in its arguments. The Cauchy–Schwartz inequality was used in the second inequality, Lemma 5.3 and Lemma 5.4 in the third, and $\|dk_r^a\|_2 \leqq \|dj_a\|_2 + \|dj_a^r\|_2$, Lemma 6.1 and $r \leqq 1$ in the last inequality.

To estimate $|B|$ we will have to divide the proof into two cases corresponding to the two cases of the theorem. However, first note that $(J, k_a^r)(\omega)$ depends only on the bond variables outside a ball of radius (cr/a), where c is a positive constant. At this time choose $r = a^{(1/2)}$, hence $r/a \to \infty$ as $a \to 0$.

Case (a) $|G(h)| = 1$: By the strong cluster property (Theorem 6.1), there exists a function $O_{p_0}(a)$ as in the statement of the theorem such that

$$|B| = |\mathrm{cov}_\mu(f, e^{(J,k_a^r)})| \leqq O_{p_0}(a) \cdot \mu(|e^{(J,k_a^r)}|). \tag{6.22}$$

Using Lemma 5.3, Lemma 6.2 and Eq. (6.22) we conclude that

$$|\mathrm{cov}_\mu(f, e^{(J,k_a^r)})| \leqq O_{p_0}(a) \cdot K(\|j\|_\infty, \|dj\|_\infty, \|dj_a\|_2). \tag{6.23}$$

Case (b) $\mu \in G(h)$ is extreme and j is purely imaginary: The observation that $|e^{(J,k_a^r)}| = 1$ and the cluster property to Theorem 6.1 asserts again an estimate of type (6.23) holds (with K independent of j in this case).

The estimate (6.19) follows from combining the estimates (6.21), (6.23), and (6.20). Q.E.D.

It is now an easy matter to extend Theorem 6.2 to allow for the fixed plaquette p_0 on the left-hand side of (6.19) to be arbitrary.

Lemma 6.2. *Suppose that $\mu \in G(h)$ is an invariant (translation and 90° rotation invariant) Gibbs state satisfying an estimate of the form (6.19), then*

$$|\mathrm{cov}_\mu(h''(d\omega(p)), e^{(J,j_a)})| \leqq K(\|j\|_\infty, \|dj\|_\infty, \|dj_a\|_2) \cdot O_{p_0}(a) \tag{6.24}$$

holds for all $p \in \mathscr{P}$.

Proof. Let T_x denote the natural translation operators on differential forms and on lattice forms, see Appendix. Then if j is a test 1-form

$$(T_{ax} j)_a = T_x j_a. \tag{6.25}$$

So using (6.25) and the fact that both sides of the estimate (6.19) are invariant under translations of j, allows us to conclude that inequality (6.19) is valid if the plaquette (p_0) in the left-hand side of (6.19) is replaced by any of its translates. Using a similar argument for rotations, we conclude that the plaquette (p_0) on the left-hand side of the estimate (6.19) may be replaced by any plaquette (p). Q.E.D.

Proof of Theorems 4.2 and 4.3. Combining Proposition 6.1, Theorem 6.2, and Lemma 6.2 with the energy function h replaced by $g^{-2}h$ gives the limits in Eq. (4.2) and (4.3) of Theorems 4.2 and 4.3. So it only remains to show that $\alpha \geqq 0$ with equality if and only if h is constant. This is a consequence of the following lemma.

Lemma 6.3. *Let μ be an invariant Gibbs state, then*

$$\alpha \equiv \mu(h''(d\omega(p))) = (2(d-1))^{-1} \cdot \mu(J(b)^2), \tag{6.26}$$

where b is any bond, and d is the dimension. Furthermore, $\alpha > 0$ if h is not a constant.

Proof.

$$\mu(J(b)^2)=\mu(\partial J(b)/\partial\omega(b))=\mu\left(\frac{\partial}{\partial\omega(b)}\Big\{\sum_{p:b\in\partial p}F(p)\Big\}\right)$$
$$=\sum_{p:b\in\partial p}\mu(h''(d\omega(p)))=2(d-1)\alpha.$$

The Schwinger–Dyson equations were used in the first equality, the definition of J in the second, the definition of F in the third, and the fact that $2(d-1)$ is the number of plaquettes with a given bond in their boundaries. So the validity of Eq. (6.26) has been shown.

Fix a bond $b\in\mathscr{B}_+$, and let $B\subset\subset\mathscr{B}_+$ be such that $b\in B$ and $J(b)$ depends only on the bond variables over B. If h is not constant, it is easy to check that $J(b)$ is not identically zero. So by continuity of $J(b)$, there is a neighborhood in Ω with $J(b)^2>0$. It then follows that $\Pi^h_B(\omega,J(b)^2)>0$ for *all* configurations ω, since finite dimensional Lebesgue measure charges open sets. Since $\mu(J(b)^2)=\mu\Pi^h_B(\cdot,J(b)^2)$ by the D.L.R. equations, we conclude that $\mu(J(b)^2)>0$, and thus $\alpha>0$. Q.E.D.

