Proceedings of Symposia in Pure Mathematics
Volume 57 (1995)

A12

Towards Calculus and Geometry on Path Spaces

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ABSTRACT. In this note I will discuss some differential and geometric problems on path space $W(M)$ of a finite-dimensional manifold M. (The interested reader should also consult the lecture notes in this volume given by H. Airault, L. Gross, E. Hsu, P. Malliavin, and I. Shigekawa.) The organizations of this paper is as follows. Section 1 is a review of some facts from finite-dimensional manifold theory. Section 2 is a discussion of calculus on classical Wiener space. In Section 3, more general path spaces are considered. Section 4 considers the question of whether certain natural Dirichlet forms are comparable.

Disclaimer. Although a number of references to the literature are given, the references are by no means complete. I have only included references which I thought most directly pertained to the discussion in the paper.

1. Some facts, tools, and properties for Riemannian manifolds

1.1. Notations. Let (W, G, ν) be the triple consisting of a smooth compact (for simplicity) N–dimensional manifold W, a Riemannian metric G on TW, and a smooth volume measure ν on W. (TW denotes the tangent bundle, and for $\omega \in W$, $T_\omega W$ denotes the tangent space to W at ω.) Let $\Lambda \equiv \Lambda(W)$ be the exterior bundle of the dual bundle T^*W, and let $C^\infty(\Lambda)$ be the smooth differential forms on W. The metric G on W induces a metric on Λ and hence we may also talk about L^2-differential forms. The space of L^2-differential forms will be denoted by $\mathrm{L}^2(\Lambda) = \mathrm{L}^2(\Lambda, \nu)$. Let d denote the exterior derivative which we view as a closed unbounded operator on $\mathrm{L}^2(\Lambda)$ which preserves $C^\infty(\Lambda)$. Let d^* denote the L^2-adjoint of d, and notice that d^* also preserves $C^\infty(\Lambda)$. Finally set $L = -(dd^* + d^*d)$. If ν is the volume measure on W then L is the Laplacian on (Δ) on W. In general $L = \Delta + R$, where R is a first-order differential operator.

1991 *Mathematics Subject Classification.* Primary 60D05, 28D05; Secondary 58D20, 58D25.
This research was partially supported by NSF Grant No. DMS 92 - 21377.
This paper is in final form and no version of it will be submitted for publication elsewhere.

1.2. Vector fields, flows, and quasi-invariance. Given a smooth vector field X on W, there is a unique flow $e^{tX} : W \to W$ such that $\frac{d}{dt}|_0 e^{tX} = X$. Furthermore, e^{tX} leaves ν quasi-invariant; i.e. $e^{tX}_* \nu = Z_t \cdot \nu$, where $Z_t : W \to (0, \infty)$ is a smooth density.

1.3. Divergence and integration by parts. Given X as above, there is a unique function $\operatorname{div}_\nu(X)$ (called the divergence of X) on W such that, for all Borel subsets $B \subset W$,

$$\frac{d}{dt}|_0 \nu(e^{tX}(B)) = \int_B \operatorname{div}_\nu(X) d\nu. \tag{1.1}$$

So $\operatorname{div}_\nu(X)$ measures the rate of spreading of the flow e^{tX} as seen by the measure ν. The relationship between Z_t and $\operatorname{div}_\nu(X)$ is

$$\operatorname{div}_\nu(X) = -\frac{d}{dt}|_0 Z_t. \tag{1.2}$$

From this fact it is easy to show the L^2-adjoint (X^*) of X is given by

$$X^* = -X - \operatorname{div}_\nu(X) \tag{1.3}$$

when acting on smooth functions. (This last equation is of course an integration by parts formula.) Applying X^* to the function 1 gives another formula for $\operatorname{div}_\nu(X)$:

$$\operatorname{div}_\nu(X) = -X^* 1. \tag{1.4}$$

It is also possible to recover the density Z_t from knowledge of $\operatorname{div}_\nu(X)$ and the flow $e^{tX.}$ Indeed, from (1.3) one shows that Z_t satisfies the first-order p.d.e.,

$$\dot{Z}_t = X^* Z_t = -XZ_t - \operatorname{div}_\nu(X) Z_t \text{ with } Z_0 \equiv 1 \text{ on } W.$$

The differential equation for Z_t is easily solved using

$$\frac{d}{dt}(Z_t \circ e^{tX}) = -\operatorname{div}_\nu(X) \circ e^{tX} \cdot Z_t \circ e^{tX}.$$

The result is

$$Z_t(\omega) = e^{-\int_0^t \operatorname{div}_\nu(X) \circ e^{-\tau X}(\omega) d\tau}. \tag{1.5}$$

REMARK 1.1. Another common formula for $\operatorname{div}_\nu(X)$ is

$$\operatorname{div}_\nu(X) = -D^* X,$$

where D is the gradient on $C^\infty(W)$, and D^* is the L^2-adjoint. This formula is the least desirable method of computing $\operatorname{div}_\nu(X)$ in the infinite-dimensional setting. This is because D^* depends on the metric in two ways: (i) in the definition of D, and (ii) in the definition of D^*. In a sense, because the metric is used an even number of times, the dependence on the metric "cancels" out leaving $\operatorname{div}_\nu(X)$ to depend only on ν. The problem in the infinite-dimensional context is that the domain of D^* will fail to contain a large class of vector fields which nevertheless do have a divergence in the previous sense.

1.4. Spectral properties of L. L is a second-order elliptic differential operator and the general properties of the spectrum $\sigma(L)$ of L are well understood. For example one knows that $\dim(\mathrm{Nul}(L)) < \infty$ and $\sigma(L) \setminus \{0\} \subset [m, \infty)$ for some $m > 0$. The best m is called the spectral gap or the mass gap.

