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Classifications of Bundle Connection Pairs by Parallel Translation and Lassos*

BRUCE K. DRIVER

Department of Mathematics, C-012, University of California, San Diego, La Jolla, California 92093

Let M be a connected manifold and G be a closed Lie subgroup of $GL(V)$—the general linear group on a finite dimensional vector space V. Denote by (Ω_m) the space of H^1-loops in M starting at a fixed point (m). Let $\mathscr{M}$ be the set of $P \in C^\infty(\Omega_m, G)$, modulo conjugation by an element of G, such that $P(\sigma\tau) = P(\sigma)P(\tau)$ for $\sigma, \tau \in \Omega_m$ ($\sigma\tau$ is the concatenation of the loops σ and τ), $P(\sigma') = P(\sigma)$ if σ' is reparameterization of σ, and the differential of P satisfies a "locality" condition. It is shown that the bundle connection pairs (E, ∇) (up to equivalence), with structure group G and fiber V, are in one to one correspondence with $\mathscr{M}$—a similar result has been announced by Kobayashi. The correspondence is induced by the parallel translation operators of connections. Furthermore, if the manifold (M) is simply connected, then the space of bundle connection pairs can be classified by a collection of Lie algebra valued 1-forms on the manifold Ω_m (called integrated lassos). These 1-forms are related to the differentials of elements of $\mathscr{M}$. This last result generalizes Weil's characterization of $U(1)$—line bundle connection pairs by the curvature 2-form. It is also a generalization of Gross' results to base manifolds (M) other than $\mathbb{R}^n$. © 1989 Academic Press, Inc.

1. INTRODUCTION

Let F be a closed imaginary valued ($i\mathbb{R}$ is the Lie algebra of $U(1)$) 2-form on a connected manifold M. Set $\omega = (2\pi i)^{-1} F$, and denote by $[\omega]$, the De Rham cohomology class of ω. Then it is known [We] that F is the curvature 2-form for a $U(1)$-connection ∇ on some complex line bundle E over M if and only if $[\omega]$ is integral. (For a short review see Pressley and Segal [PrS] Section 4.5 and for a detailed treatment see Kostant [Ko]. That is $\int_C \omega$ should be an integer for all integral 2-chains (c). Alternatively $[\omega]$ should be in the image of the natural homomorphism $\iota\colon H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$, where $H^2(M, \mathbb{R})$ stands for any one of the equivalent cohomologies—Cech, De Rham, or simplicial—and $H^2(M, \mathbb{Z})$ is either Cech or simplicial. The theory of the first Chern class gives a 1–1 correspondence

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dence with the topological types of $U(1)$-line bundles E over M and elements of $H^2(M, \mathbb{Z})$. Let $\mathscr{E} = \mathscr{E}(M, \mathbb{C}, U(1))$ denote the space of $U(1)$-line bundle connection pairs (E, ∇) modulo equivalence (see Definition 5.1).

By the above comments the map $\rho: \mathscr{E} \to 2\pi i$ (integral closed 2-forms on M) defined by

$$\rho([E, \nabla]) \equiv F^{\nabla} \tag{1.1}$$

is onto (see Proposition 2.1.1 of [Ko]). In (1.1), $[E, \nabla]$ denotes the equivalence class containing (E, ∇) and F^{∇} is the curvature of the connection ∇. If M is simply connected, the map in (1.1) is also injective (Theorem 2.2.1 of [Ko]). On the other hand (Theorem 2.5.1 of [Ko]), if M is not simply connected, $\rho^{-1}(\{F\})$ is in one to one correspondence with $\Pi_1(M)^*$–the homomorphisms from $\Pi_1(M)$ to $U(1)$. In the physics literature, the property that ρ is not $1-1$ is called the gauge field copy problem.

In [Ko], there is another description of the space $\mathscr{E}$ in terms of the parallel translation operators on loops. Let $\Omega = \{\sigma \in PC^{\infty}([0, 1], M)\colon \sigma(0) = \sigma(1)\}$ be the space of piecewise smooth loops on M. Let P^{∇} be parallel translation on Ω with respect to the connection ∇. Since, $P^{\nabla}(\sigma)$ is an endomorphism of E_m (E_m = the fiber of E at m), which is a 1-dimensional complex vector space, $P^{\nabla}(\sigma)$ can be identified with a complex number. Proposition 1.12.3 of [Ko] (also see Theorem 5.1 below) states that two $U(1)$-bundles with connections, (E, ∇) and (E', ∇'), are equivalent if and only if $P^{\nabla} = P^{\nabla'}$ on Ω.

The purpose of this paper is to prove analogous statements for more general vector bundles. We are interested in considering G-vector bundles, where G is a closed subgroup of $GL(V)$ (see Section 2). The goal is to classify the space, $\mathscr{E} = \mathscr{E}(M, V, G)$, of G-vector bundle connection pairs (E, ∇) modulo equivalence.

The first question is what should play the role of the curvature 2-form when the structure group (G) is not $U(1)$. The obvious choice of using the curvature tensor as before is not satisfactory. The main reason being that the curvature tensor (F^{∇}) is a 2-form on M with values in $\mathrm{End}(E)$—the endomorphism bundle associated to E. So in order to specify an F, one has to first specify the bundle. Leaving this issue aside for a moment, the natural analogue of $dF = 0$ is the Bianci identity $D^{\nabla}F = 0$, where D^{∇} is the covariant differential associated to ∇. As discussed in Gross [G1], when G is not commutative, the equation $D^{\nabla}F = 0$ depends on the connection ∇ (which we are trying to classify) unlike the $U(1)$ case. A third drawback of the curvature is the "gauge copy problem." For example, Wu and Yang [WY] have shown when G is not commutative that the curvature F^{∇} is not sufficient to determine the connection ∇ up to the equivalence even when $M = \mathbb{R}^n$. (See also [MS] and its bibliography.) Nevertheless, K. Mackenzie [Ma] has announced a criterion for the existence of principal bundle connections with prescribed curvature form. This result is not in the spirit of this paper, since the formulation requires the bundle to be prescribed at the outset.

In [G1], Gross proposes to use "lassos" as an analogue for the curvature tensor. In order to define a lasso, let $\mathscr{P}_m$ be the space of H^1-paths on M starting at m (see Section 3). The lasso associated to a connection ∇ on a vector bundle E is $L^{\nabla}(\sigma)\langle u, v\rangle \equiv P^{\nabla}(\sigma)^{-1} F^{\nabla}\langle u, v\rangle P^{\nabla}(\sigma) \in \mathrm{End}(E_m)$, for $\sigma \in \mathscr{P}_m$ and $u, v \in T_{\sigma(1)}M$ (see Definition 4.3). It should be noted, if $G = U(1)$, that $L^{\nabla}(\sigma)\langle u, v\rangle = F^{\nabla}(\sigma(1))\langle u, v\rangle$ is essentially the curvature 2-form again. We refer the reader to the discussion in [G1] motivating the study of these objects as related to "quantized gauge fields." Such path dependent objects have also been discussed in the physics literature, see Birula [B] and Mandelstam [Man1–3].

Now choose a local trivialization (ψ) over a neighborhood of m, and use ψ to identify E_m with the model space V. Because of this identification we may consider $L^{\nabla}(\sigma)\langle u, v\rangle \in \mathscr{G} \equiv$ Lie Algebra of G which is a subspace of $\mathrm{End}(V)$. (L^{∇} is now only defined up to conjugation by an element of G.) A lasso is an example of a more general object called a path 2-form. A path 2-form is a (smooth) function (L) with values in $\mathscr{G}$, defined on triples $(\sigma, u, v) \in \mathscr{P}_m \times TM \times TM$ with $u, \in T_{\sigma(1)}M$—$L(\sigma)\langle u, v\rangle$ is assumed skew symmetric in u and v. The main theorems of [G1] give necessary and sufficient conditions for a path 2-form L to be a lasso in case $M = \mathbb{R}^n$. These conditions generalize the equation $dF = 0$ for $G = U(1)$. The conditions formulated in [G1] are intrinsic conditions, i.e., not requiring a vector bundle or a connection for their formulation. Furthermore, it is shown in [G1] that the lassos modulo conjugation by elements of the structure group G are in one to one correspondence with elements of $\mathscr{E}(\mathbb{R}^n, V, G)$.

In Theorem 6.1 and Theorem 6.2 of this paper we extend this result to general simply connected manifolds M. This requires an added condition on a path 2-form which is an analogue of the integrality condition for F if $G = U(1)$. In fact, for $U(1)$-bundles, the added condition reduces to the integrality condition, see Corollary 6.1.

In case where M is not simply connected, the lassos are no longer in one to one correspondence with $\mathscr{E}$. In close analogy to the $U(1)$ case, given an "irreducible" lasso (L), the set of pairs (E, ∇) such that $L^{\nabla} = L$ modulo conjugation by elements of G is in one to one correspondence with the set of homomorphisms of $\Pi_1(M)$ to the center of G, see Theorem 7.1. Therefore, lassos suffer from a gauge copy problem only in the case M is not simply connected, just as in the case $G = U(1)$.

Unfortunately if M is not simply connected, an intrinsic characterization of the lassos is still unknown. The criteria given in Theorem 6.1 for M

simply connected are not sufficient to guarantee that a path 2-form is a lasso on $\mathscr{P}_m$ (Example 7.1). However (Theorem 7.2 and Corollary 7.1), the criteria of Theorem 6.1 do imply that the path 2-form is the "pullback" of a lasso on the path space of the universal cover of M. The author feels the non-simply connected case may be a good application for the non-abelian cohomology theory in Decker [D1–2] and Brown [Br].

Despite the difficulty with the lassos for non-simply connected manifolds–in analogy with the $U(1)$ case–the space $\mathscr{E}(M, V, G)$ can always be classified by the parallel translation operators. In Theorem 5.1 (also see [Kob]), we show that $\mathscr{E}(M, V, G)$ is in one to one correspondence with a subset $\mathscr{M}$ of $C^\infty(\Omega_m, G)$ (see Definitions 5.2–5.4).

It should be remarked that by Theorem 4.1 (see also Theorem 2.2 and Corollary 2.16 of [G1]) the differential of the parallel translation operator P^∇ may be expressed in terms of the lasso L^∇. So Theorem 6.1 involving the lassos may be thought of as the infinitesimal version of Theorem 5.1.

This paper is divided into seven sections. In section 2, some basic definitions and notations are introduced. Section 3 is a review of some basic properties about the Hilbert manifold of H^1-paths ($\mathscr{P}$) on a manifold M. This section also contains some technical results which are needed for determining when a function, which has $\mathscr{P}$ as either the domain or the range space, is smooth. In Section 4, the parallel translation operator is shown to be smooth, and its differential is computed (Theorem 4.1) in terms of lassos and integrated lassos (see Definitions 4.2 and 4.3). Section 5 gives the loop characterization of $\mathscr{E}$, see Theorem 5.1. Section 6 deals with the lasso characterization of $\mathscr{E}$, see Theorems 6.1 and 6.2 and Corollary 6.1. Section 7 contains some remarks for non-simply connected manifolds. The results of Sections 5–7 have already been discussed in this introduction.

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2. Notation

For the purposes of this paper all manifolds will be C^∞; however, there will be numerous occasions for using functions of different degrees of smoothness. The following prefixs will be used to denote the smoothness of a particular function: C^r for r-continuous derivatives ($C = C^0$), AC for absolutely continuous, H^1 for absolutely continuous and the derivative in L^2, and PC^r for piecewise C^r.

Throughout this paper (E, V, π, M, G) or E for short will denote a vector bundle E over M with fiber V (V is a complex or real finite dimensional vector space), $\pi: E \to M$ is the projection map, and G is the structure group. The group G is assumed to be a closed Lie subgroup of $Gl(V)$, the

linear automorphisms of V. To say that E has structure group G means that there is a distinguished class of "admissible" local tivializations (ψ, U) of E covering M for which the transition functions are G-valued. (Here, U is an open subset of M and $(\pi, \psi): \pi^{-1}(U) \to U \times V$ is a diffeomorphism.) More explicitly, if (ψ, U) and (φ, W) are admissible, then there is a C^∞ function $g: U \cap W \to G$ such that $(\pi, \varphi) \circ (\pi, \varphi)^{-1}(m, \xi) = (m, g(m)\xi)$ for all $(m, \xi) \in (U \cap W) \times V$. (In the sequel, the phrase local trivialization will always mean an admissible local trivialization.) To simplify such statements, it is often convenient to write E_m for $\pi^{-1}(\{m\})$, and ψ_m for $\psi|_{E_m}$ if ψ is any function on E.

Given a covariant derivative (or connection) ∇ on E and a local trivialization (ψ, U), the associated connection one form (A^ψ) is defined by

$$A^\psi\langle v\rangle \xi = \nabla_v((m \to \psi_m^{-1}\xi): U \to \pi^{-1}(U)) \qquad \text{for} \quad v \in TU.$$

(As a general rule, an argument of a function which is enclosed by the brackets $\langle\cdot\rangle$ will indicate that the function is linear or fiber linear in this variable.) The terminology covariant derivative and connection will be used interchangebly in this paper. We will only consider connections on E compatible with the structure group G. This means for all admissible (ψ, U), A^ψ is a 1-form on TU taking values in the Lie algebra $\mathscr{G}$ of G, which may be considered to be a subspace of $\mathrm{End}(V)$.

Remark 2.1. For later purposes we note that A^ψ is related to A^φ by

$$\begin{aligned} A^\psi &= g^{-1}A^\varphi g + g^{-1}\,dg \\ &= g^{-1}A^\varphi g - d(g^{-1})\cdot g \qquad \text{on} \quad T(U \cap V), \end{aligned} \tag{2.1}$$

where (ψ, U) and (φ, V) are two local trivializations of E.

The curvature of a connection (∇) is $F^\nabla \in \Lambda^2(T^*M) \otimes \mathrm{End}(E)$ defined by

$$F^\nabla\langle X, Y\rangle = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \tag{2.2}$$

for X and Y vector fields of M. In terms of a local trivialization (ψ, U), the local expression for the curvature is

$$\psi_m^{-1} F^\nabla(m)\, \psi_m = (dA^\psi + A^\psi \wedge A^\psi)(m) \tag{2.3}$$

for $m \in U$, where the following notation is being used. If ω is a k-form, then $\omega(m)$ denotes restriction of ω to $\Lambda^k(T_mM)$. The term $A^\psi \wedge A^\psi$ in (2.3) denotes the 2-form with values in $\mathscr{G}$ given by

$$A^\psi \wedge A^\psi\langle u, v\rangle \equiv [A^\psi\langle u\rangle, A^\psi\langle v\rangle] \tag{2.4}$$

for $u, v \in T_mM$.

DEFINITION 2.1. Given a 1-form A on an open subset (U) of M taking values in $\mathscr{G}(A \in \Gamma(T^*U \otimes \mathscr{G}))$, the curvature (F^A) of A is

$$F^A \equiv dA + A \wedge A. \tag{2.5}$$

Of course in terms of (2.5) and (2.3) one may write $F^\nabla(m) = \psi_m F^{A^\psi}(m) \psi_m^{-1}$ for $m \in U$.

DEFINITION 2.2. If A is a $\mathscr{G}$-valued 1-form on an open set U of M, and $\sigma: [0, 1] \to U$ is a C^1-path, then parallel translation along σ with respect to A up to time s is the element $P_s^A(\sigma) \in G$, where $P^A(\sigma)$ satisfies

$$\frac{d}{ds} P_s^A(\sigma) + A\langle\sigma'(s)\rangle P_s^A(\sigma) = 0 \quad \text{and} \quad P_0^A(\sigma) = 1 \in G.$$

Remark 2.2. If $A = A^\psi$ is the connection 1-form of a covariant derivative (∇) with respect to a local trivialization (ψ, U), the parallel translation operator $P_s^\nabla(\sigma)$ can be expressed as

$$P_s^\nabla(\sigma) = \psi_{\sigma(s)}^{-1} P_s^A(\sigma) \psi_{\sigma(0)}, \tag{2.6}$$

for any C^1-path in U.

If $f: N \to M$ is a map, the notation $\Gamma_f(E)$ will be used to denote the C^∞-sections $S: N \to E$ along f. As indicated above $C\Gamma_f(E)$ would then denote the continuous sections along f, with analogous statements for different prefixs.

3. HILBERT MANIFOLDS OF BASED LOOPS

This section introduces notation and reviews some basic facts about certain submanifolds of the Hilbert manifold of H^1-curves on a manifold M. (For a thorough treatment of this material see [Kl1].)

Let $(V, (\cdot,\cdot))$ be a finite dimensional inner product space, and J be a subinterval of $I \equiv [0, 1]$. Put $H^1(J, V) = \{\sigma \in AC(J, V)$: and $\|\sigma\|_1 < \infty\}$ where $\|\sigma\|_1^2 = \int_J (|\sigma'(t)|^2 + |\sigma(t)|^2)\, dt - H^1(J, V)$ is an infinite dimensional Hilbert space. Let M be a fixed n-dimensional manifold. A path $\sigma: J \to M$ will be called H^1, if $x \circ \sigma|_K \in H^1(K, \mathbb{R}^n)$ for each subinterval $K \subset J$ and coordinate chart (x, U) of M for which $\sigma(K) \subset U$. (The notion of a curve being H^1 is chart independent.) The set of H^1 curves $\sigma: J \to M$ will be denoted by $H^1(J, M)$.

It is fairly standard that $H^1(J, M)$ is Hilbert manifold modeled on $H^1(J, \mathbb{R}^n)$. (The manifold structure will be described presently.) For the basic definitions and facts about infinite dimensional manifolds, see Lang [L] and Eells [E2]. For discussions on spaces of maps between manifolds as infinite dimensional manifolds, see [E1–2], [ES1–2], [E1], [P] and [PS] and especially [KL1–3].

In order to describe the manifold structure on $H^1(J, M)$, fix a Riemannian metric on M, which will be denoted by (g) or $\langle\cdot,\cdot\rangle$. Let exp denote the geodesic flow associated to the Levi-Civita connection of g, and $d(\cdot,\cdot)$ be the induced metric on M. It is easy to check that the space $H^1(J, M)$ may alternately be characterized as

$$H^1(J, M) = \{\sigma \in AC(J, M): \|\sigma'\|_0 < \infty\} \tag{3.1}$$

where

$$\|X\|_0^2 \equiv \int_I \langle X(t), X(t)\rangle\, dt, \tag{3.2}$$

for any Borel measurable $X: I \to TM$. Let D denote the covariant derivative on M of the Levi-Civita connection for g. For any $\sigma \in H^1(J, V)$, let σ^*D be the pullback of D to $AC\Gamma_\sigma(TM)$, the absolutely continuous sections of TM along σ. To be more explicit, suppose that Γ is the connection form for D with respect to some local trivialization of TM, and $X: I \to \mathbb{R}^n$ is the local expression of a section along σ, then

$$(\text{the local form of } \sigma^*DX)(t) \equiv X'(t) + \Gamma\langle\sigma'(t)\rangle X(t). \tag{3.3}$$

For any $\sigma \in H^1(J, M)$, let

$$T_\sigma H^1(J, M) = \{X \in AC\Gamma_\sigma(TM): \|X\|_1 < \infty\}, \tag{3.4}$$

where $\|X\|_1$ is defined as

$$\|X\|_1^2 \equiv \|X\|_0^2 + \|\sigma^*DX\|_0^2.$$

As the notation suggests this will be the tangent space to $H^1(J, M)$. In view of (3.3) $T_\sigma H^1(J, M)$ may alternately be described as the set of H^1 sections along σ, a notion independent of the metric g.

