A11

CENTRO VITO VOLTERRA

UNIVERSITÀ DEGLI STUDI DI ROMA II

Loop space, curvature, and quasi–invariance

Bruce Driver

Department of Mathematics, 0112

University of California, San Diego

La Jolla, California, 92093–0112

Abstract

These notes consist of two topics. The first topic is concerned with the classification of vector bundle covariant derivative pairs by the corresponding parallel translation operator restricted to based loops on the base manifold. The second topic is concerned with a generalization of the classical Cameron–Martin quasi–invariance theorem for classical Wiener spaces to Wiener spaces associated to a compact manifold.

October '92

N. 123

Via Fontanile di Carcaricola 00133, Rome Italy Tel. 39/6/20.22.498 – 202.906.86 Fax 2022539 e–mail – Volterra@ vaxtvm.infn.it

Centro Vito Volterra

Universita Degli Studi Di Roma II

LOOP SPACE, CURVATURE, AND QUASI-INVARIANCE

Bruce Driver*
Department of Mathematics, 0112
University of California, San Diego
La Jolla, California, 92093-0112

Abstract

These notes consist of two topics. The first topic is concerned with the classification of vector bundle covariant derivative pairs by the corresponding parallel translation operator restricted to based loops on the base manifold. The second topic is concerned with a generalization of the classical Cameron-Martin quasi-invariance theorem for classical Wiener spaces to Wiener spaces associated to a compact manifold.

Section 0: Introduction.

These notes represent an expanded version of a series of four lectures that I gave for the Semestre Di Analisi A Infinite Dimensioni, at Dipartimento di Matematic - Centro Vito Volterra (Rome II) in October of 1992. The author would like to thank Professor Accardi and Centro Volterra for the opportunity to give these talks and for a very pleasent and productive stay in Rome.

These notes are organized as follows. In Part I (Sections 1. and 2.) the problem of classification of bundle covariant derivative pairs by parallel translation is discussed. In Part II (Sections 3. and 4.) a generalization of the Cameron-Martin theorem to manifolds is discussed. The remainder of this introduction is devoted to the motivation of the results in Part I.

The results to be described in part I of these talks were motivated by the problem of constructing Yang-Mills Quantum field theories. This problem may be described as defining a certain probability measure (μ) on a space of connection 1-forms $\mathcal{A}$. (An element $A \in \mathcal{A}$ is of the form $A = \sum_{i=1}^{d} A_i dx^i$ where each component A_i is a function on $\mathbf{R}^d$ taking values in $U(N)$-the $N \times N$ complex unitary matrices.) The measure μ is to be "given" by the following heuristic expression:

$$d\mu(A) = Z^{-1} \exp[\sum_{i<j} \int tr(R^A_{i,j}(x))^2 dx] DA, \qquad (YM)$$

* Financial support from Centro Vito Volterra and NSF Grant DMS 91-01720 is gratefully acknowledged

where $R^A_{i,j} \equiv \partial_i A_j - \partial_j A_i + [A_i, A_j]$ is the curvature of A, DA denotes "infinite dimensional" Lebesgue measure on $\mathcal{A}$, and Z is a normalization constant chosen to make μ a probability measure. It turns out that even at this informal level the expression in (YM) is not well defined. This is because (heuristically) both the exponent in (YM) and Lebesgue measure DA are invariant under the action of the infinite dimensional gauge group G of functions g from $\mathbf{R}^d$ to $U(N)$. This action of G on $\mathcal{A}$ is given by $(g, A) \to A^g \equiv g^{-1}dg + g^{-1}Ag$. Thus using (YM) to define a probability measure on $\mathcal{A}$ would be similar to defining a probabilty measure ν on R^2 by the formula $d\nu(x,y) = Z^{-1}\exp-(x^2)dxdy$. For this reason it is standard to reinterpret the measure μ given in (YM) as a probability measure on the quotient space $\mathcal{A}/\mathcal{G}$ rather than a measure on $\mathcal{A}$. The reader interested in more information on this topic may consult Driver [D2,D3] and Gross [G2] and the references thererin.

With the above comments in mind, in order to define the measure μ it should be helpful to have "nice" parameterizations of the "moduli space" $\mathcal{A}/\mathcal{G}$. Such parameterizations are the topics of the first two lectures.

Lecture 1. Trivial Bundle Case.

We start this lecture with a review of connections, curvature and parallel translation for trivial vector bundles. The main theorem in this section is Theorem 1.15 which states that a connection on a trivial bundle can be recovered upto a "gauge transformation" from the knowledge of its parallel translation operator on based loops. The basic reference for this section is Gross [G1], see also Driver [D1] The discussion here is more geometrical than that of [G1] or [D1].

Let (M, o) be a connected manifold with base point $o \in M$. Let $W^1(M)$ be the C^1-paths on M starting at $o \in M$ which are parametrized by the unit interval $I \equiv [0,1]$. Let $\mathcal{L}^1(M) \subseteq W^1(M)$ be the subset of loops based at o. For the purposes of this lecture the reader may assume that $M = \mathbf{R}^d$ and $o = 0 \in \mathbf{R}^d$.

We now suppose the $\mathcal{V}$ is a given finite dimensional vector space. Let us recall the notation of connections, covariant derivatives, and parallel translation for the trivial vector bundle $E \equiv M \times \mathcal{V}$

Definition 1.1. A connection 1-form (A) on E is a $gl(\mathcal{V})$-valued 1-form on M, where $gl(\mathcal{V})$ is the Lie algebra of $GL(\mathcal{V})$ – the general linear group on $\mathcal{V}$.

Definition 1.2. Corresponding to the connection 1-form (A) is the covariant derivative (∇) acting on sections (S) of E. More explicitly let $S : M \longrightarrow \mathcal{V}$ be a smooth function and $v_m \in T_m M$ - the tangent space to M at m. Then $\nabla_v S$ is given as:

$$\nabla_v S = dS < v > + A < v > S(m), \qquad (1.1)$$

where $dS < v > \equiv \frac{d}{dt}|_0 S(\sigma(t))$ provided $\frac{d}{dt}\sigma(t)|_{t=0} = v$.

