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Finite Dimensional Approximations to Wiener Measure and Path Integral Formulas on Manifolds

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Certain natural geometric approximation schemes are developed for Wiener measure on a compact Riemannian manifold. These approximations closely mimic the informal path integral formulas used in the physics literature for representing the heat semi-group on Riemannian manifolds. The path space is approximated by finite dimensional manifolds $\mathrm{H}_{\mathscr{P}}(M)$ consisting of piecewise geodesic paths adapted to partitions $\mathscr{P}$ of $[0, 1]$. The finite dimensional manifolds $\mathrm{H}_{\mathscr{P}}(M)$ carry both an H^1 and a L^2 type Riemannian structures, $G^1_{\mathscr{P}}$ and $G^0_{\mathscr{P}}$, respectively. It is proved that $(1/Z^i_{\mathscr{P}})\, e^{-(1/2)\, E(\sigma)}\, d\,\mathrm{Vol}_{G^i_{\mathscr{P}}}(\sigma) \to \rho_i(\sigma)\, d\nu(\sigma)$ as $\mathrm{mesh}(\mathscr{P}) \to 0$, where $E(\sigma)$ is the energy of the piecewise geodesic path $\sigma \in \mathrm{H}_{\mathscr{P}}(M)$, and for $i=0$ and 1, $Z^i_{\mathscr{P}}$ is a "normalization" constant, $\mathrm{Vol}_{G^i_{\mathscr{P}}}$ is the Riemannian volume form relative to $G^i_{\mathscr{P}}$, and ν is Wiener measure on paths on M. Here $\rho_1(\sigma) \equiv 1$ and $\rho_0(\sigma) = \exp(-\frac{1}{6}\int_0^1 \mathrm{Scal}(\sigma(s))\, ds)$ where Scal is the scalar curvature of M. These results are also shown to imply the well known integration by parts formula for the Wiener measure. © 1999 Academic Press

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1. INTRODUCTION

Let (M, g, o) be a Riemannian manifold M of dimension d, with Riemannian metric g (we will also use $\langle \cdot, \cdot \rangle$ to denote the metric) and a given base point $o \in M$. Let ∇ be the Levi–Civita covariant derivative, $\Delta = \operatorname{tr} \nabla^2$ denote the Laplacian acting on $C^\infty(M)$ and $p_s(x, y)$ be the fundamental solution to the heat equation, $\partial u/\partial s = \frac{1}{2}\Delta u$. More explicitly, $p_s(x, y)$ is the integral kernel of the operator $e^{(s/2)\Delta}$ acting on $L^2(M, dx)$, where dx denotes the Riemannian volume measure.

For simplicity we will restrict our attention to the case where M is either compact or M is $\mathbb{R}^d$. If $M = \mathbb{R}^d$, we will always take $o = 0$ and $\langle \cdot, \cdot \rangle$ to be the standard inner product on $\mathbb{R}^d$. In either of these cases M is stochastically complete, i.e, $\int_M p_s(x, y)\, dy = 1$ for all $s > 0$ and $x \in M$. Recall, for s small and x and y close in M, that

$$p_s(x, y) \approx \left(\frac{1}{2\pi s}\right)^{d/2} e^{-(1/2s)\, d(x, y)^2}, \tag{1.1}$$

where $d(x, y)$ is the Riemannian distance between x and y. Moreover if $M = \mathbb{R}^d$, then $\Delta = \sum_{i=1}^d \partial^2/\partial x_i^2$, $d(x, y) = |x - y|$ and Eq. (1.1) is exact.

Definition 1.1. The *Wiener space* $\mathrm{W}([0, T]; M)$, $T > 0$ is the path space

$$\mathrm{W}([0, T]; M) = \{\sigma\colon [0, T] \to M : \sigma(0) = o \text{ and } \sigma \text{ is continuous}\}. \tag{1.2}$$

The *Wiener measure* ν associated to $(M, \langle \cdot, \cdot \rangle, o)$ is the unique probability measure on $\mathrm{W}([0, T]; M)$ such that

$$\int_{\mathrm{W}([0, T]; M)} f(\sigma)\, d\nu_T(\sigma) = \int_{M^n} F(x_1, ..., x_n) \prod_{i=1}^n p_{\Delta_i s}(x_{i-1}, x_i)\, dx_1 \cdots dx_n, \tag{1.3}$$

for all functions f of the form $f(\sigma) = F(\sigma(s_1), ..., \sigma(s_n))$, where $\mathcal{P} := \{0 = s_0 < s_1 < s_2 < \cdots < s_n = T\}$ is a partition of $I := [0, T]$, $\Delta_i s := s_i - s_{i-1}$, and $F\colon M^n \to \mathbb{R}$ is a bounded measurable function. In Eq. (1.3), dx

denotes the Riemann volume measure on M and by convention $x_0 := o$. For convenience we will usually take $T=1$ and write $\mathrm{W}(M)$ for $\mathrm{W}([0,1]; M)$ and ν for ν_1.

As is well known, there exists a unique probability measure ν_T on $\mathrm{W}([0,T]; M)$ satisfying (1.3). The measure ν_T is concentrated on continuous but *nowhere* differentiable paths. In particular we get the following path integral representation for the heat semi-group in terms of the measure ν_T,

$$e^{(s/2)\Delta} f(o) = \int_{\mathrm{W}([0,T]; M)} f(\sigma(s))\, d\nu_T(\sigma), \tag{1.4}$$

where f is a continuous function on M and $0 \leqslant s \leqslant T$.

Notation 1.2. When $M = \mathbb{R}^d$, $\langle \cdot, \cdot \rangle$ is the usual dot product and $o = 0$, the measure ν defined in Definition 1.1 is standard Wiener measure on $\mathrm{W}(\mathbb{R}^d)$. We will denote this standard Wiener measure by μ rather than ν. We will also let $B(s)\colon \mathrm{W}(\mathbb{R}^d) \to \mathbb{R}^d$ be the coordinate map $B(s)(\sigma) := B(s, \sigma) := \sigma(s)$ for all $\sigma \in \mathrm{W}(\mathbb{R}^d)$.

Remark 1.3 (Brownian Motion). The process $\{B(s)\}_{s \in [0,1]}$ is a standard $\mathbb{R}^d$-valued Brownian motion on the probability space $(\mathrm{W}(\mathbb{R}^d), \mu)$.

1.1. *A Heuristic Expression for Wiener Measure.* Given a partition $\mathscr{P} := \{0 < s_1 < s_2 < \cdots < s_n = 1\}$ of $[0,1]$ and $\mathbf{x} := (x_1, ..., x_n) \in M^n$, let $\sigma_{\mathbf{x}}$ denote a path in $\mathrm{W}(M)$ such that $\sigma_{\mathbf{x}}(s_i) = x_i$ and such that $\sigma_{\mathbf{x}}|_{[s_{i-1}, s_i]}$ is a geodesic path of shortest length for $i = 1, 2, ..., n$. (As above, $x_0 := o \in M$.) With this notation and the asymptotics for $p_s(x, y)$ in Eq. (1.1), we find

$$\begin{aligned} \prod_{i=1}^n p_{\Delta_i s}(x_{i-1}, x_i) &\approx \prod_{i=1}^n \left(\frac{1}{2\pi \Delta_i s}\right)^{d/2} \exp\left\{-\frac{1}{2\Delta_i s}\, d(x_{i-1}, x_i)^2\right\} \\ &= \frac{1}{Z_{\mathscr{P}}} \exp\left\{-\frac{1}{2}\int_0^1 |\sigma'_{\mathbf{x}}(x)|^2\, ds\right\}, \end{aligned}$$

where $\sigma'_{\mathbf{x}}(s) := (d/ds)\, \sigma_{\mathbf{x}}(s)$ for $s \notin \mathscr{P}$ and $Z_{\mathscr{P}} := \prod_{i=1}^n (2\pi \Delta_i s)^{d/2}$. Using this last expression in Eq. (1.3) and letting the mesh of the partition $\mathscr{P}$ tend to zero we are lead to the *heuristic* expression

$$d\nu(\sigma) \text{ “}=\text{” } \frac{1}{Z} e^{-(1/2)\, E(\sigma)} \mathscr{D}\sigma, \tag{1.5}$$

where

$$E(\sigma) := \int_0^1 \langle \sigma'(s), \sigma'(s) \rangle\, ds \tag{1.6}$$

is the *energy* of σ, $\mathscr{D}\sigma$ denotes a "Lebesgue" like measure on $\mathrm{W}(M)$ and Z is a "normalization constant" chosen so as to make ν a probability measure.

Let V be a continuous function on M. Then Eq. (1.5) and Trotter's product formula leads to the following heuristic path integral formula for the parabolic heat kernel of the Schrödinger operator $\frac{1}{2}\Delta - V$,

$$e^{s((1/2)\,\Delta - V)}f(o)$$
$$\text{``}=\text{''}\ \frac{1}{Z}\int_{\mathrm{W}(M)} f(\sigma(1))\, e^{-((1/2s)\,E(\sigma) + s\int_0^1 V(\sigma(r))\,dr)}\,\mathscr{D}\sigma. \tag{1.7}$$

Equation (1.7) can be interpreted as a prescription for the path integral quantization of the Hamiltonian $\frac{1}{2}g^{ij}p_i p_j + V$. The use of "path integrals" in physics including heuristic expressions like those in Eqs. (1.5) and (1.7) started with Feynman in [47] with very early beginnings being traced back to Dirac [26]. See Gross [54] for a brief survey of the role of path integrals in constructive quantum field theory and Glimm and Jaffe [52] for a more detailed account.

The heuristic interpretation of the "measure" $\mathscr{D}\sigma$ is somewhat ambiguous in the literature. Some authors, for example, [21, 23–25] tend to view $\mathrm{W}(M)$ as the infinite product space M^I and $\mathscr{D}\sigma$ as an infinite product of Riemann volume measures on this product space. This is the interpretation which is suggested by the "derivation" of Eq. (1.5) which we have given above.

Other authors, [4, 11] interpret $\mathscr{D}\sigma$ as a Riemannian "volume form" on $\mathrm{W}(M)$. We prefer this second point of view. One reason for our bias towards the volume measure interpretation is the fact that the path space $\mathrm{W}(M)$ is topologically trivial whereas the product space M^I is not. This fact is reflected in the ambiguity (which we have glossed over) in assigning a path $\sigma_{\mathbf{x}}$ to a point $\mathbf{x} = (x_1, ..., x_n) \in M^n$ as above in the case when there are multiple distinct shortest geodesics joining some pair (x_{i-1}, x_i). However, from the purely measure theoretic considerations in this paper we shall see that the two interpretations of $\mathscr{D}\sigma$ are commensurate.

Of course Eqs. (1.5) and (1.7) are meaningless as they stand because: (1) infinite dimensional Lebesgue measures do not exist and (2) Wiener measure ν concentrates on nowhere differentiable paths which renders the exponent in (1.5) meaningless. Nevertheless, in Theorem 1.8 we will give two precise interpretations of Eq. (1.5).

1.2. *Volume Elements on Path Space.* To make the above discussion more precise, let $\mathrm{H}(M) \subset \mathrm{W}(M)$ be the Hilbert manifold modeled on the space $\mathrm{H}(\mathbb{R}^d)$ of finite energy paths:

$$\mathrm{H}(M) = \{\sigma \in \mathrm{W}(M) : \sigma \text{ is absolutely continuous and } E(\sigma) < \infty\}. \tag{1.8}$$

Recall that $\sigma \in \mathrm{W}(M)$ is said to be absolutely continuous if $f \circ \sigma$ is absolutely continuous for all $f \in C^\infty(M)$. (It is easily checked that the space $\mathrm{H}(M)$ is independent of the choice of Riemannian metric on M.) The tangent space $T_\sigma \mathrm{H}(M)$ to $\mathrm{H}(M)$ at σ may be naturally identified with the space of absolutely continuous vector fields $X: [0, 1] \to TM$ along σ (i.e., $X(s) \in T_{\sigma(s)} M$ for all s) such that $X(0) = 0$ and $G^1(X, X) < \infty$, where

$$G^1(X, X) := \int_0^1 \left\langle \frac{\nabla X(s)}{ds}, \frac{\nabla X(s)}{ds} \right\rangle ds, \tag{1.9}$$

$$\frac{\nabla X(s)}{ds} := //_s(\sigma) \frac{d}{ds} \{//_s(\sigma)^{-1} X(s)\}, \tag{1.10}$$

and $//_s(\sigma): T_o M \to T_{\sigma(s)} M$ is parallel translation along σ relative to the Levi–Civita covariant derivative ∇. See [35, 36, 48, 64, 85] for more details.

By polarization, Eq. (1.9) defines a Riemannian metric on $\mathrm{H}(M)$. Similarly we may define a "weak" Riemannian metric G^0 on $\mathrm{H}(M)$ by setting

$$G^0(X, X) := \int_0^1 \langle X(s), X(s) \rangle \, ds \tag{1.11}$$

for all $X \in T\mathrm{H}(M)$. Given these two metrics it is natural to interpret $\mathscr{D}\sigma$ as either of the (non-existent) "Riemannian volume measures" Vol_{G^1} or Vol_{G^0} with respect to G^1 and G^0 respectively. Both interpretations of $\mathscr{D}\sigma$ are *formally* the same modulo an infinite multiplicative constant, namely the "determinant" of d/ds acting on $\mathrm{H}(T_o M)$.

As will be seen below in Theorem 1.8, the precise version of the heuristic expressions (1.5) and (1.7) shows that *depending on the choice of volume form on the path space*, we get a scalar curvature correction term.

1.3. *Statement of the Main Results.* In order to state the main results, it is necessary to introduce finite dimensional approximations to $\mathrm{H}(M)$, G^1, G^0, Vol_{G^1} and Vol_{G^0}.

Notation 1.4. $\mathrm{H}_{\mathscr{P}}(M) = \{\sigma \in \mathrm{H}(M) \cap C^2(I \backslash \mathscr{P}): \nabla\sigma'(s)/ds = 0 \text{ for } s \notin \mathscr{P}\}$—the piecewise geodesics paths in $\mathrm{H}(M)$ which change directions only at the partition points.

It is possible to check that $\mathrm{H}_{\mathscr{P}}(M)$ is a finite dimensional submanifold of $\mathrm{H}(M)$. Moreover by Remark 4.3 below, $\mathrm{H}_{\mathscr{P}}(M)$ is diffeomorphic to $(\mathbb{R}^d)^n$. For $\sigma \in \mathrm{H}_{\mathscr{P}}(M)$, the tangent space $T_\sigma \mathrm{H}_{\mathscr{P}}(M)$ can be identified with elements $X \in T_\sigma \mathrm{H}_{\mathscr{P}}(M)$ satisfying the Jacobi equations on $I \backslash \mathscr{P}$, see Proposition 4.4 below for more details. We will now introduce Riemann sum approximations to the metrics G^1 and G^0.

DEFINITION 1.5 (The $\mathscr{P}$-Metrics). For each partition $\mathscr{P} = \{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$ of $[0, 1]$, let $G^1_{\mathscr{P}}$ be the metric on $T\mathrm{H}_{\mathscr{P}}(M)$ given by

$$G^1_{\mathscr{P}}(x, Y) := \sum_{i=1}^{n} \left\langle \frac{\nabla X(s_{i-1}+)}{ds}, \frac{\nabla Y(s_{i-1}+)}{ds} \right\rangle \Delta_i s \tag{1.12}$$

for all $X, Y \in T_\sigma \mathrm{H}_{\mathscr{P}}(M)$ and $\sigma \in \mathrm{H}_{\mathscr{P}}(M)$. (We are writing $\nabla X(s_{i-1}+)/ds$ as a shorthand for $\lim_{s \searrow s_{i-1}} (\nabla X(s)/ds)$.) Similarly, let $G^0_{\mathscr{P}}$ be the *degenerate* metric on $\mathrm{H}_{\mathscr{P}}(M)$ given by

$$G^0_{\mathscr{P}}(X, Y) := \sum_{i=1}^{n} \langle X(s_i), Y(s_i) \rangle \Delta_i s, \tag{1.13}$$

for all $X, Y \in T_\sigma \mathrm{H}_{\mathscr{P}}(M)$ and $\sigma \in \mathrm{H}_{\mathscr{P}}(M)$.

If N^p is an oriented manifold equipped with a possibly degenerate Riemannian metric G, let Vol_G denote the p-form on N determined by

$$\mathrm{Vol}_G(v_1, v_2, ..., v_p) := \sqrt{\det(\{G(v_i, v_j)\}_{i, j=1}^{p})}, \tag{1.14}$$

where $\{v_1, v_2, ..., v_p\} \subset T_n N$ is an oriented basis and $n \in N$. We will often identify a p-form on N with the Radon measure induced by the linear functional $f \in C_c(N) \to \int_N f \,\mathrm{Vol}_G$.

DEFINITION 1.6 ($\mathscr{P}$-Volume Forms). Let $\mathrm{Vol}_{G^0_{\mathscr{P}}}$ and $\mathrm{Vol}_{G^1_{\mathscr{P}}}$ denote the volume forms on $\mathrm{H}_{\mathscr{P}}(M)$ determined by $G^0_{\mathscr{P}}$ and $G^1_{\mathscr{P}}$ in accordance with Eq. (1.14).

Given the above definitions, there are now two natural finite dimensional "approximations" to v in Eq. (1.5) given in the following definition.

DEFINITION 1.7 (Approximates to Wiener Measure). For each partition $\mathscr{P} = \{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$ of $[0, 1]$, let $v^0_{\mathscr{P}}$ and $v^1_{\mathscr{P}}$ denote measures on $\mathrm{H}_{\mathscr{P}}(M)$ defined by

$$v^0_{\mathscr{P}} := \frac{1}{Z^0_{\mathscr{P}}} e^{-(1/2)E} \mathrm{Vol}_{G^0_{\mathscr{P}}}$$

and

$$v^1_{\mathscr{P}} = \frac{1}{Z^1_{\mathscr{P}}} e^{-(1/2)E} \mathrm{Vol}_{G^1_{\mathscr{P}}},$$

where E: $\mathrm{H}(M) \to [0, \infty)$ is the energy functional defined in Eq. (1.6) and $Z^0_{\mathscr{P}}$ and $Z^1_{\mathscr{P}}$ are normalization constants given by

$$Z^0_{\mathscr{P}} := \prod_{i=1}^{n} (\sqrt{2\pi}\, \Delta_i s)^d \qquad \text{and} \qquad Z^1_{\mathscr{P}} := (2\pi)^{dn/2}. \tag{1.15}$$

We are now in a position to state the main results of this paper.

THEOREM 1.8. *Suppose that f: $W(M) \to \mathbb{R}$ is a bounded and continuous, then*

$$\lim_{|\mathscr{P}| \to 0} \int_{\mathrm{H}_{\mathscr{P}}(M)} f(\sigma)\, d\nu^1_{\mathscr{P}}(\sigma) = \int_{\mathrm{W}(M)} f(\sigma)\, d\nu(\sigma) \tag{1.16}$$

and

$$\lim_{|\mathscr{P}| \to 0} \int_{\mathrm{H}_{\mathscr{P}}(M)} f(\sigma)\, d\nu^0_{\mathscr{P}}(\sigma)$$

$$= \int_{\mathrm{W}(M)} f(\sigma)\, e^{-(1/6) \int_0^1 \mathrm{Scal}(\sigma(s))\, ds}\, d\nu(\sigma), \tag{1.17}$$

where Scal *is the scalar curvature of* (M, g).

Equation (1.16) is a special case of Theorem 4.17 which is proved in Subsection 4.1 and Eq. (1.17) is a special case of Theorem 6.1 which is proved in Section 6 below. An easy corollary of Eq. (1.17) of this theorem is the following "Euler approximation" construction for the heat semi-group $e^{s\Delta/2}$ on $L^2(M, dx)$. The following corollary is a special case of Corollary 6.7

COROLLARY 1.9. *For $s > 0$ let Q_s be the symmetric integral operator on $L^2(M, dx)$ defined by the kernel*

$$Q_s(x, y) := (2\pi s)^{-d/2} \exp\left(-\frac{1}{2s} d^2(x, y) + \frac{s}{12} \mathrm{Scal}(x) + \frac{s}{12} \mathrm{Scal}(y) \right)$$

$$\text{for all} \quad x, y \in M.$$

Then for all continuous functions F: $M \to \mathbb{R}$ and $x \in M$,

$$(e^{(s/2)\Delta} F)(x) = \lim_{n \to \infty} (Q^n_{s/n} F)(x).$$

1.4. *Remarks on the Main Theorems.* Let us point out that the idea of approximating Wiener measure by measures on spaces of piecewise geodesics is not new, see for example [18, 86]. What we feel is novel about our approach is the interpretation of $\mathscr{D}\sigma$ in Eq. (1.7) as a volume form on

$H_{\mathcal{P}}(M)$ relative to a suitable metric. However (as will be shown in Propositions 5.6 and Proposition 5.14 below), the measure $d\nu^0_{\mathcal{P}}(\sigma)$ is, up to small errors, equivalent to a product measure on M^n where n is the number appearing in Definition 1.5. Reformulated in this guise, there is a large literature pertaining to Eq. (1.17) and especially Corollary[1] 1.9, see [15, 49, 58, 60, 94] to give a very small sampling of the literature. These papers along with [18, 86] are based on using a Trotter product or Euler approximation methods which are well explained in [16]. Moreover, once $d\nu^0_{\mathcal{P}}(\sigma)$ is replaced by a product measure, it would be possible to invoke weak convergence arguments to give a proof of Eq. (1.17), see, for example, Section 10 in Stroock and Varadhan [90, 91] and Ethier and Kurtz [45]. We will not use the weak convergence arguments in this paper, rather we will make use of Wong and Zakai [96] type approximation theorems for stochastic differential equations. This allows us to get the stronger form of convergence which is stated in Theorems 4.17 and 6.1 below. This stronger form of convergence is needed in the proof of the integration by parts Theorem 1.10 stated at the end of this introduction.

In the literature one often finds "verifications" (or rather tests) of path integral formulas like (1.7) by studying the small s asymptotics. This technique, known as "loop expansion" or "WKB approximation," when applied in the manifold case leads to the insight that the operator constructed from the Hamiltonian $\frac{1}{2}g^{ij}p_i p_j + V$ depends sensitively on choices made in the approximation scheme for the path integral. Claims have been made that the correct form of the operator which is the path integral quantization of the Hamiltonian $\frac{1}{2}g^{ij}p_i p_j + V$ is of the form $-\hbar^2(\frac{1}{2}\Delta - \kappa\,\mathrm{Scal}) + V$ where $\hbar$ is Planck's constant, Scal is the scalar curvature of (M, g) and κ is a constant whose value depends on the authors and their interpretation of the path integrals. Values given in the literature include $\kappa = \frac{1}{12}$, $\kappa = \frac{1}{6}$ [22], $\kappa = \frac{1}{8}$, [20, Eq. (6.5.25)] all of which are computed by formal expansion methods. The ambiguity in the path integral is analogous to the operator ordering ambiguity appearing in pseudo-differential operator techniques for quantization, see the paper by Fulling [50] for a discussion of this point. In [50] it is claimed that depending on the choice of covariant operator ordering, the correction term has κ ranging from 0 (for Weyl quantization) to $\frac{1}{6}$. For a discussion in the context of geometric quantization, see [97, Sect. 9.7], where the value $\kappa = \frac{1}{12}$ is given for the case of a real polarization. In addition to the above one also finds in the literature claims, based on perturbation calculations, that noncovariant correction terms are necessary in path integrals, see, for example, [19] and references therein.