In order to describe the coordinate charts on $H^1(J, M)$, we recall a basic fact from Riemannian geometry.

THEOREM 3.1. *Let (M, g) be a Riemannian manifold and $K \subset M$ be a compact set, then there exists an $\varepsilon > 0$ such that for each point $p \in K$ the exponential map restricted to $\{X \in T_pM: \langle X, X\rangle < \varepsilon^2\}$ is a diffeomorphism onto the neighborhood $B_p(\varepsilon) = \{m \in M: d(m, p) < \varepsilon\}$ which is geodesically convex.* (*Recall that a neighborhood N is geodesically convex if for any pair* *of points $p, q \in N$ there exists a unique geodesic of minimal length joining p to q which lies entirely in N.) Furthermore, let $K_\varepsilon = \{p \in M: d(p, K) < \varepsilon\}$ ($d(p, K) = \inf\{d(p, k): k \in K\}$) and $U_\varepsilon = \{(p, q) \in K_\varepsilon \times K_\varepsilon: d(p, q) < \varepsilon\}$ (an open subset of $M \times M$), then there is a C^∞ map $v: U_\varepsilon \to TM$ such that $v(p, q) \in T_pM$ and $\exp_p v(p, q) = q$ (i.e., $v(p, q) = \exp_p^{-1}(q)$). As discussed in Section 2, $\exp_p$ denotes exp restricted to its domain intersected with T_pM.*

Proof. This is a slight generalization of the method outlined in Spivak [Sp, pg. 491, problem 32f] for the case where K is a point. Also see pages 32–36 of [H]. To generalize this to arbitrary K compact, it is enough to show that each point of M has a neighborhood for which the theorem holds. (This fact is theorem 1.9.10 of [KL1].) But for this case the proof outlined in Spivak still goes through, if one observes by smoothness that the key estimates hold uniformly in a neighborhood of a point. Q.E.D.

We are now in a position to describe the coordinate neighborhoods on $H^1(J, M)$. Let $\sigma \in C^1(J, M)$; choose $\varepsilon > 0$ as in Theorem 3.1 for the compact set $K =$ image of σ. Put $W(\sigma, \varepsilon) = \{\tau \in H^1(J, M): d(\tau(t), \sigma(t)) < \varepsilon$ for $t \in I\}$, then for each $\tau \in W(\sigma, \varepsilon)$ and $t \in I$ the point $(\sigma(t), \tau(t))$ is in U_ε with U_ε as above. Hence the map $V_\sigma: W(\sigma, \varepsilon) \to \tilde{W}(\sigma, \varepsilon) \equiv \{X \in T_\sigma H^1(J, M): \|X\|_\infty < \varepsilon\}$ given by $V_\sigma(\tau)(t) \equiv v(\sigma(t), \tau(t)) = \exp_{\sigma(t)}^{-1}(\tau(t))$ is well defined (by Lemma 3.1 below) and is bijective, where

$$\|X\|_\infty \equiv \sup\{(\langle X(t), X(t)\rangle)^{1/2}: t \in J\}.$$

THEOREM 3.2 (Restatement of 2.3.23 Theorem of [KL1]). *The collection of charts $\{(V_\sigma, W(\sigma, \varepsilon))\}_{\sigma \in C^1(J, M)}$, induces a C^∞ manifold structure on $H^1(J, M)$ which is modeled on any one of the equivalent Hilbert spaces $T_\sigma H^1(J, M)$ for any $\sigma \in C^1(J, M)$. In particular the charts $(V_\sigma, W(\sigma, \varepsilon))$ are C^∞ related.*

For a proof of this theorem we refer the reader to [KL1–3] or [E1].

Remark 3.1. The sets $W(\sigma, \varepsilon)$ do not form a basis for the topology on $H^1(J, M)$, but sets of the form $V_\sigma(\{X \in T_\sigma H^1(J, M): \|X\|_1 < \varepsilon\})$ do form a basis.

Remark 3.2. $T_\sigma H^1(J, M)$ is naturally isomorphic to $H^1(I, T_mM)$, where $m = \sigma(a)$, $J = [a, b]$, and the inner product on T_mM is the metric (g) restricted to T_mM. To demonstrate this let $q(t) =$ parallel translation in TM along σ with respect to the Levi Civita connection D. For $X \in H^1(I, T_mM)$, put $X^q(t) = q(t)X(t)$ so that $X^q \in T_\sigma H^1(J, M)$. Then $(\sigma^* DX^q)(t) = q(t)X'(t)$, hence $\|X^q\|_1 = \|X\|_1$ for all $X \in H^1(I, T_m)$ because $q(t)$ preserves the metric $g(\sigma(t))$ on $T_{\sigma(t)}M$. Thus $X \to X^q: H^1(I, T_mM) \to T_\sigma H^1(J, M)$ is a Hilbert space isomorphism.

For later purposes it is convenient to develop some techniques for showing that maps having $H^1(J, M)$ as the range or the domain space are smooth. The reader might wish to skip to Lemma 3.2 and only refer to these results when necessary. The next proposition is useful for computing derivatives of functions on H^1.

PROPOSITION 3.1. *Let $f: B \to K$ be a continuous map between two Banach spaces B and K with norms both denoted by $|\cdot|$. Let $D \subset B$ be a dense subspace and assume there exists a continuous function $F: B \to \mathrm{Hom}(B, K)$ such that for all d and d' in D, $(\partial_{d'} f)(d) \equiv (d/ds)|_0 f(d + sd')$ exists and is equal to $F(d)\langle d'\rangle$. Then f is in fact C^1 and the differential of f is F.*

Proof. By the fundamental theorem of calculus,

$$f(d + d') - f(d) = \int_0^1 (\partial_{d'} f)(d + td')\, dt = \int_0^1 F(d + td')\langle d'\rangle\, dt$$

for all d and d' in D. Both sides of this last equation are continuous function on all of $B \times B$, so in fact the last equation hold for all d and d' in B. So for d fixed,

$$\begin{aligned} &|f(d + d') - f(d) - F(d)\langle d'\rangle| \\ &\quad \leqslant \int_0^1 |F(d + td')\{d'\rangle - F(d)\langle d'\rangle| dt \\ &\quad \leqslant \sup\{|F(d + sd') - F(d)|: s \in I\} \cdot |d'| \\ &\quad = 0(|d'|)|d'|, \qquad \text{for} \quad d' \text{ in } B. \end{aligned}$$

This shows that f is differentiable with differential equal to F. Q.E.D.

Remark 3.3. It is clear that Proposition 3.1 is true if the domain of $f(D(f))$ is an open convex subset of B and D is a dense convex subset of $D(f)$.

LEMMA 3.1. *Suppose $X \in H^1(I, \mathbb{R}^d)$, and f is a $C^{1,\infty}$-function defined in a neighborhood (N) of the graph of X taking values in $\mathbb{R}^d$. (A function $f(t, x)$ is $C^{1,\infty}$ if all partial derivatives of f with respect to the x variables are jointly C^1 in (t, x).) Then for $\varepsilon > 0$ sufficiently small, the map $Y \to F(Y) \equiv f(\cdot, Y(\cdot)): B(X, \varepsilon) \to H^1(I, \mathbb{R}^d)$ is C^∞. ($B(X, \varepsilon)$ is the ε-ball about X inside $H^1(I, \mathbb{R}^d)$.)*

Proof. This is a special case of 1.2.5 Lemma of [KL3]. A direct proof is easily given using Proposition 3.1. The fact that F is continuous on $B(X, \varepsilon)$ is fairly straightforward by uniform continuity arguments. Let

$D = C^\infty B(X, \varepsilon) \equiv B(X, \varepsilon) \cap C^\infty(I, \mathbb{R}^d)$, then it is easy to compute $(\partial_Z F)(Y)(t) = D_2 f(t, Y(t))\langle Z(t)\rangle$ for all Y and Z in D, where $D_2 f(t,\cdot)$ is the differential of $f(t,\cdot)$. The mapping $(Y \to D_2 f(\cdot, Y(\cdot))\langle\cdot\rangle)$: $B(X,\varepsilon) \to \mathrm{End}(H_0^1(I, \mathbb{R}^d))$ is continuous for the same reasons f was continuous. Thus by Proposition 3.1, F is C^1. All higher derivatives may be computed in a similar fashion. Q.E.D.

PROPOSITION 3.2. *Let J be a subinterval of I. Let $r_J: H^1(I, M) \to H^1(J, M)$ be defined by $r_J(\sigma) \equiv \sigma|_J$. Then*

(a) *r_J is C^∞.*

Also let $f: N \to H^1(I, M)$, where N is a smooth manifold, then:

(b) *The map f is smooth if and only if there exists $J_i = [t_{i-1}, t_i]$ for $i = 0$ to n with $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $r_{J_i} \circ f: N \to H^1(J_i, M)$ is smooth.*

(c) *Analogous statements hold if $H^1(I, M)$ is replaced by the tangent space $TH^1(I, M)$.*

Proof. The fact that r_J is smooth follows directly upon localization to the case where M is $\mathbb{R}^n$ ($n = \dim(M)$). But in this case r_J is linear and continuous and hence C^∞.

If f is C^∞ then $r_{J_i} \circ f$ is C^∞, being the composition of two C^∞ maps. Conversely, let $n \in N$ and choose a coordinate chart $(V_\sigma, W(\sigma, \varepsilon))$ about $f(n) \in H^1(I, M)$. Then $(V_{\sigma_{J_i}}, W(\sigma_{J_i}, \varepsilon))$ is a coordinate neighborhood of $f(n)|_{J_i}$. By the use of these charts and parallel translation in TM along the curve σ as in Remark 3.2, one may assume that $M = \mathbb{R}^n$. In this case it is easy to show that the derivatives of f exist and are found by piecing together the derivatives for $f(\cdot)|_{J_i}$.

The last statement (c) has a proof similar to that of (b). Q.E.D.

PROPOSITION 3.3. *Let N be a smooth manifold, $f \in C(I \times N, M)$, and $\{J_i\}$ be a collection of subintervals of I as in Proposition 3.2. If $f|_{J_i \times N}$ is $C^{1,\infty}$ for each i, then $\bar{f}: N \to H^1(I, M)$ defined by $\bar{f}(n) = f(\cdot, n)$ is C^∞. Furthermore the differential of $\bar{f}$ is described by $(\bar{f}_*\langle v\rangle)(t) = f(t,\cdot)_*\langle v\rangle$.*

Proof. By Proposition 3.2, it suffices to assume $f \in C^\infty(I \times N, M)$. By using a coordinate chart and Remark 3.2, we can localize to the case where $M = \mathbb{R}^n$. But this case is covered by Lemma 3.1. Q.E.D.

The remainder of this section will be devoted to describing two submanifolds of $H^1(I, M)$—the based path space and the based loop space. For this purpose, we will fix a distinguished point $m \in M$.

LEMMA 3.2. *The map $(t, \sigma) \in I \times H^1(I, M) \to \sigma(t) \in M$ is $C^{0,\infty}$.*

Proof. Let $\sigma \in C^1(I, M)$, then the lemma is equivalent to showing that $(t, X) \to \exp_{\sigma(t)}(q(t)X(t))$ is $C^{0,\infty}$ for X near zero in $H^1(I, T_{\sigma(0)}M)$, where $q(\cdot)$ is parallel translation along σ as in Remark 3.2. But this is easily seen to be the case by the same techniques of Lemma 3.1. Q.E.D.

DEFINITION 3.1. For each $t \in I$, put $\beta_t(\sigma) = \sigma(t) \in M$ for $\sigma \in H^1(I, M)$—β_t is C^∞ by Lemma 3.2. If $t = 1$, write β for β_1.

DEFINITION 3.2. The based paths at m are $\mathscr{P}_m \equiv \beta_0^{-1}(\{m\}) = \{\sigma \in H^1(I, M): \sigma(0) = m\}$.

DEFINITION 3.3. The loop space based at m is $\Omega_m = \{\sigma \in \mathscr{P}_m: \sigma(1) = m\} = \beta|_{\mathscr{P}_m}^{-1}(\{m\})$.

The space $\mathscr{P}_m$ is a submanifold of $H^1(I, M)$ and the space Ω_m is a submanifold of $\mathscr{P}_m$. This is seen by the implicit function theorem (see Corollary 2s of [L]) coupled with the fact that the differential of β_t is surjective. The tangent bundle of $\mathscr{P}_m$ may be identified with the subbundle of $TH^1(I, M)$ given by $\{X \in TH^1(I, M): X(0) = 0\}$, and the tangent bundle of Ω_m may be identified with the subbundle of $T\mathscr{P}_m$ given by $\{X \in T\mathscr{P}_m: X(1) = 0\}$.

4. PARALLEL TRANSLATION, LASSOS, AND INTEGRATED LASSOS

Our next goal is to show that parallel translation along a curve $\sigma \in \mathscr{P}_m$ is well defined (Corollary 4.1). In fact, it will be shown that the parallel translation operator $P^\nabla \equiv P_1^\nabla$ is a C^∞-function on $\mathscr{P}_m$. We first prove a local version of this fact. This discussion closely parallels the discussion in [G1] in the case $M = \mathbb{R}^n$.

LEMMA 4.1. *Let V be a finite dimensional inner product space. Then there exists a unique C^∞-function $P: L^2(I, \mathrm{End}(V)) \to H^1(I, \mathrm{Aut}(V))$ ($H^1(I, \mathrm{Aut}(V)) \equiv \{\sigma \in H^1(I, \mathrm{End}(V)): \sigma(t) \in \mathrm{Aut}(V)$ for all $t\}$, an open subset of $H^1(I, \mathrm{End}(V))$) which satisfies:*

DE1. *$(d/dt)P(A)(t) + A(t)P(A)(t) = 0$ for almost all t,*

DE2. *$P(A)(0) = Id \in \mathrm{Aut}(V)$.*

Furthermore, the differential of P is

$$DP(A)\langle B\rangle(t) = -P(A)(t)\int_0^t P(A)(\tau)^{-1} B(\tau) P(A)(\tau)\, d\tau. \qquad (4.1)$$

(Duhamel's Principle)

Remark 4.1. The interval I may be replaced by any other interval with the obvious changes in notation.

Proof. By standard theorems on linear ordinary differential equations, for each $A \in C^\infty(I, \mathrm{End}(V))$ there is a unique continuous (in fact C^∞ in t) function $(P(\cdot)(\cdot))$ satisfying DE1 and DE2. We will now show that this solution satisfies the estimates

$$|P(A)(t) - P(B)(t)| \leqslant K(\|A\|_2 \cdot \|B\|_2) \cdot \|A - B\|_2 \tag{4.2}$$

and

$$\left| \frac{d}{dt} \{P(A)(t) - P(B)(t)\} \right| \leqslant K(\|A\|_2 \cdot \|B\|_2) \cdot \|A - B\|_2, \tag{4.3}$$

where $|\cdot|$ denotes any of the equivalent norms on $\mathrm{End}(V)$, $\|\cdot\|_2$ is the L^2-norm on $C^\infty(I, \mathrm{End}(V))$ where the norm on $\mathrm{End}(V)$ is taken to be the Hilbert Schmitt norm. The function K denotes a continuous positive function increasing in each of its arguments. This last estimate shows that the function P extends uniquely to all of $L^2(I, \mathrm{End}(V))$ from $C^\infty(I, \mathrm{End}(V))$. The extended function will again be called P.

In the argument below, K will denote a continuous function which is increasing in any of its arguments. We will now prove the assertion in (4.3). Set $f(t) = |h(t)| = |P(A)(t) - P(B)(t)|$, and then by DE1 we can estimate $h'(t)$ by

$$|h'(t)| = \leqslant |B(t) - A(t)|\, |P(B)(t)| + |A(t)|\, f(t).$$

Upon integration and by the fact that $h(0) = 0$ we find the inequality

$$f(t) \leqslant \int_0^t |A(s) - B(s)|\, |P(B)(s)|\, ds + \int_0^t |A(s)|\, f(s)\, ds. \tag{4.4a}$$

Now first suppose that $A = 0$ in (4.4a); using Gronwall's lemma (see [Di]) we find the estimate

$$|P(B)(t)| \leqslant K(\|B\|) = [1 + \|B\| \exp(\|B\|)].$$

Putting this last inequality into (4.4a) one finds

$$f(t) \leqslant \|A - B\|\, K(\|B\|) + \int_0^t |A(s)|\, f(s)\, ds. \tag{4.4b}$$

Applying Gronwall's lemma to (4.4b) yields (4.2). The inequality in (4.3) is easily derived from (4.2) by integration.

Now to prove (4.1) holds, start with A and B in $C^\infty(I, \mathrm{End}(V))$. Note that

$$\begin{aligned} \partial_t P(A)(t)^{-1} &= -P(A)(t)^{-1} (\partial_t P(A)(t))\, P(A)(t)^{-1} \\ &= P(A)(t)^{-1} A(t) \end{aligned} \tag{4.5}$$

by DE1, and that

$$\begin{aligned} &\partial_t\, \partial_s P(A + sB)(t) + B(s) P(A + sB)(t) \\ &\qquad + (A(t) + sB(t))\, \partial_s P(A + sB)(t) = 0 \end{aligned} \tag{4.6}$$

by differentiating the differential equation (DE1) for $P(A + sB)$ with respect to s. Using (4.5) and (4.6), it follows that

$$\begin{aligned} &\partial_\tau [P(A + sB)(\tau)^{-1}\, \partial_s P(A + sB)(\tau)] \\ &\qquad = -P(A + sB)(\tau)^{-1}\, B(\tau) P(A + sB)(\tau). \end{aligned} \tag{4.7}$$

Integrating equation (4.7) from 0 to t with respect to τ and then left multiplying by $P(A + sB)(t)$ imply

$$\partial_B P(A)(t) = -P(A)(t) \int_0^t P(A)(\tau)^{-1}\, B(\tau)\, P(A)(\tau)\, d\tau. \tag{4.8}$$

The relation $\partial_s P(A + sB)(0) = 0$ has been used in this last step.

We now want to apply Proposition 3.1. In order to apply this proposition we note the following maps are C^∞.

S1. $(g \to g^{-1})$: $H^1(I, \mathrm{Aut}(V)) \to H^1(I, \mathrm{Aut}(V))$ where $g^{-1}(t) = g(t)^{-1}$, since $h \to h^{-1}$: $\mathrm{Aut}(V) \to \mathrm{Aut}(V)$ is a C^∞ map so that Lemma 3.1 is applicable.

S2. $((g, h) \to gh)$: $H^1(I, \mathrm{End}(V) \times H^1(I, \mathrm{End}(V)) \to H^1(I, \mathrm{End}(V))$ where $gh(t) = g(t) h(t)$.