Proof of Theorem 4.1. For notational ease, let $\varphi\equiv dj$,

$$\Pi^a_B(\omega',\cdot)\equiv\Pi^{a^{(d-4)}h}_B(\omega',\cdot),\quad\text{and}\quad u_a(s)=\mu_a(e^{sa^{(d-4)}(J,j_a)}).$$

for $s\in[0,1]$. Then by Lemma 5.2 with h replaced by $a^{(d-4)}h$,

$$u_a'(s)=(s/2)a^{(d-4)}\sum_{p\in\mathscr{P}}\varphi_a(p)^2\mu_a(h''(d\omega(p))e^{sa^{(d-4)}(J,j_a)}).\tag{6.27}$$

Since $(s/2)a^{(d-4)}\sum_{p\in\mathscr{P}}\varphi_a(p)^2\to s(\varphi,\varphi)$ as $a\to0$ (Lemma 2.2), it suffices to show that $\mu_a(h''(d\omega(p))e^{sa^{(d-4)}(J,j_a)})\to0$ uniformly in p as $a\to0$, because $u_a(0)=1$.

Let p be a plaquette based at zero. Choose a subset $B\subset\subset\mathscr{B}_+$ which contains all the bonds (disregarding orientation) of any plaquettes having a bond in common with p. By the finite volume Schwinger–Dyson equations (see Eq. (5.2)) with h replaced by $a^{(d-4)}h$,

$$\Pi^a_B(\omega',a^{(d-4)}h''(d\omega(p)))=\Pi^a_B(\omega',a^{(d-4)})\frac{\partial}{\partial\omega(b)}[h'(d\omega(p))]$$
$$=a^{2(d-4)}\Pi^a_B(\omega',J(b)F(p)),\tag{6.28}$$

where b is any bond in the ∂p_0, and $\omega'\in\Omega$ is any configuration. Dividing Eq. (6.28) by $a^{(d-4)}$, using the fact that J and F are uniformly bounded, we find

$$|\Pi^a_B(\omega',h''d\omega(p))|\leq Ka^{(d-4)},\tag{6.29}$$

where K is a constant depending on $\|h\|_\infty$.

As in Theorem 6.2, split the 1-form j_a into its "near" and "far" pieces,

$$j^n_a(b)=\begin{cases}j_a(b) & \text{if } b \text{ or } -b \text{ is in } B\\ 0 & \text{otherwise}\end{cases}\tag{6.30}$$

and

$$j^f_a=j_a-j^n_a.\tag{6.31}$$

We do not have to be so careful in this case.

Now

$$
\begin{aligned}
|\mu_a(h''(d\omega(p))e^{sa^{(d-4)}(J,j_a)})| &= |\mu_a(h''(d\omega(p))e^{sa^{(d-4)}(J,j_a^n)}\cdot e^{sa^{(d-4)}(J,j_a^f)})| \\
&= |\mu_a\{\Pi_B^a[\cdot,h''(d\omega(p))e^{sa^{(d-4)}(J,j_a^n)}]\cdot e^{sa^{(d-4)}}(J,j_a^f)\}| \\
&\leqq e^{K\|J\|_\infty\cdot a^{(d-3)}}\cdot O(a^{(d-4)})\cdot \mu_a(|e^{sa^{(d-4)}(J,j_a^f)}|),
\end{aligned}
\tag{6.32}
$$

where K is a constant depending on $|B|$ and $\|h\|_\infty$. The D.L.R. equations were used in the second equality along with the fact that (J,j_a^f) only depends on the bond variables outside of B. The inequality is a consequence of Eq. (6.29) and the easy estimate

$$
\|e^{sa^{(d-4)}(J,j_a^n)}\|_\infty \leqq e^{Ka^{(d-3)}\cdot\|j\|_\infty}, \tag{6.33}
$$

where K only depends on $|B|$. But by Lemma 5.3 with h replaced by $a^{(d-4)}h$,

$$
\begin{aligned}
\mu_a(|e^{sa^{(d-4)}(J,j_a^f)}|) &\leqq \exp\{a^{(d-4)}\|dj_a^f\|_2^2\|h\|_\infty/2\} \\
&\leqq \exp\{a^{(d-4)}\|dj_a\|_2^2\|h\|_\infty/2\}.
\end{aligned}
\tag{6.34}
$$

Combine the estimates (6.34) and (6.32) to get

$$
|\mu_a(h''(d\omega(p))e^{sa^{(d-4)}(J,j_a)})| \leqq K(\|j\|_\infty, a^{(d-4)}\|dj_a\|_2)\cdot a^{(d-4)}, \tag{6.35}
$$

where K is an increasing function in its arguments. As in Lemma 6.2 the estimate (6.35) remains valid when (p) is any plaquette, since for small a the Gibbs state μ_a is unique and hence invariant, see Lemma 4.1 and Remark 4.2. Hence we have shown that $\mu_a(h''(d\omega(p))e^{sa^{(d-4)}(J,j_a)})\to 0$ uniformly in p as $a\to 0$, so the theorem is proved. Q.E.D.