[1] After finishing this manuscript, we received the paper of Jyh-Yang Wu [98] where the Trotter product formula method is carried out in detail to give a proof of Corollary 1.9.

It should be stressed that in contrast to the informal calculations mentioned in the previous paragraph, the results presented in Theorem 1.8 and Corollary 1.9 involve only well defined quantities. Let us emphasize that the scalar curvature term appearing in Eq. (1.17) has the nature of a Jacobian factor relating the two volume forms $\mathrm{Vol}_{G^0_{\mathcal{P}}}$ and $\mathrm{Vol}_{G^1_{\mathcal{P}}}$ on path space. This scalar curvature factor would also be found using the Trotter–Euler product approximation methods as a result of the fact that the right hand side of Eq. (1.1) is a parametrix for $e^{t(\Delta/2 - \mathrm{Scal}/6)}$—not $e^{t\Delta/2}$.

We conclude this discussion by mentioning the so called Onsager-Machlup function of a diffusion process. The Onsager-Machlup function can be viewed as an attempt to compute an "ideal density" for the probability measure on path space induced by the diffusion process. In the paper [93], the probability for a Brownian path to be found in a small tubular ε-neighborhood of a *smooth* path σ was computed to be asymptotic to

$$Ce^{-\lambda_1/\varepsilon^2} \cdot \exp\left(-\frac{1}{2} E(\sigma(s)) + \frac{1}{12} \int_0^1 \mathrm{Scal}(\sigma(r))\, dr \right),$$

where λ_1 is the first eigenvalue for the Dirichlet problem on the unit ball in $\mathbb{R}^d$ and C is a constant. The expression $\frac{1}{2}E(\sigma) - \frac{1}{12}\int_0^1 \mathrm{Scal}(\sigma(r))\, dr$ thus recovered from the Wiener measure on $\mathrm{W}(M)$ is in this context viewed as the action corresponding to a Lagrangian for the Brownian motion. It is intriguing to compare this formula with Eqs. (1.16) and (1.17).

1.5. *Integration by Parts on Path Space.* An important result in the analysis on path space, is the formula for partial integration. Here we use the approximation result in Theorem 1.8 to give an alternative proof of this result.

THEOREM 1.10. *Let* $k \in \mathrm{H}(\mathbb{R}^d) \cap C^1([0,1]; \mathbb{R}^d)$, $\sigma \in W(M)$ *and* $X_s(\sigma) \in T_{\sigma(s)}M$ *be the solution to*

$$\frac{\nabla}{ds} X_s(\sigma) + \frac{1}{2} \mathrm{Ric} X_s(\sigma) = \widetilde{/\!/}_s(\sigma)\, k'(s) \qquad \text{with} \quad X_0(\sigma) = 0,$$

where $\widetilde{/\!/}_s(\sigma)$ *denotes stochastic parallel translation along* σ, *see Definition* 4.15. *Then for all smooth cylinder functions* f (*see Definition* 7.15) *on* $\mathrm{W}(M)$,

$$\int_{\mathrm{W}(M)} Xf\, d\nu = \int_{\mathrm{W}(M)} f\left(\int_0^1 \langle k', d\tilde{b} \rangle \right) d\nu.$$

Here $\tilde{b}$ is the $\mathbb{R}^d$-valued Brownian motion which is the anti-development of σ, see Definition 4.15 *and Xf is the directional derivative of f with respect to X, see Definition* 7.15.

Section 7 is devoted to the proof of this result whose precise statement may be found in Theorem 7.16.

Remark 1.11. This theorem first appeared in Bismut [10] in the special case where $f(\sigma)=F(\sigma(s))$ for some $F\in C^\infty(M)$ and $s\in[0,1]$ and then more generally in [30]. Other proofs of this theorem may be found in [1, 2, 31, 40–42, 44, 46, 56, 57, 70, 73, 75, 84].

2. BASIC NOTATIONS AND CONCEPTS

2.1. *Frame Bundle and Connections.* Let $\pi: O(M)\to M$ denote the bundle of orthogonal frames on M. An element $u\in O(M)$ is an isometry $u: \mathbb{R}^d\to T_{\pi(u)}M$. We will make $O(M)$ into a pointed space by fixing $u_0\in\pi^{-1}(o)$ once and for all. We will often use u_0 to identify the tangent space T_oM of M at o with $\mathbb{R}^d$.

Let θ denote the $\mathbb{R}^d$-valued form on $O(M)$ given by $\theta_u(\xi)=u^{-1}\pi_*\xi$ for all $u\in O(M)$, $\xi\in T_uO(M)$ and let ω be the $\mathfrak{so}(d)$-valued connection form on $O(M)$ defined by ∇. Explicitly, if $s\to u(s)$ is a smooth path in $O(M)$ then $\omega(u'(0)):=u(0)^{-1}\,\nabla u(s)/ds|_{t=0}$, where $\nabla u(s)/ds$ is defined as in Eq. (1.10) with X replaced by u. The forms (θ,ω) satisfy the structure equations

$$d\theta=-\omega\wedge\theta, \tag{2.1a}$$

$$d\omega=-\omega\wedge\omega+\Omega, \tag{2.1b}$$

where Ω is the $\mathfrak{so}(d)$-valued curvature 2-form on $O(M)$. The horizontal lift $\mathscr{H}_u: T_{\pi(u)}M\to T_uO(M)$ is uniquely defined by

$$\theta\mathscr{H}_u u=\mathrm{id}_{\mathbb{R}^d},\qquad \omega_u\mathscr{H}_u=0. \tag{2.1c}$$

DEFINITION 2.1. The *curvature tensor* R of ∇ is

$$R(X,Y)\,Z=\nabla_X\nabla_Y Z-\nabla_X\nabla_Y Z-\nabla_{[X,Y]}Z \tag{2.2}$$

for all vector fields X, Y and Z on M. The *Ricci tensor* of (M,g) is $\mathrm{Ric}X:=\sum_{i=1}^d R(X,e_i)\,e_i$ and the *scalar curvature* Scal is $\mathrm{Scal}=\sum_{i=1}^d\langle \mathrm{Ric}e_i, e_i\rangle$, where $\{e_i\}$ is an orthonormal frame.

The relationship between Ω and R is

$$\Omega(\xi,\eta)=u^{-1}R(\pi_*\xi,\pi_*\eta)\,u=\Omega(\mathscr{H}_u\pi_*\xi,\mathscr{H}_u\pi_*\eta) \tag{2.3}$$

for all $u \in O(M)$ and $\xi, \eta \in T_u O(M)$. The second equality in Eq. (2.3) follows from the fact that Ω is horizontal, i.e., $\Omega(\xi, \eta)$ depends only on the horizontal components of ξ and η.

2.2. *Path Spaces and the Development Map.* Let $(M, o, \langle \cdot, \cdot \rangle, \nabla)$, $(O(M), u_0)$, $\mathrm{W}(M)$, and $\mathrm{H}(M)$ be as above. We also let $\mathrm{H}(O(M))$ be the set of finite energy paths u: $[0, 1] \to O(M)$ as defined in Eq. (1.8) with M replaced by $O(M)$ and o by u_0.

For $\sigma \in \mathrm{H}(M)$, let $s \mapsto u(s)$ be the horizontal lift of σ starting at u_0, i.e., u is the solution of the ordinary differential equation

$$u'(s) = \mathscr{H}_{u(s)} \sigma'(s), \qquad u(0) = u_0.$$

Notice that this equation implies that $\omega(u'(s)) = 0$ or equivalently that $\nabla u(s)/ds = 0$. Hence $u(s) = //_s(\sigma)\, u_0$, where as before $//_s(\sigma)$ is the parallel translation operator along σ. Again since $u_0 \in O(M)$ is fixed in this paper we will use u_0 to identify $T_o M$ with $\mathbb{R}^d$ and simply write $u(s) = //_s(\sigma)$. By smooth dependence of solutions of ordinary differential equations on parameters, the map $\sigma \in \mathrm{H}(M) \mapsto //(\sigma) \in \mathrm{H}(O(M))$ is smooth. A proof of this fact may be given using the material in Palais [85], see also Corollary 4.1 in [28].

Definition 2.2 (Cartan's Development Map). The *development map* Φ: $\mathrm{H}(\mathbb{R}^d) \to \mathrm{H}(M)$ is defined, for $b \in \mathrm{H}$, by $\phi(b) = \sigma \in \mathrm{H}(M)$ where σ solves the functional differential equation,

$$\sigma'(s) = //_s(\sigma)\, b'(s), \qquad \sigma(0) = o, \tag{2.4}$$

see [13, 34, 65].

It will be convenient to give another description of the development map ϕ. Namely, if $b \in \mathrm{H}(\mathbb{R}^d)$ and $\sigma = \phi(b) \in \mathrm{H}(M)$ as defined in equation (2.4) then $\sigma = \pi(w)$ where $w(s) \in O(M)$ is the unique solution to the ordinary differential equation

$$w'(s) = \mathscr{H}_{w(s)} w(s)\, b'(s), \qquad w(0) = u_0. \tag{2.5}$$

From this description of ϕ and smooth dependence of solutions of ordinary differential equations on parameters it can be seen that ϕ: $\mathrm{H}(\mathbb{R}^d) \to \mathrm{H}(M)$ is smooth. Furthermore, ϕ is injective by uniqueness of solutions to ordinary differential equations.

The *anti-development map* ϕ^{-1}: $\mathrm{H}(M) \to \mathrm{H}(\mathbb{R}^d)$ is given by $b = \phi^{-1}(\sigma)$ where

$$b(s) = \int_0^s //_r^{-1}(\sigma)\, \sigma'(r)\, dr. \tag{2.6}$$

This inverse map ϕ^{-1} is injective and smooth by the same arguments as above. Hence ϕ: $\mathrm{H}(\mathbb{R}^d) \to \mathrm{H}(M)$ is a diffeomorphism of infinite dimensional Hilbert manifolds, see [34]. However, as can be seen from Eq. (3.5) below, ϕ is not an isometry of the Riemannian manifolds $\mathrm{H}(M)$ and $\mathrm{H}(\mathbb{R}^d)$ unless the curvature Ω of M is zero. So the geometry of $\mathrm{H}(\mathbb{R}^d)$ and that of $(\mathrm{H}(M), G^1)$ are not well related by ϕ.

For each $h \in C^\infty(\mathrm{H}(M) \to H)$ and $\sigma \in \mathrm{H}(M)$, let $X^h(\sigma) \in T_\sigma \mathrm{H}(M)$ be given by

$$X_s^h(\sigma) := //_s(\sigma)\, h_s(\sigma) \qquad \text{for all} \quad s \in I, \tag{2.7}$$

where for notational simplicity we have written $h_s(\sigma)$ for $h(\sigma)(s)$. The vector field X^h is a smooth vector field on $\mathrm{H}(M)$ for all $h \in \mathrm{H}$. The reader should also note that the map

$$((\sigma, h) \to X^h(\sigma)) : \mathrm{H}(M) \times \mathrm{H} \to T\mathrm{H}(M) \tag{2.8}$$

is an isometry of vector bundles.

3. DIFFERENTIALS OF THE DEVELOPMENT MAP

For $u \in O(M)$ and $v, w \in T_{\pi(u)}M$, let

$$R_u(v, w) = \Omega(\mathscr{H}_u v, \mathscr{H}_u w) = u^{-1} R(v, w)\, u$$

and for $a, b \in \mathbb{R}^d$ let

$$\Omega_u(a, b) := \Omega(\mathscr{H}_u ua, \mathscr{H}_u ub) = u^{-1} R(ua, ub)\, u.$$

For $\sigma \in \mathrm{H}(M)$ and $X \in T_\sigma \mathrm{H}(M)$, define $q_s(X) \in \mathfrak{so}(d)$ by

$$q_s(X) = \int_0^s R_{u(r)}(\sigma'(r), X(r))\, dr, \tag{3.1}$$

where $u = //(\sigma)$ is the horizontal lift of σ.

Remark 3.1. The one form q_s in Eq. (3.1) naturally appears as soon as one starts to compute the differential of parallel translation operators, see, for example, Theorem 2.2 in Gross [53] and Theorem 4.1 in [28] and Theorem 3.3 below.

Notation 3.2. Given $A \in \mathfrak{so}(d)$ and $u \in O(M)$, let $u \cdot A \in T_u O(M)$ denote the vertical tangent vector defined by $u \cdot A := (d/dr)|_0\, ue^{rA}$.

THEOREM 3.3. *Let $\sigma \in \mathrm{H}(M)$, let $u = //(\sigma)$ be the horizontal lift of σ and let $b = \phi^{-1}(\sigma)$. Then for $X \in T_\sigma \mathrm{H}(M)$,*

$$(//_s^* \omega)(X) = q_s(X), \tag{3.2}$$

$$(//_s^* \theta)(X) = u^{-1}(s)\, X(s), \tag{3.3}$$

$$(//_* X)(s) = u(s) \cdot q_s(X) + \mathscr{H}_{u(s)} X(s), \tag{3.4}$$

and

$$(\phi^* X)(s) = u^{-1}(s)\, X(s) - \int_0^s q_r(X)\, b'(r)\, dr, \tag{3.5}$$

where $\phi^* X(b) := \phi_*^{-1} X(\phi(b))$.

Remark 3.4. The results of this theorem may be found in one form or another in [10, 17, 29, 30, 53, 72]. We will nevertheless supply a proof to help fix our notation and keep the paper reasonably self contained.

Proof. Choose a one parameter family $t \mapsto \sigma_t$ of curves in $\mathrm{H}(M)$ such that $\sigma_0 = \sigma$ and $\dot{\sigma}_0(s) = X(s)$ where $\dot{\sigma}_t(s) = (d/dt)\, \sigma_t(s)$. Let $u_t(s) := //_s(\sigma_t)$, be the horizontal lift of σ_t, $u(s) = //_s(\sigma)$, $u_t'(s) := du_t(s)/ds$, $\dot{u}_t(s) := du_t(s)/dt$ and $\dot{u}(s) := du_t(s)/dt|_{t=0}$. (In general t-derivatives will be denoted by a "dot" and s-derivatives will be denoted by a "prime.") Notice, by definition, that

$$\dot{u}(s) = (//_s)_*\, X = (//_* X)(s)$$

and $\omega(u_t'(s)) = 0$ for all (t, s). The Cartan identity

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]), \tag{3.6}$$

valid for any 1-form α and vector fields X, Y, gives

$$0 = \frac{d}{dt}\, \omega(u') = d\omega(\dot{u}, u') + \frac{d}{ds}\, \omega(\dot{u}) = \Omega(\dot{u}, u') + \frac{d}{ds}\, \omega(\dot{u}),$$

where we have used the structure equations (2.1b) and $0 = \omega(u')$ in the second equality. Setting $t = 0$ and integrating the previous equation relative to s yields

$$\begin{aligned}(//_s^* \omega)(X) := \omega((//_s)_* X) &= \int_0^s \Omega(u'(0, r), \dot{u}(0, r))\, dr \\ &= \int_0^s R_{u_0(r)}(\pi_* u'(0, r), \pi_* \dot{u}(0, r))\, dr \\ &= \int_0^s R_{u_0(r)}(\sigma'(r), X(r))\, dr,\end{aligned}$$

where we have made use of the fact that Ω is horizontal and the relation $\sigma_t(s) = \pi(u_t(s))$. This proves Eq. (3.2). Equation (3.3) is verified as follows:

$$(//_s^* \theta)(X) = \theta((//_s)_* X) = \theta(\dot{u}(s)) = u_0^{-1}(s) \left.\frac{d}{dt}\right|_{t=0} \pi(u_t(s))$$

$$= //_s^{-1}(\sigma) \left.\frac{d}{dt}\right|_{t=0} \sigma_t(s) = //_s^{-1}(\sigma)\, X(s).$$

Recall that for $u \in O(M)$, (θ, ω): $T_u O(M) \to \mathbb{R}^d \times \mathfrak{so}(d)$ is an isomorphism. Therefore Eqs. (3.2) and (3.3) imply (3.4), after taking into account the definition of θ and the identity,

$$\omega(u \cdot A) := u^{-1} \left.\frac{\nabla}{dr}\right|_{r=0} u e^{rA} = A.$$

To prove Eq. (3.5), let $b = \phi^{-1}(\sigma)$ and $u(s) = //_s(\sigma)$. Then

$$b(s) = \int_0^s u^{-1}(r)\, \sigma'(r)\, dr = \int_0^s \theta(u'(r))\, dr,$$

or equivalently,

$$b'(s) = \theta(u'(s)).$$

Therefore

$$\frac{d}{ds} \phi_*^{-1} X(s) = \frac{d}{dt} \theta(u_t'(s))|_{t=0}$$

$$= \frac{d}{ds} \theta(\dot{u}(s)) + d\theta(\dot{u}(s), u'(s))$$

$$= \frac{d}{ds} (u^{-1}(s)\, X(s)) - \omega \wedge \theta(\dot{u}(s), u'(s))$$

$$= \frac{d}{ds} (u^{-1}(s)\, X(s)) - \omega(\dot{u}(s))\, \theta(u'(s))$$

$$= \frac{d}{ds} (u^{-1}(s)\, X(s)) - q_s(X)\, b'(s),$$

where we have used Eqs. (3.6), (2.1a), (3.2), and the fact that $\omega(u'(s)) = 0$. Integrating the last equation relative to s proves (3.5). ∎

3.1. *Bracket Computation.*

THEOREM 3.5 (Lie Brackets). *Let $h, k\colon \mathrm{H}(M) \to \mathrm{H}(\mathbb{R}^d)$ be smooth functions. (We will write $h_s(\sigma)$ for $h(\sigma)(s)$.) Then $[X^h, X^k] = X^{f(h,k)}$, where $f(h,k)$ is the smooth function $\mathrm{H}(M) \to \mathrm{H}(\mathbb{R}^d)$ defined by*

$$f_s(h,k)(\sigma) := X^h(\sigma)\,k_s - X^k(\sigma)\,h_s + q_s(X^k(\sigma))\,h_s - q_s(X^h(\sigma))\,k_s,$$

where $q = //^*\omega$ as in Eq. (3.2) and $X^h(\sigma)\,k_s$ denotes derivative of $\sigma \to k_s(\sigma)$ by the tangent vector $X^h(\sigma)$.

Remark 3.6. This theorem also appears in Eq. (1.32) in Leandre [71], Eq. (6.2.2) in Cruzeiro and Malliavin [17], and is Theorem 6.2 in [32]. To some extent it is also contained in [48]. Again for the readers convenience will supply a short proof.

Proof. The vector fields X^h and X^k on $\mathrm{H}(M)$ are smooth, hence $[X^h, X^k]$ is well defined. In order to simplify notation, we will suppress the arguments σ and s from the proof of Eq. (3.7).

According to Eq. (3.3), $h = (//^*\theta)(X^h)$, $k = (//^*\theta)(X^k)$, and $f(h,k) = (//^*\theta)([X^h, X^k])$. Using Eqs. (3.1)–(3.6) we find that

$$\begin{aligned} f(h,k) &= X^h[(//^*\theta)(X^k)] - X^k[(//^*\theta)(X^h)] - (d(//^*\theta))(X^h, X^k) \\ &= X^h k - X^k h - (//^*\,d\theta)(X^h, X^k) \\ &= X^h k - X^k h + (//^*(\omega \wedge \theta))(X^h, X^k) \\ &= X^h k - X^k h + (//^*\omega \wedge //^*\theta)(X^h, X^k) \\ &= X^h k - X^k h + q(X^h)\,k - q(X^k)\,h. \quad \blacksquare \end{aligned}$$

4. FINITE DIMENSIONAL APPROXIMATIONS

DEFINITION 4.1. Let $\mathscr{P} = \{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$ be a *partition* of $[0,1]$ and let $|\mathscr{P}| = \max_i |s_i - s_{i-1}|$ be the *norm* of the partition, $J_i := (s_{i-1}, s_i]$ for $i = 1, 2, ..., n$ and $s = s_{i-1}$ when $s \in J_i$. For a function k, let $\Delta_i k := k(s_i) - k(s_{i-1})$ and $\Delta_i s = s_i - s_{i-1}$. For a piecewise continuous function on $[0,1]$, we will use the notation $f(s+) = \lim_{r \searrow s} f(r)$.

Notation 4.2. $\mathrm{H}_{\mathscr{P}} = \{x \in \mathrm{H} \cap C^2(I \backslash \mathscr{P}) : x''(s) = 0 \text{ for } s \notin \mathscr{P}\}$—the piecewise linear paths in $\mathrm{H} := \mathrm{H}(\mathbb{R}^d)$; which change directions only at the partition points.

Remark 4.3 (Development). The development map $\phi: \mathrm{H} \to \mathrm{H}(M)$ has the property that $\phi(\mathrm{H}_{\mathscr{P}}) = \mathrm{H}_{\mathscr{P}}(M)$, where $\mathrm{H}_{\mathscr{P}}(M)$ has been defined in Notation 1.4 above. Indeed, if $\sigma = \phi(b)$ with $b \in \mathrm{H}_{\mathscr{P}}$, then differentiating Eq. (2.4) give

$$\frac{\nabla \sigma'(s)}{ds} = \frac{\nabla}{ds}(//_s(\sigma)\, b'(s)) = //_s(\sigma)\, b''(s) = 0 \qquad \text{for all} \quad s \notin \mathscr{P}.$$

We will write $\phi_{\mathscr{P}}$ for $\phi|_{\mathrm{H}_{\mathscr{P}}}$.

Because $\phi: \mathrm{H} \to \mathrm{H}(M)$ is a diffeomorphism and $\mathrm{H}_{\mathscr{P}} \subset \mathrm{H}$ is an embedded submanifold, it follows that $\mathrm{H}_{\mathscr{P}}(M)$ is an embedded submanifold of $\mathrm{H}(M)$. Therefore for each $\sigma \in \mathrm{H}_{\mathscr{P}}(M)$, $T_\sigma \mathrm{H}_{\mathscr{P}}(M)$ may be viewed as a subspace of $T_\sigma \mathrm{H}(M)$. The next proposition explicitly identifies this subspace.

PROPOSITION 4.4 (Tangent Space). *Let $\sigma \in \mathrm{H}_{\mathscr{P}}(M)$, then $X \in T_\sigma \mathrm{H}(M)$ is in $T_\sigma \mathrm{H}_{\mathscr{P}}(M)$ if and only if*

$$\frac{\nabla^2}{ds^2} X(s) = R(\sigma'(s), X(s))\, \sigma'(s) \qquad on \quad I \backslash \mathscr{P}. \tag{4.1}$$

Equivalently, letting $b = \phi^{-1}(\sigma)$, $u = //(\sigma)$ and $h \in \mathrm{H}$, then $X^h \in T_\sigma \mathrm{H}(M)$ defined in Eq. (2.7) is in $T_\sigma \mathrm{H}_{\mathscr{P}}(M)$ if and only if

$$h''(s) = \Omega_{u(s)}(b'(s), h(s))\, b'(s) \qquad on \quad I \backslash \mathscr{P}.$$

Proof. Since $\mathrm{H}_{\mathscr{P}}(M)$ consists of piecewise geodesics, it follows that for $\sigma \in \mathrm{H}_{\mathscr{P}}(M)$, any $X \in T_\sigma \mathrm{H}_{\mathscr{P}}(M)$ must satisfy the Jacobi Eq. (4.1) for $s \notin \mathscr{P}$. Equation 4.2 is a straightforward reformulation of this using the definitions.