S3. $(g, h, A) \to \int_0^\cdot g(\tau) A(\tau) h(\tau)\, d\tau$: $H^1(I, \mathrm{End}(V))^2 \times L^2(I, \mathrm{End}(V)) \to H^1(I, \mathrm{End}(V))$.

Since the maps in S2. and S3. are multilinear, it is enough to check continuity, but this is quite straight forward using the fact that the $\|\cdot\|_\infty \leqslant \sqrt{2}\, \|\cdot\|_1$ (2.3.5 Proposition of [KL1]).

Using S1–S3, it is easy to see that the right hand side of (4.8) is continuous as a mapping from $L^2(I, \mathrm{End}(V))$ to $\mathrm{Hom}(L^2(I, \mathrm{End}(V)$, $H^1(I, \mathrm{Aut}(V))$. So by Proposition 3.1, P is differentiable with differential as in (4.1). Higher derivatives of P may be computed by differentiating (4.1). This may easily be done first on $C^\infty(I, \mathrm{End}(V))$, and then extended to $H^1(I, \mathrm{End}(V))$ by Proposition 3.1, using simple properties like S1–S3 above to conclude the formal expression for the differential is actually continuous. Q.E.D.

From this result follows a number of corollaries.

COROLLARY 4.1. *Let P^∇ be the parallel translation operator on E with respect to the connection ∇. Then $P^\nabla: \mathscr{P}_m(M) \to \mathrm{Hom}(E_m, E)$ is C^∞, where $\mathrm{Hom}(E_m, E)$ is the vector bundle over M with the fiber over $p \in M$ given by $\mathrm{Hom}(E_m, E_p)$.*

Proof. Let $\sigma \in \mathscr{P}_m$, and choose local trivializations $\{(\psi_i, U_i)\}_{i=1}^k$, and subintervals $J_i = [t_{i-1}, t_i]$ for $i = 1$ to k such that $0 = t_0 < t_1 < \cdots < t_k = 1$ and $\sigma(J_i) \subset U_i$. Let $A^i = A^{\psi_i}$ be the connection form associated to the local trivalization (ψ_i, U_i), see Section 2. Choose $\varepsilon > 0$, such that $W(\sigma, \varepsilon)$ is a coordinate neighborhood of $\mathscr{P}_m$ as in Theorem 3.2, and such that for $\alpha \in W(\sigma, \varepsilon)$; $\alpha(J_i) \subset U_i$. Then by using a proof similar to that for Lemma 3.1, the map $\alpha \in W(\sigma, \varepsilon) \to A^i\langle \alpha'|_{J_i}\rangle \in L^2(J_i, \mathrm{End}(V))$ is C^∞. Hence by Lemma 4.1, the functions $\tilde{P}^i(\alpha) \equiv P^i(\alpha)(t_i)$ are C^∞ where $P^i(\alpha)$ is the solution to the differential equation:

$$\frac{d}{dt} P^i(\alpha)(t) + A^i\langle \alpha|'_{J_i}(t)\rangle \cdot P^i(\alpha)(t) = 0 \quad \text{with} \quad P^i(\alpha)(t_{i-1}) = \mathrm{Id}. \tag{4.9}$$

From this it follows that P^∇ is C^∞, since P^∇ may be written (see (2.6)) as

$$P^\nabla(\alpha) = \psi_k|^{-1}_{\alpha(t_k)} \tilde{P}^k(\alpha)\, h_{k-1}(\alpha(t_{k-1}))\, \tilde{P}^{k-1}(\alpha) \cdots h_1(\alpha(t_1))\, \tilde{P}^1(\alpha)\, \psi_1|_{m'} \tag{4.10}$$

where $h_i(x) \equiv (\psi_{i+1})_x (\varphi_i)_x^{-1}$ for $x \in U_{i+1} \cap U_i$. Q.E.D.

COROLLARY 4.2. *$T\mathscr{P}_m$ is trivial. More precisely let $Q_t(\sigma)$ denote the parallel translation operator on TM with respect to the Levi Civita connection D along a curve σ up to time t. Then $F: T\mathscr{P}_m \to \mathscr{P}_m \times H_0^1(I, T_m M)$ given by $F(X) = (\sigma, Q(\sigma)^{-1} X)$ for $X \in T_\sigma \mathscr{P}_m$ is a vector bundle isomorphism where $(Q(\sigma)^{-1} X)(t) \equiv Q_t(\sigma)^{-1} X(t)$, and $H_0^1(I, T_m M) = \{X \in H^1(I, T_m M): X(0) = 0\}$.*

Proof. It is clear that F is bijective and fiber linear, so the only issue is the smoothness of F and F^{-1}. Let $\tau \in C^1\mathscr{P}_m$, and $\varepsilon > 0$ such that $W(\tau, \varepsilon)$ is a coordinate neighborhood of $\mathscr{P}_m$. To make notation manageable, let $\Sigma(X)(t) \equiv \exp_\tau(t)(Q_t(\tau) X(t))$ for $X \in H_0^1(I, T_m M)$ with $\|X\|_\infty < \varepsilon$. (Note that Σ is C^∞, since it is the inverse of a coordinate chart.) Also let $q(X)(t) = Q_t(\Sigma(X))$. The local expression $(\tilde{F})$ for F with respect to the coordinate patch $W(\tau, \varepsilon)$ is

$$\tilde{F}(X, Y) = (X, q(X)^{-1} \Sigma_{*X}\langle Y\rangle) \in H_0^1(I, T_m M) \times H_0^1(I, T_m M)$$

for X, Y in $H_0^1(I, T_m M)$ such that $\|X\|_\infty < \varepsilon$. In this last expression $\Sigma_{*X}\langle Y\rangle \equiv (d/ds)|_0\, \Sigma(X + sY) \in T_{\Sigma(X)}\mathscr{P}_m$, and $(q(X)^{-1} \Sigma_{*X}\langle Y\rangle)(t) \equiv q_t(X)^{-1} \Sigma_{*X}\langle Y\rangle(t)$.

Due to Proposition 3.2, it is enough to prove that

$$G(X, Y) \equiv (q(X)^{-1} \Sigma_{*X}\langle Y\rangle)|_{J_i}$$

is C^∞ if the J_i are as in Corollary 4.1. Let ψ_i be a local trivialization of TM over the open set U_i, where now $V = \mathbb{R}^n$. Again using the notation of Corollary 4.1 (applied to the vector bundle TM) we may write

$$G(X, Y) = q(X)(t_{i-1})^{-1} \cdot (\psi_i)^{-1}_{\Sigma(X)(t_{i-1})} \cdot P^i(\Sigma(X))^{-1} \cdot \psi_i \circ \Sigma_{*X}\langle Y\rangle.$$

By Lemma 3.1, the map $(X, Y) \to \varphi_i \circ \Sigma_{*X}\langle Y\rangle|_{J_i}: H^1(J_i, \mathbb{R}^n) \times H^1(J_i, \mathbb{R}^n) \to H^1(J_i, V)$ is C^∞ and by the proof of Corollary 4.1, P^i is C^∞. Hence by Corollary 4.1 applied to $(\psi_i)_{\Sigma(X)(t_{i-1})} q(X)(t_{i-1})$ and properties S1 and S2 stated in the proof of Lemma 4.1, we conclude that $G|_{J_i}$ is C^∞. Similarly one can show that F^{-1} is also C^∞. Q.E.D.

COROLLARY 4.3. *Let E be any vector bundle over M with covariant derivative ∇. Put $\beta: \mathscr{P}_m \to M$ equal to $\beta(\sigma) = \sigma(1)$. Then the pullback bundle $\beta^* E$ over $\mathscr{P}_m$ is trivial. The map $H: \beta^* E \to \mathscr{P}_m \times E_m$ given by $H(\sigma, \xi) \equiv P^\nabla(\sigma)^{-1} \xi$ is a fiber linear isomorphism. Recall that $\beta^* E$ is the bundle over $\mathscr{P}_m$ with fibers $\beta^* E_\sigma = \{(\sigma, \xi): \xi \in E_{\beta(\sigma)} = E_{\sigma(1)}\}$.*

Proof. Similar to Corollary 4.2 but easier. Q.E.D.

Now that we know P^∇ is C^∞, it is natural to compute its differential. This computation of the differential is done in Theorem 2.2 of [G1] for the case where the base manifold is $\mathbb{R}^d$, and in Corollary 2.16 for general M. To be complete I will rederive this result using a slightly different derivation. In order to state the results it is necessary to introduce the main objects of study for this paper.

DEFINITION 4.1. Let W be a finite dimensional vector space. The elements of $\Gamma_\beta(\Lambda^2(T^*M) \otimes W)$ will be called W-valued path two forms. More explicitly, a path two form is a smooth function (L) such that for each $\sigma \in \mathscr{P}_m$, $L(\sigma)\langle \cdot, \cdot \rangle$ is an alternating bi-linear function of $T_{\sigma(1)}M$ with values in W. Particular cases of interest will be for $W = \mathrm{End}(E_m)$ and $W = \mathscr{G}$, the Lie algebra of the structure group G.

DEFINITION 4.2. Given a connection ∇ on E as in section 2, then the Lasso associated to ∇ is the End(E_m)-valued path two form L^∇ defined by

$$L^\nabla(\sigma)\langle u, v\rangle \equiv P^\nabla(\sigma)^{-1} F^\nabla\langle u, v\rangle P^\nabla(\sigma) \tag{4.11}$$

for $u, v \in T_{\sigma(1)}M$. (It is clear that L^∇ is C^∞ since P^∇ and F^∇ are C^∞.)

In order to define the "integrated lassos" we need the following result.

PROPOSITION 4.1. *For each $\sigma \in \mathscr{P}_m$ and $r \in I$, put $\sigma^r(t) = \sigma(rt)$, then the map $(r, \sigma) \to \sigma^r: I \times \mathscr{P}_m \to \mathscr{P}_m$ is $C^{0,\infty}$.*

Proof. Fix $r \in I$, and $\sigma \in \mathscr{P}_m$. Choose $\alpha \in C^\infty \mathscr{P}_m$ and $\varepsilon > 0$ such that $W(\alpha, \varepsilon)$ is a neighborhood of σ. Then $W(\alpha^r, \varepsilon)$ is a neighborhood of σ^r. Let $q(t)$ be parallel translation along the curve α in TM with respect to the connection D. Then locally the operation $(s, \sigma) \to \sigma^s$ may be expressed as $X \to (t \to f(s, t, X(st)))$ from a neighborhood of 0 in $H_0^1(I, T_m M)$ to $H_0^1(I, T_m M)$, where

$$f(s, t, u) \equiv \exp^{-1}_{\alpha(rt)} o \exp_{\alpha(st)}(q(st)u)$$

for $s, t \in I$ and u in a neighborhood of 0 in $T_m M$. By Lemma 2.7 of [G1], the map $(s, X) \to X^s: I \times H_0^1(I, T_m M) \to H_0^1(I, T_m M)$ is jointly continuous. Since for fixed s, the map $X \to X^s$ from $H_0^1(I, T_m M)$ to $H_0^1(I, T_m M)$ is linear, it follows that $(s, X) \to X^s$ is $C^{0,\infty}$. Now $f(s, t, u)$ is C^∞ on its domain, so by techniques similar to those of Lemma 3.1, the map $(s, X) \to f(s, \cdot, X(\cdot))$ is jointly C^∞ in a neighborhood of $(r, 0) \in I \times H_0^1(I, TM)$. Therefore, the composite map $(s, X) \to f(s, \cdot, X^s(\cdot))$ of $(s, X) \to f(s, \cdot, X(\cdot))$ composed with $(s, X) \to X^s$ is also $C^{0,\infty}$. Q.E.D.

Given an End(E_m)-valued path two form L on $\mathscr{P}_m$ we can define a 1-form on $\mathscr{P}_m$ with values in End(E_m) by

$$B^L\langle X\rangle \equiv \int_0^1 L(\sigma^t)\langle \sigma'(t), X(t)\rangle\, dt \qquad \text{for} \quad X \in T_\sigma \mathscr{P}_m. \tag{4.12}$$

The fact that B^L is C^∞ may be deduced with the aid of Propositions 3.2 and 4.1 and properties S1–S3 used in Proposition 4.1. There are no additional ideas introduced, so I will omit the details.

DEFINITION 4.3. If $L = L^\nabla$ is a path 2-form which is a lasso, then we write B^∇ for B^L and in this case we call B^∇ an integrated lasso.

Remark 4.2. The reason for calling L^∇ a lasso is described in [G1]. Basically, $L^\nabla(\sigma)\langle u, v\rangle$ may be computed by taking appropriate limits of the parallel translation operators as one traverses a collection of paths which have the shape of a lasso.

We now restate Theorem 2.2 and Corollary 2.16 of [G1]. For some history on this theorem see Remark 2.12 of [G1].

THEOREM 4.1. *Assume the notation above, then P^∇ is C^∞ on $\mathscr{P}_m$. Let $\sigma \in \mathscr{P}_m$, $X \in T_\sigma \mathscr{P}_m$, ψ be a local trivialization of the bundle E about $\sigma(1)$, and $f(\sigma) \equiv \psi|_{\sigma(1)} P^\nabla(\sigma)$. The derivative of f in the direction X is*

$$Xf = f(\sigma)\, B^\nabla\langle X\rangle - A^\psi\langle X(1)\rangle f(\sigma). \tag{4.13}$$

Proof. We will use the notation introduced in Corollary 4.1. We can assume that $\psi = \psi_k$. Then

$$X\tilde{P}^i = -\tilde{P}^i(\sigma) \int_{J_i} P^i(\sigma)(\tau)^{-1} \frac{d}{ds}\bigg|_{s=0} A\left\langle \frac{d}{d\tau} \exp_{\sigma(\tau)} sX(\tau)\right\rangle P^i(\sigma)(\tau)\, d\tau$$

$$= -\tilde{P}^i(\sigma) \int_{J_i} P^i(\sigma)(\tau) \Big\{ dA^i\langle X(\tau), \sigma'(\tau)\rangle$$

$$+ \frac{d}{d\tau} A^i\langle X(\tau)\rangle \Big\} P^i(\sigma)(\tau)\, d\tau,$$

where (4.1) (Duhamel's principle) was used to derive the first equality and the definition of the exterior derivative (d) was used for the second. Integrating by parts on the second term of this last equation yields

$$X\tilde{P}^i = -\tilde{P}^i(\sigma) \int_{J_i} \{P^i(\sigma)(\tau)^{-1}\{dA^i\langle X(\tau), \sigma'(\tau)\rangle$$

$$- [A^i\langle \sigma'(\tau), \langle X(\tau)\rangle]\}\, P^i(\sigma)(\tau)\}\, d\tau$$

$$- A^i\langle X(t_i)\rangle\, \tilde{P}^i(\sigma) + \tilde{P}^i(\sigma)\, A^i\langle X(t_{i-1})\rangle$$

or

$$X\tilde{P}^i = \tilde{P}^i(\sigma)\, \tilde{B}_i - A^i\langle X(t_i)\rangle\, \tilde{P}^i(\sigma) - \tilde{P}^i(\sigma)\, A^i\langle X(t_{i-1})\rangle, \tag{4.14}$$

where

$$\tilde{B}_i \equiv \int_{J_i} P^i(\sigma)(\tau)^{-1}\, \tilde{F}^i\langle \sigma'(\tau), X(\tau)\rangle P^i(\sigma)(\tau)\, d\tau \tag{4.15}$$

and

$$\tilde{F}^i\langle u, v\rangle = dA^i\langle u, v\rangle + [A^i\langle u\rangle, A^i\langle v\rangle] = (\psi_i)_x F^\nabla\langle u, v\rangle (\psi_i)_x^{-1} \tag{4.16}$$

for u and $v \in T_x U_i$. Now recall (see equation (4.10) that

$$f(\alpha) \equiv \psi P^\nabla(\alpha) = \tilde{P}^k(\alpha)\, h_{k-1}(\alpha(t_{k-1}))\, \tilde{P}^{k-1}(\alpha) \cdots h_1(\alpha(t_1))\, \tilde{P}^1(\alpha), \tag{4.17}$$

where $h_i(x) \equiv (\psi_{i+1})_x (\psi_i)_x^{-1}$ for $x \in U_{i+1} \cap U_i$. To ease notation, let

$$R_i \equiv h_i(\sigma(t_i))\, \tilde{P}^i(\sigma) \cdots h_1(\sigma(t_1))\, \tilde{P}^1(\sigma)\, \psi_1|_m$$

and

$$K_i(\sigma) \equiv h_i(\sigma(t_i))\, \tilde{P}^i(\sigma).$$

Then by the product rule

$$\begin{aligned} Xf = {} & (X\tilde{P}^k)\, R_{k-1} + \tilde{P}^k(\sigma)(XK_{k-1})\, R_{k-2} \\ & + \tilde{P}^k(\sigma)\, K_{k-1}(XK_{k-2})\, R_{k-1} + \cdots + \tilde{P}^k(\sigma)\, K_{k-1} K_{k-2} \cdots K_2(XK_1). \end{aligned} \tag{4.18}$$

So by (4.14)

$$\begin{aligned} XK_i = {} & h_i(\sigma(t_i))\{\tilde{P}^i(\sigma)\, \tilde{B}_i - A^i\langle X(t_i)\rangle\, \tilde{P}^i(\sigma) \\ & + \tilde{P}^i(\sigma)\, A^i\langle X(t_{i-1})\rangle\} + dh_i\langle X(t_i)\rangle\, \tilde{P}^i(\sigma) \end{aligned}$$

which may be written by (2.1) as

$$XK_i = K_i \tilde{B}_i - A^{i+1}\langle X(t_{i+1})\rangle\, K_i + K_i A^i \langle X(t_{i-1})\rangle. \tag{4.19}$$

Plugging (4.19) into (4.18) one finds that the "boundary terms" involving $A^i\langle X(t_i)\rangle$ all cancel except for the term $A^k\langle X(t_k)\rangle$. Noting that

$$\begin{aligned} & \tilde{P}^k(\sigma)\, K_{k-1} K_{k-2} \cdots K_i \tilde{B}_i K_{i-1} \cdots K_1 \\ & = f(\sigma) \int_{J_i} P^\nabla(\sigma)(\tau)^{-1}\, F^\nabla\langle \sigma'(\tau), X(\tau)\rangle\, P^\nabla(\sigma)(\tau)\, d\tau, \end{aligned} \tag{4.20}$$

the expression in (4.18) sums to (4.13). Q.E.D.

As an immediate corollary we have:

COROLLARY 4.4. *If $X \in T_\sigma \mathscr{P}_m$ such that $X(0) = X(1) = 0$, then*

$$XP^\nabla = P^\nabla(\sigma)\, B^\nabla\langle X\rangle. \tag{4.21}$$

The derivative of L^∇ may also be computed.