Definition 1.3. Given a curve $\sigma : \mathbf{R} \longrightarrow M$ and a function $\overline{S} : \mathbf{R} \longrightarrow \mathcal{V}$, we define $\frac{\nabla \overline{S}}{dt}$

for all $v, w \in T_m M$ and $m \in M$.

Proof: Direct computation using the fact that

$$dA < \dot{\sigma}, \sigma' > = \frac{\partial}{\partial t} A < \sigma' > - \frac{\partial}{\partial s} A < \dot{\sigma} > .$$

Q.E.D.

Theorem 1.9. Let $\{\gamma(t)\}_{t \in \mathbf{R}}$ be a one parameter family of paths in $W^1(M)$ such that $\gamma(t, s) \equiv \gamma(t)(s)$ is smooth in (t, s). Then

$$\frac{\nabla}{dt} P_1^A(\gamma(t)) = P_1^A(\gamma(t)) B^A < \dot{\gamma}(t) >, \tag{1.9}$$

where B^A is the $GL(V)$ -valued 1-form on $W^1(M)$ given by

$$B^A < X > \equiv \int_0^1 P_s^A(\sigma)^{-1} R^A < \sigma'(s), X(s) > P_s^A(\sigma) ds \tag{1.10}$$

for each $X \in T_\sigma W^1(M)$ and $\sigma \in W^1(M)$.

Remark 1.10. Notice that a tangent vector $X \in T_\sigma W^1(M)$ is by definition the derivative $(\dot{\gamma}(0))$ of a 1-parameter family of curves $\{\gamma(t)\}_{t \in \mathbf{R}}$ such that $\gamma(0) = \sigma$. Therefore $X(s) = \frac{d}{dt}|_0 \gamma(t, s) \in T_{\sigma(s)} M$ - that is $X(s)$ is a vector field along σ.

Proof of Theorem 1.9: By the definition of parallel transport we know $\nabla P_s^A(\sigma)/ds = 0$ for all (s, t). Hence

$$\frac{\nabla}{ds} \frac{\nabla}{dt} P_s^A(\gamma(t)) = [\frac{\nabla}{ds}, \frac{\nabla}{dt}] P_s^A(\gamma(t)) = R^A < \gamma'(t, s), \dot{\gamma}(t, s) > P_s^A(\gamma(t)), \tag{1.11}$$

using Lemma 1.8. in the last equality. From (1.11) and Lemma 1.7. it follows that

$$\frac{d}{ds} [P_s^A(\gamma(t))^{-1} \frac{\nabla P_s^A(\gamma(t))}{dt}] = P_s^A(\gamma(t))^{-1} R^A < \gamma'(t, s), \dot{\gamma}(t, s) > P_s^A(\gamma(t)) \tag{1.12}$$

The theorem is now easily proved by integrating (1.12) relative to s using $\nabla P_0^A(\gamma(t))/dt = 0$. Q.E.D..

Remark 1.11. Equation (1.9) may also be written as

$$\frac{d}{dt} P_1^A(\gamma(t)) = P_1^A(\gamma(t)) B^A < \dot{\gamma}(t) > - A < \dot{\gamma}(t, 1) > P_1^A(\gamma(t)) \tag{1.13}$$

Remark 1.12. It is easy to check that knowledge of B^A on $T\mathcal{L}^1(M)$ (the set of vector fields (X) along loops $\sigma \in \mathcal{L}^1(M)$ such that $X(0) = X(1) = 0$) uniquely determines B^A on all of $TW^1(M)$.

Corollary 1.13. The 1-form B^A restricted to $\mathcal{L}^1(M)$ may be computed as:

$$B^A = (P_1^A)^{-1} d(P_1^A) \text{ on } \mathcal{L}^1(M).$$

Remark 1.14. If $g : M \longrightarrow GL(\mathcal{V})$ is a *restricted* gauge transform (*i.e.* $g(0) = I$) then by Lemma 1.6, $P^{A^g} = P^A$ on $\mathcal{L}^1(M)$. Hence by Remark 1.12 and Corollary 1.13 it follows that $B^{A^g} = B^A$ on $TW^1(M)$. This is also possible to check by direct computation.

Theorem 1.15. Let A be a given connection 1-form, $M = \mathbf{R}^d$, and set $P(\sigma) \equiv P_1^A(\sigma)$ for $\sigma \in \mathcal{L}^1(M)$. The connection 1-form A may be recovered upto gauge equivalence from the function $P : \mathcal{L}^1(M) \longrightarrow GL(\mathcal{V})$.

Proof: Define the 1-form B on $T\mathcal{L}^1(M)$ by $B \equiv P^{-1}dP$. By Remark 1.12 this can be extended to $TW^1(M)$. Now for each $x \in M = \mathbf{R}^d$ choose a path $\Sigma_x \in W^1(M)$ such $\Sigma_x(1) = x$ and the map $(x, s) \longrightarrow \Sigma_x(s)$ is smooth. For example let $\Sigma_x(s) = sx$. Define $g(x) \equiv P^A(\Sigma_x)$, and notice by Lemma 1.6 that

$$I = g(x)^{-1} P_1^A(\Sigma_x) = P_1^{A^g}(\Sigma_x) \tag{1.14}$$

Let $\alpha(t)$ be a curve in $M = \mathbf{R}^d$, so by (1.14) and (1.13) with A replaced by A^g we find

$$\begin{aligned} 0 = \frac{d}{dt} I = \frac{d}{dt} P_1^{A^g}(\Sigma_{\alpha(t)}) &= B^{A^g} < \frac{d}{dt}\Sigma_{\alpha(t)} > - A^g < \dot\alpha(t) > \\ &= B < \frac{d}{dt}\Sigma_{\alpha(t)} > - A^g < \dot\alpha(t) >, \end{aligned} \tag{1.15}$$

where we have used Remark 1.14 to replace B^{A^g} by $B^A = B$. But Eq. (1.15) shows $A^g < \dot\alpha(t) > = B < \frac{d}{dt}\Sigma_{\alpha(t)} >$, that is B determines A modulo the gauge transformation g.
Q.E.D.