It is instructive to give a direct proof of Eq. (4.1). Since $\mathrm{H}_{\mathscr{P}}$ is a vector space, $T_b \mathrm{H}_{\mathscr{P}} \cong \mathrm{H}_{\mathscr{P}}$ for all $b \in \mathrm{H}_{\mathscr{P}}$. Since $\phi_{\mathscr{P}}: \mathrm{H}_{\mathscr{P}} \to \mathrm{H}_{\mathscr{P}}(M)$ is a diffeomorphism, we must identify those vectors $X \in T_\sigma \mathrm{H}(M)$ such that $\phi^* X \in \mathrm{H}_{\mathscr{P}}$, i.e., those X such that $(\phi^* X)'' := 0$ on $I \backslash \mathscr{P}$. Because $b \in \mathrm{H}_{\mathscr{P}}$ and hence $b''(s) = 0$ on $I \backslash \mathscr{P}$, it follows from Eq. (3.5) that $(\phi^* X)'' = 0$ on $I \backslash \mathscr{P}$ is equivalent to

$$0 = h''(s) - \Omega_{u(s)}(b'(s), h(s))\, b'(s) \qquad \text{on} \quad I \backslash \mathscr{P}. \quad \blacksquare$$

Remark 4.5. The metric $G^1_{\mathscr{P}}$ in Definition 1.5 above is easily seen to be non-degenerate because if $G^1_{\mathscr{P}}(X, X) = 0$ then $\nabla X(s_i +)/ds = 0$ for all i. It then follows from the continuity of X and the fact that X solves the Jacobi Eq. (4.1) that X is zero. Also note that $G^1_{\mathscr{P}}$ is a "belated" Riemann sum approximation to the metric on $\mathrm{H}_{\mathscr{P}}(M)$ which is inherited from G^1 on $\mathrm{H}(M)$. Moreover, in the case $M = \mathbb{R}^d$, the metric $G^1_{\mathscr{P}}$ is equal to G^1 on $T\mathrm{H}_{\mathscr{P}}(M)$.

DEFINITION 4.6. Let $\mathrm{Vol}_{\mathcal{P}}$ be the Riemannian volume form on $\mathrm{H}_{\mathcal{P}}$ equipped with the H^1-metric, $(h, k) := \int_0^1 \langle h'(s), k'(s)\rangle\, ds$.

Notation 4.7. Let $\mathcal{P} = \{0 = {}_0 < s_1 < s_2 < \cdots < s_n = 1\}$ be a partition of $[0, 1]$. For each $i = 1, 2, ..., n$, and $s \in (s_{i-1}, s_i]$, define

$$\hat{q}_s^{\mathcal{P}}(X) = q_{s_{i-1}}(X) \tag{4.3}$$

and

$$\tilde{q}_s^{\mathcal{P}}(X) = q_s(X) - q_{s_{i-1}}(X) = \int_{s_{i-1}}^{s} \Omega_u(\sigma'(r), X(r))\, dr. \tag{4.4}$$

Note that $q = \hat{q}^{\mathcal{P}} + \tilde{q}^{\mathcal{P}}$ and hence Eq. (3.5) becomes

$$\begin{aligned}(\phi^* X^h)'(s) &= h'(s) - q_s(X^h)\, b'(s)\\ &= h'(s) - \hat{q}_s(X^h)\, b'(s) - \tilde{q}_s(X^h)\, b'(s)\end{aligned} \tag{4.5}$$

for all $h \in \mathrm{H}(\mathbb{R}^d)$.

THEOREM 4.8. $\phi_{\mathcal{P}}^* \mathrm{Vol}_{G_{\mathcal{P}}^1} = \mathrm{Vol}_{\mathcal{P}}$.

Proof. Let $\{h_k\}$ be an orthonormal basis for $\mathrm{H}_{\mathcal{P}}$, $b \in \mathrm{H}_{\mathcal{P}}$, $\sigma = \phi(b)$ and $u = //(\sigma)$. Using the definitions of the volume form on a Riemannian manifold we must show that

$$\det(G_{\mathcal{P}}^1(\phi_* h_k, \phi_* h_j)) = 1,$$

where $\phi_* h_k := (d/dt)|_0\, \phi(b + t h_k)$.

Let $H_k(s) = u^{-1}(s)(\phi_*(h_k))(s)$ and set

$$\langle H, K\rangle_{\mathcal{P}} := \sum_{i=1}^{n} \langle H'(s_{i-1}+), K'(s_{i-1}+)\rangle\, \Delta_i s.$$

Then $X^{H_k} = \phi_*(h_k)$ and

$$\det(G_{\mathcal{P}}^1(\phi_*(h_k), \phi_*(h_j))) = \det(\langle H_k, H_j\rangle_{\mathcal{P}}).$$

By Eq. (4.5)

$$h_k' = (\phi^*(X^{H_k}))' = H_k' - q(X^{H_k})\, b' = H_k' - \hat{q}(X^{H_k})\, b' - \tilde{q}(X^{H_k})\, b'$$

so that

$$h_k' + \hat{q}(X^{H_k})\, b' = H_k' - \tilde{q}(X^{H_k})\, b'. \tag{4.6}$$

Noting that h_k', $\hat{q}(X^{H_k})$, and b' are all constant on (s_{i-1}, s_i) and that $\tilde{q}_{s_{i-1}}(X^{H_k})=0$, it follows that both sides of Eq. (4.6) are constant on (s_{i-1}, s_i) and the constant value is $H_k'(s_{i-1}+)$. Therefore

$$\langle H_k, H_j\rangle_{\mathscr{P}} = \int_0^1 \langle H_k' - \tilde{q}(X^{H_k})\, b', H_j' - \tilde{q}(X^{H_j})\, b'\rangle\, ds$$

$$= \int_0^1 \langle h_k' + \hat{q}(X^{H_k})\, b', h_j' + \hat{q}(X^{H_j})\, b'\rangle\, ds.$$

Define the linear transformation, $T\colon H_{\mathscr{P}} \to \mathrm{H}_{\mathscr{P}}$ by

$$(Th)(s) = \int_0^s \hat{q}_r(\phi_* h)\, b'(r)\, dr.$$

We have just shown that

$$\begin{aligned}\det(G^1_{\mathscr{P}}(\phi_*(h_k), \phi_*(h_j))) &= \det(\{\langle (I+T)\, h_k, (I+T)\, h_j\rangle_{\mathscr{P}}\}_{j,k})\\ &= \det(\{\langle h_k, (I+T)^*\, (I+T)\, h_j\rangle_{\mathscr{P}}\}_{j,k})\\ &= \det((I+T)^*\, (I+T)) = [\det(I+T)]^2.\end{aligned}$$

So to finish the proof it suffices to show that $\det(I+T)=1$. This will be done by showing that T is nilpotent. For this we will make a judicious choice of orthonormal basis for $\mathrm{H}_{\mathscr{P}}$. Let $\{e_a\}_{a=1}^d$ be an orthonormal basis for $T_o M \cong \mathbb{R}^d$ and define

$$h_{i,a}(s) = \left(\frac{1}{\sqrt{\Delta_i s}} \int_0^s 1_{J_{i-1}}(r)\, dr\right) e_a$$

for $i=1, 2, ..., n$, $a=1, ..., d$. Using the causality properties of ϕ and $\hat{q}$, it follows that $\phi_* h_{i,a} := 0$ on $[0, s_{i-1}]$ and hence $\hat{q}(\phi_*(h_{i,a})) := 0$ on $[0, s_i)$. Thus for any a, b, $\langle Th_{i,a}, h_{j,b}\rangle = 0$ if $j \leqslant i$. This shows that T is nilpotent and hence finishes the proof. ∎

Definition 4.9. Let $E_{\mathbb{R}^d}(b) := \int_0^1 |b'(s)|^2\, ds$ denote the energy of a path $b \in \mathrm{H}$. For each partition $\mathscr{P} = \{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$ of $[0, 1]$, let $\mu^1_{\mathscr{P}}$ denote the volume form

$$\mu^1_{\mathscr{P}} = \frac{1}{Z^1_{\mathscr{P}}}\, e^{-(1/2)\, E_{\mathbb{R}^d}}\, \mathrm{Vol}_{\mathrm{H}_{\mathscr{P}}}$$

on $\mathrm{H}_{\mathscr{P}}$, where $Z^1_{\mathscr{P}} := (2\pi)^{dn/2}$. (By Lemma 4.11 below, $\mu^1_{\mathscr{P}}$ is a probability measure on $\mathrm{H}_{\mathscr{P}}$.)

Let $b \in \mathrm{H}$ and $\sigma := \phi(b) \in \mathrm{H}(M)$. Because parallel translation is an isometry, it follows from Eq. (2.4) that $E_{\mathbb{R}^d}(b) = E(\sigma)$. As an immediate consequence of this identity and Theorem 4.8 is the following theorem.

THEOREM 4.10. *Let* $\mu^1_{\mathscr{P}}$ *(Definition 4.9) and* $\nu^1_{\mathscr{P}}$ *(Definition 1.7) be as above. Then* $\mu^1_{\mathscr{P}}$ *is the pull back of* $\nu^1_{\mathscr{P}}$ *by* $\phi_{\mathscr{P}}$, *i.e.,* $\mu^1_{\mathscr{P}} = \phi^*_{\mathscr{P}} \nu^1_{\mathscr{P}}$.

Before exploring the consequences of this last theorem, we will make a few remarks about the measure $\mu^1_{\mathscr{P}}$. Let $\pi_{\mathscr{P}}: \mathrm{W}(\mathbb{R}^d) \to (\mathbb{R}^d)^n$ be given by $\pi_{\mathscr{P}}(x) := (x(s_1), x(s_2), ..., x(s_n))$. Note that $\pi_{\mathscr{P}}: \mathrm{H}_{\mathscr{P}} \to (\mathbb{R}^d)^n$ is a linear isomorphism of finite dimensional vector spaces. We will denote the inverse of $\pi_{\mathscr{P}}|_{\mathrm{H}_{\mathscr{P}}}$ by $i_{\mathscr{P}}$.

LEMMA 4.11. *Let* $dy_1\, dy_2 \cdots dy_n$ *denote the standard volume form on* $(\mathbb{R}^d)^n$ *and* $y_0 := 0$ *by convention. Then*

$$i^*_{\mathscr{P}} \mu^1_{\mathscr{P}} = \frac{1}{Z^1_{\mathscr{P}}} \left(\prod_{i=1}^n (\Delta_i s)^{-d/2} \exp\left\{ -\frac{1}{2\Delta_i s} |y_i - y_{i-1}|^2 \right\} \right) dy_1\, dy_2 \cdots dy_n, \tag{4.7}$$

where $Z^1_{\mathscr{P}}$ *is defined in Eq. (1.15). Using the explicit value on* $Z^1_{\mathscr{P}}$, *this equation may also be written as*

$$i^*_{\mathscr{P}} \mu^1_{\mathscr{P}} = \left(\prod_{i=1}^n p_{\Delta_i s}(y_{i-1}, y_i) \right) dy_1\, dy_2 \cdots dy_n, \tag{4.8}$$

where $p_s(x, y) := (2\pi s)^{-d/2} \exp\{-|x-y|^2/2s\}$ *is the heat kernel on* $\mathbb{R}^d$. *In particular* $i^*_{\mathscr{P}} \mu^1_{\mathscr{P}}$ *and hence* $\mu^1_{\mathscr{P}}$ *are probability measures.*

Proof. Let $x \in \mathrm{H}_{\mathscr{P}}$, then

$$E(x) = \int_0^1 |x'(s)|^2\, ds = \sum_{i=1}^n \left| \frac{\Delta_i x}{\Delta_i s} \right|^2 \Delta_i s = \sum_{i=1}^n \frac{1}{\Delta_i s} |\Delta_i x|^2.$$

Hence if $x = i_{\mathscr{P}}(y)$, then

$$\int_0^1 |x'(s)|^2\, ds = \sum_{i=1}^n \frac{1}{\Delta_i s} |y_i - y_{i-1}|^2 = \sum_{i=1}^n |\xi_i|^2, \tag{4.9}$$

where $\xi_i := (\Delta_i s)^{-1/2} (y_i - y_{i-1})$. This last equation shows that the linear transformation

$$x \in \mathrm{H}_{\mathscr{P}} \to \{(\Delta_i s)^{-1/2} (x(s_i) - x(s_{i-1}))\}_{i=1}^n \in (\mathbb{R}^d)^n$$

is an isometry of vector spaces and therefore

$$i_{\mathscr{P}}^{*}\,\mathrm{Vol}_{\mathscr{P}} = d\xi_1\, d\xi_2 \cdots d\xi_n. \tag{4.10}$$

Now an easy computation shows that

$$d\xi_1\, d\xi_2 \cdots d\xi_n = \left(\prod_{i=1}^{n} (\Delta_i s)^{-d/2} \right) dy_1\, dy_2 \cdots dy_n. \tag{4.11}$$

From Eqs. (4.9)–(4.11), we see that Eq. (4.7) is valid. ∎

Notation 4.12. Let $\{B(s)\}_{s\in[0,1]}$ be the standard $\mathbb{R}^d$-valued Brownian motion on $(\mathrm{W}(\mathbb{R}^d), \mu)$ as in Notation 1.2. Given a partition $\mathscr{P}$ of $[0,1]$ as above, set $B_{\mathscr{P}} := i_{\mathscr{P}} \circ \pi_{\mathscr{P}}(B)$. The explicit formula for $B_{\mathscr{P}}$ is

$$B_{\mathscr{P}}(s) = B(s_{i-1}) + (s - s_{i-1})\,\frac{\Delta_i B}{\Delta_i s} \qquad \text{if} \quad s \in (s_{i-1}, s_i],$$

where $\Delta_i B := B(s_i) - B(s_{i-1})$. We will also denote the expectation relative to μ by $\mathbb{E}$, so that $\mathbb{E}[f] = \int_{\mathrm{W}(\mathbb{R}^d)} f\, d\mu$.

Note that $B_{\mathscr{P}}$ is the unique element in $\mathrm{H}_{\mathscr{P}}$ such that $B_{\mathscr{P}} = B$ on $\mathscr{P}$. We now have the following easy corollary of Lemma 4.11 and the fact that the right side of Eq. (4.8) is the distribution of $(B(s_1), B(s_2), ..., B(s_n))$.

COROLLARY 4.13. *The law of $B_{\mathscr{P}}$ and the law of $\phi(B_{\mathscr{P}})$ (with respect to μ) is $\mu_{\mathscr{P}}^1$ and $\nu_{\mathscr{P}}^1$, respectively.*

4.1. *Limits of the Finite Dimensional Approximations.* Let us recall the following Wong and Zakai type approximation theorem for solutions to Stratonovich stochastic differential equations.

THEOREM 4.14. *Let $f\colon \mathbb{R}^d \times \mathbb{R}^n \to \mathrm{End}(\mathbb{R}^d, \mathbb{R}^n)$ and $f_0\colon \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ be twice differentiable with bounded continuous derivatives. Let $\xi_0 \in \mathbb{R}^n$ and $\mathscr{P}$ be a partition of $[0,1]$. Further let B and $B_{\mathscr{P}}$ be as in Notation* 4.12 *and $\xi_{\mathscr{P}}(s)$ denote the solution to the ordinary differential equation*

$$\xi'_{\mathscr{P}}(s) = f(\xi_{\mathscr{P}}(s))\, B'_{\mathscr{P}}(s) + f_0(\xi_{\mathscr{P}}(s)), \qquad \xi_{\mathscr{P}}(0) = \xi_0 \tag{4.12}$$

and ξ denote the solution to the Stratonovich stochastic differential equation,

$$d\xi(s) = f(\xi(s))\, \delta B(s) + f_0(\xi(s))\, ds, \qquad \xi(0) = \xi_0. \tag{4.13}$$

(we are using $\delta B(s)$ for the Stratonovich differential of B and $dB(s)$ for the Itô differential.)

Then, for any $\gamma \in (0, \frac{1}{2})$, $p \in [1, \infty)$, *there is a constant* $C(p, \gamma) < \infty$ *depending only on* f, f_0 *and* M, *so that*

$$\lim_{|\mathscr{P}| \to 0} \mathbb{E}\,[\sup_{s \leqslant 1} |\xi_{\mathscr{P}}(s) - \xi(s)|^p] \leqslant C(p, \gamma)\, |\mathscr{P}|^{\gamma p}. \tag{4.14}$$

This theorem is a special case of Theorem 5.7.3 and Example 5.7.4 in Kunita [66]. Theorems of this type have a long history starting with Wong and Zakai [95, 96]. The reader may also find this and related results in the following *partial* list of references: [3, 5, 6, 9, 12, 27, 39, 55, 59, 61, 62, 68, 69, 74, 76, 79–83, 86, 88, 89, 90, 92]. The theorem as stated here may be found in [33].

Definition 4.15. (1) Let u be the solution to the Stratonovich stochastic differential equation

$$\delta u = \mathscr{H}_u u\, \delta B, \qquad u(0) = u_0.$$

Notice that u may be viewed as μ-a.e. defined function from $\mathrm{W}(\mathbb{R}^d) \to \mathrm{W}(O(M))$.

(2) Let $\tilde{\phi} := \pi \circ u\colon \mathrm{W}(\mathbb{R}^d) \to \mathrm{W}(M)$. This map is will be called the stochastic development map.

(3) Let $\widetilde{//}_{\cdot}(\sigma)$ denote stochastic parallel translation relative to the probability space $(\mathrm{W}(M), \nu)$. That is, $\widetilde{//}_{\cdot}(\sigma)$ is a stochastic extension of $//_{\cdot}(\sigma)$.

(4) Let $\tilde{b}(s) = \int_0^s \widetilde{//}_r^{-1}(\sigma)\, \delta\sigma(r)$, where $\delta\sigma(r)$ denotes the Stratonovich differential.

Remark 4.16. Using Theorem 4.14, one may show that $\tilde{\phi}$ is a "stochastic extension" of ϕ, i.e., $\tilde{\phi} = \lim_{|\mathscr{P}| \to 0} \phi(B_{\mathscr{P}})$. Moreover, the law of $\tilde{\phi}$ (i.e., $\mu\tilde{\phi}^{-1}$) is the Wiener measure ν on $\mathrm{W}(M)$. It is also well known that $\tilde{b}$ is a standard $\mathbb{R}^d$-valued Brownian motion on $(\mathrm{W}(M), \nu)$ and that the law of u under μ on $\mathrm{W}(\mathbb{R}^d)$ and the law of $\widetilde{//}$ under ν are equal.

The fact that $\tilde{\phi}$ has a "stochastic extension" seems to have first been observed by Eells and Elworthy [34] who used ideas of Gangolli [51]. The relationship of the stochastic development map to stochastic differential equations on the orthogonal frame bundle $O(M)$ of M is pointed out in Elworth [37–39]. The frame bundle point of view has also been developed by Malliavin, see, for example, [77, 76, 78]. For a more detailed history of the stochastic development map, see pp. 156–157 in Elworthy [39]. The results in the previous remark are all standard and may be found in the previous references and also in [43, 59, 66, 79]. For a fairly self contained short exposition of these results the reader may wish to consult Section 3 in [30]. Using Theorem 4.14 and Corollary 4.13 above, we get the following limit theorem for $\nu^1_{\mathscr{P}}$.

THEOREM 4.17. *Suppose that* $F\colon W(O(M))\to\mathbb{R}$ *is a continuous and bounded function and for* $\sigma\in \mathrm{H}(M)$ *we let* $f(\sigma):=F(//.(\sigma))$. *Then*

$$\lim_{|\mathscr{P}|\to 0}\int_{\mathrm{H}_{\mathscr{P}}(M)} f(\sigma)\,d\nu^1_{\mathscr{P}}(\sigma)=\int_{\mathrm{W}(M)}\tilde{f}(\sigma)\,d\nu(\sigma), \tag{4.15}$$

where $\tilde{f}(\sigma):=F(\widetilde{//}.(\sigma))$.

Proof. By Remark 4.16

$$\int_{\mathrm{W}(M)}\tilde{f}(\sigma)\,d\nu(\sigma)=\mathbb{E}[\tilde{f}(u)]. \tag{4.16}$$

By embedding $O(M)$ into $\mathbb{R}^D$ for some $D\in\mathbb{N}$ and extending the map $v\mapsto\mathscr{H}_u uv$ to a compact neighborhood of $O(M)\subset\mathbb{R}^D$, we may apply Theorem 4.14 to conclude that

$$\lim_{|\mathscr{P}|\to 0}\mathbb{E}[\sup_{0\leqslant s\leqslant 1}|u_{\mathscr{P}}(s)-u(s)|^p_{\mathbb{R}^D}]=0, \tag{4.17}$$

where $u_{\mathscr{P}}$ solves Eq. (2.5) with b replaced by $B_{\mathscr{P}}$. But the law of $u_{\mathscr{P}}$ is equal to the law of $//(\cdot)$ under $\nu^1_{\mathscr{P}}$, see Corollary 4.13. Therefore,

$$\int_{\mathrm{H}_{\mathscr{P}}(M)} f(\sigma)\,d\nu^1_{\mathscr{P}}(\sigma)=\mathbb{E}[f(u_{\mathscr{P}})]. \tag{4.18}$$

The limit in Eq. (4.15) now easily follows from (4.16)–(4.18) and the dominated convergence theorem. ∎

5. THE L^2 METRIC

In Section 4 we considered the metric $G^1_{\mathscr{P}}$ (see Definition 1.5) on $\mathrm{H}_{\mathscr{P}}(M)$ and the associated finite dimensional approximations of the Wiener measure ν on $\mathrm{W}(M)$. It was found that under the development map $\phi_{\mathscr{P}}$, the volume form with respect to. $G^1_{\mathscr{P}}$ pulls back to the volume form of a flat metric on $\mathrm{H}_{\mathscr{P}}(\mathbb{R}^d)$, see Theorem 4.8. As a consequence, we found that under the development map $\phi_{\mathscr{P}}$, the volume form $\nu^1_{\mathscr{P}}$ on $\mathrm{H}_{\mathscr{P}}(M)$ pulls back to the Gaussian density $\mu^1_{\mathscr{P}}$ on $\mathrm{H}_{\mathscr{P}}(\mathbb{R}^d)$.

DEFINITION 5.1. Let $M^{\mathscr{P}} := M^n$ and $\pi_{\mathscr{P}}\colon \mathrm{W}(M) \to M^{\mathscr{P}}$ denote the projection

$$\pi_{\mathscr{P}}(\sigma) := (\sigma(s_1), ..., \sigma(s_n)). \tag{5.1}$$

We will also use the same notation for the restriction of $\pi_{\mathscr{P}}$ to $\mathrm{H}(M)$ and $\mathrm{H}_{\mathscr{P}}(M)$.

In this section we will consider two further models for the geometry on path space, namely the degenerate L^2-"metric" $G^0_{\mathscr{P}}$ defined in Definition 1.5 on $\mathrm{H}_{\mathscr{P}}(M)$ and the product manifold $M^{\mathscr{P}}$ with its "natural" metric.

Remark 5.2. The form $G^0_{\mathscr{P}}$ is non-negative but fails to be definite precisely at $\sigma \in \mathrm{H}_{\mathscr{P}}(M)$ for which $\sigma(s_i)$ is conjugate to $\sigma(s_{i-1})$ along $\sigma([s_{i-1}, s_i])$ for some i. In this case there exists a nonzero $X \in T\mathrm{H}_{\mathscr{P}}(M)$ for which $G^0_{\mathscr{P}}(X, X) = 0$. Hence, $\mathrm{Vol}_{G^0_{\mathscr{P}}}$ will also be zero for such $\sigma \in \mathrm{H}_{\mathscr{P}}(M)$.