1.5. Sobolev inequalities. Sobolev inequalities involving L are well known. These Sobolev inequalities are qualitatively the same as the Sobolev inequalities involving the standard Laplacian on $\mathbb{R}^N$.

1.6. Diffusions associated to Δ and L. There exist diffusions on W with infinitesimal generators Δ and L. A process associated to the Laplacian (Δ) will be called a Brownian motion on W and its path space measure will be called a Wiener measure on W.

1.7. Geometry and topology.

THEOREM 1.2 (De Rham's theorem). *The De Rham cohomology of W is isomorphic to the real topological cohomologies of W. Recall that the De Rham cohomology is the cohomology of the differential complex* $(C^\infty(\Lambda), d)$.

THEOREM 1.3 (Hodge's theorem). *The De Rham cohomology of W is isomorphic to the kernel of L, i.e. the L-harmonic forms.*

THEOREM 1.4 (Riemann's theorem). *The Riemannian manifold (W, G) is flat iff the Riemannian curvature tensor R is identically* 0. (*Recall that (W, G) is said to be flat if there is a* (*local*) *diffeomorphism of Ψ of $\mathbb{R}^N$ to W such that the differential of Ψ is an isometry at all points in $\mathbb{R}^N$.*)

2. Classical Wiener space as a model

Our general program (which was initiated by L. Gross) is to consider the analogous properties described in the last section when W is an infinite-dimensional manifold. In this section, the path space of $\mathbb{R}^d$ will be considered.

2.1. Preliminaries. Let

$$W = W(\mathbb{R}^d) = \{\omega \in C([0, 1], \mathbb{R}^d) | \omega(0) = 0\},$$

let $\nu = \mu$ be Wiener measure on W, $H \subset W$ denote the Cameron-Martin space, and $G(h, h) \equiv \int_0^1 |h'(s)|^2 ds$ for all $h \in H$. Since W is a vector space, $T_\omega W$ can be naturally identified with W. Hence, the inner product G may be thought of as a Riemannian metric on TW. There is of course a problem with this definition since G is not well defined on W. One way around this is to interpret $T_\omega W$ to be H rather than all of W. The Cameron-Martin theorem supports this point of view. (Recall, the Cameron-Martin theorem states that μ is quasi-invariant under the transformation $\omega \to \omega + h$ iff $h \in H \subset W$.) I hope to convince the reader that H should, in fact, **not** be considered as the tangent space to W.

The following theorem is an attempt to motivate the notion of adapted tangent vector fields on $W = W(\mathbb{R}^d)$ to be defined below. In this theorem and the sequel, $O(d)$ will denote the Lie group of $d \times d$-real orthogonal matrices.

THEOREM 2.1. *Let $\phi : W \to W$ be adapted and assume*: (i) *$\phi_* \mu$ is equivalent to μ (let $Z = \frac{d\phi_*\mu}{d\mu}$), and* (ii) *there is an adapted map ϕ^{-1} : $W \to W$ such that $\phi \circ \phi^{-1}$ and $\phi^{-1} \circ \phi$ are both equal to the identity map μ-a.s.*

Then the following hold:

(i) *$\phi_*^{-1}\mu$ is also equivalent to μ.*

(ii) *There exist two $O(d) \times \mathbb{R}^d$-valued predictable processes (O, α) and $(\bar{O}, \bar{\alpha})$ on W such that $\int_0^1 |\alpha_{s'}|^2 ds' < \infty$, $\int_0^1 |\bar{\alpha}_{s'}|^2 ds' < \infty$ μ-a.s.,*

$$\phi(\omega) = \int O(\omega) d\omega + \int \alpha(\omega) ds, \tag{2.1}$$

and

$$\phi^{-1}(\omega) = \int \bar{O}(\omega) d\omega + \int \bar{\alpha}(\omega) ds. \tag{2.2}$$

(iii) *Let $\tilde{O} \equiv O \circ \phi^{-1}$, and $\tilde{\alpha} \equiv \alpha \circ \phi^{-1}$. Then $\tilde{O}\bar{O} = I$ and $\tilde{\alpha} = -\tilde{O}\bar{\alpha}$.*

(iv) *$Z \equiv \frac{d\phi_*\mu}{d\mu}$ is given by*

$$\begin{aligned} Z &= \exp\left[\int_0^1 \alpha \circ \phi^{-1}(\omega) \cdot d\omega - \tfrac{1}{2} \int_0^1 |\alpha \circ \phi^{-1}(\omega)|^2 ds\right] \\ &= \exp\left[-\int_0^1 \bar{\alpha} \cdot \bar{O} d\omega - \tfrac{1}{2} \int_0^1 |\bar{\alpha}(\omega)|^2 ds\right]. \end{aligned} \tag{2.3}$$

PROOF. Since μ and $\phi_*\mu$ are absolutely continuous with respect to one another we know that $Z > 0$ μ-a.s. Let f be a nonnegative measurable function on W; then

$$\phi_*^{-1}\mu(f)=\mu(f\circ\phi^{-1})=\phi_*\mu(Z^{-1}f\circ\phi^{-1})=\mu(\frac{1}{Z\circ\phi}f).$$

($\phi_*^{-1}\mu$ denotes the law of the process ϕ^{-1}.) It follows that $\frac{d\phi_*^{-1}\mu}{d\mu}=\frac{1}{Z\circ\phi}$ or $\phi_*^{-1}\mu=\frac{1}{Z\circ\phi}\cdot\mu$. This proves the first assertion of the theorem.