7. Proof of Lemma 4.1 and Theorem 4.4

Lemma 4.1 is a special case of the following (specialized) version of Dobrushin's uniqueness theorem (Dobrushin [1]) which is a combination of Dobrushin's result and an estimate of B. Simon [2]. Also see Gross [1–2] and Follmer [1] for other proofs and related results to Dobrushin's theorem.

Theorem 7.1. *Suppose that φ is a finite range interaction potential on Ω such that*

$$
\sup_{b\in\mathscr{B}_+}\sum_{B:b\in B}(|B|-1)\|\varphi_B\|_\infty < 1, \tag{7.1}
$$

then there is only one Gibbs state associated to the potential φ.

Proof of Lemma 4.1. Without any loss of generality, assume that $g=1$. We first note that $G(h-c)=G(h)$ for any real constant (c), since $\Pi_B^{(h-c)}=\Pi_B^h$ for all real c and $B\subset\subset\mathscr{B}_+$ ($B\subset\mathscr{B}_+$ and $|B|<\infty$). Thus we may assume that h is normalized such that $\|h\|_\infty=\frac{1}{2}[\sup(h)-\inf(h)]$.

An easy computation using the Definition 3.1 shows

$$
\sup_{b\in\mathscr{B}_+}\sum_{B:b\in B}(|B|-1)\|\varphi_B^h\|_\infty = 3(d-1)\|h\|_\infty \tag{7.2}
$$

in d-dimensions. The factor 3 is from the fact that $|\partial p|=4$ if p is a plaquette. The factor d-1 counts the number of positively oriented plaquettes with b in the boundary. The lemma now follows from Theorem 7.1. Q.E.D.

Proof of Theorem 4.4. The fact that the limit μ_g^0 defined in (4.4) exists and defines an extreme Gibbs state is a special case of Theorem 1 of Messager et al. [1] applied to our model. The invariance of the measure μ^0 easily follows from the invariance of $\{\Pi_B^{g^{-2}h}\}_{B\subset\subset\mathscr{B}_+}$:

$$\Pi_{T(B)}^{g^{-2}h}(\omega, f) = \Pi_B^{g^{-2}h}(\omega\circ T, f\circ T), \tag{7.3}$$

where T denotes the action of a translation by an element of Z^d (see Appendix) or a 90°-rotation on $\mathscr{B}_+$, Ω, and $C(\Omega)$. Hence, if ω in (7.3) is the zero configuration ($\omega(b)\equiv 0$) so that $\omega\circ T=\omega$, the invariance of μ^0 follows by letting $B\uparrow\mathscr{B}_+$ in (7.3). Q.E.D.

8. Independence of the Limit (Proof of Theorem 4.5)

Our first goal will be to find a criteria on a Gibbs state which insures that the limit in (4.5) exists and is the desired value. The following theorem is closely related to Theorem 6.2, and will take its place in this setting. Throughout this section h will denote a Wilson-like energy function.

Theorem 8.1. ($g=1$) *Let h be a Wilson-like energy function, p_0 be a fixed plaquette based at* 0, *and $\mu\in G(h)$ be any Gibbs state such that $\mu^0(h''(d\omega(p_0))=\mu(h''(d\omega(p_0)))$, where $\mu^0=\mu_1^0$ as defined in Eq. (4.4). Then*

$$|\mathrm{cov}_\mu(h''d\omega(p_0)), e^{i(J,j_a)}| \leqq K(\|j\|_\infty, \|dj\|_\infty, \|dj_a\|_2)\cdot O_{p_0}(a), \tag{8.1}$$

where K is an increasing function in its arguments, $O_{p_0}(a)\to 0$ as $a\to 0$, and j is any real test 1-*form on* $\mathbb{R}^4$ (i.e. *the estimate (6.19) is still valid).*

For the proof of this theorem we will need a fact which is Proposition 1 of Messager et al. [1] applied to this model.

Proposition 8.1. (*Messager et al.* [1]) *Let h be a Wilson-like energy function. Let μ^0 denote the Gibbs state defined in (4.4) with $g=1$, and $\mu\in G(h)$ be any other Gibbs state. Then $\mu^0(\cos(\sum_b m(b)\omega(b)))\geqq\mu(\cos(\sum_b m(b)\omega(b)))$ for any function $m:\mathscr{B}_+\to Z$ of finite support.*