DEFINITION 5.3. Let $M^{\mathscr{P}}$ be as in Definition 5.1. For $\mathbf{x} = (x_1, x_2, ..., x_n) \in M^{\mathscr{P}}$, let

$$E_{\mathscr{P}}(\mathbf{x}) := \sum_{i=1}^{n} \frac{d^2(x_{i-1}, x_i)}{\Delta_i s}, \tag{5.2}$$

where d is the geodesic distance on M. Let $g_{\mathscr{P}}$ be the Riemannian metric on $M^{\mathscr{P}}$ given by

$$g_{\mathscr{P}} = (\Delta_1 s)\, g \times (\Delta_2 s)\, g \times \cdots \times (\Delta_n s)\, g, \tag{5.3}$$

i.e., if $\mathbf{v} = (v_1, v_2, ..., v_n) \in TM^n = (TM)^n$ then

$$g_{\mathscr{P}}(\mathbf{v}, \mathbf{v}) := \sum_{i=1}^{n} g(v_i, v_i)\, \Delta_i s.$$

Let the normalizing constant $Z^0_{\mathscr{P}}$ be given by Eq. (1.15) and let $\gamma_{\mathscr{P}}$ denote the measure on $M^{\mathscr{P}}$ defined by

$$\gamma_{\mathscr{P}}(d\mathbf{x}) := \frac{1}{Z^0_{\mathscr{P}}} \exp\left(-\frac{1}{2} E_{\mathscr{P}}(\mathbf{x})\right) \mathrm{Vol}_{g_{\mathscr{P}}}(d\mathbf{x}), \tag{5.4}$$

where $\mathrm{Vol}_{g_{\mathscr{P}}}$ denotes volume form on $M^{\mathscr{P}}$ defined with respect to $g_{\mathscr{P}}$.

Remark 5.4. An easy computation shows that

$$\mathrm{Vol}_{g_{\mathscr{P}}} = \left(\prod_{i=1}^{n} (\Delta_i s)^{d/2}\right) \times \mathrm{Vol}_g^n, \tag{5.5}$$

where Vol_g is the volume measure on (M, g) and Vol_g^n denotes the n-fold product measure of Vol_g with itself.

The next proposition shows the relationship between $\nu_{\mathscr{P}}^0$ (defined in Definition 1.7 above) and $\gamma_{\mathscr{P}}$. For the statement we need to define a subset of paths σ in $\mathrm{H}_{\mathscr{P}}(M)$ such that each geodesic piece $\sigma([s_{i-1}, s_i])$ is short. The formal definition is as follows.

DEFINITION 5.5. (1) For any $\varepsilon > 0$, let

$$\mathrm{H}_{\mathscr{P}}^{\varepsilon}(M) := \left\{\sigma \in \mathrm{H}_{\mathscr{P}}(M) : \int_{s_{i-1}}^{s_i} |\sigma'(s)|\, ds < \varepsilon \text{ for } i = 1, 2, ..., n\right\}.$$

(2) For any $\varepsilon > 0$, let

$$M_{\varepsilon}^{\mathscr{P}} = \{\mathbf{x} \in M^{\mathscr{P}} : d(x_{i-1}, x_i) < \varepsilon \text{ for } i = 1, 2, ..., n\},$$

where d is the geodesic distance on (M, g) and $x_0 := o$.

PROPOSITION 5.6. *For $\varepsilon > 0$ less than the injectivity radius of M, we have*

(1) *$G_{\mathscr{P}}^0$ is a Riemannian metric on $\mathrm{H}_{\mathscr{P}}^{\varepsilon}(M)$.*

(2) *The image of $\mathrm{H}_{\mathscr{P}}^{\varepsilon}(M)$ under $\pi_{\mathscr{P}}$ is $M_{\varepsilon}^{\mathscr{P}}$ and the map*

$$\pi_{\mathscr{P}} : (\mathrm{H}_{\mathscr{P}}^{\varepsilon}(M), G_{\mathscr{P}}^0) \to (M_{\varepsilon}^{\mathscr{P}}, g_{\mathscr{P}})$$

is an isometry, where $g_{\mathscr{P}}$ is the metric on $M^{\mathscr{P}}$ in Eq. (5.3).

(3) *$\pi_{\mathscr{P}}^* \gamma_{\mathscr{P}} = \nu_{\mathscr{P}}^0$ on $\mathrm{H}_{\mathscr{P}}^{\varepsilon}(M)$.*

Proof. Because ε is less than the injectivity radius of M, it follows that any $X \in T_\sigma \mathrm{H}_{\mathscr{P}}(M)$ is determined by its values on the partition points $\mathscr{P}$. Therefore, if $G_{\mathscr{P}}^0(X, X) = 0$ for $X \in T_\sigma \mathrm{H}_{\mathscr{P}}^{\varepsilon}(M)$, then $X := 0$. This proves the first item. The second item is a triviality. The last item is proved by noting that for $\sigma \in \mathrm{H}_{\mathscr{P}}^{\varepsilon}(M)$, $\sigma|_{[s_{i-1}, s_i]}$ is a minimal length geodesic joining $\sigma(s_{i-1})$ to $\sigma(s_i)$, and therefore

$$\int_{s_{i-1}}^{s_i} |\sigma'(s)|^2\, ds = \left(\frac{d(\sigma(s_{i-1}), \sigma(s_i))}{\Delta_i s}\right)^2 \Delta_i s = \frac{d^2(\sigma(s_{i-1}), \sigma(s_i))}{\Delta_i s}. \tag{5.6}$$

Summing this last equation on i shows,

$$E(\sigma) = \int_0^1 |\sigma'(s)|^2\, ds = \sum_{i=1}^{n} \frac{d^2(\sigma(s_{i-1}), \sigma(s_i))}{\Delta_i s} = E_{\mathscr{P}}(\pi_{\mathscr{P}}(\sigma)). \tag{5.7}$$

Hence by the definition of $\gamma_{\mathcal{P}}$, the fact that $\pi_{\mathcal{P}}$ is an isometry on $\mathrm{H}^{\varepsilon}_{\mathcal{P}}(M)$ (point(2) above), and (5.7) above, we find that on $\mathrm{H}^{\varepsilon}_{\mathcal{P}}(M)$,

$$\pi^*_{\mathcal{P}}\gamma_{\mathcal{P}} = \frac{1}{Z^0_{\mathcal{P}}}\, e^{-E/2}\,\mathrm{Vol}_{G^0_{\mathcal{P}}} = \nu^0_{\mathcal{P}}. \quad \blacksquare$$

Note that in general, for $\mathbf{x} \in M^{\mathcal{P}}$, $\pi^{-1}_{\mathcal{P}}(\mathbf{x})$ has more than one element, and may even fail to be a discrete subset. Therefore using the product manifold $M^{\mathcal{P}}$ as a model for $\mathrm{H}_{\mathcal{P}}(M)$ requires some care. The important aspect of the isometric subsets $M^{\mathcal{P}}_{\varepsilon}$ and $\mathrm{H}^{\varepsilon}_{\mathcal{P}}(M)$ is that in a precise sense they have nearly full measure with respect to $\gamma_{\mathcal{P}}$, $\nu^1_{\mathcal{P}}$ and $\nu^0_{\mathcal{P}}$. This will be proved in Section 5.1 below.

Before carrying out these estimates we will finish this section by comparing $\nu^0_{\mathcal{P}}$ to $\nu^1_{\mathcal{P}}$.

Notation 5.7. Let $\mathbb{R}^{d\mathcal{P}}$ denote the Euclidean space $(\mathbb{R}^d)^n$ equipped with the product inner product defined in the same way as $g_{\mathcal{P}}$ in Eq. (5.3) with $\mathbb{R}^d$ replacing TM.

To simplify notation throughout this section, let

$$\begin{aligned} &\sigma \in \mathrm{H}_{\mathcal{P}}(M), \qquad b := \phi^{-1}(\sigma), \qquad u := //(\sigma), \qquad \text{and} \\ &A(s) := \Omega_{u(s)}(b'(s), \cdot)\, b'(s). \end{aligned} \tag{5.8}$$

Since $b \in \mathrm{H}_{\mathcal{P}}(\mathbb{R}^d)$,

$$b'(s) = \Delta_i b/\Delta_i s \qquad \text{and} \qquad A(s) = \Omega_{u(s)}\left(\frac{\Delta_i b}{\Delta_i s}, \cdot\right)\frac{\Delta_i b}{\Delta_i s} \tag{5.9}$$

for $s \in (s_{i-1}, s_i]$. Let us also identify $X \in T_{\sigma}\mathrm{H}_{\mathcal{P}}(M)$ with $h := u^{-1}X$. Recall from Proposition 4.4 that $h\colon [0,1] \to \mathbb{R}^d$ is a piecewise smooth function such that $h(0) = 0$ and Eq. (4.2) holds, i.e.,

$$h'' = Ah \qquad \text{on} \quad I\backslash\mathcal{P} \qquad \text{and} \qquad h(0) = 0 \in \mathbb{R}^d. \tag{5.10}$$

In order to compare $\mathrm{Vol}_{G^0_{\mathcal{P}}}$ and $\mathrm{Vol}_{G^1_{\mathcal{P}}}$ it is useful to define two linear maps

$$J_0 : (T_{\sigma}\mathrm{H}_{\mathcal{P}}(M), G^0_{\mathcal{P}}) \to \mathbb{R}^{d\mathcal{P}}$$
$$J_1 : (T_{\sigma}\mathrm{H}_{\mathcal{P}}(M), G^1_{\mathcal{P}}) \to \mathbb{R}^{d\mathcal{P}}$$

by

$$J_0(X) = (h(s_1), h(s_2), ..., h(s_n))$$

and

$$J_1(X) = (h'(s_0+), h'(s_1+), ..., h'(s_{n-1}+)),$$

where $h := u^{-1}X$ as above.

It follows from the definition of $G^0_{\mathscr{P}}$ and the metric on $\mathbb{R}^{d\mathscr{P}}$ that if σ is such that J_0 is injective, then J_0 is an isometry. By point (2) of Proposition 5.6 this holds on $\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)$. However, by Remark 5.2 there is in general a nonempty subset of $\mathrm{H}_{\mathscr{P}}(M)$ where J_0 fails to be injective. Clearly, J_0 fails to be injective precisely where $G^0_{\mathscr{P}}$ fails to be positive definite. Similarly, it is immediate from the definitions and the fact that $G^1_{\mathscr{P}}$ is a nondegenerate Riemann metric, see Remark 4.5, that J_1 is an isometry at all $\sigma \in \mathrm{H}_{\mathscr{P}}(M)$.

To simplify notation, let V denote the vector space $(\mathbb{R}^d)^n$ and let $T = T_{\mathscr{P}}(\sigma)$ be defined by $T := J_0 \circ J_1^{-1}$. Thus $T\colon V \to V$ is the unique linear map such that

$$T(h'(x_0+), h'(s_1+), ..., h'(s_{n-1}+)) = (h(s_1), h(s_2), ..., h(s_n)) \quad (5.11)$$

for all $h = u^{-1}X$ with $X \in T_\sigma \mathrm{H}_{\mathscr{P}}(M)$. With this notation it follows that

$$\begin{aligned} \mathrm{Vol}_{G^0_{\mathscr{P}}} &= J_0^* \mathrm{Vol}_{\mathbb{R}^{d\mathscr{P}}} = (T \circ J_1)^* \, \mathrm{Vol}_{\mathbb{R}^{d\mathscr{P}}} \\ &= J_1^* T^* \, \mathrm{Vol}_{\mathbb{R}^{d\mathscr{P}}} = \det(T) \, J_1^* \mathrm{Vol}_{\mathbb{R}^{d\mathscr{P}}} \\ &= \det(T) \, \mathrm{Vol}_{G^1_{\mathscr{P}}}. \end{aligned} \quad (5.12)$$

Note that in this computation $\sigma \in \mathrm{H}_{\mathscr{P}}(M)$ is fixed and we treat $\mathrm{Vol}_{G^0_{\mathscr{P}}}$, $\mathrm{Vol}_{G^1_{\mathscr{P}}}$ as elements of the exterior algebra $\bigwedge^{dn}(T^*_\sigma \mathrm{H}_{\mathscr{P}}(M))$ and $\mathrm{Vol}_{\mathbb{R}^{d\mathscr{P}}}$ as an element of $\bigwedge^{dn}((\mathbb{R}^{d\mathscr{P}})^*)$.

Our next task is to compute $\det(T)$.

LEMMA 5.8. *Let $Z_{i-1}(s)$ denote the $d \times d$ matrix-valued solution to*

$$Z''_{i-1}(s) = A(s)\, Z_{i-1}(s)$$

$$\textit{with} \quad Z_{i-1}(s_{i-1}) = 0 \quad \textit{and} \quad Z'_{i-1}(s_{i-1}) = I. \quad (5.13)$$

Then

$$\det(T_{\mathscr{P}}(\sigma)) = \prod_{i=1}^{n} \det(Z_{i-1}(s_i)).$$

Proof. We start by noting that for $\sigma \in \mathrm{H}_{\mathscr{P}}(M)$ such that $G^0_{\mathscr{P}}$ is nondegenerate, then $\det(Z_{i-1}) \neq 0$ for $i = 1, 2, ..., n$. To see this assume that $\det(Z_{i-1}) = 0$ for some i. In view of the fact that Z solves the Jacobi Eq. (5.13), this is equivalent to the existence of a vector field X_{i-1} along $\sigma([s_{i-1}, s_i])$ which solves (4.1) for $s \in [s_{i-1}, s_i]$ and which satisfies

$$X_{i-1}(s_{i-1}) = 0, \qquad X_{i-1}(s_i) = 0.$$

Define X by

$$X(s)=\begin{cases} X_{i-1}(s), & s\in[s_{i-1},s_i] \\ 0 & s\in[0,1]\backslash[s_{i-1},s_i]. \end{cases}$$

Then $X\in T_\sigma \mathrm{H}_{\mathscr{P}}(M)$ and it is clear from the construction that $G^0_{\mathscr{P}}(X,X)=0$. Thus for such σ, $\mathrm{Vol}_{G^0_{\mathscr{P}}}|_\sigma=0$. Hence we may without loss of generality restrict our considerations to the case when $\det(Z_{i-1})\neq 0$ for all i.

Let $C_{i-1}(s)$ be the $d\times d$ matrix-valued solutions to

$$C''_{i-1}(s)=A(s)\,C_{i-1}(s)$$
$$\text{with}\quad C_{i-1}(s_{i-1})=I \qquad\text{and}\qquad C'_{i-1}(s_{i-1})=0.$$

For $i\in\{1,2,...,n\}$ and $h=u^{-1}X$ with $X\in T_\sigma \mathrm{H}_{\mathscr{P}}(M)$ let

$$k(s):=C_{i-1}(s)\,h(s_{i-1})+Z_{i-1}(s)\,h'(s_{i-1}+).$$

Then $k''=Ak$ on (s_{i-1},s_i), $k(s_{i-1})=h(s_{i-1})$ and $k'(s_{i-1})=h'(s_{i-1}+)$. Since h satisfies the same linear differential equation with initial conditions at s_{i-1}, it follows that $h=k$ on $[s_{i-1},s_i]$ and in particular that

$$h(s_i)=C_{i-1}(s_i)\,h(s_{i-1})+Z_{i-1}(s_i)\,h'(s_{i-1}+).$$

Solving this equation for $h'(s_{i-1}+)$ gives

$$h'(s_{i-1}+)=Z_{i-1}(s_i)^{-1}\,(h(s_i)-C_{i-1}(s_i)\,h(s_{i-1}))$$

from which it follows that $T^{-1}(\xi_1,\xi_2,...,\xi_n)=(\eta_1,\eta_2,...,\eta_n)$ where

$$\eta_i=\alpha_i\xi_i-\beta_i\xi_{i-1}\qquad\text{for}\quad i=1,2,...,n,$$

$\alpha_i:=Z_{i-1}(s_i)^{-1}$ and $\beta_i:=Z_{i-1}(s_i)^{-1}\,C_{i-1}(s_i)$. (In the previous displayed equation ξ_0 should be interpreted as 0.) Thus the linear transformation $T^{-1}\colon V\to V$ may be written in block lower triangular form as

$$T^{-1}=\begin{bmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ \beta_2 & \alpha_2 & 0 & \cdots & 0 \\ 0 & \beta_3 & \alpha_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \beta_n & \alpha_n \end{bmatrix}$$

and hence for $\sigma \in H_{\mathcal{P}}(M)$ such that $G^0_{\mathcal{P}}$ is nondegenerate,

$$\det(T^{-1}) = \prod_{i=1}^{n} \det(\alpha_i) = \prod_{i=1}^{n} \det(Z_{i-1}(s_i)^{-1}).$$

It follows by the above arguments that for all $\sigma \in H_{\mathcal{P}}(M)$

$$\det(T) = \sum_{i=1}^{n} \det(Z_{i-1}(s_i)). \quad \blacksquare$$

As a consequence, we have the key theorem relating $\nu^0_{\mathcal{P}}$ to $\nu^1_{\mathcal{P}}$.

THEOREM 5.9. *Let*

$$\rho_{\mathcal{P}}(\sigma) := \prod_{i=1}^{n} \det\left(\frac{Z_{i-1}(s_i)}{\Delta_{i-1} s}\right), \tag{5.14}$$

then $\nu^0_{\mathcal{P}} = \rho_{\mathcal{P}} \nu^1_{\mathcal{P}}$.

Proof. From Definition 1.6 for $\nu^0_{\mathcal{P}}$, Eq. (5.12), and Lemma 5.10 we find that

$$\begin{aligned}
\nu^0_{\mathcal{P}} &= \frac{1}{Z^0_{\mathcal{P}}} e^{-(1/2) E} \operatorname{Vol}_{G^0_{\mathcal{P}}} \\
&= \frac{1}{Z^0_{\mathcal{P}}} e^{-(1/2) E} \prod_{i=1}^{n} \det(Z_{i-1}(s_i)) \operatorname{Vol}_{G^1_{\mathcal{P}}} \\
&= \frac{1}{Z^0_{\mathcal{P}}} e^{-(1/2) E} \prod_{i=1}^{n} (\Delta_{i-1} s)^d \cdot \prod_{i=1}^{n} \det\left(\frac{1}{\Delta_{i-1} s} Z_{i-1}(s_i)\right) \operatorname{Vol}_{G^1_{\mathcal{P}}}.
\end{aligned}$$

Equation (5.14) now follows from Definition 1.7 (for $\nu^1_{\mathcal{P}}$) and the expressions for $Z^1_{\mathcal{P}}$ and $Z^0_{\mathcal{P}}$ in Eq. (1.15). $\blacksquare$

Using this result and Bishop's Comparison Theorem we have the following estimate on $\rho_{\mathcal{P}}(\sigma)$.

COROLLARY 5.10. *Let* $K > 0$ *be such that* $\operatorname{Ric} \geqslant -(d-1) KI$ (*for example take* K *to be a bound on* Ω) *then*

$$\rho_{\mathcal{P}}(\sigma) \leqslant \prod_{i=1}^{n} \left(\frac{\sinh(\sqrt{K}\,|\Delta_i b|)}{\sqrt{K}\,|\Delta_i b|}\right)^{d-1}. \tag{5.15}$$

Proof. The proof amounts to applying Theorem 3.8 on p. 120 of [14] to each of the $Z_{i-1}(s_i)$'s above. In order to use this theorem one must keep in mind that $\Delta_i b / \Delta_i s$ is not a unit vector and the estimate given in [14] corresponds to the determinant of $Z_{i-1}(s_i)$ restricted $\{\xi := \Delta_i b / \Delta_i s\}^{\perp}$. Noting that $Z_{i-1}(s_i)\,\xi = \Delta_i s \cdot \xi$ and accounting for the aforementioned discrepancies, Theorem 3.8 in [14] gives the estimate

$$\det(Z_{i-1}(s_i)) \leqslant \left(\frac{\sinh(\sqrt{K}\,|\Delta_i b|)}{\sqrt{K}\,|\Delta_i b|/\Delta_i s}\right)^{d-1} \Delta_i s$$

or equivalently that

$$\det\left(\frac{1}{\Delta_{i-1} s} Z_{i-1}(s_i)\right) \leqslant \left(\frac{\sinh(\sqrt{K}\,|\Delta_i b|)}{\sqrt{K}\,|\Delta_i b|}\right)^{d-1}.$$

This clearly implies the estimate in Eq. (5.15). ∎

5.1. *Estimates of the Measure of* $\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)$ *and* $M^{\mathscr{P}}_{\varepsilon}$. We will need the following lemma, which is again a consequence of Bishop's comparison theorem.

LEMMA 5.11. *Let* ω_{d-1} *denote the surface area of the unit sphere in* $\mathbb{R}^d$, *R be the diameter of M and let* $K \geqslant 0$ *be such that* $\mathrm{Ric} \geqslant -(d-1)\,KI$. *Then for all* $F\colon [0, R] \to [0, \infty]$,

$$\int_M F(d(o,\cdot))\, dvol \leqslant \omega_{d-1} \int_0^R r^{d-1} F(r) \left(\frac{\sinh(\sqrt{K}\,r)}{\sqrt{K}\,r}\right)^{d-1} dr.$$

Proof. See Chavel [14, Eqs. (2.48) on p. 72, (3.15) on p. 113, and Theorem 3.8 on p. 120]. ∎

We are now ready to estimate the measures of $M^{\mathscr{P}}_{\varepsilon}$ and $\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)$. We start by considering $\gamma_{\mathscr{P}}(M^{\mathscr{P}} \backslash M^{\mathscr{P}}_{\varepsilon})$.