Section 2: The General Case.

In this section we review the notions of vector bundles, covariant derivatives, curvature, and parallel translation. The main theorem is the classification Theorem 2.15. The basic reference for this section is [D1].

In this lecture we keep the same notation as in Lecture 1 with the exception that E is now a general vector bundle. We recall the definition.

Definition 2.1. A vector bundle (E) over M with model space $\mathcal{V}$ is an assignment of a vector space (E_m) for each $m \in M$ such that E_m is isomorphic to $\mathcal{V}$ and

i) $E_m \cap E'_m = \emptyset$ for $m \neq m'$ and $E = \bigcup_{m\in M} E_m$ is a smooth manifold.

ii) The map $\pi : E \longrightarrow M$ given by $\pi(E_m) = \{m\}$ is smooth.

iii) $\exists$ an open cover $\{\mathcal{O}_a\}$ of M and functions $\{U_a\}$ defined on $\mathcal{O}_a$ such that for all $x \in \mathcal{O}_a$, $U_a(x) : \mathcal{V} \longrightarrow E_x$ is an isomorphism and the map $x \longrightarrow U_a(x)\xi$ from $\mathcal{O}_a$ to E is a smooth $\forall \xi \in \mathcal{V}$. (The U_a's are called local frames.)

Example 2.2a. $E = M \times \mathcal{V}$, $E_m = \{m\} \times \mathcal{V}$, and $U(x)\xi = (x, \xi)$ for all $x \in M$. $E = M \times \mathcal{V}$ is called a trivial bundle.

Example 2.2b. Tangent bundle TM of M with model space $\mathbf{R}^d$ where d is the dimension of M.

Definition 2.3. Given a system of local frames of E as in Definition 2.1. we may form the transition functions

$$g_{ab}(x) \equiv U_a(x)^{-1} U_b(x) \in GL(\mathcal{V}) \text{ for } x \in \mathcal{O}_a \cap \mathcal{O}_b.$$

Proposition 2.4. The functions $\{g_{ab}\}$ satisfy the *cocycle condition*:

$$g_{ac}(x) = g_{ab}(x) g_{bc}(x) \quad \forall x \in \mathcal{O}_a \cap \mathcal{O}_b \cap \mathcal{O}_c.$$

Conversley to each collection of smooth functions $g_{ab} : \mathcal{O}_a \cap \mathcal{O}_b \longrightarrow GL(\mathcal{V})$ satisfying the cocycle condition above there exists a vector bundle E over M modeled on $\mathcal{V}$ with a collection of local frames $\{U_a\}$ such that $g_{ab}(x) = U_a(x)^{-1}U_b(x)$.

Proof: The proof of the first statement is easy. For a proof of the second statement see Steenrod [S].

Definition 2.5. a) A section (S) of E is a smooth function $S : M \longrightarrow E$ such that $S(m) \in E_m \quad \forall m \in M$. Let $\Gamma(E)$ denote the sections of E.

b) A section along a parametrized curve $(s \longrightarrow \sigma(s))$ in M is a function $(s \longrightarrow S(s))$ such that $S(s) \in E_{\sigma(s)}$ for all s.

c) A section along a parametrized surface $((s,t) \longrightarrow \gamma(s,t))$ in M is a function $((s,t) \longrightarrow S(s,t))$ such that $S(s,t) \in E_{\gamma(s,t)}$ for all (s,t).

Definition 2.6. A covariant derivative ∇ on E is a map $\nabla : TM \times \Gamma(E) \longrightarrow E$ (we will write $\nabla_v S$ for $\nabla(v, S)$) such that

i) For $v \in T_m M$ and $S \in \Gamma(E), \nabla_v S \in E_m$.

ii) $v \longrightarrow \nabla_v S$ is linear on $T_m M$ for all $m \in M$.

iii) For sections $S_1, S_2 \in \Gamma(E)$, $f \in C^\infty(M)$ and $v \in T_m M$; ∇_v satisfies

$$\nabla_v(S_1 + fS_2) = \nabla_v S_1 + (vf)S_2(m) + f(m)\nabla_v S_2.$$

Definition 2.14. P_1 and $P_2 \in \mathcal{P}$ are said to be equivalent if there exists a $g \in GL(\mathcal{V})$ such that $gP_2(\sigma)g^{-1} = P_1(\sigma)$ for all $\sigma \in \mathcal{L}^1(M)$. Denote the equivalence class containing $P \in \mathcal{P}$ by $[P]$, and the set of all such equivalence classes by $\overline{\mathcal{P}}$.

Theorem 2.15 Given a bundle covariant derivative pair (E, ∇), define $[P^\nabla] \in \overline{\mathcal{P}}$ by $[u_o^{-1}P^\nabla u_o]$ where $u_o : \mathcal{V} \longrightarrow E_o$ is any linear isomorphism. Then the map $[(E, \nabla)] \longrightarrow [P^\nabla]$ of $\mathcal{E}$ to $\overline{\mathcal{P}}$ is a 1-1 correspondence.

Proof: (Sketch) We will restrict our attenetion to the proof that $P \in \mathcal{P}$ can be used to construct a bundle covariant derivative pair (E, ∇). For this purpose choose an open cover $\{\mathcal{O}_a\}$ of M such that each open set $\mathcal{O}_a$ has the following property. For each $x \in \mathcal{O}_a$ there exists a path $\Sigma_x^a \in W^1(M)$ such that $\Sigma_x^a(1) = x \quad \forall x \in \mathcal{O}_a$ and $(x, s) \longrightarrow \Sigma_x^a(s) \in M$ is smooth. Now define $g_{ab}(x) = P(\overline{\Sigma_x^a}\Sigma_x^b)$ for $x \in \mathcal{O}_a \cap \mathcal{O}_b$, where $\overline{\Sigma_x^a}$ denotes the path Σ_x^a traversed in the reverse direction. It can be shown that the functions g_{ab} satisfy the cocycle condition in Proposition 2.4. and hence define a vector bundle E over M.