Set $U\equiv\frac{1}{Z\circ\phi}$ and notice that $\mu(U)=\phi_*\mu(\frac{1}{Z})=\mu(\frac{Z}{Z})=1$. By Protter [27, Corollary 4, p. 157], there exists an $\mathbb{R}^d$-valued predictable process J on W such that $\int_0^1|J_s|^2ds<\infty$ μ-a.s. and

$$U_s=\exp\left[\int_0^s J\cdot d\omega-\frac{1}{2}\int_0^s|J|^2ds\right], \tag{2.4}$$

where $U_s\equiv E_\mu(U|\mathscr{F}_s)$. By the first assertion of the theorem it follows that $\mu=\phi_*\phi_*^{-1}\mu=\phi_*(U\cdot\mu)$. In other words ϕ is a $U\cdot\mu$ Brownian motion. Hence, by Girsanov's and Lévy's theorem it follows that $\phi=b+A$, where b is a μ-Brownian motion and

$$A\equiv\int Ud[\frac{1}{U},\phi]=\int Ud[\frac{1}{U},b]=\int-(J\cdot d\omega)db \tag{2.5}$$

is a process of bounded variation. Because we are on the canonical path space with the natural filtration it follows that

$$b(\omega)=\int O(\omega)d\omega \tag{2.6}$$

for some predictable process O; see for example [27, Corollary 2, p. 156]. Since the process b is a μ-Brownian motion it follows by computing the quadratic variation of b that the process O is $O(d)$–valued μ-a.s.. It is now easy to conclude from (2.4),(2.5),(2.6) that

$$A=-\int(J\cdot d\omega)O(\omega)d\omega=-\int OJds.$$

So if we define $\alpha\equiv-OJ$, it follows that (2.1) holds. The same logic implies the existence of a process $(\bar{O},\bar{\alpha})$ such that (2.2) holds.

Now use $\phi\circ\phi^{-1}=id$ μ-a.s. and the computation

$$\phi\circ\phi^{-1}(\omega)=\int O\circ\phi^{-1}(\omega)[\bar{O}(\omega)d\omega+\bar{\alpha}(\omega)ds]+\int\alpha\circ\phi^{-1}(\omega)ds$$

to conclude that $\tilde{O}\bar{O}=I$ μ-a.s. and that $\tilde{O}\bar{\alpha}+\tilde{\alpha}=0$ μ-a.s. This proves assertion (iii) of the theorem.

Now insert $J=-O^{-1}\alpha$ into (2.4) to get

$$U_s=\exp\left[-\int_0^s O^{-1}\alpha\cdot d\omega-\frac{1}{2}\int_0^s|\alpha|^2ds\right],$$

from which it follows that

$$Z \circ \phi = \frac{1}{U_1} = \exp\left[\int_0^1 O^{-1}\alpha \cdot d\omega + \frac{1}{2}\int_0^1 |\alpha|^2 ds\right]. \tag{2.7}$$

By right composing (2.7) with ϕ^{-1} we learn that

$$Z = \exp\left[\int_0^1 \tilde{O}^{-1}\tilde{\alpha} \cdot [\bar{O}d\omega + \bar{\alpha}ds] + \frac{1}{2}\int_0^1 |\tilde{\alpha}|^2 ds\right].$$

Because of item (iii) of the theorem the above equation may be rewritten as

$$\begin{aligned} Z &= \exp\left[\int_0^1 \tilde{\alpha} \cdot d\omega - \tilde{\alpha} \cdot \tilde{\alpha} ds] + \frac{1}{2}\int_0^1 |\tilde{\alpha}|^2 ds\right] \\ &= \exp\left[\int_0^1 \tilde{\alpha} \cdot d\omega - \frac{1}{2}\int_0^1 |\tilde{\alpha}|^2 ds\right], \end{aligned}$$

which is the same as the first equality in (2.3). The second equality in (2.3) follows easily from the first equality using $\tilde{\alpha} = -\bar{O}^{-1}\bar{\alpha}$ — by item (iii) again. □

Now suppose that $\phi_t : W \to W$ is a flow on Wiener space such that, for each $t \in \mathbb{R}$, ϕ_t is an adapted process on W and ϕ_t leaves μ quasi-invariant. Then by the above theorem, ϕ_t necessarily has the form

$$\phi_t(\omega) = \int O_t(\omega)d\omega + \int \alpha_t(\omega)ds,$$

where for each $t \in \mathbb{R}$, (O_t, α_t) is an $O(d) \times \mathbb{R}^d$-valued adapted process on W. Assuming sufficient regularity on the coefficients (O_t, α_t), we learn the "adapted vector-field" $X \equiv \frac{d}{dt}|_0 \phi_t$ must have the form

$$X(\omega) = \int C(\omega)d\omega + \int r(\omega)ds \tag{2.8}$$

where (C, r) is an adapted $so(d) \times \mathbb{R}^d$-valued process. ($so(d)$ is the Lie algebra of $O(d)$—i.e. the space of skew symmetric $d \times d$-matrices.)

DEFINITION 2.2. An adapted process $X : W \to W$ is an **adapted vector field** on W if X has a representation as in (2.8) such that $\int_0^1 |r(\omega)(s)|^2 ds < \infty$ a.s. (μ).

Open problem. Determine the set of nonadapted tangent vector fields on W. That is, find all infinitesimal generators of flows on W which leave μ quasi-invariant. See, for example, [7, **26**, **32**] for some examples of non-adapted vector fields.

2.2. Flow and quasi-invariance. Although the following theorem does not explicitly appear in [**8**] it is implicitly proved in course of proving Theorem 6.1 of [**8**]. Before stating this theorem we need the following norms.