PROPOSITION 5.12. *Fix* $\varepsilon > 0$ *and let* $M^{\mathscr{P}}_{\varepsilon}$ *be as in Definition* 5.5 *and let* $\gamma_{\mathscr{P}}$ *be the measure on* $M^{\mathscr{P}}$ *defined by* (5.4). *Then there is a constant* $C < \infty$ *such that*

$$\gamma_{\mathscr{P}}(M^{\mathscr{P}} \backslash M^{\mathscr{P}}_{\varepsilon}) \leqslant C \exp\left(-\frac{\varepsilon^2}{4\,|\mathscr{P}|}\right).$$

Proof. Let $f\colon [0,\infty)^n \to [0,\infty)$ be a measurable function. Let $d\mathbf{x} = \prod_{i=1}^n \mathrm{Vol}_g(dx_i)$ and note that

$$d\mathbf{x} = \prod_{i=1}^n (\Delta_i s)^{-d/2}\, d\,\mathrm{Vol}_{g_{\mathscr{P}}}(\mathbf{x}). \tag{5.16}$$

An application of Lemma 5.11 and Fubini's theorem proves

$$\int_{M^{\mathscr{P}}} f(d(o, x_1), d(x_1, x_2), ..., d(x_{n-1}, x_n))\, \gamma_{\mathscr{P}}(d\mathbf{x})$$
$$\leqslant \int_{[0,\infty)^n} f(r_1, r_2, ..., r_n) \exp\left(-\sum_{i=1}^n \frac{r_i^2}{2\Delta_i s}\right)$$
$$\times \prod_{i=1}^n \left(\frac{\sinh(\sqrt{K}\, r_i)}{\sqrt{K}\, r_i}\right)^{d-1} \frac{\omega_{d-1} r_i^{d-1}\, dr_i}{(2\pi\, \Delta_{i-1} s)^{d/2}}.$$

As usual let $\{B(s)\}_{s \in [0,1]}$ be a standard $\mathbb{R}^d$-valued Brownian motion in Notation 4.12 and $\Delta_i B = B(s_i) - B(s_{i-1})$. Noting that

$$\exp\left(-\sum_{i=1}^n \frac{r_i^2}{2\Delta_i s}\right) \prod_{i=1}^n \frac{\omega_{d-1} r_i^{d-1}\, dr_i}{(2\pi\, \Delta_{i-1} s)^{d/2}}$$

is the distribution of $(|\Delta_1 B|, |\Delta_2 B|, ..., |\Delta_n B|)$, the above inequality may be written as

$$\int_{M^{\mathscr{P}}} f(d(o, x_1), d(x_1, x_2), ..., d(x_{n-1}, x_n))\, \gamma_{\mathscr{P}}(d\mathbf{x})$$
$$\leqslant \mathbb{E}\left[f(|\Delta_1 B|, |\Delta_2 B|, ..., |\Delta_n B|) \prod_{i=1}^n \left(\frac{\sinh(\sqrt{K}\, |\Delta_i B|)}{\sqrt{K}\, |\Delta_i B|}\right)^{d-1}\right]. \tag{5.17}$$

For $i \in \{1, 2, ..., n\}$, let $\mathscr{A}_i := \{\mathbf{x} \in M^{\mathscr{P}} : d(x_{i-1}, x_i) \geqslant \varepsilon\}$ so that $M^{\mathscr{P}} \backslash M^{\mathscr{P}}_{\varepsilon} = \bigcup_{i=1}^n \mathscr{A}_i$ and

$$\gamma_{\mathscr{P}}(M^{\mathscr{P}} \backslash M^{\mathscr{P}}_{\varepsilon}) \leqslant \sum_{i=1}^n \gamma_{\mathscr{P}}(\mathscr{A}_i). \tag{5.18}$$

Since $1_{\mathscr{A}_i}(\mathbf{x}) = \chi_\varepsilon(d(x_{i-1}, x_i))$, where $\chi_\varepsilon(r) = 1_{r \geqslant \varepsilon}$, we find from Eq. (5.17) that

$$\gamma_{\mathscr{P}}(\mathscr{A}_i) \leqslant \mathbb{E}\left[\chi_\varepsilon(|\Delta_i B|) \prod_{j=1}^n \left(\frac{\sinh(\sqrt{K}\, |\Delta_j B|)}{\sqrt{K}\, |\Delta_j B|}\right)^{d-1}\right]$$
$$= \mathbb{E}\left[\chi_\varepsilon(|\Delta_i B|) \left(\frac{\sinh(\sqrt{K}\, |\Delta_i B|)}{\sqrt{K}\, |\Delta_i B|}\right)^{d-1}\right] \prod_{j \neq i} \psi(\sqrt{\Delta_j s}), \tag{5.19}$$

where ψ is defined in Eq. (8.19) of the Appendix. An application of Lemma 8.7 of the Appendix now completes the proof in view of (5.18) and (5.19). ∎

We also have the following analogue of Proposition 5.12.

PROPOSITION 5.13. *For any $\varepsilon > 0$ there is a constant $C < \infty$ such that*

$$\nu^1_{\mathscr{P}}(\mathrm{H}_{\mathscr{P}}(M)\backslash \mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)) \leqslant C \exp\left(-\frac{\varepsilon^2}{4\,|\mathscr{P}|}\right).$$

Proof. Let us recall that $\phi(\mathrm{H}_{\mathscr{P}}(\mathbb{R}^d)) = \mathrm{H}_{\mathscr{P}}(M)$ and let us note that $\phi(\mathrm{H}^{\varepsilon}_{\mathscr{P}}(\mathbb{R}^d)) = \mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)$. By Theorem 4.10 and Corollary 4.13 this implies that

$$\begin{aligned}\nu^1_{\mathscr{P}}(\mathrm{H}_{\mathscr{P}}(M)\backslash \mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)) &= \mu^1_{\mathscr{P}}(\mathrm{H}_{\mathscr{P}}(\mathbb{R}^d)\backslash \mathrm{H}^{\varepsilon}_{\mathscr{P}}(\mathbb{R}^d))\\ &= \mu(\{\max\{|\Delta_{i-1}B| : i = 1, 2, ..., n\} \geqslant \varepsilon\})\\ &\leqslant \sum_{i=1}^{n} \mu(|\Delta_{i-1}B| \geqslant \varepsilon)\\ &= \sum_{i=1}^{n} \mathbb{E}[\chi_{\varepsilon}(|\Delta_i B|)]\\ &\leqslant C e^{-\varepsilon^2/4\,|\mathscr{P}|},\end{aligned}$$

where as above $\chi_{\varepsilon}(r) = 1_{r \geqslant \varepsilon}$. The last inequality follows from Lemma 8.7 with $K = 0$. ∎

Finally we consider $\nu^0_{\mathscr{P}}(\mathrm{H}_{\mathscr{P}}(M)\backslash \mathrm{H}^{\varepsilon}_{\mathscr{P}}(M))$.

PROPOSITION 5.14. *Let $\varepsilon > 0$. Then there is a constant $C < \infty$ such that*

$$\nu^0_{\mathscr{P}}(\mathrm{H}_{\mathscr{P}}(M)\backslash \mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)) \leqslant C \exp\left(-\frac{\varepsilon^2}{4\,|\mathscr{P}|}\right). \tag{5.20}$$

Proof. Let B be the standard $\mathbb{R}^d$ valued Brownian motion. For $i = 1, 2, ..., n$, let $\mathscr{A}_i = \{|\Delta_i B| > \varepsilon\}$ and set $\mathscr{A} = \bigcup_{i=1}^{n} \mathscr{A}_i$. Then $\mathrm{H}_{\mathscr{P}}(M)\backslash \mathrm{H}^{\varepsilon}_{\mathscr{P}}(M) = \phi_{\mathscr{P}}(\mathscr{A})$ where $\phi_{\mathscr{P}} \colon \mathrm{H}_{\mathscr{P}}(\mathbb{R}^d) \to \mathrm{H}_{\mathscr{P}}(M)$ denotes the development map.

By Theorem 5.9, $\nu^0_{\mathscr{P}} = \rho_{\mathscr{P}} \nu^1_{\mathscr{P}}$, where $\rho_{\mathscr{P}}$ is given by (5.14). By Theorem 4.10 and Corollary 5.10 above,

$$\begin{aligned}\nu^0_{\mathscr{P}}(\mathrm{H}_{\mathscr{P}}(M)\backslash \mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)) &= \int_{\mathscr{A}} \rho_{\mathscr{P}}(\phi(B_{\mathscr{P}}))\, d\mu^1_{\mathscr{P}}\\ &\leqslant \int_{\mathscr{A}} \prod_{i=1}^{n} \left(\frac{\sinh(\sqrt{K}\,|\Delta_i B|)}{\sqrt{K}\,|\Delta_i B|}\right)^{d-1} d\mu,\end{aligned}$$

wherein we have used the fact that the distribution of $\{\Delta_i B_{\mathcal{P}}\}_i$ under $\mu^1_{\mathcal{P}}$ is the same as the distribution of $\{\Delta_i B\}_i$ under μ. Thus arguing as in the proof of Proposition 5.12 we have with $\chi_\varepsilon = 1_{r \geqslant \varepsilon}$,

$$\begin{aligned} \nu^0_{\mathcal{P}}(\phi_{\mathcal{P}}(\mathcal{A})) &\leqslant \sum_{i=1}^n \nu^0_{\mathcal{P}}(\phi_{\mathcal{P}}(\mathcal{A}_i)) \\ &\leqslant \sum_{i=1}^n \mathbb{E}\left[\chi_\varepsilon(|\Delta_i B|) \prod_{j=1}^n \left(\frac{\sinh(\sqrt{K}\,|\Delta_j B|)}{\sqrt{K}\,|\Delta_j B|}\right)^{d-1}\right] \\ &= \sum_{i=1}^n \mathbb{E}\left[\chi_\varepsilon(|\Delta_i B|) \left(\frac{\sinh(\sqrt{K}\,|\Delta_i B|)}{\sqrt{K}\,|\Delta_i B|}\right)^{d-1}\right] \prod_{j\neq i} \psi(\sqrt{\Delta_j s}), \end{aligned}$$

where ψ is defined in Eq. (8.19) of the Appendix. An application of Lemma 8.7 in the Appendix completes the proof. ∎

6. CONVERGENCE OF $\nu^0_{\mathcal{P}}$ TO WIENER MEASURE

This section is devoted to the proof of the following theorem.

THEOREM 6.1. *Let F: $W(O(M)) \to \mathbb{R}$ be a continuous and bounded function and set $f(\sigma) := F(//.(\sigma))$ for $\sigma \in \mathrm{H}(M)$. Then*

$$\lim_{|\mathcal{P}| \to 0} \int_{\mathrm{H}_{\mathcal{P}}(M)} f(\sigma)\, d\nu^0_{\mathcal{P}}(\sigma) = \int_{W(M)} \tilde{f}(\sigma)\, e^{-(1/6)\int_0^1 \mathrm{Scal}(\sigma(s))\,ds}\, d\nu(\sigma),$$

where $\tilde{f}(\sigma) := F(\widetilde{//}.(\sigma))$ and $\widetilde{//}_r(\sigma)$ is stochastic parallel translation, see Definition 4.15.

Because of Theorem 4.17, in order to prove this theorem it will suffice to compare $\nu^1_{\mathcal{P}}$ with $\nu^0_{\mathcal{P}}$. Of course the main issue is to compare $\mathrm{Vol}_{G^0_{\mathcal{P}}}$ with $\mathrm{Vol}_{G^1_{\mathcal{P}}}$. In view of Proposition 5.14 and the boundedness of f and Scal,

$$\left|\int_{\mathrm{H}_{\mathcal{P}}(M)\setminus \mathrm{H}^\varepsilon_{\mathcal{P}}(M)} f(\sigma)\, d\nu^0_{\mathcal{P}}(\sigma)\right| \leqslant C\, \|f\|_\infty\, e^{-\varepsilon^2/4|\mathcal{P}|}$$

which tends to zero faster than any power of $|\mathcal{P}|$. Therefore, it suffices to prove that for any $\varepsilon > 0$ smaller than the injectivity radius of M,

$$\lim_{|\mathcal{P}| \to 0} \int_{\mathrm{H}^\varepsilon_{\mathcal{P}}(M)} f(\sigma)\, d\nu^0_{\mathcal{P}}(\sigma) = \int_{W(M)} \tilde{f}(\sigma)\, e^{-(1/6)\int_0^1 \mathrm{Scal}(\sigma(s))\,ds}\, d\nu(\sigma). \quad (6.1)$$

6.1. *Estimating the Radon Nikodym Derivative.* In this section we will continue to use the notation set out in Eq. (5.8).

PROPOSITION 6.2. *Suppose that A is given by Eq. (5.9) and that Z_{i-1} is defined as in Lemma 5.8. Let Λ be an upper bound for both the norms of the curvature tensor R (or equivalently Ω) and its covariant derivative ∇R. Then*

$$Z_{i-1}(s_i) = \Delta_i s (I + \tfrac{1}{6} \Omega_{u(s_{i-1})}(\Delta_i b, \cdot)\, \Delta_i b + \mathscr{E}_{i-1}), \tag{6.2}$$

where

$$|\mathscr{E}_{i-1}| \leqslant \tfrac{1}{6}(2\Lambda\, |\Delta_i b|^3 + \tfrac{1}{2}\Lambda^2\, |\Delta_i b|^4) \cosh(\sqrt{\Lambda}\, |\Delta_i b|). \tag{6.3}$$

In particular, if $\varepsilon > 0$ is given and it is assumed that $|\Delta_i b| \leqslant \varepsilon$ for all i, then

$$|\mathscr{E}_{i-1}| \leqslant C\, |\Delta_i b|^3, \tag{6.4}$$

where $C = C(\varepsilon, R, \nabla R) = \frac{1}{6}(2\Lambda + \frac{1}{2}\Lambda^2 \varepsilon) \cosh(\sqrt{\Lambda}\, \varepsilon)$.

Proof. By Lemma 8.3 of the Appendix,

$$Z_{i-1}(s_i) = \Delta_i s I + \frac{\Delta_i s^3}{6} \Omega_{u(s_{i-1})} \left(\frac{\Delta_i b}{\Delta_i s}, \cdot \right) \frac{\Delta_i b}{\Delta_i s} + \Delta_i s \mathscr{E}_{i-1}, \tag{6.5}$$

with $\mathscr{E}_{i-1}$ satisfying the estimate

$$|\mathscr{E}_{i-1}| = \tfrac{1}{6}(2K_1 (\Delta_i s)^3 + \tfrac{1}{2} K^2 (\Delta_i s)^4) \cosh(\sqrt{K}\, \Delta_i s), \tag{6.6}$$

where $K := \sup_{s \in (s_{i-1}, s_i)} |A(s)|$ and $K_1 := \sup_{s \in (s_{i-1}, s_i)} |A'(s)|$.
By (5.9), for $s \in [s_{i-1}, s_i]$,

$$|A(s)| \leqslant \Lambda\, |\Delta_i b|^2\, (\Delta_i s)^{-2}$$

and hence $K(\Delta_i s)^2 \leqslant \Lambda\, |\Delta_i b|^2$.
Since $u'(s) = \mathscr{H}_{u(s)} u(s)\, b'(s)$, we see for $s_{i-1} < s \leqslant s_i$ that

$$\begin{aligned} A'(s) &= (D\Omega)_{u(s)}\, (b'(s), b'(s), \cdot)\, b'(s) \\ &= (\Delta_i s)^{-3}\, (D\Omega)_{u(s)}\, (\Delta_i b, \Delta_i b, \cdot)\, \Delta_i b, \end{aligned}$$

where $(D\Omega)_{u(s)}\, (b'(s), \cdot, \cdot) := (d/ds)\, \Omega_{u(s)}$. Therefore $|A'(s)| \leqslant \Lambda(\Delta_i s)^{-3}\, |\Delta_i b|^3$ which combined with Eq. (6.6) proves Eq. (6.3). ∎

PROPOSITION 6.3. *Let $\Psi(U)$ be given as in Lemma* 8.1 *of the Appendix and define*

$$U_{i-1} := \tfrac{1}{6} \Omega_{u(s_{i-1})}(\Delta_i b, \cdot)\, \Delta_i b + \mathscr{E}_{i-1}, \tag{6.7}$$

where $\mathscr{E}_{i-1}$ is defined in Proposition 6.2. *Then*

$$\rho_{\mathscr{P}}(\sigma) = \exp(W_{\mathscr{P}}(\sigma)) \exp\left(-\tfrac{1}{6} \mathscr{R}_{\mathscr{P}}(\sigma)\right), \tag{6.8}$$

where

$$\mathscr{R}_{\mathscr{P}}(\sigma) := \sum_{i=1}^{n} \langle \mathrm{Ric}_{u(s_{i-1})} \Delta_i b, \Delta_i b \rangle$$

and

$$W_{\mathscr{P}}(\sigma) := \sum_{i=1}^{n} (tr\, \mathscr{E}_{i-1} + \Psi(-U_{i-1})). \tag{6.9}$$

Moreover there exists $\varepsilon_0 > 0$ *and* $C_1 < \infty$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$,

$$|W_{\mathscr{P}}(\sigma)| \leqslant C_1 \sum_{i=1}^{n} |\Delta_i b|^3 \qquad \text{for all} \quad \sigma \in H^{\varepsilon}_{\mathscr{P}}(M). \tag{6.10}$$

Proof. Recall that by definition, the trace of the linear map $v \mapsto \Omega_{u(s_{i-1})}(\Delta_i b, v)\, \Delta_i b$ equals $-\langle \mathrm{Ric}_{u(s_{i-1})} \Delta_i b, \Delta_i b \rangle$ and hence

$$\mathrm{tr}\, U_{i-1} = -\tfrac{1}{6} \langle \mathrm{Ric}_{u(s_{i-1})} \Delta_i b, \Delta_i b \rangle + \mathrm{tr}\, \mathscr{E}_{i-1}.$$

From the definitions of $\mathscr{R}_{\mathscr{P}}$ and $W_{\mathscr{P}}$, we get using Lemma 5.8 and Lemma 8.1,

$$\begin{aligned} \rho_{\mathscr{P}}(\sigma) &= \prod_{i=1}^{n} \exp(-\tfrac{1}{6} \langle \mathrm{Ric}_{u(s_{i-1})} \Delta_i b, \Delta_i b \rangle + \mathrm{tr}\, \mathscr{E}_{i-1} + \Psi(-U_{i-1})) \\ &= \exp\left(-\tfrac{1}{6} \sum_{i=1}^{n} \langle \mathrm{Ric}_{u(s_{i-1})} \Delta_i b, \Delta_i b \rangle\right) \\ &\quad \times \exp\left(\sum_{i=1}^{n} \mathrm{tr}\, \mathscr{E}_{i-1} + \sum_{i=1}^{n} \Psi(-U_{i-1})\right) \end{aligned}$$

which proves Eq. (6.8).

Letting Λ be a bound on the curvature tensor Ω, it follows using Eq. (6.4) that

$$\begin{aligned} |U_{i-1}| &\leqslant \frac{1}{6} |\Omega_{u(s_{i-1})} (\Delta_i b, \cdot)\, \Delta_i b| + |\mathscr{E}_{i-1}| \\ &\leqslant \frac{\Lambda}{6} |\Delta_i b|^2 + C\, |\Delta_i b|^3 \\ &\leqslant \left(C\varepsilon + \frac{\Lambda}{6}\right) |\Delta_i b|^2 \leqslant \left(C\varepsilon + \frac{\Lambda}{6}\right) \varepsilon^2 \leqslant \frac{1}{2} \end{aligned}$$

for ε sufficiently small. So, using Lemma 8.1 of the Appendix, $W_{\mathscr{P}}$ satisfies the estimate

$$
\begin{aligned}
|W_{\mathscr{P}}(\sigma)| &\leqslant \sum_{i=1}^{n} (|\operatorname{tr} \mathscr{E}_{i-1}| + |\Psi(-U_{i-1})|) \\
&\leqslant d \sum_{i=1}^{n} (|\mathscr{E}_{i-1}| + |U_{i-1}|^2 (1-|U_{i-1}|)^{-1}) \\
&\leqslant d \sum_{i=1}^{n} \left[C\,|\Delta_i b|^3 + 2\left(\left(C\varepsilon + \frac{\Lambda}{6}\right) |\Delta_i b|^2\right)^2 \right] \\
&\leqslant C_1 \sum_{i=1}^{n} |\Delta_i b|^3. \quad \blacksquare
\end{aligned}
$$

Let $\mathscr{S}_{\mathscr{P}}\colon \mathrm{H}_{\mathscr{P}}(M) \to \mathbb{R}$ be given as

$$
\mathscr{S}_{\mathscr{P}}(\sigma) := \sum_{i=1}^{n} \operatorname{Scal}(\sigma(s_{i-1}))\, \Delta_i s, \tag{6.11}
$$

where Scal is the scalar curvature of $(M, \langle\cdot, \cdot\rangle)$.

Proposition 6.4. *Let $p \in \mathbb{R}$ and $\varepsilon > 0$. Then there exists $C = C(p, \varepsilon, M) < \infty$ such that*

$$
\begin{aligned}
1 - Ce^{-2/4\,|\mathscr{P}|} &\leqslant \int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} e^{p(\mathscr{R}_{\mathscr{P}}(\sigma) - \mathscr{S}_{\mathscr{P}}(\sigma))}\, d\nu^1_{\mathscr{P}}(\sigma) \\
&\leqslant e^{CK^2\,|\mathscr{P}|} - Ce^{-\varepsilon^2/4\,|\mathscr{P}|},
\end{aligned} \tag{6.12}
$$

and hence

$$
\left| \int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} e^{p(\mathscr{R}_{\mathscr{P}}(\sigma) - \mathscr{S}_{\mathscr{P}}(\sigma))}\, d\nu^1_{\mathscr{P}}(\sigma) - 1 \right| \leqslant e^{CK^2\,|\mathscr{P}|} - 1 + Ce^{-\varepsilon^2/4|\mathscr{P}|} \leqslant C\,|\mathscr{P}| \tag{6.13}
$$

for all partitions $\mathscr{P}$ with $|\mathscr{P}|$ sufficiently small.

Proof. Let $u_{\mathscr{P}}$ be the solution to Eq. (2.5) with b replaced by $B_{\mathscr{P}}$, $\mathbf{R}_i := \operatorname{Ric}_{u_{\mathscr{P}}(s_{i-1})}$, and

$$
Y := e^{p \sum_{i=1}^{n} (\langle \mathbf{R}_i \Delta_i B,\, \Delta_i B\rangle - \operatorname{tr}(\mathbf{R}_i)\, \Delta_i s)}.
$$

By Theorem 4.10, the distribution of $e^{p(\mathscr{R}_{\mathscr{P}} - \mathscr{S}_{\mathscr{P}})}$ under $\nu^1_{\mathscr{P}}$ is the same as the distribution of Y under μ. Therefore,

$$
\int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} e^{p(\mathscr{R}_{\mathscr{P}}(\sigma) - \mathscr{S}_{\mathscr{P}}(\sigma))}\, d\nu^1_{\mathscr{P}}(\sigma) = \int_{\mathscr{A}^c} Y\, d\mu,
$$

where $\mathscr{A} := \bigcup_{i=1}^{n} \mathscr{A}_i$ and $\mathscr{A}_i := \{|\Delta_i B| \geqslant \varepsilon\}$ as in the proof of Proposition 5.14. By Proposition 8.8 of the Appendix

$$1 \leqslant \int_{\mathbf{W}(\mathbb{R}^d)} Y\,d\mu = \int_{\mathscr{A}^c} Y\,d\mu + \int_{\mathscr{A}} Y\,d\mu \leqslant e^{dp^2 K^2 |\mathscr{P}|},$$

where K is a bound on Ric. Therefore,

$$1 - \int_{\mathscr{A}} Y\,d\mu \leqslant \int_{\mathbf{H}^{\varepsilon}_{\mathscr{P}}(M)} e^{p(\mathscr{R}_{\mathscr{P}}(\sigma) - \mathscr{S}_{\mathscr{P}}(\sigma))}\,d\nu^1_{\mathscr{P}}(\sigma) \leqslant e^{dp^2 K^2 |\mathscr{P}|} - \int_{\mathscr{A}} Y\,d\mu.$$

So to finish the proof it suffices to show that $\int_{\mathscr{A}} Y\,d\mu \leqslant C\exp(-\varepsilon^2/4|\mathscr{P}|)$.
Since

$$\left|\sum_{i=1}^{n} (\langle \mathbf{R}_i\,\Delta_i B, \Delta_i B\rangle - \operatorname{tr}(\mathbf{R}_i)\,\Delta_i s)\right| \leqslant K\left(\sum_{i=1}^{n} |\Delta_i B|^2 + d\right),$$

it follows that

$$\begin{aligned}
\int_{\mathscr{A}} Y\,d\mu &\leqslant \int_{\mathscr{A}} \exp\left(K\,|p|\left(\sum_{j=1}^{n} |\Delta_j B|^2 + d\right)\right) d\mu \\
&\leqslant \sum_i \int_{\mathscr{A}_i} \exp\left(K\,|P|\left(\sum_{j=1}^{n} |\Delta_j B|^2 + d\right)\right) d\mu \\
&\leqslant \sum_i \mathbb{E}\left[\exp\left(K\,|p|\left(\sum_{j=1,\,j\neq i}^{n} |\Delta_j b|^2 + d\right)\right)\right] \\
&\quad \times \mathbb{E}[\chi_\varepsilon(|\Delta_i B|)\,e^{K\,|p|\,(|\Delta_i B|^2 + d)}],
\end{aligned} \tag{6.14}$$

where $\chi_\varepsilon(r) = 1_{r \geqslant \varepsilon}$. The first factor of each term in the sum is bounded by Lemma 8.5. Using the same type of argument as in the proof of Lemma 8.6 one shows for $|\mathscr{P}|$ sufficiently small that there is a constant $C < \infty$ such that

$$\begin{aligned}
\mathbb{E}[\chi_\varepsilon(|\Delta_i B|)\,e^{K\,|P|\,(|\Delta_i B|^2 + d)}] &= \mathbb{E}[\chi_\varepsilon(\sqrt{\Delta_i s}\,|B(1)|)\,e^{K\,|p|\,(\Delta_i s\,|B(1)|^2 + d)}] \\
&\leqslant C(\Delta_i s)\,e^{-\varepsilon^2/4|\mathscr{P}|}.
\end{aligned}$$

Hence the sum in Eq. (6.14) may be estimated to give $\int_{\mathscr{A}} Y\,d\mu \leqslant C\exp(-\varepsilon^2/4|\mathscr{P}|)$. ∎

Corollary 6.5. *Let $\mathscr{S}_{\mathscr{P}} \colon \mathrm{H}_{\mathscr{P}}(M) \to \mathbb{R}$ be given as in Eq. (6.11). Then for all $\varepsilon > 0$ sufficiently small there is a constant $C = C(\varepsilon)$ such that*

$$\int_{\mathbf{H}^{\varepsilon}_{\mathscr{P}}(M)} |\rho_{\mathscr{P}} - e^{-(1/6)\,\mathscr{S}_{\mathscr{P}}}|\,d\nu^1_{\mathscr{P}} \leqslant C\sqrt{|\mathscr{P}|} \tag{6.15}$$

for all partitions $\mathscr{P}$ with $|\mathscr{P}|$ sufficiently small.

