JOURNAL OF FUNCTIONAL ANALYSIS 133, 69–128 (1995)

On the Kakutani–Itô–Segal–Gross and Segal–Bargmann–Hall Isomorphisms

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Communicated by L. Gross

Received July 27, 1994

Recently, Gross has shown that the Kakutani–Itô–Segal isomorphism theorem has an extension from the setting of Gaussian measure on a vector space to "heat kernel" measure (p_t) on a simply connected Lie group (G) of compact type. The isomorphism relates $L^2(p_t)$ to a certain completion of the universal enveloping algebra of $\mathfrak{g} \doteq \mathrm{Lie}(G)$. Gross proves this result using the Kakutani–Itô–Segal theorem and an infinite dimensional calculus associated to G-valued Brownian motion. Hijab has greatly simplified and clarified Gross' proof. Hijab's proof avoids most, but not all, of the "infinite dimensional" analysis in the original proof. In this paper, we will build on Hijab's proof to give a completely "finite dimensional" non-probabilistic proof of Gross' isomorphism theorem. The proof given here relies heavily on Hall's beautiful "extension" of the Segal–Bargmann transformation to the setting of compact Lie groups. This theorem relating $L^2(p_t)$ to a certain L^2-space of holomorphic functions $(L^2(G^{\mathbb{C}}) \cap \mathscr{H})$ on the complexified Lie group $G^{\mathbb{C}}$ will also be generalized to Lie groups of compact type. In the process, it is shown how to characterize, in terms of summability conditions on all the derivatives at the identity in $G^{\mathbb{C}}$, those holomorphic functions which are in $L^2(G^{\mathbb{C}})$.

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0022-1236/95 $12.00

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1. Introduction

1.1. *Notation*

Throughout this paper G will denote a connected Lie group of compact type, with fixed Ad_G-invariant inner product $(\cdot, \cdot)$ on $\mathfrak{g}$—the Lie algebra of G. (Recall that G is of compact type if there exists an Ad_G-invariant inner product on $\mathfrak{g}$. This is equivalent to assuming that G is isomorphic to $K \times \mathbb{R}^d$ for some compact Lie group K, see Corollary 2.2 below.) Let $G^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}$ denote the complexifications of G and $\mathfrak{g}$ respectively.

Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be a fixed orthonormal basis for $\mathfrak{g}$. Denote by $\mathscr{J} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ the operation of multiplication by $\sqrt{-1}$, i.e.

$$\mathscr{J}(X \otimes z) \doteq X \otimes (\sqrt{-1}z)\ \forall X \in \mathfrak{g},\ z \in \mathbb{C}.$$

The Laplacians on G and $G^{\mathbb{C}}$ are (respectively)

$$\Delta \doteq \sum_{A \in \mathfrak{g}_0} \tilde{A}^2 \tag{1.1}$$

and

$$\Delta_{\mathbb{C}} \doteq \sum_{A \in \mathfrak{g}_0} \{\tilde{A}^2 + (\widetilde{\mathscr{J}A})^2\}, \tag{1.2}$$

where $\tilde{A}$ in (1.1) ((1.2)) denotes the unique left invariant vector field on $G(G^{\mathbb{C}})$ which agrees with A at the identity.

For $t > 0$, let p_t and μ_t denote the convolution heat kernels on G and $G^{\mathbb{C}}$ associated to $\Delta/2$ and $\Delta_{\mathbb{C}}/4$ respectively, i.e., p_t and μ_t are the fundamental solutions to the heat equations:

$$\partial p_t/\partial t = \tfrac{1}{2}\Delta p_t \quad \text{on } G$$

and

$$\partial \mu_t/\partial t = \tfrac{1}{4}\Delta_{\mathbb{C}}\mu_t \quad \text{on } G^{\mathbb{C}}$$

respectively. (See Section 2 for more details.) Write $L^2(p_t)$ $(L^2(\mu_t))$ for the Hilbert space of complex valued square integrable functions on G $(G^{\mathbb{C}})$

relative to the measure $p_t(x)\,dx$ ($\mu_t(g)\,dg$), where dx (dg) is a Haar measure on G ($G^{\mathbb{C}}$).

Let $T \doteq \bigoplus_{n=0}^{\infty} (\mathfrak{g}^{\mathbb{C}})^{\otimes n}$ ($(\mathfrak{g}^{\mathbb{C}})^0 \equiv \mathbb{C}$) be the tensor algebra of $\mathfrak{g}^{\mathbb{C}}$, T' be the *algebraic* dual, and J be the two sided ideal generated by

$$\{\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] \mid \xi, \eta \in \mathfrak{g}^{\mathbb{C}}\} \subset T.$$

Notation 1.1. Throughout this paper, we will use $\langle \cdot, \cdot \rangle$ to denote the pairing between a vector space and its dual.

Let $\mathbf{1}$ denote the element in T' defined by $\langle \mathbf{1}, a \rangle = a$ for all $a \in \mathbb{C} \subset T$ and $\langle \mathbf{1}, \mathfrak{g} \otimes T \rangle \doteq 0$. Following Gross (when $t = 1$), for each $t > 0$, let $((\cdot, \cdot))_t$ denote the inner product on T uniquely determined by

$$((h_1 \otimes \cdots \otimes h_n, g_1 \otimes \cdots \otimes g_m))_t \doteq \frac{n!}{t^n} \delta_{nm} \prod_{k=1}^{n} (h_k, g_k)_{\mathfrak{g}^{\mathbb{C}}} \tag{1.3}$$

where $\{h_i\}_{i=1}^n \cup \{g_j\}_{j=1}^m \subset \mathfrak{g}^{\mathbb{C}}$, $n, m \in \{0, 1, 2 \ldots\}$, and $(\cdot, \cdot)_{\mathfrak{g}^{\mathbb{C}}}$ is the extension of $(\cdot, \cdot)$ on $\mathfrak{g}$ to a complex inner product on $\mathfrak{g}^{\mathbb{C}}$. (We assume that $(\cdot, \cdot)_{\mathfrak{g}^{\mathbb{C}}}$ is conjugate linear in the second varable.) Let T_t be the completion of T relative to $((\cdot, \cdot))_t$. Let T_t^* be the *topological* dual of T_t and $(\cdot, \cdot)_t$ denote the dual inner product on T_t^*; i.e., $\alpha \in T_t^* \subset T'$ iff

$$(\alpha, \alpha)_t = \sum_{n=0}^{\infty} \sum_{\xi_1, \ldots, \xi_n \in \mathfrak{g}_0} |\langle \alpha, \xi_1 \otimes \cdots \otimes \xi_n \rangle|^2 \, t^n/n! < \infty. \tag{1.4}$$

Remark 1.2. If $\mathfrak{g}_0 \subset \mathfrak{g}$ is an orthonormal basis for $\mathfrak{g}$ (as a real inner product space), then

$$\beta \equiv \{1\} \cup \bigcup_{n=1}^{\infty} \{t^n(\xi_1 \otimes \cdots \otimes \xi_n)/\sqrt{n!} \mid \xi_1, \ldots, \xi_n \in \mathfrak{g}_0\}$$

is an orthonormal basis for T_t as a complex Hilbert space. Hence for $\alpha \in T_t^*$,

$$\|\alpha\|_t^2 = \sum_{n=0}^{\infty} \sum_{\xi_1, \ldots, \xi_n \in \mathfrak{g}_0} |\langle \alpha, \xi_1 \otimes \cdots \otimes \xi_n \rangle|^2 \, t^n/n!.$$

Alternatively,

$$\|\alpha\|_t^2 = \sum_{n=0}^{\infty} \frac{t^n}{n!} |\alpha_n|_{\mathfrak{g}^{\otimes n}}^2,$$

where

$$|\alpha_n|_{\mathfrak{g}^{\otimes n}}^2 \equiv \sum_{\xi_1, \ldots, \xi_n \in \mathfrak{g}_0} |\langle \alpha, \xi_1 \otimes \cdots \otimes \xi_n \rangle|^2.$$

Finally, let ${}^{\perp}J = \{\alpha \in T' \mid \alpha(J) = \{0\}\}$ (the annihilator of J) and

$${}^{\perp}J_t \doteq {}^{\perp}J \cap T_t^* = \{\alpha \in T' \mid \alpha(J) \doteq 0 \text{ and } (\alpha, \alpha)_t < \infty\}.$$

Notice that the natural isometric isomorphism of T_t^* with T_t restricts to an isometric isomorphism of ${}^{\perp}J_t$ with $J_t^{\perp}$—the orthogonal complement of J.

EXAMPLE 1.3. Suppose that $G = \mathbb{R}^d$ with the usual inner product. Then

1. $G^{\mathbb{C}} = \mathbb{C}^d$,
2. $\Delta = \sum_{i=1}^d \partial^2/\partial^2 x_i$ and $\Delta_{\mathbb{C}} = \sum_{i=1}^d \partial^2/\partial x_i^2 + \sum_{i=1}^d \partial^2/\partial y_i^2$, where $z_i = x_i + \sqrt{-1}\, y_i$ are the standard linear coordinates on $\mathbb{C}^d = \mathbb{R}^d + i\mathbb{R}^d$.
3. $p_t(x) = (2\pi t)^{-d/2} \exp[-|x|^2/2t]$.
4. $\mu_t(z) = (\pi t)^{-d} \exp[-|z|^2/t]$.
5. $J \perp \mathscr{S}$(relative to $(\cdot, \cdot)_t$ for all t) where $\mathscr{S} \subset T$ are the symmetric tensors.
6. ${}^{\perp}J_t$ is naturally isomorphic to $\mathscr{S}_t$—the completion of the symmetric tensor algebra $\mathscr{S}$ of $\mathbb{C}^d$ with respect to the inner product, $((\cdot, \cdot))_t$, on T. The isomorphism is given explicitly by:

$$\beta \in \mathscr{S}_t \to ((\beta, \cdot))_t \in {}^{\perp}J_t \subset T'.$$

DEFINITION 1.4. For each $u \in C^{\infty}(G, \mathbb{C})$ or $C^{\infty}(G^{\mathbb{C}}, \mathbb{C})$ and $g \in G$ or $G^{\mathbb{C}}$ respectively, let $(1-D)_g^{-1} u$ denote the element $\alpha \in T'$ determined uniquely by: (i) $\langle \alpha, 1 \rangle = u(g)$ and (ii) for all $n = 1, 2, 3, \ldots$ and $\{\xi_i\}_{i=1}^n \subset \mathfrak{g}$,

$$\langle \alpha, \xi_1 \otimes \cdots \otimes \xi_n \rangle = \langle D^n u(g), \xi_1 \otimes \cdots \otimes \xi_n \rangle \equiv (\tilde{\xi}_1 \cdots \tilde{\xi}_n u)(g) \quad (1.5)$$

where $\tilde{\xi}_i$ denotes unique left invariant vector field on G respectively $G^{\mathbb{C}}$ such that $\tilde{\xi}_i(e) = \xi_i$. (The form α is found by extending the above formulas to all of T by complex linearity.)

Remark 1.5. Notice in the above definition that the vectors ξ_i were required to be in the real Lie algebra $\mathfrak{g}$ even in the case that u was a smooth function on $G^{\mathbb{C}}$. If $u \in \mathscr{H}(G^{\mathbb{C}})$ (the holomorphic functions on $G^{\mathbb{C}}$) and α is defined as in Definition 1.4, (1.5) is valid for all $\{\xi_i\}_{i=1}^n \subset \mathfrak{g}^{\otimes \mathbb{C}}$.

Remark 1.6. Because $\tilde{\xi}\tilde{\eta} - \tilde{\eta}\tilde{\xi} - \widetilde{[\xi, \eta]} \equiv 0$ for all $\xi, \eta \in \mathfrak{g}$, it is easy to check that $\alpha \doteq (1-D)_g^{-1} u$ in Definition 1.4 automatically lies in

$${}^{\perp}J \doteq \{\alpha \in T' \mid \alpha(J) \doteq 0\} \subset T'.$$

1.2. *A Review of the Euclidean Results*

The Kakutani–Itô–Segal isometry [19, 18, 25] may be described explicitly in finite dimensions by the map in the following theorem.

THEOREM 1.7 (Kakutani–Itô–Segal). *Let $G=\mathbb{R}^d$ and keep the notation as in Example 1.3. Then the map*

$$(1-D)_0^{-1}\, e^{t\Delta/2} : L^2(p_t) \to {}^{\perp}J$$

has range ${}^{\perp}J_t$ and

$$(1-D)_0^{-1}\, e^{t\Delta/2} : L^2(p_t) \to {}^{\perp}J_t \cong \mathscr{S}_t$$

is an isometric isomorphism of Hilbert spaces.

Remark 1.8. Actually, the Kakutani–Itô–Segal theorem deals with the context where $d=\infty$, that is $G=H$ is a real Hilbert space. In the finite dimensional setting the Kakutani–Itô–Segal theorem is closely related to the mathematics of quantum harmonic oscillators. With this in mind, there are certainly precursors to the Kakutani–Itô–Segal theorem in the mathematics and physics literature. The formal study of the "Fock–Cook" space ($\mathscr{S}_t$) was started by Fock [10] and carried out mathematically by Cook [7]. For a more detailed history of this subject, the reader is referred to [28] and to Chapter 1 of [1].

The Segal–Bargmann Theorem [27, 28, 29, 2, 3, 4] in the context of a finite dimensional vector space may be stated as follows.

THEOREM 1.9 (Segal–Bargmann). *Let $G=\mathbb{R}^d$, $\mathscr{H}$ be the holomorphic functions on $\mathbb{C}^d$. Also keep the same notation as in Example 1.3. Then for each $f\in L^2(p_t)$, $e^{t\Delta/2}f$ has an analytic continuation to a holomorphic function (still denoted by $e^{t\Delta/2}f$) on $G^{\mathbb{C}}=\mathbb{C}^d$. Moreover, $e^{t\Delta/2}f\in L^2(\mu_t)\cap\mathscr{H}$ and the linear map*

$$e^{t\Delta/2} : L^2(p_t) \to L^2(\mu_t)\cap\mathscr{H}$$

is an isometric isomorphism of Hilbert spaces.

The following result is an immediate corollary of Theorems 1.7 and 1.9.

COROLLARY 1.10. *Let $G=\mathbb{R}^d$ as above, then*

$$(1-D)_0^{-1} : L^2(\mu_t)\cap\mathscr{H} \to {}^{\perp}J_t$$

is an isometric isomorphism of Hilbert spaces.

Theorems 1.7, 1.9, and Corollary 1.10 may be summarized by the assertion that all maps in the following commutative diagram below are isometric isomorphisms.

$$\begin{array}{ccc} L^2(p_t) & \xrightarrow{(1-D)_0^{-1} e^{t\Delta/2}} & {}^\perp J_t \\ & {\scriptstyle e^{t\Delta/2}} \searrow \quad \nearrow {\scriptstyle (1-D)_0^{-1}} & \\ & L^2(\mu_t) \cap \mathscr{H} & \end{array} \tag{1.6}$$

In this context where $G = \mathbb{R}^d$, the notation we have been using is very close to the notation which has been used in the white noise analysis literature, see for example Y.-J. Lee [22] and the references therein.

Remark 1.11. The significance of the isomorphism in Theorem 1.7 is that it intertwines directional derivatives with "annihilation" operators. Similarly, the isomorphism in Theorem 1.9 intertwines the directional derivatives on $L^2(p_t)$ with those on $L^2(\mu_t) \cap \mathscr{H}(G^{\mathbb{C}})$. In fact, these intertwining properties are often used to uniquely characterize the isomorphisms in Theorem 1.7 and Theorem 1.9. This will be discussed below in the general context of Lie groups of compact type.

1.3. *Statement of Results*

DEFINITION 1.12. Let $\xi \in \mathfrak{g}$.

1. Define the *annihilation* operator $A_\xi : T' \to T'$ by

$$\langle A_\xi \alpha, \eta \rangle \doteq \langle \alpha, \eta \otimes \xi \rangle$$

for all $\eta \in T$. (It is easily checked that $A_\xi({}^\perp J) \subset {}^\perp J$.)

2. For each $t > 0$ and $\xi \in \mathfrak{g}$, let

$$\mathscr{D}(A_\xi^t) \doteq \{\alpha \in {}^\perp J_t \mid A_\xi \alpha \in T_t^*\}$$

and for $\alpha \in \mathscr{D}(A_\xi^t)$ set

$$A_\xi^t \alpha \doteq A_\xi \alpha \in {}^\perp J_t.$$

3. Define $\tilde{\xi}_t$ to be the $L^2(p_t)$-closure of $\tilde{\xi}|_{C_c^\infty(G)}$.

4. Define $\tilde{\xi}_t$ to be the linear operator on $L^2(\mu_t) \cap \mathscr{H}(G^{\mathbb{C}})$ determined by $\tilde{\xi}_t f = \tilde{\xi} f$ for each $f \in \mathscr{D}(\tilde{\xi}_t)$, where

$$\mathscr{D}(\tilde{\xi}_t) \doteq \{f \in L^2(\mu_t) \cap \mathscr{H}(\mathrm{G}^{\mathbb{C}}) \mid \tilde{\xi} f \in L^2(\mu_t)\}.$$

The reader may readily check that A_ξ^t is a closed operator on ${}^\perp J_t$ and using Lemma 3.2 below that $\bar{\xi}_t$ is a closed operator on $L^2(\mu_t) \cap \mathscr{H}(G^{\mathbb{C}})$. With this notation, L. Gross' theorem extending Theorem 1.7 may be stated as follows.

THEOREM 1.13 (Gross [12, Theorem 2.1]). *Assume that G is a simply connected Lie group of compact type. There exists a unique isometry U_t from $L^2(G, p_t)$ onto ${}^\perp J_t$ such that*

1. $U_t 1 = \mathbf{1} \in {}^\perp J_t$.
2. $U_t \bar{\xi}_t = A_\xi^t U_t$, *for all* $\xi \in \mathfrak{g}$.

THEOREM 1.14 (Hijab's Formula [16]). *Keeping the hypothesis and notation of the above theorem the isometry U_t is given explicitly as:*

$$U_t = (1 - D)_e^{-1} e^{t\Delta/2}.$$

This explicit formula for Gross' isometry (in the case that G is a compact Lie group) is due to Hijab [16]. The formula given here differs slightly from that in [16] because ${}^\perp J_t$ is not identified with $J_t^\perp$ as was done there.

Proofs of these two theorems are contained in the statements of Theorem 4.1 (the isometry assertion), Corollary 6.5 (the surjectivity assertion), and Corollary 7.3 (the intertwining property.)

Hall's "extension" of the Segal–Bargmann theorem is as follows.

THEOREM 1.15 (Hall [15]). *Suppose that G is a compact Lie group. Let $\mathscr{H}$ denote the holomorphic functions on $G^{\mathbb{C}}$. Then for each $f \in L^2(p_t)$, $e^{t\Delta/2} f$ exists and has an analytic continuation to $G^{\mathbb{C}}$, which is still denoted by $e^{t\Delta/2} f$. Moreover, $e^{t\Delta/2} f \in L^2(\mu_t) \cap \mathscr{H}$ and $e^{t\Delta/2} : L^2(p_t) \to L^2(\mu_t) \cap \mathscr{H}$ is an isometric isomorphism of Hilbert spaces.*

Corollary 4.5 and Theorem 7.2 below generalizes Hall's theorem to the statement.

THEOREM 1.16. *Suppose that G is a Lie group of compact type. Let $\mathscr{H}$ denote the holomorphic functions on $G^{\mathbb{C}}$. Then for each $f \in L^2(p_t)$, $e^{t\Delta/2} f$ exists and has an analytic continuation to $G^{\mathbb{C}}$, which is still denoted by $e^{t\Delta/2} f$. Moreover, $e^{t\Delta/2} f \in L^2(\mu_t) \cap \mathscr{H}$,*

$$e^{t\Delta/2} : L^2(p_t) \to L^2(\mu_t) \cap \mathscr{H}$$

is an isometric isomorphism of Hilbert spaces, and

$$e^{t\Delta/2} \bar{\xi}_t = \xi_t e^{t\Delta/2}. \tag{1.7}$$

COROLLARY 1.17. *Let G be a compact simply connected Lie group. Then*

$$(1-D)_e^{-1} : L^2(\mu_t) \cap \mathscr{H} \to {}^{\perp}J_t$$

is an isometric isomorphism of Hilbert spaces such that

$$(1-D)_e^{-1} \check{\xi}_t = A_\xi^t (1-D)_e^{-1}.$$

One of the main goals of this paper is to give a direct proof of this corollary, see Theorem 4.4 (the isometry assertion), Theorem 6.4 (the surjectivity assertion), and Theorem 7.1 (the intertwining property). The key to the proof is Corollary 5.9 below which asserts (for general G of compact type), a holomorphic function u on $G^{\mathbb{C}}$ is in $L^2(\mu_t)$ iff $(1-D)_e^{-1} u \in {}^{\perp}J_t$.

The main results quoted above may be summarized as follows. The maps in the commutative diagram in (1.6) (with 0 replaced by $e \in G \subset G^{\mathbb{C}}$) are all isometric isomorphisms of Hilbert space provided that G is a simply connected Lie group of compact type. Moreover, these maps intertwine the operators defined in Definition 1.12.

As mentioned in the abstract, Gross' proof of Theorem 1.14 is rather involved and uses heavily the machinery of stochastic analysis for G-valued Brownian motion. Omar Hijab [16] greatly simplified Gross' original proof. Hijab's proof also clarified the structure of Gross' isomorphism. However, Hijab's proof of the fact that the map $(1-D)_e^{-1} e^{\Delta/2}$ is surjective relied on a technical result about analytic vectors from Gross' original paper. This technical result was proved using an infinite dimensional calculus associated to G-valued Brownian motion. The main aim of this paper is to give a "finite dimensional" and "non-probabilistic" proof of Gross' isomorphism theorem. This will be done by giving a direct proof of Corollary 1.17. By combining this Corollary with Hall's Theorem 1.15, we will produce a "finite dimensional" proof of Gross' Theorem 1.13 and Hijab's Formula, Theorem 1.14.[1]

In order to keep the paper essentially self contained, I will give a slight variation of the proof of Hall's theorem and at the same time extend his result to Lie groups of compact type. One step in Hall's proof of Theorem 1.15 is to show that p_t has an analytic continuation to $G^{\mathbb{C}}$. This is done using some detailed results about the representations of G. The proof given here observes that it is possible to show directly that $e^{t\Delta/2} f$ has an analytic continuation to $G^{\mathbb{C}}$ for all $f \in L^2(p_t)$. As a consequence, p_t has an analytic continuation on $G^{\mathbb{C}}$.

[1] O. Hijab [17], using a technique similar to one in this paper, has found a proof of Theorem 1.14 which avoids the complexified Lie group and Hall's theorem.

2. Notation and Preliminaries

2.1. *Compact-Type Lie Groups and Complexifications*

THEOREM 2.1 (Structure Theorem). *Suppose that G is a connected Lie group of compact type with Ad_G invariant inner product $(\cdot, \cdot)$. Let*

$$\mathfrak{k} \doteq [\mathfrak{g}, \mathfrak{g}] \doteq \{[A, B] \mid A, B \in \mathfrak{g}\}$$

and $\mathfrak{z}$ be the center of $\mathfrak{g}$. Then $\mathfrak{g}$ is the orthogonal direct sum of $\mathfrak{k}$ and $\mathfrak{z}$, and both $\mathfrak{k}$ and $\mathfrak{z}$ are ideals. Let K and Z be the connected Lie-subgroups of G such that $\mathrm{Lie}(K) = \mathfrak{k}$ *and* $\mathrm{Lie}(Z) = \mathfrak{z}$. *Then K is compact, Z is the connected component of the center of G (hence Z is closed), $G = KZ$, and $K \cap Z$ is finite. Furthermore, the map $\phi: K \times Z \to G$ given by $\phi(k, z) \doteq kz$ is a Lie group homomorphism onto G with*

$$\ker \phi = \{(z, z^{-1}) \mid z \in K \cap Z\} \doteq D.$$

For a proof see Lemma 4.6.9 and Theorem 3.6.6 in Wallach [32].