Proof of Theorem 8.1. For ease of notation set $f_0=h''(d\omega(p_0))$. It is well known that any Gibbs state (μ) may be decomposed into its extreme states;

$$\mu(f)=\int_{G_e(h)} \nu(f)P(d\nu), \tag{8.2}$$

where P is a probability measure on the extreme Gibbs states ($G_e(h)$) and f is any continuous function on Ω (see for example Ruelle [1], Preston [1] or Dynkin [1]). Set $f=f_0$ in (8.2) and use $\mu(f_0)=\mu^0(f_0)\geqq\nu(f_0)$ for all $\nu\in G(h)$ (Proposition 8.1) to conclude that

$$P\{\nu\in G_e(h):\nu(f_0)=\mu^0(f_0)\}=1. \tag{8.3}$$

If f is any continuous function on Ω, let $\hat{f}:G_e(h)\to\mathbb{R}$ be the function $\hat{f}(\nu)=\nu(f)$. By definition, the function $\hat{f}$ is measurable on $G_e(h)$—the σ-algebra on $G_e(h)$ is taken to be the smallest σ-algebra for which the functions $\hat{f}$ are measurable (Dynkin [1]). With this notation we may restate (8.3) as $\hat{f}_0 = \mu^0(f_0)$ P-almost surely.

Now if f and g are two continuous functions on Ω and μ is given by Eq. (8.2), an easy computation shows that

$$\mathrm{cov}_\mu(f,g) = \int_{G_e(h)} \mathrm{cov}_\nu(f,g)P(d\nu) + \mathrm{cov}_P(\hat{f},\hat{g}). \tag{8.4}$$

In particular if $f = f_0$, then

$$\mathrm{cov}_\mu(f_0,g) = \int_{G_e(h)} \mathrm{cov}_\nu(f_0,g)P(d\nu), \tag{8.5}$$

since $\hat{f}_0$ is a constant P-almost surely. By Theorem 6.2, for each $\nu \in G_e(h)$ there exist functions K and O_ν such that

$$|\mathrm{cov}_\nu(f_0, e^{i(J,j_a)})| \leqq K(\|j\|_\infty, \|dj\|_\infty, \|dj_a\|_2)\cdot O_\nu(a), \tag{8.6}$$

where K is increasing in its arguments, $O_\nu(a) \to 0$ as $a \to 0$, and j is any real test 1-form. Furthermore, looking at the proof of Theorem 6.2, the function K may be chosen to be continuous and independent of the Gibbs state ν. For later convenience the function K is also chosen to be larger than one. It will be shown below (Lemma 8.3) that the functions $O_\nu(a)$ may be chosen so that for each fixed a the map $\nu \to O_\nu(a)$ is measurable on $G_e(h)$ and the map is bounded by $2\|h''\|_\infty$. So use the estimate (8.6) in Eq. (8.5) with $g = e^{i(J,j_a)}$ to get

$$|\mathrm{cov}_\mu(f_0, e^{i(J,j_a)})| \leqq K(\|j\|_\infty, \|dj\|_\infty, \|dj_a\|_2)\cdot O_{p_0}(a), \tag{8.7}$$

where

$$O_{p_0}(a) = \int_{G_e(h)} O_\nu(a)P(d\nu). \tag{8.8}$$

An application of the dominated convergence theorem shows that $O_{p_0}(a) \to 0$ as $a \to 0$. Q.E.D.

Corollary 8.1. *Let h be a Wilson-like energy function and $\mu \in G(h)$ be an invariant Gibbs state such that $\mu(h''(d\omega(p))) = \mu^0(h''(d\omega(p)))$ for some plaquette p (and hence all p's). Then for each real test 1-form on $\mathbb{R}^4$,*

$$\lim_{a \downarrow 0} \mu(e^{i(J,j_a)}) = \exp\left(\frac{-\alpha}{2}(dj, dj)\right), \tag{8.9}$$

where $\alpha = \mu^0(h''(d\omega(p_0)))$.

Proof. The proof is the same as the proof of Theorem 4.3 after using Theorem 8.1 in place of Theorem 6.2 part (b). Q.E.D.

This corollary is the desired convergence criteria that we were seeking. The technical detail of measurability will be completed at the end of this section.

The next objective is the proof of Theorem 4.5. In view of Corollary 8.1, it is enough to show the following proposition.

Proposition 8.2. (*Dimension* $= d$.) *Let h be a Wilson-like energy function and μ_β^0 denote the limit in (4.4) with $g^{-2} = \beta$. Then each translation invariant measure*

$\mu \in G(\beta h)$ satisfies $\mu(h''(d\omega(p))) = \mu^0_\beta(h''(d\omega(p)))$ (p is any plaquette), for all but at most a countable number of $\beta > 0$.

Remark 8.1. This result is modeled on Corollary 4.3 of Pfister [1], and Proposition 3.5 of Fröhlich and Pfister [1].

