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The Lie Bracket of Adapted Vector Fields on Wiener Spaces*

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Abstract. Let $W(M)$ be the based (at $o \in M$) path space of a compact Riemannian manifold M equipped with Wiener measure ν. This paper is devoted to considering vector fields on $W(M)$ of the form $X^h_s(\sigma) = P_s(\sigma)h_s(\sigma)$ where $P_s(\sigma)$ denotes stochastic parallel translation up to time s along a Wiener path $\sigma \in W(M)$ and $\{h_s\}_{s\in[0,1]}$ is an adapted T_oM-valued process on $W(M)$. It is shown that there is a large class of processes h (called adapted vector fields) for which we may view X^h as first-order differential operators acting on functions on $W(M)$. Moreover, if h and k are two such processes, then the commutator of X^h with X^k is again a vector field on $W(M)$ of the same form.

Key Words. Wiener measure, Itô development map, Lie bracket, Integration by parts.

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1. Introduction

Let $(M^d, \langle\cdot,\cdot\rangle, \nabla, o)$ be given, where M is a compact connected manifold (without boundary) of dimension d, $\langle\cdot,\cdot\rangle$ is a Riemannian metric on M, ∇ is a $\langle\cdot,\cdot\rangle$-compatible covariant derivative, and o is a fixed base point in M. Let $T = T^\nabla$ and $R = R^\nabla$ denote the torsion and curvature of ∇, respectively.

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Standing Assumption. The covariant derivative (∇) is assumed to be *Torsion Skew Symmetric* or *TSS* for short. That is to say $\langle T(X,Y),Y\rangle \equiv 0$ for all vector fields X and Y on M. (With the TSS condition, the Laplacian on functions ($\Delta f = \mathrm{tr}(\nabla \mathrm{grad}\, f)$ associated to ∇ is the same as the usual Levi–Civita Laplacian.)

Let ν denote Wiener measure on the path space

$$W(M) \equiv \{\sigma \in C^0([0,1],M) | \sigma(0) = o\}.$$

To be more explicit, let $\Sigma_s(\sigma) \equiv \sigma(s)$ for $\sigma \in W(M)$ and let $\mathcal{F}_s$ be the σ-field on $W(M)$ generated by $\{\Sigma_{s'} : s' \leq s\}$. Then ν is the unique probability measure on $(W(M), \{\mathcal{F}_s\})$ such that $\{\Sigma_s\}_{s\in[0,1]}$ is a diffusion process with $\frac{1}{2}\Delta$ as the generator.

Let P_s denote stochastic parallel translation along Σ up to time s relative to the covariant derivative ∇. Given h in the Cameron–Martin space H,

$$H \equiv \left\{ h : [0,1] \to T_oM | h \text{ is absolutely continuous and } \int_0^1 |h'(s)|^2\, ds < \infty \right\},$$

let X^h denote the *Cameron–Martin vector field* on $W(M)$ given by $X^h_s \equiv P_s h(s)$. It was shown in [15] that X^h may indeed be considered as a vector field on $W(M)$ in the sense that X^h generates a quasi-invariant flow, at least when h is C^1. This theorem was extended by Hsu [34], [35] to include all $h \in H$. See also [44] and [24] for other approaches.

It was also shown in Theorem 9.1, p. 363, of [15] (where X^h was written as ∂_h), that X^h may be viewed as a densely defined closed operator on $L^2(W(M),\nu)$. This last result relies on an integration by parts formula which in the special case of X^h acting on functions of the form $f(\sigma) = F(\sigma(s))$ is due to Bismut [9]. There have been numerous proofs and extensions of integration by parts formulas on $W(M)$, see, for example, [4], [7], [22], [28], [29], [31], [40]–[44], and the references therein for some of the more recent articles. See also Proposition 4.10 below.

The purpose of this paper is to consider the commutator $[X^h, X^k]$ of two vector fields X^h and X^k. It has been known for some time that, in general, the commutator between two Cameron–Martin vector fields on $W(M)$ is no longer a Cameron–Martin vector field. This is explicitly pointed out in Section 6.5 of [14] and in the case that M is a homogeneous manifold by Aida in [4]. Since so much of differential geometry relies on the use of the commutator of vector fields, it is highly desirable to have a class of vector fields which is stable under the Lie bracket operation.

In this paper we study the "adapted vector fields" on $W(M)$ introduced in [18], see also [14]. (Cruzeiro and Malliavin call the adapted vector fields by the suggestive name of tangent processes.) Intuitively, X is an adapted vector field (or tangent process) on $W(M)$ iff $X = (d/dt)|_0\varphi_t$, where $\varphi_t : W(M) \to W(M)$ is a one-parameter family of quasi-invariant adapted maps on $W(M)$ such that $\varphi_0 = id$, see Definitions 3.2 and 4.1 for the precise definition. We call $\{\varphi_t\}_{t\in\mathbb{R}}$ as above an approximate flow for X.

The main result of this paper is Theorem 7.4 and Corollary 8.4 both of which state:

Theorem 1.1 (Informal Version). *The Lie bracket of sufficiently regular "adapted vector fields" on $W(M)$ is an "adapted vector field."*

We may give an informal proof of this theorem as follows. Let X and Y be two adapted vector fields on $W(M)$ and let φ_t and ψ_t be approximate flows for X and Y, respectively. Then (formally) $[X, Y] = (d/dt)|_{0^+}\eta_t$ where η_t is the approximate flow on $W(M)$ defined by

$$\eta_t \equiv \psi_{-\sqrt{t}} \circ \varphi_{-\sqrt{t}} \circ \Psi_{\sqrt{t}} \circ \varphi_{\sqrt{t}}. \tag{1.1}$$

Thus $[X, Y]$ is also an adapted vector field.

The remainder of this paper is devoted to a precise formulation and proof of the above stability result. Along the way we develop explicit formulas for the bracket $[X, Y]$. Our approach here is at the infinitesimal level, viewing vector fields as first-order differential operators. Making the argument given in the above paragraph rigorous would involve a more delicate global analysis. See [10] for the beginnings of such an analysis. In [10] the differential of the flow e^{tX} is computed for certain adapted vector fields X.

As stated above, the reason for wanting vector fields to be stable under the Lie bracket operation is related to the desire to develop calculus and geometry on $W(M)$. Recall that the Lie bracket typically enters into the coordinate-free definition of differential and geometric objects. At this time, the coordinate-free approach seems to be essential when working on path and loop spaces. For **some** of the recent developments on the calculus and geometry of path and loop spaces, see [1]–[6], [8], [11]–[18], [20]–[22], [24]–[36], [38], and [44].

2. Background and Notation

2.1. *Geometric Notation*

Let $O(M)$ denote the orthogonal frame bundle of M. We choose T_oM ($o \in M$ is the fixed base point) as the model fiber of TM so that the fiber of $O(M)$ above $m \in M$ is

$$O_m(M) \equiv \{u : T_oM \to T_mM | u \text{ is an isometry}\}.$$

The structure group of this bundle is the group $O(d)$ of isometries of T_oM. Let $so(d)$ be the Lie algebra of $O(d)$ consisting of skew-symmetric linear transformations on T_oM. Given smooth paths u in $O(M)$ and σ in M such that $u(s) \in O_{\sigma(s)}(M)$, let $\nabla u(s)/ds : T_oM \to T_{\sigma(s)}M$ denote the linear operator defined by $(\nabla u(s)/ds)a = \nabla(u(s)a)/ds$ for all $a \in T_oM$. Notice that $V(s) \equiv u(s)a$ is a vector field along σ so that $\nabla V(s)/ds = \nabla(u(s)a)/ds$ makes sense.

Definition 2.1 (Connection 1-Form). Let ω be the $so(d)$-valued connection 1-form on $O(M)$ given by $\omega\langle u'(0)\rangle \equiv u(0)^{-1}\nabla u(s)/ds|_{s=0}$, where $s \to u(s)$ is any smooth curve in $O(M)$. Notice that a path u is parallel or horizontal in $O(M)$ iff $\omega\langle u'\rangle \equiv 0$.

Definition 2.2 (Horizontal Vector Fields). For $a \in T_oM$ and $u \in O(M)$ let $B\langle a\rangle(u) \in T_uO(M)$ be defined by $\omega\langle B\langle a\rangle(u)\rangle = 0$ (i.e., $B\langle a\rangle(u)$ is horizontal) and $\pi_*B\langle a\rangle(u) = ua$, where $\pi: O(M) \to M$ is the canonical projection map. We often write $B_a(u)$ instead of $B\langle a\rangle(u)$.

Let $T = T^{\nabla}$ and $R = R^{\nabla}$ denote the torsion and curvature of ∇, respectively, and let $\{e_i\}_{i=1}^d$ be an orthonormal basis for T_oM.

Notation 2.3. For $a, b \in T_oM$ and an isometry $u: T_oM \to T_mM$ (i.e., $u \in O_m(M)$), define

$$\Omega_u\langle a, b\rangle \equiv u^{-1}R^{\nabla}\langle ua, ub\rangle u \in End(T_oM),$$

$$Ric_u\langle a\rangle \equiv \sum_{i=1}^{d} \Omega_u\langle a, e_i\rangle e_i,$$

$$\Theta_u\langle a, b\rangle \equiv u^{-1}T^{\nabla}\langle ua, ub\rangle \in T_oM,$$

and

$$\bar{\Theta}_u\langle a\rangle \equiv \sum_{i=1}^{d} B_{e_i}(u)(\Theta_{\cdot}\langle a, e_i\rangle).$$

So Ω, Ric, and Θ are the equi-invariant forms of the curvature tensor, the Ricci tensor, and the torsion tensor, respectively. Similarly, $\bar{\Theta}_u$ is the equi-invariant form of a contraction of ∇T^{∇}.

2.2. *Path Spaces and Development Maps*

In this subsection we introduce a number of path spaces and connecting maps between these path spaces. The reader is referred to [15] and [19] for a more leisurely discussion of this material.

Definition 2.4 (Path Spaces). Let

$$W \equiv W(T_oM) \equiv \{\omega \in C^0([0,1], T_0M) | \omega(0) = 0\},$$

$$W(so(d)) \equiv \{A \in C^0([0,1], so(d)) | A(0) = 0\},$$

$$W(M) \equiv \{\sigma \in C^0([0,1], M) | \sigma(0) = o\},$$

and

$$W(O(M)) \equiv \{u \in C^0([0,1], O(M)) | u(0) = id\}.$$

Let μ denote the Wiener measure on W and let ν denote the Wiener measure on $W(M)$.

A map $b: W(M) \to W$ may be viewed as a continuous process on $W(M)$ with values in T_oM. This is done by writing b_s for the function from $W(M)$ to T_oM given by $b_s(\sigma) = b(\sigma)(s)$. Conversely, given a continuous process b_s on $W(M)$ with values in T_oM, we may define a map $b\colon W(M) \to W$ by $b(\sigma)(s) \equiv b_s(\sigma)$. Similar identifications may be made for maps between any of the two path spaces defined above. In what follows we use both points of view interchangeably. Finally recall from the Introduction that $\Sigma\colon W(M) \to W(M)$ is the process defined by $\Sigma_s(\sigma) = \sigma(s)$ for all $\sigma \in W(M)$.

Definition 2.5 (Connecting Maps). There are the following maps connecting the above path spaces:

1. **(Canonical Brownian Motion)** Let $\beta\colon W \to W$ be the canonical Brownian motion on W given by $\beta_s(\omega) = \omega(s)$ for all $\omega \in W$ and $s \in [0, 1]$.
2. **(Itô Map)** $I\colon W \to W(O(M))$ is defined by the Stratonovich differential equation:

   $$\delta I_s = B\langle \delta\beta_s\rangle(I_s) \qquad \text{with} \qquad I_0 = id \in O_o(M),$$

   where the stochastic integrals are computed relative to the Wiener measure μ on W. Notice that I is horizontal, in the sense that $\omega\langle\delta I\rangle = 0$.
3. **(Projection)** $\pi\colon W(O(M)) \to W(M)$ is an abuse of notation by which we mean $\pi(u) \equiv \pi \circ u$.
4. **(Horizontal Lift/Parallel Translation)** $P\colon W(M) \to W(O(M))$ is the process defined by $\pi(P) = \Sigma$, $P_0 = id|_{T_0M}$, and $\omega\langle\delta P\rangle = 0$, where the stochastic integrals are computed relative to the Wiener measure ν on $W(M)$.
5. **(Development Map)** Let $\varphi\colon W \to W(M)$ denote the composite map $\varphi = \pi \circ I$.
6. **(Inverse Brownian Motion)** Let $b\colon W(O(M)) \to W$ be defined as a version of $\int \theta\langle\delta P\rangle$ (relative to ν), where $\theta\langle\xi_u\rangle \equiv u^{-1}\pi_*\xi$ for $\xi \in T_uO(M)$.