COROLLARY 2.2. *Let G be a connected Lie group of compact type, then G is isomorphic to $K' \times \mathbb{R}^d$ where K' is a connected compact Lie group.*

Proof. From Theorem 2.1, G is isomorphic to $(K \times Z)/D$ where K is compact and connected, Z is abelian and connected, and $D = \{(z^{-1}, z) \mid z \in K \cap Z\}$ is finite. Since the Lie group Z is abelian and connected, there are integers k and d, and a Lie group isomorphism $\rho: Z \to T^k \times \mathbb{R}^d$, where T^k is a k-dimensional torus. (See, for example, Theorem 3.6, p. 25 in [5].) Since $K \cap Z$ is a finite subgroup of Z, it follows that $\rho(K \cap Z) \subset T^k \times \{0\}$. It is now a simple matter to check that G is isomorphic to $K' \times \mathbb{R}^d$, where $K' \doteq (K \times T^k)/D'$, $D' \doteq \{(z, \rho'(z^{-1})) : z \in K \cap D\}$, and $\rho(z) = (\rho'(z), 0)$ for all $z \in K \cap Z$. Q.E.D.

COROLLARY 2.3. *Let G be a connected, simply connected Lie group of compact type. Then there is a simply connected compact Lie group K (with Lie algebra $\mathfrak{k}$), integer $d \geq 0$, and Lie group isomorphism $\phi: K \times \mathbb{R}^d \to G$ such that* $\phi_*(\mathrm{Lie}(K) \times \{0\}) = [\mathfrak{g}, \mathfrak{g}]$, *and* $\phi_*(\{0\} \times \mathbb{R}^d) = \mathfrak{z}$*—the center of $\mathfrak{g}$. In particular, the pull back of the inner product $(\cdot, \cdot)$ on $\mathfrak{g}$ via ϕ is an inner product on $\mathfrak{k} \times \mathbb{R}^d$ such that $\mathfrak{k} \times \mathbb{R}^d$ is the orthogonal direct sum of $\mathfrak{k} \times \{0\}$ and $\{0\} \times \mathbb{R}^d$.*

Proof. Let $\phi: K \times Z \to G$ be the Lie group homomorphism in Theorem 2.1. Since $\ker \phi$ is finite (in particular discrete), ϕ is a covering map. Hence ϕ induces an injection from $\pi_1(K \times Z) \cong \pi_1(K) \times \pi_1(Z)$ to $\pi_1(G)$. Since G is assumed to be simply connected, it follows that both K

and Z are simply connected. Since Z is abelian and simply connected, Z is isomorphic to $\mathbb{R}^d$ for some d. Because there are no non-trivial finite subgroups of $\mathbb{R}^d$, we must have that $\ker \phi = \{e\}$. Therefore, ϕ is a Lie group isomorphism. Q.E.D.

Let $\mathfrak{g}^{\mathbb{C}} \equiv \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of $\mathfrak{g}$, and extend the Ad_G invariant inner product on $\mathfrak{g}$ to a complex inner product on $\mathfrak{g}^{\mathbb{C}}$ which is conjugate linear in the second variable. Let $K^{\mathbb{C}}$ denote the complexification of K, where K is a compact Lie group. (See [5] pp. 151–156 for precise definitions and existence of complexifications.)

EXAMPLE 2.4. 1. If $K = SU(n, \mathbb{C})$, then $K^{\mathbb{C}} = SL(n, \mathbb{C})$.
2. If $K = T^k \doteq \mathbb{R}^k/\mathbb{Z}^k$, the $K^{\mathbb{C}} = \mathbb{C}^k/\mathbb{Z}^k$.

DEFINITION 2.5. If $Z = T^k \times \mathbb{R}^d$, set $Z^{\mathbb{C}} \doteq (T^k)^{\mathbb{C}} \times \mathbb{C}^d$. Given a Lie group G of compact type, set $G^{\mathbb{C}} \doteq (K^{\mathbb{C}} \times Z^{\mathbb{C}})/D$, where K, Z and D are as in the Structure Theorem 2.1.

LEMMA 2.6. *The complexified Lie group $G^{\mathbb{C}}$ is unimodular.*

Proof. Using the fact that $ad_X : \mathfrak{g} \to \mathfrak{g}$ is skew symmetric for all $X \in \mathfrak{g}$, it is an easy exercise to verify that $\mathrm{Tr}(ad_X : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}) = 0$ for all $X \in \mathfrak{g}^{\mathbb{C}}$. It is well known that this implies that $G^{\mathbb{C}}$ is uni-modular. Q.E.D.

2.2. *Laplacians and Heat Kernels on Uni-modular Lie Groups*

This section reviews some basic facts about heat kernels on uni-modular Lie groups. In the next section, we will return to the Lie groups G and $G^{\mathbb{C}}$ defined above.

Let H be a unimodular Lie group, dh denote a bi-invariant Haar measure on H, $\mathfrak{h} = T_e H$ be the Lie algebra of H, and $\mathfrak{h}_0 \subset \mathfrak{h}$ be a basis for $\mathfrak{h}$. For $A \in \mathfrak{h}$, let $\tilde{A}(\hat{A})$ denote the unique left (right) invariant vector-field on H which agrees with A at $e \in H$. The left and right invariant Laplacian relative to the basis $\mathfrak{h}_0$ is given by $\Delta \doteq \sum_{A \in \mathfrak{h}_0} \tilde{A}^2$ and $\Delta' \doteq \sum_{A \in \mathfrak{h}_0} \hat{A}^2$ respectively. Since H is unimodular, it is easy to check the formal adjoint, relative to $L^2(\mathrm{Haar})$, of $\tilde{A}(\hat{A})$ is $-\tilde{A}(-\hat{A})$. Hence, $\Delta/2$ and $\Delta'/2$ are symmetric operators on the smooth functions with compact support on H. It is well known, see for example Theorem 2.1, p. 152 of [24], that $\Delta/2$ and $\Delta'/2$ are essentially self-adjoint and the closures of $\Delta/2$ and $\Delta'/2$ generate strongly continuous, self-adjoint contraction semi-groups $e^{t\Delta/2}$ and $e^{t\Delta'/2}$ on $L^2(\mathrm{Haar})$. From standard elliptic regularity theory, the heat semi-groups have integral kernels k_t and k'_t respectively, i.e.

$$e^{t\Delta/2} f(x) = \int_H k_t(x, h) f(h) \, dh \qquad \text{and} \qquad e^{t\Delta'/2} f(x) = \int_H k'_t(x, h) f(h) \, dh$$

for all $f \in L^2(\text{Haar})$. Because $\Delta/2$ is invariant under left translations and hence so is $e^{t\Delta/2}$, it follows easily that $k_t(x, h) = k_t(e, x^{-1}h)$. Set $p_t(h) \doteq k_t(e, h)$, so that $k_t(x, h) = p_t(x^{-1}h)$. Since $e^{t\Delta/2}$ is self-adjoint, k_t is symmetric and hence $p_t(x^{-1}) = p_t(x)$ for all $x \in H$. Also notice that $p_t(x) \doteq k_t(x, e)$ satisfies the "left" heat equation:

$$\partial p_t/\partial t = \tfrac{1}{2}\Delta p_t \qquad \text{with} \quad \lim_{t \to 0} p_t = \delta_e, \tag{2.1}$$

where δ_e denotes the delta function at the identity. Similarly, one shows that $k'_t(x, h) = p'_t(hx^{-1}) = p'_t(xh^{-1})$, where p'_t is a solution to "right" heat equation

$$\partial p'_t/\partial t = \tfrac{1}{2}\Delta' p'_t \qquad \text{with} \quad \lim_{t \to 0} p'_t = \delta_e. \tag{2.2}$$

Let $\kappa(h) \doteq h^{-1}(\kappa : H \to H)$, then it is easy to check that $\kappa_* \tilde{A} = -\hat{A} \circ k$ for all $A \in \mathfrak{h}$. From this is follows that $\Delta(f \circ \kappa) = (\Delta' f) \circ \kappa$ for all sufficiently smooth f. Therefore $e^{t\Delta/2}(f \circ \kappa) = (e^{t\Delta'/2} f) \circ \kappa$ for all $f \in L^2(dh)$. Combining this fact together with the fact that Haar measure on H is invariant under κ, it follows that $k_t(x, h^{-1}) = k'_t(x^{-1}, h)$. Consequently, $p_t(x) = p'_t(x^{-1}) = p'_t(x)$.

The following theorem summarizes the above discussion.

THEOREM 2.7. *Assuming the above notation, let p_t denote the fundamental solution to the left heat equation* (2.1). *Then*

1. $p_t(x) = p_t(x^{-1})$ *for all* $x \in H$,
2. p_t *solves the right heat equation* (2.2),
3. *for* $f \in L^2(dh)$,

$$e^{t\Delta/2} f(x) = \int_H p_t(x^{-1}h)\, f(h)\, dh = \int_H p_t(h^{-1}x)\, f(h)\, dh,$$

4. *for* $f \in L^2(dh)$,

$$e^{t\Delta'/2} f(x) = \int_H p_t(hx^{-1})\, f(h)\, dh = \int_H p_t(xh^{-1})\, f(h)\, dh.$$

In particular, for $f \in L^2(dh)$:

$$e^{t\Delta/2} f(e) = e^{t\Delta'/2} f(e) = \int_H p_t(h)\, f(h)\, dh. \tag{2.3}$$

In the sequal, we will only consider left invariant vector fields and Laplacians. From the above discussion, only minor modifications are necessary to change from "left" to "right."

The basic Gaussian upper bound for the heat kernel that will be used in the sequel is summarized in the following theorem. For a proof see Theorem 2.1, p. 257 of Robinson [24].

THEOREM 2.8 (Gaussian Bound). *Let H be a unimodular Lie group, $(\cdot,\cdot)$ be an arbitrary inner product on $\mathfrak{h} \doteq T_e H$, $\mathfrak{h}_0 \subset \mathfrak{h}$ be an orthonormal basis for $\mathfrak{h}$, $\Delta \doteq \sum_{A \in \mathfrak{h}_0} \tilde{A}^2$, p_t be the corresponding heat kernel, and for $x \in H$*

$$|x| \doteq d(x, e),$$

where $d(x, e)$ is Riemannian distance from x to e relative to the left (or right) invariant Riemannian metric on H which agrees with $(\cdot,\cdot)$ at $T_e H$. Then for all $0 < t < \tau < \infty$, there is a constant $C(t, \tau) < \infty$ such that

$$p_t(x) \leqslant C(t, \tau)\, e^{-|x|^2/2\tau} \qquad \forall x \in H. \tag{2.4}$$

(See the Appendix for some remarks on the definition of $|x|$.)

EXAMPLE 2.9. Suppose that $H = \mathbb{R}^n/(\mathbb{Z}^k \times \{0\})$ $(k \leqslant d)$ and let $\pi: \mathbb{R}^n \to H$ denote the canonical projection. Since π is a local diffeomorphism, we may identify $\mathfrak{h}$ with $\mathbb{R}^n$ using π_{*0}. Moreover $L^2(H, \text{Haar})$ may be identified with $L_p^2(\mathbb{R}^n)$—the set of measurable functions F on $\mathbb{R}^n$ such that;

$$F(x+m) = F(x) \text{ a.e. } x,\ \forall m \in \mathbb{Z}^k \times \{0\},$$

and

$$\int_{C \times \mathbb{R}^{n-k}} |F(x)|^2\, dx < \infty,$$

where $C \doteq [0, 1]^k$ is a "unit cell." The identification is given explicitly by the map

$$f \in L^2(H, \text{Haar}) \to f \circ \pi \in L_p^2(\mathbb{R}^n).$$

Let Q be a positive definite $n \times n$ matrix and set $(x, y) \doteq x \cdot Qy$. Suppose that $\mathfrak{h}_0$ is an orthonormal basis relative to $(\cdot,\cdot)$, then $\Delta \doteq \sum_{A \in \mathfrak{h}_0} \tilde{A}^2$, thought of as an operator on $L_p^2(\mathbb{R}^n)$, is given by

$$\Delta = \sum_{i,j=1}^{n} (Q^{-1})_{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j},$$

where $x=(x^1, \ldots, x^n)$ are the standard linear coordinates on $\mathbb{R}^n$. For $F \in L_p^2(\mathbb{R}^n)$,

$$(e^{t\Delta/2}F)(x) = (2\pi t)^{-n/2}\sqrt{\det Q}\int_{\mathbb{R}^n} e^{-(x-y)\cdot Q(x-y)/2t}F(y)\,dy$$

$$= \int_{C\times\mathbb{R}^{n-k}} p_t \circ \pi(x-y)\,F(y)\,dy$$

$$= \int_{C\times\mathbb{R}^{n-k}} \bar{p}_t(x-y)\,F(y)\,dy,$$

where

$$\bar{p}_t(x) \doteq p_t \circ \pi(x) = \sum_{m\in\mathbb{Z}^k\times\{0\}} (2\pi t)^{-n/2}\sqrt{\det Q}\,e^{-(x-m)\cdot Q(x-m)/2t}. \quad (2.5)$$

To understand $\bar{p}_t$ better, let us decompose $x \in \mathbb{R}^n$ into $x=(u, v) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. We also decompose Q into block matrix form as:

$$Q = \begin{bmatrix} A & B \\ B' & C \end{bmatrix}$$

where A and C are symmetric positive-definite $k\times k$ and $(n-k)\times(n-k)$ matrices respectively, and B is a $k\times(n-k)$ matrix.

PROPOSITION 2.10. *Keeping the notation of the above example, then*

$$\bar{p}_t(u, v) = P(u + A^{-1}Bv)\,E(v), \quad (2.6)$$

where

$$E(v) \doteq \exp[(-Cv\cdot v + A^{-1}Bv\cdot Bv)/2t], \quad (2.7)$$

and P is a positive smooth periodic function on $\mathbb{R}^k$. In particular there is a constant $0 < M < \infty$, such that

$$M^{-1}\bar{p}_t(u, v) \leqslant \bar{p}_t(u-\theta, v) \leqslant M\bar{p}_t(u, v) \quad (2.8)$$

for all $(u, v) \in \mathbb{R}^d$ and $\theta \in \mathbb{R}^k$. Consequently, $L_p^2(\bar{p}_t) = L_p^2(\hat{p}_t)$, where

$$\hat{p}_t(u, v) \doteq \int_C \bar{p}_t(u-\theta, v)\,d\theta.$$

$$= (2\pi t)^{-(n-k)/2}\sqrt{C - B'A^{-1}B}\cdot E(v). \quad (2.9)$$

Remark 2.11. Equation (2.8) is a special case of B. Hall's "averaging lemma" which appears in the next section.

Proof of Proposition 2.10. A completion of the squares argument yields:

$$\begin{aligned} x \cdot Qx &= Au \cdot u + 2u \cdot Bv + Cv \cdot v \\ &= (u + A^{-1}Bv) \cdot A(u + A^{-1}Bv) + Cv \cdot v - A^{-1}Bv \cdot Bv. \end{aligned} \tag{2.10}$$

For $u \in \mathbb{R}^k$ set

$$P(u) \doteq (2\pi t)^{-n/2} \sqrt{\det Q} \sum_{m \in \mathbb{Z}^k} e^{-A(u+m) \cdot (u+m)/2t}, \tag{2.11}$$

which is convergent since A is positive definite. Using the definition of $\bar{p}_t$ in (2.5) and the above two displayed equations it follows that (2.6) holds where $E(v)$ is given in (2.7). It is easy to see that P defined in (2.11) is smooth, positive, and $\mathbb{Z}^k$-periodic. Hence there is a constant $M_1 > 0$ such that $M_1^{-1} \leqslant P \leqslant M_1$. Thus

$$M_1^{-1} E(v) \leqslant \bar{p}_t(u, v) \leqslant M_1 E(v),$$

and hence (2.8) holds with $M = M_1^2$.

The explicit formula for $\hat{p}_t(u, v)$ is found as follows.

$$\begin{aligned} \hat{p}_t(u, v) &\doteq \int_C \bar{p}_t(u - \theta, v)\, d\theta\, x) \\ &= \int_C \sum_{m \in \mathbb{Z}^k} (2\pi t)^{-n/2} \sqrt{\det Q}\, e^{-(u-m-\theta, v) \cdot Q(u-m-\theta, v)/2t}\, d\theta \\ &= \int_{\mathbb{R}^k} (2\pi t)^{-n/2} \sqrt{\det Q}\, e^{-(u,v) \cdot Q(u,v)/2t}\, du \\ &= E(v) \int_{\mathbb{R}^k} (2\pi t)^{-n/2} \sqrt{\det Q}\, e^{-u \cdot Au/2t}\, du \\ &= (2\pi t)^{-(n-k)/2} \sqrt{\det Q / \det A} \cdot E(v). \end{aligned}$$

Since $\hat{p}_t$ is a probability measure, it follows that

$$\sqrt{\det Q / \det A} = \sqrt{\det(C - B' A^{-1} B)}.$$

This may also be seen directly using basic properties of the determinant and the identity:

$$Q = M' \begin{bmatrix} A & 0 \\ 0 & C - B'A^{-1}B \end{bmatrix} M,$$

where

$$M \doteq \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}. \qquad \text{Q.E.D.}$$

Remark 2.12. The Gaussian upper bound (Theorem 2.8) for the p_t in the abelian case of the above example and proposition follows from (2.6) and (2.7). Indeed, by (2.10) the square of the distance between $\pi(x)$ $(x=(u,v)\in \mathbb{R}^k\times\mathbb{R}^{n-k}=\mathbb{R}^n)$ and $\pi(\mathbb{R}^k\times\{0\})$ is given by

$$d(\pi(u,v),\pi(\mathbb{R}^k\times\{0\})=Cv\cdot v-A^{-1}vBv\cdot Bv.$$

Therefore, according to (2.6) and (2.7),

$$p_t\circ\pi(x)=P(u+A^{-1}Bv)\exp[-d^2(\pi(x),\pi(\mathbb{R}^k\times\{0\})/2t].$$

The Gaussian upper bound follows from this equation, because $d(\pi(x),\pi(\mathbb{R}^k\times\{0\})$ and $|\pi(x)|\doteq d(\pi(x),\pi(0))$ are comparable in size, since $\pi(\mathbb{R}^k\times\{0\})$ is compact.

We finish this section by recalling a specil case of Langland's theorem. This theorem asserts that the heat kernel p_t can be used to construct solutions for the "abstract heat equation" associated to any "reasonable" representation of H.

THEOREM 2.13 (Langland's Theorem). *Assume the same hypothesis of the previous theorem. Let (π, V_π) be a finite dimensional representation of H. Then for each $h\in H$, the* End(V_π)*-valued function*

$$F(t)\doteq\int_H p_t(x)\,\pi(hx)\,dx$$

satisfies the abstract heat equation:

$$\dot F(t)=F(t)\,\pi(\Delta)/2 \qquad \textit{with} \quad F(0)=\pi(h), \tag{2.12}$$

where

$$\pi(\Delta)\doteq\sum_{A\in\mathfrak{h}_0}\frac{d^2}{ds^2}\bigg|_0\pi(e^{sA}),$$

and $\mathfrak{h}_0\subset\mathfrak{h}$ is an orthonormal basis for $\mathfrak{h}$. Alternatively,

$$\int_H p_t(x)\,\pi(hx)\,dx=(e^{t\Delta/2}\pi)(h), \tag{2.13}$$

where

$$(e^{t\Delta/2}\pi)(h) \doteq \sum_{n=0}^{\infty} \frac{t^n}{2^n n!} (\Delta^n \pi)(h).$$

Proof. The assertion in (2.12) is a special case of Theorem 2.1 (p. 152) in [24]. See also Nelson [23] and Lemma 8 in [15]. Ignoring technical difficulties the proof of this theorem follows easily from (2.1) along with integration by parts.

To show that (2.13) is equivalent to the assertion in (2.12), one need only observe that $(\Delta^n \pi)(h) = \pi(h)[\pi(\Delta)]^n$. From this observation, it follows that

$$(e^{t\Delta/2}\pi)(h) = \pi(h)\, e^{t\pi(\Delta)/2},$$

which is precisely the unique solution to (2.12). Q.E.D.

2.3. *Heat Kernels on $G^{\mathbb{C}}$ and the Averaging Lemma*

Recall that any Lie group G of compact type may be viewed as $G = K \times \mathbb{R}^d$, where K is compact and d is a non-negative integer.

DEFINITION 2.14. 1. The heat kernel on $G = K \times \mathbb{R}^d$ is the unique function (p_t) on G such that

$$(e^{t\Delta/2} f)(e) = \int_G f(g)\, p_t(g)\, dg \tag{2.14}$$

for all $f \in L^2(G, \text{Haar})$.

2. The heat kernel on $G^{\mathbb{C}}$ is the unique function μ_t on $G^{\mathbb{C}}$ such that

$$(e^{t\Delta_{\mathbb{C}}/4} f)(e) = \int_{G^{\mathbb{C}}} f(x)\, \mu_t(x)\, dx \tag{2.15}$$

for all $f \in L^2(G^{\mathbb{C}}, \text{Haar})$. (*Note well* in (2.15) that $\Delta_{\mathbb{C}}/4$ appears *not* $\Delta_{\mathbb{C}}/2$.)

3. The K-averaged heat kernel on $G^{\mathbb{C}}$ is the function

$$\nu_t(x) \doteq \int_K \mu_t(xk^{-1})\, dk. \tag{2.16}$$

Remark 2.15. It is easy to check that $\Delta_{\mathbb{C}}$ commutes with the adjoint action of G, and hence so does $e^{t\Delta_{\mathbb{C}}/4}$. Consequently $\mu_t(gx) = \mu_t(xg)$ for all $g \in G$ and $x \in G^{\mathbb{C}}$. In particular, ν_t is also given by:

$$\nu_t(x) \doteq \int_K \mu_t(k^{-1}x)\, dk. \tag{2.17}$$

Also note that

$$\nu_t(xk) = \nu_t(kx) = \nu_t(x) \quad \forall x \in G^{\mathbb{C}}, \quad k \in K.$$

Hence the function ν_t is the heat kernel on $G^{\mathbb{C}}/K$ composed with the canonical projection from $G^{\mathbb{C}}$ to $G^{\mathbb{C}}/K$.

LEMMA 2.16 (Averaging Lemma). *For each $t > 0$ there is a constant $0 < C = C(t) < \infty$ such that for all $k \in K$, and $x \in G^{\mathbb{C}}$,*

$$C^{-1}\mu_t(kx) \leqslant \mu_t(x) \leqslant C\mu_t(kx).$$

In particular,

$$C^{-1}\mu_t \leqslant \nu_t \leqslant C\mu_t$$

which implies that $L^2(\mu_t) = L^2(\nu_t)$.

Proof. See Lemma 11 of B. Hall [15] where this is proved in the case that $G = K$. However, the proof is valid in this generality. It is only ncessary to read Hall's proof with G replaced by $G^{\mathbb{C}}$ everywhere. The proof for the special case that G is abelian was already given in Proposition 2.10. Q.E.D.