Proof. Since h is a Wilson-like energy,

$$h''(x) = \sum_{k=1}^{N} k^2 b_k \cos(kx). \tag{8.10}$$

Let

$$P(\beta) \equiv P(\varphi^{\beta h}) \tag{8.11}$$

be the pressure $P(\varphi^{\beta h})$ as defined in Eq. (A.10) of the appendix. By Theorem A.1, the function $P(\beta)$ is a convex continuous function. So by standard facts about convex functions, $P'(\beta)$ exists for all but a countable number of $\beta > 0$. By Corollary A.1, Definition 3.1, and $\varphi^{\beta h} = \beta \varphi^h$;

$$\begin{aligned} P'(\beta) &= -\mu\Big(\sum_{B \subset \mathscr{B}_+ : 0 \in B^0} |B^0|^{-1} \cdot \varphi^h_B\Big) = -\mu\Big(\sum_{p \subset \mathscr{P}_+ : 0 \in \partial p^0} h(d\omega(p))/4\Big) \\ &= -\mu\Big(\sum_{p \subset \mathscr{P}_+ : 0 \in \mathrm{base}(p)} h(d\omega(p))\Big) = \mu\Big(\sum_{p \subset \mathscr{P}_+ : 0 \in \mathrm{base}(p)} h''(d\omega(p))\Big), \end{aligned} \tag{8.12}$$

where $\mu \in G(\beta h)$ is any translation invariant Gibbs state (which includes μ^0_β) and β is a point where $P'(\beta)$ exists. By Eq. (8.10), h'' is a sum of cosine terms with positive coefficients, and hence by Proposition 8.1.

$$\mu^0_\beta(h''(d\omega(p))) \geqq \mu(h''(d\omega(p))) \tag{8.13}$$

for all plaquettes p and $\mu \in G(\beta h)$. In view of Eq. (8.12) and (8.13), we conclude that

$$\mu^0_\beta(h''(d\omega(p))) = \mu(h''(d\omega(p))) \tag{8.14}$$

for all plaquettes p and translation invariant $\mu \in G(\beta h)$. Equation (8.14) is valid for all β for which $P'(\beta)$ exists, that is for all but a countable number of β's. Q.E.D.

Proof of Theorem 4.5. As already noted, Theorem 4.5 is a direct consequence of Corollary 8.1 and Proposition 8.2—take $\beta = g^{-2}$. Q.E.D.

We now finish with the technical measurability detail which was left open in the proof of Theorem 8.1.

Lemma 8.1. *The collection of real continuous functions on $\mathbb{R}^4$ with compact support ($C_c(\mathbb{R}^4)$) is separable in the sub-norm topology. Furthermore, a countable dense set $D \subset C_c(\mathbb{R}^4)$ may be chosen to have the following property. If $f \in C_c(\mathbb{R}^4)$ and n is sufficiently large such that the* $\mathrm{supp}(f) \subset B(0,n) \equiv \{x \in \mathbb{R}^4 : |x| \leqq n\}$, *then there exists $g \in D$ arbitrarily close to f with the* $\mathrm{supp}(g) \subset B(0, n+1)$.

Proof. Let D' be the collection of continuous functions formed by taking polynomials with rational coefficients of the functions $x \to |x-y| : \mathbb{R}^4 \to \mathbb{R}$, where $y \in \mathbb{R}^4$ with rational components. The collection D' is a countable set. By the Stone–Weierstrass theorem the collection D' when restricted to any compact set $K \subset \mathbb{R}^4$ is dense in $C(K)$.

For each positive integer n, choose $g_n \in C_c(\mathbb{R}^4)$ such that $\mathrm{supp}(g_n) \subset B(0, n+1)$, $g_n \equiv 1$ on $B(0,n)$, and $0 \leqq g_n \leqq 1$. We now define the countable collection of functions (D) as $D = \{fg_n : f \in D'$ and n a positive integer$\}$. Then if $f \in C_c(\mathbb{R}^4)$ with $\mathrm{supp}(f) \subset B(0,n)$ there exists $h_k \in D'$ such that $\| f - h_k \|_{L^\infty(B(0,n+1))} \to 0$ as $k \to \infty$. Hence $\| f - g_n h_k \|_\infty \to 0$ as $k \to \infty$, since $\| f - g_n h_k \|_\infty \leqq \| f - h_k \|_{L^\infty(B(0,n+1))}$. Q.E.D.

Definition 8.1. A continuous k-form (φ) on $\mathbb{R}^d$ is a continuous function from $\mathbb{R}^d$ to the degree-k exterior algebra over $\mathbb{R}^d$. In other words, φ is a differential k-form except that the standard coefficients are only required to be continuous rather than smooth.

Lemma 8.2. *Let $a > 0$ be fixed. Let X denote the space of pairs $\langle j, \varphi \rangle$, where j is a continuous 1-form and φ is a continuous 2-form on $\mathbb{R}^4$ both with compact support. The space X is given the norm*

$$\| \langle j, \varphi \rangle \| \equiv \| j \|_\infty + \| \varphi \|_\infty + \| \varphi_a \|_2 . \tag{8.15}$$

With the above notation, the space $(X, \|\cdot\|)$ is a separable space.