The next theorem is well known. It recalls how the maps above are all related. The proof may be found in many places, see, for example, Theorem 3.3, p. 297, in [15] and also [19] and [23].

Theorem 2.6. *The following identities hold*:

1. $\pi \circ P = id_{W(M)}$ ν*-a.s.*,
2. $\varphi \circ b = id_{W(M)}$ ν*-a.s.*,
3. $b \circ \varphi = id_W$ μ*-a.s.*,
4. $I \circ b = P$ ν*-a.s.*, *and*
5. $I = P \circ \varphi$ μ*-a.s.*

Moreover $\varphi_*\mu = Law(\varphi) = \nu$ *and* $b_*\nu = Law(b) = \mu$.

2.3. *A Collection of Norms*

In this section $(\Omega, \mathcal{F}, P)$ is a probability space, $1 \le p \le \infty$, and $(V, |\cdot|)$ is a finite-dimensional normed vector space.

Definition 2.7. Given a jointly measurable function $f : \Omega \times [0, 1] \to V$ we define:

1. The root mean square norm in L^p:

$$\|f\|_{R^p(V)} \equiv \left\| \left(\int_0^1 |f(\cdot, s)|^2 \, ds \right)^{1/2} \right\|_{L^p}.$$

2. The supremum norm in L^p:

$$\|f\|_{S^p(V)} \equiv \|f^*\|_{L^p},$$

where $f^*(\omega)$ is the essential supremum of $s \to f(\omega, s)$ relative to Lebesgue measure on [0,1]. (Notice that $f^*(\omega) = \lim_{n\to\infty} \|f(\omega, \cdot)\|_{L^n(ds)}$ so that $f^*(\omega)$ is measurable.)

Let $R^p(V)$ denote those $f : \Omega \times [0, 1] \to V$ such that $\|f\|_{R^p(V)} < \infty$. Let $S^p(V)$ denote those $f : \Omega \times [0, 1] \to V$ such that $s \to f(s, \omega)$ is continuous for almost every $\omega \in \Omega$ and $\|f\|_{S^p(V)} < \infty$.

In what follows, we write R^p for $R^p(V)$ and S^p for $S^p(V)$ since the appropriate vector space V may be determined by looking at the range of the function f. We write $f_s(\omega)$, $f(\omega, s)$, $f(\omega)(s)$, and $f(s)(\omega)$ interchangeably.

Now suppose that $(\Omega, \{\mathcal{F}_s\}, \{X_s\}, \mathcal{F}, P)$ is a filtered probability space equipped with a T_oM-valued Brownian motion X. A function $f : \Omega \times [0, 1] \to V$ is a *Brownian semimartingale* if f may be represented in the form

$$f(s, \cdot) = \int_0^s Q_{s'} \, dX(s') + \int_0^s r_{s'} \, ds', \tag{2.1}$$

where (Q_s, r_s) is a predictable process with values in $Hom(T_oM, V) \times V$. ($Hom(T_oM, V)$ denotes the set of linear transformations from T_oM to V.) We call the processes Q_s and r_s the *kernels* of f and write Q_s^f for Q_s and r_s^f for r_s if we are considering more than one Brownian semimartingale at a time. The Brownian semimartingale f is said to have continuous kernels if $s \to (Q_s(\omega), r_s(\omega))$ is continuous for almost every $\omega \in \Omega$.

Definition 2.8. Let $\mathcal{H}^p(X)$ denote the set of Brownian semimartingales f such that

$$\|f\|_{\mathcal{H}^p(X)} \equiv \|Q\|_{R^p} + \|r\|_{R^p} < \infty,$$

and let B^p denote the set of Brownian semimartingales f such that f has continuous kernels (Q_s, r_s) and

$$\|f\|_{B^p(X)} \equiv \|Q\|_{S^p} + \|r\|_{S^p} < \infty.$$

Remark 2.9. The B^p-norms are the same as those used in [15]. However, the $\mathcal{H}^p$-norm differs slightly from the $\mathcal{H}^p$-norm used in [15]. In [15] the $\mathcal{H}^p$-norm was the **weaker**

norm given by

$$\left\| \left(\int_0^1 |Q_s|^2\, ds \right)^{1/2} + \int_0^1 |r_s|\, ds \right\|_{L^p}.$$

To avoid notational clutter, if $(\Omega, X, P) = (W, \beta, \mu)$ and $f: W \times [0,1] \to V$, then we let $\|f\|_{\mathcal{H}^p}$ and $\|f\|_{B^p}$ denote $\|f\|_{\mathcal{H}^p(\beta)}$ and $\|f\|_{B^p(\beta)}$, respectively, where β is the canonical Brownian motion on W given by $\beta_s(\omega) = \omega(s)$. Similarly, if $(\Omega, X, P) = (W(M), b, \nu)$ and $f: W(M) \times [0,1] \to V$, then we let $\|f\|_{\mathcal{H}^p}$ and $\|f\|_{B^p}$ denote $\|f\|_{\mathcal{H}^p(b)}$ and $\|f\|_{B^p(b)}$, respectively, where b is the Brownian motion on $W(M)$ defined in Definition 2.5.

The next lemma is proved by unwinding the definitions and applying Theorem 2.6.

Lemma 2.10. *Suppose that $f_s: W \to V$ is a process and $\tilde{f}_s: W(M) \to V$ is the process defined by $\tilde{f}_s = f_s \circ b$, then $\|f\|_{R^p} = \|\tilde{f}\|_{R^p}$, $\|f\|_{S^p} = \|\tilde{f}\|_{S^p}$, $\|f\|_{\mathcal{H}^p} = \|\tilde{f}\|_{\mathcal{H}^p}$, and $\|f\|_{B^p} = \|\tilde{f}\|_{B^p}$.*

Note, for all $p \in [1, \infty]$, that $\|f\|_{R^p} \le \|f\|_{S^p}$ and $\|f\|_{\mathcal{H}^p} \le \|f\|_{B^p}$. Also it follows from Burkholder's inequality that for each $p \in [1, \infty)$ there is a constant $c_p < \infty$ such that $\|f\|_{S^p} \le c_p \|f\|_{\mathcal{H}^p}$.

Notation 2.11. Given $p \in [1, \infty)$, let $L^{p+} \equiv \bigcup_{q>p} L^q$ and $L^{p-} \equiv \bigcap_{q<p} L^q$. We say that $\lim_{t\to 0} f(t) = f$ in L^{p+} (resp. L^{p-}) if $\lim_{t\to 0} f(t) = f$ in L^q for some $q > p$ (resp. for all $q < p$). Analogous definitions for $R^{p\pm}$, $S^{p\pm}$, $\mathcal{H}^{p\pm}$, and $B^{p\pm}$ are also used.

3. Vector Fields on W

For motivational purposes, recall Theorem 2.1 on p. 408 of [18].

Theorem 3.1 (Structure Theorem). *Let $\Psi: W \to W$ be an adapted map (i.e., $s \to \Psi_s$ is an adapted process) such that $\Psi_* \mu$ is equivalent to μ. Also assume there is an adapted map $\Psi^{-1}: W \to W$ such that $\Psi \circ \Psi^{-1}$ and $\Psi^{-1} \circ \Psi$ are both equal to the identity map μ-a.s. Then there exist $(O(d) \times T_o M)$-valued predictable processes (O, γ) on W such that*

$$\Psi(\omega) = \int O(\omega)\, d\omega + \int \gamma(\omega)\, ds, \tag{3.1}$$

and $\int_0^1 |\gamma_{s'}|^2\, ds' < \infty$ μ–a.s.

With this in mind, if $\Psi_t: W \to W$ is an adapted flow on W, then

$$\Psi_t(\omega)(s) = \int_0^s O_t(\omega)(s')\, d\omega(s') + \int_0^s \gamma_t(\omega)(s')\, ds', \tag{3.2}$$

where, for each $t \in \mathbb{R}$, (O_t, γ_t) are $(O(d) \times T_oM)$-valued predictable processes on W. Differentiating this equation at $t = 0$ gives the form of the vector fields which can generate adapted flows on W. This motivates the following definition.

Definition 3.2 (Adapted Vector Fields on W). An adapted vector field h is a T_oM-valued Brownian semimartingale on W with predictable kernels $Q_s^h \in so(d)$ and $r_s^h \in T_oM$ such that $\int_0^1 |r_s^h|^2\, ds < \infty$ a.s. Let $\mathcal{V}$ denote the collection of adapted vector fields on W and $\mathcal{V}^p = \mathcal{V} \cap \mathcal{H}^p$.

Such processes were called adapted tangent vector fields on W in Definition 2.2, p. 410, of [18] and tangent processes in [12]–[14]. Given an adapted vector field h as above, following Fang and Malliavin [31], let

$$E_0(th)(s) \equiv \int_0^s e^{tQ_{s'}^h}\, d\beta(s') + t\int_0^s r_{s'}^h\, ds'. \tag{3.3}$$

Notice that formally, $(d/dt)|_0 E_0(th) = h$, hence if $f \in L^p(W, d\mu)$ is a function, it is reasonable to try to define the directional derivative $\partial_h f$ of f by h by $\partial_h f \equiv (d/dt)|_0 f \circ E_0(th)$. A minimal requirement for this to make sense is that the law of $E_0(th)$ must be equivalent to μ, since otherwise the composition $f \circ E_0(th)$ is not well defined.

Proposition 3.3. *Suppose that $h \in \mathcal{V}^p$ (an adapted vector field on W in $\mathcal{H}^p$) and $r_s^h(\omega)$ is bounded by a nonrandom constant k. Then $E_0(th)$ has its law equivalent to μ and if $f \in L^{p+}(\mu)$, then $f(E_0(th)) \in L^p(\mu)$ for all $t \in \mathbb{R}$. Moreover,*

$$(d/dt)|_0 E_0(th) = h \qquad \text{in} \quad \mathcal{H}^p. \tag{3.4}$$

Proof. Girsanov's theorem shows that, for all $f \in L^1(W, \mu)$,

$$\mathbb{E}_\mu(f(E_0(th))e^{F(t)}) = \mathbb{E}_\mu f, \tag{3.5}$$

where

$$F(t) \equiv -t\int_0^1 r^h \cdot e^{tQ^h}\, d\beta - (t^2/2)\int_0^1 |r^h|^2\, ds. \tag{3.6}$$

From this it follows that $E_0(th)$ has its law equivalent to Brownian motion, see Lemma 8.2, p. 347, of [15] for details.

Let $q > 1$ and q' be the conjugate exponent to q. Then, by Hölder's inequality,

$$\begin{aligned} \mathbb{E}_\mu|f(E_0(th))|^p &= \mathbb{E}_\mu(|f(E_0(th))|^p e^{F(t)/q} \cdot e^{-F(t)/q}) \\ &\le (\mathbb{E}_\mu\{|f(E_0(th))|^{pq} e^{F(t)}\})^{1/q} \cdot \|e^{-F(t)/q}\|_{q'} \\ &= \|f\|_{pq}^p \cdot \|e^{-F(t)/q}\|_{q'}, \end{aligned}$$

where in the last line we have used (3.5). Since $f \in L^{p+}$, q may be chosen sufficiently close to 1 such that $\|f\|_{pq}^p < \infty$. By Remark 8.1 of [15], $e^{\pm F(t)} \in L^{\infty-}$ and hence $\|e^{-F(t)/q}\|_{q'} < \infty$. Thus $f(E_0(th)) \in L^p$ for all $t \in \mathbb{R}$.

By the fundamental theorem of calculus

$$e^{tQ^h} - I = tQ^h \int_0^1 e^{utQ^h}\, du. \tag{3.7}$$

Therefore

$$\begin{aligned}\|(e^{tQ^h} - I)/t - Q^h\|_{R^p} &= \left\| Q^h \int_0^1 (e^{utQ^h} - I)\, du \right\|_{R^p} \\ &= \left\| \left(\int_0^1 |Q^h_s R_s(t)|^2\, ds \right)^{1/2} \right\|_{L^p},\end{aligned}$$

where

$$R_s(t) = \int_0^1 (e^{utQ^h_s} - I)\, du. \tag{3.8}$$

Since $R_s(t) \to 0$ as $t \to 0$ and $R_s(t)$ is uniformly bounded ($e^{utQ^h_s}$ is orthogonal since Q^h_s is in $so(d)$), the dominated convergence theorem shows that $\|(e^{tQ^h} - I)/t - Q^h\|_{R^p} \to 0$ as $t \to 0$, that is,

$$(d/dt)|_0 e^{tQ^h} = Q^h \qquad \text{in } R^p. \tag{3.9}$$

This proves the proposition since

$$\|(E_0(th) - E_0(0h))/t - h\|_{\mathcal{H}^p} = \|[(1/t)(e^{tQ^h} - I) - Q^h]\|_{R^p}. \qquad \square$$

For $h \in \mathcal{V}^2$ set

$$z(h) \equiv \int_0^1 r^h_s \cdot d\beta(s). \tag{3.10}$$

With this notation we have the following integration by parts formula. The idea of the proof already appears in [9], see also [40] and [31].