2.4. *The Frechét Tensor Albegra*

DEFINITION 2.17. Let $T_+ \equiv \bigcap_{t>0} T_t$, J_t be the closure of J in T_t, and $J_+ \equiv \bigcap_{t>0} J_t$. (We call T_+ the Frechét tensor algebra of $G^{\mathbb{C}}$.)

For $\xi \in T_t$ and $\alpha \in T_t^*$, we will write $\|\xi\|_t^2$ for $((\xi, \xi))_t$ (defined in (1.3)) and $\|\alpha\|_t^2$ for $(\alpha, \alpha)_t$ (defined in (1.4)). Hopefully no confusion will arise from this abuse of notation. Notice that for $0 < s \leqslant t$, $T_+ \subset T_s \subset T_t$. The following lemma shows that T_+ is an algebra and that J_+ is a two sided ideal in T_+.

LEMMA 2.18. *Suppose that $t, s > 0$, $A \in T_t$ and $B \in T_s$. Then for all $r > (t+s)$, $A \otimes B \equiv \sum_{n=0}^{\infty} (\sum_{k=0}^{n} A_k \otimes B_{n-k})$ is in T_r and*

$$\|A \otimes B\|_r \leqslant C((t+s)/r)\, \|A\|_t\, \|B\|_s,$$

where $C(\alpha) \equiv \sqrt{(1+\alpha)(1-\alpha)^{-3}}$.

Proof. Write $A=\sum_{k=0}^{\infty} A_k$ and $B=\sum_{k=0}^{\infty} B_k$, where $A_k, B_k \in (\mathfrak{g}^{\mathbb{C}})^{\otimes n}$. Then

$$\begin{aligned}
\|A\otimes B\|_r^2 &= \sum_{n=0}^{\infty} \frac{n!}{r^n} \left| \sum_{k=0}^{n} A_k \otimes B_{n-k} \right|_{\mathfrak{g}^{\otimes n}}^2 \\
&\sum_{n=0}^{\infty} \frac{n!}{r^n} \left\{ \sum_{k=0}^{n} |A_k|\,|B_{n-k}| \right\}^2 \\
&\leqslant \sum_{n=0}^{\infty} \frac{n!}{r^n} (n+1)^2 \left\{ \sum_{k=0}^{n} |A_k|^2\,|B_{n-k}|^2 \right\} \qquad \text{(Cauchy Schwartz)} \\
&= \sum_{n=0}^{\infty} \frac{n!\cdot(n+1)^2}{r^n} \sum_{k=0}^{n} \left\{ \frac{t^k}{k!} \|A_k\|_t^2 \frac{s^{(n-k)}}{(n-k)!} \|B_{n-k}\|_s^2 \right\} \\
&\leqslant \|A\|_t^2\,\|B\|_s^2 \sum_{n=0}^{\infty} \frac{(n+1)^2}{r^n} \sum_{k=0}^{n} \binom{n}{k} t^k s^{(n-k)} \\
&= \|A\|_t^2\,\|B\|_s^2 \sum_{n=0}^{\infty} (n+1)^2 \frac{(t+s)^n}{r^n} \\
&= C((t+s)/r)^2\, \|A\|_t^2\,\|B\|_s^2,
\end{aligned}$$

where

$$\begin{aligned}
C(\alpha)^2 &\equiv \sum_{n=0}^{\infty} (n+1)^2 \alpha^n = \alpha^{-1} \sum_{n=0}^{\infty} n^2 \alpha^n \\
&= \alpha^{-1}\cdot\left(\alpha \frac{d}{d\alpha}\right)^2 (1-\alpha)^{-1} = (1+\alpha)(1-\alpha)^{-3}. \qquad \text{Q.E.D.}
\end{aligned}$$

In the sequel we will be most concerned with the following subspaces of the algebraic dual T'.

DEFINITION 2.19. 1. $T_+^* = \bigcup_{t>0} T_t^*$, where T_t^* is the topological dual of T_t.

2. ${}^{\perp}J \equiv \{\alpha \in T' : \alpha(J) = \{0\}\}$.

3. ${}^{\perp}J_t \equiv T_t^* \cap ({}^{\perp}J) = \{\alpha \in T_t^* : \alpha(J) = \{0\}\} = \{\alpha \in T_t^* : \alpha(J_t) = \{0\}\}$, where $t > 0$.

4. ${}^{\perp}J_+ \equiv T_+^* \cap ({}^{\perp}J) = \{\alpha \in T_+^* : \alpha(J) = \{0\}\} = \{\alpha \in T_+^* : \alpha(J_+) = \{0\}\}$.

3. DENSITY THEOREMS

Given a Lie group G of compact type, let $\mathscr{F}(G)$ ($\mathscr{F}(G^{\mathbb{C}})$) denote the linear span of all functions of the form

$$f(g) = \mathrm{Tr}(A\pi(g)), \tag{3.1}$$

where (π, V_π) is a finite dimensional (finite dimensional holomorphic) representation of G ($G^{\mathbb{C}}$) such that π ($\pi|_G$) is unitary and $A \in \mathrm{End}(V_\pi)$. By definition of the complexification of G, each finite dimensional representation (π, V_π) of G has a unique extension to a holomorphic representation of $G^{\mathbb{C}}$. We will continue to denote this representation by (π, V_π). Therefore, a holomorphic function f on $G^{\mathbb{C}}$ is in $\mathscr{F}(G^{\mathbb{C}})$ iff $f|_G$ is in $\mathscr{F}(G)$.

3.1. $\mathscr{F}(G)$ *Is Dense in* $L^2(p_t)$

LEMMA 3.1. *For each* $t>0$, $\mathscr{F}(G)$ *is dense in*

$$L^2(p_t) \doteq \left\{ f : G \to \mathbb{C} \,\middle|\, \int_G |f(g)|^2 \, p_t(g) \, dg < \infty \right\}.$$

Proof. By the structure theorem, we may take G to be of the form $K \times \mathbb{R}^d$, where K is a compact Lie group. Given $h \in L^2(p_t)$, it suffices to show $h \equiv 0$ if

$$\int_G h(g) \bar{f}(g) \, p_t(g) \, dg = 0, \qquad \forall f \in \mathscr{F}(G). \tag{3.2}$$

Suppose that (π, V_π) is a finite dimensional representation of K, and $\lambda \in \mathbb{R}^d$. Setting $g = (k, x) \in K \times \mathbb{R}^d$, then $f(g) \doteq e^{i\lambda \cdot x} \operatorname{Tr}(A\pi(k))$ is in $\mathscr{F}(G)$. Thus by Fubini's theorem and standard facts about the Fourier transform, (3.2) implies that

$$\int_K h(k, x) (\overline{\operatorname{Tr}(A\pi(k))}) \, p_t(k, x) \, dk = 0 \qquad \text{for a.e.} \quad x \in \mathbb{R}^d. \tag{3.3}$$

The Peter–Weyl theorem guarantees that the matrix elements of irreducible representations on K are dense in $L^2(dk)$. Since the number of irreducible representations are countable it follows from (3.3) and Fubini's theorem that $h(k, x) = 0$ for a.e. (k, x). Q.E.D.

3.2. $\mathscr{F}(G^{\mathbb{C}})$ *Is Dense in* $\mathscr{H}(G^{\mathbb{C}}) \cap L^2(\mu_t)$

Let us start by recalling the fact that $\mathscr{H}(G^{\mathbb{C}}) \cap L^2(\mu_t)$ is a closed subspace of $L^2(\mu_t)$. The proof is standard but for the readers convenience I will sketch a proof in the next lemma.

LEMMA 3.2. *Let* M *be a complex analytic manifold and* ρ *be a smooth positive measure on* M. *Let* $\mathscr{H}(M)$ *denote the holomorphic functions on* M. *Then* $\mathscr{H}(M) \cap L^2(\rho)$ *is a closed subspace of* $L^2(\rho)$. *Moreover, if* $f_n \to f$ *in* $L^2(\rho)$ *as* $n \to \infty$, *then* f_n *and* df_n *converges to* f *and* df *respectively uniformly on compact subsets of* M.

Proof. Since the property that a function on M is holomorphic is local, it suffices to prove the lemma in the case that $M \doteq D_1$ and $\rho = 1$, where for any $R > 0$,

$$D_R \doteq \{z \in \mathbb{C}^d \mid |z_i| < R \; \forall i = 1, 2, ..., d\}. \tag{3.4}$$

Let f be a holomorphic function on D_1, $0 < \alpha < 1$, and $z \in D_\alpha$. By the mean value theorem for holomorphic functions;

$$f(z) = (2\pi)^{-d} \int_{[0, 2\pi]^d} f_n(\{z_j + r_j e^{\sqrt{-1}\theta_j}\}_{j=1}^d) \prod_{j=1}^d d\theta_j, \tag{3.5}$$

where $r = (r_1, r_2, ..., r_d) \in \mathbb{R}^d$ such that $0 \leqslant r_i < \varepsilon \doteq 1 - \alpha$ for all $i = 1, 2, ..., d$. Multiplying (3.5) by $r_1 \cdots r_d$ and integrating each r_i over $[0, \varepsilon)$ shows

$$f(z) = (\pi\varepsilon^2)^{-d} \int_{D_\varepsilon} f(z + \xi) \, \lambda(d\xi),$$

where λ denotes Lebesgue measure on $\mathbb{C}^d$. In particular, for each $\alpha < 1$,

$$\sup_{z \in D_\alpha} |f(z)| \leqslant (\pi(1-\alpha)^2)^{-d} \, \|f\|_{L^2}.$$

Therefore, an L^2-convergent sequence of holomorphic functions is uniformly convergent on compact subsets of D_1 and so the limit is also holomorphic. Since the derivatives of uniformly convergent holomorphic functions are uniformly convergent, it follows that L^2 convergence also implies uniform convergence of the differentials on compact sets. Q.E.D.

The remainder of this section is devoted to the proof of the following key density theorem. The case where G is compact has already been proven by B. Hall in [15], see Lemma 10 and 11. We will give a slight variant Hall's proof.

THEOREM 3.3. *Let G be a Lie group of compact type equipped with an Ad_G invariant inner product $(\cdot, \cdot)$ on $\mathfrak{g}$. Then $\mathscr{F}^{\mathbb{C}} \doteq \mathscr{F}(G^{\mathbb{C}})$ is a dense subspace of $L^2(\mu_t) \cap \mathscr{H}(G^{\mathbb{C}})$.*

The following lemma is well known from the "classical" Segal–Bargmann theory, see for example [1] Theorem 1.13 and [2]. The short proof in this finite dimansional context is given for the readers convenience.

LEMMA 3.4. *Let $G = \mathbb{R}^d$ with the standard inner product $(z, w) \doteq z \cdot w$. Suppose that $f \in \mathscr{H}(G^{\mathbb{C}} = \mathbb{C}^d)$, then for all $\lambda \in \mathbb{C}^d$*

$$\lim_{R \to \infty} \int_{D_R} f(z) \, e^{\lambda \cdot \bar{z}} \mu_t(dz) = f(t\lambda), \tag{3.6}$$

where D_R is defined in (3.4). *In particular, if $f(z)\,e^{\lambda\cdot\bar{z}}$ is in $L^1(\mu_t)$, then*

$$\int_{\mathbb{C}^d} f(z)\,e^{\lambda\cdot\bar{z}}\mu_t(dz)=f(t\lambda).$$

Proof. As in Example 1.3,

$$\mu_t(z)=(\pi t)^{-d}\exp\{-|z|^2/t\}.$$

Write $f(z)=\sum_{\alpha\in\mathbb{N}^d} f^{(\alpha)}(0)\,z^\alpha/\alpha!$, where $z^\alpha\equiv\prod_{i=1}^d z_i^{\alpha_i}$, $f^{(\alpha)}(0)=(\partial^\alpha f)(0)$, $\partial^\alpha\equiv\prod_{i=1}^d(\partial/\partial z^i)^{\alpha_i}$, and $\alpha!\equiv\prod_{i=1}^d\alpha_i!$. Then

$$\int_{D_R} f(z)\,e^{\lambda\cdot\bar{z}}\mu_t(dz)=\sum_{\alpha\in\mathbb{N}}(\alpha!)^{-1}f^{(\alpha)}(0)\int_{D_R} z^\alpha\,e^{\lambda\cdot\bar{z}}\mu_t(dz).$$

Similarly

$$\int_{D_R} z^\alpha e^{\lambda\cdot\bar{z}}\mu_t(dz)=\sum_{\beta\in\mathbb{N}^d}(\beta!)^{-1}\,\lambda^\beta\int_{D_R} z^\alpha\bar{z}^\beta\mu_t(dz).$$

Because $\mu_t(dz)$ is invariant under the transformation $z\to(e^{i\theta_1}z_1,\ldots,e^{i\theta_d}z_d)$, it is easy to conclude that $\int_{D_R} z^\alpha\bar{z}^\beta\mu_t(dz)=0$ if $\alpha\neq\beta$. Therefore,

$$\int_{D_R} f(z)\,e^{\lambda\cdot\bar{z}}\mu_t(dz)=\sum_{\alpha\in\mathbb{N}^d}(\alpha!)^{-2}f^{(\alpha)}(0)\,\lambda^\alpha c(\alpha,R),$$

where

$$c(\alpha,R)\equiv\int_{D_R}|z^\alpha|^2\,\mu_t(dz).$$

Notice that $c(\alpha, R)$ increases as R increases and $\lim_{R\to\infty}c(\alpha,R)=t^{|\alpha|}\alpha!$ by standard Gaussian integral formulas. Therefore, by the dominated convergence theorem (applied to the sum);

$$\lim_{R\to\infty}\int_{D_R} f(z)\,e^{\lambda\cdot\bar{z}}\mu_t(dz)=\sum_{\alpha\in\mathbb{N}^d}(\alpha!)^{-2}f^{(\alpha)}(0)\,\lambda^\alpha t^{|\alpha|}\alpha!=f(t\lambda).\qquad\text{Q.E.D.}$$

COROLLARY 3.5. *Let $G=\mathbb{R}^d$, Q be a symmetric positive definite matrix on $\mathbb{R}^d$, and $(z,w)_Q\doteq z\cdot Qw$. Suppose that $f\in\mathscr{H}(\mathbb{C}^d)\cap L^2(\mu_t)$, where μ_t is the heat kernel measure on $\mathbb{C}^d$ associated to $(\cdot,\cdot)_Q$. Then for all $\lambda\in\mathbb{C}^d$*

$$\int_{\mathbb{C}^d} f(z)\,e^{\lambda\cdot\bar{z}}\mu_t(dz)=f(tQ^{-1}\lambda). \tag{3.7}$$

Proof. The heat kernel μ_t is now given by:

$$\mu_t(z) = (\pi t)^{-d} \det Q \cdot \exp\{-z \cdot Q\bar{z}/t\}. \tag{3.8}$$

Suppose that $f \in L^2(\mu_t) \cap \mathscr{H}(G^{\mathbb{C}})$. Using the holomorphic change of variable $w = \sqrt{Q}z$ and (3.6) one has

$$\begin{aligned}\int_{G^{\mathbb{C}}} f(z)\, e^{\lambda \cdot \bar{z}}\, d\mu_t(z) &= \int_{G^{\mathbb{C}}} f(Q^{-1/2}w)\, e^{Q^{-1/2}\lambda \cdot \bar{w}} (\pi t)^{-d} \exp\{-|w|^2/t\}\, \lambda(dw)\\ &= f(tQ^{-1/2}Q^{-1/2}\lambda) = f(tQ^{-1}\lambda).\end{aligned}$$

Q.E.D.

LEMMA 3.6 ($G = K \times Z$). *Suppose that $G = K \times Z$, where K is a connected compact Lie group, $Z = \mathbb{R}^d/(\mathbb{Z}^k \times \{0\})$, and the inner product $(\cdot, \cdot)$ on $\mathfrak{g}$ is such that $\mathfrak{k} \doteq \mathrm{Lie}(K)$ is perpendicular to $\mathbb{R}^d = \mathrm{Lie}(Z)$. Then $\mathscr{F}^{\mathbb{C}} \doteq \mathscr{F}(G^{\mathbb{C}})$ is dense in $L^2(\mu_t) \cap \mathscr{H}(G^{\mathbb{C}})$.*

Proof. Let

$$p : \mathbb{C}^d \to \mathbb{C}^d/(\mathbb{Z}^k \times \{0\})$$

denote the canonical projection,

$$C \doteq [0, 1]^k \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{d-k} \subset \mathbb{C}^d$$

be the "unit cell," and Q be the real positive definite symmetric $d \times d$ matrix such that $((0, a), (0, b)) = a \cdot Qb$ for all $a, b \in \mathbb{R}^d$. We will identify a function $f : G^{\mathbb{C}} \to \mathbb{C}$ with the function $F : K \times \mathbb{C}^d \to \mathbb{C}$ determined by $F(k, z) \doteq f(k, p(z))$. Let μ_t be the heat kernel density on $G^{\mathbb{C}}$, $\mu_t^Z(z)$ denote the Gaussian density in Eq. (3.8), and μ_t^K denote the heat kernel density on K associated to $(\cdot, \cdot)|_{\mathfrak{k} \times \{0\}}$. With this notation

$$\int_{G^{\mathbb{C}}} f(g)\, \mu_t(g)\, dg = \int_{K^{\mathbb{C}} \times \mathbb{C}^d} F(k, z)\, \mu_t^K(k)\, \mu_t^Z(z)\, dk\, dz,$$

and

$$L^2(\mu_t) \cong L^2_p(\mu_t^K \otimes \mu_t^Z),$$

where

$$L^2_p(\mu_t^K \otimes \mu_t^Z) \doteq \{F \in L^2(\mu_t^K \otimes \mu_t^Z) \mid F(\cdot, \cdot + n) = F(\cdot, \cdot) \text{ a.e., } \forall n \in \mathbb{Z}^k \times \{0\}\}.$$

Because of the Averaging Lemma 2.16 (also see Proposition 2.10), $L^2(\mu_t) = L^2(\nu_t)$, where ν_t is defined in Definition 2.14. Under the above identifications this is equivalent to the assertion:

$$L^2_p(\mu_t^K \otimes \mu_t^Z) = L^2_p(\nu_t^K \otimes \nu_t^Z),$$

where

$$\nu_t^K(x) \doteq \int_K \mu_t^K(k^{-1}x)\, dk \tag{3.9}$$

and

$$\nu_t^Z(z) = \int_{C \times \{0\}} \mu_t^Z(z - \theta)\, d\theta. \tag{3.10}$$

Now assume that $f \in L^2(\nu_t)$ and f is $L^2(\nu_t)$-perpendicular to $\mathscr{F}^{\mathbb{C}}$. The proof will be completed by showing that $f \equiv 0$. The assumption $f \perp \mathscr{F}^{\mathbb{C}}$ implies

$$0 = \int_{K^{\mathbb{C}} \times \mathbb{C}^d} F(x, z) \exp(i(2\pi n) + \lambda) \cdot z)\, \overline{\operatorname{Tr}(\pi(x) A)}\, \nu_t^K(x)\, \nu_t^Z(z)\, dx\, dz, \tag{3.11}$$

where $n \in \mathbb{Z}^k \times \{0\} \subseteq \mathbb{R}^d$, $\lambda \in \{0\} \times \mathbb{R}^d$, (π, V_π) is a finite dimensional holomorphic representation of $K^{\mathbb{C}}$, and $A \in \operatorname{End}(V_\pi)$. Using Fubini's theorem, the definition ν_t^Z, the change of variables $z \to z + \theta$, and Corollary 3.5 one easily shows that (3.11) is equivalent to:

$$\begin{aligned} 0 &= \int_{K^{\mathbb{C}} \times C} F(x, tQ^{-1}(i2\pi n + i\lambda) + \theta) \exp(i(2\pi n + \lambda) \cdot \theta)\, \overline{\operatorname{Tr}(\pi(x) A)}\, \nu_t^K(x)\, dx\, d\theta \\ &= \int_{K^{\mathbb{C}} \times C} F(x, tQ^{-1}(i2\pi n + i\lambda) + \theta)\, e^{i2\pi n \cdot \theta}\, \overline{\operatorname{Tr}(\pi(x) A)}\, \nu_t^K(x)\, dx\, d\theta, \end{aligned} \tag{3.12}$$

where in the second equality we have used the fact that $\theta \cdot \lambda = 0$.

To simplify notation, let $w \doteq itQ^{-1}(2\pi n + \lambda)$ and decompose w into $w = w_1 + w_2$, where $w_1 \in \mathbb{C}^k \times \{0\}$ and $w_2 \in \{0\} \times \mathbb{C}^{d-k}$. I assert that

$$\int_C F(x, w_1 + w_2 + \theta)\, e^{i2\pi n \cdot \theta}\, d\theta = \int_C F(x, w_2 + \theta)\, e^{i2\pi n \cdot (\theta - w_1)}\, d\theta. \tag{3.13}$$

To see this it suffices to notice that both sides of (3.13) are holomorphic functions of w_1 and that (3.13) holds for $w_1 \in \mathbb{R}^k \times \{0\}$ because of the

$\mathbb{Z}^k \times \{0\}$-periodicity in the functions $\theta \to F(x, w+\theta)$ and $\theta \to e^{i2\pi n \cdot \theta}$. Combining (3.12) and (3.13) gives

$$0 = \int_{K^{\mathbb{C}} \times C} F(x, w_2 + \theta)\, e^{i2\pi n \cdot \theta}\, \overline{\mathrm{Tr}(\pi(x) A)}\, v_t^K(x)\, dx\, d\theta. \qquad (3.14)$$

Since K is compact, we may (and do) choose an inner product on V_π for which $\pi|_K$ is a unitary representation of K. Using Fubini's theorem, the definition of v_t^K and the invariance of Haar measure on $K^{\mathbb{C}}$, it easily follows from (3.14) that

$$\begin{aligned} 0 &= \int_{K \times K^{\mathbb{C}} \times C} F(kx, w_2 + \theta)\, e^{i2\pi n \cdot \theta}\, \overline{\mathrm{Tr}(\pi(kx) A)}\, \mu_t^K(x)\, dk\, dx\, d\theta \\ &= \int_{K \times K^{\mathbb{C}} \times C} F(kx, w_2 + \theta)\, e^{i2\pi n \cdot \theta}\, \mathrm{Tr}(A^* \pi(x)^* \pi(k^{-1}))\, \mu_t^K(x)\, dk\, dx\, d\theta. \end{aligned}$$

Using an argument similar to the proof of Eq. (3.13), we may make the "change of variables" $k \to kx^{-1}$ in the above equation to conclude that:

$$\begin{aligned} 0 &= \int_{\times K \times K^{\mathbb{C}} \times C} F(k, w_2 + \theta) \\ &\quad \times e^{i2\pi n \cdot \theta}\, \mathrm{Tr}(A^* \pi(x)^* \pi(x)\, \pi(k^{-1}))\, \mu_t^K(x)\, dk\, dx\, d\theta \\ &= \int_{K \times C} F(k, w_2 + \theta)\, e^{i2\pi n \cdot \theta}\, \mathrm{Tr}(A^* Q_\pi \pi(k^{-1}))\, dk\, d\theta, \end{aligned} \qquad (3.15)$$

where Q_π is the positive definite element of $\mathrm{End}(V_\pi)$ defined by

$$Q_\pi \equiv \int_{K^{\mathbb{C}}} \pi(x)^* \pi(x)\, \mu_t(x)\, dx. \qquad (3.16)$$

Recall that w_2 is the $\{0\} \times \mathbb{C}^{d-k}$ component of $itQ^{-1}(2\pi n + \lambda)$. Because Q^{-1} is positive definite, one easily shows that w_2 ranges over $i(\{0\} \times \mathbb{R}^{d-k})$ as λ varies over $\{0\} \times \mathbb{R}^{d-k}$. This observation and the fact that the right member of (3.15) is holomorphic in w_2 implies that (3.15) holds for all $w_2 \in \{0\} \times \mathbb{C}^{d-k}$. Since Q_π is positive definite, $A^* Q_\pi$ ranges over $\mathrm{End}(V_\pi)$ as A varies over $\mathrm{End}(V_\pi)$. Since π and n are arbitrary, (3.15) and the Peter–Weyl (Fourier's) Theorem implies that $F(k, \theta + w_2) \equiv 0$ for all $k \in K$, $\theta \in \mathbb{R}^k \times \{0\}$, and $w_2 \in \{0\} \times \mathbb{C}^{d-k}$. Because F is holomorphic, it follows that $F \equiv 0$ and hence $f \equiv 0$. Q.E.D.