Proof. Put $\| \langle j, \varphi \rangle \|_\infty = \| j \|_\infty + \| \varphi_a \|_\infty$. Let $D_0 \subset X$ be the collection of pairs which have all of their components in the set D of Lemma 8.1. The countable set D_0 is clearly dense in the space $(X, \| \langle \cdot, \cdot \rangle \|_\infty)$. So let $\varepsilon > 0$, and $\langle j, \varphi \rangle \in X$ be given, and suppose that $\mathrm{supp}(\langle j, \varphi \rangle) \subset B(0,n)$. Then by Lemma 8.1, there exist $\langle j_k, \varphi_k \rangle \in D_0$ supported in $B(0, n+1)$ converging to $\langle j, \varphi \rangle$ in the sup-norm. It follows by the easy estimate

$$\| (\varphi - \varphi_k)_a \|_2 \leqq K[n + 1 + 2a]^4 \| \varphi - \varphi_k \|_\infty , \tag{8.16}$$

that $\| \langle j_k, \varphi_k \rangle - \langle j, \varphi \rangle \| \to 0$ as $n \to \infty$, where K is the volume of the unit sphere in 4-dimensions. Hence, D_0 is also a countable dense set in $(X, \|\cdot\|)$. Q.E.D.

Lemma 8.3. *Let K be a continuous function, increasing in its arguments, $K \geqq 1$, and such that an estimate of form (6.19) of Theorem 6.2 is valid for all extreme Gibbs states ν. Then for each $a \in (0,1)$, the function*

$$O_\nu(a) \equiv \sup \left\{ \frac{|\mathrm{cov}_\nu(f_0, e^{i(J, j_a)})|}{K(\| j \|_\infty, \| dj \|_\infty, \| dj_a \|_2)} : j \text{ is a test 1-form.} \right\} \tag{8.17}$$

is measurable as a function of $\nu \in G_e(h)$. Furthermore $O_\nu(a)$ is uniformly bounded by $2 \| h'' \|_\infty$, and $O_\nu(a) \to 0$ as $a \to 0$.

Proof. Each real test 1-form (j) may be identified with the element $\langle j, dj \rangle \in X$, where X is the space defined in Lemma 8.2. The space of test 1-forms given the norm

$$\| j \| \equiv \| \langle j, dj \rangle \| \tag{8.18}$$

is a subspace of the separable normed space X, and hence is separable. The expression in the bracket of Eq. (8.17) is easily seen to be continuous in the $\|\cdot\|$-topology of (8.15). So it suffices to take the supremum in (8.17) over a countable set. But the expression in the braces of (8.17) when considered as a function of ν is measurable (by definition), and hence so is $\nu \to O_\nu(a)$.

The estimate that $O_\nu(a) \leqq 2 \| h'' \|_\infty$ is trivial. The fact that $O_\nu(a) \to 0$ as $a \to 0$ is Theorem 6.2. Q.E.D.

Appendix: Pressure

For this appendix, $\varphi = \{\varphi_B\}_{B\subset\subset\mathscr{B}_+}$ ($\varphi_B:\Omega\to\mathbb{R}$ such that $\varphi_B(\omega)$ only depends on ω_B) is an arbitrary finite range translation invariant (defined below) interaction potential on $\Omega=(S^1)^{\mathscr{B}_+}$. The case of interest is the interaction potential in Definition 3.1. The norm of such an interaction potential is defined to be

$$\|\varphi\| \equiv \sup_{b\in\mathscr{B}_+} \sum_{B:b\in B\subset\subset\mathscr{B}_+} |B|^{-1}\|\varphi_B\|_\infty. \tag{A.1}$$

The group Z^d naturally acts on the lattice $\mathscr{B}_+$ via

$$T_x[(e_i)_y] \equiv (e_i)_{(x+y)}. \tag{A.2}$$

So T_x acts on a bond simply by translating the base point. The action on Z^d naturally induces actions on Ω, and the functions on Ω,

$$T_x(\omega)=\omega\circ T_{-x}, \tag{A.3}$$

$$T_x(f)(\omega)=f(\omega\circ T_x). \tag{A.4}$$

Definition A.1. The interaction potential (φ) is translation invariant if

$$\varphi_B(\omega\circ T_x)=\varphi_{T_x(B)}(\omega), \tag{A.5}$$

for all $x\in Z^d$ and $B\subset\subset\mathscr{B}_+$.

Definition A.2. Let $\Lambda\subset Z^d$. The set of positively oriented bonds associated to Λ is

$$\Lambda^{(1)}=\{(e_i)_x | x\in\Lambda^d \text{ and } i=1,2,\dots,d\}. \tag{A.6}$$

Definition A.3. The pressure ($P_\Lambda(\omega,\varphi)$) of a translation invariant, finite range interaction potential (φ) on Ω, given a configuration (ω) and $\Lambda\subset\subset Z^d$, is

$$P_\Lambda(\omega,\varphi)=|\Lambda|^{-1}\ln[Z^\varphi_{\Lambda^{(1)}}(\omega)], \tag{A.7}$$

where

$$Z^\varphi_{\Lambda^{(1)}}(\omega)=\int_{(S^1)^{\mathscr{B}_+}} \exp(-H^\varphi_{\Lambda^{(1)}}(\omega'|\omega))d\lambda^{\Lambda^{(1)}}(\omega'_{\Lambda^{(1)}}) \tag{A.8}$$

is the normalization constant for the associated specification, and

$$H^\varphi_{\Lambda^{(1)}}(\omega'|\omega)=\sum_{B\subset\subset\mathscr{B}_+:B\cap\Lambda^{(1)}\neq\phi} \varphi_B(\omega'_{\Lambda^{(1)}}\times\omega_{\mathscr{B}_+\setminus\Lambda^{(1)}}) \tag{A.9}$$

is the energy of the configuration ω' in $\Lambda^{(1)}$ given the "boundary conditions" ω.