DEFINITION 2.3. Suppose $\phi : W \to W$ is a semimartingale of the form

$$\phi(\omega) = \int A(\omega)d\omega + \int \alpha(\omega)ds,$$

where (A, α) is an $\mathrm{End}(\mathbb{R}^d) \times \mathbb{R}^d$-valued adapted process. ($\mathrm{End}(\mathbb{R}^d)$ denotes the real $d \times d$-matrices.) For each $p \geq 1$, set

$$\|\phi\|_{B^p} \equiv \| \sup_s |A(\cdot)(s)| \|_{L^p(\mu)} + \| \sup_s |\alpha(\cdot)(s)| \|_{L^p(\mu)}.$$

THEOREM 2.4. *Suppose $X(\omega) = \int C(\omega)d\omega + \int r(\omega)ds$ is a adapted tangent vector field on W such that*

$$\| \sup_s |r(\cdot)(s)| \|_{L^\infty(\mu)} < \infty, \tag{2.9}$$

and for some $p \geq 2$ there is a function $K : \mathbb{R}_+^2 \to (0, \infty)$ such that

$$\|X \circ S - X \circ \bar{S}\|_{B^p} \leq K(\|S\|_{B^\infty}, \|\bar{S}\|_{B^\infty})\|S - \bar{S}\|_{B^p}$$

where S and $\bar{S}$ are semimartingales of the form

$$S(\omega) = \int O(\omega)d\omega + \int \alpha(\omega)ds,$$

and

$$\bar{S}(\omega) = \int \bar{O}(\omega)d\omega + \int \bar{\alpha}(\omega)ds,$$

where O and $\bar{O}$ are both assumed to be $O(d)$-valued and $\|S\|_{B^\infty}, \|\bar{S}\|_{B^\infty} < \infty$ (Notice by Girsanov's theorem, S and $\bar{S}$ have laws equivalent to μ, so $X \circ S$ and $X \circ \bar{S}$ are well defined independent of the version chosen for X.)

***Conclusion**: The vector field X admits an adapted flow e^{tX} on W which leaves Wiener measure (μ) quasi-invariant.*

PROOF. (Sketch) If $\phi_t \equiv e^{tX}$ exists, it must be of the form

$$\phi_t = e^{tX}(\omega) = \int O_t(\omega)d\omega + \int \alpha_t(\omega)ds, \tag{2.10}$$

where for each $t \in \mathbb{R}$, (O_t, α_t) is an $O(d) \times \mathbb{R}^d$-valued adapted process. Inserting (2.10) into the differential equation for e^{tX},

$$\frac{d}{dt}e^{tX} = X \circ e^{tX} \text{ with } e^{0X} = \mathrm{Id}_W, \tag{2.11}$$

yields the equations:

$$\begin{aligned} \dot{O}_t &= (C \circ \phi_t)O_t & \text{with } & O_0 = I, \\ \dot{\alpha}_t &= (C \circ \phi_t)\alpha_t + r \circ \phi_t & \text{with } & \alpha_0 = 0, \end{aligned} \tag{2.12}$$

where the dot above a function always refers to a derivative relative to t. The equations in (2.12) can be solved by iteration as follows. Let $O_t^0 \equiv I$, and $\alpha_t^0 \equiv 0$. We will now define O^n and α^n inductively. Assume O^n and

α^n have already been defined, and set $\phi_t^n \equiv \int O_t^n(\omega)d\omega + \int \alpha_t^n(\omega)ds$. Then define O^{n+1} and α^{n+1} as the unique solutions (path-wise) to the ordinary differential equations:

$$\begin{aligned} \dot{O}_t^{n+1}(\omega) &= C \circ \phi_t^n(\omega) O_t^{n+1}(\omega) && \text{with } O_0^{n+1}(\omega) \equiv I, \\ \dot{\alpha}_t^{n+1}(\omega) &= C \circ \phi_t^n(\omega)\alpha_t^{n+1}(\omega) + r \circ \phi_t^n(\omega) && \text{with } \alpha_0^{n+1}(\omega) = 0. \end{aligned}$$

It is now possible to show that the $\lim_{n\to\infty} \phi_t^n$ exists in $\|\cdot\|_{B^p}$, and the resulting process satisfies the conclusion of the theorem. □

It may be somewhat surprising that in the above theorem no uniform boundedness assumption is made on $C(\omega)$. The reason no such bound is needed is related to the fact that $C(\omega)$ is skew symmetric. To understand the basic idea involved, let $C_1(t)$ and $C_2(t)$ be two deterministic curves in $so(d)$ and assume that O_1 and O_2 are solutions to the following differential equations:

$$\dot{O}_i(t) = C_i(t)O_i(t) \text{ with } O_i(0) = I, \quad i = 1, 2.$$

Because C_1 and C_2 are skew symmetric it follows that O_1 and O_2 are orthogonal matrices. An easy computation shows

$$\frac{d}{dt}(O_1^{-1}O_2) = O_1^{-1}(C_2 - C_1)O_2. \tag{2.13}$$

Now for any $d \times d$-matrix A, let $|A| \equiv \mathrm{tr}^{1/2}(A^*A)$ denote the Hilbert-Schmidt norm. Notice for this norm, $|OA| = |AO|$ for any orthogonal matrix O. Now take the norm of both sides of (2.13) and integrate to find the estimate:

$$|O_2(t) - O_1(t)| = |O_1^{-1}(t)O_2(t) - I| \le \left| \int_0^t |C_1(\tau) - C_2(\tau)| d\tau \right|. \tag{2.14}$$

The important point is that neither $|C_1|$ nor $|C_2|$ appears in the above estimate, only $|C_1 - C_2|$. More general estimates of this nature can be found in Lemma 6.1. of **[8]**.

REMARK 2.5. Elton Hsu in **[19]** makes the following very nice observation. By replacing **everywhere** the measure $d\Lambda(s) = ds$ by another measure it is possible to lessen the restrictive hypothesis in (2.9). For example, if $d\Lambda(s) = \rho(s)ds$ with $\rho \in L^1(ds)$ and $\rho \ge 1$, one can replace the bound in (2.9) by the bound

$$\| \sup_s |r(\cdot)(s)|/\rho(s)\|_{L^\infty(\mu)} < \infty.$$

For the quasi-invariance property of the flow, it will still be necessary to assume some boundedness for the function $\omega \to \int_0^1 |r(\omega)(s)|^2 ds$.