Theorem 3.4 (Integration by Parts). *Suppose that* $h \in \mathcal{V}^8$ *with* $r^h_s(\omega)$ *bounded by a nonrandom constant* k, $f, g \in \mathrm{L}^{4+}(d\mu)$, *and the directional derivatives* $\partial_h f \equiv (d/dt)|_0 f \circ E_0(th)$ *and* $\partial_h g \equiv (d/dt)|_0 g \circ E_0(th)$ *exist in* $\mathrm{L}^4(d\mu)$. *Then*

$$\mathbb{E}_\mu(\partial_h f \cdot g) = \mathbb{E}_\mu(f \cdot \partial_h^\dagger g),$$

where

$$\partial_h^\dagger g \equiv [-\partial_h g + z(h) g].$$

Proof. The idea of the proof is simply to compute the derivative of the identity

$$\mathbb{E}_\mu(f(E_0(th)) g(E_0(th)) e^{F(t)}) = \mathbb{E}_\mu(fg),$$

with respect to t at $t = 0$, where $F(t)$ is given in (3.6).

Set $j = fg$, then by standard arguments involving Hölder's inequality, $j \in L^{2+}$ and the derivative $\partial_h j \equiv (d/dt)|_0 j(E_0(th))$ exists in $L^2(\mu)$ and is given by

$$\partial_h j = \partial_h f \cdot g + f \cdot \partial_h g.$$

Using Lemma 3.5 below and standard Hölder's inequality arguments it follows that

$$\begin{aligned} 0 = (d/dt)|_0 \mathbb{E}_\mu(j(E_0(th))e^{F(t)}) &= \mathbb{E}_\mu(\partial_h j + j(-z(h))) \\ &= \mathbb{E}_\mu(\partial_h f \cdot g + f \cdot \partial_h g - fgz(h)). \end{aligned}$$

□

Lemma 3.5. *Suppose that* $p \geq 1$, $h \in \mathcal{V}^{4p}$, *and assume that* r^h *is bounded by a nonrandom constant* k, *then with* $F(t)$ *as* (3.6),

$$(d/dt)|_0 e^{F(t)} = -z(h) = -\int_0^1 r_s^h \cdot d\beta(s) \qquad in \quad L^p.$$

Proof. To simplify notation, let $\|f\|_p$ denote the $L^p(\mu)$ norm of f. By Burkholder's inequality,

$$\begin{aligned} \left\| F(t)/t + \int_0^1 r^h \cdot d\beta \right\|_{4p} &= \left\| \int_0^1 r^h \cdot (I - e^{tQ^h})\, d\beta - (t/2)\int_0^1 |r^h|^2\, ds \right\|_{4p} \\ &\leq k^2|t|/2 + c_{4p}k \left\| \left(\int_0^1 |e^{tQ_s^h} - 1|^2 ds \right)^{1/2} \right\|_{4p}, \end{aligned}$$

where $|A|^2 \equiv \operatorname{tr} A^*A$ if A is a matrix. By the fundamental theorem of calculus we have

$$|e^{tQ^h} - 1|^2 = \left| tQ^h \int_0^1 e^{utQ^h} du \right|^2 \leq t^2 |Q^h|^2,$$

where we have used the fact that e^{utQ^h} is orthogonal in this last inequality. Combining the two above displayed inequalities shows that

$$\left\| F(t)/t + \int_0^1 r^h \cdot d\beta \right\|_{4p} \leq |t|(k^2/2 + c_{4p}k\|h\|_{\mathcal{H}_{4p}}) \to 0 \qquad \text{as} \quad t \to 0. \quad (3.11)$$

By the fundamental theorem of calculus and Hölder's inequality the quantity

$$\varepsilon_p(t) \equiv \left\| (e^{F(t)} - e^{F(0)})/t + \int_0^1 r^h \cdot d\beta \right\|_p$$

is bounded by

$$\begin{aligned} \varepsilon_p(t) &= \left\| (F(t)/t) \int_0^1 e^{uF(t)} du + \int_0^1 r^h \cdot d\beta \right\|_p \\ &\leq \left\| F(t)/t + \int_0^1 r^h \cdot d\beta \right\|_p + \left\| (F(t)/t) \cdot [\int_0^1 e^{uF(t)}\, du - 1] \right\|_p \end{aligned}$$

$$\leq \left\| F(t)/t + \int_0^1 r^h \cdot d\beta \right\|_p + \|F(t)/t\|_{2p} \left\| \int_0^1 e^{uF(t)}\, du - 1 \right\|_{2p}$$

$$\leq \left\| F(t)/t + \int_0^1 r^h \cdot d\beta \right\|_p + \|F(t)/t\|_{2p} \int_0^1 \|e^{uF(t)} - 1\|_{2p}\, du.$$

Because of (3.11) it follows that

$$\limsup_{t\to 0} \varepsilon_p(t) \leq K \cdot \limsup_{t\to 0} \left\| \int_0^1 e^{uF(t)} du - 1 \right\|_{2p}$$

$$\leq K \cdot \limsup_{t\to 0} \int_0^1 \|e^{uF(t)} - 1\|_{2p}\, du, \tag{3.12}$$

where K is a bound on $\|F(t)/t\|_{2p}$ for t near 0. Now

$$\|e^{uF(t)} - 1\|_{2p} = \left\| \int_0^u F(t) e^{vF(t)}\, dv \right\|_{2p} \leq \|F(t)\|_{4p} \cdot \int_0^u \|e^{vF(t)}\|_{4p}\, dv$$

and by (3.11) $\|F(t)\|_{4p} \leq C|t|$ for some constant C. These observations coupled with Remark 8.1, p. 348, of [15], which shows that $\|e^{vF(t)}\|_{4p}$ is bounded (by say K_1) for $0 \leq v \leq 1$ and t near zero, shows that $\|e^{uF(t)} - 1\|_{2p} \leq K_1 C|t|$. Hence the limit in (3.12) is zero. □

3.1. *Pull-Back Vector Fields*

In the next section certain vector fields on $W(M)$ are studied. These vector fields may be pulled back to W by the map $\varphi\colon W \to W(M)$ in Definition 2.5. In this section we study these pulled-back vector fields. As above $\delta\beta$ is used to denote the Stratonovich differential of β and $d\beta$ the Itô differential.

Definition 3.6 (Pull-Back Fields). For an adapted vector field $h \in \mathcal{V}$, let Y^h denote the adapted vector field on W given by

$$Y^h_s \equiv \int_0^s C^h_{s'} \delta\beta(s') + h_s, \tag{3.13}$$

where

$$C^h_s \equiv A_s\langle h\rangle + \Theta_{I_s}\langle h_s, \cdot\rangle \tag{3.14}$$

and

$$A_s\langle h\rangle \equiv \int_0^s \Omega_{I_{s'}}\langle h_{s'}, \delta\beta(s')\rangle. \tag{3.15}$$

In what follows, we abbreviate (3.13)–(3.15) by

$$Y^h \equiv \int C^h \delta\beta + h,$$

$$C^h \equiv A\langle h\rangle + \Theta_I\langle h, \cdot\rangle,$$

and

$$A\langle h\rangle \equiv \int \Omega_I\langle h, \delta\beta\rangle,$$

respectively. For future reference, the Itô form of Y^h is

$$Y^h \equiv \int (C^h + Q^h)\, d\beta + \int \left[r^h + \tfrac{1}{2}\left(Ric_I\langle h\rangle + \bar{\Theta}_I\langle h\rangle + \sum_i \Theta_I\langle Q^h e_i, e_i\rangle \right)\right] ds, \tag{3.16}$$

where $\{e_i\}_{i=1}^d$ is an orthonormal basis for T_oM and Ric, Θ, and $\bar{\Theta}$ are defined in Notation 2.3. The proof of (3.16) is straightforward, for details see the proof of Proposition 6.1, p. 323 in [15].

Equation (3.16) is equivalent to

$$Y^h \equiv \int Q^{Y^h}\, d\beta + \int r^{Y^h}\, ds, \tag{3.17}$$

where

$$Q^{Y^h} = C^h + Q^h = A\langle h\rangle + \Theta_I\langle h, \cdot\rangle + Q^h \tag{3.18}$$

and

$$r^{Y^h} = r^h + \tfrac{1}{2}\left(Ric_I\langle h\rangle + \bar{\Theta}_I\langle h\rangle + \sum_i \Theta_I\langle Q^h e_i, e_i\rangle \right). \tag{3.19}$$

Lemma 3.7. *For all $p \in [2, \infty)$ there is a constant K depending only on p and the geometry of M such that*

1. $\|A\langle h\rangle\|_{S^p} \le K\|h\|_{\mathcal{H}^p}$ *for all* $h \in \mathcal{H}^p$,
2. $\|Y^h\|_{\mathcal{H}^p} \le K\|h\|_{\mathcal{H}^p}$ *for all* $h \in \mathcal{H}^p$, *and*
3. $\|Y^h\|_{B^p} \le K\|h\|_{B^p}$ *for all* $h \in B^p$.

Proof. In the following argument K denotes a generic finite constant which only depends on p and the geometry of M.

Express $A\langle h\rangle$ in Itô form as

$$A\langle h\rangle = \int \Omega_I\langle h, d\beta\rangle + \tfrac{1}{2}\sum_i \int \{\Omega_I\langle Q^h e_i, e_i\rangle + (B_i\Omega)_I\langle h, e_i\rangle\}\, ds, \tag{3.20}$$

where $(B_i\Omega)_u \equiv (d/dt)|_0 \Omega_{e^{tB_i}(u)}$, $B_i \equiv B\langle e_i\rangle$, and $\{e_i\}_{i=1}^d$ is an orthonormal basis for T_oM. Using the compactness of M and Burkholder's inequality, we show that

$$\|A\langle h\rangle\|_{S^p} = K\{\|h\|_{S^p} + \|Q^h\|_{R^p}\} \le K\|h\|_{\mathcal{H}^p}. \tag{3.21}$$

Similarly,

$$\|r^{Y^h}\|_{R^p} \le \|r^h\|_{R^p} + K(\|h\|_{S^p} + \|Q^h\|_{R^p}) \le K\|h\|_{\mathcal{H}^p}, \tag{3.22}$$

$$\begin{aligned}\|r^{Y^h}\|_{S^p} &\le \|r^h\|_{S^p} + K(\|h\|_{S^p} + \|Q^h\|_{S^p})\\ &\le \|r^h\|_{S^p} + K(c_p\|h\|_{\mathcal{H}^p} + \|Q^h\|_{S^p})\\ &\le K\|h\|_{B^p},\end{aligned} \tag{3.23}$$

and, by (3.14) and (3.18),

$$\begin{aligned}\|Q^{Y^h}\|_{R^p} &= \|A\langle h\rangle + \Theta_I\langle h,\cdot\rangle + Q^h\|_{R^p}\\ &\le \|A\langle h\rangle\|_{S^p} + \|h\|_{S^p} + \|Q^h\|_{R^p}\\ &\le K\|h\|_{\mathcal{H}^p}\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\|Q^{Y^h}\|_{S^p} &= \|A\langle h\rangle + \Theta_I\langle h,\cdot\rangle + Q^h\|_{S^p}\\ &\le \|A\langle h\rangle\|_{S^p} + \|h\|_{S^p} + \|Q^h\|_{S^p}\\ &\le K\|h\|_{\mathcal{H}^p} + \|h\|_{B^p} \le K\|h\|_{B^p}.\end{aligned} \tag{3.25}$$

Item 2 of the lemma follows from (3.22) and (3.24) while item 3 follows from (3.23) and (3.25). □

4. Geometric Vector Fields

Definition 4.1 (Geometric Vector Fields). Given an adapted vector field (h) on W, let X^h denote the **adapted vector field on** $W(M)$ given by

$$X^h(\sigma) \equiv P(\sigma)h(b(\sigma)),$$

i.e., $X^h(\sigma)$ is the vector field along σ such that $X^h_s(\sigma) = P_s(\sigma)h_s(b(\sigma))$ for all $s \in [0,1]$.

We wish to have the vector fields X^h act as first-order differential operators on a large class of functions on $W(M)$. Our starting point will be to differentiate functions along an "approximate" flow to X^h for nice $h \in C\mathcal{V} \equiv \mathcal{V} \cap S^\infty \cap B^\infty$. The next lemma guarantees that $C\mathcal{V}$ is sufficiently large.