Now set $B = P^{-1}dP$ on $\mathcal{L}^1(M)$. It can be shown that B may be extended to $TW^1(M)$ in a natural way. After doing this define $A^a < v > \equiv B < \Sigma_*^a v >$ for all $v \in T\mathcal{O}_a$. Here $\Sigma_*^a v \equiv \frac{d}{dt}|_0 \Sigma_{\alpha(t)}^a \in TW^1(M)$, where $\alpha(t)$ is any curve in M such that $\frac{d}{dt}|_0\, \alpha(t) = v$. It is then possible to prove that the A^a's constructed above are local connection 1-forms for the bundle E constructed above and hence defines a covariant derivative on E. For more details see Driver [D1].

I will end this lecture with another more explicit method for construction the bundle E and the parallel translation operators on $W^1(M)$. The construction is as follows.

Let $E = W^1(M) \times \mathcal{V}/\sim$ where a pair (σ_1, ξ_1) is said to be equivalent to (σ_2, ξ_2) provided that end points of σ_1 and σ_2 agree and $\xi_2 = P(\overline{\sigma_2}\sigma_1)\xi_1$, where $\overline{\sigma_2}$ denotes the path σ_2 traversed in the reverse direction. We will denote the equivalence class of (σ, ξ) by $P(\sigma)\cdot\xi$. The fiber E_m of E over a point $m \in M$ is

$$E_m = \{P(\sigma)\cdot\xi : \sigma(1) = m \text{ and } \xi \in \mathcal{V}\}.$$

With this notation given an arbitrary C^1-curve $\alpha : [0,1] \longrightarrow M$ we may define "parallel translation" (P^∇) along α by:

$$P^\nabla(\alpha)(P(\sigma)\cdot\xi) = P(\tau)\cdot(P(\overline{\tau}\alpha\sigma)\xi), \qquad (2.1)$$

where σ and τ are arbitrary curves in $W^1(M)$ such that $\alpha(0) = \sigma(1)$ and $\alpha(1) = \tau(1)$.

The above construction is motivated by the following considerations. Suppose that (E, ∇) is a give bundle covariant derivative pair and P^∇ is the corresponding parallel translation operator along C^1-curves in M. For notational simplicity we will identify E_o with V. Now consider the map $\psi : W^1(M) \times E_o \longrightarrow E$ given by

$$\psi(\sigma, \xi) = P^\nabla(\sigma)\xi \qquad (2.2)$$

Writing P for P^∇ restricted to $\mathcal{L}^1(M)$ one easily observes that (σ_1, ξ_1) and (σ_2, ξ_2) are mapped to same point in E under ψ iff they are equivalent in the sense described above. It is also trivial to see that ψ is a surjective map.

Now let α, σ, and τ be curves as above. Given an $\eta \in E_{\alpha(0)}$, set $\xi = P^\nabla(\sigma)^{-1}\eta$ so that $\eta = P^\nabla(\sigma)\xi$. Then by the multiplicative property of parallel transport,

$$P^\nabla(\alpha)\eta = P^\nabla(\alpha)(P^\nabla(\sigma)\xi) = P^\nabla(\alpha\sigma)\xi = P^\nabla(\tau)(P(\bar{\tau}\alpha\sigma)\xi). \tag{2.3}$$

Eq. (2.3) should convince the reader that the definition in (2.1) is the correct one. Of course in general one has to check a number of details.

Section 3. Cameron-Martin Theorem for Manifolds.

In this section we first review the notion of Wiener measure on the path space of a compact manifold. The Cameron-Martin theorem for manifolds (Theorem 3.6) is then stated. Finaly we end this section with a review of the Itô development map for the case of smooth curves. The basic reference for this section is Driver [D4]. The reader will find related topics in [D5] and Driver and Röckner [DR].

The notations for Sections 3. and 4. are the same as above with $E = TM$. We also assume that M is equipped with a Riemannian metric (g) and a covariant derivative (∇) on (TM). The covariant derivative ∇ is assumed to be compatible with g. This means that $P_s^\nabla(\sigma)$ is an isometry for all $\sigma \in W^1(M)$ and $s \in [0,1]$ and in particular $P_s^\nabla(\sigma)^* = P_s^\nabla(\sigma)^{-1}$. Given $v, w \in T_m M$, we will denote $g(v, w)$ by $v \cdot w$. Finally, for a smooth function f on M we will denote the gradient of f by either Grad$f$ or by ∇f.

Definition 3.1. Let $\Delta f(m) = \mathrm{Tr}(\nabla \mathrm{Grad} f)(m) = \sum_{i=1}^d (\nabla_{e_i}(\mathrm{Grad} f)\cdot e_i)$, where $\{e_i\}_{i=1}^d$ is an orthonormal basis for $T_m M$. The Laplacian (Δ) is a second order elliptic differential operator. Denote by $p_s(x,y)$ the solution to the heat equation:

$$\frac{\partial}{\partial s} p_s(x,y) = \frac{1}{2}\Delta_x p_s(x,y) \text{ with } p_0(x,y) = \delta_y(x), \tag{3.1}$$

where δ_y denotes the δ-function (relative to the volume form on M) concentrated at $y \in M$. In other words, $p_s(x,y) = \text{kernel } (e^{\frac{s}{2}\Delta})(x,y)$.