The other differences between **[8]** and **[19]** are in technical points in the proof. In particular, the ideas involved in (2.14) are not used in **[19]**. Instead, for the particular $C(\omega)$ of interest, a truncation argument is used coupled with an estimate showing C has Gaussian-like tails.

Elton Hsu [**20**] has also outlined another proof of Theorem 2.4 using an Euler method. The idea of the proof is to first choose a "good approximation" ($\Psi_t : W \to W$) to the flow e^{tX} when t is small. More explicitly, choose Ψ_t such that $\Psi_0(\omega) = \omega$ and $\frac{d}{dt}|_0 \Psi_t(\omega) = \omega$ for $\omega \in W$, and such that $\Psi_t \circ \Psi_\tau$ is approximately equal to $\Psi_{t+\tau}$ for t and τ small. Then construct e^{tX} as the limit:

$$e^{tX} = \lim_{n\to\infty} \Psi_{t/2^n}^{(2^n)},$$

where $\Psi_\tau^{(2^n)}$ denotes Ψ_τ composed with itself 2^n-times. Following Bismut [**6**] (also see Fang and Malliavin [**14**]), Hsu chooses the following approximate flow:

$$\Psi_t(\omega) = \int e^{tC(\omega)} d\omega + t \int r(\omega) ds. \tag{2.15}$$

The idea of using Ψ_t defined above for proving integration by parts formulas is due to Bismut [**6**]. Leandre [**21**] and Fang and Malliavin [**14**] use Bismut's ideas to prove integration by parts formulas in the geometric setting described in §3. We now discuss the integration by parts formulas in the context of Theorem 2.4.

2.3. Divergence and integration by parts.

COROLLARY 2.6 (Integration by parts). *Let X be an adapted vector field as in Theorem 2.4 and define*

$$\operatorname{div}_\mu(X) = -\int_0^1 \langle r(\omega), d\omega\rangle.$$

Then $\operatorname{div}_\mu(X)$ is the unique function on W such that, for all Borel subsets $B \subset W$,

$$\frac{d}{dt}|_0 \mu(e^{tX}(B)) = \int_B \operatorname{div}_\mu(X) d\mu.$$

Furthermore, if Z_t denotes the density of $e_*^{tX}\mu$ relative to μ, then

$$\operatorname{div}_\mu(X) = -\frac{d}{dt}|_0 Z_t,$$

$$Z_t(\omega) = e^{-\int_0^t \operatorname{div}_\mu(X)\circ e^{-\tau X}(\omega) d\tau}, \tag{2.16}$$

and acting on cylinder functions (for example)

$$X^* = -X - \operatorname{div}_\mu(X)$$

where X^* denotes the $L^2(\mu)$ adjoint of the vector field X thought of as a differential operator on $L^2(\mu)$. Given a smooth cylinder function ($f \in L^2(\mu)$)

with at most polynomial growth, we may define Xf *by*

$$Xf = L^2 - \lim_{t\to 0}(f\circ e^{tX} - f)/t$$

or

$$Xf = L^2 - \lim_{t\to 0}(f\circ \Psi_t - f)/t.$$

PROOF. Let us check that (2.16) holds. We know from Theorem 2.1 that

$$Z_t(\omega) = \exp\left[\int_0^1 \alpha_t \circ e^{-tX}(\omega)\cdot d\omega - \frac{1}{2}\int_0^1 |\alpha_t\circ e^{-tX}(\omega)|^2 ds\right],$$

so that

$$Z_t\circ e^{tX}(\omega) = \exp\left[\int_0^1 \alpha_t(\omega)\cdot de^{tX}(\omega) - \frac{1}{2}\int_0^1 |\alpha_t(\omega)|^2 ds\right]. \tag{2.17}$$

Let W_t denote the right-hand member of (2.16). Then

$$\begin{aligned} W_t\circ e^{tX} &= e^{-\int_0^t \operatorname{div}_\mu(X)\circ e^{(t-\tau)X}d\tau} \\ &= e^{-\int_0^t \operatorname{div}_\mu(X)\circ e^{\tau X}d\tau} \\ &= \exp^{\int_0^t\left[\int_0^1 \langle r\circ e^{\tau X},\, de^{\tau X}\rangle\right]d\tau}. \end{aligned} \tag{2.18}$$

Comparing the above two displayed equations we see it suffices to show that

$$\int_0^1 \langle r\circ e^{tX},\, de^{tX}\rangle = \frac{d}{dt}\left[\int_0^1 \alpha_t\cdot de^{tX} - \frac{1}{2}\int_0^1 |\alpha_t|^2 ds\right] =: A. \tag{2.19}$$

We now compute the right-hand member of (2.19):

$$\begin{aligned} A := &\frac{d}{dt}\left[\int_0^1 \langle \alpha_t,\, de^{tX}\rangle - \frac{1}{2}\int_0^1 |\alpha_t|^2 ds\right] \\ = &\frac{d}{dt}\left[\int_0^1 \langle \alpha_t,\, O_t db + \alpha_t ds\rangle - \frac{1}{2}\int_0^1 |\alpha_t|^2 ds\right] \\ = &\frac{d}{dt}\left[\int_0^1 \langle O_t^{-1}\alpha_t,\, db\rangle + \frac{1}{2}\int_0^1 |\alpha_t|^2 ds\right]. \end{aligned}$$

Using the differential equations for O_t and α_t, one finds that

$$\frac{d}{dt}[O_t^{-1}\alpha_t] = O_t^{-1} r\circ e^{tX},$$

and

$$\frac{d}{dt}\frac{1}{2}\int_0^1 |\alpha_t|^2 ds = \int_0^1 \langle \alpha_t,\, Coe^{tX}\alpha_t + r\circ e^{tX}\rangle ds = \int_0^1 \langle r\circ e^{tX},\, \alpha_t\rangle ds.$$