Proof of Theorem 3.3. By the structure Theorem 2.1, $G = KZ$, where K is the connected (compact) Lie subgroup of G with $\mathrm{Lie}(K) = \mathfrak{k} \doteq [\mathfrak{g}, \mathfrak{g}]$, Z is the connected (abelian) Lie subgroup of G which $\mathrm{Lie}(Z) = \mathfrak{z} = \mathfrak{z}(\mathfrak{g})$ (the center of $\mathfrak{g}$), $K \cap Z$ is finite, and $\mathfrak{g}$ is the orthogonal direct sum of $\mathfrak{k}$ and $\mathfrak{z}$. Let μ_t, μ_t^K and μ_t^Z denote the heat kernels on the complexified Lie groups $G^{\mathbb{C}}$, $K^{\mathbb{C}}$ and $Z^{\mathbb{C}}$ respectively. It is not difficult to show: for all non-negative measurable functions f on $G^{\mathbb{C}}$,

$$\int_{G^{\mathbb{C}}} f(g)\,\mu_t(g)\,dg = \int_{K^{\mathbb{C}} \times Z^{\mathbb{C}}} f(kz)\,\mu_t^K(k)\,\mu_t^Z(z)\,dk\,dz$$

$$= \int_{K^{\mathbb{C}} \times Z^{\mathbb{C}}} f(kz)\,\tilde{\mu}_t(k, z)\,dk\,dz,$$

where, with $N = \#(K \cap Z)$,

$$\tilde{\mu}_t(k, z) \doteq \frac{1}{N} \sum_{\xi \in K \cap Z} \mu_t^K(\xi k)\,\mu_t^Z(\xi^{-1} z).$$

By the averaging Lemma 2.16, it follows that $L^2(\tilde{\mu}_t) = L^2(\mu_t^K \otimes \mu_t^Z)$.

Suppose that $f \in L^2(\mu_t) \cap \mathscr{H}(G^{\mathbb{C}})$ such that $f \perp \mathscr{F}(G^{\mathbb{C}})$. For each $h \in \mathscr{F}(K^{\mathbb{C}} \times Z^{\mathbb{C}}) = \mathscr{F}(K^{\mathbb{C}}) \otimes \mathscr{F}(Z^{\mathbb{C}})$ it is readily checked that

$$\tilde{h}(kz) \doteq \frac{1}{N} \sum_{\xi \in K \cap Z} h(\xi k, \xi^{-1} z)$$

is well defined.

Claim. $\tilde{h} \in \mathscr{F}(G^{\mathbb{C}})$.

In order to prove the claim, we may assume that

$$h|_{K \times Z}(k, z) \doteq \mathrm{Tr}(A\pi(k) \otimes \chi(z))$$

where (π, V_π) and (χ, V_χ) are unitary representations of K and Z respectively and $A \in \mathrm{End}(V_\pi \otimes V_\chi)$. As usual the holomorphic extensions of π and χ to $K^{\mathbb{C}}$ and $Z^{\mathbb{C}}$ respectively are still denoted by π and χ. Let

$$P \doteq \frac{1}{N} \sum_{\xi \in K \cap Z} \pi(\xi) \otimes \chi(\xi^{-1})$$

The reader may check that P is an orthogonal projection which commutes with $\pi \otimes \chi$, and therefore $\rho(kz) \doteq \pi(k) \otimes \chi(z)|_{\mathrm{ran}(P)}$ is a unitary representation of G. The claim is proved upon noting that

$$\tilde{h}(g) \doteq \mathrm{Tr}_{\mathrm{ran}(P)}(PAP\rho(g)).$$

Since $\tilde{h} \in \mathscr{F}(G^{\mathbb{C}})$,

$$\begin{aligned} 0 &= \int_{G^{\mathbb{C}}} \tilde{h}(g)\, \bar{f}(g)\, \mu_t(g)\, dg \\ &= \int_{K^{\mathbb{C}} \times Z^{\mathbb{C}}} \left(\frac{1}{N} \sum_{\xi \in K \cap Z} h(\xi k, \xi^{-1} z) \right) \bar{f}(kz)\, \mu_t^K(k)\, \mu_t^Z(z)\, dk\, dz \\ &= \int_{K^{\mathbb{C}} \times Z^{\mathbb{C}}} h(k, z)\, \bar{f}(kz)\, \tilde{\mu}_t(k, z)\, dk\, dz. \end{aligned}$$

But by the above Lemma $\mathscr{F}(K^{\mathbb{C}} \times Z^{\mathbb{C}}) = \mathscr{F}(K^{\mathbb{C}}) \otimes \mathscr{F}(Z^{\mathbb{C}})$ is dense in $L^2(\tilde{\mu}_t) \cap \mathscr{H}(K \times Z) = L^2(\mu_t^K \otimes \mu_t^Z) \cap \mathscr{H}(K \times Z)$. Therefore $F(k, z) = f(kz) \equiv 0$. Q.E.D.

4. Three Isometries

4.1. *Hijab's Formula for Gross' Isometry*

THEOREM 4.1. *Let G be a Lie group of compact type with Ad_G invariant inner product $(\cdot, \cdot)$ on $\mathfrak{g}$. As above define the heat kernel p_t on G by*

$$(e^{t\Delta/2} f)(e) = \int_G f(g)\, p_t(g)\, dg$$

Then the map

$$(1 - D)_e^{-1}\, e^{t\Delta/2} : L^2(G, dp_t) \to {}^{\perp}J_t \tag{4.1}$$

is an isometry.

Proof. For $f \in \mathscr{F} = \mathscr{F}(G)$ and $z \in \mathbb{C}$, define $e^{z\Delta} f$ by

$$e^{z\Delta} f = \sum z^n \Delta^n f / n!.$$

The above series is convergent since f is the finite linear combination of matrix elements of finite dimensional representations of G. It is an easy consequence of Langland's Theorem (Theorem 2.13), that for all $t > 0$ and $f \in \mathscr{F}(G)$:

$$(e^{t\Delta/2} f)(g) = \int_G p_t(x)\, f(gx)\, dx. \tag{4.2}$$

Now fix $f \in \mathscr{F}(G)$ and $\tau > 0$, and let $F(t, \cdot) : G \to \mathbb{R}$ be defined by

$$F(t, \cdot) \equiv e^{t\Delta/2}[\,|e^{(\tau - t)\Delta/2} f|^2]. \tag{4.3}$$

Claim. The function $t \to F(t, e)$ is analytic in t.

Let $\mathscr{F}_0$ denote the linear span of all functions of the form $\{\Delta^k f \cdot \Delta^n \bar{f}\}_{n,k=0}^{\infty}$. By basic representation theory, $\dim \mathscr{F}_0 < \infty$. It is now easy to check that

$$|e^{(\tau - t)\Delta/2} f|^2 = e^{(\tau - t)\Delta/2} f \cdot e^{(\tau - t)\Delta/2} \bar{f}$$

is an $\mathscr{F}_0$-vector valued analytic function of t, say

$$|e^{(\tau - t)\Delta/2} f|^2 = \sum_{k=0}^{\infty} t^k c_k(\cdot),$$

where $c_k \in \mathscr{F}_0$. Hence

$$F(t, \cdot) = \sum_{k=0}^{\infty} t^k e^{t\Delta/2} c_k(\cdot) = \sum_{n=0}^{\infty} t^n \sum_{l=0}^{n} (2^l l!)^{-1} \Delta^l c_{n-l},$$

where the above sum converges in $\mathscr{F}_0$, i.e., $F(t, \cdot)$ is an $\mathscr{F}_0$-valued analytic function. The claim is proved because evaluation at $e \in G$ is a linear functional on $\mathscr{F}_0$ and hence $t \to F(t, e)$ is also analytic in t.

Because $F(t, e)$ is analytic,

$$\int_G f^2(g)\, p_\tau(g)\, dg = F(\tau, e) = \sum_{n=0}^{\infty} F^{(n)}(0, e)\, \tau^n/n!, \tag{4.4}$$

where $F^{(n)}$ denotes nth t-derivative of F.

To finish the proof, it is only necessary to compute all the derivatives of F relative to t. To simplify notation set $U(t, g) \equiv e^{(\tau - t)\Delta/2} f$, so that $F(t, \cdot) = e^{t\Delta/2}[\,|U(t, \cdot)|^2](g)$ and $\dot{U} = -\frac{1}{2}\Delta U$. Then it follows by the chain rule and the product rule that

$$\begin{aligned} dF/dt &= e^{t\Delta/2}\left[\frac{\Delta}{2}(U\bar{U}) - U\frac{\Delta}{2}\bar{U} - \bar{U}\frac{\Delta}{2}U\right] \\ &= \sum_{\xi \in \mathfrak{g}_0} e^{t\Delta/2}[\,|\tilde{\xi}U|^2] \\ &= e^{t\Delta/2}[\,|DU|^2], \end{aligned} \tag{4.5}$$

where $|DU|^2 = \sum_{\xi \in \mathfrak{g}_0} |\tilde{\xi} U|^2$, $\mathfrak{g}_0 \subset \mathfrak{g}$ is an orthonormal basis for $\mathfrak{g}$, and $\tilde{\xi}$ is the left invariant vector-field associated to $\xi \in \mathfrak{g}_0$. Using $[\tilde{\xi}, \Delta] = 0$, it is easy to continue computing derivatives of F in this fashion to learn:

$$F^{(n)}(t, \cdot) = e^{t\Delta/2}[|D^n U|^2] \doteq e^{t\Delta/2} \sum_{\xi_1, ..., \xi_n \in \mathfrak{g}_0} |\tilde{\xi}_1 \cdots \tilde{\xi}_n U|^2. \tag{4.6}$$

Combining (4.6) with (4.4) and the fact that $u(g) \doteq U(0, g) = (e^{\tau\Delta/2} f)(g)$ shows that

$$\begin{aligned}\int_G f^2(g)\, p_\tau(g)\, dg &= \sum_{n=0}^{\infty} \frac{\tau^n}{n!} |D^n u(e)|_{\mathfrak{g}^{*\otimes n}} \\ &= \sum_{n=0}^{\infty} \frac{\tau^n}{n!} |(D^n e^{\tau\Delta/2} f)(e)|_{\mathfrak{g}^{*\otimes n}} \\ &= \|((1-D)^{-1} e^{\tau\Delta/2} f)(e)\|_\tau \end{aligned} \tag{4.7}$$

which is the desired isometry property. Since $\mathscr{F}(G)$ is dense in $L^2(p_t)$, the map in (4.1) extends uniquely to an isometry of all of $L^2(p_t)$. Q.E.D.

4.2. *The Isometry* $(1-D)_e^{-1} : L^2(G^{\mathbb{C}}, \mu_t) \cap \mathscr{H}(G^{\mathbb{C}})$ *into* ${}^{\perp}J_t$

Let $\{X_i\}_{i=1}^{\dim \mathfrak{g}}$ be an orthonormal basis for $\mathfrak{g}$ and for each i set $Y_i \doteq \mathscr{J} X_i$. ($\mathscr{J}$ denotes multiplication by $\sqrt{-1}$ in $\mathfrak{g}^{\mathbb{C}}$.) As usual let $\tilde{X}_i$ and $\tilde{Y}_i$ denote the left invariant vector fields on $G^{\mathbb{C}}$ which agree with X_i and Y_i at $e \in G^{\mathbb{C}}$ respectively. For the next theorem, it will be convenient to introduce the complex left invariant vector fields Z_i and $\bar{Z}_i$ on $G^{\mathbb{C}}$ by:

$$Z_i = (\tilde{X}_i - \sqrt{-1}\, \tilde{Y}_i)/2$$

and

$$\bar{Z}_i = (\tilde{X}_i + \sqrt{-1}\, \tilde{Y}_i)/2.$$

respectively.

EXAMPLE 4.2. Suppose that $G = \mathbb{R}^d$ and $X_i = e_i$, where $\{e_i\}_{i=1}^d$ is the standard basis for $\mathbb{R}^d$. Then $\tilde{X}_i = \partial/\partial x^i$, $\tilde{Y}_i = \partial/\partial y^i$,

$$Z_i = \partial/\partial z^i \doteq (\partial/\partial x^i - \sqrt{-1}\, \partial/\partial y^i)/2,$$

and

$$\bar{Z}_i = \partial/\partial \bar{z}^i \doteq (\partial/\partial x^i + \sqrt{-1}\, \partial/\partial y^i)/2,$$

where $\{z^i = x^i + \sqrt{-1}\, y^i\}_{i=1}^d$ are the standard holomorphic coordinates on $\mathbb{C}^d$.

Remark 4.3. If f is holomorphic on $G^{\mathbb{C}}$, then $\tilde{Y}_i f = \sqrt{-1}\,\tilde{X}_i f$ and $\tilde{Y}_i \bar{f} = -\sqrt{-1}\,\tilde{X}_i \bar{f}$ and hence $Z_i \bar{f} = \bar{Z}_i f = 0$. This is easily seen using: (i) exponential coordinates are holomorphic coordinates and (ii) left translations on $G^{\mathbb{C}}$ are holomorphic maps. See Chapter 2 of [21] for more details.

THEOREM 4.4. *The map* $(1-D)_e^{-1} : L^2(\mu_t) \cap \mathscr{H}(G^{\mathbb{C}}) \to {}^{\perp}J_t$ *is a well defined isometry.*

Proof. The key observation is that $\Delta_{\mathbb{C}}$ may be written as:

$$\Delta_{\mathbb{C}} = 4 \sum_i Z_i \bar{Z}_i =: 4Z \cdot \bar{Z}. \tag{4.8}$$

To see this, write out $\sum_i Z_i \bar{Z}_i$ as

$$\sum_i Z_i \bar{Z}_i = \tfrac{1}{4} \sum_i (\tilde{X}_i - \sqrt{-1}\,\tilde{Y}_i)(\tilde{X}_i + \sqrt{-1}\,\tilde{Y}_i)$$

$$= \tfrac{1}{4} \sum_i \{ \tilde{X}_i^2 + (\tilde{Y}_i)^2 + \sqrt{-1}[\tilde{X}_i, \tilde{Y}_i] \} = \Delta_{\mathbb{C}}/4, \tag{4.9}$$

where in the last equality we have used $[\tilde{X}_i, \tilde{Y}_i] = 0$. This holds because the Lie algebra of $G^{\mathbb{C}}$ is the same as the complexification of $\mathfrak{g}$ and X_i and $Y_i = \mathscr{J} X_i = \sqrt{-1}\,X_i$ commute in $\mathfrak{g}^{\mathbb{C}}$.

Let $f \in \mathscr{F}(G^{\mathbb{C}}) \subset \mathscr{H}(G^{\mathbb{C}}) \cap L^2(\mu_t)$. Notice by elementary representation theory that $|f|^2 = f\bar{f} \in \mathscr{F}(G^{\mathbb{C}})$. Hence by Langland's Theorem (Theorem 2.13),

$$\|f\|^2_{L^2(\mu_t)} = \int_{G^{\mathbb{C}}} f\bar{f}(g)\, \mu_t(g)\, dg = e^{tZ \cdot \bar{Z}}(f\bar{f})|_e = \sum_{n=0}^{\infty} \frac{t^n}{n!} (Z \cdot \bar{Z})^n (f\bar{f})|_e. \tag{4.10}$$

From Remark 4.3, it easily follows that $(Z \cdot \bar{Z})(f\bar{f}) = \sum_i Z_i f \cdot \overline{Z_i f}$ and more generally

$$(Z \cdot \bar{Z})^n (f\bar{f}) = \sum_{i_1, \ldots, i_n = 1}^{d} |Z_{i_1} \cdots Z_{i_n} f|^2 = \sum_{i_1, \ldots, i_n = 1}^{d} |X_{i_1} \cdots X_{i_n} f|^2. \tag{4.11}$$

Combining (4.10) and (4.11) shows that

$$\|f\|^2_{L^2(\mu_t)} = \|(1-D)_e^{-1} f\|^2_{J_t}. \tag{4.12}$$

This shows that $(1-D)_e^{-1}$ is an isometry on $\mathscr{F}(G^{\mathbb{C}})$. Since $\mathscr{F}(G^{\mathbb{C}})$ is a dense subspace of $\mathscr{H}(G^{\mathbb{C}}) \cap L^2(\mu_t)$, $(1-D)_e^{-1}$ extends uniquely to an isometry of $\mathscr{H}(G^{\mathbb{C}}) \cap L^2(\mu_t)$. Q.E.D.

4.3. *Halls Isometry*

Hall's analogue (Theorem 1 in [15]) of the Segal–Bargmann transform is now an easy corollary of Theorems 4.4 and 4.1. From the proof of Theorem 2.13, $e^{t\Delta/2}\mathscr{F}(G)\subset\mathscr{F}(G)$. We also know by definition of $G^{\mathbb{C}}$ that each $f\in\mathscr{F}(G)$ has an analytic continuation to an element of $\mathscr{F}(G^{\mathbb{C}})$. Hence, we may and do view $e^{t\Delta/2}$ as an operator from $\mathscr{F}(G)$ to $\mathscr{F}(G^{\mathbb{C}})$.

COROLLARY 4.5 (Hall). *For each $t>0$, the map $e^{t\Delta/2}:\mathscr{F}(G)\to\mathscr{F}(G^{\mathbb{C}})$ extends uniquely to an isometry of $L^2(G,p_t)$ onto $L^2(G^{\mathbb{C}},\mu_t)\cap\mathscr{H}(G^{\mathbb{C}})$. (The extension to $L^2(G,p_t)$ will still be denoted by $e^{t\Delta/2}$.)*

Proof. For $f\in\mathscr{F}(G)$,

$$\|e^{t\Delta/2}f\|^2_{L^2(\mu_t)}=\|(1-D)_e^{-1}\,e^{t\Delta/2}f\|_t^2=\|f\|^2_{L^2(p_t)},$$

where the first equality is the content of Theorem 4.4 and the second equality is the content of Theorem 4.1. Therefore

$$e^{t\Delta/2}:\mathscr{F}(G)\subset L^2(p_t)\to\mathscr{F}(G^{\mathbb{C}})\subset L^2(G^{\mathbb{C}},\mu_t)\cap\mathscr{H}(G^{\mathbb{C}})$$

is an isometry. This finishes the proof, since by Lemma 3.1 and Theorem 3.3 $\mathscr{F}(G)$ is dense in $L^2(p_t)$ and $\mathscr{F}(G^{\mathbb{C}})$ is dense in $L^2(G^{\mathbb{C}},\mu_t)\cap\mathscr{H}(G^{\mathbb{C}})$.
Q.E.D.

COROLLARY 4.6. *For each $s>0$, the heat kernel p_s on G has an analytic continuation to $G^{\mathbb{C}}$.*

Proof. Choose $s'\in(0,s)$ and set $t\doteq s-s'$. Then $p_{s'}\in L^2(p_t)$ and $p_s=e^{t\Delta/2}p_{s'}$. By Corollary 4.5, $p_s\in L^2(\mu_t)\cap\mathscr{H}(G^{\mathbb{C}})$. In particular, p_s has an analytic continuation to $G^{\mathbb{C}}$. Q.E.D.

The isometry in Corollary 4.5 has a "K-averaged" version. In order to state the result, write $G\doteq K\times\mathbb{R}^d$, where K is a compact Lie group. This is permissible by Corollary 2.2.

LEMMA 4.7. *Suppose that p_t is the heat kernel on $G=K\times\mathbb{R}^d$, then there exists a finite constant $C>1$ such that*

$$C^{-1}p_t(g,x)\leqslant p_t(k,x)\leqslant Cp_t(g,x)\ \forall k,g\in K \text{ and } x\in\mathbb{R}^d. \tag{4.13}$$

In particular $L^2(\hat{p}_t)=L^2(p_t)$, where

$$\hat{p}_t(x)\doteq\int_K p_t(k,x)\,dk. \tag{4.14}$$

Moreover,

$$\hat{p}_t(x) = (2\pi t)^{-d/2} \sqrt{\det q}\, e^{-x \cdot qx/2t} \tag{4.15}$$

where q is the positive definite $d \times d$ matrix such that

$$qx \cdot x = (P(0_{\mathfrak{k}}, x), P(0_{\mathfrak{k}}, x)) \tag{4.16}$$

and P is the orthogonal projection of $\mathfrak{g}$ onto $(\mathfrak{k} \times \{0\})^{\perp} \subset \mathfrak{g}$.

Proof. By Corollary 2.2, we may assume that $K = (K' \times \mathbb{R}^l/\mathbb{Z}^l)/D$ where D is a finite subgroup of the center of $(K' \times \mathbb{R}^l/\mathbb{Z}^l)$, and $\mathfrak{g} \doteq \mathrm{Lie}(G)$ is the orthogonal direct sum of $\mathrm{Lie}(K')$ and $\mathbb{R}^l \times \mathbb{R}^d = \mathrm{Lie}(\mathbb{R}^l/\mathbb{Z}^l) \times \mathrm{Lie}(\mathbb{R}^d)$. (Throughout this proof, $\mathfrak{k}$, $\mathbb{R}^d$, and $\mathbb{R}^l \times \mathbb{R}^d$ will be identified with $\mathfrak{k} \times \{0_{\mathbb{R}^d}\}$, $\{0_{\mathfrak{k}}\} \times \mathbb{R}^d$, and $\{0_{\mathfrak{k}'}\} \times \mathbb{R}^l \times \mathbb{R}^d$ in $\mathfrak{g}$ respectively.) Let $\tilde{p}_t$ be the heat kernel on $(\mathbb{R}^l/\mathbb{Z}^l) \times \mathbb{R}^d$ relative to $(\cdot, \cdot)_{\mathfrak{g}}|_{\mathbb{R}^l \times \mathbb{R}^d}$, $\check{p}_t$ be the heat kernel on K' relative to $(\cdot, \cdot)|_{\mathrm{Lie}(K')}$, and $\rho_t(k', \theta, x) \doteq \check{p}_t(k')\, \tilde{p}_t(\theta, x)$—the heat kernel on $(K' \times \mathbb{R}^l/\mathbb{Z}^l) \times \mathbb{R}^d$. By Proposition 2.10 (applied to $\tilde{p}_t$) and the fact that $\check{p}_t$ is a continuous positive function on the compact set K', it follows that there exists a finite constant $C > 1$ such that

$$C^{-1} \rho_t(k', \theta, x) \leqslant \rho_t(k'', \theta', x) \leqslant C \rho_t(k', \theta, x) \tag{4.17}$$

for all $k', k'' \in K'$ and $\theta, \theta' \in \mathbb{R}^l/\mathbb{Z}^l$. The inequality in (4.13) follows from (4.17) since

$$p_t(\pi(k', \theta), x) = \frac{1}{|D|} \sum_{(k'', \theta') \in D} \rho_t(k'k'', \theta + \theta', x), \tag{4.18}$$

where

$$\pi : K' \times \mathbb{R}^l/\mathbb{Z}^l \to K = (K' \times \mathbb{R}^l/\mathbb{Z}^l)/D$$

is the canonical projection and $|D|$ is the number of elements in D.