Example 3.2. If $M = \mathbf{R}^d$ with the usual Riemannian metric and $\nabla = d$ then $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ and

$$p_s(x,y) = \left(\frac{1}{2\pi s}\right)^{d/2} e^{-\frac{1}{2s}|x-y|^2}. \tag{3.2}$$

Definition 3.3. Wiener measure ν^∇ on $W(M)$ is the unique probability measure ν^∇ on $W(M)$ such that

$$\int_{W(M)} F(\sigma)d\nu^\nabla(\sigma) = \int_{M^k} f(x_1, \ldots, x_k)\prod_{i=0}^{k-1} p_{(s_{i+1}-s_i)}(x_i, x_{i+1})dx_1 \cdots dx_k, \tag{3.3}$$

where $F(\sigma) = f(\sigma(s_1), \ldots, \sigma(s_k))$ with $0 < s_1 < s_2 < \ldots < s_k = 1$, $f : M^k \longrightarrow \mathbf{R}$ is a bounded measurable function, $x_0 \equiv o$, dx denotes the volume measure on M, and $k = 1, 2, 3 \ldots$.

Fact 3.4. There exists a unique measure ν^∇ on $W(M)$ satisfying (3.3). However, the measure is concentrated on continuous but nowhere differentiable paths.

Remark 3.5. It is interesting to notice for $M = \mathbf{R}^d$ that

$$\begin{aligned}\prod_{i=0}^{k-1} p_{(s_{i+1}-s_i)}(x_i, x_{i+1}) &= \exp -\frac{1}{2}\sum_{i=0}^{k-1}\frac{|x_{i+1}-x_i|^2}{(s_{i+1}-s_i)^2}(s_{i+1}-s_i)\\ &= \exp -\frac{1}{2}\int_0^1 |\, x'(s)\,|^2\, ds,\end{aligned} \tag{3.4}$$

where

$$x(s) \equiv x_i + \frac{s - s_i}{s_{i+1}-s_i}(x_{i+1}-x_i) \text{ if } s \in [s_i, s_{i+1}].$$

Notice that x is the piecewise linear path connecting (s_i, x_i) to (s_{i+1}, x_{i+1}) for $i = 0, \ldots, k-1$. Because of (3.4) and (3.3) one is motivated to write heuristicly

$$d\mu(x) = \frac{1}{Z}e^{-\frac{1}{2}\int_0^1|x'(s)|^2 ds}\mathcal{D}x, \tag{3.5}$$

where μ is supposed to be Wiener measure on $W(\mathbf{R}^d)$ and $\mathcal{D}x$ is supposed to be "Lebesgue measure" on $W(\mathbf{R}^d)$ and Z is a normalization constant so that $\mu(W(\mathbf{R}^d)) = 1$.

Theorem 3.6. (Cameron-Martin 1945) Let $H = \{h : [0,1] \longrightarrow \mathbf{R}^d \in \int_0^1 |\, h'(s)\,|^2\, ds < \infty\}$, the Cameron- Martin space, and μ be Wiener on $W(\mathbf{R}^d)$. Then for each $h \in H$ and

$$\frac{d\mu(h+x)}{d\mu(x)} = e^{-\int_0^1 h'(s)\cdot dx(s) - \frac{1}{2}\int_0^1|h'(s)|^2 ds}, \tag{3.6}$$

where $\int_0^1 h'(s)\cdot dx(s)$ is to be interpreted as a stochastic integral.

Exercise 3.7. Give a heuristic derivation of (3.6) using (3.5).

We would like to generalize the Cameron-Martin theorem to Wiener measure on $W(M)$. This is contained in the following theorem.

Theorem 3.8. Let $h : [0,1] \longrightarrow T_o M$ be a C^1 -function such that $h(0) = 0$. Let $P_s^\nabla(\sigma)$ denote stochastic parallel transport along the Wiener trajectory $\sigma \in W(M)$. (Note: $P_s^\nabla(\sigma)$ is only almost every-where defined function of $\sigma \in W(M)$). Define the vector-field (X^h) on $W(M)$ by

$$X_s^h(\sigma) = P_s^\nabla(\sigma)h(s) \text{ for } s \in [0,1] \tag{3.7}$$

Proof: Compute:

$$\frac{d}{ds} f(\sigma(s)) = \nabla f(\sigma(s)) \cdot P_s^{\nabla}(\sigma) b'(s) = P_s^{\nabla}(\sigma)^{-1} \nabla f(\sigma(s)) \cdot b'(s), \tag{3.12}$$

and

$$\begin{aligned} \frac{d^2}{ds^2} f(\sigma(s)) &= P_s^{\nabla}(\sigma)^{-1} \nabla_{\sigma'(s)} \nabla f \cdot b'(s) + P_s^{\nabla}(\sigma)^{-1} \nabla f(\sigma(s)) \cdot b''(s) \\ &= (\nabla_{P_s^{\nabla}(\sigma) b'(s)} \nabla f) \cdot P_s^{\nabla}(\sigma) b'(s) + P_s^{\nabla}(\sigma)^{-1} \nabla f(\sigma(s)) \cdot b''(s) \end{aligned} \tag{3.13}$$

Add (3.12) times ds to 1/2 of (3.13) times ds^2 to get (3.11). We have used $db(s) \otimes db(s) = b'(s) \otimes b'(s) ds^2$. Because we only keep terms upto second order. Q.E.D.

Section 4: On the Proof of Theorem 3.8.

This section is devoted to a sketch of the proof of Theorem 3.8. of the last section. We start by reviewing the stochastic Itô map and its properties.

Definition 4.1. The stochastic Ito map $\varphi^{\nabla} : W(\mathbf{R}^n) \longrightarrow W(M)$ is given by $\varphi_s^{\nabla}(\omega) = \sigma(s)$ where σ solves the Stratonovich stochastic differential equation:

$$\delta\sigma(s) = P_s^{\nabla}(\sigma) \delta\omega(s) \text{ with } \sigma(0) = o. \tag{4.1}$$

(Here δ is always used to denote the Stratonovich differential in the "s" variable.) Eq. (4.1) is to be interpreted as follows. For all $f \in C^{\infty}(M)$ the real process $f(\sigma(s))$ should be a semi-martingale and

$$\delta f(\sigma(s)) = P_s^{\nabla}(\sigma)^{-1} \nabla f(\sigma(s)) \cdot \delta\omega(s). \tag{4.2a}$$