Lemma 4.2. *For each $p \in [2,\infty)$, $C\mathcal{V} \equiv \mathcal{V} \cap S^\infty \cap B^\infty$ is dense in $\mathcal{V}^p$.*

Proof. First suppose that $h \in \mathcal{V} \cap B^\infty$, i.e., (Q^h, r^h) is an $(so(d) \times T_oM)$-valued continuous and bounded adapted process. For each integer n choose $\varphi_n \in C^\infty(T_oM, T_oM)$ such that $\varphi_n(x) = x$ if $|x| \le n$, $\varphi_n'(x) = 0$ if $|x| \ge 2n$, $|\varphi_n(x)| \le 2n$, and $\sup_{x,n}\{|\varphi_n'(x)| + |\varphi_n''(x)|\} < \infty$. To construct such functions, let $\Psi \in C_c^\infty(\mathbb{R})$ such that $0 \le \Psi(x) \le 1$ for all x and

$$\Psi(x) \equiv \begin{cases} 1 & \text{if} \quad |x| \le 1,\\ 0 & \text{if} \quad |x| \ge 2.\end{cases}$$

Then define

$$\varphi_n(x) \equiv \Psi(|x|^2/n^2)x.$$

It is now a simple matter to verify that the functions φ_n satisfy all of the requirements.

Set $h^n \equiv \varphi_n(h)$, then, by Itô's lemma,

$$dh^n = \varphi_n'(h)\{Q^h\, d\beta + r^h\, ds\} + \tfrac{1}{2}\sum_i \varphi_n''(h)\langle Q^h e_i, Q^h e_i\rangle\, ds.$$

So h^n is bounded with bounded kernels:

$$Q^{h^n} = \varphi_n'(h) Q^h \quad \text{and} \quad r^{h^n} = \varphi_n'(h) r^h + \tfrac{1}{2}\sum_i \varphi_n''(h)\langle Q^h e_i, Q^h e_i\rangle,$$

i.e., $h^n \in C\mathcal{V}$. It easily follows from the construction of the φ_n's that (Q^{h^n}, r^{h^n}) converges boundedly to (Q^h, r^h) as $n \to \infty$. Therefore, by the dominated convergence theorem, $h^n \to h$ in $\mathcal{H}^p$ as $n \to \infty$.

Because of the above paragraph, it suffices to show that $bC\mathcal{P}$ (the bounded continuous adapted processes) are dense in $\mathcal{P}R^p$ (the predictable processes in R^p). Using the fact that continuous adapted processes generate the predictable σ-algebra (see Remark 2.3, p. 16, of [37]) one may mimic the proof of Theorem 2 on p. 126 of [45] to show that $bC\mathcal{P}$ is dense in $b\mathcal{P}R^p$—the bounded predictable processes endowed with the R^p-norm. This proves the theorem, since it is easily shown that $b\mathcal{P}R^p$ is dense in $\mathcal{P}R^p$ using a standard truncation argument and the dominated convergence theorem. □

4.1. *Differentials of I and P*

Notation 4.3. Suppose that $I \in W(O(M))$ and $A \in W(so(d))$, let $I \cdot A \in W(TO(M))$ be defined by

$$(I \cdot A)(s) \equiv (d/dt)|_0 I(s) e^{tA(s)}. \tag{4.1}$$

Also if $I \in W(O(M))$ and $h \in W$, let $B\langle h\rangle(I) \in W(TO(M))$ be defined by

$$(B\langle h\rangle(I))(s) \equiv B\langle h(s)\rangle(I(s)). \tag{4.2}$$

Theorem 4.4. *Suppose that $h \in C\mathcal{V} \equiv \mathcal{V} \cap S^\infty \cap B^\infty$. For $t \in \mathbb{R}$ let*

$$u(t) \equiv I \circ E_0(tY^h)\colon W \to W(O(M)). \tag{4.3}$$

Then the process $u(t)$ is B^p-differentiable for all $1 \le p < \infty$ and

$$\dot u(0) = -I \cdot A\langle h\rangle + B\langle h\rangle(I). \tag{4.4}$$

We summarize this formula by writing

$$I_* Y^h = -I \cdot A\langle h\rangle + B\langle h\rangle(I).$$

(*The B^p norms for manifold-valued processes are defined with the aid of an embedding of the manifold, see Definition* 4.1, *Proposition* 4.1 (*p.* 301), *and Notation* 5.1 (*p.* 319) *of* [15] *for more details*.)

The following lemma is needed in the proof of Theorem 4.4.

Lemma 4.5. *Suppose that $h \in B^{\infty-} \cap \mathcal{V}$, then*

$$(d/dt)e^{tQ^h} = Q^h e^{tQ^h} \qquad in \;\; S^{\infty-} \tag{4.5}$$

and

$$(d/dt)E_0(th) = \int Q^h e^{tQ^h} d\beta + \int r^h \, ds \qquad in \;\; B^{\infty-}. \tag{4.6}$$

Moreover, $E_0(th)$ and $dE_0(th)/dt$ are B^p-Lipschitz in t for all $p \in [1, \infty)$.

Proof. Using (3.7) and Hölder's inequality, for any $q < \infty$ we have

$$\|[(1/t)(e^{tQ^h} - I) - Q^h]\|_{S^q} = \|Q^h R_\cdot(t)\|_{S^q} \le \|Q^h\|_{S^r} \cdot \|R_\cdot(t)\|_{S^p}, \tag{4.7}$$

where $1/q = 1/r + 1/p$ and $R_s(t)$ is given in (3.8). Since e^{tQ^h} is an orthogonal matrix, it easily follows from (3.7) and (3.8) that

$$\|R_\cdot(t)\|_{S^p} \le \left\| \int_0^1 |utQ^h| du \right\|_{S^p} \le \tfrac{1}{2}|t| \|Q^h\|_{S^p} \to 0 \qquad \text{as} \quad t \to 0. \tag{4.8}$$

Therefore $(d/dt)|_0 e^{tQ^h} = Q^h$ in $S^{\infty-}$. For general $t_0 \in \mathbb{R}$,

$$(d/dt)|_{t_0} e^{tQ^h} = (d/dt)|_0 e^{(t_0+t)Q^h} = (d/dt)|_0 e^{tQ^h} e^{t_0 Q^h}. \tag{4.9}$$

Since $e^{t_0 Q^h}$ is an orthogonal matrix-valued process, it follows from the case $t_0 = 0$ above that the derivative in (4.9) exists in S^p and is given by $Q^h e^{t_0 Q^h}$. This proves (4.5) and because $(d/dt)(tr^h) = r^h$ in S^p for all $p \in [0, \infty)$ holds trivially we have also proved (4.6).

Again since e^{tQ^h} is an orthogonal matrix, it follows from (4.6) that $\|(d/dt)E_0(th)\|_{B^p} = \|h\|_{B^p} < \infty$ for all $1 \le p < \infty$. Hence $t \to E_0(th)$ is B^p-Lipschitz. By similar calculations to above, we show that $d^2E_0(th)/dt^2 = \int (Q^h)^2 e^{tQ^h} d\beta$ in $B^{\infty-}$ and hence by Hölder's inequality,

$$\|d^2E_0(th)/dt^2\|_{B^p} = \|(Q^h)^2\|_{S^p} \le \|Q^h\|^2_{S^{2p}} \le \|h\|^2_{B^{2p}} < \infty.$$

This shows that $t \to (d/dt)E_0(th)$ is also B^p-Lipschitz. □

Proof of Theorem 4.4. Because $h \in C\mathcal{V}$, it follows by Lemma 3.7 and (3.19) that $Y^h \in B^{\infty-}$ and r^{Y^h} is bounded. By Proposition 3.3, $w(t) \equiv E_0(tY^h) \in B^\infty$ has a law equivalent to μ so that $u(t)$ in (4.3) is well defined. By Lemma 4.5 $\dot{w}(t) \equiv dw(t)/dt = Y^h$ in $B^{\infty-}$ and $w(t)$ and $\dot{w}(t)$ are B^p-Lipschitz.

Using these observations, the proof of (4.4) may now be given using exactly the same argument as the proof of Theorem 5.2, p. 321, of [15]. It is only necessary for the reader to replace $w(t)$ used in [15] by $w(t) \equiv E_0(tY^h)$ and then to evaluate all t's appearing in the proof at $t = 0$. □

Corollary 4.6 (Approximate Flow for X^h). *Assume that $h \in C\mathcal{V}$. For each $t \in \mathbb{R}$, let*

$$E(tX^h) \equiv \varphi \circ E_0(tY^h) \circ b. \tag{4.10}$$

Then

1. $(d/dt)|_0 P \circ E(tX^h) = -A\langle h \circ b\rangle \cdot P + B\langle h \circ b\rangle(P)$ *in* $B^{\infty-}(b)$, *and*
2. $(d/dt)|_0 E(tX^h) = X^h$ *in* $B^{\infty-}(b)$.

Proof. By Theorem 2.6,

$$P \circ E(tX^h) = P \circ \varphi \circ E_0(tY^h) \circ b = I \circ E_0(tY^h) \circ b = u(t) \circ b.$$

This equation shows that the first assertion is a direct consequence of Theorem 4.4 and Lemma 2.10. The second assertion follows from the first assertion, since $\pi \circ P = id_{W(M)}$, $\pi_*(A \cdot P) = 0$, and $\pi_* B_a(u) = ua$ so that

$$(d/dt)|_0 E(tX^h) = (d/dt)|_0 \pi \circ P \circ E(tX^h) = P(h \circ b) = X^h. \qquad \square$$

4.2. *First-Order Differential Operators*

We now wish to have X^h act as a first-order differential operator on functions on $W(M)$. We begin with the action of X^h on smooth cylinder functions (see Definition 4.9 below) based on $W(O(M))$. We first need some more notation.

Notation 4.7. Suppose that V is a finite-dimensional vector space, $F\colon O(M)^n \to V$ is a smooth function, and $\mathcal{Q} \equiv \{0 < s_1 < s_2 < \cdots < s_n \leq 1\}$ is a partition of [0, 1].

1. For each path $u\colon [0,1] \to O(M)$ set

   $F_{\mathcal{Q}}(u) \equiv F(u(s_1), u(s_2), \ldots, u(s_n)).$
2. Suppose also that $A\colon [0,1] \to so(d)$ and $h\colon [0,1] \to T_oM$. Let

   $F'_{\mathcal{Q}}(u)\langle A + h\rangle \equiv (d/dt)|_0 F_{\mathcal{Q}}(ue^{tA}) + (d/dt)|_0 F_{\mathcal{Q}}(e^{tB\langle h\rangle}(u)),$

   where $(ue^{tA})(s) \equiv u(s)e^{tA(s)}$, and $(e^{tB\langle h\rangle}(u))(s) \equiv e^{tB\langle h(s)\rangle}(u(s))$. (We view $A + k$ as a path from $[0,1] \to so(d) \oplus T_oM$.)
3. Similarly, if $C\colon [0,1] \to so(d)$ and $k\colon [0,1] \to T_oM$, set

   $$F''_{\mathcal{Q}}(u)\langle A + h, C + k\rangle \equiv (d/dt)|_0 F'_{\mathcal{Q}}(ue^{tA})\langle C + k\rangle + (d/dt)|_0 F'_{\mathcal{Q}}(e^{tB\langle h\rangle}(u))\langle C + k\rangle.$$

Remark 4.8. The notion of B^p-differentiability used in Theorem 4.4 is very strong. In particular, with F as in the above notation and $u(t)$ as in Theorem 4.4, $(d/dt)|_0 F_{\mathcal{Q}}(u(t))$ exists in $L^p(d\mu)$ and

$$(d/dt)|_0 F_{\mathcal{Q}}(u(t)) = F'_{\mathcal{Q}}(I)\langle -A\langle h\rangle + h\rangle. \tag{4.11}$$

Definition 4.9 (Cylinder Functions). A function $f\colon W(M) \to \mathbb{R}$ is a smooth cylinder function if there is an integer n, a C^∞-function $F\colon O(M)^n \to \mathbb{R}$, and a partition $\mathcal{Q} = \{0 < s_1 < s_2 < \cdots < s_n \leq 1\}$ of [0, 1] such that

$$f = F_{\mathcal{Q}}(P_{s_1}, \ldots, P_{s_n}) = F_{\mathcal{Q}} \circ P \quad \text{a.s.} \tag{4.12}$$

We denote the collection of smooth cylinder functions by $\mathcal{F}C^\infty$.

Suppose that $\tilde{f}: M^n \to \mathbb{R}$ is a smooth function, then

$$f(\sigma) \equiv \tilde{f}(\sigma_{s_1}, \ldots, \sigma_{s_n}) \tag{4.13}$$

is in $\mathcal{F}C^\infty$. Indeed, $f = F_{\mathcal{Q}}(P_{s_1}, \ldots, P_{s_n})$ a.s. where $F \equiv \tilde{f} \circ \pi$, i.e., $F(u_1, \ldots, u_n) = \tilde{f}(\pi(u_1), \ldots, \pi(u_n))$. We call a cylinder function f as in (4.13) a *restricted cylinder function* and denote the collection of restricted cylinder functions by $\mathcal{RF}C^\infty$.