Finally;

$$\begin{aligned}
\hat{p}_t(x) &= \int_{K' \times \mathbb{R}^l/\mathbb{Z}^l} p_t(\pi(k', \theta), x)\, dk'\, d\theta \\
&= \frac{1}{|D|} \int_{K' \times \mathbb{R}^l/\mathbb{Z}^l} \sum_{(k'', \theta') \in D} \rho_t(k'k'', \theta + \theta', x)\, dk'\, d\theta \\
&= \int_{K' \times \mathbb{R}^l/\mathbb{Z}^l} \rho_t(k', \theta, x)\, dk'\, d\theta
\end{aligned}$$

$$=\int_{K'\times\mathbb{R}^l/\mathbb{Z}^l} p_t(k',\theta,x)\,dk'\,d\theta$$

$$=\int_{K'\times\mathbb{R}^l/\mathbb{Z}^l} p_t(k')\,\tilde{p}_t(\theta,x)\,dk'\,d\theta$$

$$=\int_{\mathbb{R}^l/\mathbb{Z}^l} \tilde{p}_t(\theta,x)\,d\theta.$$

This last integral was done in Proposition 2.10 where it was found that

$$\hat{p}_t(x)=\int_{\mathbb{R}^l/\mathbb{Z}^l}\tilde{p}_t(\theta,x)\,d\theta=(2\pi t)^{-d/2}\sqrt{\det q}\,e^{-x\cdot qx/2t}$$

where $q \doteq C-B'A^{-1}B$ and the matrix (Q) of $(\cdot,\cdot)_{\mathfrak{g}}$ restricted to $\mathbb{R}^l\times\mathbb{R}^d\cong([\mathfrak{g},\mathfrak{g}])^\perp$ is of the block form $Q=\left[\begin{smallmatrix}A & B\\ B' & C\end{smallmatrix}\right]$. It is not hard to verify that the Q-perpendicular subspace to $\mathbb{R}^l\times\{0\}\subset\mathbb{R}^l\times\mathbb{R}^d$ is

$$\{(-A^{-1}Bx,x)\mid x\in\mathbb{R}^d\}$$

and that the orthogonal projection P' of $\mathbb{R}^l\times\mathbb{R}^d$ onto this subspace is given by the matrix:

$$P'=\begin{bmatrix}0 & -A^{-1}B\\ 0 & I\end{bmatrix}.$$

Since $\mathfrak{k}'\doteq \mathrm{Lie}(K')$ is $(\cdot,\cdot)_{\mathfrak{g}}$-perpendicular to $\mathbb{R}^l\times\mathbb{R}^d$, it follows that

$$(P(0_{\mathfrak{k}},x),(0_{\mathfrak{k}},x))_{\mathfrak{g}}=QP'\begin{bmatrix}0_{\mathbb{R}^l}\\ x\end{bmatrix}\cdot\begin{bmatrix}0_{\mathbb{R}^l}\\ x\end{bmatrix}$$

which is equal to $qx\cdot x=(C-B'A^{-1}B)x\cdot x$.

Alternative Proof of (4.15) *and* (4.16). Since Δ commutes with left translations by $k\in K$ and $\hat{p}_t$ is an average of left translation of p_t, $\hat{p}_t$ is still a solution to the heat equation

$$\partial\hat{p}_t/\partial t=\Delta\hat{p}_t/2. \tag{4.19}$$

Let $\{\xi_i\}_{i=1}^{\dim\mathfrak{k}}$ be an orthonormal basis for $\mathfrak{k}$ and choose $\{v_i\}_{i=1}^d\subset\mathfrak{g}$ such that $\{\xi_i\}_{i=1}^{\dim\mathfrak{k}}\cup\{v_i\}_{i=1}^d$ is an orthonormal basis for $\mathfrak{g}$. Decompose ech v_i as:

$$v_i=\eta_i+u_i\in\mathfrak{k}\oplus\mathbb{R}^d.$$

(Notice that $\{u_i\}_{i=1}^d$ is a basis for $\mathbb{R}^d$.) Then

$$\Delta = \sum_{i=1}^{\dim \mathfrak{k}} \tilde{\xi}_i^2 + \sum_{i=1}^{d} (\tilde{\eta}_i + \tilde{u}_i)^2,$$

and hence

$$\Delta \hat{p}_t = \Delta' \hat{p}_t, \tag{4.20}$$

where

$$\Delta' \doteq \sum_{i=1}^{d} \tilde{u}_i^2. \tag{4.21}$$

Because p_t converges to δ-function on $G = K \times \mathbb{R}^d$ as $t \to 0$, it follows that $\hat{p}_t$ converges to a δ-function on $\mathbb{R}^d$ as $t \to 0$. Therefore, by this observation, (4.19), and (4.20), $\hat{p}_t$ is the fundamental solution to the heat equation $\partial u/\partial t = \Delta' u/2$ on $\mathbb{R}^d$. Hence $\hat{p}_t$ is given by (4.15) where q is the unique $d \times d$ positive definite matrix such that $qu_i \cdot u_j = \delta_{ij}$. To identify q more explicitly, noice that $v_i = Pu_i$ and that $(v_i, v_j)_{\mathfrak{g}} = \delta_{ij}$. Therefore q satisfies

$$qu_i \cdot u_j = (Pu_i, Pu_j)_{\mathfrak{g}} \, \forall i, j = 1, ..., d.$$

That is to say $qx \cdot x = (Px, Px)_{\mathfrak{g}}$ for all $x \in \mathbb{R}^d$. Q.E.D.

The following Corollary is due to B. Hall in the case that $G = K$ is a compact Lie group.

Corollary 4.8. *Let $G = K \times \mathbb{R}^d$, $\hat{p}_t$ be the K-averaged heat kernel on G as given in* (4.15) *and ν_t be the K-averaged heat kernel on $G^{\mathbb{C}}$ as defined in* (2.16). *Then*

$$e^{t\Delta/2} : L^2(\hat{p}_t) \to L^2(\nu_t) \cap \mathscr{H}(G^{\mathbb{C}}) \tag{4.22}$$

is an isometric isomorphism of Hilbrt spaces.

Proof. Because of Lemma 2.16 and Lemma 4.7, the $L^2(\hat{p}_t)$ and the $L^2(p_t)$-norms and the $L^2(\nu_t)$ and $L^2(\mu_t)$-norms are equivalent. So by Corollary 4.5, $e^{t\Delta/2} : L^2(\hat{p}_t) \to L^2(\nu_t) \cap \mathscr{H}(G^{\mathbb{C}})$ is a homeomorphism of Hilbert spaces. So it suffices to show that $e^{t\Delta/2}$ is an isometry.

Let $f \in L^2(\hat{p}_t) = L^2(p_t)$. For each $k \in K$, set $f_k(g, x) \doteq f(kg, x)$. Because of Lemma 4.7, $f_k \in L^2(p_t)$ for all $k \in K$. By Corollary 4.5

$$\|f_k\|_{L^2(p_t)} = \|e^{t\Delta/2} f_k\|_{L^2(\mu_t)}.$$

This equality, the facts that left translations by elements in K commutes with both the Laplacian (Δ) and the operation of analytic continuation, and the invariance of the Haar measures on G and $G^{\mathbb{C}}$ implies that

$$\int_{G=K\times\mathbb{R}^d} |f(g,x)|^2\, p_t(k^{-1}g,x)\,dg\,dx$$

$$=\int_{G^{\mathbb{C}}=K^{\mathbb{C}}\times\mathbb{C}^d} |(e^{t\Delta/2}f)(g,z)|^2\,\mu_t(k^{-1}g,z)\,dg\,dz.$$

The assertion that $e^{t\Delta/2}$ is an isometry from $L^2(\hat{p}_t)$ to $L^2(\nu_t)$ now follows by integrating both sides of the above equation relative to $k\in K$. Q.E.D.

5. A Generalized Power Series and Its Consequences

In this section, it will be shown that a holomorphic function on $G^{\mathbb{C}}$ may be expanded in a generalized power series expansion. This expansion will be used to show: a function $u\in\mathscr{H}(G^{\mathbb{C}})$ is in $L^2(\mu_t)$ iff $(1-D)_e^{-1}u\in{}^{\perp}J_t$. Let θ be the Maurer–Cartan form on $G^{\mathbb{C}}$. That is θ is the $\mathfrak{g}^{\mathbb{C}}$-valued 1-form on $G^{\mathbb{C}}$ defined by $\theta\langle A\rangle\equiv L_{g^{-1}*}A$ when $A\in T_gG^{\mathbb{C}}$. Given any smooth $g:[0,1]\to G^{\mathbb{C}}$ such that $g(0)=e\in G^{\mathbb{C}}$, let

$$c(s)\equiv\theta\langle g'(s)\rangle=L_{g(s)^{-1}*}g'(s),$$

so that $c: I\equiv[0,1]\to\mathfrak{g}^{\mathbb{C}}$ is a smooth path in $\mathfrak{g}^{\mathbb{C}}$.

Proposition 5.1. *Suppose that $u:G^{\mathbb{C}}\to\mathbb{C}$ is a holomorphic function, $g:[0,1]\to G^{\mathbb{C}}$ is a smooth path such that $g(0)=e$, and $c(s)\equiv\theta\langle g'(s)\rangle$. Then*

$$u(g(s))=\sum_{n=0}^{\infty}\left\{\int_{\Delta_n(s)}\langle D^nu(e), c(s_1)\otimes\cdots\otimes c(s_n)\rangle\,ds_1\cdots ds_n\right\},\quad(5.1)$$

where

$$\Delta_n(s)\equiv\{(s_1,...,s_n)\in\mathbb{R}^n: 0\leqslant s_1\leqslant s_2\cdots\leqslant s_n\leqslant s\}.$$

The series in (5.1) *converges absolutely.*

The proof relies on:

Lemma 5.2. *Suppose that $u:G^{\mathbb{C}}\to\mathbb{C}$ is a C^∞-function, $g:[0,1]\to G^{\mathbb{C}}$ is a smooth path such that $g(0)=e$, and $c(s)\equiv\theta\langle g'(s)\rangle$. Then*

$$u(g(s))=\sum_{n=0}^{N-1}\int_{\Delta_n(s)}\langle D^nu(e), c(s_1)\otimes\cdots\otimes c(s_n)\rangle\,\mathbf{ds}+R_N(s),\quad(5.2)$$

where

$$R_N(s) \equiv \int_{\Delta_N(s)} \langle D^N u(g(s_1)), c(s_1) \otimes \cdots \otimes c.(s_N) \rangle \, \mathbf{ds}. \tag{5.3}$$

Proof. By the fundamental theorem of calculus;

$$u(g(s)) = u(e) + \int_0^s \langle Du(g(s_1)), c(s_1) \rangle \, ds_1. \tag{5.4}$$

For fixed s_1, let h be the smooth function on $G^{\mathbb{C}}$ given by $h(x) \equiv \langle Du(x), c(s_1) \rangle$. Applying (5.4) with u replaced by h and s replaced by s_1 gives:

$$\langle Du(g(s_1)), c(s_1) \rangle = h(g(s_1)) = h(e) + \int_0^{s_1} \langle Dh(g(s_2)), c(s_2) \rangle \, ds_2. \tag{5.5}$$

It is clear by the definition that

$$\langle Dh(g(s_2)), c(s_2) \rangle = \langle D^2 u(g(s_2)), c(s_2) \otimes c(s_1) \rangle. \tag{5.6}$$

Combining (5.4–5.6) yields:

$$u(g(s)) = u(e) + \int_0^s \langle Du(g(e), c(s_1) \rangle \, ds_1 \\ + \int_0^s ds_1 \int_0^{s_1} ds_2 \langle D^2 u(g(s_2), c(s_2) \otimes c(s_1) \rangle. \tag{5.7}$$

After relabeling s_1 and s_2 so that s_1 becomes s_2 and s_2 becomes s_1, Eq. (5.7) may be written as:

$$u(g(s)) = u(e) + \int_0^s \langle Du(g(e), c(s_1) \rangle \, ds_1 \\ + \int_{\Delta_2(s)} \langle D^2 u(g(s_1), c(s_1) \otimes c(s_2) \rangle \, ds_1 \, ds_2.$$

Repeated application of this argument shows the truth of (5.2) and (5.3) for all integrs $N \geqslant 0$. Q.E.D.

Proof of Proposition 5.1. For each $z = x + iy \in \mathbb{C}$, let $X_z(s, g) \equiv L_{g*}(zc(s))$, where $zc(s) \doteq xc(s) + y\mathscr{J}c(s)$. Let $\sigma(s, z)$ denote the solution to the ordinary differential equation:

$$d\sigma(s, z)/ds = X_z(s, \sigma(s, z)) \qquad \text{with} \quad \sigma(0, z) = e. \tag{5.8}$$

By definition of c, $g(s) = \sigma(s, 1)$.

Because exponential map and the group operations are holomorphic, one can show (for any $s \in [0, 1]$) that the map

$$((z, g) \to X_z(s, g)f) : \mathbb{C} \times G^{\mathbb{C}} \to \mathbb{C}$$

is holomorphic provided f is a (locally) defined holomorphic function on $G^{\mathbb{C}}$. Hence, in a holomorphic coordinate chart (w), the differential equation in (5.8) translates into an equation of the form

$$dW(s, z)/ds = F(s, z, W(s, z)), \tag{5.9}$$

where

$$((z, W) \to F(s, z, W)) : \mathbb{C} \times U \to U$$

is holomorphic and $U \subset \mathbb{C}^d$ is the range of w. By standard O.D.E. theorems, solutions W to (5.9) are holomorphic in z. These local results can be pieced together to show that $z \to \sigma(s, z)$ is holomorphic for each $s \in [0, 1]$. The interested reader may provide a more intrinsic proof along the lines of the material in Chapter 2 of [21].

Now fix $s \in [0, 1]$. We have shown that $U(z) \equiv u(\sigma(s, z))$ is holomorphic on $\mathbb{C}$. Therefore

$$u(\sigma(s, z)) = \sum_{n=0}^{\infty} \frac{z^n}{n!} U^{(n)}(0), \tag{5.10}$$

where $U^{(n)}(z) = (d/dz)^n U^{(n)}(z)$. On the other hand by Lemma 5.2, with $g(s)$ replaced by $\sigma(s, z)$,

$$\begin{aligned} u(\sigma(s, z)) &= \sum_{n=0}^{N-1} \int_{\Delta_n(s)} \langle D^n u(e), zc(s_1) \otimes \cdots \otimes zc(s_n) \rangle \, \mathbf{ds} + O(z^N) \\ &= \sum_{n=0}^{N-1} z^n \int_{\Delta_n(s)} \langle D^n u(e), c(s_1) \otimes \cdots \otimes c(s_n) \rangle \, \mathbf{ds} + O(z^N), \end{aligned} \tag{5.11}$$

where in the second equality we have made use of the fact that u is holomorphic so that $D^n u(e)$ is complex linear on $(\mathfrak{g}^{\mathbb{C}})^{\otimes n}$. Comparing equations (5.10) and (5.11), we learn that

$$U^{(n)}(0)/n! = \int_{\Delta_n(s)} \langle D^n u(e), c(s_1) \otimes \cdots \otimes c(s_n) \rangle \, \mathbf{ds}.$$

Using this expression back in (5.10) and then setting $z = 1$ yields Eq. (5.1). Q.E.D.

We may write Eq. (5.1) as

$$u(g(s)) = \langle (1-D)_e^{-1} u, \Psi_s(g) \rangle \tag{5.12}$$

where

$$\Psi_s(g) \equiv \sum_{n=0}^{\infty} \int_{\Delta_n(s)} c(s_1) \otimes \cdots \otimes c(s_n)\, \mathbf{ds}, \tag{5.13}$$

and $c(s) \equiv \theta\langle g'(s) \rangle$.

PROPOSITION 5.3. *Let $g : [0, 1] \to G^{\mathbb{C}}$ be a smooth path such that $g(0) = e$. Then $\Psi(g) \equiv \Psi_1(g)$ satisfies the estimate*:

$$\|\Psi(g)\|_t^2 \leqslant \exp\{\|c\|_1^2/t\} = \exp\{l(g)^2/t\}, \tag{5.14}$$

where $\|c\|_1 \equiv \int_0^1 |c(s)|\, ds$ and $l(g) = \int_0^1 |\theta\langle g'(s)\rangle|\, ds$—the length of the path g relative to the left invariant metric determined by $\mathrm{Re}(\cdot,\cdot)_{\mathfrak{g}^{\mathbb{C}}}$ *on* $\mathfrak{g}^{\mathbb{C}} = \mathrm{Lie}(G^{\mathbb{C}})$.

Remark 5.4. Notice that $\mathrm{Re}(\cdot,\cdot)_{\mathfrak{g}^{\mathbb{C}}}$ is a real inner product on $\mathfrak{g}^{\mathbb{C}}$ for which $\mathfrak{g}_0 \cup \mathscr{I}\mathfrak{g}_0$ is an orthonormal basis for $\mathfrak{g}^{\mathbb{C}}$ whenever $\mathfrak{g}_0$ is an orthonormal basis for $\mathfrak{g}$.

Proof. The result follows by summing (on n) the following elementary estimate:

$$\begin{aligned} &\left\| \int_{\Delta_n} c(s_1) \otimes \cdots \otimes c(s_n)\, \mathbf{ds} \right\|_t^2 \\ &\quad = n!/t^n \int_{\Delta_n \times \Delta_n} \prod_{i=1}^{n} (c(s_i), c(t_i))\, \mathbf{dt}\, \mathbf{ds} \\ &\quad \leqslant \frac{n!}{t^n} \left[\int_{\Delta_n} \prod_{i=1}^{n} |c(s_i)|_{\mathfrak{g}^{\mathbb{C}}}\, \mathbf{ds} \right]^2 = t^{-n}\, \|c\|_1^{2n}/n!. \end{aligned}$$

Q.E.D.

COROLLARY 5.5. *Let $u \in \mathscr{H}(G^{\mathbb{C}})$ and set $\alpha \doteq (1-D)_e^{-1} u \in {}^{\perp}J$. If $\alpha \in {}^{\perp}J_t$, then*

$$|u(x)| \leqslant \|\alpha\|_t \exp\{|x|^2/2t\}, \tag{5.15}$$

where $|x| = d(x, e)$—the distance relative to the left-invariant Riemannian metric on $G^{\mathbb{C}}$.

Proof. Let $g : [0, 1] \to G^{\mathbb{C}}$ be a smooth path joining e to $x \in G^{\mathbb{C}}$. Then by (5.14)

$$|u(x)| = |\langle \alpha, \Psi(g) \rangle| \leqslant \|\alpha\|_t \cdot \|\Psi(g)\|_t \leqslant \|\alpha\|_t \exp\{l^2(g)/2t\}.$$

The proof is completed by taking the infimum of both sides of the above equation over all paths (g) joining e to $x \in G^{\mathbb{C}}$. Q.E.D.

Remark 5.6. Suppose that $c(s) = \xi \in \mathfrak{g}^{\mathbb{C}}$ in the above theorem so that $g(1) = e^{\xi}$. Then

$$u(e^{\xi}) = \sum_{n=0}^{\infty} \langle D^n u(e), \xi^{\otimes n} \rangle / n! = \langle (1-D)_e^{-1} u, \operatorname{Exp}(\xi) \rangle, \tag{5.16}$$

where

$$\operatorname{Exp}(\xi) \equiv \Psi(\xi) = \sum_{n=0}^{\infty} \xi^{\otimes n}/n!. \tag{5.17}$$

THEOREM 5.7. *Let $u \in \mathscr{H}(G^{\mathbb{C}})$ such that $\alpha \equiv (1-D)_e^{-1} u \in {}^{\perp}J_t$. Then $u \in L^2(G^{\mathbb{C}}, \mu_t)$ and $\|u\|_{L^2(\mu_t)} = \|\alpha\|_t$.*

The proof will be given after the following measure theoretic lemma.

LEMMA 5.8. *Let $V(r)$ denote the volume of the Riemannian ball of radius r in $G^{\mathbb{C}}$ relative to the left invariant metric on $G^{\mathbb{C}}$ which agrees with* $\operatorname{Re}(\cdot, \cdot)$ *at $e \in G^{\mathbb{C}}$. Then there is a constant $C > 0$ such that $V(r) \leqslant C \exp(Cr)$.*

Proof. For each $r > 0$, let B_r denote the open ball of radius r centered at $e \in G^{\mathbb{C}}$. Choose $\varepsilon > 0$ sufficiently small such that the closure K of the ball $B_{2\varepsilon}$ is compact. By compactness, it is possible to choose a positive integer N and $\{x_i\}_{i=1}^{N} \subset K$ such that $K \subset \bigcup_{i=1}^{N} (x_i B_\varepsilon)$.

Claim. Let $U_n \equiv B_{n\varepsilon}$ for $n \in \{1, 2, ...\}$ then

$$U_n \subset W_n \equiv \bigcup_{l=1}^{n} \bigcup_{i_1, ..., i_l = 1}^{N} (x_{i_1} \cdots x_{i_n} B_\varepsilon).$$

The claim will be proved by induction. Clearly it holds by definition when $n = 1$ or 2. So assume the claim holds for some positive integer $n \geqslant 2$. Let $x \in U_{n+1}$. Then $r \equiv |x| = d(x, e) < (n+1)\varepsilon$ by definition of $U_{(n+1)}$. We may, (and do) assume that $r \geqslant n\varepsilon$, otherwise $x \in U_n$ which is in $W_n \subset W_{n+1}$ by the induction hypothesis. Choose $\rho \in (r, (n+1)\varepsilon)$ and a path $g : [0, \rho] \to G^{\mathbb{C}}$ such that $|g'(s)| \equiv 1$, $g(0) = e$, and $g(\rho) = x$. Choose $t \in (0, n\varepsilon)$ such that $|t - r| < \varepsilon$ and set $y \equiv g(t)$. Then $y \in U_n$ and $|y^{-1}x| = d(y, x) \leqslant r - t < \varepsilon$. Since $y \in U_n$, there exists (by induction hypothesis) an $l \leqslant n$ and $i_1, ..., i_l \in \{1, 2, ..., N\}$ such that $y \in x_{i_1} \cdots x_{i_l} B_\varepsilon$. Set $w \equiv x_{i_1} \cdots x_{i_l}$, then

$$|w^{-1}x| = d(x, w) \leqslant d(x, y) + d(y, w) < \varepsilon + \varepsilon = 2\varepsilon.$$

Thus $w^{-1}x \in K$ and hence there exist $i_{l+1} \in \{1, 2, ..., N\}$ such that $w^{-1}x \in x_{i_{l+1}} B_\varepsilon$, i.e.

$$x \in wx_{i_{l+1}} B_\varepsilon = x_{i_1} \cdots x_{i_l} \cdot x_{i_{l+1}} B_\varepsilon.$$

This proves $x \in W_{l+1} \subset W_{n+1}$, and hence the claim.