COROLLARY 4.5. *Under the assumptions above,*

$$(\beta^* D)_X L^\nabla = [L^\nabla(\sigma), B^\nabla\langle X\rangle] + P^\nabla(\sigma)^{-1}\, ((\nabla \otimes D_{X(1)} F^\nabla)(\sigma(1))\, P^\nabla(\sigma), \tag{4.22}$$

where $\beta\colon \mathscr{P}_m \to M$ is defined by $\beta(\sigma) = \sigma(1)$, and $\beta^* D$ is the pullback of the Levi-Civita connection (D) to sections along β. (Both sides of equation (4.21) are in $\Lambda^2(T^*_{\sigma(1)} M)$ where $X \in T_\sigma \mathscr{P}_m$.)

Before proving this corollary we will pause to introduce some notation which will be useful in the proof.

DEFINITION 4.4. Let F be a function on $\mathscr{P}_m$ taking values in some set, then $F_s(\sigma) \equiv F(\sigma^s)$ for all $s \in I$.

Remark 4.3. If P^∇ is the parallel translation operator of a connection ∇, then $P^\nabla_s(\sigma)$ is parallel translation along σ up to time s. We also write $P^\nabla_s(\sigma)$ for parallel translation along a curve σ up to time s for a curve which is defined on some interval $(-\varepsilon, \varepsilon)$ about the origin. This should not cause confusion, since when there is any ambiguity the two definitions agree.

DEFINITION 4.5. Let σ and τ be in $\mathscr{P}$ such that $\tau(1) = \sigma(0)$. Then $\sigma\tau$ is defined to be the element in $\mathscr{P}$ given by,

$$\sigma\tau(t) = \begin{Bmatrix} \tau(2t) & 0 \leqslant t \leqslant 1/2 \\ \sigma(2t-1) & 1/2 \leqslant t \leqslant 1 \end{Bmatrix}. \tag{4.23a}$$

Similarly if $X \in T_\sigma \mathscr{P}$ and $Y \in T_\tau \mathscr{P}$ with $X(0) = Y(1)$, then $XY \in T_{\sigma\tau} \mathscr{P}$ is defined by

$$XY(t) = \begin{Bmatrix} X(2t) & 0 \leqslant t \leqslant 1/2 \\ Y(2t-1) & 1/2 \leqslant t \leqslant 1 \end{Bmatrix}. \tag{4.23b}$$

PROPOSITION 4.2. *Suppose that γ and η are C^k maps of $(-\varepsilon, \varepsilon)$ to $\mathscr{P}$ such that $\gamma(s)(1) = \eta(s)(0)$ for all $s \in (-\varepsilon, \varepsilon)$, then the map*

$$(s \to \eta(s)\gamma(s))\colon (-\varepsilon, \varepsilon) \to \mathscr{P} \tag{4.24}$$

is also of class C^k and furthermore

$$\left.\frac{d}{ds}\right|_{s=0} \eta(s)\gamma(s) = \eta'(s)\, \gamma'(s).$$

Proof. An easy application of Proposition 3.2. Q.E.D.

Proof of Corollary 4.5. Put $\gamma(s)(t) \equiv \exp_{\sigma(t)}(sX(t))$ for s small and $t \in I$.

Then $\gamma\colon(-\varepsilon,\varepsilon)\to\mathscr{P}_m$ is C^∞ (for some $\varepsilon>0$) and $(d/ds)\gamma(s)|_0=X$. Set $p(s)\equiv P_s^\nabla(\beta\circ\gamma)=P_s^\nabla(\gamma(\cdot)(1))$ and $q(s)\equiv Q_s(\beta\circ\gamma)$ where $Q_s=Q_s^D$ is parallel translation on TM defined by the Levi-Civita connection (D) on M). Then by definition

$$(\beta^*D)_X L^\nabla=\frac{d}{ds}\bigg|_{s=0} L^\nabla(\gamma(s))\langle q(s)^{-1}\cdot, q(s)^{-1}\cdot\rangle$$

$$=\frac{d}{ds}\bigg|_{s=0}\{P^\nabla(\gamma(s))^{-1}F^\nabla(\beta\circ\gamma)(s))\langle q(s)^{-1}\cdot, q(s)^{-1}\cdot\rangle P^\nabla(\gamma(s))\}. \tag{4.25}$$

Define the function $\Gamma\colon(-\varepsilon,\varepsilon)\to\mathscr{P}_m$ by

$$\Gamma(s)(t)\equiv\begin{cases}\gamma(s)(2t) & 0\leqslant t\leqslant 1/2\\ \gamma(-2s(t-1))(1) & 1/2\leqslant t\leqslant 1\end{cases} \tag{4.26}$$

for $s\in(-\varepsilon,\varepsilon)$ and $t\in I$. It is easy to check using Proposition 4.2 and Lemma 3.2 that Γ is a C^1-function, and in fact even C^∞. We now may write out (4.25) more explicitly as

$$(\beta^*D)_X L^\nabla=\frac{d}{ds}\bigg|_{s=0}\{P^\nabla(\Gamma(s))^{-1}\,p(s)^{-1}\,F^\nabla(\beta\circ\gamma(s))$$
$$\langle q(s)^{-1}\cdot, q(s)^{-1}\cdot\rangle\,p(s)\,P^\nabla(\Gamma(s))\}, \tag{4.27}$$

since $P^\nabla(\Gamma(s))=p(s)^{-1}P^\nabla(\gamma(s))$ by the "multiplicative" property of parallel translation. But, by definition,

$$(\nabla\otimes D)_{X(1)}F=\frac{d}{ds}\bigg|_{s=0}\{p(s)^{-1}F^\nabla(\beta\circ\gamma(s))\langle q(s)^{-1}\cdot, q(s)^{-1}\cdot\rangle\,p(s)\}. \tag{4.28}$$

Therefore, by the product rule, Corollary 4.5, and equations (4.27), (4.28), and (4.5) one can easily verify equation (4.22). Q.E.D.

In the sequel we will find the following form of Corollary 4.5 to be more useful.

COROLLARY 4.6. *Let $u, w\in\Gamma(TM)$–vector fields on M; put $U(\sigma)=u(\sigma(1))$ and $W(\sigma)=w(\sigma(1))$. With this notation $L^\nabla\langle U, W\rangle$ is an* End(E_m) *valued path function. The derivative of this function is*

$$X(L^\nabla\langle U, W\rangle)=[L^\nabla\langle U, W\rangle(\sigma), B^\nabla\langle X\rangle]$$
$$+P^\nabla(\sigma)^{-1}(\nabla_{X(1)}F^\nabla)\langle u, w\rangle\,P^\nabla(\sigma), \tag{4.29}$$

where $X\in T_\sigma\mathscr{P}_m$, and u, w in (4.29) *should be evaluated at $\sigma(1)$.*

Proof. First note that

$$X(L^\nabla\langle U, W\rangle)=(\beta^*D_X L)\langle U, W\rangle+L^\nabla(\sigma)\langle\beta^*D_X U, W\rangle$$
$$+L^\nabla(\sigma)\langle U, \beta^*D_X W\rangle,$$

and that $\beta^*D_X U=D_{X(1)}u$. Equation (4.29) now follows from (4.22) and the fact that

$$\nabla_{X(1)}(F^\nabla\langle u, w\rangle)=(\nabla\otimes D_{X(1)}F^\nabla)\langle u, w\rangle+F^\nabla\langle D_{X(1)}u, w\rangle$$
$$+F^\nabla\langle u, D_{X(1)}w\rangle,$$

where u and w should be evaluated at $\sigma(1)$. Q.E.D.

5. Loop Variable Characterization of (E,∇)

This section shows that the set of equivalence classes (defined below) of vector bundle connection pairs are in one to one correspondence with a certain class of functions, $P\colon\Omega_m\to G$, defined up to conjugation by an element of G. In order to formulate this statement precisely, it is necessary to introduce some more definitions and notation. For the rest of this paper, (E,∇) or (E',∇') will denote vector bundles over M with connection ∇ or ∇', respectively, structure group G, and fiber model space V as described in Section 2.1. We also fix a distinguished base point $m\in M$.

DEFINITION 5.1. A pair (E,∇) is said to be equivalent to (E',∇') if there is a vector bundle isomorphism $K\colon E\to E'$ such that $K\nabla=\nabla'K$ and K respects the structure group G. To be more explicit, the first condition requires that for any $s\in\Gamma(E)$ and $v\in TM$, then $K\nabla_v S=\nabla'_v(K\circ S)$. The second condition requires that $\psi'_x\circ K_x\circ\psi_x^{-1}$ be in G for all admissible local trivializations ψ' of E' and ψ of E about any point $x\in M$. ($K_x\equiv K|_{E_x}$.)

The notation $[E,\nabla]$ will denote the equivalence class containing (E,∇), and $\mathscr{E}=\mathscr{E}(M,V,G)$ will denote the collection of equivalence classes $[E,\nabla]$.

DEFINITION 5.2. A function $P\colon\Omega_m\to G$ is said to be strongly differentiable if P is C^∞ on Ω_m, and there is a non-negative function c on Ω_m such that

$$|XP|\leqslant c(\sigma)\|X\|_0\qquad\text{for all}\quad X\in T_\sigma\mathscr{P}_m, \tag{5.1}$$

where $\|X\|_0^2\equiv\int_0^1\langle X(t), X(t)\rangle\,dt$ for $X\in T_\sigma\mathscr{P}_m$ as in (3.2).

DEFINITION 5.3. A function $P: \Omega_m \to G$ is said to be parametrization invariant if $P(\sigma \circ r) = P(\sigma)$ for all PC^1-functions $r: I \to I$ such that $r(0) = 0$ and $r(1) = 1$. Such a function (r) is called a reparametrization.

DEFINITION 5.4. A function $P: \Omega_m \to G$ is called multiplicative if

$$P(\sigma\tau) = P(\sigma) P(\tau) \tag{5.2}$$

for all $\sigma, \tau \in \Omega_m$, where $\sigma\tau$ is the concatenation of paths defined in (4.23a). Note that if P is multiplicative then $P(C_m) = 1$, where C_m is the constant path at m, since $C_m C_m = C_m$.

EXAMPLE 5.1. Let ψ be a local trivialization of a bundle connection pair (E, ∇), then the function $P(\sigma) = \psi_m P^\nabla(\sigma) \psi_m^{-1}$ for $\sigma \in \Omega_m$ is strongly differentiable, parametrization invariant and multiplicative, where P^∇ is the parallel translation operator on Ω_m.

Given a strongly differentiable, parametrization invariant, and multiplicative (SDPIM) function $P: \Omega_m \to G$, we can always define another by $P'(\sigma) = gP(\sigma) g^{-1}$ for any $g \in G$. Two SDPIM functions P and P' related by conjugation in this way will be called equivalent, and the equivalence class will be denoted by $[P]$. Let $\mathscr{M} = \mathscr{M}(M, m, V, G)$ be the collection of all equivalence classes $[P]$ of SDPIM functions.

DEFINITION 5.5. A linear isomorphism $\kappa: E_m \to V$ is said to be admissible if $\kappa \circ \psi_m^{-1} \in G$ for any admissible local trivialization of E about $m \in M$.

We now can state the main theorem of this section. An informal version of this theorem is mentioned in Giles [Gi].

THEOREM 5.1. *There is a one to one correspondence between $\mathscr{E}$ and $\mathscr{M}$ given by*

$$[E, \nabla] \to [P^\nabla], \tag{5.3}$$

where $[P^\nabla]$ is by definition $[\kappa \circ P^\nabla(\cdot)|_{\Omega_m} \circ \kappa^{-1}]$ for any admissible κ as in Definition 5.5. (Note that $[\kappa \circ P^\nabla(\cdot)|_{\Omega_m} \circ \kappa^{-1}]$ is independent of the choice of admissible κ.)

A statement similar to this theorem has been announced by Kobayashi [Kob]. However, the necessary condition of strong differentiability seems to be missing from his statement. The proof of this theorem will be carried out in a sequence of lemmas and propositions to follow.

PROPOSITION 5.1. *The map defined in (5.3) is well defined and is one to one.*

Proof. Suppose $K: (E, \nabla) \to (E', \nabla')$ is a bundle isomorphism as in Definition 5.1. Then it is easily checked that $P^\nabla(\sigma) = K_m^{-1} P^{\nabla'}(\sigma) K_m$ for all $\sigma \in \Omega_m$. Let ψ and ψ' be local trivializations of E, and E' respectively about m. Then

$$\begin{aligned} \psi_m P^\nabla(\sigma) \psi_m^{-1} &= (\psi_m K_m^{-1} \psi_m'^{-1}) \psi_m' P^{\nabla'}(\sigma) \psi_m'^{-1} (\psi_m' K_m \psi_m^{-1}) \\ &= g\psi_m' P^{\nabla'}(\sigma) \psi_m'^{-1} g^{-1}, \end{aligned}$$

where $g \in G$, since K respects the structure group G of E and E'. This shows that $[P^\nabla] = [P^{\nabla'}]$, so the map (5.1) is well defined.

Now suppose that (E, ∇) and (E', ∇') are two pairs such that $[P^\nabla] = [P^{\nabla'}]$. This implies, by definition, that there is a linear isomorphism $k: E_m \to E'_m$ which respects the structure group G and satisfies

$$P^{\nabla'}(\sigma) = kP^\nabla(\sigma) k^{-1}$$

for $\sigma \in \Omega_m$. We now define $K: E \to E'$ by

$$K_x \equiv P^{\nabla'}(\sigma) kP^\nabla(\sigma)^{-1}: E_x \to E'_x, \tag{5.4}$$

where σ is any path in $\mathscr{P}_m$ with $\sigma(1) = x$. To show that K is well defined, we must show that $P^{\nabla'}(\sigma) kP^\nabla(\sigma)^{-1} = P^{\nabla'}(\tau) kP^\nabla(\tau)^{-1}$ if $\tau \in \mathscr{P}_m$ such that $\tau(1) = x$. This is equivalent to showing that $P^{\nabla'}(\tau\sigma)k = kP^\nabla(\tau\sigma)$ by the multiplicative property of parallel transport, where τ is the "reverse" path in Ω_m defined by $\tau(t) = \tau(1-t)$. But now $\tau\sigma$ is in Ω_m, and so this last statement follows from the equation above (5.4).

The map K_x is a fiber linear isomorphism and respects the structure group G. Furthermore, it is easy to check that $K: E \to E'$ is C^∞, since in a small neighborhood U of a point $x \in M$ one may easily construct a smooth map $\Sigma: U \to \mathscr{P}_m$ such that $\Sigma(x)(1) = x$. Thus $K_x = P^{\nabla'}(\Sigma(x)) kP^\nabla(\Sigma(x))^{-1}$ for $x \in U$, which is C^∞, since P^∇ and $P^{\nabla'}$ are C^∞. Q.E.D.

So in order to finish the proof of Theorem 5.1, it suffices to show that the map (5.1) is onto. In order to see how to construct a bundle connection pair from an SDPIM function, it is useful to introduce the notion of an m-contraction. This will lead to a prefered class of local trivializations on a vector bundle connection pair (E, ∇).

DEFINITION 5.6. Let U be an open set of M. A smooth function $\varphi: I \times U \to M$ satisfying $\varphi(1, x) = x$ and $\varphi(0, x) = m$ for all x in U will be called an m-contraction over U or simply an m-contraction if the set U is understood or not important. (It is easy to see that there exists m-contractions over any open set U in M that is diffeomorphic to a Euclidean ball.)

Given an m-contraction over U and an admissible isomorphism $\kappa: E_m \to V$, there is a natural local trivialization E over U induced by parallel translation as

$$\varphi^\kappa|_x \equiv \kappa \circ P^\nabla(\varphi(\cdot, x))^{-1}: E_x \to V \tag{5.5}$$

for all $x \in U$. (Clearly φ^κ is C^∞.)

PROPOSITION 5.2. *Let φ be an m-contraction over U, and $v \in T_x U$, then*

$$A^{\varphi^\kappa}\langle v\rangle = \kappa \circ B^\nabla\langle X\rangle \circ \kappa^{-1}, \tag{5.6}$$

where $X \in T_\sigma \mathscr{P}_m$ is the vector field along the path $\sigma = \varphi(\cdot, x)$ given by $X(t) = \varphi(t, \cdot)_* v$, A^{φ^κ} is the connection 1-form of ∇ with respect to φ^κ, and B^∇ is the integrated lasso of Definition 4.5.

Proof. Set $f(\sigma) = \varphi^\kappa|_{\sigma(1)} P^\nabla(\sigma)\,\kappa^{-1}$ for all $\sigma \in \mathscr{P}_m$ with $\sigma(1) \in U$. Let α be a smooth curve in M such that $\alpha'(0) = v$, so that $X(t) = (d/ds)|_{s=0}\,\varphi(t, \alpha(s))$. By the definition of φ^κ, $f(\varphi(\cdot, x)) = 1 \in G$ for all $x \in U$, so that $Xf = (d/ds)|_{s=0} f(\varphi(\cdot, \alpha(s))) = 0$. Noting that $X(0) = 0$, the proposition now follows directly from Theorem 4.1. Q.E.D.

PROPOSITION 5.3. *Let $\kappa: E_m \to V$ be an admissible isomorphism, and let φ and ψ be two m-contractions over open sets U and W of M, respectively. Then the transition function between ψ^κ and φ^κ is*

$$g_{\varphi^\kappa \psi^\kappa}(x) = \kappa P^\nabla(\boldsymbol{\varphi}(\cdot, \mathbf{x})\,\psi(\cdot, x))\,\kappa^{-1} \qquad \text{for} \quad x \in U \cap W. \tag{5.7}$$

Notice that the path $\boldsymbol{\varphi}(\cdot, \mathbf{x})\,\psi(\cdot, x)$ is in Ω_m.

Proof. A simple matter of unwinding definitions. Q.E.D.

Propositions 5.2 and 5.3 will be the main motivation for the construction of a pair (E, ∇) from an SDPIM function P. In order to carry out this construction it is necessary first to develop some properties of SDPIM functions.

LEMMA 5.1. *Let P be an* SDPIM *function on Ω_m. Define a one form B on Ω_m by $B = P^{-1}\,dP$. The following properties hold.*

(1) *For each $\sigma \in \Omega_m$, there is a function $b(\sigma, \cdot)\langle\cdot\rangle \in L^2\Gamma_\sigma(T^*M)$ (the L^2-sections of T^*M along σ) such that*

$$B\langle X\rangle = \int_0^1 b(\sigma, t)\langle X(t)\rangle\, dt \qquad \textit{for all} \quad X \in T_\sigma \Omega_m.$$

(2) *Let $\sigma \in \Omega_m$, and $J = (a, b) \subset I$. If $\sigma|_J$ is C^2 then $b(\sigma, t)\langle\sigma'(t)\rangle = 0$ for almost every (with respect to Lebesque measure) $t \in J$.*

(3) *$P((\boldsymbol{\sigma\delta})(\boldsymbol{\delta\tau})) = P(\boldsymbol{\sigma\tau})$ for all $\sigma, \tau, \delta \in \mathscr{P}_m$ such that $\sigma = \tau = \delta$ at $t = 1$.*

Proof. First note that $|B\langle X\rangle| = |P(\sigma)^{-1}\,dP\langle X\rangle| \leqslant |P(\sigma)^{-1}|\,c(\sigma)\,\|X\|_0$ for all X in $T_\sigma\Omega_m$. Since $T_\sigma\Omega_m$ is a dense subspace of the Hilbert space $(L^2\Gamma_\sigma(TM))$ of L^2-sections along σ, the above estimate shows that $B(\sigma)$ extends uniquely to a bounded linear function on $L^2\Gamma_\sigma(TM)$. So (1) is an easy consequence of the Riesz Representation theorem.