The Itô form of (4.2a) is

$$\delta f(\sigma(s)) = P_s^{\nabla}(\sigma)^{-1} \nabla f(\sigma(s)) \cdot d\omega(s) + \frac{1}{2} \Delta f(\sigma(s)) ds \tag{4.2b}$$

This should be compared with Eq. (3.11). Notice that the last term in in (3.11) in the case that $b = \omega$ is a Brownian motion becomes

$$(\nabla_{P_s^{\nabla}(\sigma) d\omega(s)} \nabla f) \cdot P_s^{\nabla}(\sigma) d\omega(s) = \mathrm{Tr}(\nabla\nabla f)(\sigma(s)) ds \equiv \Delta f(\sigma(s)) ds. \tag{4.3}$$

Fact 4.2. The map $\varphi^{\nabla} : W(\mathbf{R}^n) \longrightarrow W(M)$ is a μ a.e. defined function and $\varphi_*^{\nabla}\mu \equiv \mu \circ \varphi^{-1} = \nu^{\nabla}$. Furthermore, there exists a measure theoretic inverse $(\varphi^{\nabla})^{-1}$ to φ^{∇} such that $(\varphi^{\nabla})_*^{-1} \nu^{\nabla} = \mu$.

Remark 4.3. For $X \in TW^1(M)$ let

$$G^{\nabla} < X, X > \equiv \int_0^1 | \frac{\nabla X}{ds}(s) |^2 \, ds. \tag{4.4}$$

Then G^∇ defines a metric on $TW^1(M)$ -this metric is often used by differential geometers. The vector fields $\{X^h\}_{h\in H}$ satisfy

$$G^\nabla < X^h, X^k, > (\sigma) = (h,k)_H \; \forall h,k \in H \tag{4.5}$$

That is to say $h \longrightarrow X^h$ is an isometric trivilization of the tangent bundle $TW^1(M)$.

Fact 4.2. states that the Ito map (φ^∇) is a measure theoretic isomorphism of $(W(\mathbb{R}^d), \mu)$ with $(W(M), \nu^\nabla)$. However, $\varphi^\nabla : W^1(\mathbb{R}^d) \longrightarrow W^1(M)$ is not an isometry. That is φ^∇ does not respect Riemannian geometries on $W^1(\mathbb{R}^d)$ and $W^1(M)$ described in Remark 4.3.

Theorem 4.4. Let Y^h denote the vector-field on $W(\mathbb{R}^d)$ found by pulling back X^h via the Ito map φ^∇. Then on $W^1(\mathbb{R}^d))$ one finds:

$$Y_s^h(\omega) = \int_0^s C_{s'}^h(\omega)\omega'(s')ds' + h(s), \tag{4.6}$$

where

$$C_s^h(\omega) = \int_0^s \Omega_{u(s')} < h(s'), \omega'(s') > ds' + \Theta_{u(s)} < h(s), \cdot > . \tag{4.7}$$

In this last expression $u(s) \equiv P_s^\nabla(\varphi^\nabla(\omega))$, $\Omega_u < a,b >= u^{-1}R^\nabla < ua, ub > u$ for $a,b \in T_oM$ and any frame $u : T_oM \longrightarrow T_xM$. Similarly, Θ is defined by $\Theta_u < a,b >\equiv u^{-1}T^\nabla < ua, ub >$. Here R^∇ is the curvature of ∇ and T^∇ is the torsion of ∇.

For a proof see [D4, Theorem 2.1.].

Theorem 4.5 Using the same notation as Theorem 4.4. The correct stochastic form of Eq. (4.6) is :

$$Y_s^h(\omega) = \int_0^s C_{s'}^h(\omega)d\omega(s') + \int_0^s R_{s'}^h(\omega)ds', \tag{4.8}$$

where

$$C_s^h(\omega) = \int_0^s \Omega_u(s') < h(s'), \delta\omega(s') > + \Theta_u(s) < h(s), \cdot > . \tag{4.9}$$

and

$$R_s^h(\omega) = \frac{1}{2}[\mathrm{Ric}_{u(s)} < h(s) > + \overline{\Theta}_{u(s)} < h(s) >] + h'(s). \tag{4.10}$$

(The integrals in the above equations are stochastic integrals.) In (4.10) Ric denotes the Ricci-tensor and $\overline{\Theta}$ denotes a contraction of a certain derivative of the torsion tensor Θ.

Proof: For a proof see [D4, Theorem 5.1, and Theorem 6.2.] The computations are similar to those done in Section 1. The reader will also find in [D4] a more detail explanation of the notation.

We now outline the proof of Theorem 3.8. — the Cameron-Martin Theorem for manifolds.

Step 1. By Theorem 4.4 and 4.5 we may study the vector field Y^h on $W(\mathbb{R}^d)$ instead of X^h on $W(M)$.

Step 2. So we now want to solve for a function $y : \mathbb{R} \times W(\mathbb{R}^d) \longrightarrow W(\mathbb{R}^d)$ such that

$$\dot{y}(t,\omega) = Y^h(y(t,\omega)) \text{ with } y(0,\omega) = \omega \quad \forall \omega \in W(\mathbb{R}^d). \tag{4.11}$$

We will look for a solution in the form

$$y_s(t,\omega) = \int_0^s O_{s'}(t,\omega)d\omega(s') + \int_0^s \alpha_{s'}(t,\omega)ds',$$

where $s \longrightarrow O_s(t,\omega)$ is an End(T_oM) - valued adapted process and $s \longrightarrow \alpha_s(t,\omega)$ is an T_oM -valued adapted process. Eq. (4.11) translates into the following equations for O and α:

$$\dot{O}_s(t,\omega) = C_s^h(y(t,\omega))O_s(t,\omega) \text{ with } O_s(0,\omega) = id \tag{4.12}$$

and

$$\dot{\alpha}_s(t,\omega) = C_s^h(y(t,\omega))\alpha_s(t,\omega) + R_s^h(y(t,\omega)) \text{ with } \alpha_s(0,\omega) = 0 \tag{4.13}$$

Step 3. Because $T^\nabla < X, Y > \cdot Y = 0$ for all vector fields X and Y on M, it follows $C_s^h(y(t,\omega))$ is a skew adjoint operator on T_oM. Therefore, any solution to (4.12) is necessarily orthogonal. It is now possible to solve (4.12) and (4.13) to find $O(t,\omega)$ and $\alpha(t,\omega)$. This can be done using a modified Piccard iteration scheme which is chosen to preserve the orthogonality of the process O at each stage of the iteration.