The integration by parts formula in the next proposition is a slight generalization of Theorem 3.6.1 in [31]. For the special case that h is in the Cameron–Martin space see Theorem 9.1 in [15]. Also see [22], [24], [34], [40], [41] and [44].

Proposition 4.10. *Given $f \in \mathcal{F}C^\infty$ as in* (4.12) *and $h \in \mathcal{V}^2$, then*

$$X^h f \equiv F'_{\mathcal{Q}}(P)\langle -(A\langle h \rangle) \circ b + h \circ b \rangle \tag{4.14}$$

is well defined. Moreover, if $g \in \mathcal{F}C^\infty$, then

$$\mathbb{E}_\nu(X^h f \cdot g) = \mathbb{E}_\nu(f \cdot (X^h)^\dagger g), \tag{4.15}$$

where

$$(X^h)^\dagger g \equiv -X^h g + (z(Y^h) \circ b) g \tag{4.16}$$

and $z(Y^h)$ is defined in (3.10). *We view X^h and $(X^h)^\dagger$ as unbounded operators on $L^2(W(M), \nu)$ each with $\mathcal{F}C^\infty$ as its domain.*

Proof. First suppose that $h \in C\mathcal{V}$. Because of Corollary 4.6, $(d/dt)|_0 f \circ E(tX^h)$ exists and is given by the right-hand side of (4.14). Therefore, for such h, $X^h f$ is well defined. Moreover, since

$$f \circ E(tX^h) = F_{\mathcal{Q}} \circ P \circ \varphi \circ E_0(tY^h) \circ b = F_{\mathcal{Q}} \circ I \circ E_0(tY^h) \circ b,$$

we see by Theorem 4.4 that

$$X^h f = (\partial_{Y^h}(F_{\mathcal{Q}} \circ I)) \circ b = ((d/dt)|_0 F_{\mathcal{Q}} \circ I \circ E_0(tY^h)) \circ b, \tag{4.17}$$

where the derivative exists in $L^{\infty-}$. Let $G \in C_c^\infty(O(M)^n)$ and $g = G_{\mathcal{Q}} \circ P \in \mathcal{F}C^\infty$. Now apply Theorem 3.4 with h replaced by Y^h, f by $F_{\mathcal{Q}} \circ I$, and g by $G_{\mathcal{Q}} \circ I$ to find

$$\begin{aligned}&\mathbb{E}_\mu[(\partial_{Y^h}(F_{\mathcal{Q}} \circ I)) \cdot (G_{\mathcal{Q}} \circ I)] \\ &\quad = \mathbb{E}_\mu((F_{\mathcal{Q}} \circ I) \cdot \{-\partial_{Y^h}(G_{\mathcal{Q}} \circ I) + z(Y^h) \cdot (G_{\mathcal{Q}} \circ I)\}),\end{aligned}$$

which owing to (4.17) and the fact that $Law(b) = \mu$ shows that (4.15) holds.

For general $h \in \mathcal{V}^2$, choose $h^n \in C\mathcal{V}$ such that $h^n \to h$ in $\mathcal{H}_2$ as $n \to \infty$. Then it is easy to check that

$$L^2 - \lim_{n\to\infty} X^{h^n} f = F'_{\mathcal{Q}}(P)\langle -(A\langle h \rangle) \circ b + h \circ b \rangle, \tag{4.18}$$

which shows that $X^h f$ defined in (4.14) is well defined. By (3.19) $z(Y^h) = \int_0^1 r^{Y^h} \cdot d\beta$ is linear in h and by (3.22)

$$\mathbb{E}_\mu |z(Y^h)|^2 = \mathbb{E}_\mu \int_0^1 |r_s^{Y^h}|^2 \, ds = \|r^{Y^h}\|_{R^2}^2 \le K\|h\|_{\mathcal{H}^2}^2.$$

Hence $z(Y^{h_n}) \to z(Y^h)$ in L^2 as $n \to \infty$ when $h_n \to h$ in $\mathcal{H}^2$ as $n \to \infty$. From this fact and (4.18), it is easy to conclude that (4.15) holds for general $h \in \mathcal{V}^2$. ☐

We now define a closed extension $\bar{X}^h$ of X^h using the weak derivative formulation in the next definition.

Definition 4.11. Given $h \in \mathcal{V}^2$ let $\bar{X}^h \equiv ((X^h)^\dagger)^*$. Explicitly, $f \in L^2(\nu)$ is in the domain $\mathcal{D}(\bar{X}^h)$ of $\bar{X}^h$ and $\bar{X}^h f = k \in L^2(\nu)$ iff $\mathbb{E}_\nu(k \cdot g) = \mathbb{E}_\nu(f \cdot (X^h)^\dagger g)$ for all $g \in \mathcal{F}C^\infty$.

It is reasonable to conjecture that $\bar{X}^h$ is the closure of X^h on $\mathcal{RF}C^\infty$. We do not pursue this here. However, the following elementary lemma gives a sufficient condition for a function f on $W(M)$ to be in the domain of $\bar{X}^h$. The proof is similar to that of Proposition 4.10.

Lemma 4.12. *Suppose that $f \in L^{4+}(\nu)$, and, for $h \in C\mathcal{V}$, $(d/dt)|_0 f \circ E(tX^h)$ exists in $L^4(\nu)$ and there is a constant $K < \infty$ such that*

$$\|(d/dt)|_0 f \circ E(tX^h)\|_2 \le K\|h\|_{\mathcal{H}^4}, \qquad \forall h \in C\mathcal{V}.$$

Then $f \in \bigcap_{h\in\mathcal{V}^4} \mathcal{D}(\bar{X}^h)$, $\bar{X}^h f = L^2 - (d/dt)|_0 f \circ E(tX^h)$ for all $h \in C\mathcal{V}$, and for $h \in \mathcal{V}^4$ we have $\bar{X}^h f = \lim_{n\to\infty} \bar{X}^{h^n} f$, where $h^n \in C\mathcal{V}$ is any sequence such that $h^n \to h$ in $\mathcal{H}^4$ as $n \to \infty$.

Proof. Let $h \in C\mathcal{V}$, $g \in \mathcal{F}C^\infty$, and put $\tilde{f} \equiv f \circ \varphi$ and $\tilde{g} \equiv g \circ \varphi$. Since $\varphi_*\mu \equiv \mu \circ \varphi^{-1} = \nu$, the map $f \in L^p(\nu) \to \tilde{f} \in L^p(\mu)$ is an isometric isomorphism with inverse $\tilde{f} \to f = \tilde{f} \circ b$. Since $\tilde{f} \circ E_0(tY^h) = f \circ E(tX^h) \circ \varphi$, it follows that

$$Y^h \tilde{f} \equiv (d/dt)|_0 \tilde{f} \circ E_0(tY^h) = S \circ \varphi,$$

where $S \equiv (d/dt)|_0 f \circ E(tX^h)$. Hence, by Theorem 3.4,

$$\mathbb{E}_\mu((S \circ \varphi)\tilde{g}) = \mathbb{E}_\mu(\tilde{f}(Y^h)^\dagger \tilde{g}) = \mathbb{E}_\mu(\tilde{f}(-Y^h\tilde{g} + z(Y^h)\tilde{g})). \tag{4.19}$$

From the proof of Proposition 4.10, $(Y^h)^\dagger \tilde{g} = ((X^h)^\dagger g) \circ \varphi$. Hence (4.19) shows that $\mathbb{E}_\nu[Sg] = \mathbb{E}_\nu(f((X^h)^\dagger g))$. Since $g \in \mathcal{F}C^\infty$ was arbitrary, it follows that $f \in \mathcal{D}(\bar{X}^h)$ and $S = \bar{X}^h f$.

For general $h \in \mathcal{V}^4$, choose $h^n \in C\mathcal{V}$ such that $h^n \to h$ in $\mathcal{H}^4$ as $n \to \infty$. Using the above paragraph, we have for any $g \in \mathcal{F}C^\infty$ that

$$\mathbb{E}_\nu[\bar{X}^{h^n} f \cdot g] = \mathbb{E}_\nu[f \cdot (X^{h^n})^\dagger g]. \tag{4.20}$$

With $S \equiv L^2 - \lim_{n\to\infty} \bar{X}^{h^n} f$, it follows by letting $n \to \infty$ in (4.20) that

$$\mathbb{E}_\nu[S \cdot g] = \mathbb{E}_\nu[f \cdot (X^h)^\dagger g]. \tag{4.21}$$

Therefore, $f \in \mathcal{D}(\bar{X}^h)$ and $\bar{X}^h f = L^2 - \lim_{n\to\infty} \bar{X}^{h^n} f$. □

5. Product of Two Vector Fields on Restricted Cylinder Functions

Definition 5.1. An adapted vector field k on W is said to be p-**smooth** if (i) $k \in \mathcal{H}^{p+}$, (ii) for all $h \in \mathcal{V} \cap B^{\infty-}$ with r^h bounded the derivative $(d/dt)|_0 k \circ E_0(th) =: \partial_h k$ exists in $\mathcal{H}^p$, and (iii) there is a constant $C < \infty$ such that $\|\partial_h k\|_{\mathcal{H}^{p/2}} \le C\|h\|_{\mathcal{H}^p}$ holds for all $h \in \mathcal{V} \cap B^{\infty-}$.

If k is p-smooth we may and do extend the definition of $\partial_h k$ to all $h \in \mathcal{V}^p$ by continuity. In what follows, to simplify notation we write $Y^h k$ for $\partial_{Y^h} k$.

Theorem 5.2. *Suppose that k is* 4*-smooth, $h \in \mathcal{V}^4$, $f \in \mathcal{RFC}^\infty$ is given as in* (4.13), *and $F(u_1, \dots, u_n) = \tilde{f}(\pi(u_1), \dots, \pi(u_n))$. Then $X^k f \in \mathcal{D}(\bar{X}^h)$ and* (*using Notation* 4.7)

$$\bar{X}^h X^k f = F'_{\mathcal{Q}}(P)\langle (Y^h k)) \circ b\rangle + F''_{\mathcal{Q}}(P)\langle (-A\langle h\rangle + h) \circ b, k \circ b\rangle. \tag{5.1}$$

Proof. Lemma 3.7 implies that $Y^h \in \mathcal{V}^4$. By (4.14) and the assumption $f \in \mathcal{RFC}^\infty$,

$$X^k f = F'_{\mathcal{Q}}(P)\langle k \circ b\rangle.$$

For the moment assume that $h \in C\mathcal{V}$ and consider

$$\begin{aligned}(X^k f) \circ E(tX^h) &= F'_{\mathcal{Q}}(P \circ E(tX^h))\langle k \circ b \circ E(tX^h)\rangle \\ &= F'_{\mathcal{Q}}(P \circ E(tX^h))\langle k \circ E_0(tY^h) \circ b\rangle.\end{aligned}$$

Because F is a smooth function on the compact manifold $O(M)^n$, the assumption that k is 4-smooth, and Corollary 4.6 one may show by standard arguments that

$$(d/dt)|_0 (X^k f) \circ E(tX^h) = F'_{\mathcal{Q}}(P)\langle (Y^h k) \circ b\rangle + F''_{\mathcal{Q}}(P)\langle (-A\langle h\rangle + h) \circ b, k \circ b\rangle, \tag{5.2}$$

where the derivative exists in L^4. It follows from this equation and the use of Hölder's and Burkholder's inequalities that there are constants K and $\tilde{K}$ depending on the bounds on $F'_{\mathcal{Q}}$ and $F''_{\mathcal{Q}}$ such that

$$\begin{aligned}\|(d/dt)|_0 (X^k f) \circ E(tX^h)\|_2 &\le K(\|Y^h k\|_{S^2} + \|k\|_{S^4}\{\|h\|_{S^4} + \|A\langle h\rangle\|_{S^4}\}) \\ &\le \tilde{K}(\|Y^h k\|_{\mathcal{H}_2} + \|k\|_{\mathcal{H}^4}\|h\|_{\mathcal{H}^4}),\end{aligned}$$

where in the second inequality we have used Lemma 3.7 to bound $\|A\langle h\rangle\|_{S^4}$. Hence, using the assumption that k is 4-smooth, it follows that there is a constant $C = C(k, f)$ such that

$$\|(d/dt)|_0 (X^k f) \circ E(tX^h)\|_2 \le C\|h\|_{\mathcal{H}^4} \qquad \text{for all} \quad h \in C\mathcal{V}.$$

The theorem now follows by an application of Lemma 4.12. □

6. Lie Bracket on Restricted Cylinder Functions

Our goal in this section is to compute the commutator of two geometric vector fields, see Theorem 6.2 below. The following lemma is in preparation for this result.