Let r be any C^1-reparameterization of the unit interval such that $r(t) = t$ for $t \in I\backslash J$, put $r_s(t) \equiv sr(t) + (1-s)t$. Then for each $s \in I$, r_s is also such a reparameterization of I. So by assumption $P(\sigma \circ r_s) = P(\sigma)$ for all s and by differentiation this implies that $B\langle (d/ds)|_{0+}\, \sigma \circ r_s\rangle = 0$. (Note that $(d/ds)|_{0+}\,\sigma \circ r_s$ exists in $T_\sigma\Omega_m$, since $\sigma|_J$ is C^2.) As a result of property (1), one finds that

$$\int_0^1 b(t, \sigma)\langle (r(t) - t)\,\sigma'(t)\rangle\, dt = 0 \tag{5.8}$$

for all reparametrizations of I with $r(t) = t$ on $I\backslash J$. By taking limits of different reparametrizations, one can easily show that (5.8) implies that

$$\int_{J'} b(t, \sigma)\langle \sigma'(t)\rangle\, dt = 0 \tag{5.9}$$

for all subintervals J' of J. From (5.9), and standard measure theoretic arguments the statement (2) follows.

It suffices to prove (3) for $\sigma, \delta, \tau \in C^2\Omega_m$, since the general result will then follow by continuity of P. Fix a C^2-reparametrization (r) of I such that r is strictly increasing on I, $r(1/2) = 1/2$, and $r'(1/2) = r''(1/2) = 0$. Then for any $\sigma, \tau \in C^2\mathscr{P}$, $\sigma\tau \circ r = (\sigma \circ r)(\tau \circ r)$, and $(\sigma\tau) \circ r \in C^2\mathscr{P}$ because $r'(1/2) = r''(1/2) = 0$. By the reparameterization invariance of P we know that

$$\begin{aligned} p(s) &\equiv P((\boldsymbol{\sigma}\delta^s)(\boldsymbol{\delta}^s\tau)) = P([(\boldsymbol{\sigma}\delta^s)(\boldsymbol{\delta}^s\tau)] \circ r) \\ &= P((\boldsymbol{\sigma}\delta^s) \circ r \cdot (\boldsymbol{\delta}^s\tau) \circ r) = P((\boldsymbol{\sigma} \circ r \cdot \delta^s \circ r)(\boldsymbol{\delta}^s \circ r \cdot \tau \circ r)) = P(\alpha(s)), \end{aligned}$$

where $\alpha(s) \equiv (\boldsymbol{\sigma} \circ r \cdot \delta^s \circ r)(\boldsymbol{\delta}^s \circ r \cdot \tau \circ r) \in C^1\Omega_m$. So we may compute

$$\frac{d}{ds}\, p(s) = p(s) \int_0^1 b(t, \alpha(s)) \left\langle \frac{d}{ds}\, \alpha(s)(t) \right\rangle = 0,$$

since $(d/ds)\,\alpha(s)(t)$ is proportional to $(d/dt)\,\alpha(s)(t)$ so that (2) applies. As a result of this computation $p(0) = p(1)$. But $(\boldsymbol{\sigma}\delta^0)(\boldsymbol{\delta}^0\tau)$ differs from $\boldsymbol{\sigma\tau}$ by a reparameterization, so that $P((\boldsymbol{\sigma\delta})(\boldsymbol{\delta\tau})) = p(1) = p(0) = P((\boldsymbol{\sigma}\delta^0)(\boldsymbol{\delta}^0\tau)) = P(\boldsymbol{\sigma\tau})$. Q.E.D.

The next goal is to show that $B \equiv P^{-1}\,dP$ can be extended naturally to $T\mathscr{P}_m$. First we note that B extends by continuity uniquely to $L^2\Gamma_\sigma(TM)$ for $\sigma \in \Omega_m$. Now suppose that $\sigma \in \mathscr{P}_m$ and $X \in T_\sigma\mathscr{P}_m$; the next lemma shows that $B\langle 0_\tau X\rangle$ is independent of $\tau \in \mathscr{P}_m$ such that $\tau(1) = \sigma(1)$. Here

$$0_\tau X(t) \equiv \begin{cases} X(2t) & 0 \leqslant t < 1/2 \\ 0 \in T_{\tau(2t-1)}M & 1/2 \leqslant t \leqslant 1 \end{cases} \in L^2\Gamma_\sigma(TM). \tag{5.10}$$

Thus we extend B (using the same symbol) to $T\mathscr{P}_m$ by $B\langle X\rangle \equiv B\langle 0_\tau X\rangle$.

LEMMA 5.2. *Assuming the notation above, $B\langle 0_\tau X\rangle$ is independent of $\tau \in \mathscr{P}_m$ such that $\tau(1) = \sigma(1)$. Furthermore, the extended 1-form B on $\mathscr{P}_m$ is C^∞.*

Proof. We will show that $B\langle 0_\tau X\rangle = B\langle 0_\sigma X\rangle$ in the special case that $X(1) = 0$. This restriction is easily removed by the continuity of B on $L^2\Gamma_\sigma(TM)$. Define, for small s, $\alpha(s) \equiv \exp_\sigma(sX) \in \mathscr{P}_m$. Note that $\alpha(s)(1) = \sigma(1)$ for all s. By property (3) of Lemma 5.1,

$$P(\sigma\tau)\,P(\tau\alpha(s)) = P(\sigma\alpha(s)). \tag{5.11}$$

Differentiate both sides of (5.11) at $s = 0$ to conclude that

$$P(\sigma\tau)\,P(\tau\sigma)\,B\langle 0_\tau X\rangle = P(\sigma\sigma)\,B\langle 0_\sigma X\rangle. \tag{5.12}$$

By another application of property (3) of Lemma 5.1, (5.12) implies that $B\langle 0_\tau X\rangle = B\langle 0_\sigma X\rangle$.

To show that B is C^∞ on $\mathscr{P}_m$, let $W(\sigma, \varepsilon)$ be a coordinate neighborhood of $\mathscr{P}_m$ and $X \in T_\tau W(\sigma, \varepsilon)$ for some $\tau \in W(\sigma, \varepsilon)$. Choose an m-contraction (φ, U) about $\tau(1)$ and $\delta > 0$ sufficiently small such that $\exp_{\tau(1)}(\delta X(1)) \in U$. Let $f \in C^\infty(I, [0, \delta])$ such that $f(t) = t$ for t near 0, and $f(t) = \delta$ for $t \geqslant \frac{1}{2}$. For all Y sufficiently close to X such that $\exp(tY(1)) \in U$ for $0 \leqslant t \leqslant \delta$, define $F_Y \in C^\infty\mathscr{P}$ by $F_Y(t) = (d/dt)\exp(f(t)\,Y(1))$. Then the concatenation $(\tilde{Y})$ of $0_{\varphi(\cdot,\exp(\delta Y(1)))}$, F_Y, and Y given by

$$\tilde{Y} = [0_{\varphi(\cdot,\exp(\delta Y(1)))} \cdot F_Y] \cdot Y$$

is in $T\Omega_m$. Using Proposition 3.2c, it is easy to show that the map $Y \to \tilde{Y}$ from a neighborhood of X in $T\mathscr{P}_m$ to $T\Omega_m$ is C^∞. Finally, let $\gamma \in \Omega_m$ such that $\tilde{Y} \in T_\gamma\Omega_m$, then

$$B\langle \tilde{Y}\rangle = \int_0^1 b(\gamma, t)\langle \tilde{Y}(t)\rangle\,dt = \int_0^{1/2} b(\gamma, t)\langle \tilde{Y}(t)\rangle\,dt$$

by property (2) of Lemma 5.1 and the fact that $\tilde{Y}(t)$ is proportional to $\gamma'(t)$ for $\frac{1}{2} \leqslant t \leqslant 1$. (Note that $\gamma|_{[(1/2),1]}$ is C^∞.) From this last equation it is clear that $B\langle \tilde{Y}\rangle = B\langle 0_{\text{path}}\,Y\rangle \equiv B\langle Y\rangle$. Since $B\langle Y\rangle = B\langle \tilde{Y}\rangle$ and $Y \to \tilde{Y}$ is C^∞, it follows that B is C^∞. Q.E.D.

Conclusion of the Proof of Theorem 5.1. We will construct a bundle connection pair (E, ∇) from the SDPIM function P. The pair (E, ∇) will be described by prescribing a collection of transition functions and connection 1-forms.

Let $\{(\varphi_a, U_a)\}$ be a collection of m-contractions over an open cover $\{U_a\}$ of M. In analogy with (5.7) and (5.6), define $g_{ab}(x) = P(\boldsymbol{\varphi}_{\mathbf{a}}(\cdot, \mathbf{x})\,\varphi_b(\cdot, x)) \in G$ for $x \in U_a \cap U_b$, and put $A^a\langle v\rangle = B\langle X_v\rangle \in \mathscr{G}$ for $v \in TU_a$, where $X_v(t) \equiv \varphi_a(t, \cdot)_* v$.

Claim 1. The $\{g_{ab}\}$ form the transition functions for a vector bundle E over M. Let ψ^a denote the local trivializations of E over the set U_a satisfying $\psi^a_x \circ (\psi^b_x)^{-1} = g_{ab}(x)$ for all $x \in U_a \cap U_b$.

Claim 2. The $\{A^a\}$ are the connection 1-forms with respect to the local trivializations $\{\psi^a\}$ for a connection ∇ on the bundle E constructed in Claim 1.

Claim 3. Let ψ^a be any local trivialization of E about $m \in M$. Then $P(\sigma) = \psi^a_m P^\nabla(\sigma)(\psi^a_m)^{-1}$ for all $\sigma \in \Omega_m$.

First note that the g_{ab}'s are C^∞ since the map $x \to \boldsymbol{\varphi}_{\mathbf{a}}(\cdot, \mathbf{x})\,\varphi_b(\cdot, x)$ is C^∞ from $U_a \cap U_b \to \Omega_m$. Furthermore, by property (3) of Lemma 5.1, the g's obey the cocycle condition $g_{ab}(x)\,g_{bc}(x) = g_{ac}(x)$ for $x \in U_a \cap U_b \cap U_c$. So the g_{ab}'s are the transition functions for a vector bundle E with local trivializations as described in Claim 1. By Lemma 5.2, each A^a is a smooth $\mathscr{G}$-valued 1-form on TU_a, since the map $(v \to X_v)$: $TU_a \to T\mathscr{P}_m$ is C^∞.

Suppose that $\sigma \in C^2(I, U_a)$, put $p^a_s(\sigma) = P(\alpha(s))$, where $\alpha(s)$ is the concatenation; $\alpha(s) \equiv \boldsymbol{\varphi}_{\mathbf{a}}(\cdot, \boldsymbol{\sigma}(\mathbf{s}))[\sigma^s \cdot \varphi_a(\cdot, \sigma(0))] \in \Omega_m$.

We will now show that p^a is parallel translation along σ with respect to the connection 1-form A^a by showing p^a satisfies the correct differential equation. Using the definition of B and p^a one finds that

$$\frac{d}{ds}\,p^a_s(\sigma) = \frac{d}{ds}\,P(\boldsymbol{\alpha}(\mathbf{s}))^{-1} = -B\left\langle \frac{d}{ds}\,\boldsymbol{\alpha}(\mathbf{s})\right\rangle p^a_s(\sigma).$$

So now compute

$$\begin{aligned} B\left\langle \frac{d}{ds}\,\boldsymbol{\alpha}(\mathbf{s})\right\rangle &= B\left\langle \frac{d}{ds}\,\{[\boldsymbol{\varphi}_{\mathbf{a}}(\cdot, \boldsymbol{\sigma}(\mathbf{0}))\cdot\boldsymbol{\sigma}^{\mathbf{s}}]\,\varphi_a(\cdot, \sigma(s))\}\right\rangle \\ &= B\left\langle 0_{[\boldsymbol{\varphi}_{\mathbf{a}}(\cdot,\boldsymbol{\sigma}(\mathbf{0}))\boldsymbol{\sigma}^{\mathbf{s}}]}\cdot\frac{d}{ds}\,\varphi_a(\cdot, \sigma(s))\right\rangle \\ &= B\left\langle \frac{d}{ds}\,\varphi_a(\cdot, \sigma(s))\right\rangle \equiv A^a\langle \sigma'(s)\rangle, \end{aligned}$$

where in the second equality we have used property (2) of Lemma 5.1. Because of these last two equations, $p_s^a(\sigma)$ satisfies the desired differential equation. Furthermore $p_0^a(\sigma)=1\in G$ by property (3) of Lemma 5.1.

Now suppose that σ is also in $C^2(I, U_b)$ for some other index b, then

$$p_s^b(\sigma)=g_{ba}(\sigma(s))\, p_s^a(\sigma)\, g_{ab}(\sigma(0)), \tag{5.13}$$

using the definitions of p^a, p^b, g_{ab}, and g_{ba} along with property (3) of Lemma 5.1. Using the above computation, take the derivative of both sides of (5.13) at $s=0$ to show that

$$A^b\langle v\rangle = dg_{ba}\langle v\rangle\, g_{ab}(x)+g_{ba}(x)\, A^a\langle v\rangle\, g_{ab}(x), \tag{5.14}$$

where $v=\sigma'(0)\in T_xM$. Since the curve σ was arbitrary, equation (5.14) holds for all $v\in T(U_a\cap U_b)$. In view of Remark 2.1, this shows that the connection 1-forms $\{A^a\}$ are consistently related, and hence define a connection ∇ on the bundle E as in Claim 2.

Furthermore our computations show that

$$P^\nabla(\sigma)=(\psi^a_{\sigma(1)})^{-1}\, P(\boldsymbol{\varphi}_a(\cdot, \boldsymbol{\sigma}(\mathbf{s}))[\sigma\varphi_a(\cdot, \sigma(0))])(\psi^a_{\sigma(0)}) \tag{5.15}$$

for all $\sigma\in C^2(I, U_a)$. Now suppose that $\sigma=\alpha\tau$ with $\tau\in C^2(I, U_b)$, $\alpha\in C^2(I, U_a)$, and $\tau(1)=\alpha(0)=x$. Then by (5.15), the multiplicative property of P^∇, and the relation $g_{ab}(x)=\psi^a_x\circ(\psi^b_x)^{-1}=P(\boldsymbol{\varphi}_a(\cdot, \mathbf{x})\cdot\varphi_b(\cdot, x))$ one shows

$$\begin{aligned}P^\nabla(\sigma)&=(\psi^a_{\alpha(1)})^{-1}\, P(\boldsymbol{\varphi}_a(\cdot, \boldsymbol{\alpha}(1))[\alpha\cdot\varphi_a(\cdot, x)])\, g_{ab}(x)\\&\quad\times P(\boldsymbol{\varphi}_b(\cdot, \mathbf{x})[\tau\cdot\varphi_b(\cdot, \tau(0))])(\psi^b_{\tau(0)})\\&=(\psi^a_{\alpha(1)})^{-1}\, P(\boldsymbol{\varphi}_a(\cdot, \boldsymbol{\alpha}(1))[\alpha\cdot\varphi_a(\cdot, x)])\, P(\boldsymbol{\varphi}_a(\cdot, \mathbf{x})\,\varphi_b(\cdot, x))\\&\quad\times P(\boldsymbol{\varphi}_b(\cdot, \mathbf{x})[\tau\cdot\varphi_b(\cdot, \tau(0))])(\psi^b_{\tau(0)})\\&=(\psi^a_{\sigma(1)})^{-1}\, P(\boldsymbol{\varphi}_a(\cdot, \boldsymbol{\alpha}(1))[\alpha\cdot[\tau\cdot\varphi_b(\cdot, 0))]])(\psi^b_{\sigma(0)})\\&=(\psi^a_{\sigma(1)})^{-1}\, P(\boldsymbol{\varphi}_a(\cdot, \boldsymbol{\sigma}(1))[\sigma\cdot\varphi_b(\cdot, \sigma(0))])(\psi^b_{\sigma(0)}),\end{aligned}$$

where we have used repeatedly the reparameterization invariance of P and the properties of Lemma 5.1. Now any curve $\sigma\in C^2(I, M)$ (if appropriately reparameterized) may be split into a finite product of paths $\sigma=\sigma_1\sigma_2\cdots\sigma_n$ with each $\sigma_i\in C^2(I, U_{a_i})$ for some a_i. So by repeating the above argument $n-1$ times one finds that

$$P^\nabla(\sigma)=(\psi^a_{\sigma(1)})^{-1}\, P(\boldsymbol{\varphi}_a(\cdot, \boldsymbol{\sigma}(1))[\sigma\cdot\varphi_b(\cdot, \sigma(0))])(\psi^b_{\sigma(0)}) \tag{5.16}$$

provided $\sigma(0)\in U_b$ and $\sigma(1)\in U_a$. Since both sides of (5.16) are continuous in σ, (5.16) holds for $\sigma\in\mathscr{P}$. Claim 3 is a special case of (5.16).

Because of Claim 3, $[P^\nabla]=[P]$. Hence the map in (5.1) is onto. This fact along with Proposition 5.1 proves Theorem 5.1. Q.E.D.

6. Characterization of $[E, \nabla]$ by Lassos

The main result of this section (Theorem 6.1) states that if M is simply connected, then the space $\mathscr{E}=\mathscr{E}(M, G, V)$ is parameterized by the set of path two forms L such that B^L has curvature 0 on Ω_m and L satisfies a monodromy condition associated with $\Pi_2(M, m)$. In Theorem 6.2 the zero curvature condition is reformulated so as to be in closer analogy with the condition $dF=0$ for ordinary closed 2-forms. Using Theorem 6.2, it is easy to show that the Weil's integrality condition for $U(1)$-line bundle can be recovered from Theorem 6.1. In order to state the main results of this section we need the following two propositions.

Proposition 6.1. *Let Ω be a connected Hilbert manifold and B be a $\mathscr{G}$-valued 1-form on Ω. Suppose that $F^B=dB+B\wedge B=0$, then the parallel translation operator (P^B) associated to B (see definition 2.2) induces a homomorphism from $\Pi_1(\Omega, \sigma)$ to G, where σ is a fixed point in Ω.*

Proof. Suppose that $\alpha, \beta\in C^\infty_\sigma(I, \Omega)=\{\tau\in C^\infty(I, \Omega): \tau(0)=\tau(1)=\sigma\}$ (the C^∞ loops based at σ) and that there is a C^∞-homotopy γ between α and β. We may view γ as a map from I to $C^\infty_\sigma(I, \Omega)$, such that $\gamma(0)=\alpha$ and $\gamma(1)=\beta$. Had we developed the loop space of an infinite dimensional Hilbert manifold we could apply Corollary 4.1 to see that $P^B(\gamma(s))$ is differentiable. Then by Corollary 4.4 it would follow that $(d/ds)\, P^B(\gamma(s))=0$, since the integrated lasso associated to B would be zero because B has zero curvature. To make this rigorous one only needs to note that γ may be used to pullback all objects to a bundle over $I\times I$ where one can use Corollaries 4.1 and 4.4. (The fact that $I\times I$ has boundaries and corners does not cause any difficulties.)