Step 4. We now have a solution y to (4.11) written in the form

$$y_s(t,\omega) = \int_0^s O_{s'}(t,\omega)d\omega(s') + \alpha_{s'}(t,\omega)ds'$$

with $O(t,\omega)$ an orthogonal process for all t. It is now possible to use Girsanov's theorem to show that the law of $y(t,\cdot)$ is equivalent to μ.

Step 5. Use the Itô map to push the results on $W(\mathbb{R}^d)$ to similar results on $W(M)$. For more details see [D4] Q.E.D.

References

[CM1] R. H. Cameron and W. T. Martin, Transformations of Wiener integrals under translations, Annals of Math., 45, No. 2 (1944), 386 -386.

[CM2] R. H. Cameron and W. T. Martin, Transformations of Wiener integrals under a general class of linear transformations, Trans. Amer. Math. Soc. 58, (1945) 184 - 219.

[D1] B. K. Driver, Classifications of bundle connection pairs by parallel translation and lassos, J. Func. Anal. 83 (1989), 185 231.

CENTRO MATEMATICO V.VOLTERRA

UNIVERSITÀ DI ROMA II

Elenco Preprints

1989

1. S.Buoncristiano–D.Hacon: Characteristics numbers for unoriented Z-homology manifolds.
2. T.Arak-D.Surgailis: Consistent polynomial fields.
3. A.Beretti–D.Sokal: Vectorized program for Monte Carlo simulation of self-avoiding walks.
4. M.Spera: Yang Mills equations and holomorphic structures on C^* dinamica systems.
5. Alicki: Remarks on quantum Boltzmann equation.
6. V.P.Belavkin: Quantum stochastic calculus and quantum nonlinear filtering.
7. L.Accardi: A mathematical theory of quantum noise.
8. M.Spera: Yang Mills theory in non commutative differential geometry.
9. M.Spera: A symplectic approach to Yang Mills theory in non commutative differential geometry.
10. L.Accardi–Lu Yun Gang: The low density limit: the Boson Fock case.
11. L.Accardi–A.Frigerio–Lu Yun Gang: The weak coupling limit as a quantum functional central limit.
12. L.Accardi–A.Frigerio–Lu Yun Gang: The weak coupling limit for fermions.
13. L.Accardi–A.Frigerio–Lu Yun Gang: The quantum weak coupling limit (II): Langevin equation and finite temperature case.
14. Lu Yun Gang: The boson and fermion quantum Brownian motions as quantum central limits of the quantum Bernoulli processes.
15. Massimo A. Picardello–Peter Sjögren: Boundary behaviour of eigenfunctions of the Laplacian in a bi-tree.
16. L.Accardi–R.L. Hudson: Quantum stochastic flows and non abelian cohomology.
17. F. De Blasi–G. Pianigiani: Non-convex valued differential inclusions in Banach spaces.

1990

18. L.Accardi–F.Fagnola–J.Quaegebeur: A representation free quantum stochastic calculus.
19. A.Vignoli: L'analisi nonlineare nella teoria della biforcazione (Parte I).
20. Yali Amit–Ulf Grenander–Mauro Piccioni: Structural image restoration through deformable templates.

21. M.Ferri–M.Piccioni: Optimal Selection of Observatories: An Approach via Simulated Annealing.
22. Yali Amit–Mauro Piccioni: A non-homogeneous Markov process for the estimation of Gaussian random fields with nonlinear observations.
23. L.Accardi–Lu Yun Gang: The weak coupling limit without rotating wave approximation.
24. L.Accardi–Lu Yun Gang: The weak coupling limit for nonlinear interactions.
25. L.Di Luozzo–S.Fanelli: An efficient algorithm for the characterization of sequences of 3–valent convex polyhedra.
26. L.Accardi–Lu Yun Gang: On the low density limit of Boson models: Langevin equation.
27. L.Accardi–Lu Yun Gang: Squeezing noises as weak coupling limit of Hamiltonian system.
28. V.P. Belavkin–P. Staszewski: A stochastic solution of Zeno paradox for quantum Brownian motion.
29. M. Abundo–G. Mele: Absolutely continuous invariant measures for expanding maps of a d–dimensional unit cube in $\mathbf{R}^d$ and their entropy.
30. M. Abundo: A stochastic model for predator–prey systems: basic properties, stability and computer simulation.
31. V.P. Belavkin: A nonadapted stochastic calculus and non Markovian quantum evolution.
32. M. Pavone: On the geodesic distance and group actions on trees.
33. V.P. Belavkin: Quantum stochastic calculus and quantum nonlinear filtering.
34. M. Pavone: Partially ordered groups, almost invariant sets and Toeplitz operators.
35. L. Accardi–D. Koroliuk: Quantum Markov Chains: the Recurrency Problem.
36. J.H. Elton–M. Piccioni: Iterated Function Systems Arising from Recursive Estimation Problems.
37. Franco Fagnola: On Quantum Stochastic Integration With Respect to "Free" Noises.
38. Rosa Q. Huang–Gian Carlo Rota–Joel A. Stein: Supersymmetric Bracket Algebra and Invariant Theory.
39. Tibor Vámos: Rudolf Kalman's œuvre in an epistemic mirror.
40. M. Pavone: Toeplitz operators on discrete groups.
41. L. Accardi–Y.G. Lu–R. Alicki–A. Frigerio: An invitation to the weak coupling and low density limits.
42. M. Pavone: On norm–closed ideals in von Neumann algebras.
43. Franco Fagnola: Pure birth and pure death processes as quantum flows in Fock space.