Lemma 6.1. *Let $A, C \in W(so(d))$, $u \in W(O(M))$, and $h, k \in W \equiv W(T_oM))$. Then in the notation of Notation* 4.7,

$$\begin{aligned} &F''_{\mathcal{Q}}(u)\langle A+h, C+k\rangle - F''_{\mathcal{Q}}(u)\langle C+k, A+h\rangle \\ &\quad = F'_{\mathcal{Q}}(u)\langle([A,C]-\Omega_u\langle h,k\rangle) + (Ah - Ck - \Theta_u\langle h,k\rangle)\rangle, \end{aligned} \tag{6.1}$$

where

$$([A,C]-\Omega_u\langle h,k\rangle)(s) \equiv [A(s),C(s)] - \Omega_{u(s)}\langle h(s),k(s)\rangle$$

is in $W(so(d))$ and

$$(Ah - Ck - \Theta_u\langle h,k\rangle)(s) \equiv A(s)h(s) - C(s)k(s) - \Theta_{u(s)}\langle h(s),k(s)\rangle$$

is in W.

Proof. For $\alpha \in so(d)$, let $\hat{\alpha}$ be the vertical vector field on $O(M)$ defined by $\hat{\alpha}(u) = (d/dt)|_0 ue^{t\alpha}$. For $A \in W(so(d))$ and $h \in W(T_oM)$ let $\hat{A}(s_i)$ and $B_{h(s_i)}$ denote the vector fields on $O(M)^n$ given by

$$(\hat{A}(s_i)F)(u_1,u_2,\ldots,u_n) \equiv (d/dt)|_0 F(u_1,\ldots,u_{i-1},u_ie^{tA(s_i)},u_{i+1},\ldots,u_n)$$

and

$$(B_{h(s_i)}F)(u_1,u_2,\ldots,u_n) \equiv (d/dt)|_0 F(u_1,\ldots,u_{i-1},e^{tB_{h(s_i)}}(u_i),u_{i+1},\ldots,u_n),$$

respectively. Also for $u \in W(O(M))$, let $u_{\mathcal{Q}} \equiv (u(s_1),u(s_2),\ldots,u(s_n))$. Then with this notation:

$$\begin{aligned} &F''_{\mathcal{Q}}(u)\langle A+h, C+k\rangle - F''_{\mathcal{Q}}(u)\langle C+k, A+h\rangle \\ &\quad = \sum_{i,j=1}^n ([\hat{A}(s_i)+B_{h(s_i)}, \hat{C}(s_j)+B_{k(s_j)}]F)(u_{\mathcal{Q}}) \\ &\quad = \sum_{i=1}^n ([\hat{A}(s_i)+B_{h(s_i)}, \hat{C}(s_i)+B_{k(s_i)}]F)(u_{\mathcal{Q}}). \end{aligned}$$

Using the commutator formulas in Lemma A.2 in the Appendix gives

$$\sum_{i=1}^n [\hat{A}(s_i)+B_{h(s_i)}, \hat{C}(s_i)+B_{k(s_i)}]$$

$$= \sum_{i=1}^{n} \{[\hat{A}(s_i), \hat{C}(s_i)] + [B_{h(s_i)}, \hat{C}(s_i)] + [\hat{A}(s_i), B_{k(s_i)}] + [B_{h(s_i)}, B_{k(s_i)}]\}$$
$$= \sum_{i=1}^{n} \{[A, C\hat{]}(s_i) + B_{(Ah-Ck)(s_i)}\hat{C}(s_i) - \hat{\Omega}\langle h(s_i), k(s_i)\rangle - B_{\Theta\langle h(s_i), k(s_i)\rangle}\}.$$

Combining the two above displayed equations proves (6.1). □

We are now ready to compute the commutator of two geometric vector fields.

Theorem 6.2. *Let h and k be 4-smooth adapted vector fields on W, let $f \in \mathcal{RFC}^\infty$ be as in (4.13), and let $F(u_1, \ldots, u_n) = \tilde{f}(\pi(u_1), \ldots, \pi(u_n))$. Then*

$$[\bar{X}^h, \bar{X}^k] f = F'(P)\langle\{Y^h k - Y^k h + c\langle h, k\rangle\} \circ b\rangle, \tag{6.2}$$

where $c\langle h, k\rangle$ is the process on W given by

$$c\langle h, k\rangle \equiv -A\langle h\rangle k + A\langle k\rangle h - \Theta_I\langle h, k\rangle. \tag{6.3}$$

Proof. By Theorem 5.2,

$$\begin{aligned}[\bar{X}^h, \bar{X}^k] f &= F'(P)\langle(Y^h k - Y^k h) \circ b\rangle + F''(P)\langle(-A\langle h\rangle + h) \circ b, k \circ b\rangle \\ &\quad - F''(P)\langle(-A\langle k\rangle + k) \circ b, h \circ b\rangle.\end{aligned}$$

Because of Lemma 6.1, this equation shows

$$\begin{aligned}[\bar{X}^h, \bar{X}^k] f &= F'(P)\langle(Y^h k - Y^k h - A\langle h\rangle k + A\langle k\rangle h - \Omega_I\langle h, k\rangle - \Theta_I\langle h, k\rangle) \circ b\rangle \\ &= F'(P)\langle(Y^h k - Y^k h - A\langle h\rangle k + A\langle k\rangle h - \Theta_I\langle h, k\rangle) \circ b\rangle,\end{aligned}$$

where in the second equality we have used the assumption that $f \in \mathcal{RFC}^\infty$, so that $F'(P)\langle(\Omega_I\langle h, k\rangle) \circ b\rangle = 0$. □

7. The Lie Bracket Preserves Adapted Vector Fields

Lemma 7.1. *Suppose that k is a $2p$-smooth adapted vector field on W, then, for all $h \in \mathcal{V} \cap B^{\infty-}$ with r^h bounded, $\partial_h Q^k := (d/dt)|_0 Q^k(E_0(th))$ and $\partial_h r^k := (d/dt)|_0 r^k(E_0(th))$ exist in $R^{(2p)-}$ and there exists a constant $C = C(k)$ (depending only on k and p) such that*

$$\left\| \int_0^1 |(\partial_h Q^k)_s|\, ds \right\|_{L^p(\mu)} \le C\|h\|_{\mathcal{H}^{2p}}, \tag{7.1}$$

and

$$\left\| \int_0^1 |(\partial_h r^k)_s|\, ds \right\|_{L^p(\mu)} \le C\|h\|_{\mathcal{H}^{2p}}. \tag{7.2}$$

Because of the estimates in (7.1) *and* (7.2), *we may extend by continuity the definitions of* $\partial_h Q^k$ *and* $\partial_h r^k$ *to all* $h \in \mathcal{V}^{2p}$. *Then with this notation the kernels of* $\partial_h k$ (*for all* $h \in \mathcal{V}^{2p}$) *are given by*

$$Q^{\partial_h k} = \partial_h Q^k + Q^k Q^h \tag{7.3}$$

and

$$r^{\partial_h k} = \partial_h r^k + Q^k r^h. \tag{7.4}$$

Proof. Let $h \in \mathcal{V} \cap B^{\infty-}$ with r^h bounded. To simplify notation, let $d\beta = d\beta(s)$,

$$w(t) \equiv E_0(th) = \int e^{tQ^h}\, d\beta + t \int r^h\, ds,$$

and $dw(t) = d_s w_s(t)$ so that

$$dw(t) \equiv e^{tQ^h}\, d\beta + r^h\, ds.$$

(Please note well that $dw(t)$ is the differential of $w(t)$ in the suppressed s variable—**not** the t variable.) The assumption that

$$\begin{aligned} k(w(t)) &= \int Q^k(w(t))\, dw(t) + \int r^k(w(t))\, ds \\ &= \int Q^k(w(t)) e^{tQ^h}\, d\beta + \int \{tQ^k(w(t)) r^h + r^k(w(t))\}\, ds \end{aligned}$$

is $\mathcal{H}^{2p}$ differentiable is equivalent to

$$Q(t) := Q^k(w(t)) e^{tQ^h}$$

and

$$r(t) := \{tQ^k(w(t)) r^h + r^k(w(t))\}$$

being R^{2p} differentiable. Using Lemma 4.5 and Hölder's inequality on the identity

$$[Q^k(w(t)) - Q^k]/t = Q(t)(e^{-tQ^h} - I)/t + [Q(t) - Q(0)]/t$$

we find

$$\begin{aligned} \partial_h Q^k = (d/dt)|_0 Q^k(w(t)) &= -Q^k Q^h + \dot{Q}(0) \\ &= -Q^k Q^h + Q^{\partial_h k} \qquad \text{in } R^{(2p)-}. \end{aligned} \tag{7.5}$$

Similarly,

$$\begin{aligned} (r^k(w(t)) - r^k)/t &= (r(t) - tQ^k(w(t)) r^h - r(0))/t \\ &= (r(t) - r(0))/t - Q^k(w(t)) r^h \\ &\to \dot{r}(0) - Q^k r^h \qquad \text{in } R^{(2p)-} \text{ as } \quad t \to 0, \end{aligned}$$

where in taking the limit as $t \to 0$ we have used the continuity of $t \to Q^k(w(t))$ in $R^{(2p)-}$ at $t = 0$ implied by (7.5) and the boundedness of r^h. The above displayed equation is equivalent to

$$\partial_h r^k = (d/dt)|_0 r^k(w(t)) = r^{\partial_h k} - Q^k r^h \qquad \text{in } R^{(2p)-} \text{ as } \quad t \to 0. \tag{7.6}$$

So for $h \in \mathcal{V} \cap B^{\infty-}$ with r^h bounded we have proved the differentiability assertions of the lemma and identities in (7.3) and (7.4). To finish the proof it suffices to prove the estimates in (7.1) and (7.2). Using the definition of k being $2p$-smooth and Hölder's inequality, it follows from (7.3) that

$$\begin{aligned} \left\| \int_0^1 |(\partial_h Q^k)_s|\, ds \right\|_{L^p} &\le \left\| \int_0^1 |(Q^{\partial_h k})_s|\, ds \right\|_{L^p} + \left\| \int_0^1 |Q_s^k||Q_s^h|\, ds \right\|_{L^p} \\ &\le \|Q^{\partial_h k}\|_{R^p} + \|Q^k\|_{R^{2p}} \|Q^h\|_{R^{2p}} \\ &\le (\tilde{C}(k) + \|Q^k\|_{R^{2p}}) \|h\|_{\mathcal{H}^{2p}} \\ &= C(k) \|h\|_{\mathcal{H}^{2p}}, \end{aligned}$$

wherein we have used

$$\|Q^{\partial_h k}\|_{R^p} \le \|\partial_h k\|_{\mathcal{H}^p} \le \tilde{C}(k) \|h\|_{\mathcal{H}^{2p}}.$$

Here $\tilde{C}(k)$ is a finite constant guaranteed to exist because k is $2p$-smooth, see Definition 5.1. The estimate in (7.2) is proved similarly. □

Lemma 7.2. *For $h, k \in \mathcal{H}^4$, $A\langle h\rangle k \in \mathcal{H}^2$,*

$$Q^{A\langle h\rangle k} = \Omega_I\langle h, \cdot\rangle k + A\langle h\rangle Q^k \tag{7.7}$$

and

$$r^{A\langle h\rangle k} = \left(\mathcal{R}\langle h\rangle k + A\langle h\rangle r^k + \sum_i \Omega_I\langle h, e_i\rangle Q^k e_i \right), \tag{7.8}$$

where

$$\mathcal{R}\langle h\rangle \equiv \tfrac{1}{2} \sum_i \{\Omega_I\langle Q^h e_i, e_i\rangle + (B_i\Omega)_I\langle h, e_i\rangle\}. \tag{7.9}$$

Proof. By Itô's lemma and (3.20),

$$\begin{aligned} d(A\langle h\rangle k) &= \Omega_I\langle h, d\beta\rangle k + \mathcal{R}\langle h\rangle k\, ds \\ &\qquad + A\langle h\rangle (Q^k\, d\beta + r^k\, ds) + \sum_i \Omega_I\langle h, e_i\rangle Q^k e_i\, ds \\ &= \Omega_I\langle h, d\beta\rangle k + A\langle h\rangle Q^k\, d\beta \\ &\qquad + \left(\mathcal{R}\langle h\rangle k + A\langle h\rangle r^k + \sum_i \Omega_I\langle h, e_i\rangle Q^k e_i \right) ds \end{aligned}$$

which proves (7.7) and (7.8).