Combining the last three equations gives

$$A = \int_0^1 \langle r \circ e^{tX}, O_t db \rangle + \int_0^1 \langle r \circ e^{tX}, \alpha_t \rangle ds = \int_0^1 \langle r \circ e^{tX}, de^{tX} \rangle,$$

which is the left-hand side of (2.19) as desired. □

2.4. Spectral properties of L. It is possible to define L acting on functions via the formula $L = -D^* D$, where D is the gradient operator on W. Alternatively, if $S \subset H$ is an orthonormal basis for H and

$$\partial_h f = \mathrm{L}^2\text{-}\lim_{t \to 0} (f(\cdot + h) - f(\cdot))/t,$$

then $L = \sum_{h \in S} -\partial_h^* \partial_h$. The operator L in this case is the Ornstein-Uhlenbeck operator (or number operator from quantum field theory). The spectrum of L is known to be discrete and in fact $\sigma(L)$ consists of the nonnegative integers. Each positive integer occurs with infinite multiplicity, while $0 \in \sigma(L)$ occurs with multiplicity one. In particular, L has a spectral gap.

2.5. Sobolev inequalities. Standard Sobolev inequalities do not hold in infinite dimensions. However, L does satisfy the logarithmic Sobolev inequality of L. Gross (see **[18]**).

2.6. Diffusions associated to Δ and L. Let $\Delta \equiv \sum_{h \in S} \partial_h^2$ be the Laplacian on W. It is well known that there exist diffusions B (Brownian motion) and X (Ornstein-Uhlenbeck) on W with infinitesimal generator $\frac{1}{2}\Delta$ and $\frac{1}{2}L$ respectively. In fact, when B starts at the zero path, B is the $\mathbb{R}^d$-valued Brownian sheet.

2.7. Hodge's theorem. The L^2-Hodge theorem was proved in this context by Shigekawa **[31]**.

3. Wiener space based on a manifold

We now wish to consider the issues discussed in the last section in the case that $\mathbb{R}^d$ is replaced by a compact Riemannian manifold M. As we will see there are many open problems in this setting. There are even more open problems when one uses loop space instead of path space.

3.1. Preliminaries. Our starting point is a tuple: (M^d, g, ∇, o), where M is a compact connected manifold (without boundary) of dimension d, g is a Riemannian metric on M, ∇ is a g-compatible covariant derivative, and o is a fixed base point in M. The symbol ∇ will also be used to denote the the gradient operator acting on functions on M. It will always be assumed that the covariant derivative (∇) is "Torsion Skew Symmetric" or TSS for short. That is to say, if $T = T^{\nabla}$ is the torsion tensor of ∇, then $g(T(X, Y), Y) \equiv 0$ for all vector fields X and Y on M. With the TSS condition, the Laplacian on functions ($\Delta f = \mathrm{tr}(\nabla \mathrm{grad} f)$) associated to ∇ is the usual Levi-Civita Laplacian. Let ν denote Wiener measure concentrated on the paths $(W(M))$ starting at $o \in M$:

$$W(M) \equiv \{\sigma \in C([0, 1], M) | \sigma(0) = o\}. \tag{3.1}$$

Let $\{P_s^\nabla\}_{s\in[0,1]}$ denote a fixed version of the stochastic parallel transport process on $W(M)$. So $P_s^\nabla(\sigma) : T_oM \to T_{\sigma(s)}M$ is an isometry for all $s \in [0, 1]$.

DEFINITION 3.1. The **continuous tangent space** to $W(M)$ at $\sigma \in W(M)$ is the set $CT_\sigma W(M)$ of continuous vector fields along σ which are zero at $s = 0$:

$$\begin{aligned} CT_\sigma W(M) = \{X \in C([0, 1], TM) | X(s) \in T_{\sigma(s)}M \\ \forall\, s \in [0, 1] \text{ and } X(0) = 0\}. \end{aligned} \tag{3.2}$$

To motivate the above definition, consider a differentiable curve in $W(M)$ going through σ at $t = 0$: $(t \to f(t, \cdot)) : (-1, 1) \to W(M)$. The derivative $X(s) \equiv \frac{d}{dt}|_0 f(t, s)$ of such a curve should, by definition, be a tangent vector $W(M)$ at σ. This is indeed the case.

We now wish to define a Riemannian metric on $W(M)$. We know from the case that $M = \mathbb{R}^d$ that the continuous tangent space is too large for this purpose. The continuous tangent space is also too large from the point of the view of the Cameron-Martin theorem. We will have to introduce the Riemannian structure on a subbundle which we call the Cameron-Martin tangent space. In the sequel, set

$$H \equiv \{h : [0, 1] \to T_oM : h(0) = 0, \text{ and } (h, h) \equiv \int_0^1 |h'(s)|^2_{g_o} ds < \infty\}.$$

H is just the usual Cameron-Martin space with $\mathbb{R}^d$ replaced by the isometric inner-product space (T_oM, g_o).

DEFINITION 3.2. The **Cameron-Martin tangent space** $(HT_\sigma W(M))$ to $\sigma \in W(M)$ is the set of vectors $X \in CT_\sigma W(M)$ such that the function $h(s) \equiv P_s^\nabla(\sigma)^{-1} X(s)$ is in H. We define a metric on $HT_\sigma W(M)$ by requiring

$$G^\nabla\langle X, X\rangle = (h, h). \tag{3.3}$$

REMARK 3.3. Notice, if σ is a smooth curve then the expression in (3.3) could be written as

$$G^\nabla\langle X, X\rangle = \int_0^1 g\langle \frac{\nabla}{ds} X(s), \frac{\nabla}{ds} X(s)\rangle ds,$$

where $\frac{\nabla}{ds}$ denotes the covariant derivative along the curve σ which is induced from the covariant derivative ∇. This is the usual metric used by differential geometers on path and loop spaces.