Proof. Let C be a generic constant depending on the geometry and the dimension of M. Let J denote the left side of Eq. (6.15) and let K be a constant so that $|\mathrm{Scal}| \leqslant K$. Then

$$J = \int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} |\rho_{\mathscr{P}} - e^{-(1/6)\,\mathscr{S}_{\mathscr{P}}}|\, dv^1_{\mathscr{P}}$$

$$= \int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} |e^{-(1/6)\,\mathscr{R}_{\mathscr{P}}} e^{W_{\mathscr{P}}} - e^{-(1/6)\,\mathscr{S}_{\mathscr{P}}}|\, dv^1_{\mathscr{P}}$$

$$\leqslant e^K \int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} |e^{-(1/6)(\mathscr{R}_P - \mathscr{S}_P)} e^{W_P} - 1|\, dv^1_{\mathscr{P}} \leqslant I + II,$$

where

$$I := e^K \int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} |e^{-(1/6)(\mathscr{R}_{\mathscr{P}} - \mathscr{S}_{\mathscr{P}})} - 1|\; e^{W_{\mathscr{P}}}\, dv$$

and

$$II := e^K \int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} |e^{W_{\mathscr{P}}} - 1|\, dv.$$

Since $|e^a - 1| \leqslant e^{|a|} - 1 \leqslant |a|\; e^{|a|}$ for all $a \in \mathbb{R}$,

$$\int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} |e^{W_{\mathscr{P}}} - 1|\, dv^1_{\mathscr{P}} \leqslant \int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} |W_{\mathscr{P}}|\; e^{|W_{\mathscr{P}}|}\, dv^1_{\mathscr{P}}. \tag{6.16}$$

By Proposition 6.3 there exist $\varepsilon_0 > 0$ such that $|W_{\mathscr{P}}(\sigma)| \leqslant C \sum_{i=1}^{n} |\Delta_i b|^3$ on $\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)$ for $\varepsilon < \varepsilon_0$. Therefore, with the aid of Theorem 4.10,

$$\int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} |W_{\mathscr{P}}|\; e^{|W_{\mathscr{P}}|}\, dv^1_{\mathscr{P}}$$

$$\leqslant C \sum_{i=1}^{n} \int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} |\Delta_i b|^3 \exp\left(C\varepsilon_0 \sum_{j=1}^{n} |\Delta_j b|^2 \right) dv^1_{\mathscr{P}}$$

$$\leqslant C \sum_{i=1}^{n} \int_{\mathrm{H}_{\mathscr{P}}(M)} |\Delta_i b|^3 \exp\left(C\varepsilon_0 \sum_{j=1}^{n} |\Delta_j b|^2 \right) dv^1_{\mathscr{P}}$$

$$= C \sum_{i=1}^{n} \int_{\mathrm{W}(\mathbb{R}^d)} |\Delta_i B|^3 \exp\left(C\varepsilon_0 \sum_{j=1}^{n} |\Delta_j B|^2 \right) d\mu$$

$$= C \sum_{i=1}^{n} \mathbb{E}[\,|\Delta_i B|^3 \exp(C\varepsilon_0\, |\Delta_i B|^2)]\; \mathbb{E} \exp\left(C\varepsilon_0 \sum_{j:\, j \neq i}^{n} |\Delta_i B|^2 \right).$$

By Lemma 8.5, $\limsup_{|\mathscr{P}| \to 0} \mathbb{E}[e^{C\varepsilon_0 \sum_{i:\, i \neq j}^{n} |\Delta_i B|^2}] = e^{dC\varepsilon_0} < \infty$ and hence

$$\begin{aligned} II &\leqslant 2e^K C e^{dC\varepsilon_0} \sum_{i=1}^{n} \mathbb{E}[\,|\Delta_i B|^3 \exp(C\varepsilon_0 \,|\Delta_i B|^2)] \\ &= 2e^K C e^{dC\varepsilon_0} \sum_{i=1}^{n} (\Delta_i s)^{3/2} \,\mathbb{E}[\,|B(1)|^3 \exp(C\varepsilon_0 \,\Delta_i s\, |B(1)|^2)] \\ &\leqslant 2e^K C e^{dC\varepsilon_0} \mathbb{E}[\,|B(1)|^3 \exp(C\varepsilon_0 \,|\mathscr{P}|\, |B(1)|^2)] \sqrt{|\mathscr{P}|} \\ &\leqslant C \sqrt{|\mathscr{P}|}, \end{aligned} \tag{6.17}$$

for all partitions $\mathscr{P}$ with $|\mathscr{P}|$ sufficiently small.

To estimate I, apply Holder's inequality to get

$$I^2 \leqslant e^{2K} \left(\int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} |e^{-(1/6)(\mathscr{R}_{\mathscr{P}} - \mathscr{S}_{\mathscr{P}})} - 1|^2 \, d\mu \right) \left(\int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} e^{2\,|W_{\mathscr{P}}|} \, d\mu \right).$$

The second term is bounded by the above arguments. Expanding the square gives

$$\begin{aligned} |e^{-(1/6)(\mathscr{R}_{\mathscr{P}} - \mathscr{S}_{\mathscr{P}})} - 1|^2 &= (e^{-(1/3)(\mathscr{R}_{\mathscr{P}} - \mathscr{S}_{\mathscr{P}})} - 1) - 2(e^{-(1/6)(\mathscr{R}_{\mathscr{P}} - \mathscr{S}_{\mathscr{P}})} - 1) \\ &\leqslant |e^{-(1/3)(\mathscr{R}_{\mathscr{P}} - \mathscr{S}_{\mathscr{P}})} - 1| + 2\,|e^{-(1/6)(\mathscr{R}_{\mathscr{P}} - \mathscr{S}_{\mathscr{P}})} - 1|. \end{aligned}$$

By Eq. (6.13) of Proposition 6.4 to each term above, there is a constant $C = C(\varepsilon, M) < \infty$, such that $I^2 \leqslant C\,|\mathscr{P}|$ for all partitions $\mathscr{P}$ with $|\mathscr{P}|$ sufficiently small. From this we see that

$$I \leqslant C\,|\mathscr{P}|^{1/2}$$

which together with (6.17) proves the corollary. ∎

6.2. *Proof of Theorem* 6.1. To simplify notation, let ρ: $W(M) \to (0, \infty)$ be given by

$$\rho(\sigma) := \exp\left(-\tfrac{1}{6} \int_0^1 \mathrm{Scal}(\sigma(s))\, ds \right), \tag{6.18}$$

where Scal is the scalar curvature of (M, g). Recall, by the remark following Theorem 6.1, to prove Theorem 6.1 it suffices to prove Eq. (6.1) for some $\varepsilon > 0$. Let F: $\mathrm{W}(O(M)) \to \mathbb{R}$, f: $\mathrm{H}(M) \to \mathbb{R}$, and $\tilde{f}$: $\mathrm{W}(M) \to \mathbb{R}$ be as in the statement of Theorem 6.1. Then by Corollary 6.5 and Proposition 5.13, for $\varepsilon > 0$ sufficiently small and for partitions $\mathcal{P}$ with $|\mathcal{P}|$ sufficiently small,

$$\begin{aligned}\int_{\mathbf{H}^{\varepsilon}_{\mathcal{P}}(M)} f\,dv^0_{\mathcal{P}} &= \int_{\mathbf{H}^{\varepsilon}_{\mathcal{P}}(M)} f\rho_{\mathcal{P}}\,dv^1_{\mathcal{P}}\\ &= \int_{\mathbf{H}^{\varepsilon}_{\mathcal{P}}(M)} fe^{-(1/6)\,\mathcal{S}_{\mathcal{P}}}\,dv^1_{\mathcal{P}} + \tilde{\varepsilon}_{\mathcal{P}}\\ &= \int_{\mathbf{H}_{\mathcal{P}}(M)} fe^{-(1/6)\,\mathcal{S}_{\mathcal{P}}}\,dv^1_{\mathcal{P}} + \varepsilon_{\mathcal{P}},\end{aligned}$$

and $|\varepsilon_{\mathcal{P}}| \leqslant C\,\|f\|_\infty\,|\mathcal{P}|^{1/2}$ where C is a constant independent of $\mathcal{P}$. Because of Theorem 4.17, to finish the proof, it suffices to show that

$$\lim_{|\mathcal{P}|\to 0}\int_{\mathbf{H}_{\mathcal{P}}(M)} f(e^{-(1/6)\,\mathcal{S}_{\mathcal{P}}} - \rho)\,dv^1_{\mathcal{P}} = 0.$$

As above, let B be the $\mathbb{R}^d$-Brownian motion in Notation 1.2, $B_{\mathcal{P}}$ be its piecewise linear approximation, $\sigma_{\mathcal{P}} = \phi(B_{\mathcal{P}})$ and $u_{\mathcal{P}} := //(\sigma_{\mathcal{P}})$. If Λ is a constant such that $|\mathrm{Scal}| \leqslant \Lambda$ and $|\nabla\mathrm{Scal}| \leqslant \Lambda$, then

$$\begin{aligned}&\left|\int_{\mathbf{H}_{\mathcal{P}}(M)} f(e^{-(1/6)\,\mathcal{S}_{\mathcal{P}}} - \rho)\,dv^1_{\mathcal{P}}\right|\\ &\quad\leqslant \mathbb{E}[\,|f(u_{\mathcal{P}})(e^{-(1/6)\int_0^1 \mathrm{Scal}(\sigma_{\mathcal{P}}(\underline{s}))\,ds} - e^{-(1/6)\int_0^1 \mathrm{Scal}(\sigma_{\mathcal{P}}(s))\,ds})|\,]\\ &\quad\leqslant \|f\|_\infty\, e^{\Lambda/6}\,\mathbb{E}\left[\int_0^1 |\mathrm{Scal}(\sigma_{\mathcal{P}}(s)) - \mathrm{Scal}(\sigma_{\mathcal{P}}(\underline{s}))|\,ds\right] \qquad (6.19)\end{aligned}$$

wherein the last step we used the inequality $|e^a - e^b| \leqslant e^{\max(a,\,b)}\,|a-b|$. For $s \in [s_{i-1}, s_i)$, we have

$$|\mathrm{Scal}(\sigma_{\mathcal{P}}(s)) - \mathrm{Scal}(\sigma_{\mathcal{P}}(s_{i-1}))| \leqslant \Lambda\,|\Delta_i B|$$

and hence

$$\begin{aligned}\left|\int_{\mathbf{H}_{\mathcal{P}}(M)} f(e^{-(1/6)\,\mathcal{S}_{\mathcal{P}}} - \rho)\,dv^1_{\mathcal{P}}\right| &\leqslant \|f\|_\infty\, e^{\Lambda/6}\Lambda \sum_{i=1}^n \mathbb{E}\,|\Delta_i B|\,\Delta_i s\\ &= \|f\|_\infty\, e^{\Lambda/6}\Lambda\,\mathbb{E}\,|B(1)| \sum_{i=1}^n (\Delta_i s)^{3/2}\\ &\leqslant C\,\|f\|_\infty\,|\mathcal{P}|^{1/2}.\end{aligned}$$

This finishes the proof of Theorem 6.1. ∎

DEFINITION 6.6. Let $\mathscr{P}$ be a partition of $[0, 1]$. To every point $\mathbf{x} \in M^{\mathscr{P}}$ we will associate a path $\sigma_{\mathbf{x}} \in \mathrm{H}_{\mathscr{P}}(M)$ as follows. If for each i, there is a unique minimal geodesic joining x_{i-1} to x_i, let $\sigma_{\mathbf{x}}$ be the unique path in $\mathrm{H}_{\mathscr{P}}(M)$ such that $\sigma_{\mathbf{x}}(s_i) = x_i$ and $\int_{s_{i-1}}^{s_i} |\sigma'(s)|\, ds = d(x_{i-1}, x_i)$ for $i = 1, 2, ..., n$. Otherwise set $\sigma_{\mathbf{x}}(s) := o$ for all s.

COROLLARY 6.7. *Let* $\alpha \in [0, 1]$, F: $W(O(M)) \to \mathbb{R}$ *be a continuous and bounded function and set* $f(\sigma) := F(//.(\sigma))$ *for* $\sigma \in \mathrm{H}(M)$. *Then* $\alpha \in [0, 1]$,

$$\lim_{|\mathscr{P}| \to 0} \int_{M^{\mathscr{P}}} f(\sigma_{\mathbf{x}})\, e^{(1/6) \sum_{i=1}^{n} (\alpha \operatorname{Scal}(x_{i-1}) + (1-\alpha) \operatorname{Scal}(x_i))\, \Delta_i s}\, d\gamma_{\mathscr{P}}(\mathbf{x})$$

$$= \int_{W(M)} \tilde{f}(\sigma)\, d\nu(\sigma),$$

where $\tilde{f}(\sigma) := F(\widetilde{//}.(\sigma))$ *and* $\widetilde{//}_r(\sigma)$ *is stochastic parallel translation, see Definition* 4.15.

Proof. For $\sigma \in \mathrm{H}(M)$, let

$$\chi_{\mathscr{P}, \alpha}(\sigma) = e^{(1/6) \sum^{n}{}_{i=1} (\alpha \operatorname{Scal}(\sigma(s_{i-1})) + (1-\alpha) \operatorname{Scal}(\sigma(s_i)))\, \Delta_i s}.$$

Let Λ be a constant such that $|\operatorname{Scal}| \leqslant \Lambda$ and $|\nabla \operatorname{Scal}| \leqslant \Lambda$. Then $\chi_{\mathscr{P}, \alpha}(\sigma) \leqslant e^{\Lambda/6}$ so by Proposition 5.12

$$\int_{M^{\mathscr{P}} \setminus M^{\mathscr{P}}_{\varepsilon}} f(\sigma_{\mathbf{x}})\, \chi_{\mathscr{P}, \alpha}(\sigma_{\mathbf{x}})\, d\gamma_{\mathscr{P}}(\mathbf{x}) = \varepsilon_{\mathscr{P}},$$

where $\varepsilon_{\mathscr{P}} \leqslant C\, \|f\|_{\infty}\, |\mathscr{P}|^{1/2}$. Therefore it is sufficient to consider $\int_{M^{\mathscr{P}}_{\varepsilon}} f(\sigma_{\mathbf{x}})\, \chi_{\mathscr{P}, \alpha}(\sigma_{\mathbf{x}})\, d\gamma_{\mathscr{P}}(\mathbf{x})$. By Proposition 5.6 we have

$$\int_{M^{\mathscr{P}}_{\varepsilon}} f(\sigma_{\mathbf{x}})\, \chi_{\mathscr{P}, \alpha}(\sigma_{\mathbf{x}})\, d\gamma_{\mathscr{P}}(\mathbf{x}) = \int_{\mathrm{H}^{\varepsilon}_{\mathscr{P}}(M)} f(\sigma)\, \chi_{\mathscr{P}, \alpha}(\sigma)\, d\nu^{0}_{\mathscr{P}}(\sigma).$$

Let $\rho(\sigma)$ be given by (6.18). Arguing as in the proof of Theorem 6.1, the corollary will follow if

$$\lim_{|\mathscr{P}| \to 0} \int_{\mathrm{H}_{\mathscr{P}}(M)} f(\sigma)(\chi_{\mathscr{P}, \alpha}(\sigma)\, \rho(\sigma) - 1)\, d\nu^{1}_{\mathscr{P}}(\sigma) = 0.$$

Let $\sigma_{\mathcal{P}}$, $B_{\mathcal{P}}$ be as in the proof of Theorem 6.1. We estimate, as in the proof of Theorem 6.1,

$$\left| \lim_{|\mathcal{P}| \to 0} \int_{\mathrm{H}_{\mathcal{P}}(M)} f(\sigma)(\chi_{\mathcal{P}, \alpha}(\sigma)\, \rho(\sigma) - 1)\, d\nu^1_{\mathcal{P}}(\sigma) \right|$$

$$\leqslant \|f\|_\infty\, e^{\Lambda/6}\, \mathbb{E}\left[\left| \sum_{i=1}^{n} (\alpha \operatorname{Scal}(\sigma_{\mathcal{P}}(s_{i-1})) + (1-\alpha) \operatorname{Scal}(\sigma_{\mathcal{P}}(s_i)))\, \Delta_i s \right.\right.$$

$$\left.\left. - \int_0^1 \operatorname{Scal}(\sigma_{\mathcal{P}}(s))\, ds \right| \right]$$

$$\leqslant C\, \|f\|_\infty\, |\mathcal{P}|^{1/2}$$

which completes the proof of Corollary 6.7. ∎

7. PARTIAL INTEGRATION FORMULAS

As an application of Theorem 4.17, we will derive the known integration by parts formula for the measure ν. This will be accomplished by taking limits of the finite dimensional integration by parts formulas for the measure $\nu^1_{\mathcal{P}}$. The main result appears at the end of this section in Theorem 7.16. A similar method for proving integration by parts formula for laws of solutions to stochastic differential equations has been used by Bell [7, 8].

7.1. *Integration by Parts for the Approximate Measures.* The two ingredients for computing the integration by parts formula for the form $\nu^1_{\mathcal{P}}$ is the differential of E and the Lie derivative of $\mathrm{Vol}_{G^1_{\mathcal{P}}}$. The following lemma may be found in any book on Riemannian geometry. We will supply the short proof for the readers convenience.

LEMMA 7.1. *Let $Y \in T_\sigma \mathrm{H}(M)$. Then*

$$YE = dE(Y) = 2 \int_0^1 \left\langle \sigma'(s), \frac{\nabla Y(s)}{ds} \right\rangle ds. \tag{7.1}$$

Proof. Choose a one parameter family of paths at $\sigma_t \in \mathrm{H}(M)$ such that $\sigma_0 = \sigma$ and $(d/dt)\,|_{t=0}\, \sigma_t = Y$. Then

$$YE = \frac{d}{dt}\bigg|_{t=0} \int_0^1 |\sigma_t'(s)|^2\, ds = 2 \int_0^1 \left\langle \frac{\nabla}{dt} \sigma_t'(s)|_{t=0}, \sigma'(s) \right\rangle ds.$$

Since ∇ has zero torsion,

$$\frac{\nabla}{dt}\,\sigma_t'(s)|_{t=0}=\frac{\nabla}{ds}\frac{d}{dt}\bigg|_{t=0}\sigma_t(s)=\frac{\nabla}{ds}\,Y(s).$$

The last two equations clearly imply Eq. (7.1). ∎

To compute the Lie derivative of $\mathrm{Vol}_{G^1_{\mathscr{P}}}$ is will be useful to have an orthonormal frame on $\mathrm{H}_{\mathscr{P}}(M)$ relative to $G^1_{\mathscr{P}}$. We will construct such a frame in the next lemma.

Notation 7.2. Given $\sigma\in\mathrm{H}_{\mathscr{P}}(M)$, let $\mathrm{H}_{\mathscr{P},\sigma}$ be the subspace of H given by

$$\mathrm{H}_{\mathscr{P},\sigma}:=\{v\in\mathrm{H}\colon v''(s)=\Omega_{u(s)}(b'(s),v(s))\,b'(s),\ \forall s\notin\mathscr{P}\},\tag{7.2}$$

where $u=//(\sigma)$ and $b=\phi^{-1}(\sigma)$.

Because of Eq. (4.2) of Proposition 4.4, $v\in\mathrm{H}_{\mathscr{P},\sigma}$ if and only if $X^v(\sigma):=//(\sigma)\,v\in T_\sigma\mathrm{H}_{\mathscr{P}}(M)$.

LEMMA 7.3 ($G_{\mathscr{P}}$-Orthonormal Frame). *Let $\mathscr{P}$ be a partition of $[0,1]$ and $G^1_{\mathscr{P}}$ be as in Eq. (1.12) above. Also let $\{e_a\}_{a=1}^d$ be an orthonormal frame for $T_oM\cong\mathbb{R}^d$. For $\sigma\in\mathrm{H}_{\mathscr{P}}(M)$, $i=1,2,...,n$ and $a=1,...,d$ let $h_{i,a}(s,\sigma):=v(s)$ be determined (uniquely) by*:

(1) $v\in\mathrm{H}_{\mathscr{P},\sigma}$.

(2) $v'(s_j+)=0$ *if* $j\neq i-1$.

(3) $v'(s_{i-1}+)=(1/\sqrt{\Delta_i s})\,e_a$.

Then $\{X^{h_{a,i}}, i=1,...,n, a=1,...,d\}$ *is a globally defined orthonormal frame for* $(\mathrm{H}_{\mathscr{P}}(M), G^1_{\mathscr{P}})$.