To prove $A\langle h\rangle k \in \mathcal{H}^2$ notice:

$$\begin{aligned}\|Q^{A\langle h\rangle k}\|_{R^2} &\leq \|\Omega_I\langle h,\cdot\rangle k\|_{R^2} + \|A\langle h\rangle Q^k\|_{R^2}\\ &\leq C\{\|h\|_{S^4}\|k\|_{S^4} + \|A\langle h\rangle\|_{S^4}\|Q^k\|_{R^4}\}\\ &\leq C\|h\|_{\mathcal{H}^4}\|k\|_{\mathcal{H}^4} < \infty,\end{aligned}$$

where we have used Lemma 3.7 in the last inequality. One similarly shows that $\|r^{A\langle h\rangle k}\|_{R^2} < \infty$. □

Lemma 7.3. *For $h, k \in \mathcal{H}^4$ such that either Q^h or Q^k in S^4, then $\Theta_I\langle h,k\rangle \in \mathcal{H}^2$ and*

$$Q^{\Theta_I\langle h,k\rangle} = \Theta_I\langle Q^h\cdot, k\rangle + \Theta_I\langle h, Q^k\cdot\rangle + \Theta_I'\langle\cdot, h, k\rangle \tag{7.10}$$

and

$$\begin{aligned}r^{\Theta_I\langle h,k\rangle} = &\sum_i\{\Theta_I\langle Q^h e_i, Q^k e_i\rangle + (B_i\Theta)_I\langle Q^h e_i, k\rangle + (B_i\Theta)_I\langle h, Q^k e_i\rangle\}\\ &+ \Theta_I\langle r^h, k\rangle + \Theta_I\langle h, r^k\rangle + \tfrac{1}{2}\sum_i (B_i^2\Theta)_I\langle h,k\rangle,\end{aligned} \tag{7.11}$$

where $\{e_i\}_{i=1}^d$ is an orthonormal basis for T_0M.

Proof. By Itô's lemma we have

$$\begin{aligned}d(\Theta_I\langle h,k\rangle) &= \Theta_I\langle\delta h, k\rangle + \Theta_I\langle h, \delta k\rangle + (B\langle\delta\beta\rangle\Theta)_I\langle h,k\rangle\\ &=: \alpha + \gamma + \kappa.\end{aligned}$$

We now work on the three terms separately:

$$\begin{aligned}\alpha &= \Theta_I\langle\delta h, k\rangle = \Theta_I\langle dh, k\rangle + \tfrac{1}{2}\{\Theta_I\langle dh, dk\rangle + (B\langle d\beta\rangle\Theta)_I\langle dh, k\rangle\}\\ &= \Theta_I\langle dh, k\rangle + \tfrac{1}{2}\sum_{i=1}^d\{\Theta_I\langle Q^h e_i, Q^k e_i\rangle + (B_i\Theta)_I\langle Q^h e_i, k\rangle\}\, ds\end{aligned}$$

so that

$$\begin{aligned}\alpha + \beta = &\,\Theta_I\langle dh, k\rangle + \Theta_I\langle h, \delta k\rangle\\ &+ \tfrac{1}{2}\sum_{i=1}^d\{2\Theta_I\langle Q^h e_i, Q^k e_i\rangle + (B_i\Theta)_I\langle Q^h e_i, k\rangle + (B_i\Theta)_I\langle h, Q^k e_i\rangle\}\, ds.\end{aligned}$$

Similarly,

$$\begin{aligned}\kappa &= (B\langle d\beta\rangle\Theta)_I\langle h,k\rangle\\ &\quad + \tfrac{1}{2}\{(B\langle d\beta\rangle\Theta)_I\langle dh, k\rangle + (B\langle d\beta\rangle\Theta)_I\langle h, dk\rangle + (B_{d\beta}^2\Theta)_I\langle h,k\rangle\}\\ &= (B\langle d\beta\rangle\Theta)_I\langle h,k\rangle\\ &\quad + \tfrac{1}{2}\sum_{i=1}^d\{(B_i\Theta)_I\langle Q^h e_i, k\rangle + (B_i\Theta)_I\langle h, Q^k e_i\rangle + (B_i^2\Theta)_I\langle h,k\rangle\}\, ds\end{aligned}$$

and hence

$$d(\Theta_I\langle h,k\rangle) = (\Theta_I\langle Q^h\,d\beta,k\rangle + \Theta_I\langle h,\,Q^k\,d\beta\rangle + (B\langle d\beta\rangle\Theta)_I\langle h,k\rangle) \\ + \ (\Theta_I\langle r^h,k\rangle + \Theta_I\langle h,r^k\rangle)\,ds + \tfrac{1}{2}\sum_i (B_i^2\Theta)_I\langle h,k\rangle\,ds \\ + \sum_i \{\Theta_I\langle Q^h e_i,\,Q^k e_i\rangle + (B_i\Theta)_I\langle Q^h e_i,k\rangle \\ + (B_i\Theta)_I\langle h,\,Q^k e_i\rangle\}\,ds.$$

This formula implies (7.10) and (7.11). The assertion that $\Theta_I\langle h,k\rangle \in \mathcal{H}^2$ is easily verified using (7.10) and (7.11). The term $\Theta_I\langle Q^h e_i,\,Q^k e_i\rangle$ in (7.11) is the one which requires the assumption that either Q^h or Q^k is in S^4. □

Theorem 7.4. *Suppose that h and k are* 4*-smooth adapted vector fields on W and that either Q^h or Q^k is in S^4. Then $\{Y^h k - Y^k h + c\langle h,k\rangle\} \in \mathcal{V}^2$ and*

$$[\bar{X}^h,\bar{X}^k] = \bar{X}^{\{Y^h k - Y^k h + c\langle h,k\rangle\}} \qquad on \quad \mathcal{RFC}^\infty, \tag{7.12}$$

where

$$c\langle h,k\rangle \equiv -A\langle h\rangle k + A\langle k\rangle h - \Theta_I\langle h,k\rangle$$

as in (6.3).

Proof. Once we know that $J \equiv \{Y^h k - Y^k h + c\langle h,k\rangle\} \in \mathcal{V}^2$, (7.12) follows from (4.14) and (6.2). Using Lemma 3.7 and the assumption that h and k are 4-smooth, it follows that $Y^h k$ and $Y^k h$ are in $\mathcal{H}^2$. By Lemmas 7.2 and 7.3 and the definition of $c\langle h,k\rangle$, we see that $c\langle h,k\rangle \in \mathcal{H}^2$. Therefore $J = Y^h k - Y^k h + c\langle h,k\rangle \in \mathcal{H}^2$ and it suffices to show that $J \in \mathcal{V}$, i.e., Q^J is an $so(d)$-valued process.

From (7.3) and (3.18),

$$Q^{Y^h k} = Y^h Q^k + Q^k Q^{Y^h} \\ = Y^h Q^k + Q^k(A\langle h\rangle + \Theta_I\langle h,\cdot\rangle + Q^h)$$

and hence

$$Q^{(Y^h k - Y^k h)} = (Y^h Q^k - Y^k Q^h) + [Q^k,Q^h] + Q^k A\langle h\rangle \\ - Q^h A\langle k\rangle + Q^k\Theta_I\langle h,\cdot\rangle - Q^h\Theta_I\langle k,\cdot\rangle. \tag{7.13}$$

By Lemmas 7.2 and 7.3,

$$-Q^{c\langle h,k\rangle} = (\Omega_I\langle h,\cdot\rangle k - \Omega_I\langle k,\cdot\rangle h) + (A\langle h\rangle Q^k - A\langle k\rangle Q^h) \\ + \Theta_I\langle Q^h\cdot,k\rangle + \Theta_I\langle h,Q^k\cdot\rangle + \Theta_I'\langle\cdot,h,k\rangle. \tag{7.14}$$

We write Θ_I^h for $\Theta_I\langle h,\cdot\rangle$ and note that Θ_I^h is an $so(d)$-valued process because of the standing TSS assumption on the covariant derivative ∇. Combining (7.13) and (7.14) gives

$$Q^J = (Y^h Q^k - Y^k Q^h) + [Q^k,Q^h] \\ + [Q^k,A\langle h\rangle] - [Q^h,A\langle k\rangle] + [Q^k,\Theta_I^h] - [Q^h,\Theta_I^k] \\ - (\Omega_I\langle h,\cdot\rangle k - \Omega_I\langle k,\cdot\rangle h + \Theta_I'\langle\cdot,h,k\rangle).$$

The first two lines in the above formula for Q^J are clearly in $so(d)$, so to finish the proof it suffices to show that

$$\tilde{Q} \equiv -\Omega_I\langle h, \cdot\rangle k + \Omega_I\langle k, \cdot\rangle h - \Theta_I'\langle \cdot, h, k\rangle$$
$$= \Omega_I\langle \cdot, h\rangle k - \Omega_I\langle \cdot, k\rangle h - \Theta_I'\langle \cdot, h, k\rangle$$

is an $so(d)$-valued process, i.e., $\langle \tilde{Q}b, b\rangle = 0$ for all $b \in T_oM$. However, this follows directly from Theorem A.4 in the Appendix with $a = h$ and $c = k$. □

8. Lie Bracket on General Cylinder Functions

By taking limits one expects (7.12) to hold on a much larger class of functions than $\mathcal{RFC}^\infty$. In this section, we outline how to show that (7.12) is valid on $\mathcal{FC}^\infty$. For the sake of brevity only the algebraic aspects of the proofs are given; leaving the analytic details and even the precise statements of the theorems to the reader. The next theorem is the analogue of Theorem 5.2 which is stated without proof.

Theorem 8.1. *Suppose $h, k \in \mathcal{V}^4$ are "sufficiently smooth," and $f \in \mathcal{FC}^\infty$ is given as in* (4.12). *Then $X^k f \in \mathcal{D}(\bar{X}^h)$ and*

$$\bar{X}^h X^k f = F'(P)\langle Y^h(-A\langle k\rangle + k)\circ b\rangle$$
$$+ F''(P)\langle(-A\langle h\rangle + h)\circ b, (-A\langle k\rangle + k)\circ b\rangle,$$

where $Y^h(-A\langle k\rangle + k)$ denotes the "directional derivative" of $(-A\langle k\rangle + k)$ by Y^h.

Combining this theorem with lemma 6.1 gives the following commutator formula on $\mathcal{FC}^\infty$.

Theorem 8.2. *Let $f \in \mathcal{FC}^\infty$ be represented as in* (4.12), *and let $h, k \in \mathcal{V}^4$ be "sufficiently smooth," then*

$$[\bar{X}^h, \bar{X}^k]f = F'(P)\langle(Y^h(-A\langle k\rangle + k) - Y^k(-A\langle h\rangle + h) + [A\langle h\rangle, A\langle k\rangle]$$
$$- A\langle h\rangle k + A\langle k\rangle h - \Omega_I\langle h, k\rangle - \Theta_I\langle h, k\rangle)\circ b\rangle$$
$$= F'(P)\langle(Y^h k - Y^k h - A\langle h\rangle k + A\langle k\rangle h - \Theta_I\langle h, k\rangle)\circ b\rangle$$
$$+ F'(P)\langle(-Y^h(A\langle k\rangle) + Y^k(A\langle h\rangle) + [A\langle h\rangle, A\langle k\rangle] - \Omega_I\langle h, k\rangle)\circ b\rangle.$$

The next lemma is the key identity which enables us to recognize the far right member in the above equation as $\bar{X}^{\{Y^h k - Y^k h + c\langle h,k\rangle\}} f$, see Corollary 8.4 below.

Lemma 8.3. *Again assuming that $h, k \in \mathcal{V}^4$ are "sufficiently smooth,"*

$$-Y^h(A\langle k\rangle) + Y^k(A\langle h\rangle) + [A\langle h\rangle, A\langle k\rangle] - \Omega_I\langle h, k\rangle$$
$$= -A\langle Y^h k - Y^k h - A\langle h\rangle k + A\langle k\rangle h - \Theta_I\langle h, k\rangle\rangle, \tag{8.1}$$

where

$$c\langle h,k\rangle \equiv -A\langle h\rangle k + A\langle k\rangle h - \Theta_I\langle h,k\rangle$$

as in (6.3).

The desired commutator formula is now a direct consequence of Theorem 8.2, Lemma 8.3, and (4.14).

Corollary 8.4. *Let $h, k \in \mathcal{V}^4$ be "sufficiently smooth," then*

$$[\bar{X}^h, \bar{X}^k] = \bar{X}^{\{Y^h k - Y^k h + c\langle h,k\rangle\}} \qquad on \quad \mathcal{F}C^\infty,$$

where as before $c\langle h,k\rangle \equiv -A\langle h\rangle k + A\langle k\rangle h - \Theta_I\langle h,k\rangle$.