Definition A.4. If $B\subset\subset\mathscr{B}_+$, put $B^0=\{x\in Z^d|\exists$ a bond $b\in B$ with x as its base point$\}$, the base of A.

Theorem A.1. *The infinite volume pressure (P) defined by the limit*

$$P(\varphi)=\lim_{\Lambda\uparrow Z^d} P_\Lambda(\omega,\varphi) \tag{A.10}$$

exists and is independent of the boundary conditions (ω), as Λ increases to Z^d through cubes. The limiting function P is a continuous convex function on the space of finite range translation invariant interaction potentials. Furthermore, there is a one-to-one *correspondence between tangent functionals to the pressure at φ and translation invariant Gibbs states of the interaction potential φ. (A tangent functional to P at φ is a continuous linear functional (α) such that $P(\psi)-P(\varphi)\geqq\alpha(\psi-\varphi)$ for all invariant interaction potentials ψ.) If μ is a translation invariant Gibbs state, the associated tangent functional is*

$$\alpha(\psi)=-\mu\Big[\sum_{B:0\in B^0}|B^0|^{-1}\psi_B\Big]. \tag{A.11}$$

If α is a tangent functional, the associated translation invariant Gibbs state is determined by

$$\mu(f)=-\alpha(\psi^f), \tag{A.12}$$

where f is a continuous function depending only on the bond variables over some set $A\subset\subset B_+$, and

$$\psi_B^f(\omega)=\begin{cases} f(\omega\circ T_x) & \text{if } B=T_x(A) \text{ for some } x\in Z^d \\ 0 & \text{otherwise}\end{cases}. \tag{A.13}$$

Proof. This is easily reduced to the corresponding well known statements for lattice models over Z^d, see Israel [1]. Indeed, if $T\equiv S^d$, then the map $K:S^{\mathscr{B}_+}\to T^{Z^d}$, given by $K(\omega)(x)=\{\omega((e_i)_x)\}_{i=1}^d$, can be used to map the lattice system over $S^{\mathscr{B}_+}$ to one over T^{Z^d}. Q.E.D.

Corollary B.1. *Let P and φ be as above and let $P(\beta)\equiv P(\beta\varphi)$ for all $\beta>0$. If $P'(\beta)$ exists at β, then*

$$P'(\beta)=-\mu\Big[\sum_{B:0\in B^0}|B^0|^{-1}\varphi_B\Big]\equiv\alpha_\mu(\varphi) \tag{A.14}$$

for all translation invariant Gibbs states $\mu\in G(\Pi^{\beta\varphi})$.

Proof. By Theorem A.1,

$$P(\beta+h)-P(\beta)\geqq h\alpha_\mu(\varphi) \tag{A.15}$$

for all $h\in\mathbb{R}$ and $\mu\in G(\Pi^{\beta\varphi})$ which are translation invariant. Divide both sides of (A.15) by $|h|$ and take the limit as h tends to zero from above and below to get $\pm P'(\beta)\geqq\pm\beta\alpha_\mu(\varphi)$. Q.E.D.

Acknowledgement. It is a pleasure to thank L. Gross for many useful discussions.

References

Balaban, T.:

1. $(\text{Higgs})_{2,3}$ quantum fields in a finite volume, I. A lower bound. Commun. Math. Phys. **85**, 603–636 (1982);
2. $(\text{Higgs})_{2,3}$ quantum fields in a finite volume, II. An upper bound. Commun. Math. Phys. **86**, 555–594 (1982);
3. $(\text{Higgs})_{2,3}$ quantum fields in a finite volume, III. Renormalization. Commun. Math. Phys. **88**, 411–445 (1983);
4. Regularity and decay of lattice green's functions. Commun. Math. Phys. **89**, 571–597 (1983);
5. Renormalization group methods in non-abelian gauge theories. Harvard preprint, HUTMP B134;