Let λ denote (a) Haar measure on $G^{\mathbb{C}}$. Then

$$V(n\varepsilon) \doteq \lambda(U_n) \leqslant \sum_{l=1}^{n} \sum_{i_1, ..., i_l = 1}^{N} \lambda(x_{i_1} \cdots x_{i_n} B_\varepsilon)$$

$$= \sum_{l=1}^{n} N^l V(\varepsilon) \leqslant N N^n V(\varepsilon)$$

Suppose that $r > 0$ is given. Choose n such that $r \in [n\varepsilon, (n+1)\varepsilon)$. Then

$$V(r) \leqslant V((n+1)\varepsilon) \leqslant N^2 N^n V(\varepsilon) \leqslant N^{(r/\varepsilon)} N^2 V(\varepsilon) \leqslant Ce^{Cr},$$

where $C \equiv \max\{N^2 V(\varepsilon), \log(N)/\varepsilon\}$. Q.E.D.

Proof of Theorem 5.7. Choose $\tau \in (0, t)$ and $r \in (\tau, t)$, then by Theorem 2.8,

$$\mu_\tau(x) \leqslant K' e^{-|x|^2/r}, \tag{5.18}$$

where K' is a finite constant. From (5.15) and (5.18) we find:

$$\|u\|^2_{L^2(\mu_\tau)} \leqslant K' \|\alpha\|^2_t \int_{G^{\mathbb{C}}} \exp\{|x|^2/t - |x|^2/r\} \, dx$$

$$= K \int_{G^{\mathbb{C}}} \exp\{-\delta |x|^2\} \, dx, \tag{5.19}$$

where $K \doteq K' \|\alpha\|^2_t$, $\delta \equiv r^{-1} - t^{-1} > 0$, and dx denotes Haar measure on $G^{\mathbb{C}}$. By Fubini's theorem, the identity

$$\exp\{-\delta |x|^2\} = \int_{\delta |x|^2}^{\infty} e^{-\rho} \, d\rho = \int_0^\infty 1_{\{\rho > \delta |x|^2\}} e^{-\rho} \, d\rho,$$

Lemma 5.8, and (5.19);

$$\|u\|^2_{L^2(\mu_\tau)} \leqslant K \int_0^\infty d\rho e^{-\rho} \lambda(\{x : |x| < \sqrt{\rho/\delta}\})$$

$$\leqslant C' \int_0^\infty d\rho e^{-\rho} e^{C\sqrt{\rho/\delta}} < \infty,$$

where C' and C are finite constants. In particular, it follows that $u \in L^2(\mu_\tau)$ for all $\tau \in (0, t)$. By Theorem 4.4,

$$\|u\|^2_{L^2(\mu_\tau)} = \|\alpha\|^2_\tau \equiv \sum_{n=0}^{\infty} \frac{\tau^n}{n!} |\alpha_n|^2_{\mathfrak{g}^{\otimes n}} \leqslant \sum_{n=0}^{\infty} \frac{t^n}{n!} |\alpha_n|^2_{\mathfrak{g}^{\otimes n}} \leqslant \|\alpha\|^2_t < \infty$$

for all $\tau < t$. Using the continuity of $(\tau, g) \to \mu_\tau(g)$ and Fatou's Lemma, it follows that

$$\|u\|^2_{L^2(\mu_t)} \leqslant \liminf_{\tau \uparrow t} \|u\|^2_{L^2(\mu_\tau)} = \|\alpha\|^2_t < \infty.$$

Thus $u \in L^2(\mu_t)$ and (by Theorem 4.4) $\|u\|_{L^2(\mu_t)} = \|\alpha\|_t$. Q.E.D.

COROLLARY 5.9. *Let* $u \in \mathscr{H}(G^{\mathbb{C}})$, *then* $u \in L^2(\mu_t)$ *iff* $(1-D)_e^{-1} u \in {}^{\perp}J_t$.

Proof. Combine Theorems 4.4 and 5.7. Q.E.D.

6. RECONSTRUCTION OF u FROM α

In this section, G (and hence $G^{\mathbb{C}}$) is assumed to be simply connected. We will see that to each $\alpha \in {}^{\perp}J_+$ there is a unique holomorphic function u_α on $G^{\mathbb{C}}$ such that $(1-D)_e^{-1} u_\alpha = \alpha$. It will then be an easy matter to finish the proof of Gross' theorem.

6.1. *Construction of* u_α

THEOREM 6.1 (Gross). *Suppose now that* $\pi_1(G) = \{1\}$ *(hence* $\pi_1(G^{\mathbb{C}}) = \{1\}$*) and* $\alpha \in {}^{\perp}J_+$. *Given a smooth path* $g: [0, 1] \to G^{\mathbb{C}}$ *such that* $g(0) = e$, *let* $\Psi(g) \doteq \Psi_1(g)$ *be as in* (5.13). *Then there is a unique smooth function* u_α *on* $G^{\mathbb{C}}$ *such that*

$$u_\alpha(g(1)) = \langle \alpha, \Psi(g) \rangle. \tag{6.1}$$

Proof. The proof of this result may be found in Gross [12] in the case that $G^{\mathbb{C}}$ is replaced by G, see Lemma 8.2 on p. 427. The proof given there goes through without change when G is replaced by $G^{\mathbb{C}}$ and $\mathfrak{g}$ is replaced by $\mathfrak{g}^{\mathbb{C}}$.

To indicate the "reason" this result is true, I will sketch a different proof. The main point is to show $\langle \alpha, \Psi(g) \rangle$ only depends on $g(1)$. The proof of the smoothness of u_α will be deferred until Proposition 6.2 below where it is shown that u_α is holomorphic. Since the theorem has already been proved in [12], I will let the reader verify any technical differentiability questions. (None are too severe.)

Let $\Psi_s(g)$ be efined in (5.13), then $\Psi_s(g) \in T_+$ and $\Psi_s(g)$ satisfies the differential equation

$$d\Psi_s(g)/ds = \Psi_s(g)\, \theta\langle g'(s)\rangle \qquad \text{with} \quad \Psi_0(g) = 1 \in T_+. \tag{6.2}$$

It can be shown that $\Psi_s(g)$ is invertible in T_+ and that $\Psi_s^{-1}(g)$ is given as the solution $P_s(g)$ to the differential equation

$$dP_s(g)/ds + \theta\langle g'(s)\rangle\, P_s(g) = 0 \qquad \text{with} \quad P_0(g) = 1 \in T_+. \tag{6.3}$$

To prove this one shows (with P_s defined by (6.3)) that $\Psi_s(g) \cdot P_s(g)$ and $P_s(g) \cdot \Psi_s(g)$ both satisfy linear differential equations which have 1 as a unique solution.

Let A be the End(T_+)-valued connection 1-form on $G^{\mathbb{C}}$ defined by $A = L_\theta$ where L_θ denotes left multiplication on T_+ by the Maurer–Cartan form θ. We will identify $P_s(g) \in T_+$ with $L_{P_s(g)} \in \text{End}(T_+)$—left multiplication by $P_s(g)$. Because of Eq. (6.3), P_s has the interpretation of parallel translation along g relative to the covariant derivative $\nabla \doteq d + A$ on the tirivial bundle $G^{\mathbb{C}} \times T_+$.

Let $h : [0, 1] \to \mathfrak{g}^{\mathbb{C}}$ be a smooth function such that $h(0) = 0$. Since $P_1(g)$ is parallel translation along g, it is well known how to compute its derivative as a function of g. The answer is:

$$\frac{\nabla}{dt}\bigg|_0 P_1(ge^{th}) \doteq \frac{d}{dt}\bigg|_0 P_1(ge^{th}) + A\langle \tilde{h}(1)\rangle\, P_1(g) = P_1(g)\, B(\tilde{h}), \tag{6.4}$$

where $\tilde{h}(s) \equiv L_{g(s)*} h(s)$,

$$B\langle \tilde{h}\rangle \equiv \int_0^1 P_s(g)^{-1}\, F\langle g'(s), \tilde{h}(s)\rangle\, P_s(g)\, ds, \tag{6.5}$$

and F is the curvature tensor of ∇. For this result the reader is referred to (for example) Theorem 2.2 in Gross [11], Theorem 4.1 of Driver [8], or Theorem 1.9 of Driver [9] for a very simple proof. Using the relation $\Psi_1(g) = P_1(g)^{-1}$, we know

$$\frac{d}{dt}\bigg|_0 \Psi_1(ge^{th}) = -\Psi_1(g) \left[\frac{d}{dt}\bigg|_0 P_1(ge^{th})\right] \Psi_1(g).$$

Thus, (6.4) and (6.5) may be rewritten as:

$$\frac{d}{dt}\bigg|_0 \Psi_1(ge^{th}) = \Psi_1(g) \cdot h(1) - B\langle \tilde{h}\rangle \cdot \Psi_1(g), \tag{6.6}$$

where

$$B\langle \tilde{h}\rangle \equiv \int_0^1 \Psi_s(g)\, F\langle g'(s), \tilde{h}(s)\rangle\, \Psi_s^{-1}(g)\, ds, \tag{6.7}$$

To compute F, let X and Y be in $\mathfrak{g}^{\mathbb{C}}$, and $\tilde{X}$ and $\tilde{Y}$ denote there unique extensions to left invariant vector-fields on $G^{\mathbb{C}}$. Notice that $A\langle \tilde{X}\rangle = L_X$ and $A\langle \tilde{Y}\rangle = L_Y$, where L_X and L_Y denotes left multiplication by X and Y respectively on T_+. Hence

$$dA\langle \tilde{X}, \tilde{Y}\rangle = \tilde{X}(L_Y) - \tilde{Y}(L_X) - (L_{\theta\langle[\tilde{X},\tilde{Y}]\rangle}) = -L_{[X,Y]},$$

and

$$A \wedge A\langle \tilde{X}, \tilde{Y}\rangle \equiv [L_X, L_Y] = L_{\{X\otimes Y - Y\otimes X\}}.$$

Adding these last two equations together gives,

$$F\langle \tilde{X}, \tilde{Y}\rangle = L_{\{X\otimes Y - Y\otimes X - [X,Y]\}}. \tag{6.8}$$

Assembling equations (6.6–6.8) gives

$$\left.\frac{d}{dt}\right|_0 \Psi_1(ge^{th}) = \Psi_1(g)\, h(1) - B\langle \tilde{h}\rangle\, \Psi_1(g) \tag{6.9}$$

where

$$B\langle \tilde{h}\rangle \equiv \int_0^1 \Psi_s(g)\{c(s)\otimes h(s) - h(s)\otimes c(s) - [c(s), h(s)]\}\, \Psi_s^{-1}(g)\, ds, \tag{6.10}$$

and $c(s) \equiv \theta\langle g'(s)\rangle$.

It is now easy to check that $B\langle \tilde{h}\rangle\, \Psi_1(g) \in J_+$ so that $\langle \alpha, \Psi_1(g)\, B\langle \tilde{h}\rangle\rangle = 0$. Therefore

$$\left.\frac{d}{dt}\right|_0 \langle \alpha, \Psi_1(ge^{th})\rangle = \langle \alpha, \Psi_1(g)\, h(1)\rangle, \tag{6.11}$$

and hence $(d/dt)|_0 \langle \alpha, \Psi_1(ge^{th})\rangle = 0$ if $h(1) = 0$. This fact and the assumption that $G^{\mathbb{C}}$ is simply connected implies that $\langle \alpha, \Psi_1(g)\rangle$ depends only on $g(1)$. Q.E.D.

PROPOSITION 6.2. *For each* $\alpha \in {}^{\perp}J_+$, *the function* u_α *defined above is holomorphic.*

Proof. Let $\xi \in \mathfrak{g}^{\mathbb{C}}$, $x \in G^{\mathbb{C}}$, and write u for u_α. Choose smooth paths $g : [0, 1] \to G^{\mathbb{C}}$ and $h : [0, 1] \to \mathfrak{g}^{\mathbb{C}}$ such that $g(0) = e$, $g(1) = x$, $h(0) = 0$, and $h(1) = \xi$. Then by Equations (6.1) and (6.11);

$$\langle du, \tilde{\xi}(x) \rangle = \frac{d}{dt}\bigg|_0 u(ge^{t\xi}) = \frac{d}{dt}\bigg|_0 \langle \alpha, \Psi_1(ge^{th}) \rangle = \langle \alpha, \Psi_1(g)\xi \rangle.$$

The above displayed equation shows that du_x is complex linear and therefore $u : G^{\mathbb{C}} \to \mathbb{C}$ is holomorphic. Q.E.D.

PROPOSITION 6.3. *Let α and $u \equiv u_\alpha$ be as in the above proposition. Then* $(1 - D)_e^{-1} u = \alpha$.

Proof. It suffices to show

$$\langle D^n u(e), \xi_1 \otimes \cdots \otimes \xi_n \rangle = \langle \alpha, \xi_1 \otimes \cdots \otimes \xi_n \rangle, \forall \{\xi_i\}_{i=1}^n \subset \mathfrak{g} \quad (6.12)$$

We will do this by induction on n making use of (5.16) which asserts that

$$u(e^\xi) = \langle (1 - D)_e^{-1} u, \operatorname{Exp}(\xi) \rangle$$

for all $\xi \in \mathfrak{g}^{\mathbb{C}}$. If $n = 0$, we have $u(e) = \langle \alpha, \operatorname{Exp}(0) \rangle$. So assume 6.12 holds at level $n - 1 \geqslant 0$. We will show (6.12) holds at level n.

Let $f(t) \equiv u(e^{t\xi})$. Then f is real analytic and

$$f(t) \equiv u(e^{t\xi}) = \langle \alpha, \operatorname{Exp}(t\xi) \rangle = \sum_{n=0}^{\infty} t^n \langle \alpha, \xi^{\otimes n} \rangle / n!.$$

Hence

$$\langle D^n u(e), \xi^n \rangle = f^{(n)}(0) = \langle \alpha, \xi^{\otimes n} \rangle. \quad (6.13)$$

Now assume that $D^k u(e) = \alpha|_{\mathfrak{g}^{\otimes k}}$ for all $k < n$. By polarizing (6.13) we know, for all $\{\xi_i\}_{i=1}^n \subset \mathfrak{g}$, that:

$$\frac{1}{n!} \sum_\sigma \langle D^n u(e) - \alpha, \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)} \rangle = 0, \quad (6.14)$$

where the sum is over all permutation of $\{1, 2, ..., n\}$. Using the induction hypothesis $(D^{(n-1)} u(e) = \alpha|_{\mathfrak{g}^{\otimes(n-1)}})$ and the fact that both $(1 - D)^{-1} u(e)$ and α annihilate J, one shows

$$\frac{1}{n!} \sum_\sigma \langle D^n u(e) - \alpha, \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)} \rangle = \langle D^n u(e) - \alpha, \xi_1 \otimes \cdots \otimes \xi_n \rangle.$$

Combining this equation with (6.14) shows that $D^n u(e) = \alpha|_{\mathfrak{g}^{\otimes n}}$, which completes the inductive argument. Q.E.D.

6.2. *Surjectivity of the Isometries*

THEOREM 6.4. *Suppose that G is a Lie group of compact type as above. Further assume that G is simply connected and hence so is $G^{\mathbb{C}}$. Then the map*

$$(1-D)_e^{-1}: L^2(G^{\mathbb{C}}, \mu_t) \cap \mathscr{H}(G^{\mathbb{C}}) \to {}^{\perp}J_t$$

is an isometric isomorphism. In particular this map is surjective.

Proof. Theorem 4.4 asserts that $(1-D)_e^{-1}$ is an isometry from $L^2(G^{\mathbb{C}}, \mu_t) \cap \mathscr{H}(G^{\mathbb{C}})$ *into* ${}^{\perp}J_t$. So it suffices to show $(1-D)_e^{-1}$ is surjective. Suppose that $\alpha \in {}^{\perp}J_t$. By Theorem 6.1, Proposition 6.2, and Proposition 6.3 there exists a function $u_\alpha \in \mathscr{H}(G^{\mathbb{C}})$ such that $(1-D)_e^{-1} u_\alpha = \alpha$. By Corollary 5.9, this function u_α is in $L^2(\mu_t)$. Q.E.D.

COROLLARY 6.5. *The map*

$$(1-D)_e^{-1} e^{t\Delta/2}: L^2(G, dp_t) \to {}^{\perp}J_t$$

is a surjective isometry provided that G is simply connected.

Proof. The map $(1-D)_e^{-1} e^{t\Delta/2}$ is the composition of $(1-D)_e^{-1}$ and $e^{t\Delta/2}$, each of which is a surjective isometry by Corollary 4.5 and Theorem 6.4 respectively. Q.E.D.

7. INTERTWINING PROPERTIES

Recall that G is a connected Lie group of compact type. In this section, $t>0$ will be fixed, $\mathbb{N} \doteq \{0, 1, 2, ...\}$, and the operators A_ξ, A_ξ^t, $\bar{\xi}_t$, and $\check{\xi}_t$ are those given in Definition 1.12.

7.1. *Intertwining Results*

THEOREM 7.1. *Let G be a simply connected Lie group of compact type. Then for each $\xi \in \mathfrak{g}$,*

$$(1-D)_e^{-1} \check{\xi}_t = A_\xi^t (1-D)_e^{-1}. \tag{7.1}$$

Proof. Let $f \in \mathscr{H}(G^{\mathbb{C}})$ and set $\alpha \doteq (1-D)_e^{-1} f$. Since

$$\begin{aligned}\langle (1-D)_e^{-1}(\tilde{\xi} f), \xi_1 \otimes \cdots \otimes \xi_n \rangle &\doteq (\tilde{\xi}_1 \cdots \tilde{\xi}_n \tilde{\xi} f)(e) \\ &= \langle \alpha, \xi_1 \otimes \cdots \otimes \xi_n \otimes \xi \rangle \\ &= \langle A_\xi \alpha, \xi_1 \otimes \cdots \otimes \xi_n \rangle\end{aligned}$$

for all $\xi_1, ..., \xi_n \in \mathfrak{g}$ and $n \in \mathbb{N}\backslash\{0\}$, it follows that

$$(1-D)_e^{-1}\,\tilde{\xi}f = A_\xi \alpha. \tag{7.2}$$

By this equation and the isometry property of $(1-D)_e^{-1}$, $f \in \mathscr{D}(\check{\xi}_t)$ iff $\alpha \in \mathscr{D}(A_\xi^t)$ and for $f \in \mathscr{D}(\check{\xi}_t)$,

$$(1-D)_e^{-1}\,\check{\xi}_t f = A_\xi \alpha = A_\xi^t (1-D)_e^{-1} f. \qquad \text{Q.E.D.}$$

THEOREM 7.2. *Let G be a connected Lie group of compact type, then*

$$e^{t\Delta/2}\bar{\xi}_t = \check{\xi}_t\, e^{t\Delta/2} \qquad \forall \xi \in \mathfrak{g},$$

where $e^{t\Delta/2}$ is Hall's isometry of $L^2(p_t)$ with $L^2(\mu_t) \cap \mathscr{H}(G^{\mathbb{C}})$.

To simplify notation, let U_t denote Hijab's formula for Gross' isometry, i.e.

$$U_t = (1-D)_e^{-1}\, e^{t\Delta/2} : L^2(p_t) \to L^2(\mu_t).$$

The following corollary is a direct consequence of Theorems 7.1 and 7.2.

COROLLARY 7.3. *Suppose that G is a simply connected Lie group of compact type. Then for each $\xi \in \mathfrak{g}$,*

$$U_t \bar{\xi}_t = A_\xi^t U_t. \tag{7.3}$$

Proof of Theorem 7.2. Let $f \in C_c^\infty(G)$. Because $\tilde{\xi}$ commutes with $e^{t\Delta/2}$ and analytic continuation, we see that

$$e^{t\Delta/2}\bar{\xi}_t f = e^{t\Delta/2}\tilde{\xi} f = \tilde{\xi} e^{t\Delta/2} f.$$

By the definition of $\check{\xi}_t$ and the isometry property of $e^{t\Delta/2}$, the above equation implies that $e^{t\Delta/2} f \in \mathscr{D}(\check{\xi}_t)$ and

$$\check{\xi}_t\, e^{t\Delta/2} f = e^{t\Delta/2}\bar{\xi}_t f \qquad \forall f \in C_c^\infty(G). \tag{7.4}$$

Since $C_c^\infty(G)$ is a core for $\bar{\xi}_t$, $e^{t\Delta/2}$ is an isometry, and $\check{\xi}_t$ is a closed operator, from Eq. (7.4) we may conclude that

$$e^{t\Delta/2}\bar{\xi}_t \subset \check{\xi}_t\, e^{t\Delta/2}. \tag{7.5}$$

In Corollary 7.17 below, it will be shown that the inclusion in (7.5) is an equality. Q.E.D.

The remainder of this paper will be devoted to finishing the proof Theorem 7.2. The main technical difficulties arise when the Lie group G is not compact or $\mathbb{R}^d$. There are also extra complications arising from the case where the inner product on $\mathfrak{g}$ is such that $\mathfrak{k} \times \{0\}$ is not perpendicular to $\{0\} \times \mathbb{R}^d$. So before ending this section, I will sketch a relatively easy proof of Corollary 7.3 in the case that $G = K$ is compact.

Proof of Corollary 7.3 *for G Compact.* For notational simplicity, we will prove Corollary 7.3 with $t = 1$. Let $\xi \in \mathfrak{g}$ be fixed. Define a one parameter strongly continuous group $\{T_t\}_{t \in \mathbb{R}}$ on $L^2(G, p_1)$ by:

$$(T_t f)(g) \doteq f(ge^{t\xi}),\ g \in G.$$

Let $\bar{\xi}$ denote the generator of T_t and notice that $\bar{\xi}|_{C^\infty(G)} = \tilde{\xi}$. By standard regularization arguments, $C^\infty(G)$ is a core for $\bar{\xi}$, i.e., $\bar{\xi} = \bar{\xi}_1$. Let $R_t = U_1 T_t U_1^{-1} : {}^\perp J_1 \to {}^\perp J_1$. Then R_t is a strongly continuous semi-group on ${}^\perp J_1$. Let B_ξ denote the generator of R_t. The proof will be complete upon showing that $B_\xi = A_\xi^1$.

Let $\alpha \in {}^\perp J_1$, $f \doteq U_1^{-1}\alpha \in L^2(p_1)$, and $u \doteq e^{\Delta/2} f \in L^2(\mu_1) \cap \mathscr{H}(G^{\mathbb{C}})$. Then

$$R_t \alpha = U_1 T_t f = (1 - D)_e^{-1} e^{\Delta/2} T_t f = (1 - D)_e^{-1} T_t u,$$

where by abuse of notation, $(T_t u)(x) \doteq u(xe^{t\xi})$ for all $x \in G^{\mathbb{C}}$. Since u is analytic,

$$(T_t u)(x) = \sum_{n=0}^{\infty} t^n (\tilde{\xi}^n u)(x)/n!.$$

Hence

$$\begin{aligned}\langle R_t \alpha, \xi_1 \otimes \cdots \otimes \xi_n \rangle &= \sum_{n=0}^{\infty} \tilde{\xi}_1 \cdots \tilde{\xi}_n t^n \tilde{\xi}^n u/n!\,|_e \\ &= \left\langle \alpha, \sum_{n=0}^{\infty} \xi_1 \otimes \cdots \otimes \xi_n \otimes t^n \xi^n/n! \right\rangle \\ &= \langle \alpha, \xi_1 \otimes \cdots \otimes \xi_n \otimes \operatorname{Exp}(t\xi) \rangle \qquad (7.6)\end{aligned}$$

Suppose that $\alpha \in D(B_\xi)$. Differentiating the above equation at $t = 0$ gives:

$$\langle B_\xi \alpha, \xi_1 \otimes \cdots \otimes \xi_n \rangle = \langle \alpha, \xi_1 \otimes \cdots \otimes \xi_n \otimes \xi \rangle = \langle A_\xi \alpha, \xi_1 \otimes \cdots \otimes \xi_n \rangle.$$

Since $\{\xi_i\} \subset \mathfrak{g}$ are arbitrary in this last equation; $A_\xi \alpha = B_\xi \alpha \in {}^\perp J_1$. Hence $\alpha \in \mathscr{D}(A_\xi^1)$ and $B_\xi \subset A_\xi^1$.