Therefore, if α and β are C^∞-homotopic, then $P^B(\alpha)=P^B(\beta)$. Now suppose γ is only a continuous homotopy. Since Hilbert manifolds admit partitions of unity (see [L] or [E1]), the standard smoothing techniques (see [Mu] or [St]) may be used to modify γ to produce a C^∞ homotopy from α to β. So again $P^B(\alpha)=P^B(\beta)$. Since, each $\alpha\in C_\sigma(I, \Omega)$ is homotopic to some $\alpha'\in C^\infty_\sigma(I, \Omega)$, one may define $P^B(\alpha')\equiv P^B(\alpha)$, which is well defined by the above comments. Thus, P^B is constant on homotopy classes and is multiplicative, hence defines a homomorphism from $\Pi_1(\Omega, \sigma)$ to G. Q.E.D.

Proposition 6.2. *$\Pi_1(M, m)$ is in one to one correspondence with the connected components of Ω_m–the H^1-loops on M based at m.*

Sketch of Proof. Suppose that σ and $\tau \in \Omega_m$ and there is a continuous path γ connecting σ to τ. Then σ and τ are homotopic, and so define the same element in $\Pi_1(M, m)$. Thus, any σ in the same connected component of Ω_m, gives rise to the same element of $\Pi_1(M, m)$. Conversely suppose that $\sigma, \tau \in \Omega_m$ are homotopic by a homotopy γ. If it were sufficiently smooth, the homotopy (γ) would supply in a natural way a path between σ and τ. But as noted above if σ and τ happened to be smooth then it is possible to modify γ to get a C^∞-homotopy, so that σ and τ would be path connected. Finally, any $\sigma \in \Omega_m$ is path connected, and hence homotopic, to a $\sigma' \in C^\infty \Omega_m$ with $\sigma = \exp_{\sigma'}(X)$ for some $X \in C^\infty T_{\sigma'} \mathscr{P}_m$. So if σ and τ are homotopic, then σ and τ are in the same path component of Ω_m. Q.E.D.

DEFINITION 6.1. A $\mathscr{G}$-valued path 2-form (L) is called closed if the $\mathscr{G}$-valued one form B^L has curvature zero on Ω_m. That is $F^{B^L} \equiv dB^L + B^L \wedge B^L = 0$ when restricted to $T\Omega_m$.

At the end of this section in Theorem 6.2, another characterization for L to be closed will be given. This will be closer to the usual definition for an ordinary 2-form to be closed.

DEFINITION 6.2. If L and L' are $\mathscr{G}$-valued path 2-forms, L is said to be equivalent to L' if there exists a $g \in G$ such that $gL\langle \cdot,\cdot \rangle g^{-1} = L'\langle \cdot,\cdot \rangle$. Denote the equivalence class containing L by $[L]$.

If L is a closed $\mathscr{G}$-valued path 2-form, then by Proposition 6.1, the parallel translation operator associated to B^L on Ω_m induces a homomorphism from $\Pi_1(\Omega_m, C_m) = \Pi_2(M, m)$ to G, where $C_m \in \Omega_m$ is the constant loop at $m \in M$. If this homomorphism is trivial, then L is said to have zero monodromy. We now state the main theorem of this section.

THEOREM 6.1. *Suppose that M is simply connected so that Ω_m is connected. Let $\mathscr{L} = \mathscr{L}(M, V, G) = \{[L]$: L is a $\mathscr{G}$-valued closed path 2-form with trivial monodromy$\}$. Then the mapping*

$$([E, \nabla] \to [L^\nabla]): \mathscr{E} \to \mathscr{L} \tag{6.1}$$

is a one to one correspondence, where $[L^\nabla] \equiv [\kappa \circ L^\nabla\langle \cdot,\cdot \rangle \kappa^{-1}]$ for any admissible κ: $E_m \to V$. Note that the equivalence class $[L^\nabla]$ is independent of the κ chosen.

If $\Pi_2(M, m)$ is trivial then the monodromy condition is of course vacuous. The analogous theorem for the case in which M is not simply connected is still open. The problem is to find a characterization of the set of $[L^\nabla]$'s in this case. (For an indication of the difficulties of this problem see Example 7.1.) Even if a characterization was found, we will see in

Theorem 7.1 that the correspondence in (6.1) would no longer be one to one.

The proof of Theorem 6.1 will be postponed until we have proved a number of useful propositions which will also help to motivate the theorem. Let B^∇ be the integrated lasso form for a connection ∇ as in Definition 4.3. Our main point of view is that B^∇ should be considered as a globally defined connection one form on the trivial vector bundle $\mathscr{P}_m \times E_m$. Recall from Corollary 4.3 that β^*E is isomorphic to $\mathscr{P}_m \times E_m$. Since β^*E is the pullback of a bundle with a connection ∇, there is a natural connection $\beta^*\nabla$ on β^*E. The next proposition shows that under the identification of $\mathscr{P}_m \times E_m$ with β^*E, the connection one form for $\beta^*\nabla$ is B^∇.

PROPOSITION 6.3. *Let β^*P^∇ denote the induced parallel translation operator on the bundle β^*E, and H: $\beta^*E \to \mathscr{P}_m \times E_m$ be the global trivialization defined in Corollary 4.3. Suppose that γ: $I \to \mathscr{P}_m$ is a C^∞ map, then*

$$R^\nabla(\gamma) \equiv H_{\gamma(1)} \beta^*P^\nabla(\gamma)\, H^{-1}_{\gamma(0)} = P^\nabla(\gamma(1))^{-1}\, P^\nabla(\beta \circ \gamma)\, P^\nabla(\gamma(0)), \tag{6.2}$$

and B^∇ is the connection 1-form for $\beta^*\nabla$ with respect to the trivialization (H) of the bundle $\beta^*E = \{(\sigma, \xi)$: $\xi \in E_{\sigma(1)}$ where $\sigma \in \mathscr{P}_m\}$.

Proof. Recall that $(\beta^*P^\nabla)(\gamma) \cdot (\gamma(0), \xi) = (\gamma(1),\ P^\nabla(\beta \circ \gamma)\, \xi)$, where $\xi \in E_{\beta \circ \gamma(0)}$. Hence, the fact that (6.2) holds is an immediate consequence of the definition of H in Corollary 4.3. In order to show that B^∇ is the connection 1-form for $\beta^*\nabla$ with respect to the trivialization (H), it suffices to show

$$\frac{d}{ds} R^\nabla(\gamma^s) + B^\nabla\langle \gamma'(s) \rangle R^\nabla(\gamma^s) = 0. \tag{6.3a}$$

This is equivalent to showing that

$$\frac{d}{ds} R^\nabla(\gamma^s)^{-1} = R^\nabla(\gamma^s)^{-1} B^\nabla\langle \gamma'(s) \rangle. \tag{6.3b}$$

To this end define Γ: $I \to \mathscr{P}_m$ by

$$\Gamma(s)(t) \equiv \begin{Bmatrix} \gamma(s)(2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ \gamma(s(1-2t))(1) & \frac{1}{2} \leqslant t \leqslant 1 \end{Bmatrix}, \tag{6.4}$$

which is seen to be C^∞ by Lemma 3.2, Proposition 3.3, and Proposition 4.2. In terms of (6.4), $R^\nabla(\gamma^s)^{-1} = P^\nabla(\gamma(0))^{-1}\, P^\nabla(\Gamma(s))$. We may apply

Corollary 4.4 to this last expression to find $(d/ds)\, R^\nabla(\gamma^s)^{-1} = R^\nabla(\gamma^s)^{-1}\, B^\nabla\langle \Gamma'(s)\rangle$, where by definition

$$B^\nabla\langle \Gamma'(s)\rangle = \int_0^1 L^\nabla(\Gamma(s)')\langle \partial_s\Gamma(s)(t), \partial_t\Gamma(s)(t)\rangle\, dt$$
$$= \int_0^{1/2} L^\nabla(\Gamma(s)')\langle \partial_s\Gamma(s)(t), \partial_t\Gamma(s)(t)\rangle\, dt. \tag{6.5}$$

The second equality follows from the fact that $\partial_s\Gamma(s)(t)$ and $\partial_t\Gamma(s)(t)$ are parallel for $\frac{1}{2}\leqslant t\leqslant 1$ as is easily seen from (6.4). (Note: L is a 2-form and hence zero on parallel vectors.) By a change of variables in (6.5) and the expression (6.4) for Γ, one easily shows that $B^\nabla\langle\Gamma'(s)\rangle = B^\nabla\langle\gamma'(s)\rangle$. Q.E.D.

PROPOSITION 6.4. *Let $F^{B^\nabla} = dB^\nabla + B^\nabla \wedge B^\nabla$ denote the curvature of B^∇ as in Definition* 2.1, *then*

$$F^{B^\nabla}(\sigma) = L^\nabla(\sigma)\langle \beta_*\cdot, \beta_*\cdot\rangle \tag{6.6}$$

for all $\sigma\in\mathscr{P}_m$. In particular, the curvature of B, when restricted to the submanifold Ω_m is identically zero.

Proof. The curvature $(F^{\beta^*\nabla})$ of $\beta^*\nabla$ on β^*E is simply the pullback β^*F^∇ of F^∇. So equation (6.6) now follows immediately from how $F^{\beta^*\nabla}$ and F^{B^∇} are related (see Definition 2.1) and the definition of L^∇ (see Definition 4.2). Finally, F^{B^∇} is zero when restricted to Ω_m, since β_* restricted to Ω_m is identically zero. Q.E.D.

Remark. The fact that F^{B^∇} is zero on Ω_m is also evident in the fact that the parallel translation operators (R^∇) defined by B^∇ are path independent for paths in Ω_m.

PROPOSITION 6.5. *Let Ω be a connected Hilbert manifold and B be a smooth $\mathscr{G}$-valued* 1*-form on Ω. Then there exists a smooth function $P\colon\Omega\to G$ such that*

$$XP = P(\sigma)\, B\langle X\rangle \tag{6.7}$$

for all $\sigma\in\Omega$ and $X\in T\Omega$ if and only if $F^B = dB + B\wedge B = 0$ and the homomorphism from $\Pi_1(\Omega,\sigma)$ to G described in Proposition 6.1 *is trivial. (Such a B is said to have trivial monodromy.)*

This proposition is the "curved" version of Lemma 3.6 of [G1].

Proof. Suppose that P is a solution to (6.7). Let $\alpha\colon(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\to\Omega$ be a smooth function. Let $p(s,t)\equiv P(\alpha(s,t))$, so that $p\in C^\infty((-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon), G)$. Therefore $\partial_s\partial_t p(s,t) = \partial_t\partial_s p(s,t)$. So compute

$$\partial_s\partial_t p(s,t) = \partial_s\{p(s,t)\, B\langle\partial_t\alpha(s,t)\rangle\}$$
$$= p(s,t)\{\partial_s B\langle\partial_t\alpha(s,t)\rangle + B\langle\partial_s\alpha(s,t)\rangle\, B\langle\partial_t\alpha(s,t)\rangle\}. \tag{6.8}$$

Interchange s and t in (6.8) and subtract from (6.8) to get

$$\partial_s B\langle\partial_t\alpha(s,t)\rangle - \partial_t B\langle\partial_s\alpha(s,t)\rangle + [B\langle\partial_s\alpha(s,t)\rangle, B\langle\partial_t\alpha(s,t)\rangle] = 0.$$

But the left hand side of this last equation is $F^B\langle\partial_s\alpha(s,t), \partial_t\alpha(s,t)\rangle$. Since α was arbitrary we conclude that F^B is identically zero.

Let $\gamma\in C^1(I,\Omega)$, and set $R_s(\gamma) = P(\gamma(s))^{-1}\, P(\gamma(0))$. Then an easy computation using equation (6.7) shows that

$$\frac{d}{ds} R_s(\gamma) + B\langle\gamma'(s)\rangle\, R_s(\gamma) = 0, \tag{6.9}$$

so that R is the parallel translation operator associated to B, see Definition 2.2. Clearly $R(\gamma)\equiv R_1(\gamma) = 1$ if γ is a loop. So the induced representation on $\Pi_1(\Omega)$ is trivial. (We could have deduced the curvature zero condition from the path independence of R.)

Conversely, suppose that B is given with curvature zero and trivial monodromy. In other words, if $R_s(\gamma)$ denotes the solution to (6.9) with $R_0(\gamma) = 1\in G$, then $R(\gamma) = 1\in G$ if γ is a loop based at some fixed $\sigma\in\Omega$. Suppose that $\tau\in\Omega$, choose any C^1 path (γ) from σ to τ, and then define $P(\tau)\equiv R(\gamma)^{-1}$. The function $P\colon\Omega\to G$ is well defined since R is path independent (R(closed loop) $= 1$). Suppose that $X\in T_\tau\Omega$, choose $\gamma\in C^\infty([0, 1+\varepsilon), \Omega)$ such that $\gamma(0) = \sigma$ and $\gamma'(1) = X$. Then by (6.9), $(d/ds)\, P(\gamma(s)) = (d/ds)\, R_s(\gamma)^{-1} = R_s(\gamma)^{-1}\, B\langle\gamma'(s)\rangle$. Evaluating this last expression at $s=1$ shows that $XP = P(\tau) B\langle X\rangle$ for all $\tau\in\Omega$ and $X\in T_\tau\Omega$.

The only thing left to prove is that the function P is C^∞. Let $\tau\in\Omega$, and choose a τ-contraction φ over neighborhood $U\subset\Omega$ of τ. Since P satisfies equation (6.7), it follows that

$$\frac{d}{dt} P(\varphi(t,\rho)) = P(\varphi(t,\rho))\, B\left\langle \frac{d}{dt}\varphi(t,\rho)\right\rangle \tag{6.10}$$

for all $\rho\in U$, and $t\in I$. Thus $P(\varphi(t,\rho))$ is a solution to a linear ordinary differential equation, with the coefficients $(B\langle(d/dt)\,\varphi(t,\rho)\rangle)$ depending smoothly on a parameter (ρ). So by standard regularity theorems of ordinary differential equations (see Lang [L]), it follows that $P(\rho) = P(\varphi(1,\rho))$ is smooth for $\rho\in U$. Q.E.D.

Remark 6.1. A solution to the system of equations (6.7) is uniquely determined by specifying P at one point. This is the case since $P(\tau)$ can be found by solving a differential equation of the form (6.10) once $P(\sigma)$ is specified. Furthermore, the value at any one point can be specified arbitrarily, since for any $g \in G$, $gP(\cdot)$ is a solution to the system (6.7) provided $P(\cdot)$ is a solution.

LEMMA 6.1. *Let M be simply connected so that Ω_m is connected. Suppose that L is a $\mathscr{G}$-valued path 2-form on $\mathscr{P}_m$ with trivial monodromy. Then by Proposition 6.5, there exists a solution P to $XP = P(\sigma)\, B^L\langle X\rangle$ for all $\sigma \in \Omega_m$ and $X \in T_\sigma \Omega_m$. Furthermore, by Remark 6.1, it is possible to choose P such that $P(C_m) = 1 \in G$, where C_m is the constant loop at m. This function P is a strongly differentiable, parameterization invariant, and multiplicative (i.e., SDPIM).*

Proof. It is obvious that P is strongly differentiable. Now let r be a reparameterization of I and set $r_s(t) = (1-s)\,r(t) + st$ so that $s \to \sigma \circ r_s$ is a path in Ω_m from $\sigma \circ r$ to σ, provided $\sigma \in C^2\Omega_m$. Using (6.7),

$$\frac{d}{ds} P(\sigma \circ r_s) = P(\sigma \circ r_s)\, B^L \left\langle \frac{d}{ds} \sigma \circ r_s \right\rangle = 0$$

because $(d/ds)\,\sigma \circ r_s(t)$ and $(d/ds)\,\sigma \circ r_s(t)$ are both proportional to $\sigma' \circ r_s(t)$, so that $L\langle (d/ds)\,\sigma \circ r_s(t), (d/ds)\,\sigma \circ r_s(t)\rangle = 0$ and hence $B^L\langle (d/ds)\,\sigma \circ r_s\rangle = 0$. Therefore, $P(\sigma) = P(\sigma \circ r)$ for all C^2 loops σ. By continuity, this holds for all loops $\sigma \in \Omega_m$.

Let $\sigma, \tau \in \Omega_m$, and choose a smooth path $\gamma\colon I \to \Omega_m$ starting at C_m and ending at τ. Since τC_m is a reparameterization of τ, $P(\tau C_m) = P(\tau)$. The strategy now is to show that both $P(\tau\gamma(s))$ and $P(\tau)\, P(\gamma(s))$ satisfy the same differential equation. By (6.7), this amounts to showing that $B\langle (d/ds)\,\tau\gamma(s)\rangle = B\langle (d/ds)\,\gamma(s)\rangle$. But

$$\begin{aligned} B\left\langle \frac{d}{ds} \tau\gamma(s)\right\rangle &= \int_0^1 L(\tau\gamma(s)')\left\langle \frac{d}{ds}[\tau\gamma(s)](t), \frac{d}{ds}[\tau\gamma(s)](t)\right\rangle dt \\ &= \int_0^{1/2} L(\tau\gamma(s)')\left\langle \frac{d}{dt}[\tau\gamma(s)](t), \frac{d}{ds}[\tau\gamma(s)](t)\right\rangle dt, \\ &= \int_0^{1/2} L(\tau\gamma(s)')\left\langle \frac{d}{dt}\gamma(s)(2t), \frac{d}{ds}\gamma(s)(2t)\right\rangle dt \\ &= \int_0^{1} L(\tau\gamma(s)^{t/2})\left\langle \frac{d}{dt}\gamma(s)(t), \frac{d}{ds}\gamma(s)(t)\right\rangle dt \\ &= \int_0^{1} L(\gamma(s)^{t})\left\langle \frac{d}{dt}\gamma(s)(t), \frac{d}{ds}\gamma(s)(t)\right\rangle dt \\ &= B\left\langle \frac{d}{ds}\gamma(s)\right\rangle, \end{aligned}$$

where $(d/ds)(\tau\gamma(s))(t) = 0$ for $t > \frac{1}{2}$ was used in the first equality, and a change of variables was performed to get the fourth equality. The other steps are all a matter of using the definitions. Q.E.D.

The proof of Theorem 6.1 is now an easy matter.

Proof of Theorem 6.1. By Proposition 6.3 and 6.4, if $L = \kappa L^\nabla \kappa^{-1}$, for some lasso L^∇, where $\kappa\colon E_m \to V$ is an admissible isomorphism, then L is closed and has trivial monodromy. Furthermore, suppose that (E, ∇) and (E', ∇') are equivalent and $K\colon E \to E'$ is a vector bundle isomorphism exibiting this equivalence. Then it is easy to check that $K_m L^\nabla\langle\cdot,\cdot\rangle K_m^{-1} = L^{\nabla'}$, so $[L^\nabla] = [L^{\nabla'}]$. Thus the map in (6.1) ($[E,\nabla] \to [L^\nabla]$) is well defined.