6. Propagators and renormalization transformations for lattice gauge theories. I. Commun. Math. Phys. **95**, 17–40 (1984);
7. Propagators and renormalization transformations for lattice gauge theories. II. Commun. Math. Phys. **96**, 223–250 (1984);
8. Recent results in constructing gauge fields. Physica **124A**, 79–90 (1984);
9. Averaging operations for lattice gauge theories. Commun. Math. Phys. **98**, 17–51 (1985);
10. Spaces of regular gauge field cinfigurations on a lattice and gauge fixing conditions. Commun. Math. Phys. **99**, 75–102 (1985);
11. Propogators for lattice gauge theories in a background field. Commun. Math. Phys. **99**, 389–434 (1985);
12. Ultraviolet stability of three-dimensional lattice pure gauge field theories. Commun. Math. Phys. **102**, 255–275 (1985);
13. The variational problem and background fields in renormalization group method for lattice gauge theories. Commun. Math. Phys. **102**, 277–309 (1985);
14. Renormalization group approach to lattice gauge fields theories. I. Generation of effective actions in a small fields approximation and a coupling constant renormalization in four dimensions. Commun. Math. Phys. (in press) 1987

Bott, R., Tu, L. W.:
1. Differential forms in algebraic topology. Berlin, Heidelberg, New York: Springer 1982

Dobrushin, R. L.:
1. Prescribing a system of random variables by conditional distributions. Theory Probab. Appl. **15**, 458–486 (1970)

Dynkin, E. B.:
1. Sufficient statistics and extreme points. Ann. Probab. **6**, 705–730 (1978)

Federbush, P.:
1. A phase cell approach to Yang–Mills theory, O. Introductory exposition. Univ. Mich. preprint (1984);
2. A phase cell approach to Yang–Mills theory, I. Small field modes. Univ. Mich. preprint (1984);
3. A phase cell approach to Yang–Mills theory, I. Modes, lattice-continuum duality. Commun. Math. Phys. **107**, 319–329 (1986);
4. A phase cell approach to Yang–Mills theory, III. Local stability, modified renormalization group transformation Commun. Math. Phys. (in press) 1987
5. A phase cell approach to Yang–Mills theory, IV. General modes and general stability. Univ. Mich. preprint (1985);
6. A phase cell approach to Yang–Mills theory, V. Stability and interpolation in the small field region, covariance fall-off. Univ. Mich. preprint (1986);
7. A phase cell approach to Yang–Mills theory, VI. Non-abelian lattice-continuum duality. Univ. Mich. preprint (1986)

Follmer, H.:
1. A covariance estimate for Gibbs measures. J. Funct. Anal. **46**, 387–395 (1982)

Fröhlich, J., Pfister, C.:
1. Spin waves, vortices, and the structure of equilibrium states in the classical xy model. Commun. Math. Phys. **89**, 303–327 (1983)

Gross, L.:
1. Decay of correlations in classical lattice models at high temperature. Commun. Math. Phys. **68**, 9–27 (1979);
2. Absence of second-order phase transitions in the Dobrushin uniqueness region. J. Stat. Phys. **25**, 57–72 (1980);
3. Convergence of $U(1)_3$ lattice gauge theory to its continuum limit. Commun. Math. Phys. **92**, 137–162 (1983);
4. A Poincare lemma for connection forms. J. Funct. Anal. **63**, 1–46 (1985);
5. Lattice gauge theory; Heuristics and convergence. In: Stochastic processes—Mathematics and physics. Lecture Notes in Mathematics, vol. **1158**, Albeverio, S., Blanchard Ph., Streit, L. (eds.). Berlin, Heidelberg, New York: Springer 1986

Guth A.:
1. Existence proof of a nonconfining phase in four-dimensional $U(1)$ lattice gauge theory. Phys. Rev. **D21**, 2291–2307 (1980)
Israel R. B.:
1. Convexity in the theory of lattice gases. Princeton, NJ: Princeton University Press 1979
King, C.:
1. The $U(1)$ Higgs model, I. The continuum limit. Commun. Math. Phys. **102**, 649–677 (1986);
2. The $U(1)$ Higgs model, II. The infinite volume limit. Commun. Math. Phys. **103**, 323–349 (1986)
Kobayashi, S., Nomizu, K.:
1. Foundations of differential geometry, Vol. **1**. New York: Interscience 1963
Messager, A., Miracle-Sole, S., Pfister, C.:
1. Correlation inequalities and uniqueness of the equilibrium state for the plane rotator ferromagnetic model. Commun. Math Phys. **58**, 19–29 (1978)
Ruelle, D.:
1. Thermodynamic formalism. Reading, MA: Addison–Wesley 1978
Simon, B.:
1. Functional integration and quantum physics. New York: Academic Press 1979;
2. A remark on Dobrushin's uniqueness theorem. Commun. Math. Phys. **68**, 183–185 (1979)
Spivak, M.:
1. A comprehensive introduction to differential Geometry, Vol. **1**, Wilmington, DE: Publish or Perish 1979
Wilson, K. G.:
1. Confinement of quarks. Phys. Rev. **D10**, 2445–2459 (1974);
2. Quantum chromodynamics on a lattice, in Quandum Field Theory and Statistical Mechanics, Cargese 1976. Levy M., Mitter, P. (eds.) New York: Plenum Press 1977, pp. 143–172

Communicated by A. Jaffe

Received September 22, 1986; in revised form December 16, 1986