Proof of Lemma 8.3. (Sketch! Again only the algebraic aspects of the proof are given here.) We start with the identity:

$$\begin{aligned} Y^h(A\langle k\rangle) &= Y^h\left(\int \Omega_I\langle k,\delta\beta\rangle\right) \\ &= \int \{(Y^h\Omega_I)\langle k,\delta\beta\rangle + \Omega_I\langle Y^h k,\delta\beta\rangle + \Omega_I\langle k,\delta Y^h\rangle\} \\ &= \int \Omega_I'\langle -A\langle h\rangle + h, k, \delta\beta\rangle + A\langle Y^h k\rangle \\ &\quad + \int \Omega_I\langle k, A\langle h\rangle\delta\beta + \Theta_I\langle h,\delta\beta\rangle + \delta h\rangle, \end{aligned}$$

where, for $A \in so(d)$ and $a, b, c \in T_oM$,

$$\Omega_u'\langle A + a, b, c\rangle \equiv \hat{A}(u)\Omega_\cdot\langle b,c\rangle + B_a(u)\Omega_\cdot\langle b,c\rangle$$

and

$$\begin{aligned} \hat{A}(u)\Omega_\cdot\langle b,c\rangle &\equiv (d/dt)|_0\Omega_{ue^{tA}}\langle b,c\rangle = (d/dt)|_0 Ad_{e^{-tA}}\Omega_u\langle e^{tA}b, e^{tA}c\rangle \\ &= -[A,\Omega_u\langle b,c\rangle] + \Omega_u\langle Ab,c\rangle + \Omega_u\langle b,Ac\rangle. \end{aligned}$$

Therefore,

$$\int \Omega_I'\langle -A\langle h\rangle, k, \delta\beta\rangle = \int [A\langle h\rangle, \delta(A\langle k\rangle)] - A\langle A\langle h\rangle k\rangle - \int \Omega_I\langle k, A\langle h\rangle\delta\beta\rangle$$

which combined with the first displayed equation in the proof gives

$$\begin{aligned} Y^h(A\langle k\rangle &= \int [A\langle h\rangle, \delta(A\langle k\rangle)] + A\langle Y^h k - A\langle h\rangle k\rangle + \int \Omega_I'\langle h,k,\delta\beta\rangle \\ &\quad + \int \Omega_I\langle k, \Theta_I\langle h,\delta\beta\rangle + \delta h\rangle. \end{aligned}$$

Hence using

$$\delta(\Omega_I\langle h,k\rangle) = \Omega_I\langle \delta h, k\rangle + \Omega_I\langle h, \delta k\rangle + \Omega_I'\langle \delta\beta, h, k\rangle$$

we find

$$\begin{aligned}
Y^h(A\langle k\rangle) - Y^k(A\langle h\rangle) &= \int \{[A\langle h\rangle, \delta(A\langle k\rangle)] + [\delta(A\langle h\rangle), A\langle k\rangle]\} \\
&\quad + A\langle Y^h k - Y^k h - A\langle h\rangle k + A\langle k\rangle h\rangle \\
&\quad + \int \{\Omega_I'\langle h, k, \delta\beta\rangle + \Omega_I\langle k, \Theta_I\langle h, \delta\beta\rangle\rangle \\
&\qquad - \Omega_I'\langle k, h, \delta\beta\rangle - \Omega_I\langle h, \Theta_I\langle k, \delta\beta\rangle\rangle\} \\
&\quad + \int \{\Omega_I\langle k, \delta h\rangle + \Omega_I\langle \delta k, h\rangle\} \\
&= [A\langle h\rangle, A\langle k\rangle] + A\langle Y^h k - Y^k h - A\langle h\rangle k + A\langle k\rangle h\rangle \\
&\quad + \int \{\Omega_I'\langle h, k, \delta\beta\rangle + \Omega_I'\langle k, \delta\beta, h\rangle\} \\
&\quad + \int \{\Omega_I\langle k, \Theta_I\langle h, \delta\beta\rangle\rangle + \Omega_I\langle h, \Theta_I\langle \delta\beta, k\rangle\rangle\} \\
&\quad - \Omega_I\langle h, k\rangle + \int \Omega_I'\langle \delta\beta, h, k\rangle \\
&= [A\langle h\rangle, A\langle k\rangle] + A\langle Y^h k - Y^k h - A\langle h\rangle k + A\langle k\rangle h\rangle \\
&\quad - \Omega_I\langle h, k\rangle \\
&\quad + \int \{(\Omega_I'\langle h, k, \delta\beta\rangle + \Omega_I\langle k, \Theta_I\langle h, \delta\beta\rangle\rangle) + (\text{cyclic})\} \\
&\quad - \Omega_I\langle h, k\rangle + \int \Omega_I\langle \delta\beta, \Theta_I\langle h, k\rangle\rangle,
\end{aligned}$$

where in the last equality the term "(cyclic)" indicates there are two more terms obtained from the preceding term by performing cyclic permutation in h, k, and $\delta\beta$. Combining this equation with the geometric identity in Lemma A.3 shows that

$$\begin{aligned}
Y^h(A\langle k\rangle) - Y^k(A\langle h\rangle) &= [A\langle h\rangle, A\langle k\rangle] + A\langle Y^h k - Y^k h - A\langle h\rangle k + A\langle k\rangle h\rangle \\
&\quad -\Omega_I\langle h, k\rangle + \int \Omega_I\langle \delta\beta, \Theta_I\langle h, k\rangle\rangle,
\end{aligned}$$

which is the same as (8.1) upon noting that

$$\int \Omega_I\langle \delta\beta, \Theta_I\langle h, k\rangle\rangle = -A\langle \Theta_I\langle h, k\rangle\rangle. \qquad \square$$

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Appendix. Geometric Identities

Remark A.1. Let θ denote the T_oM-valued 1-form on $O(M)$ defined by $\theta\langle\xi_u\rangle = u^{-1}\pi_*\xi_u$, where $\xi_u \in T_uO(M)$. Set

$$\Omega \equiv d\omega + \omega \wedge \omega \tag{A.1}$$

and

$$\Theta \equiv d\theta + \omega \wedge \theta, \tag{A.2}$$

then

$$\Omega_u\langle a, b\rangle = \Omega\langle B_a(u), B_b(u)\rangle \quad \text{and} \quad \Theta_u\langle a, b\rangle = \Theta\langle B_a(u), B_b(u)\rangle.$$

For $A \in so(d)$, let $\hat{A}$ be the vertical vector field on $O(M)$ defined by $\hat{A}(u) = (d/dt)|_0 ue^{tA} \in T_uO(M)$ for all $u \in O(M)$. The tangent vector $\hat{A}(u)$ is also denoted by $u \cdot A$ when convenient. We also write $\Omega\langle a, b\rangle$ and $\Theta\langle a, b\rangle$ for the functions $u \to \Omega_u\langle a, b\rangle$ and $u \to \Theta_u\langle a, b\rangle$.

Lemma A.2 (Commutator Formulas). *Let $a, b \in T_oM$ and $A, C \in so(d)$, then*

1. $[\hat{A}, B_a] = B_{Aa}$,
2. $[\hat{A}, \hat{C}] = [A, C]\hat{}$, *and*
3. $[B\langle a\rangle, B\langle b\rangle](u) = -\hat{\Omega}_u\langle a, b\rangle - B\langle\Theta_u\langle a, b\rangle\rangle(u)$, *where* $\hat{\Omega}_u\langle a, b\rangle \equiv (d/dt)|_0\, ue^{t\Omega_u\langle a,b\rangle} \in T_uO(M)$. *We abbreviate this last equation as*

$$[B_a, B_b] = -\hat{\Omega}\langle a, b\rangle - B_{\Theta\langle a,b\rangle}. \tag{A.3}$$

Proof. The proof may be found in [39]. The short proof is given here for the readers convenience.

The proof of the first two assertions relies on the fact that $e^{t\hat{A}} = R_{e^{tA}}$, where $R_g(u) \equiv ug$ for $u \in O(M)$ and $g \in O(d)$. It is easy to verify that

$$R_{e^{-tA}*}B_a \circ R_{e^{tA}} = B_{e^{tA}a},$$

and hence $[\hat{A}, B_a] = (d/dt)|_0 B_{e^{tA}a} = B_{Aa.}$ Similarly,

$$\begin{aligned}[\hat{A}, \hat{C}] &= (d/dt)|_0 R_{e^{-tA}*}\hat{C} \circ R_{e^{tA}} = (d/dt)|_0(d/ds)|_0 R_{e^{-tA}} \circ R_{e^{sC}} \circ R_{e^{tA}} \\ &= (d/dt)|_0(d/ds)|_0 R_{e^{tA}e^{sC}e^{-tA}} = (d/dt)|_0(d/ds)|_0 R_{e^{sAd_{e^{tA}}C}} \\ &= (d/dt)|_0(Ad_{e^{tA}}C)\hat{} = ([A, C])\hat{}.\end{aligned}$$

For the proof of the third item recall that all tangent vectors $\eta_u \in T_uO(M))$ can be written as $u \cdot \omega\langle\eta_u\rangle + B\langle\theta\langle\eta_u\rangle\rangle$. Using the structure equation $\Omega = d\omega + \omega \wedge \omega$ it is easy to conclude that $\omega\langle\eta_u\rangle = -\Omega_u\langle a, b\rangle$. Similarly, using the structure equation $\Theta = d\theta + \omega \wedge \theta$, we show that $\theta\langle\eta_u\rangle = -\Theta_u\langle a, b\rangle$.

Lemma A.3. *For all $a, b, c \in T_o M$ and $u \in O(M)$,*

$$(\Omega_u'\langle a, b, c\rangle + \Omega_u\langle\Theta_u\langle a, b\rangle, c\rangle) + (\textit{cyclic permutations in } a, b, c) = 0, \tag{A.4}$$

where

$$\Omega_u'\langle a, b, c\rangle \equiv (B_a\Omega)_u\langle b, c\rangle = (d/dt)|_0\Omega_{e^{tB_a}(u)}\langle b, c\rangle.$$

Proof. The Bianchi identity states that $(d\Omega)^H = 0$. That is,

$$0 = d\Omega\langle B_a, B_b, B_c\rangle = B_a\Omega\langle B_b, B_c\rangle - \Omega\langle [B_a, B_b], B_c\rangle + (\text{cyclic}).$$

Because of (A.3) and the fact that Ω annihilates vertical vectors, the above equation may be written as

$$0 = B_a\Omega\langle B_b, B_c\rangle + \Omega\langle\Theta\langle B_a, B_b\rangle, B_c\rangle,$$

which, in view of Remark A.1, is equivalent to (A.4). □

Theorem A.4. *Let $a, b, c \in T_o M$, then*

$$\Omega\langle b, a\rangle c \cdot b - \Omega\langle b, c\rangle a \cdot b - B_b\Theta\langle a, c\rangle \cdot b = 0, \tag{A.5}$$

where $B_b \equiv B\langle b\rangle$ and $a \cdot b \equiv \langle a, b\rangle$.

Proof. The identity (see [39])

$$d\Theta + \omega \wedge \Theta = \Omega \wedge \theta$$

applied to B_a, B_b, and B_c gives

$$\begin{aligned} 0 &= \{B_a\Theta\langle b, c\rangle - \Theta\langle [B_a, B_b], B_c\rangle - \Omega\langle a, b\rangle c\} + \text{cyclic}\{a, b, c\} \\ &= \{B_a\Theta\langle b, c\rangle + \Theta\langle\Theta\langle a, b\rangle, c\rangle - \Omega\langle a, b\rangle c\} + \text{cyclic}\{a, b, c\}, \end{aligned} \tag{A.6}$$

where (A.3) and the fact that Θ annihilates vertical vectors was used in the second equality. Take the inner product of (A.6) with b and use the TSS assumption on Θ to conclude that

$$\begin{aligned} 0 &= \{B_a\Theta\langle b, c\rangle + \Theta\langle\Theta\langle a, b\rangle, c\rangle - \Omega\langle a, b\rangle c\} \cdot b \\ &\quad + \{B_b\Theta\langle c, a\rangle + \Theta\langle\Theta\langle b, c\rangle, a\rangle - \Omega\langle b, c\rangle a\} \cdot b \\ &\quad + \{B_c\Theta\langle a, b\rangle + \Theta\langle\Theta\langle c, a\rangle, b\rangle - \Omega\langle c, a\rangle b\} \cdot b \\ &= -\Theta\langle b, c\rangle \cdot \Theta\langle a, b\rangle - \Omega\langle a, b\rangle c \cdot b \\ &\quad + B_b\Theta\langle c, a\rangle \cdot b - \Theta\langle b, a\rangle \cdot \Theta\langle b, c\rangle - \Omega\langle b, c\rangle a \cdot b \\ &\quad + 0 \\ &= \Omega\langle a, b\rangle b \cdot c - B_b\Theta\langle c, b\rangle \cdot a - \Omega\langle c, b\rangle b \cdot a \\ &= \Omega\langle b, a\rangle c \cdot b - B_b\Theta\langle a, c\rangle \cdot b - \Omega\langle b, c\rangle a \cdot b. \end{aligned}$$

□

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