Now suppose that $[L] \in \mathscr{L}$. By Proposition 6.5 and Lemma 6.1, there is a unique SDPIM function (P) such that $XP = P(\sigma)\, B^L\langle X\rangle$ for all $X \in T_\sigma\Omega_m$ and $\sigma \in \Omega_m$. By Theorem 5.1, there is an $[E,\nabla] \in \mathscr{E}$ such that $[P^\nabla] = [P]$. Thus, there is an admissible isomorphism $\kappa\colon E_m \to V$ such that

$$\kappa \circ P^\nabla(\cdot) \circ \kappa^{-1} = P \text{ on } \Omega_m. \tag{6.11}$$

Differentiating (6.11) by $X \in T_\sigma\Omega_m$ implies that $\kappa \circ B^\nabla\langle X\rangle \circ \kappa = B^L\langle X\rangle$ for $X \in T\Omega_m$. Let $\tau \in \mathscr{P}_m$ and $u, v \in T_x M$ where $x \equiv \tau(1)$. Choose a path $\sigma \in \Omega_m$ such that $\sigma^{1/2} = \tau$ and $(d/dt)|_{1/2}\,\tau = u$. Choose $X_n \in T_\sigma\Omega_m$, such that $X_n(t) = 0$ if $t \leqslant \frac{1}{2}$, and $X_n \to v\delta_{1/2}$. In other words, choose X_n such that $\int_0^1 f(t)\langle X_n(t)\rangle\, dt \to f(1/2)\langle v\rangle$ for any $f \in C\Gamma_\sigma(T^*M)$—the continuous sections of T^*M along σ. Then $L(\sigma)\langle u, v\rangle = \lim_{n\to\infty} B^L\langle X_n\rangle = \lim_{n\to\infty} \kappa \circ B^\nabla\langle X_n\rangle \circ \kappa^{-1} = \kappa \circ L^\nabla(\sigma)\langle u, v\rangle\, \kappa^{-1}$. This shows that $[L] = [L^\nabla]$, and hence $[E,\nabla] \to [L^\nabla]$ is onto.

Now suppose that $[L^\nabla] = [L^{\nabla'}]$; that is suppose there exists an admissible $k\colon E_m \to E'_m$ such that $kL^\nabla(\cdot)\langle\cdot,\cdot\rangle k^{-1} = L^{\nabla'}(\cdot)\langle\cdot,\cdot\rangle$. This implies that $kB^\nabla\langle\cdot\rangle k^{-1} = B^{\nabla'}\langle\cdot\rangle$. By the uniqueness of solutions to the system of equations (6.7) (see Remark 6.1), it follows that $kP^\nabla(\cdot)\, k^{-1} = P^{\nabla'}$ on Ω_m. That is $[P^\nabla] = [P^{\nabla'}]$. So again by Theorem 5.1, it follows that $[E,\nabla] = [E',\nabla']$. Hence the map in (6.1) is one to one. Q.E.D.

From the proof it is fairly evident that Theorem 6.1 could be formulated directly in terms of $\mathscr{G}$-valued 1-forms B on $\mathscr{P}_m$. That is the path 2-forms L could be eliminated altogether. I refer the reader to Theorem 3.13 of [G1] for a result along these lines which has an obvious generalization to this setting.

We will finish this section by discussing characterizations for a $\mathscr{G}$-valued path 2-form L to be closed. Gross, in [G1], gives a number of characterizations using the notion of "end point derivative." The definitions and characterization used in [G1] all carry over to this more general case with only minor changes. So I will concentrate on one specific characterization of closedness given in Corollary 4.12 and Remark 4.13 of [G1]. This is not a serious omission, since the other characterizations can be easily deduced from the results to follow. The following lemma allows us to define the endpoint derivative. The reader is urged to consult [G1] for a more natural definition of this derivative.

LEMMA 6.2. *Let* $u, z \in \Gamma(TM)$, *and* $U = u \circ \beta$ *and* $Z = z \circ \beta$ *be as in Corollary* 4.6. *Suppose that* L *is a* $\mathscr{G}$*-valued path* 2*-form on* $\mathscr{P}_m$ *which satisfies*

$$X(L\langle U, Z\rangle) + [B\langle X\rangle, L\langle U, Z\rangle] = 0 \tag{6.12}$$

for all $X \in T\mathscr{P}_m$ *for which* $X(1) = 0$, *where* $B = B^L$. *Also suppose that* $\{X_n\}$ *is a sequence in* $T_\sigma \mathscr{P}_m$ *for which*

$$X_n(t) \to \chi_{\{1\}}(t)\, w \equiv \begin{Bmatrix} 0 & \text{if } 0 \leqslant t < 1 \\ w \in T_{\sigma(1)}M & \text{if } t = 1 \end{Bmatrix} \quad \text{as } n \to \infty,$$

and

$$|X_n(t)| \leqslant C < \infty$$

for all n, and some constant C. Then the limit $\lim_{n\to\infty} X_n(L\langle U, Z\rangle)$ *exists and is independent of the particular sequence converging to* $\chi_{\{1\}} w$. *This limit will be denoted by* $w(L\langle U, Z\rangle)$, *and will be called the endpoint derivative of* $L\langle U, Z\rangle$ *with respect to w. The endpoint derivative satisfies the relation*

$$X(1)\, L\langle U, Z\rangle(\sigma) = X(L\langle U, Z\rangle) + [B\langle X\rangle, L\langle U, Z\rangle(\sigma)], \tag{6.13}$$

where $X \in T_\sigma \mathscr{P}_m$ *and* $X(1)$ *need not be zero.*

Remark. The endpoint derivative is interpreted to be the variation of a function as one varies the endpoint of a path. Equation (6.13) may be interpreted as saying that the end point derivative of $L\langle U, V\rangle$ is the same as the covariant derivative with respect to the connection determined by the $\mathscr{G}$-valued 1-form (B) on $\mathscr{P}_m$.

Proof. Fix $X \in T_\sigma \mathscr{P}_m$ such that $X(1) = w \in T_{\sigma(1)}M$. Then $Y_n \equiv X - X_n$ satisfies $Y_n(1) = 0$. So by hypothesis

$$Y_n(L\langle U, Z\rangle) + [B\langle Y_n\rangle, L\langle U, Z\rangle(\sigma)] = 0 \qquad \text{for all } n.$$

Or in other words

$$\begin{aligned} &X_n(L\langle U, Z\rangle) + [B\langle X_n\rangle, L\langle U, Z\rangle(\sigma)] \\ &\quad = X(L\langle U, Z\rangle) + [B\langle X\rangle, L\langle U, Z\rangle(\sigma)] \end{aligned}$$

for all n. By the dominated convergence theorem, $B\langle X_n\rangle$ converges to zero

as $n \to \infty$. This shows that the $\lim_{n\to\infty} X_n(L\langle U, Z\rangle)$ exists and satisfies (6.12). Q.E.D.

Remark 6.2. If the path 2-form L is a lasso, then by Corollary 4.6, L^∇ satisfies condition (6.12), and furthermore the end point derivative of L^∇ with respect to $w \in T_{\sigma(1)}M$ $(\sigma \in \mathscr{P}_m)$ is

$$w(L^\nabla\langle U, Z\rangle)(\sigma) = P^\nabla(\sigma)^{-1}\, \nabla_w(F^\nabla\langle u, z\rangle)\, P^\nabla(\sigma). \tag{6.14}$$

DEFINITION 6.3. Let L be a path 2-form satisfying (6.12). The endpoint differential of L is a path 3-form defined by

$$(d^e L)\langle U, Z, W\rangle \equiv UL\langle Z, W\rangle + L\langle [U, Z], W\rangle + \text{cyclic combinations} \tag{6.15}$$

where U, Z, W are $u \circ \beta$, $z \circ \beta$, $w \circ \beta$, respectively, and u, z, w are vector fields on M. The notation $[U, Z]$ denotes $[u, z] \circ \beta$.

Remark 6.3. One can easily check that $d^e L\langle U, Z, W\rangle$ is antisymmetric and tensorial in U, V, W. If $x = (x^1, x^2, ..., x^n)$ is chart for M, then write $L_{ij}(\sigma) \equiv L\langle \partial_i, \partial_j\rangle(\sigma)$ and $d^e L_{ijk}(\sigma) = (d^e L)\langle \partial_i, \partial_j, \partial_k\rangle(\sigma)$ where $\partial_i = (\partial/\partial x^i) \circ \beta$. In these local coordinates (6.15) becomes

$$d^e L_{ijk} = \partial_i L_{jk} + \text{cyclic}, \tag{6.16}$$

valid for σ such that $\sigma(1)$ in the domain of the chart x.

By Remarks 6.2 and the definition of d^e, if $L = L^\nabla$ is a lasso, then $d^e L^\nabla = 0$, since F^∇ satisfies the Bianchi identity $d^\nabla F^\nabla = 0$. This suggests the following theorem, see Theorem 4.5 of [G1] for the case $M = \mathbb{R}^n$.

THEOREM 6.2. *Let L be a path 2-form which satisfies condition* (6.12) *and the "Bianchi" identity* $d^e L = 0$. *Then L is closed—that is the curvature of* $B = B^L$ *is zero on* Ω_m.

Proof. The proof will be quite similar to the proof of Theorem 4.5 of [G1]. We must show that $F^B\langle X, Y\rangle = 0$ for X and Y in $T_\sigma \Omega_m$ and $\sigma \in \Omega_m$, where $F^B \equiv (dB + B \wedge B)$. By continuity it is sufficient to prove it for the special case where X and Y in $T_\sigma \Omega_m$ are C^∞-sections along a C^∞-path $\sigma \in \Omega_m$. Let $\gamma(s, t)(r)$ be the C^∞-map given by $\exp_{\sigma(r)}(sX(r) + tY(r))$ for s and t near zero. Then $\gamma = \gamma(\cdot,\cdot)(\cdot) : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to \Omega_m$ is C^∞ and $X = \gamma_s(0, 0)$ and $Y = \gamma_t(0, 0)$, where γ_s and γ_t denote partial derivatives of γ with respect to s and t, respectively. Since the differential d commutes with pullbacks, $dB\langle \gamma_s(s, t), \gamma_t(s, t)\rangle$ may be computed as $\partial_s B\langle \gamma_t(s, t)\rangle - \partial_t B\langle \gamma_s(s, t)\rangle$.

In order to simplify notation, the s and t parameters will be suppressed. If f is a function of $f(s, t, r)$, then f_t will denote $\partial_t f$, $f_s = \partial_s f$, and $\dot{f} = \partial_r f$.

Also let $\gamma^r \equiv \gamma(s,t)^r$, $\gamma_s^r \equiv \gamma_s(s,t)^r$, $\gamma_t^r \equiv \gamma_t(s,t)^r$, $\gamma_t = \gamma(s,t)(r)$, $\gamma_s = \gamma_s(s,t)(r)$, and $\dot\gamma = \partial_r\gamma(s,t)(r)$. Then a straightforward but tedious calculation using the fact that $d^e L = 0$ (see the appendix to this section) shows that

$$\begin{aligned}&\partial_s(L(\gamma^r)\langle\dot\gamma,\gamma_t\rangle) - \partial_t(L(\gamma^r)\langle\dot\gamma,\gamma_s\rangle)\\ &\quad = \partial_r(L(\gamma^r)\langle\gamma_s,\gamma_t\rangle) + [L\langle\dot\gamma,\gamma_t\rangle, B\langle\gamma_s^r\rangle] - [L\langle\dot\gamma,\gamma_s\rangle, B\langle\gamma_t^r\rangle]. \quad (6.17)\end{aligned}$$

Now observe

$$\partial_s B\langle\gamma_t\rangle = \int_0^1 \partial_s(L(\gamma^r)\langle\dot\gamma(r),\gamma_t(r)\rangle)\,dr \quad (6.18)$$

and

$$\int_0^1 \partial_r(L(\gamma^r)\langle\gamma_s,\gamma_t\rangle)\,dr = \partial_r(L(\gamma^r)\langle\gamma_s,\gamma_t\rangle)|_0^1 = 0,$$

since $\gamma_s(s,t)(r) = 0$ at $r=0$ and $r=1$. Therefore, by integrating (6.17) with respect to r over the unit interval (I) we find

$$\begin{aligned}&\partial_s B\langle\gamma_t\rangle - \partial_t B\langle\gamma_s\rangle\\ &\quad = \int_I \{[L(\gamma^r)\langle\dot\gamma(r),\gamma_t(r)\rangle, B\langle\gamma_s^r\rangle]\\ &\qquad - [L(\gamma^r)\langle\dot\gamma(r),\gamma_s(r)\rangle, B\langle\gamma_t^r\rangle]\}\,dr. \quad (6.19)\end{aligned}$$

But by the definition of B and a simple change of variables, we get

$$\begin{aligned}B\langle\gamma_s^r\rangle &= \int_0^1 L(\gamma^{ru})\langle r\dot\gamma(ru),\gamma_s(ru)\rangle\,du\\ &= \int_0^r L(\gamma^u)\langle\dot\gamma(u),\gamma_s(u)\rangle\,du.\end{aligned}$$

So by this last equation, (6.19) may be rewritten as

$$\begin{aligned}&\partial_s B\langle\gamma_t\rangle - \partial_t B\langle\gamma_s\rangle\\ &\quad = \int_0^1\int_0^r \{[L(\gamma^r)\langle\dot\gamma(r),\gamma_t(r)\rangle, L(\gamma^u)\langle\dot\gamma(u),\gamma_s(u)\rangle]\\ &\qquad - [L(\gamma^r)\langle\dot\gamma(r),\gamma_s(r)\rangle, L(\gamma^u)(\dot\gamma(u),\gamma_t(u)\rangle]\}\,du\,dr\\ &\quad = \int_0^1\int_0^r \{[L(\gamma^r)\langle\dot\gamma(r),\gamma_t(r)\rangle, L(\gamma^u)\langle\dot\gamma(u),\gamma_s(u)\rangle]\\ &\qquad + [L(\gamma^u)\langle\dot\gamma(u),\gamma_t(u)\rangle, L(\gamma^r)\langle\dot\gamma(r),\gamma_s(r)\rangle]\}\,du\,dr. \quad (6.20)\end{aligned}$$

After interchanging the letters u and r and the order of integration in the second term, one shows that the two terms in (6.20) add up to give $[B\langle\gamma_t\rangle, B\langle\gamma_s\rangle]$. This proves B has zero curvature on Ω_m. Q.E.D.

As a consequence of this last characterization for a path 2-form to be closed, we can easily see that Theorem 6.1 reduces for the $U(1)$ structure group to the standard result of Weil's stated in the introduction.

Corollary 6.1. *Let F be an imaginary valued 2-form on a simply connected manifold M for which $[(2\pi i)^{-1}F]$ is integral, then there exists a $U(1)$-line bundle connection pair (E,∇) such that the curvature 2-form $F^\nabla = F$. This correspondence characterizes the pairs (E,∇) up to equivalence.*

Proof. Let $L(\sigma) = L^F(\sigma) \equiv F(\sigma(1))$, a path 2-form on $\mathscr{P}_m$. Condition (6.12) is easily seen to hold in this case and it is easy to check that $(d^eL)(\sigma) = dF(\sigma(1))$ which is zero by assumption. Set $B\langle X\rangle = B^L\langle X\rangle = \int_0^1 F(\sigma(s)\langle\sigma'(s), X(s)\rangle\,ds$. Now suppose that $\gamma: I\to\Omega_m$ is a smooth path such that $\gamma(0)=\gamma(1)=C_m$—the constant path at m. Since $U(1)$ is commutative, the operators $B\langle\gamma'(s)\rangle$ all commute among themselves, so that differential equation for parallel translation may be explicitly solved to give

$$\begin{aligned}P^B(\gamma) &= \exp\left\{\int_0^1 B\langle\gamma'(s)\rangle\,ds\right\}\\ &= \exp\left\{\int_0^1\int_0^1 F(\gamma(s)(t))\langle\partial_t\gamma(s)(t), \partial_s\gamma(s)(t)\rangle\right\}.\end{aligned}$$

Now thinking of γ as a map from $I\times I$ to M, the last double integral may be written as $\int_\gamma F \equiv \int_{I\times I}\gamma^* F$—using the standard differential form notation. According to Theorem 6.1, L is a lasso if and only if $P^B(\gamma) = \exp(\int_\gamma F) = 1 \in U(1)$ for all such γ. That is $\int_\gamma F\in 2\pi i\mathbb{Z}$. Now by the Hurewicz isomorphism theorem (see Bott and Tu [BT] Theorem 17.21), the second fundamental group $\Pi_2(M,m)$ is isomorphic to $H_2(M,\mathbb{Z})$, since M is simply connected. Making use of this isomorphism, the map $\gamma: I\to\Omega_m$ with $\gamma(0)=\gamma(1)=C_m$ may be identified as representative of a homology class $[\gamma]$ in $H_2(M,\mathbb{Z})$. Since this mapping from $\Pi_2(M,m)$ to $H_2(M,\mathbb{Z})$ is onto it follows that $\int_\gamma F\in(2\pi i)\mathbb{Z}$ for all $\gamma: I\to\Omega_m$ is equivalent to the statement that $[(2\pi i)^{-1}F]$ is integral. Hence, F is closed and integral if and only if L^F is closed and has trivial monodromy. This shows that $F\to L^F$ is a map from the space of closed imaginary valued 2-forms such that $[(2\pi i)^{-1}F]$ is integral onto the space of lassos. This map is also easily seen to be one to one. The Corollary now follows from Theorem 6.1, and the fact that the conjugacy classes [L] contain only the single element (L) because of the commutativity of $U(1)$. Q.E.D.

Appendix to Section 6

The calculation showing (6.17) holds will be presented in this appendix. Introduce a chart $(x=(x^1, x^2 \cdots x^n))$ on M in a neighborhood of $\gamma(s,t)(r)$ and let ${}^i\gamma = x^i \circ \gamma$. We continue the notation introduced in the proof of theorem 6.2 with obvious extensions to the functions ${}^i\gamma$, for example ${}^i\gamma^r = (x^i \circ \gamma)^r$.