44. S. Abeasis: Poincaré series of n–tuples of 3×3 matrices.
45. M. Abundo: The simulated annealing algorithm viewed as an iterated function system.
46. M. Abundo: Sequencing DNA fragments by using a simulated annealing algorithm.
47. C. Plini–A. Schiaffino: The sentinel problem for stationary solutions of nonlinear heat equations.
48. F.S. De Blasi–J. Myjak–P.L. Papini: Best approximation in spaces of convex sets.
49. L. Accardi–R. Longo: Martingale convergence of generalized conditional expectations.
50. Antonio Ruberti: Address by Antonio Ruberti to the Seminar in honour of R.E. Kalman.
51. F. Fagnola–Yun Gang Lu: The weak coupling limit for free noise.
52. L. Accardi–Yun Gang Lu: The low density limit of quantum systems.
53. Yun Gang Lu: The weak coupling limit for quadratic interaction.
54. L. Accardi–Yun Gang Lu: Quantum central limit theorems for weakly dependent maps.
55. L. Accardi–Yun Gang Lu: The Fermion number processes as a functional central limit of quantum Hamiltonian models.
56. Yun Gang Lu: On the solution of the quantum stochastic differential equation in the squeezing case.
57. Yun Gang Lu: The describtion of the classical processes which can be regarded as a quantum Markov chain
58. L.Accardi: Quantum probability and the foundations of quantum theory.
59. L. Accardi–Yun Gang Lu: A continuous version of De Finetti's theorem.
60. L. Accardi–Yun Gang Lu: The low density limit in finite temperature case.
61. L. Accardi–Yun Gang Lu: The low density limit: without rotating wave approximation.

<u>1991</u>

62. Boris P. Kovachev: Least squares estimation in almost-linear regression model.
63. G. Galati: Signal processing in modern radar systems (Invited Paper).
64. Luigi Accardi: Vito Volterra and the developement of functional analysis.
65. Luigi Accardi–D. Koroliuk: Stopping times for quantum Markov chains.
66. M. Abundo, L. Accardi, A. Auricchio: Hyperbolic automorphisms of Tori and pseudo-Random sequences.
67. L. Accardi, A. Frigerio, Y.G. Lu: On the relation between the singular and the weak coupling limits.

90. Takeyuki Hida: The impact of classical functional analysis on white noise calculus.
91. F.S. De Blasi, J. Myiak: On compact connected sets in Banach spaces
92. P. Baldi, A. Frigessi, M. Piccioni: Importance sampling for Gibbs Random Fields.
93. Marek Bożejko: Harmonic analysis on discrete groups and noncommutative probability.
94. Mu-Fa Chen: Selected topics in probability theory.
95. F. Fagnola, Kalyan B. Sinha: Scattering theory for unitary cocycles.
96. P. Gibilisco: Induced representations of non-locally compact groups.
97. V.M. Zolotarev: Method of Metric Distances (MMD).
98. V.P. Belavkin: The unified Ito formula has the pseudo-Poisson structure $df(x) = [f(x+b) - f(x)]^{\mu}_{\nu} da^{\nu}_{\mu}$.
99. Ulrich Oberst: Galois theory and the fast gelfand transform (FGT).
100. Leonard Gross: Logarithmic Sobolev Inequalities and Contractivity Properties of Semigroups.
101. Stéphane Attal: Characterizations of operators commuting with some conditional expectations on multiple Fock spaces.
102. Stéphane Attal: Characterizations of some operators on Fock space.
103. Carlo Cecchini: A non commutative Markov equivalence theorem.
104. V. Aloisio, A. Di Vito, G. Galati: Optimization of doppler filters for fluctuating radar targets.
105. Carlo Cecchini: On the structure of quantum Markov processes.
106. Y.G. Lu: Quantum stochastic calculus on Hilbert module.
107. L. Accardi, Y.G. Lu: Quantum electro-dynamics and semi-circle noises.
108. A.M. Chebotarev, F. Fagnola: Sufficient conditions for conservativity of quantum dynamical semigroups.
109. Carlo Cecchini: State Extentions and ω conditional Expectations on Von Neumann Algebras.
110. V.P. Belavkin: Nondemolition principle of quantum measurement and filtering theory.
111. Emanuela Fachini, Angelo Monti: Regular sets, context-free languages and Systolic Y-tree Automata.
112. L. Accardi, D. Koroliuk: Statistical package VLTSTAT: a new approach to robustness.
113. Y.G. Lu: Passage from quantum systems with continuous spectrum to quantum Poisson processes on Hilbert modules.
114. Y.G. Lu: Quantum stochastic calculus with respect to Poisson processes on Hilbert modules.
115. F. De Blasi, G. Pianigiani: Solution sets of boundary value problems for nonconvex differential inclusions.

116. F. De Blasi, G. Pianigiani, V. Staicu: Topological properties of nonconvex differential inclusions of evolution type.

117. F. De Blasi, G. Pianigiani, V. Staicu: On the solution sets of some nonconvex hyperbolic differential inclusions.

118. L. Accardi, K. Koroliuk: Statistical package VLTSTAT: processing of random samples of small volumes (from 5 to 100 dates). Statistical principles and a description of algorithms.

119. D. Koroliuk: Approccio termodinamico generale al problema di scambio e ridistribuzione di risorse tra mercati economici.

120. N.A. Greggio, M. Cameran, C. Giaquinto, F. Zacchello, D. Koroliuk, V. Colizzi: DNA HLA-DRB1 analysis in children of positive mothers and estimated risk of vertical HIV transmission.

121. L. Accadi, D. Koroliuk, M. Regoli, S. Vovk: Software package REGDES: Design, Analysis and Optimization of Regression Multifactor Experiments

122. G. Galati, L. Cipriani, A. Di Vito, S. Galli, A. Veredice: Generazione al calcolatore di sequenze pseudocasuali con assegnata autocorrelazione.

123. B. Driver: Loop space, curvature, and quasi-invariance.