The function G^∇ is to be interpreted as a Riemannian metric on $W(M)$. Notice that the vector bundle $HTW(M) \equiv \cup_{\sigma \in W(M)} HT_\sigma W(M)$ over $W(M)$ is trivial and the map

$$((\sigma, h) \to P^\nabla(\sigma)h) : W(M) \times H \to HTW(M) \tag{3.4}$$

is an isometric trivialization of $HTW(M)$.

3.2. Flows, and quasi-invariance. The following theorem is the main result in Driver [**8**] (see Theorem 8.5).

THEOREM 3.4. *Let $h \in H \cap C^1$ and let X^h be the Cameron-Martin vector field on $W(M)$ given by $X^h(\sigma)(s) \equiv P_s^\nabla(\sigma)h(s)$. Then the vector field X^h admits a flow $(e^{tX^h})_{t\in\mathbb{R}}$ on $W(M)$. This flow leaves Wiener measure ν quasi-invariant.*

REMARK 3.5. The unnatural restriction that h be in $H \cap C^1$ rather than in H has been removed by Elton Hsu; see [**19**, **20**]. Also see Remark 2.5.

In the case that the manifold $M = G$ is a Lie group or a homogeneous space, the quasi-invariance of left and right multiplication by finite energy paths has been extensively studied; see [**4**, **16**, **25**, **29**, **30**].

The idea of the proof of this theorem is to use the stochastic development $\Psi^\nabla : W(T_oM) \to W(M)$ map (see Section 4.1. below) to pull back the vector field X^h on $W(M)$ to a vector field Y^h on $W(T_oM)$. The vector field Y^h then has the form

$$Y^h(\omega) = \int C(\omega)d\omega + \int r(\omega)ds, \tag{3.5}$$

with (C, r) an $so(d) \times \mathbb{R}^d$-valued adapted process on $W(T_oM)$. One then applies Theorem 2.4 to Y^h to construct a flow e^{tY^h} on $W(T_oM)$. The desired flow e^{tX^h} is then given as

$$e^{tX^h} = \Psi^\nabla \circ e^{tY^h} \circ (\Psi^\nabla)^{-1}.$$

3.3. Divergence and integration by parts. The following integration by parts formula is a corollary of the previous theorem (see [**8**]). This theorem is essentially in Bismut [**6**]. See also Leandre [**21**], Hsu [**19**], and Fang and Malliavin [**14**].

THEOREM 3.6 (Integration by parts). *Let f be a smooth cylinder function on $W(M)$ and $X \equiv X^h$ be the vector field on $W(M)$ in the above theorem.*

Define the action of X on f by the formula

$$Xf = L^2\text{-}\lim_{t\to 0}(f\circ e^{tX} - f)/t.$$

Then the L^2-adjoint of X on cylinder functions is given by

$$X^* = -X - \operatorname{div}_\nu(X),$$

where

$$\operatorname{div}_\nu(X)(\sigma) = -\int_0^1 \langle h'(s), d\omega(s)\rangle - \frac{1}{2}\int_0^1 \langle \operatorname{Ric}(P_s^\nabla(\sigma)h(s)), P_s^\nabla(\sigma)d\omega(s)\rangle$$
$$-\frac{1}{2}\int_0^1 \langle \hat{T}^\nabla(P_s^\nabla(\sigma)h(s)), P_s^\nabla(\sigma)d\omega(s)\rangle,$$

Ric *denotes the Ricci tensor, $\hat{T}^\nabla$ is a contraction of the covariant derivative (∇T^∇) of the torsion tensor T^∇, and $\omega = (\Psi^\nabla)^{-1}(\sigma)$. Recall, Ψ^∇ is the stochastic development map; see Section 4.1. Furthermore, as before,* $\operatorname{div}_\nu(X)$ *is the unique function on $W(M)$ such that*

$$\frac{d}{dt}|_0 \nu(e^{tX}(B)) = \int_B \operatorname{div}_\nu(X)d\nu$$

for all Borel subsets $B\subset W(M)$.

3.4. Spectral properties of L. One may again construct an " Ornstein-Uhlenbeck" operator L on $L^2(\nu)$ by setting

$$L = \sum_{h\in S} -(X^h)^* X^h$$

(see Driver and Röckner **[10]**). Detailed information about the spectrum of L is not known except in special cases where L is unitarily equivalent to the Ornstein-Uhlenbeck operator on $W(\mathbb{R}^d)$. (This can occur in the case that M is a compact Lie group.) However, there has been a recent beautiful paper of S. Fang **[13]**, in which he shows L does have a mass gap and the kernel of L consists of the constant functions.

3.5. Sobolev inequalities. Except in the special case of a compact Lie group, it is not known whether L satisfies a log-Sobolev inequalities. The Krée-Meyer inequalities are also unknown. One possible method of proof here is to use the Bakry-Emery method **[5]**. However, in these path space situations the Bakry-Emery conditions are hard to check. It is also quite possible that their hypothesis is not generally satisfied in these examples. See **[17]** for an example in the loop group case.

3.6. Diffusions associated to Δ and L. Because $(W(M), G^\nabla)$ is formally a Riemannian manifold, one can formally construct a Laplacian on $W(M)$ (there will be domain questions). Again, for general M, the existence of a Brownian motion associated to the Laplacian on $W(M)$ is still up in the air—but see [15] for the local theory. (I have heard rumors that some version of this problem may be in the process of being solved.) A diffusion associated to L has been shown to exist (see Driver and Röckner [10]).

4. Noncomparable Dirichlet forms

Using the stochastic development map Ψ^∇ (described below) it is possible to pull-back Dirichlet forms on $W(M)$ to Dirichlet forms on $W(T_oM)$. The question I wish to address is whether the pull backed Dirichlet form is comparable with the "natural" Dirichlet form on $W(T_oM)$.