By Eq. (7.6) and the fundamental theorem of calculus, for all $\alpha \in {}^\perp J_1$

$$\begin{aligned}&\langle R_t \alpha, \xi_1 \otimes \cdots \otimes \xi_n \rangle \\ &\quad = \langle \alpha, \xi_1 \otimes \cdots \otimes \xi_n \rangle + \int_0^t \langle R_\tau A_\xi \alpha, \xi_1 \otimes \cdots \otimes \xi_n \rangle \, d\tau.\end{aligned}$$

For $\alpha \in \mathscr{D}(A_\xi^1)$, this equation and the strong continuity of R_t implies that

$$R_t\alpha = \alpha + \int_0^t R_\tau A_\xi \alpha \, d\tau \tag{7.7}$$

holds in ${}^\perp J_1$. Differentiating this equation at $t=0$ shows that $\alpha \in \mathscr{D}(B_\xi)$ and that $B_\xi\alpha = A_\xi\alpha$. That is $A_\xi^1 \subset B_\xi$. Q.E.D.

7.2. *The Homogeneous Chaos Expansion*

In this section, we may (by Corollary 2.2) and do assume that $G = K \times \mathbb{R}^d$ where K is compact. Also let $\hat{K}$ denote a collection of finite dimensional irreducible representations of K such that any finite dimensional irreducible representation of K is equivalent to exactly one elementl of $\hat{K}$. Our immediate goal is to develop the "Homogeneous Chaos" expansion for $L^2(p_t)$. As in Lemma 4.7 let

$$\hat{p}_t(x) \doteq \int_K p_t(k, x) \, dk = (2\pi t)^{-d/2} \sqrt{\det q} \, e^{-x \cdot qx/2t},$$

where q is the positive definite $d \times d$ matrix determined by Eq. (4.16). Let $\{u_i\}_{i=1}^d$ be a basis for $\mathbb{R}^d$ such that $u_i \cdot qu_j = \delta_{ij}$ and define the q-Laplacian on $\mathbb{R}^d$ by:

$$\Delta_q \doteq \sum_{i=1}^d \tilde{u}_i^2 = \sum_{i,j=1}^d q_{ij}^{-1} \, \partial^2/\partial x^i \, \partial x^j,$$

where $(x^1, ..., x^d)$ are the standard coordinates on $\mathbb{R}^d$. Notice that $\hat{p}_t$ is the heat kernel associated to Δ_q.

Definition 7.4. Given $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ let

$$h_\alpha(x) \doteq \prod_{i=1}^d (qx \cdot u_i)^{\alpha_i}, \tag{7.8}$$

and

$$H_\alpha(x) \doteq e^{-t\Delta_q/2} h_\alpha(x) \doteq \sum_{m=0}^\infty \frac{(-t)^m}{2^n m!} (\Delta_q^n h_\alpha)(x). \tag{7.9}$$

Remark 7.5. The funtion H_α is the "Wick ordering" of the monomial h_α, and is a Hermite polynomial up to a normalization constant. Also notice that

$$H_\alpha(x) = h_\alpha(x) + (\text{lower order terms}). \tag{7.10}$$

DEFINITION 7.6. To each finite dimensional unitary representation (π, V_π) of K and $n \in \mathbb{N}$, let

$$\mathscr{H}_{\pi,n} \doteq \operatorname{span}\{f^\pi_{A,\alpha} \mid A \in \operatorname{End}(V_\pi),\ |\alpha| = n\}, \tag{7.11}$$

where

$$f^\pi_{A,\alpha}(k, x) \doteq \operatorname{Tr}(A\pi(k))\, H_\alpha(x), \tag{7.12}$$

$\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$, and $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

For $a \in \mathbb{R}^d$, $\eta \in \mathfrak{k} \equiv \operatorname{Lie}(K)$ and $f \in C^\infty(G)$, let

$$(\tilde{a}f)(k, x) \doteq df(k, x + ta)/dt\,|_0$$

and

$$(\tilde{\eta}f)(k, x) \doteq df(ke^{t\eta}, x)/dt\,|_0,$$

i.e. $\tilde{a}$ and $\tilde{\eta}$ denote the left invariant vector-fields on $G = K \times \mathbb{R}^d$ such that $\tilde{a}|_{(e,0)} = (0, a)$ and $\tilde{\eta}|_{(e,0)} = (\eta, 0)$.

Remark 7.7. Notice that $\tilde{a}(\mathscr{H}_{\pi,n}) \subset \mathscr{H}_{\pi,n-1}$ and $\tilde{\eta}(\mathscr{H}_{\pi,n}) \subset \mathscr{H}_{\pi,n}$. Since Δ_q commutes with $\tilde{\eta}$ and $\tilde{a}$,

$$(\tilde{\eta}f^\pi_{A,\alpha})(k, x) = f^\pi_{A\pi(\eta),\alpha}, \tag{7.13}$$

and

$$(\tilde{a}f^\pi_{A,\alpha})(k, x) = \sum_{i=1}^{d} (u_i \cdot qa)\, \alpha_i f^\pi_{A,\alpha - e_i}, \tag{7.14}$$

where $\pi(\eta) = d\pi(e^{t\eta})/dt\,|_{t=0}$ and e_i denotes the i'th standard basis vector for $\mathbb{R}^d$.

THEOREM 7.8 (Homogeneous Chaos Expansion). *$L^2(\hat{p}_t) \equiv L^2(K \times \mathbb{R}^d$, $dk\, \hat{p}_t(x)\, dx)$ is the orthogonal direct sum of the subspaces $\{\mathscr{H}_{\pi,n}\}_{\pi \in \hat{K}, n \in \mathbb{N}}$. Define $\mathscr{H}^n$ to be the* algebraic *direct sum of $\{\mathscr{H}_{\pi,n}\}_{\pi \in \hat{K}}$. Then for each $a \in \mathbb{R}^d$, the operator $\tilde{a}|_{\mathscr{H}^n} : \mathscr{H}^n \to \mathscr{H}^{n-1}$ is a bounded operator with operator norm equal to $\sqrt{(a \cdot qa)n/t}$.*

Proof. Let $\pi, \tau \in \hat{K}$, $A \in \operatorname{End}(V_\pi)$, $B \in \operatorname{End}(V_\tau)$, and $\alpha, \beta \in \mathbb{N}^d$. Letting $d_\pi \doteq \dim(V_\pi)$, it follows from the Schur orthogonality relations that

$$\begin{aligned}(f^\pi_{A,\alpha}, f^\tau_{B,\beta})_{L^2(\hat{p}_t)} &= \int_K \operatorname{Tr}(A\pi(k))\, \overline{\operatorname{Tr}(B\tau(k))}\, dk \cdot \int_{\mathbb{R}^d} H_\alpha(x)\, H_\beta(x)\, \hat{p}_t(x)\, dx \\ &= d_\pi\, \delta_{\pi,\tau} \operatorname{Tr}(AB^*) \cdot \int_{\mathbb{R}^d} H_\alpha(x)\, H_\beta(x)\, \hat{p}_t(x)\, dx.\end{aligned}$$

Using Theorem 4.1 applied to $L^2(\mathbb{R}^d, \hat{p}_t(x)\,dx)$ and some calculus and combinatorics, one shows that

$$\begin{aligned}\int_{\mathbb{R}^d} & H_\alpha(x)\, H_\beta(x)\, \hat{p}_t(x)\, dx \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{i_1, \ldots, i_n = 1}^{d} (\tilde{u}_{i_1} \cdots \tilde{u}_{i_n} h_\alpha)(0) \cdot (\tilde{u}_{i_1} \cdots \tilde{u}_{i_n} h_\beta)(0) \\ &= \delta_{\alpha,\beta}\, t^{|\alpha|} \alpha!, \end{aligned} \tag{7.15}$$

where $\alpha! \doteq \prod_{i=1}^{d} \alpha_i!$. Combining the above two displayed equations gives:

$$(f^\pi_{A,\alpha}, f^\pi_{B,\beta})_{L^2(\hat{p}_t)} = \alpha!\, d_\pi t^{|\alpha|}\, \delta_{\pi,\tau}\, \delta_{\alpha,\beta} \operatorname{Tr}(AB^*). \tag{7.16}$$

This last equation clearly suffices to prove the pairwise $L^2(\hat{p}_t)$-orthogonality of the subspaces $\{\mathscr{H}_{\pi,n}\}$. The assertion that

$$L^2(\hat{p}_t) = \bigoplus_{\pi \in \hat{K},\, n \in \mathbb{N}} \mathscr{H}_{\pi,n} \quad \text{(Hilbert space direct sum)}$$

now follows from the Peter–Weyl theorem and the well known fact that polynomials are dense in $L^2(\mathbb{R}^d, \hat{p}_t(x)\,dx)$.

Now let $f \in \mathscr{H}^n$, then

$$f(k, x) = \sum_{|\alpha| = n} f_\alpha(k)\, H_\alpha(x)$$

where $f_\alpha \in C^\infty(K)$. Using (7.14) and (7.15), one shows:

$$\|f\|^2_{L^2(\hat{p}_t)} = t^n \sum_{|\alpha| = n} \alpha!\, \|f_\alpha\|^2_{L^2(dk)}$$

and

$$\begin{aligned}\|\tilde{a}f\|^2_{L^2(\hat{p}_t)} &= t^{n-1} \left\| \sum_{|\alpha| = n} \sum_{i=1}^{d} (a \cdot qu_i)\, \alpha_i f_\alpha H_{\alpha - e_i} \right\|^2_{L^2(\hat{p}_t)} \\ &\leqslant t^{n-1} \left[\sum_{i=1}^{d} |(a \cdot qu_i)| \cdot \left\| \sum_{|\alpha| = m} \alpha_i f_\alpha H_{\alpha - e_i} \right\|_{L^2(\hat{p}_t)} \right]^2 \\ &= t^{n-1} \sum_{i=1}^{d} |(a \cdot qu_i)|^2 \cdot \sum_{i=1}^{d} \left\| \sum_{|\alpha| = n} \alpha_i f_\alpha H_{\alpha - e_i} \right\|^2_{L^2(\hat{p}_t)}\end{aligned}$$

$$= t^{n-1}(a \cdot qa) \cdot \sum_{i=1}^{d} \sum_{|\alpha|=n} \alpha_i^2 (\alpha - e_i)! \, \|f_\alpha\|^2_{L^2(dk)}$$

$$= t^{n-1}(a \cdot qa) \cdot \sum_{i=1}^{d} \sum_{|\alpha|=n} \alpha_i \alpha! \, \|f_\alpha\|^2_{L^2(dk)}$$

$$= (a \cdot qa) \cdot nt^{-1} \, \|f\|^2_{L^2(\hat{p}_t)}.$$

This last inequality shows that the operator norm on $\mathscr{H}^n$ is less than or equal to $\sqrt{(a \cdot qa)n/t}$. By choosing $f(k, x) \doteq e^{-t\Delta/2}(qa \cdot x)^n \in \mathscr{H}^n$ one easily shows that this bound is sharp. Q.E.D.

Notation 7.9. Given $f \in L^2(\hat{p}_t) \equiv L^2(K \times \mathbb{R}^d, dk\, \hat{p}_t(x)\, dx)$, let $f_{\pi,n}$ denote the orthogonal projection of f onto $\mathscr{H}_{\pi,n}$ so that

$$f = \sum_{\pi,n} f_{\pi,n}.$$

7.3. *Explicit Description of* $\bar{\xi}_t$

The following theorem plays a role analogous to Lemma 4.3 in Gross [12]. The technique of proof is also very similar.

THEOREM 7.10. *Let* $\mathscr{D}$ *denote the* algebraic *direct sum of the subspaces* $\{\mathscr{H}_{\pi,n}\}_{\pi \in \hat{K}, n \in \mathbb{N}}$, *and set* $\|\cdot\| \doteq \|\cdot\|_{L^2(\hat{p}_t)}$. *For* $\xi = (\eta, a) \in \mathfrak{g} = \mathfrak{k} \times \mathbb{R}^d$, *the operator* $\xi|_{\mathscr{D}} = (\tilde{\eta} + \tilde{a})|_{\mathscr{D}}$ *is closable in* $L^2(\hat{p}_t)$ *and hence also in* $L^2(p_t)$. *Let* $\bar{\xi}_t$ *denote the* $L^2(p_t) = L^2(\hat{p}_t)$*-closure of* $\tilde{\xi}|_{\mathscr{D}}$. (*Item* 2 *below asserts this definition of* $\bar{\xi}_t$ *agrees with that given in Definition* 1.12.) *Then*:

1. *The operator* $\bar{\xi}_t$ *is given explicitly as*:

$$\mathscr{D}(\bar{\xi}_t) = \left\{ f = \sum_{\pi,n} f_{\pi,n} \in L^2(\hat{p}_t) \,\middle|\, \sum_{\pi,n} \|\tilde{\eta} f_{\pi,n} + \tilde{a} f_{\pi,n+1}\|^2_{L^2(\hat{p}_t)} < \infty \right\} \tag{7.17}$$

and for $f = \sum_{\pi,n} f_{\pi,n} \in \mathscr{D}(\bar{\xi}_t)$,

$$\bar{\xi}_t f = \sum_{\pi,n} (\tilde{\eta} f_{\pi,n} + \tilde{a} f_{\pi,n+1}). \tag{7.18}$$

2. $C_c^\infty(G) \subset \mathscr{D}(\bar{\xi}_t)$ *and* $C_c^\infty(G)$ *is a core for* $\bar{\xi}_t$.

Proof. Let

$$\xi^\dagger = (-\tilde{\eta} - \tilde{a} + M_a)|_{\mathscr{D}},$$

where M_a denotes multiplication by the function $(k, x) \to qa \cdot x$. Elementary integration by parts shows for all $h, f \in \mathscr{D}$ that $(\tilde{\eta} f, h) = (f, -\tilde{\eta} h)$, $(\tilde{a} f, h) = (f, (-\tilde{a} + M_a) h)$, and hence $(\tilde{\xi} f, h) = (\xi^\dagger f, h)$, where $(\cdot, \cdot) \doteq (\cdot, \cdot)_{L^2(\hat{p}_t)}$.

For the purposes of this proof let $\bar{\xi}_t$ be the linear operator determined by equations (7.17) and (7.18). Then for $h \in \mathscr{D}$ and $f \in \mathscr{D}(\bar{\xi}_t)$,

$$\begin{aligned}(\bar{\xi}_t f, h) &= \sum_{\pi,n} (\tilde{\eta} f_{\pi,n} + \tilde{a} f_{\pi,n+1}, h_{\pi,n}) \qquad \text{(finite sum)} \\ &= \sum_{\pi,n} \{(f_{\pi,n}, -\tilde{\eta} h_{\pi,n}) + (f_{\pi,n+1}, (-\tilde{a} + M_a) h^{\pi,n})\} \\ &= \sum_{\pi,n} (f_{\pi,n}, -\tilde{\eta} h_{\pi,n} + (-\tilde{a} + M_a) h_{\pi,n-1}) \qquad (h_{\pi,-1} \doteq 0) \\ &= \sum_{\pi,n} (f, -\tilde{\eta} h_{\pi,n} + (-\tilde{a} + M_a) h_{\pi,n-1}) \\ &= (f, (-\tilde{\eta} - \tilde{a} + M_a) h) = (f, \xi^\dagger h).\end{aligned}$$

Therefore $\bar{\xi}_t \subset (\xi^\dagger)^*$.

Claim. $\bar{\xi}_t = (\xi^\dagger)^*$.

To prove the claim let $f \in \mathscr{D}((\xi^\dagger)^*)$ and $(\xi^\dagger)^* f = h$, so that

$$(h, r) = (f, \xi^\dagger r), \qquad \forall r \in \mathscr{D} = \mathscr{D}(\xi^\dagger). \tag{7.19}$$

A similar computation to the one above shows (7.19) is equivalent to

$$\sum_{\pi,n} (h_{\pi,n}, r_{\pi,n}) = \sum_{\pi,n} (-\tilde{\eta} f_{\pi,n} - \tilde{a} f_{\pi,n+1}, r_{\pi,n}). \qquad \text{(finite sum)} \tag{7.20}$$

From this equation it follows that $-\tilde{\eta} f_{\pi,n} - \tilde{a} f_{\pi,n+1} = h_{\pi,n}$ for all $\pi \in \hat{K}$ and $n \in \mathbb{N}$. Since $h \in L^2(\hat{p}_t)$, we see that $f \in \mathscr{D}(\bar{\xi}_t)$ and that hence $\bar{\xi}_t = (\xi^\dagger)^*$. This proves the claim.

Because of the above claim, it follows that $\bar{\xi}_t$ is a closed operator. (This could have also been seen directly.) In order to finish the proof of the Lemma it suffices to show that $C_c^\infty(G) \subset \mathscr{D}(\bar{\xi}_t)$ and that both $C_c^\infty(G)$ and $\mathscr{D}$ are cores for $\bar{\xi}_t$. Let us first show that $\mathscr{D}$ is a core for $\bar{\xi}_t$.

For each $f \in \mathscr{D}(\bar{\xi}_t)$, $\Lambda \Subset \hat{K}$ (i.e., $\Lambda \subset \hat{K}$ and Λ is a finite set), and $N \in \mathbb{N} \setminus \{0\}$, set

$$f^{\Lambda,N} \doteq \sum_{\pi \in \Lambda} \sum_{n=0}^{N-1} f_{\pi,n}. \tag{7.21}$$

The $f^{\Lambda,N} \to f$ as Λ increases to $\hat{K}$ and $N \to \infty$. We also have

$$\bar{\xi}_t(f - f^{\Lambda,N}) = \sum_{\{\pi \notin \Lambda \text{ or } n \geq N\}} (\tilde{\eta} f_{\pi,n} + \tilde{a} f_{\pi,n+1}) + \sum_{\pi \in \Lambda} \tilde{a} f_{\pi,N}. \tag{7.22}$$

Because $f \in \mathscr{D}(\bar{\xi}_t)$, the first sum on the right of Eq. (7.22) converges to zero as Λ increases to $\hat{K}$ and $N \to \infty$. So to finish the proof that $\mathscr{D}$ is a core it suffices to show

$$\forall \Lambda \Subset \hat{K}, \liminf_{N \to \infty} \left\| \sum_{\pi \in \Lambda} \tilde{a} f_{\pi,N} \right\| = 0. \tag{7.23}$$

For sake of contradiction suppose that (7.23) does not hold. Then there exists $\varepsilon > 0$, $\Lambda \Subset \hat{K}$, and $N_0 \in \mathbb{N}$ such that

$$\left\| \tilde{a} \sum_{\pi \in \Lambda} f_{\pi,N} \right\|^2 = \left\| \sum_{\pi \in \Lambda} \tilde{a} f_{\pi,N} \right\|^2 \geqslant \varepsilon, \qquad \forall N \geqslant N_0.$$

But by Theorem 7.8 there exists $C > 0$ such that

$$\left\| \sum_{\pi \in \Lambda} \tilde{a} f_{\pi,N} \right\|^2 \leqslant CN \left\| \sum_{\pi \in \Lambda} f_{\pi,N} \right\|^2 = CN \sum_{\pi \in \Lambda} \|f_{\pi,N}\|^2.$$

From the two above displayed equations it follows that

$$\sum_{N \geqslant N_0} \sum_{\pi \in \Lambda} \|f_{\pi,N}\|^2 \geqslant \sum_{N \geqslant N_0} \varepsilon/CN = \infty,$$

which contradicts the assumption that $f \in L^2(\hat{p}_t)$. Thus (7.23) is valid and $\mathscr{D}$ is a core for $\bar{\xi}_t$.

To show $C_c^\infty(G) \subset \mathscr{D}(\bar{\xi}_t)$, it suffices to notice by integration by parts that $(f, \xi^\dagger h) = (\tilde{\xi} f, h)$ for all $f \in C_c^\infty(G)$ and $h \in \mathscr{D} = \mathscr{D}(\xi^\dagger)$. This shows that $f \in \mathscr{D}((\xi^\dagger)^*) = \mathscr{D}(\bar{\xi}_t)$ and that $\bar{\xi}_t f = \tilde{\xi} f$, i.e., $\tilde{\xi}|_{C_c^\infty(G)} \subset \bar{\xi}_t$.

To see that $C_c^\infty(G)$ is a core for $\bar{\xi}_t$, we will show that any $f \in \mathscr{D}$ may be approximated in the $\bar{\xi}_t$-graph norm by functions in $C_c^\infty(G)$. To this end, let $\phi : \mathbb{R} \to [0, 1]$ be a smooth function such that $\phi(s) = 1$ if $|s| \leqslant 1$ and $\phi(s) = 0$ if $|s| \geqslant 2$. Set $f^l(k, x) \doteq \phi(|x|/l) f(k, x) \in C_c^\infty(G)$, then it is routine to show that $f^l \to f$ in the $\bar{\xi}_t$-graph norm. Q.E.D.

7.4. *Explicit Description of* $\check{\xi}_t$

DEFINITION 7.11. For $\pi \in \hat{K}$ and $n \in \mathbb{N}$, let

$$\tilde{\mathscr{H}}_{\pi,n} = e^{t\Delta/2} \mathscr{H}_{\pi,n}. \tag{7.24}$$

The next lemma examines the structure of functions in $\tilde{\mathscr{H}}_{\pi,n}$.

LEMMA 7.12. *Let* $f(k, z) \doteq \operatorname{Tr}(\pi(k) A)\, p(z)$, *where* $\pi \in \hat{K}$, $A \in \operatorname{End}(V_\pi)$, *and* p *is a polynomial on* $\mathbb{R}^d$. (*The holomorphic extensions of* π *and* p *to* $K^{\mathbb{C}}$

and $\mathbb{C}^d$ *respectively will continue to be denoted by* π *and* p.) *Then there exists* $C_\pi(t) > 0$ *and* $\rho_\pi \in \mathbb{R}^d$ *such that on* $G^{\mathbb{C}} = K^{\mathbb{C}} \times \mathbb{C}^d$,

$$(e^{t\Delta/2} e^{-t\Delta_q/2} f)(g, z) = C_\pi(t)\, \mathrm{Tr}(\pi(g) A)\, p(z + t\sqrt{-1}\rho_\pi). \quad (7.25)$$

In particular, the general element $F \in \tilde{\mathscr{H}}_{\pi,n}$ *is of the form*:

$$F(g, z) = \sum_{|\alpha| = n} \mathrm{Tr}(\pi(g) A_\alpha)\, h_\alpha(z + t\sqrt{-1}\rho_\pi). \quad (7.26)$$

Remark 7.13. As will be seen by the proof, if $\mathfrak{k} \times \{0\}$ and $\{0\} \times \mathbb{R}^d$ are $(\cdot, \cdot)_{\mathfrak{g}}$-orthogonal then $\rho_\pi \equiv 0$ for all $\pi \in \hat{K}$.