Proof. This lemma is easily verified using the definition of $G^1_{\mathscr{P}}$ in Eq. (1.12), the identity

$$\frac{\nabla X^v(\sigma)(s+)}{ds}=//_s(\sigma)\,v'(s+),$$

and the fact that $//_s(\sigma)$ is an isometry. ∎

DEFINITION 7.4. Let PC^1 denote the set of $k\in\mathrm{H}$ which are piecewise C^1. Given $k\in PC^1$, define $k_{\mathscr{P}}\colon\mathrm{H}_{\mathscr{P}}(M)\to\mathrm{H}$ by requiring $k_{\mathscr{P}}(\sigma)\in\mathrm{H}_{\mathscr{P},\sigma}$ for all $\sigma \in \mathrm{H}_{\mathscr{P}}(M)$ and $k'_{\mathscr{P}}(\sigma, s+) = k'(s+)$ for all $s \in \mathscr{P} \backslash \{1\}$. Note that with this definition of $k_{\mathscr{P}}$, $X^{k_{\mathscr{P}}}$ is the unique tangent vector field on $\mathrm{H}_{\mathscr{P}}(M)$ such that

$$\frac{\nabla X^{k_{\mathscr{P}}}(s+)}{ds} = \frac{\nabla X^{k}(s+)}{ds} \qquad \text{for all} \quad s \in \mathscr{P} \backslash \{1\}.$$

LEMMA 7.5. *If $k \in PC^1$, then $L_{X^{k_{\mathscr{P}}}} \mathrm{Vol}_{G^1_{\mathscr{P}}} = 0$.*

Proof. Recall that on a general Riemannian manifold

$$L_X \mathrm{Vol} = -\sum_i \langle L_X e_i, e_i \rangle \mathrm{Vol} = \sum_i \langle [e_i, X], e_i \rangle \mathrm{Vol},$$

where $\{e_i\}$ is an orthonormal frame. Therefore we must show that

$$\sum_{i=1}^{n} \sum_{a=1}^{d} G^1_{\mathscr{P}}([X^{h_{a,i}}, X^{k_{\mathscr{P}}}], X^{h_{a,i}}) = 0. \tag{7.3}$$

Suppressing $\sigma \in \mathrm{H}_{\mathscr{P}}(M)$ from the notation and using Theorem 3.5 to expand the Lie bracket, we find

$$\begin{aligned} &G^1_{\mathscr{P}}([X^{h_{a,i}}, X^{k_{\mathscr{P}}}], X^{h_{a,i}}) \\ &\quad = \sum_{j=1}^{n} \langle (X^{h_{a,i}} k_{\mathscr{P}} - X^{k_{\mathscr{P}}} h_{a,i})', h'_{a,i} \rangle |_{(s_{j-1}+)} \Delta_j s \\ &\qquad + \sum_{j=1}^{n} \langle (q(X^{k_{\mathscr{P}}})\, h_{a,i} - q(X^{h_{a,i}})\, k_{\mathscr{P}})', h'_{a,i} \rangle |_{(s_{j-1}+)} \Delta_j s. \end{aligned}$$

For $s \in \mathscr{P} \backslash \{1\}$, $(X^{h_{a,i}} k_{\mathscr{P}})'(s+) = X^{h_{a,i}} k'_{\mathscr{P}}(s+) = 0$, since $k'(s+)$ is independent of σ. For the same reason, $(X^{k_{\mathscr{P}}} h_{a,i})'(s+) = 0$ for $s \in \mathscr{P} \backslash \{1\}$. Moreover for $s \in \mathscr{P} \backslash \{1\}$,

$$\langle (q(X^{k_{\mathscr{P}}})\, h_{a,i})', h'_{a,i} \rangle |_{s+} = \langle q(X^{k_{\mathscr{P}}})\, h'_{a,i} + R_u(\sigma', X^{k_{\mathscr{P}}})\, h_{a,i}, h'_{a,i} \rangle |_{s+} = 0,$$

because $q(X^{k_{\mathscr{P}}})$ is skew symmetric and because either $h_{a,i}(s+)$ or $h'_{a,i}(s+)$ are equal to zero for all $s \in \mathscr{P} \backslash \{1\}$. Similarly,

$$\langle (q(X^{h_{a,i}})\, k_{\mathscr{P}})', h'_{a,i} \rangle |_{s+} = \langle q(X^{h_{a,i}})\, k'_{\mathscr{P}} + R_u(\sigma', X^{h_{a,i}})\, k_{\mathscr{P}}, h'_{a,i} \rangle |_{s+} = 0$$

because for all $s \in \mathscr{P} \backslash \{1\}$, either $q_{s+}(X^{h_{a,i}}) = 0$ or $h'_{a,i}(s+) = 0$ and either $h_{a,i}(s+) = 0$ or $h'_{a,i}(s+) = 0$. Thus every term in the sum in Eq. (7.3) is zero. ∎

THEOREM 7.6. *Suppose that* $k \in PC^1$, $\mathscr{P}$ *is a partition of* $[0,1]$, $b \in \mathrm{H}_{\mathscr{P}}$ *and* $\sigma = \phi(b) \in \mathrm{H}_{\mathscr{P}}(M)$. *Then*

$$(L_{X^{k_{\mathscr{P}}}} v^1_{\mathscr{P}})_\sigma = -\left(\sum_{i=1}^{n} \langle k'(s_{i-1}+), \Delta_i b \rangle \right) (v^1_{\mathscr{P}})_\sigma, \tag{7.4}$$

i.e., the divergence of $X^{k_{\mathscr{P}}}$ *relative to the volume form* $v^1_{\mathscr{P}}$ *is*

$$(\operatorname{div}_{v^1_{\mathscr{P}}} X^{k_{\mathscr{P}}})(\sigma) = -\sum_{i=1}^{n} \langle k'(s_{i-1}+), \Delta_i b \rangle. \tag{7.5}$$

Proof. By Lemma 7.5,

$$(L_{X^{k_{\mathscr{P}}}} v^1_{\mathscr{P}})_\sigma = [-\tfrac{1}{2}(X^{k_{\mathscr{P}}} E)(\sigma)] \cdot (v^1_{\mathscr{P}})_\sigma$$

and by Lemma 7.1,

$$\begin{aligned}(X^{k_{\mathscr{P}}} E)(\sigma) &= 2 \int_0^1 \left\langle \sigma'(s), \frac{\nabla X^{k_{\mathscr{P}}}(\sigma)(s)}{ds} \right\rangle ds \\ &= 2 \int_0^1 \langle //_s(\sigma)\, b'(s), //_s(\sigma)\, k'_{\mathscr{P}}(\sigma, s) \rangle \, ds \\ &= 2 \sum_{i=1}^{n} \int_{J_i} \langle b'(s), k'_{\mathscr{P}}(\sigma, s) \rangle \, ds.\end{aligned}$$

Now for $s \in J_i := (s_{i-1}, s_i]$,

$$\begin{aligned}\langle b'(s), k'_{\mathscr{P}}(\sigma, s) \rangle &= \langle b'(s_{i-1}+), k'_{\mathscr{P}}(\sigma, s_{i-1}+) \rangle + \int_{s_{i-1}}^{s} b'(r) \cdot k''_{\mathscr{P}}(\sigma, r) \, dr \\ &= \langle b'(s_{i-1}+), k'_{\mathscr{P}}(s_{i-1}+) \rangle \\ &\quad + \int_{s_{i-1}}^{s} \langle b'(s), \Omega_{u(r)}(b'(r), k_{\mathscr{P}}(\sigma, r))\, b'(r) \rangle \, dr \\ &= \langle b'(s_{i-1}+), k'_{\mathscr{P}}(s_{i-1}+) \rangle,\end{aligned}$$

wherein the last equality we used the skew adjointness of $\Omega_{u(r)}(b'(r), k_{\mathscr{P}}(\sigma, r))$ and the fact that $b'(s) = b'(r) = \Delta_i b / \Delta_i s$ for all $s, r \in J_i$. Combining the previous three displayed equations proves Eq. (7.4). ∎

COROLLARY 7.7. *Let* $k \in PC^1$, $\mathscr{P}$ *be a partition of* $[0,1]$ *as above, and let* $f\colon \mathrm{H}_{\mathscr{P}}(M) \to \mathbb{R}$ *be a* C^1 *function for which* f *and its differential is bounded, then*

$$\int_{\mathrm{H}_{\mathscr{P}}(M)} (X^{k_{\mathscr{P}}} f)\, v^1_{\mathscr{P}} = \int_{\mathrm{H}_{\mathscr{P}}(M)} f \left(\sum_{i=1}^{n} \langle k'(s_{i-1}+), \Delta_i b \rangle \right) v^1_{\mathscr{P}}, \tag{7.6}$$

wherein this formula $\Delta_i b$ is to be understood as the function on $\mathrm{H}(M)$ *defined by*

$$\Delta_i b(\sigma) := \phi^{-1}(\sigma)(s_i) - \phi^{-1}(\sigma)(s_{i-1}). \tag{7.7}$$

Proof. First assume that f has compact support. Then by Stoke's theorem

$$0 = \int_{\mathrm{H}_{\mathcal{P}}(M)} d[i_{X^{k_{\mathcal{P}}}}(f v^1_{\mathcal{P}})] = \int_{\mathrm{H}_{\mathcal{P}}(M)} L_{X^{k_{\mathcal{P}}}}(f v^1_{\mathcal{P}})$$

$$= \int_{\mathrm{H}_{\mathcal{P}}(M)} [(X^{k_{\mathcal{P}}} f)\, v^1_{\mathcal{P}} + f L_{X^{k_{\mathcal{P}}}} v^1_{\mathcal{P}}]$$

which combined with Eq. (7.4) proves Eq. (7.6). For the general case choose $\chi \in C_c^\infty(\mathbb{R})$ such that χ is one in a neighborhood of 0. Define $\chi_n := \chi(\frac{1}{n} E(\cdot)) \in C_c^\infty(\mathrm{H}_{\mathcal{P}}(M))$ and $f_n := \chi_n f \in C_c^\infty(\mathrm{H}_{\mathcal{P}}(M))$. Observe that

$$(X^{k_{\mathcal{P}}} f_n) = \chi_n \cdot X^{k_{\mathcal{P}}} f + \frac{1}{n} f \cdot \chi' \left(\frac{1}{n} E(\cdot)\right) X^{k_{\mathcal{P}}} E$$

$$= \chi_n \cdot X^{k_{\mathcal{P}}} f + \frac{1}{n} f \cdot \chi' \left(\frac{1}{n} E(\cdot)\right) \left(\sum_{i=1}^{n} \langle k'(s_{i-1}+), \Delta_i b \rangle \right),$$

wherein the last equality we have used the formula for $X^{k_{\mathcal{P}}} E$ computed in the proof of Lemma 7.6. Because of Theorem 4.10, $\sum_{i=1}^n \langle k'(s_{i-1}+), \Delta_i b \rangle$ is a Gaussian random variable on $(\mathrm{H}_{\mathcal{P}}(M), v^1_{\mathcal{P}})$ and hence is in L^p for all $p \in [1, \infty)$. Also

$$|X^{k_{\mathcal{P}}} f| \leqslant C \sqrt{G^1_{\mathcal{P}}(X^{k_{\mathcal{P}}}, X^{k_{\mathcal{P}}})}$$

$$= C \sqrt{\sum_{i=1}^{n} \langle k'(s_{i-1}+), k'(s_{i-1}+) \rangle \, \Delta_i s} \leqslant C \, \|k'\|_\infty,$$

where C is bound on the differential of f. Using these remarks and the dominated convergence theorem, we may replace f by f_n in Eq. (7.6) and pass to the limit to conclude that Eq. (7.6) holds for bounded f with bounded derivatives. ∎

Remark 7.8. Obviously Corollary 7.7 holds for more general functions f. For example the above proof works if f and df are in $L^p(\mathrm{H}_{\mathcal{P}}(M), v^1_{\mathcal{P}})$ for some $p > 1$.

We would like to pass to the limit as $|\mathcal{P}| \to 0$ in Eq. (7.6) of Corollary 7.7. The right side of this equation is easily dealt with using Theorem 4.17. In order to pass to the limit on the left side of Eq. (7.6) it will be necessary to understand the limiting behavior of $k_{\mathcal{P}}$ as $|\mathcal{P}| \to 0$. This is the subject of the next subsection.

7.2. The Limit of $k_{\mathscr{P}}$.

Notation 7.9. Let $\mathscr{P}=\{0=s_0<s_1<s_2<...<s_n=1\}$ be a partition of $[0,1]$ and for $r\in(s_{j-1},s_j]$, let $\underline{r}:=s_{j-1}$. For $k\in PC^1$, define $\|k'\|_{1,\mathscr{P}}$ and $|||k'|||_{\mathscr{P}}$ by

$$\|k'\|_{1,\mathscr{P}}=\sum_{i=1}^{n}|k'(s_{i-1}+)|\,\Delta_i s, \tag{7.8}$$

and

$$|||k'|||_{\mathscr{P}}=\int_0^1|k'(r)-k'(\underline{r})|\,dr. \tag{7.9}$$

Note that $|||k'|||_{\mathscr{P}}=0$ if $k\in \mathrm{H}_{\mathscr{P}}$.

LEMMA 7.10. *Let $\mathscr{P}$ be a partition of $[0,1]$, $\sigma\in\mathrm{H}_{\mathscr{P}}(M)$, $b=\phi^{-1}(\sigma)$, $u=//(\sigma)$, $k\in PC^1$ and $k_{\mathscr{P}}(\sigma,\cdot)$ be as in Definition* 7.4. *Then with $\Delta_i b$ given by* (7.7) *and $\|k'\|_{1,\mathscr{P}}$ given by* (7.8),

$$|k_{\mathscr{P}}(\sigma,s)|\leqslant\|k'\|_{1,\mathscr{P}}\,e^{(1/2)\,\Lambda\sum_{j=1}^{n}|\Delta_j b|^2}\qquad\forall s\in[0,1] \tag{7.10}$$

and

$$\begin{aligned}&|k_{\mathscr{P}}(\sigma,s)-k_{\mathscr{P}}(\sigma,s_{i-1})|\\&\qquad\leqslant(|k'(s_{i-1}+)|\,\Delta_i s+\tfrac{1}{2}\,|k_{\mathscr{P}}(\sigma,s_{i-1})|\,\Lambda\,|\Delta_i b|^2)\cosh\sqrt{\Lambda}\,|\Delta_i b|,\end{aligned} \tag{7.11}$$

and

$$\begin{aligned}&|k_{\mathscr{P}}(\sigma,s)-k_{\mathscr{P}}(\sigma,s_{i-1})|\\&\qquad\leqslant|k'(s_{i-1}+)|\,\Delta_i s+\tfrac{1}{2}\Lambda\,|\Delta_i b|^2\,\|k_{\mathscr{P}}(\sigma,\cdot)\|_\infty\qquad\forall s\in(s_{i-1},s_i],\end{aligned} \tag{7.12}$$

where Λ is a bound on the curvature tensor.

Proof. Let $\kappa(\cdot):=k_{\mathscr{P}}(\sigma,\cdot)\in\mathrm{H}_{\mathscr{P},\sigma}$ and $A(s):=\Omega_{u(s)}(b'(s),\cdot)\,b'(s)$. By Definition 7.4 of $k_{\mathscr{P}}$, κ satisfies

$$\kappa''(s)=A(s)\,\kappa(s)\qquad\text{for all}\quad s\notin\mathscr{P} \tag{7.13}$$

and

$$\kappa'(s+)=k'(s+)\qquad\forall s\in\mathscr{P}\backslash\{1\}. \tag{7.14}$$

Noting that $|\Omega_{u(s)}(b'(s),\cdot)\, b'(s)| \leqslant \Lambda\, |b'(s)|^2 = \Lambda(|\Delta_i b|^2/\Delta_i s^2)$ for $s \in (s_{i-1}, s_i]$, Lemma 8.2 of the Appendix implies Eq. (7.12) and that

$$|\kappa(s)-\kappa(s_{i-1})| \leqslant |\kappa(s_{i-1})|\, (\cosh \sqrt{\Lambda}\, |\Delta_i b| - 1) + |k'(s_{i-1}+)|\, \Delta_i s \,\frac{\sinh \sqrt{\Lambda}\, |\Delta_i b|}{\sqrt{\Lambda}\, |\Delta_i b|}$$
$$\leqslant \left(|k'(s_{i-1}+)|\, \Delta_i s + \frac{1}{2}\, |\kappa(s_{i-1})|\, \Lambda\, |\Delta_i b|^2 \right) \cosh \sqrt{\Lambda}\, |\Delta_i b|,$$

where we have made use of the elementary inequalities

$$\cosh(a) - 1 \leqslant \frac{1}{2}\, a^2 \cosh(a), \qquad \text{and}$$
$$\frac{\sinh(a)}{a} \leqslant \cosh(a) \qquad \forall a \in \mathbb{R}. \tag{7.15}$$

In particular, Eq. (7.11) is valid and

$$|\kappa(s)| \leqslant |\kappa(s_{i-1})| \cosh \sqrt{\Lambda}\, |\Delta_i b| + |k'(s_{i-1}+)|\, \Delta_i s \,\frac{\sinh \sqrt{\Lambda}\, |\Delta_i b|}{\sqrt{\Lambda}\, |\Delta_i b|}$$
$$\leqslant (|\kappa(s_{i-1})| + |k'(s_{i-1}+)|\, \Delta_i s) \exp\left\{\frac{1}{2}\, \Lambda\, |\Delta_i b|^2\right\}, \tag{7.16}$$

since $\cosh(a) \leqslant e^{a^2/2}$ for all a. Using the fact that $\kappa(s_0) = \kappa(0) = 0$ and an inductive argument, Eq. (7.16) with $s = s_i$ implies that

$$|\kappa(s_i)| \leqslant \left(\sum_{j=1}^{i} |k'(s_{j-1}+)|\, \Delta_j s \right) e^{(1/2)\, \Lambda \sum_{j=1}^{i} |\Delta_j b|^2}.$$

Combining this last equation with Eq. (7.16) proves the bound in Eq. (7.10). ∎

In the rest of this section, unless otherwise stated, C will be a generic constant depending only on the geometry of M and $C(\gamma, p)$ will be a generic constant depending only on γ, p and the geometry of M.

THEOREM 7.11. *Let $k \in PC^1$ and B and $B_{\mathcal{P}}$ be the $\mathbb{R}^d$-valued processes defined in Notation* 1.2 *and Notation* 4.12, *respectively. Also let u be the $O(M)$-valued process which solves the Stratonovich stochastic differential equation*

$$\delta u = \mathcal{H}_u u \, \delta B, \qquad u(0) = u_0, \tag{7.17}$$

$u_{\mathcal{P}} = //(\phi(B_{\mathcal{P}}))$ and $z_{\mathcal{P}} = k_{\mathcal{P}}(\phi(B_{\mathcal{P}}), \cdot)$. (Note by Theorem 4.14 *that $u = \lim_{|\mathcal{P}| \to \infty} //(\phi(B_{\mathcal{P}}))$ is a stochastic extension of ϕ.) Let z denote the solution to the (random) ordinary differential equation*

$$z'(s) + \tfrac{1}{2} \mathrm{Ric}_{u(s)} \, z(s) = k'(s), \qquad z(0) = 0. \tag{7.18}$$

Then for $\gamma \in (0, \frac{1}{2})$, $p \in [1, \infty)$,

$$\mathbb{E}[\sup_{s \in [0,1]} |z_{\mathcal{P}}(s) - z(s)|^2] \leqslant C(\gamma, p)(\|k'\|^p_{1,\mathcal{P}} \, |\mathcal{P}|^{\gamma p} + |||k'|||^p_{\mathcal{P}}).$$

We will prove this theorem after the next two lemmas. Before doing this let us note that $z_{\mathcal{P}}$ in Theorem 7.11 above is determined by

$$z''_{\mathcal{P}}(s) = A(s) \, z_{\mathcal{P}} \quad \text{for } s \notin \mathcal{P}, \qquad z_{\mathcal{P}}(0) = 0, \qquad \text{and}$$

$$z'_{\mathcal{P}}(s+) = k'(s+) \quad \forall s \in \mathcal{P} \setminus \{1\}, \tag{7.19}$$

where

$$A(s) := \Omega_{u_{\mathcal{P}}(s)} \left(\frac{\Delta_i B}{\Delta_i s}, \cdot \right) \frac{\Delta_i B}{\Delta_i s} \qquad \text{when} \quad s \in (s_{i-1}, s_i]. \tag{7.20}$$

LEMMA 7.12. *Let δ_i be defined by*

$$\delta_i := z_{\mathcal{P}}(s_i) + \int_0^{s_i} (\tfrac{1}{2} \mathrm{Ric}_{u_{\mathcal{P}}(\underline{r})} \, z_{\mathcal{P}}(r) - k'(\underline{r}+)) \, dr.$$

Then for all $p \in [1, \infty)$ and $\gamma \in (0, 1/2)$ there is a constant $C = C(p, \gamma, \Lambda) < \infty$ such that

$$\mathbb{E}[\max_i |\delta_i|^p] \leqslant C \, \|k'\|^p_{1,\mathcal{P}} \, |\mathcal{P}|^{\gamma p},$$

where Λ is a bound on Ω and its horizontal derivative.

Proof. Without loss of generality, we can assume that $p \geqslant 2$. Throughout the proof, C will denote generic constant depending only on p,

γ, Λ, and possibly the dimension of M. By Taylor's theorem with integral remainder and Eq. (7.19) and Eq. (7.20) we have

$$\begin{aligned} z_{\mathcal{P}}(s_i) &= z_{\mathcal{P}}(s_{i-1}) + z'_{\mathcal{P}}(s_{i-1}+)\,\Delta_i s + \int_{s_{i-1}}^{s_i} (s_i - r)\, z''_{\mathcal{P}}(r)\, dr \\ &= z_{\mathcal{P}}(s_{i-1}) + k'(s_{i-1}+)\,\Delta_i s \\ &\quad + \int_{s_{i-1}}^{s_i} (s_i - r)\, \Omega_{u_{\mathcal{P}}(r)}(B'_{\mathcal{P}}(r), z_{\mathcal{P}}(r))\, B'_{\mathcal{P}}(r)\, dr \\ &= z_{\mathcal{P}}(s_{i-1}) + k'(s_{i-1}+)\,\Delta_i s \\ &\quad + \tfrac{1}{2}\Omega_{u_{\mathcal{P}}(s_{i-1})}(\Delta_i B, z_{\mathcal{P}}(s_{i-1}))\,\Delta_i B + \beta_i, \end{aligned} \tag{7.21}$$

where

$$\begin{aligned} \beta_i = \int_{s_{i-1}}^{s_i} (s_i - r)(\Omega_{u_{\mathcal{P}}(r)}(B'_{\mathcal{P}}(r), z_{\mathcal{P}}(r)) \\ - \Omega_{u_{\mathcal{P}}(s_{i-1})}(B'_{\mathcal{P}}(r), z_{\mathcal{P}}(s_{i-1})))\, B'_{\mathcal{P}}(r)\, dr. \end{aligned} \tag{7.22}$$

By Itô's lemma,

$$\begin{aligned} &\Omega_{u_{\mathcal{P}}(s_{j-1})}(\Delta_j B, z_{\mathcal{P}}(s_{j-1}))\,\Delta_j B \\ &\quad = \int_{s_{j-1}}^{s_j} \Omega_{u_{\mathcal{P}}(s_{j-1})}(B(r) - B(s_{j-1}), z_{\mathcal{P}}(s_{j-1}))\, dB(r) \\ &\qquad + \int_{s_{j-1}}^{s_j} \Omega_{u_{\mathcal{P}}(s_{j-1})}(dB(r), z_{\mathcal{P}}(s_{j-1}))(B(r) - B(s_{j-1})) \\ &\qquad - \mathrm{Ric}_{u_{\mathcal{P}}(\underline{r})}\, z_{\mathcal{P}}(z_{j-1})\,\Delta_j s. \end{aligned}$$

Using this equation and the fact that $z_{\mathcal{P}}(0) = 0$, we may sum Eq. (7.21) on i to find

$$z_{\mathcal{P}}(s_i) = \int_0^{s_i} (k'(\underline{r}+) - \tfrac{1}{2}\mathrm{Ric}_{u_{\mathcal{P}}(\underline{r})}\, z_{\mathcal{P}}(r))\, dr + M^{\mathcal{P}}_{s_i} + \sum_{j=1}^{i} \beta_j, \tag{7.23}$$

where $M^{\mathcal{P}}$ is the $\mathbb{R}^d$-valued Martingale,

$$\begin{aligned} M^{\mathcal{P}}_s := \int_0^s \Omega_{u_{\mathcal{P}}(\underline{r})}(B(r) - B(\underline{r}), z_{\mathcal{P}}(\underline{r}))\, dB(r) \\ + \int_0^s \Omega_{u_{\mathcal{P}}(\underline{r})}(dB(r), z_{\mathcal{P}}(\underline{r}))(B(r) - B(\underline{r})). \end{aligned}$$

Therefore $\delta_i = M^{\mathcal{P}}_{s_i} + \sum_{j=1}^{i} \beta_j$.