4.1. Stochastic development. There is a well-known measure-theoretic isomorphism (Ψ^∇) between $(W(T_oM), \mu)$ and $(W(M), \nu)$, where μ denotes standard Wiener measure on $W(T_oM)$. The map $\Psi \equiv \Psi^\nabla$ is defined uniquely up to μ-equivalence as the solution to the functional stochastic differential equation

$$\delta\Psi_s(\omega) = P_s^\nabla(\Psi_.(\omega))\delta\omega \text{ with } \Psi_0(\omega) = o, \tag{4.1}$$

where δ is used to denote the Stratonovich differential and ω is a Wiener path in $W(T_oM)$ (see Malliavin [22] and Eells and Elworthy [11]).

REMARK 4.1. For convex surfaces embedded in $\mathbb{R}^3$ equipped with the induced Riemannian structure, the map Ψ has the interpretation of transferring the path ω in $\mathbb{R}^2$ to a path on the surface by rolling the surface along ω without slipping.

It is well known that Ψ carries the measure μ to ν and that Ψ is invertible up to equivalence. However, as pointed out by Malliavin [23, 24], the map Ψ does **not** preserve the Riemannian metrics on $W(T_oM)$ and $W(M)$ except in the case that (M, ∇) itself has trivial geometry. In fact, the map Ψ does not even preserve the notion of Cameron-Martin tangent spaces. This fact has already manifested itself in the appearance of the $\int C(\omega)d\omega$ term in (3.5).

Let Q denote the usual Dirichlet form on $\mathrm{L}^2(W(T_oM))$ determined by

$$Q(f, f) = \int_{W(T_oM)} (Df, Df)_H d\mu = \sum_{h\in S}\int_{W(T_oM)} (\partial_h f)^2 d\mu, \tag{4.2}$$

where D denotes the gradient operator determined by $(\cdot, \cdot)_H$ and S is an orthonormal basis for H. Let $\mathscr{E}^\nabla$ denote the usual Dirichlet form on $\mathrm{L}^2(W(M))$ determined by

$$\mathscr{E}^\nabla(F, F) = \int_{W(M)} G^\nabla(D^\nabla F, D^\nabla F)d\nu = \sum_{h\in S}\int_{W(M)} (X^h F)^2 d\nu, \tag{4.3}$$

where D^{∇} denotes the gradient operator determined by G^{∇}. We now wish to consider the Dirichlet form Q^{∇} on $L^2(W(T_oM))$ given by

$$Q^{\nabla}(f, f) \equiv \mathscr{E}^{\nabla}(f \circ \phi^{\nabla}, f \circ \phi^{\nabla}),$$

where $\phi^{\nabla} \equiv (\Psi^{\nabla})^{-1}$ is the inverse of Ψ^{∇}. In particular, is it possible that Q^{∇} is comparable to Q? This will be the topic of the next section in the case that $M = K$ is a compact Lie group.

4.2. Compact Lie group examples. For the rest of this paper let $M = K$ be a compact Lie group, which, for simplicity of notation, is assumed to be a matrix group. The base point o is taken to be the identity element (e) in K. I will write kA and Ak for $L_{k*}A$ and $R_{k*}A$ respectively, where $A \in \mathfrak{k} \equiv \mathrm{Lie}(K) = T_eK$, $k \in K$, and L_k (R_k) is left (right) translation by k. We construct a metric on K by translating over the group K a fixed Ad_K–invariant inner product $(\langle\cdot,\cdot\rangle_{\mathfrak{k}})$ on $\mathfrak{k}$.

Let ∇^L (∇^R) denote the covariant derivative on K determined uniquely by requiring left (right) invariant vector fields are covariantly constant. Given a path $\sigma \in W(K)$, the left and right parallel translations operators are given by $P_s^{\nabla^L}(\sigma) = L_{\sigma(s)*}$ and $P_s^{\nabla^R}(\sigma) = R_{\sigma(s)*}$ respectively. The development map $\Psi^L \equiv \Psi^{\nabla^L}$ is determined by $\Psi^L(\omega) = \sigma$ where σ solves the stochastic differential equation

$$\delta\sigma = \sigma\delta\omega \text{ with } \sigma(0) = e. \tag{4.4}$$

Similarly $\Psi^R \equiv \Psi^{\nabla^R}$ is determined by the stochastic differential equation

$$\delta\sigma = \delta\omega\sigma \text{ with } \sigma(0) = e. \tag{4.5}$$

THEOREM 4.2. *Let $Q^L \equiv Q^{\nabla^L}$, $Q^R \equiv Q^{\nabla^R}$, $\mathscr{E}^L \equiv \mathscr{E}^{\nabla^L}$, and $\mathscr{E}^R \equiv \mathscr{E}^{\nabla^R}$. Assume that $\mathfrak{k}$ is non-commutative; then the following hold.*

(i) $\sup_f Q(f)/Q^L(f) = \infty$ *and* $\sup_f Q^L(f)/Q(f) = \infty$.

(ii) $\sup_f Q(f)/Q^R(f) = \infty$ *and* $\sup_f Q^R(f)/Q(f) = \infty$.

(iii) $\sup_F \mathscr{E}^R(F)/\mathscr{E}^L(F) = \infty$ *and* $\sup_F \mathscr{E}^L(F)/\mathscr{E}^R(F) = \infty$.

In the above expressions, the supremum is taken over the intersection of the domain of the quadratic form in the numerator with that in the denominator.

For a proof and a more detailed statement of the above theorem, see [**9**]. I conjecture that analogues of the above result hold for arbitrary Riemannian manifolds (M) with nontrivial geometry. For related issues concerning Sobolev spaces determined by L and domains of essential selfadjointness for L, see [**1, 2, 3**].

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