With this notation and the notation of Remark 6.3,

$$L(\gamma^r)\langle \dot\gamma, \gamma_t\rangle = \Sigma L_{ij}(\gamma^r)\cdot {}^i\dot\gamma \cdot {}^j\gamma_t,$$

where all sums are on the indicies i and j from 1 to n. Therefore,

$$\begin{aligned}
&\partial_s(L(\gamma^r)\langle\dot\gamma(r), \gamma_t(r)\rangle)\\
&\quad= \partial_s(\Sigma L_{ij}(\gamma^r)\, {}^i\dot\gamma(r), {}^j\gamma_t(r))\\
&\quad= \Sigma\{(\gamma^r_s L_{ij})\, {}^i\dot\gamma\, {}^j\gamma_t + L_{ij}(\gamma^r)\, {}^i\dot\gamma_s\, {}^j\gamma_t + L_{ij}(\gamma^r)\, {}^i\dot\gamma\, {}^j\gamma_{ts}\}\\
&\quad= \Sigma\{\gamma_s(r)\, L_{ij}\, {}^i\dot\gamma\, {}^j\gamma_t + L_{ij}(\gamma^r)\, {}^i\dot\gamma_s\, {}^j\gamma_t + L_{ij}(\gamma^r)\, {}^i\dot\gamma\, {}^j\gamma_{ts}\}\\
&\qquad + [L(\gamma^r)\langle\dot\gamma(r), \gamma_t(r)\rangle, B\langle\gamma_s(r)\rangle],
\end{aligned} \tag{6.21}$$

where (6.13) was used in the last equality. Taking (6.21) and subtracting the same expression with s replaced by t one finds

$$\begin{aligned}
&\partial_s(L(\gamma^r)\langle\dot\gamma, \gamma_t\rangle - \partial_t(L(\gamma^r)\langle\dot\gamma, \gamma_s\rangle\\
&\quad= \Sigma\{(\dot\gamma(r)\, L_{ij})\cdot {}^i\gamma_s \cdot {}^j\gamma_t + L_{ij}(\gamma^r)\, {}^i\dot\gamma_s\, {}^j\gamma_t - L_{ij}(\gamma^r)\, {}^i\dot\gamma_t\, {}^j\gamma_s\}\\
&\qquad + [L(\gamma^r)\langle\dot\gamma(r), \gamma_t(r)\rangle, B\langle\gamma_s(r)\rangle] - [L(\gamma^r)\langle\dot\gamma(r), \gamma_s(r)\rangle, B\langle\gamma_t(r)\rangle],
\end{aligned} \tag{6.22}$$

where the Bianchi identity (6.16) has been used in the form

$$(\dot\gamma(r)\, L_{ij})\cdot {}^i\gamma_s \cdot {}^j\gamma_t = (\gamma_s(r)\, L_{ij})\cdot {}^i\dot\gamma \cdot {}^j\gamma_t - (\gamma_t(r)\, L_{ij})\cdot {}^i\dot\gamma \cdot {}^j\gamma_s$$

to get the first term. (All derivatives in this last expression are end point derivatives). Now notice that

$$\begin{aligned}
&\partial_r(L(\gamma^r)\langle\gamma_s, \gamma_t\rangle)\\
&\quad= \Sigma\{(\partial_r\, L_{ij}(\gamma^r))\cdot {}^i\gamma_s \cdot {}^j\gamma_t + L_{ij}(\gamma^r)\cdot {}^i\dot\gamma_s \cdot {}^j\gamma_t + L_{ij}(\gamma^r)\cdot {}^i\gamma_s \cdot {}^j\dot\gamma_t\}\\
&\quad= \Sigma\{(\dot\gamma(r)\, L_{ij})\cdot {}^i\gamma_s \cdot {}^j\gamma_t + L_{ij}(\gamma^r)\, {}^i\dot\gamma_s \cdot {}^j\gamma_t + L_{ij}(\gamma^r)\cdot {}^i\gamma_s \cdot {}^j\dot\gamma_t\}.
\end{aligned} \tag{6.23}$$

Where, in the second equality, we have used the fact that

$$\partial_r\, L_{ij}(\gamma^r) = \dot\gamma(r)\, L_{ij}$$

which follows from (6.13) using $B\langle\partial_r \gamma^r\rangle = 0$, and $\partial_r \gamma^r(1) = \dot\gamma(r)$. ($B\langle\partial_r \gamma^r\rangle = 0$, since $\partial_r \gamma^r$ is parallel to $\dot\gamma$.) So (6.23) can be used to replace the sum in (6.22) by $\partial_r(L(\gamma^r)\langle\gamma_s, \gamma_t\rangle)$, which results in the desired equation (6.17).

7. Remarks for M Not Simply Connected

In this section, the manifold M is no longer assumed to be simply connected. For the purposes of this section let $\mathscr{L} = \mathscr{L}(M, V, G) \equiv \{[L^\nabla]: (E, \nabla)$ is a G-bundle connection pair$\}$. If M is not simply connected, we no longer have an intrinsic characterization of $\mathscr{L}$ even in the case $G = U(1)$ (see Example 7.1 below). We first show (Theorem 7.1) that in general the space $\mathscr{L}$ is no longer in 1–1 correspondence with $\mathscr{E} = \mathscr{E}(M, V, G)$. This "gauge copy" problem is intimately related with our difficulty in characterizing the space $\mathscr{L}$. However, in Theorem 7.2 and Corollary 7.1, it is shown that the closed path 2-forms (L) (modulo conjugation) with trivial monodromy are in one to one correspondence with $\mathscr{E}^1 = \mathscr{E}(M^1, V, G)$ where M^1 is the universal covering space of M.

Theorem 7.1. *Suppose that* $[E, \nabla] \in \mathscr{E}$, *and* $P(\sigma) \equiv \kappa \circ P^\nabla(\sigma)\, \kappa^{-1}$ *for* $\sigma \in \Omega_m$, *where* $\kappa: E_m \to V$ *is an admissible isomorphism. Set* $\rho([E, \nabla]) = [L^\nabla]$, *then* $\rho^{-1}([L^\nabla])$ *is in one to one correspondence with the set of functions* $h: \Pi^1(M, m) \to G$, *modulo conjugation by elements of* G, *which satisfy*

$$h([\sigma][\tau])\, P(\sigma) = h([\sigma])\, P(\tau)\, h([\tau]) \tag{7.1}$$

for all $\sigma, \tau \in \Omega_m$. *The notation* $[\sigma] \in \Pi^1(M, m)$ *denotes the homotopy class (i.e., path component of* Ω_m*) containing* σ. *In particular any* $h \in \mathrm{Hom}(\Pi^1(M, m), Z(G))$ *(the homomorphisms from* $\Pi^1(M, m)$ *to the center* $(Z(G))$ *of* G*) satisfies (7.1). Conversely if* G *is connected and* ∇ *is irreducible in the sense that the* $\mathrm{span}\{\kappa \circ L^\nabla(\sigma)\langle u, v\rangle\, \kappa^{-1}: \sigma \in \mathscr{P}_m\ u, v \in T_{\sigma(1)}M\} = \mathscr{G}$, *then every* h *satisfying (7.1) is actually in* $\mathrm{Hom}(\Pi^1(M, m), Z(G))$.

Proof. Set $B\langle\cdot\rangle \equiv \kappa \circ B^\nabla\langle\cdot\rangle\, \kappa^{-1}$. So by Corollary 4.4

$$XP = P(\sigma)\, B\langle X\rangle \tag{7.2}$$

for all $X \in T_\sigma \Omega_m$ and $\sigma \in \Omega_m$. Using Theorem 5.1, it is easy to see that $\rho^{-1}([L^\nabla])$ is in 1–1 correspondence with $\{[P'] \in \mathscr{M}$: for which $gP'(\cdot)\, g^{-1}$ satisfies (7.2) for some $g \in G\}$. This is the same as saying that $\rho^{-1}([L^\nabla])$ is in 1–1 correspondence with $\{[P'] \in \mathcal{M}: XP' = P'(\sigma)\, B\langle X\rangle \text{ for } X \in T\Omega_m\}$. Because of Remark 6.1, the most general solution to (7.2) is of the form

$$P'(\sigma) = h([\sigma])\, P(\sigma) \tag{7.3}$$

where $h: \Pi_1(M, m) \to G$, since Π_1 indexes the path components of Ω_m. Any function of the form (7.3) is strongly differentiable and parameterization invariant. Thus we need only find conditions on h for P' to be multiplicative. It is easily checked, by demanding that $P'(\sigma\tau) = P'(\sigma)\, P'(\tau)$ holds for all σ and τ, that the condition (7.1) is precisely what is needed to make P' defined by (7.3) multiplicative.

Now suppose that G is connected and ∇ is irreducible. If $[\tau] = 1 \in \Pi_1(M, m)$, then (7.1) reduces to $P(\tau) h([\sigma]) = h([\sigma])\, P(\tau)$ for all $\sigma \in \Omega_m$. By the Ambrose Singer Theorem (see for example Corollary 2.17 of [G1] for a proof in the spirit of this paper), we can conclude that $\{P(\tau): [\tau] = 1 \in \Pi^1(M, m)\} = G$, since G is connected. Therefore, $h([\sigma])$ must be in the center of G for all $\sigma \in \Omega_m$, in which case (7.1) requires h to be a homomorphism of groups. Q.E.D.

Let M be a connected but non-simply connected manifold, with universal cover M^1 and covering map $\theta: M^1 \to M$. Choose $m^1 \in M^1$ and set $m = \theta(m^1)$. Let $\mathscr{P}_m \equiv \mathscr{P}_m(M)$ and $\mathscr{P}_{m^1} \equiv \mathscr{P}_{m^1}(M^1)$. If $\sigma \in \mathscr{P}_m$, set $\tilde{\sigma} \in \mathscr{P}_{m^1}$ to be the unique curve such that $\tilde{\sigma}(0) = m^1$, and $\theta \circ \tilde{\sigma} = \sigma$.

Suppose that L^1 is a path 2-form on $\mathscr{P}_{m^1}$. Using L^1 we may define a path 2-form on $\mathscr{P}_m$ by

$$L(\sigma)\langle u, v\rangle = L^1(\tilde{\sigma})\langle \tilde{u}, \tilde{v}\rangle, \tag{7.4}$$

where $\tilde{u} \equiv \theta_{*\tilde{\sigma}(1)}^{-1}(u)$. In the future, $\tilde{u}$ may be written as u, since by context it will be clear when u should be lifted to TM^1. One can easily check that

$$B^L\langle X\rangle = B^{L^1}\langle \tilde{X}\rangle \tag{7.5}$$

where $\tilde{X}$ is the unique element of $T_{\tilde{\sigma}}\mathscr{P}_{m^1}$ such that $\theta_* \tilde{X} = X \in T_\sigma \mathscr{P}_m$.

THEOREM 7.2. *Let L be a $\mathscr{G}$-valued closed path 2-form on $\mathscr{P}_m(M)$, such that L has trivial monodromy in the sense that the representation it induces on $\Pi^2(M, m) = \Pi^1(\Omega_m(M), C_m)$ is trivial. Then there exists a unique element $[E^1, \nabla^1] \in \mathscr{E}^1 \equiv \mathscr{E}(M^1, V, G)$ such that $[\theta^* L] = [L^{\nabla^1}]$, where*

$$(\theta^* L)(\alpha)\langle u, v\rangle \equiv L(\theta \circ \alpha)\langle \theta_* u, \theta_* v\rangle \tag{7.6}$$

for $\alpha \in \mathscr{P}_{m^1}$ and $u, v \in T_{\alpha(1)} M^1$.

Proof. Let $L^1 \equiv \theta^* L$ as in equation (7.6). Then if $B^1 \equiv B^{L^1}$ and $B \equiv B^L$, it is easy to check that

$$B^1 = \theta^* B \tag{7.7}$$

where $(\theta^* B)\langle X\rangle \equiv B\langle \theta_* X\rangle$ for all $X \in T\mathscr{P}_{m^1}$. Note that the map $\theta: M^1 \to M$ naturally induces a map, which is again called θ, from $\mathscr{P}_{m^1}$ to $\mathscr{P}_m$ and consequently from Ω_{m^1} to Ω_m. It is this latter map that θ represents in (7.7). Since exterior derivatives and pullbacks commute, it follows that $dB^1 = d\theta^* B = \theta^* dB$. Also pullbacks respect wedge products so that $\theta^*(B \wedge B) = \theta^* B \wedge \theta^* B$. Therefore,

$$F^{B^1} = \theta^* F^B, \tag{7.8}$$

from which it follows that F^{B^1} is 0 on Ω_{m^1}. Furthermore it is easy to check that

$$P^{B^1} = P^B \circ \theta \tag{7.9}$$

where P^B, and P^{B^1} are the parallel translation operators with respect to the connection 1-forms B and B^1 respectively. So suppose that $\gamma: I \to \Omega_{m^1}$ is a smooth loop based at C_{m^1}. Then $\theta \circ \gamma: I \to \Omega_m$ is a smooth loop based at C_m. Since B has trivial monodromy, $P^{B^1}(\gamma) = P^B(\theta \circ \gamma) = 1$, so that B^1 has trivial monodromy. Thus L^1 satisfies the hypothesis of Theorem 6.1, which may now be used to conclude the proof. Q.E.D.

COROLLARY 7.1. *Let $\mathscr{L}'$ be the collection of $\mathscr{G}$-valued closed path 2-forms (L) on $\mathscr{P}_m$ such that B^L has zero monodromy on the path component of Ω_m containing C_m. Then the map*

$$([L] \to [\theta^* L]): \mathscr{L}' \to \mathscr{L}^1, \tag{7.10}$$

is a 1–1 correspondence where $\mathscr{L}^1 \equiv \mathscr{L}(M^1, V, G)$. Hence by Theorem 6.1, $\mathscr{L}'$ is in 1–1 correspondence with $\mathscr{E}^1 = \mathscr{E}(M^1, V, G)$.

Proof. This follows immediately from Theorem 7.2, and the relation (7.4) which enables one to define the inverse map to (7.10). Q.E.D.

We now give an example which demonstrates that the characterization of the set of lassos given in Theorem 6.1 is no longer sufficient if the manifold (M) is not simply connected.

EXAMPLE 7.1. Let $G = U(1)$ and M be a connected manifold for which $\Pi_1(M, m)$ is non-trivial but $\Pi_2(M, m)$ is trivial. Using the notation above, let F^1 be a closed imaginary valued 2-form on M^1 for which F^1 is not θ^*F for some 2-form on M. (All of this is easily accomplished on $S^1 \times \mathbb{R}$ for instance.) Using F^1, construct a path 2-form (L) on $\mathscr{P}_m$ by

$$L(\sigma) = F^1(\tilde{\sigma}(1)). \tag{7.11}$$

So if $X \in T_\sigma \mathscr{P}_m$ such that $X(1) = 0$, then $XL = 0$, since the variation of σ by X does not change the homotopy class of σ, and hence does not change $\tilde{\sigma}(1)$. So L satisfies the condition in equation (6.12). Furthermore, it is easy to check that $d^eL(\sigma) = dF^1(\tilde{\sigma}(1))$ which is zero by assumption. So by Theorem 6.2, L is closed. The path 2-form L also has trivial monodromy, since $\Pi_2(M, m) = \{1\}$. Nevertheless, L need not be a lasso, because if it were it would imply that $L(\sigma) = F(\sigma(1))$, for some 2-form F. But this would imply that $F^1 = \theta^*F$.

This last example shows that the conditions of Theorem 6.1 no longer characterize the lassos. Furthermore, using this same example wth $M = S^1 \times \mathbb{R}$, one may easily show that the representations induced by $B = B^L$ on the other fundamental groups $\Pi_1(\Omega_m, \sigma)$ are trivial, where $\sigma \in \Omega_m$ is a path not homotopic to C_m. This shows that requiring trivial monodromy on the fundamental groups $\Pi_1(\Omega_m, \sigma)$ for each path component of Ω_m is still not enough to guarantee that L is a lasso.

To finish this section we record a result which may be useful for future considerations of non-simply connected M. In order to state the result, let $\mathrm{Cov}(M^1)$ denote the space of covering transformations (μ) of M^1 with respect to the covering map $\theta: M^1 \to M$. Recall that $\mathrm{Cov}(M^1)$ is a group under composition which is isomorphic to $\Pi_1(M, m)$.

THEOREM 7.3. *Let $\theta: M^1 \to M$, and $m = \theta(m^1)$ be as above. Suppose that (E^1, ∇^1) is a vector bundle over M^1 with connection ∇^1, fiber model space V and structure group G. Then there exists a bundle connection pair (E, ∇) over M such that (E^1, ∇^1) is equivalent to the pullback of (E, ∇) by θ if and only if the following conditions hold:*

(1) *For each $\mu \in \mathrm{Cov}(M^1)$ there exists a lifting to a smooth map $\tilde{\mu}: E^1 \to E^1$ for which $\tilde{\mu}|_p \equiv \tilde{\mu}|_{E^1_p}$ is a linear isomorphism from E^1_p to $E^1_{\mu(p)}$. Furthermore, the lifts should satisfy $\tilde{\mu} \circ \tilde{\nu} = (\mu \circ \nu)^\sim$ for all μ and $\nu \in \mathrm{Cov}(M^1)$.*

(2) *If $S \in \Gamma(E^1)$ is a smooth section, then the connection ∇^1 should satisfy*

$$\nabla^1_v S = \tilde{\mu}^{-1}(\nabla^1_{\mu_* v}(\tilde{\mu} \circ S \circ \mu^{-1})), \tag{7.12}$$

for any $v \in TM^1$.

Remark. The theorem gives a criterion for when one can "push forward" the bundle (E^1, ∇^1) over M^1 to a bundle connection pair (E, ∇) over M.

Sketch of Proof. First suppose that (E^1, ∇^1) is the pullback of (E, ∇). Then by definition of the pullback, the fibers of E^1 are $E^1_x = \{(x, \xi): \xi \in E_{\theta(x)}\}$. So the desired lifts may be defined by $\tilde{\mu}((x, \xi)) = (\mu(x), \xi) \in E^1_{\mu(x)}$. For this choice of lifts, it is easily checked that conditions (1) and (2) hold.

Conversely assume that (E^1, ∇^1) is a bundle connection pair for which conditions (1) and (2) hold. Define an equivalence relation on E^1 by $\xi \approx \eta$ if there is a $\mu \in \mathrm{Cov}(M^1)$ such that $\xi = \tilde{\mu}(\eta)$. It is clear that E should be defined as $E^1/\approx$ with the obvious projection map onto M. To define the local trivializations of E, let (ψ^1, U^1) be a local trivialization of E^1 over an open set of M^1 such that $\theta|_{U^1}$ is a diffeomorphism of U^1 onto an open set $U = \theta(U^1)$ of M. For any $\xi \in E^1$, let $[\xi]$ denote the element of E given by the equivalence class containing ξ. Set $\psi([\xi]) = \psi^1(\xi)$ for all $\xi \in \pi^{-1}(U^1)$, where π is the projection map from E^1 to M^1. It is easy to check using property (1) that these local trivializations define a vector bundle structure on E with structure group G. Now suppose that $S \in \Gamma(E)$ is a section of E, and $v \in T_m M$. Let U be a neighborhood of $\{m\}$, which is covered by U^1 in such a way that $\alpha^{-1} \equiv \theta|_{U^1}$ is a diffeomorphism. The section S can be written as $S(p) = [S^1(\alpha(p))]$ for $p \in U$, where S^1 is a local section of E^1 over U^1. With this notation, define

$$\nabla_v S = [\nabla^1_{\alpha_* v} S^1].$$

One can check using properties (1) and (2) that this last expression is well defined and in fact defines a covariant derivative on E. Furthermore, the bundle pair (E^1, ∇^1) is equivalent to the pullback of the bundle pair (E, ∇). Q.E.D.

So in order to characterize the lassos it is sufficient to find conditions on a path 2-form L such that the bundle connection pair (E^1, ∇^1) constructed in Theorem 7.2 satisfies the hypothesis of Theorem 7.3.

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