Proof. Let $\{\xi_i\}_{i=1}^{\dim \mathfrak{k}}$ be a basis for $\mathfrak{k}$ such that $\{(\xi_i, 0)\}_{i=1}^{\dim \mathfrak{k}}$ is a $(\cdot, \cdot)_{\mathfrak{g}}$-orthonormal set. Let $\{u_i\}_{i=1}^d$ be a basis for $\mathbb{R}^d$ such that $u_i \cdot q u_j = \delta_{ij}$, or equivalently $\{P(0, u_i)\}_{i=1}^d$ is a $(\cdot, \cdot)_{\mathfrak{g}}$-orthonormal set, where P is orthogonal projection of $\mathfrak{g}$ onto the orthogonal compliment of $\mathfrak{k} \times \{0\}$ in $\mathfrak{g}$. Let $\gamma_i \in \mathfrak{k}$ be defined by $P(0, u_i) = (\gamma_i, u_i)$ for $i = 1, ..., d$. From the proof of Lemma 4.7 or using Theorem 2.1 it is not difficult to check that γ_i is in the center of $\mathfrak{k}$. (Recall that K is isomorphic to $(K' \times \mathbb{R}^k/\mathbb{Z}^k)/D$, where D is a discrete subgroup in the center of $(K' \times \mathbb{R}^k/\mathbb{Z}^k)$. The γ_i "live" in the Lie algebra of $\mathbb{R}^k/\mathbb{Z}^k$.) Since (π, V_π) is irreducible and unitary,

$$\pi(\gamma_i) \doteq \frac{d}{dt}\bigg|_0 \pi(e^{t\gamma_i}) = \sqrt{-1}\rho_i\, Id,$$

for some $\rho_i \in \mathbb{R}$. Define

$$\rho_\pi \doteq \sum_{i=1}^d \rho_i u_i \in \mathbb{R}^d.$$

Let $\Delta_{\mathfrak{k}} \doteq \sum_{i=1}^{\dim \mathfrak{k}} \tilde{\xi}_i^2$, then

$$\Delta = \sum_{j=1}^{\dim \mathfrak{k}} \tilde{\xi}_j^2 + \sum_{i=1}^d (\tilde{u}_i + \tilde{\gamma}_i)^2 = \Delta_{\mathfrak{k}} + \Delta_q + \sum_{i=1}^d \tilde{\gamma}_i^2 + 2 \sum_{i=1}^d \tilde{u}_i \tilde{\gamma}_i$$

and all terms on the far right side of the above equality commute. Notice that $\Delta_{\mathfrak{k}} + \sum_{i=1}^d \tilde{u}_i^2$ is in the center of the universal enveloping algebra of $\mathfrak{k}$, so that

$$\pi\left(\Delta_{\mathfrak{k}} + \sum_{i=1}^d \tilde{\gamma}_i^2\right) = K_\pi \cdot Id$$

where $K_\pi < 0$. Therefore

$$e^{t\Delta/2} e^{-t\Delta_q/2} = e^{t(\Delta - \Delta_q)/2} = e^{t(\Delta_{\mathfrak{k}} + \sum_{i=1}^d \tilde{u}_i^2)/2} e^{t \sum_{i=1}^d \tilde{u}_i \tilde{\gamma}_i}, \quad (7.27)$$

and hence

$$e^{t\Delta/2}\, e^{-t\Delta_{\mathscr{G}}/2} f(g, z) = C_\pi(t)\, e^{t\sum_{i=1}^d \tilde{u}_i \tilde{y}_i} \operatorname{Tr}(\pi(k) A)\, p(z), \tag{7.28}$$

where $C_\pi(t) \doteq e^{K_\pi t/2}$.

Using the above information,

$$\begin{aligned}
& e^{t\sum_{i=1}^d \tilde{u}_i \tilde{y}_i} \operatorname{Tr}(\pi(k) A)\, p(z) \\
&= \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{i_1, \ldots, i_m = 1}^{d} \tilde{y}_{i_1} \cdots \tilde{y}_{i_n} \operatorname{Tr}(\pi(k) A) \cdot \tilde{u}_{i_1} \cdots \tilde{u}_{i_n} p(z) \\
&= \operatorname{Tr}(\pi(k) A) \sum_{m=0}^{\infty} \frac{(\sqrt{-1}t)^m}{m!} \sum_{i_1, \ldots, i_m = 1}^{d} \rho_{i_1} \cdots \rho_{i_n} \tilde{u}_{i_1} \cdots \tilde{u}_{i_n} p(z) \\
&= \operatorname{Tr}(\pi(k) A) \sum_{m=0}^{\infty} \frac{(\sqrt{-1}t)^m}{m!} \tilde{\rho}_\pi^m p(z) \\
&= \operatorname{Tr}(\pi(k) A)\, p(z + \sqrt{-1}t\rho_\pi),
\end{aligned}$$

where the last equality is Taylor's Theorem. Q.E.D.

By Theorem 7.8 and Corollary 4.8,

$$L^2(\nu_t) \cap \mathscr{H}(G^{\mathbb{C}}) = \bigoplus_{\pi \in \hat{K},\, n \in \mathbb{N}} \tilde{\mathscr{H}}_{\pi,n}. \qquad \text{(Orthogonal Direct Sum).} \tag{7.29}$$

For $f \in L^2(\nu_t) \cap \mathscr{H}(G^{\mathbb{C}})$, let $f_{\pi,n}$ denote the orthogonal projection of f onto $\tilde{\mathscr{H}}_{\pi,n}$, so that

$$f = \sum_{\pi,n} f_{\pi,n} \qquad (\text{in } L^2(\nu_t) = L^2(\mu_t)) \tag{7.30}$$

The next lemma will be used shortly to relate L^2-notion of convergence of the above sum with the notion of uniform convergence on compact subsets of $G^{\mathbb{C}}$.

LEMMA 7.14. *Suppose that $F_{\pi,n} \in \tilde{\mathscr{H}}_{\pi,n}$ for all $\pi \in \hat{K}$ and $n \in \mathbb{N}$ and that $\sum_{\pi,n} F_{\pi,n}$ converges to 0 uniformly on compact subsets of $G^{\mathbb{C}}$. Then $F_{\pi,n} \equiv 0$ for all $\pi \in \hat{K}$ and $n \in \mathbb{N}$.*

Proof. By Lemma 7.12, there exists $A_\alpha^\pi \in \operatorname{End}(V_\pi)$ and $\rho_\pi \in \mathbb{R}^d$ such that

$$F_{\pi,n}(g, z) = \sum_{|\alpha| = n} \operatorname{Tr}(A_\alpha^\pi \pi(g))\, h_\alpha(z + \sqrt{-1}t\rho_\pi). \tag{7.31}$$

Hence, using the uniform convergence on compacts and the Schur orthogonality relations, for each $\tau \in \hat{K}$ and $B \in \mathrm{End}(V_\tau)$

$$
\begin{aligned}
0 &= \sum_{\pi, n} \int_K F_{\pi,n}(k, z)\, \overline{\mathrm{Tr}(B\tau(k))}\, dk \\
&= \sum_{n=0}^{\infty} \int_K F_{\tau,n}(k, z)\, \overline{\mathrm{Tr}(B\tau(k))}\, dk \\
&= d_\tau \sum_{n=0}^{\infty} \sum_{|\alpha| = n} \mathrm{Tr}(A_\alpha^\tau B^*)\, h_\alpha(z + \sqrt{-1}\, t\rho_\tau).
\end{aligned}
$$

The last sum converges uniformly for z in compact subsets of $\mathbb{C}^d$. Replacing z by $rz - \sqrt{-1}\, t\rho_\tau$ and using the homogeneity of h_α, it follows that

$$0 = \sum_{n=0}^{\infty} r^n \sum_{|\alpha| = n} \mathrm{Tr}(A_\alpha^\tau B^*)\, h_\alpha(z),$$

for all $r \in \mathbb{C}$ and $z \in \mathbb{C}^d$. It is now elementary to conclude that

$$\mathrm{Tr}(A_\alpha^\tau B^*) = 0 \ \forall \alpha \in \mathbb{N}^d.$$

Since $B \in \mathrm{End}(V_\tau)$ and $\tau \in \hat{K}$ is arbitrary, $A_\alpha^\tau = 0$ and hence $F_{\tau,n} = 0$ for all $\alpha \in \mathbb{N}^d$ and $\tau \in \hat{K}$. Q.E.D.

PROPOSITION 7.15. *Let $f \in \mathscr{H}(G^{\mathbb{C}})$ and $f_{\pi,n} \in \tilde{\mathscr{H}}_{\pi,n}$ for $\pi \in \hat{K}$ and $n \in \mathbb{N}$. Assume that*

$$f = \sum_{\pi, n} f_{\pi,n} \ (\textit{uniformly on compact subsets of } G^{\mathbb{C}}). \tag{7.32}$$

Then

$$\|f\|_{L^2(\nu_t)}^2 = \sum_{\pi, n} \|f_{\pi,n}\|_{L^2(\nu_t)}^2, \tag{7.33}$$

where this equation is to be interpreted as an equality on $[0, \infty]$. Moreover,, if either side of (7.33) is finite then the sum in (7.32) is also $L^2(\nu_t) = L^2(\mu_t)$-convergent.

Proof. By Fatou's Lemma and the $L^2(\nu_t)$-orthogonality of the terms in the sum in (7.32),

$$\|f\|_{L^2(\nu_t)}^2 \leqslant \sum_{\pi, n} \|f_{\pi,n}\|_{L^2(\nu_t)}^2. \tag{7.34}$$

Suppose that the right member of (7.34) is finite, and hence $f \in L^2(\nu_t)$. Let $F \doteq L^2 - \sum_{\pi,n} f_{\pi,n}$. By Lemma 3.2, there is a version of F which is holomorphic and for this version

$$F = \sum_{\pi,n} f_{\pi,n} \text{ (uniformly on compact subsets of } G).$$

Thus $F = f$, the sum in (7.32) is L^2-convergent, and (7.33) holds.

Conversely assume that $\|f\|^2_{L^2(\nu_t)} < \infty$, then

$$f = \sum_{\pi,n} \tilde{f}_{\pi,n} \text{ in } L^2(\nu_t) = L^2(\mu_t). \tag{7.35}$$

where $\tilde{f}_{\pi,n} \in \tilde{\mathscr{H}}_{\pi,n}$ for all $\pi \in \hat{K}$ and $n \in \mathbb{N}$. Again by Lemma 3.2, the sum in (7.35) is also uniformly convergent on compact subsets of $G^{\mathbb{C}}$. Let $F_{\pi,n} \doteq f_{\pi,n} - \tilde{f}_{\pi,n} \in \tilde{\mathscr{H}}_{\pi,n}$, then $\sum_{\pi,n} F_{\pi,n}$ converges to zero uniformly on compact subsets of $G^{\mathbb{C}}$. So by Lemma 7.14, $F_{\pi,n} \equiv 0$, i.e., $f_{\pi,n} = \tilde{f}_{\pi,n}$. Therefore the sum in (7.32) converges in $L^2(\nu_t) = L^2(\mu_t)$ and Eq. (7.33) holds.

Q.E.D.

THEOREM 7.16. *Let $\xi = (\eta, a) \in \mathfrak{g} = \mathfrak{k} \times \mathbb{R}^d$, and let $\tilde{\eta}$ and $\tilde{a}$ denote the left invariant vector fields on $G^{\mathbb{C}}$ which agree with $(\eta, 0)$ and $(0, a)$ at $e \in G^{\mathbb{C}}$. Then*

$$\mathscr{D}(\check{\xi}_t) = \left\{ f \in L^2(\nu_t) \cap \mathscr{H}(G^{\mathbb{C}}) \;\middle|\; \sum_{\pi,n} \|\tilde{\eta} f_{\pi,n} + \tilde{a} f_{\pi,n+1}\|^2_{L^2(\nu_t)} < \infty \right\}$$

and for $f \in \mathscr{D}(\check{\xi}_t)$,

$$\check{\xi}_t f = \sum_{\pi,n} (\tilde{\eta} f_{\pi,n} + \tilde{a} f_{\pi,n+1}).$$

Proof. Let $f \in L^2(\nu_t) \cap \mathscr{H}(G^{\mathbb{C}})$, then by Lemma 3.2 the $L^2(\nu_t)$-convergent sum

$$f = \sum_{\pi,n} f_{\pi,n}$$

is uniformly convergent on compact subsets of $G^{\mathbb{C}}$ and also $\sum_{\pi,n} df_{\pi,n}$ converges to df uniformly on compacts. In particular for all $f \in L^2(\nu_t) \cap \mathscr{H}(G^{\mathbb{C}})$,

$$\tilde{\xi} f = \sum_{\pi,n} \tilde{\xi} f_{\pi,n} = \sum_{\pi,n} (\tilde{\eta} f_{\pi,n} + \tilde{a} f_{\pi,n+1}), \tag{7.36}$$

with the sums converging uniformly on compacts. By Proposition 7.15, $\bar{\bar{\xi}} f \in L^2(v_t)$ iff $\sum_{\pi,n} \|\tilde{\eta} f_{\pi,n} + \tilde{a} f_{\pi,n+1}\|^2_{L^2(v_t)} < \infty$, i.e., iff $f \in \mathscr{D}(\check{\xi}_t)$. It also follows from Proposition 7.15 that

$$\check{\xi}_t f = \bar{\bar{\xi}} f = \sum_{\pi,n} (\tilde{\eta} f_{\pi,n} + \tilde{a} f_{\pi,n+1}),$$

where the sum is convergent in L^2. Q.E.D.

The following corollary is an easy consequence of the explicit description of $\bar{\xi}_t$ and $\check{\xi}_t$ in Theorems 7.10 and 7.16.

COROLLARY 7.17. *The domains of* $\bar{\xi}_t$ *and* $\check{\xi}_t$ *are related by*:

$$\mathscr{D}(\check{\xi}_t) = e^{t\Delta/2} \mathscr{D}(\bar{\xi}_t). \tag{7.37}$$

Proof. Let $f \in L^2(p_t) = L^2(\hat{p}_t)$. By Theorem 7.16, $e^{t\Delta/2} f \in \mathscr{D}(\check{\xi}_t)$ iff

$$\sum_{\pi,n} \|\tilde{\eta} e^{t\Delta/2} f_{\pi,n} + \tilde{a} e^{t\Delta/2} f_{\pi,n+1}\|^2_{L^2(v_t)} < \infty. \tag{7.38}$$

Notice that for all $\xi \in \mathfrak{g}$,

$$e^{t\Delta/2} \tilde{\xi} = \tilde{\xi} e^{t\Delta/2} \text{ on } \mathscr{H}_{\pi,n}, \forall \pi \in \hat{K}, n \in \mathbb{N},$$

where $\tilde{\xi}$ on the left (right) side of the equality is the left invariant vector field on G ($G^{\mathbb{C}}$) which agrees with ξ at the identity in G ($G^{\mathbb{C}}$). From this observation and isometry property of $e^{t\Delta/2}$ in Corollary 4.8, Eq. (7.38) is equivalent to

$$\sum_{\pi,n} \|\tilde{\eta} f_{\pi,n} + \tilde{a} f_{\pi,n+1}\|^2_{L^2(\hat{p}_t)} < \infty. \tag{7.39}$$

By Theorem 7.10, Eq. (7.39) is equivalent to the assertion that $f \in \mathscr{D}(\bar{\xi}_t)$. Q.E.D.

Remark 7.18. The interested reader may easily continue the above argument to give a direct proof of Theorem 7.2 without reference to Eq. (7.5).

8. APPENDIX ON METRICS

Suppose that G is an arbitrary Lie group and $(\cdot, \cdot)$ is a given inner product on $\mathfrak{g} = \text{Lie}(G) \doteq T_e G$. Let $(\cdot, \cdot)_L$ ($(\cdot, \cdot)_R$) be the unique left (right) invariant Riemannian metric on G which agrees with $(\cdot, \cdot)$ at $e \in G$.

DEFINITION 8.1. The left distance metric $d_L : G \times G \to G$ is defined by

$$d_L(g, h) = \inf \int_0^1 \sqrt{(\sigma'(s), \sigma'(s))_L}\, ds,$$

where the infimum is taken over all C^1-paths σ in G such that $\sigma(0) = g$ and $\sigma(1) = h$. The right distance metric d_R is defined similarly with $(\cdot, \cdot)_L$ replaced by $(\cdot, \cdot)_R$.

Notice that

$$d_L(xg, xh) = d_L(g, h)$$

for all $g, h, x \in G$, since if σ is a curve joining g to h, $x\sigma(\cdot)$ is a curve joining xg to xh which has the same length as σ. Set $|g| \equiv d_L(g, e) = d_L(e, g)$, then because of the above displayed equation,

$$d_L(g, h) = |g^{-1}h| = |h^{-1}g|.$$

Setting $h = e$ in this equation shows that $|g| = |g^{-1}|$ for all $g \in G$.

To relate d_L to d_R, let $\kappa(g) = g^{-1}$ and consider

$$\begin{aligned}(\kappa_* L_{g*} A, \kappa_* L_{g*} B)_R &= (R_{g^{-1}*} \kappa_* A, R_{g^{-1}*} \kappa_* B)_R \\ &= (-R_{g^{-1}*} A, -R_{g^{-1}*} B)_R = (A, B) \\ &= (L_{g*} A, L_{g*} B)_L.\end{aligned}$$

From this equation it follows that if σ is a curve joining g to h, then

$$\begin{aligned}\int_0^1 \sqrt{(\sigma'(s), \sigma'(s))_L}\, ds &= \int_0^1 \sqrt{(\kappa_* \sigma'(s), \kappa_* \sigma'(s))_R}\, ds \\ &= \int_0^1 \sqrt{(\tau'(s), \tau'(s))_R}\, ds,\end{aligned}$$

where $\tau \equiv \kappa \circ \sigma$. Notice that τ is now a curve in G joining g^{-1} to h^{-1}. Hence

$$d_L(g, h) = d_R(g^{-1}, h^{-1}).$$

Therefore,

$$|g| = |g^{-1}| = d_L(g^{-1}, e) = d_R(g, e)$$

and hence

$$d_R(g, h) = |gh^{-1}| = |hg^{-1}|.$$

The next proportion summarizes the observations of this appendix.

PROPOSITION 8.2. *Let* $|x| \doteq d_L(x, e)$ *for all* $x \in G$. *Then for all* $x, k, g \in G$:

1. $d_R(x, e) = |x|$,
2. $|x^{-1}| = |x|$,
3. $d_L(g, k) = |g^{-1}k|$,
4. $d_R(g, k) = |gk^{-1}|$.

ACKNOWLEDGMENTS

It is a pleasure to thank Leonard Gross, Hakön Gjessing, Bian Hall, and Nolan Wallach for useful discussions related to this work. I would also like to thank Omar Hijab for providing me with preprints of his papers. It should be noted by the reader that the proof of the isometry property in Theorem 1.14 (see Theorem 4.1) is related to the proof given in Hijab [16]. Finally, I would like to thank L. Gross for his suggestions which led to improvements in this manuscript.

REFERENCES

1. J. C. BAEZ, I. E. SEGAL, Z. F. ZHOU, "Introduction to Algebraic and Constructive Quantum Filed Theory," Princeton Univ. Press, Princeton, NJ, 1992.
2. V. BARGMANN, On a Hilbert space of analytic functions and an associated integral transform, I, *Comm. Pure Appl. Math.* **24** (1961), 187–214.
3. V. BARGMANN, Remarks on a Hilbert space of analytic functions, *Proc. Natl. Acad. Sci.* **48** (1962), 199–204.
4. V. BARGANN, Acknowledgement, *Proc. Natl. Acad. Sci.* **48** (1962), 2204.
5. T. BRÖCKER AND T. T. DIECK, "Representations of Compact Lie Groups," Springer-Verlag, Berlin/New York, 1985.
6. E. A. CARLEN, Some integral identities and inequalities for entire functions and their applications to the coherent state transform, *J. Funct. Anal.* **97** (1991), 231–149.
7. J. COOK, The mathematics of second quantization, *Trans. Amer. Math. Soc.* **74** (1953), 222–245.
8. B. K. DRIVER, Classifications of bundle connection pairs by parallel translation and lassos, *J. Funct. Anal.* **83** (1989), 185–231.
9. B. K. DRIVER, Loop space, curvature, and quasi-invariance, lecture notes from "Centro Vito Volterra for Lectures" given in the fall of 1992.
10. V. FOCK, Verallgemeinerung und Lösung der Diracschen statistischen Gleichung, *Z. Phys.* **49** (1928), 339–357.
11. L. GROSS, A Poincaré lemma for connection forms, *J. Funct. Anal.* **63** (1985), 1–46.
12. L. GROSS, Uniqueness of ground states for Schrödinger operators over loop groups, *J. Funct. Anal.* **112** (1993), 373–441.
13. L. GROSS, The homogeneous chaos over compact Lie groups, *in* "A Festschrift in Honour of Gopinath Kallianpur" (S. Cambanis *et al.*, Eds.), pp. 117–123, Springer-Verlag, New York, 1993.
14. L. GROSS, Harmonic analysis for the heat kernel measure on compact homogeneous spaces, *in* "Stochastic Analysis on Infinite Dimensional Spaces" (Kunita and Kuo, Eds.) pp. 99–110, Longman House, Essex, England, 1994.

15. B. HALL, The Segal–Bargmann "coherent state" transform for compact Lie groups, *J. Funct. Anal.* **122** (1994), 103–151.
16. O. HIJAB, Hermite functions on compact lie groups. I, *J. Funct. Anal.* **125** (1994), 480–492.
17. O. HIJAB, Hermite functions on compact lie groups, II, Temple Univ. preprint, 1994.
18. K. ITÔ, Multiple Wiener integral, *J. Math. Soc. Japan* **3** (1951), 157–169.
19. S. KAKUTANI, Determination of the spectrum of the flow of Brownian motion, *Proc. Natl. Acad. Sci. U.S.A.* **36** (1950), 319–323.
20. KOBAYASHI AND NOMIZU, "Foundations of Differntial Geometry, Vol. 1," Interscience/Wiley, New York/London, 1963.
21. KOBAYASHI AND NOMIZU, "Foundations of Differential Geometry," Vol. 2, Interscience/Wiley, New York/London/Sydney, 1969.
22. Y.-J. LEE, Transformation and Wiener–Itô decomposition of white noise functionals, *Bul. Inst. Math. Acad. Sinica* **21** (1993), 279–291.
23. E. NELSON, Analytic vectors, *Ann. of Math.* **70**, No. 3 (1959).
24. D. W. ROBINSON, "Elliptic Operators and Lie Groups," Clarendon Press, Oxford/New York/Tokyo, 1991.
25. I. E. SEGAL, Tensor algebras over Hilbert spaces, *Trans. Amer. Mathh. Soc.* **81** (1956), 106–134.
26. I. E. SEGAL, Distributions in Hilbert space and canonical systems of operators, *Trans. Amer. Math. Soc.* **88** (1958), 12–41.
27. I. E. SEGAL, Mathematical characterization of the physical vacuum for a linear Bose–Einstein field, *Illinois J. Math.* **6** (1962), 500–523.
28. I. E. SEGAL, Mathematical problems in relativitic quantum mechanics, *in* "Proc., AMS Summer Seminar on Applied Mathematics, Bounder, Colorado, 1960," Amer. Math. Soc., Providence, RI, 1963.
29. I. E. SEGAL, Construction of non-linear local quantum processes, I, *Ann. of Math.* **92** (1970), 462–481.
30. N. TH. VAROPOULOS, Small time Gaussian estimates of heat diffusion kernels. I. The semigroup technique, *Bull. Sci. Math. Sér. 2* **113** (1989), 253–277.
31. N. TH. VAROPOULOS, Small time Gaussian estimates of heat diffusion kernels. II. The theory of large deviations, *J. Funct. Anal.* **93** (1990), 1–33.
32. N. R. WALLACH, "Harmonic Analysis on Homogeneous Spaces," Decker, New York, 1973.
33. N. WIENER, The homogeneous chaos, *Amer. J. Math.* **60** (1938), 897–936.