By the martingale moment inequality [63, Proposition 3.26],

$$\mathbb{E}[\sup_s |M_s^{\mathcal{P}}|^p] \leqslant C_p \mathbb{E}[\langle M^{\mathcal{P}} \rangle_1^{p/2}], \tag{7.24}$$

where C_p is a constant and $\langle M^{\mathcal{P}} \rangle$ is the quadratic variation of $M^{\mathcal{P}}$. It is easy to estimate $\langle M^{\mathcal{P}} \rangle_1$ by

$$\langle M^{\mathcal{P}} \rangle_1 \leqslant 2d\Lambda^2 \int_0^1 |B(r) - B(\underline{r})|^2 \, |z_{\mathcal{P}}(\underline{r})|^2 \, dr$$

and hence by Jensen's inequality

$$\langle M^{\mathcal{P}} \rangle_1^{p/2} \leqslant (2d)^{p/2} \, \Lambda^p \int_0^1 |B(r) - B(\underline{r})|^p \, |z_{\mathcal{P}}(\underline{r})|^p \, dr.$$

Because $\{z_{\mathcal{P}}(\underline{r})\}_{r \in [0,1]}$ is adapted to the filtration generated by B we may use the independence of the increments of B along with scaling to find

$$\begin{aligned}
\mathbb{E}\langle M^{\mathcal{P}} \rangle_1^{m/2} &\leqslant (2d)^{p/2} \, \Lambda^p \int_0^1 \mathbb{E}\, |B(r) - B(\underline{r})|^p \cdot \mathbb{E}\, |z_{\mathcal{P}}(\underline{r})|^p \, dr \\
&= C_p (2d)^{d/2} \, \Lambda^p \int_0^1 |r - \underline{r}|^{p/2} \cdot \mathbb{E}\, |z_{\mathcal{P}}(\underline{r})|^p \, dr \\
&\leqslant C_p (2d)^{p/2} \, \Lambda^p \, \|k'\|_{1,\mathcal{P}}^p \int_0^1 |r - \underline{r}|^{p/2} \\
&\qquad \times \mathbb{E} e^{(p/2)\, \Lambda \sum_{j=1}^n |\Delta_j B|^2} \, dr,
\end{aligned}$$

where Eq. (7.10) was used in the last equality. By Lemma 8.5 of the Appendix, $\mathbb{E}[e^{(p/2)\, \Lambda \sum_{j=1}^n |\Delta_j B|^2}]$ is bounded independent of $\mathcal{P}$ when $|\mathcal{P}|$ is sufficiently small. Hence we have shown that

$$\mathbb{E}[\sup_s |M_s^{\mathcal{P}}|^p] \leqslant C_p(\Lambda) \, \|k'\|_{1,\mathcal{P}}^p \int_0^1 |r - \underline{r}|^{p/2} \, dr \leqslant C_p(\Lambda) \, \|k'\|_{1,\mathcal{P}}^p \, |\mathcal{P}|^{p/2}.$$

So finish the proof it suffices to show that

$$\mathbb{E}\left(\sum_{j=1}^n |\beta_j| \right)^p \leqslant C \, \|k'\|_{1,\mathcal{P}}^p \, |\mathcal{P}|^{\gamma p}. \tag{7.25}$$

By assumption, $u_{\mathcal{P}}$ solves the differential equation

$$u'_{\mathcal{P}}(s) = \mathscr{H}_{u_{\mathcal{P}}(s)} u_{\mathcal{P}}(s) \, B'_{\mathcal{P}}(s), \qquad u_{\mathcal{P}}(0) = u_0,$$

so that for any $F \in C^1(O(M))$, $r \in (s_{i-1}, s_i]$,

$$|F(u_{\mathscr{P}}(r)) - F(u_{\mathscr{P}}(s_{i-1}))| \leqslant C \left| \int_{s_{i-1}}^{r} B'_{\mathscr{P}}(s)\, ds \right| \leqslant C\, |\Delta_i B|, \tag{7.26}$$

where C bounds the horizontal derivatives of F. Applying this estimate to Ω implies

$$|\Omega_{u_{\mathscr{P}}(r)} - \Omega_{u_{\mathscr{P}}(s_{i-1})}| \leqslant \Lambda\, |\Delta_i B|. \tag{7.27}$$

Using the inequalities in (7.12) and (7.27) and Eq. (7.22) we find that

$$\begin{aligned} |\beta_i| &\leqslant \tfrac{1}{2} \Lambda \max_{s_{i-1} \leqslant s \leqslant s_i} |z_{\mathscr{P}}(s) - z_{\mathscr{P}}(s_{i-1})|\, |\Delta_i B|^2 + \Lambda\, |z_{\mathscr{P}}(s_{i-1})|\, |\Delta_i B|^3 \\ &\leqslant \tfrac{1}{2} \Lambda (|k'(s_{i-1}+)|\, \Delta_i s + \tfrac{1}{2}\, |z_{\mathscr{P}}(s_{i-1})|\, \Lambda\, |\Delta_i B|^2) \\ &\quad \times \cosh(\sqrt{\Lambda}\, |\Delta_i B|)\, |\Delta_i B|^2 + \Lambda\, |z_{\mathscr{P}}(s_{i-1})|\, |\Delta_i B|^3. \end{aligned} \tag{7.28}$$

Letting K_γ denote the random variable defined in Eq. (8.15) of Fernique's Lemma 8.3, the above estimate implies that

$$\begin{aligned} |\beta_i| &\leqslant \frac{\Lambda}{2}\, |k'(s_{i-1}+)|\, \Delta_i s \cosh(\sqrt{\Lambda}\, K_\gamma\, |\mathscr{P}|)\, K_\gamma^2\, |\mathscr{P}|^{2\gamma} \\ &\quad + \left(\frac{\Lambda^2}{4} K_\gamma^4\, |\Delta_i s|^{4\gamma} \cosh(\sqrt{\Lambda}\, K_\gamma\, |\mathscr{P}|) + C_2 K_\gamma^3\, |\Delta_i s|^{3\gamma} \right) |z_{\mathscr{P}}(s_{i-1})|, \end{aligned}$$

where $\gamma \in (0, 1/2)$. We will now suppose that γ is close to 1/2. Then by Eq. (7.10) of Lemma 7.10, we find that

$$\begin{aligned} \sum_{i=1}^{n} |\beta_i| &\leqslant \frac{\Lambda}{2}\, \|k'\|_{1, \mathscr{P}} \cosh(\sqrt{\Lambda}\, K_\gamma\, |\mathscr{P}|)\, K_\gamma^2\, |\mathscr{P}|^{2\gamma} + C\, \|k'\|_{1, \mathscr{P}}\, |\mathscr{P}|^{3\gamma - 1} \\ &\quad \times (K_\gamma^4 \cosh(\sqrt{\Lambda}\, K_\gamma\, |\mathscr{P}|) + K_\gamma^3)\, e^{(1/2)\, \Lambda \sum_{j=1}^{n} |\Delta_j B|^2} \\ &\leqslant C\, \|k'\|_{1, \mathscr{P}}\, |\mathscr{P}|^{3\gamma - 1} \\ &\quad \times ((K_\gamma^4 + K_\gamma^2) \cosh(\sqrt{\Lambda}\, K_\gamma\, |\mathscr{P}|) + K_\gamma^3)\, e^{(1/2)\, \Lambda \sum_{j=1}^{n} |\Delta_j B|^2}. \end{aligned}$$

Using Lemma 8.4 and 8.5 of the Appendix, it follows that

$$((K_\gamma^4 + K_\gamma^2) \cosh(\sqrt{\Lambda}\, K_\gamma\, |\mathscr{P}|) + K_\gamma^3)\, e^{(1/2)\, \Lambda \sum_{j=1}^{n} |\Delta_j B|^2}$$

is bounded in all L^p for $|\mathscr{P}|$ small. This proves $\mathbb{E}(\sum_{j=1}^{n} |\beta_j|)^p \leqslant C\, \|k'\|_{1, \mathscr{P}}^p\, |\mathscr{P}|^{(3\gamma - 1)\, p}$ which proves Eq. (7.25) since $(3\gamma - 1)$ approaches 1/2 when γ approaches 1/2. ∎

LEMMA 7.13. *Let* $\varepsilon_{\mathscr{P}}$ *be defined by*

$$\varepsilon_{\mathscr{P}}(s) := z_{\mathscr{P}}(s) + \int_0^s (\tfrac{1}{2}\mathrm{Ric}_{u_{\mathscr{P}}(r)}\, z_{\mathscr{P}}(r) - k'(r))\, dr. \tag{7.29}$$

Then for all $\gamma \in (0, \frac{1}{2})$ *and* $p \in [1, \infty)$,

$$\mathbb{E}[\max_s |\varepsilon_{\mathscr{P}}(s)|^p] \leqslant C(\gamma, p)(\|k'\|_{1,\mathscr{P}}^p\, |\mathscr{P}|^{\gamma p} + |||k'|||_{\mathscr{P}}^p). \tag{7.30}$$

Proof. Let δ_i be as in the previous lemma and set $\delta_{\mathscr{P}}(s) := \sum_{i=1}^n \delta_i 1_{(s_{i-1}, s_i]}(s)$. By the definitions of $\varepsilon_{\mathscr{P}}$, (7.29) and $\delta_{\mathscr{P}}$, we have for $s \in (s_{i-1}, s_i]$,

$$\begin{aligned}
\varepsilon_{\mathscr{P}}(s) - \delta_{\mathscr{P}}(s) = {} & z_{\mathscr{P}}(s) - z_{\mathscr{P}}(s_i) \\
& + \tfrac{1}{2}\left(\int_0^s \mathrm{Ric}_{u_{\mathscr{P}}(r)}\, z_{\mathscr{P}}(r)\, dr - \int_0^{s_i} \mathrm{Ric}_{u_{\mathscr{P}}(\underline{r})}\, z_{\mathscr{P}}(\underline{r})\, dr\right) \\
& + \int_0^{s_i} k'(\underline{r}+)\, dr - \int_0^s k'(r)\, dr \\
= {} & \tfrac{1}{2}\int_0^{s_i} (\mathrm{Ric}_{u_{\mathscr{P}}(r)}\, z_{\mathscr{P}}(r) - \mathrm{Ric}_{u_{\mathscr{P}}(\underline{r})}\, z_{\mathscr{P}}(\underline{r}))\, dr \\
& - \tfrac{1}{2}\int_s^{s_i} \mathrm{Ric}_{u_{\mathscr{P}}(r)}\, z_{\mathscr{P}}(r)\, dr \\
& + (z_{\mathscr{P}}(s) - z_{\mathscr{P}}(s_i)) + (k(s_i) - k(s)) \\
& - \int_0^{s_i} (k'(r) - k'(\underline{r}+))\, dr \\
=: {} & \tfrac{1}{2}A_i + \tfrac{1}{2}B_i + C_i(s) + E_i,
\end{aligned}$$

where for $r \in (s_{j-1}, s_j]$, $\underline{r} := s_{j-1}$. We will now prove the estimate

$$\mathbb{E}[\sup_s |\varepsilon_{\mathscr{P}}(s) - \delta_{\mathscr{P}}(s)|^p] \leqslant C(\gamma, p)(\|k'\|_{1,\mathscr{P}}^p\, |\mathscr{P}|^{2\gamma p} + |||k'|||_{\mathscr{P}}^p)$$

This will complete the proof (7.30) in view Lemma 7.12.

By definition of $|||k'|||_{\mathscr{P}}$ in Eq. (7.9)

$$\max_i |E_i| \leqslant |||k'|||_{\mathscr{P}}. \tag{7.31}$$

In the argument to follow let $\{K_{\mathscr{P}}\}_{\mathscr{P}}$ denote a collection functions on $(W(\mathbb{R}^d), \mu)$ such that $\sup_{\mathscr{P}} \|K_{\mathscr{P}}\|_{L^p(\mu)} < \infty$ for all $p \in [1, \infty)$. Using Eq. (7.10) with $b = B$ and $\sigma = \phi(B_{\mathscr{P}})$ and Lemma 8.5 of the Appendix,

$$|B_i| \leqslant \|\mathrm{Ric}\|_\infty\ |\mathscr{P}|\ \|z_{\mathscr{P}}\|_\infty \leqslant \|\mathrm{Ric}\|_\infty\ K_{\mathscr{P}}\ |\ \|k'\|_{1,\mathscr{P}}\ |\mathscr{P}|.$$

So for $p \in [1, \infty)$,

$$\mathbb{E}[\max_i |B_i|^p] \leqslant \|\mathrm{Ric}\|_\infty^p\ |\ \|k'\|_{1,\mathscr{P}}^p\ \mathbb{E}[K_{\mathscr{P}}^p]\ |\mathscr{P}|^p \leqslant C\ |\mathscr{P}|^p.$$

Next we consider C_i. We have $C_i(s_i) = 0$ and by (7.13) and (7.14) with $b = B$ and $\sigma = \phi(B_{\mathscr{P}})$ for $s \in (s_{i-1}, s_i]$,

$$\begin{aligned} C_i'(s) &= z_{\mathscr{P}}'(s) - k'(s) \\ &= k'(s_{i-1}) - k'(s) + \int_{s_{i-1}}^{s} \Omega_{u_{\mathscr{P}}(r)}(B'(r), z_{\mathscr{P}}(r))\, dr\ B'(s) \end{aligned}$$

which implies after integrating

$$|C_i(s)| \leqslant \Lambda\ |\Delta_i B|^2\ \|z_{\mathscr{P}}\|_\infty + |\!|\!|k'|\!|\!|_{\mathscr{P}} \leqslant \Lambda K_\gamma^2\ |\Delta_i s|^{2\gamma}\ \|z_{\mathscr{P}}\|_\infty + |\!|\!|k'|\!|\!|_{\mathscr{P}},$$

where Λ is a bound on Ω and K_γ is defined in Lemma 8.4. Therefore, again by (7.10) and Lemma 8.5, if $p \in [1, \infty)$ and $\gamma \in (0, 1/2)$ then

$$\mathbb{E}[\max_{i,s} |C_i(s)|^p] \leqslant C(\gamma, p, \Lambda)(\|k'\|_{1,\mathscr{P}}^p\ |\mathscr{P}|^{2\gamma p} + |\!|\!|k'|\!|\!|_{\mathscr{P}}^p). \quad \blacksquare$$

So to finish the proof it only remains to consider the A_i term. Applying the estimate in Eq. (7.26) with $F = \mathrm{Ric}$ gives, for $r \in (s_{j-1}, s_j]$,

$$|\mathrm{Ric}_{u_{\mathscr{P}}(r)} - \mathrm{Ric}_{u_{\mathscr{P}}(\underline{r})}| \leqslant C\ |\Delta_j B| \leqslant C K_\gamma\ |\mathscr{P}|^\gamma,$$

where C is a bound on the horizontal derivative of Ric. Therefore,

$$\begin{aligned} |A_i| &\leqslant C K_\gamma\ |\mathscr{P}|^\gamma\ \|z_{\mathscr{P}}\|_\infty + \|\mathrm{Ric}\|_\infty \int_0^1 |z_{\mathscr{P}}(r) - z_{\mathscr{P}}(\underline{r})|\, dr \\ &\leqslant C K_\gamma\ |\mathscr{P}|^\gamma\ \|k'\|_{1,\mathscr{P}}\ e^{(1/2)\Lambda \sum_{j=1}^n |\Delta_j B|^2} \\ &\quad + \|\mathrm{Ric}\|_\infty\ (\|k'\|_{1,\mathscr{P}}\ |\mathscr{P}| + \tfrac{1}{2}\Lambda \max_i |\Delta_i B|^2\ \|z_{\mathscr{P}}\|_\infty) \\ &\leqslant C\ \|k'\|_{1,\mathscr{P}}\ \{e^{(1/2)\Lambda \sum_{j=1}^n |\Delta_j B|^2}(K_\gamma\ |\mathscr{P}|^\gamma + K_\gamma^2\ |\mathscr{P}|^{2\gamma}) + |\mathscr{P}|\} \\ &\leqslant K_{\mathscr{P}}\ \|k'\|_{1,\mathscr{P}}\ |\mathscr{P}|^\gamma, \end{aligned}$$

wherein we have made use of Eqs. (7.10) and (7.12) of Lemma 7.10 in the second inequality, Eq. (7.10) and the definition of K_γ in Eq. (8.15) in the third inequality, and Lemmas 8.4 and 8.5 in the last inequality. Thus

$$\mathbb{E}[\max_i |A_i|^p] \leqslant C(\gamma, p)\, \|k'\|_{1,\mathcal{P}}^p\, |\mathcal{P}|^{\gamma p}$$

for $p \in [1, \infty)$ and $\gamma \in (0, 1/2)$. This completes the proof of Lemma 7.13. ∎

Proof of Theorem 7.11. Let $\varepsilon_{\mathcal{P}}$ be defined as in Eq. (7.29) and let $y_{\mathcal{P}}(s)$ denote the solution to the differential equation,

$$y'_{\mathcal{P}}(s) + \tfrac{1}{2}\mathrm{Ric}_{u_{\mathcal{P}}(s)}\, y_{\mathcal{P}}(s) = k'(s) \qquad \text{with} \quad y_{\mathcal{P}}(0) = 0.$$

Then

$$z_{\mathcal{P}}(s) - y_{\mathcal{P}}(s) = -\int_0^s \tfrac{1}{2}\mathrm{Ric}_{u_{\mathcal{P}}(r)}(z_{\mathcal{P}}(r) - y_{\mathcal{P}}(r))\, dr + \varepsilon_{\mathcal{P}}(s)$$

and hence

$$|z_{\mathcal{P}}(s) - y_{\mathcal{P}}(s)| \leqslant \int_0^s C\, |(z_{\mathcal{P}}(r) - y_{\mathcal{P}}(r))|\, dr + \varepsilon_{\mathcal{P}}(s),$$

where C is a bound on $\frac{1}{2}\mathrm{Ric}$. So by Gronwall's inequality,

$$|z_{\mathcal{P}}(s) - y_{\mathcal{P}}(s)| \leqslant \max_s(|\varepsilon_{\mathcal{P}}(s)|\, e^{Cs}) \leqslant \max_s |\varepsilon_{\mathcal{P}}(s)|\, e^C,$$

which combined with Eq. (7.30) of Lemma 7.12 shows that

$$\mathbb{E}[\max_s |z_{\mathcal{P}}(s) - y_{\mathcal{P}}(s)|^p] \leqslant C(\gamma, p)(\|k'\|_{1,\mathcal{P}}^p\, |\mathcal{P}|^{\gamma p} + |||k'|||_{\mathcal{P}}^p).$$

for $p \in [1, \infty)$, $\gamma \in (0, 1/2)$.

To finish the proof of the theorem it is sufficient to prove

$$\mathbb{E}[\max_s |y_{\mathcal{P}}(s) - z(s)|^p] \leqslant C(\gamma, p)(\|k'\|_{1,\mathcal{P}}^p\, |\mathcal{P}|^{\gamma p} + |||k'|||_{\mathcal{P}}^p\, |\mathcal{P}|^{\gamma p}). \tag{7.32}$$

First note that a Gronwall estimate gives

$$\max_s |z(s)| \leqslant \|k'\|_{L^1(ds)}\, e^{\|\mathrm{Ric}\|_\infty s} \leqslant C\, \|k'\|_{L^1(ds)} \tag{7.33}$$

and similarly

$$\max_s |y_{\mathcal{P}}(s)| \leqslant C\, \|k'\|_{L^1(ds)},$$

where $\|k'\|_{L^1(ds)} = \int_0^1 |k'(s)|\, ds$. Let $w = y_{\mathscr{P}} - z$. Then

$$w'(s) = \tfrac{1}{2}\mathrm{Ric}_{u_{\mathscr{P}}(s)}\, w(s) + \tfrac{1}{2}(\mathrm{Ric}_{u_{\mathscr{P}}(s)} - \mathrm{Ric}_{u(s)})\, z(s).$$

Letting

$$A_{\mathscr{P}} = \max_s \tfrac{1}{2}\, |\mathrm{Ric}_{u_{\mathscr{P}}(s)} - \mathrm{Ric}_{u(s)}|$$

the inequality (7.33) and an application of Gronwall's inequality give

$$|w(s)| \leqslant CA_{\mathscr{P}}\, \|k'\|_{L^1}\, e^{Cs}. \tag{7.34}$$

Theorem 4.14 implies

$$\mathbb{E}[\,|A_{\mathscr{P}}|^p] \leqslant C(\gamma, p)\, |\mathscr{P}|^{\gamma p}$$

and hence by (7.34),

$$\mathbb{E}[\max_s |y_{\mathscr{P}}(s) - z(s)|^p] \leqslant C(\gamma, p)\, \|k'\|_{L^1}^p\, |\mathscr{P}|^{\gamma p}.$$

This implies (7.32) in view of the fact that

$$\|k'\|_{L^1} \leqslant \|k'\|_{1,\mathscr{P}} + |||k'|||_{\mathscr{P}}.$$

This completes the proof of Theorem 7.11. ∎

7.3. *Integration by Parts for Wiener Measure.*

PROPOSITION 7.14. *Let* $|\mathscr{P}| := \max\{|\Delta_i s| : i = 1, 2, ..., n\}$ *denote the mesh size of the partition* $\mathscr{P}$ *and* f *be a function on* $\mathrm{H}(M)$ *and* $\tilde{f}$ *on* $\mathrm{W}(M)$ *as in Theorem* 4.17. *Then*

$$\begin{aligned}\lim_{|\mathscr{P}| \to 0} \int_{\mathrm{H}_{\mathscr{P}}(M)} f\left(\sum_{i=1}^n \langle k'(s_{i-1}+), \Delta_i b\rangle\right) \nu_{\mathscr{P}}^1 \\ = \int_{\mathrm{W}(M)} \left(\tilde{f} \int_0^1 \langle k', d\tilde{b}\rangle\right) d\nu \end{aligned} \tag{7.35}$$

where $\Delta_i b$ *is to be interpreted as a function on* $\mathrm{H}(M)$ *as in Eq.* (7.7) *and* $\tilde{b}$ *is the anti-development map. Recall that* $\tilde{b}$ *is an* $\mathbb{R}^d$*-valued Brownian motion on* $(\mathrm{W}(M), \nu)$ *which was defined in Definition* 4.15. *Here* $\int_0^1 \langle k', d\tilde{b}\rangle$ *denotes the Itô integral of* k' *relative to* $\tilde{b}$.

Proof. Let B denote the standard $\mathbb{R}^d$-valued Brownian motion in Notation 1.2 and u denote the solution to the Stratonovich stochastic differential Eq. (7.17). By Lemma 4.11 and Theorem 4.10,

$$\int_{\mathrm{H}_{\mathscr{P}}(M)} f\left(\sum_{i=1}^{n} \langle k'(s_{i-1}+), \Delta_i b\rangle\right) \nu_{\mathscr{P}}^1$$

$$= \mathbb{E}\left[f(\phi(B_{\mathscr{P}}))\left(\sum_{i=1}^{n} \langle k'(s_{i-1}+), \Delta_i B\rangle\right)\right]. \tag{7.36}$$

By the isometry property of the Itô integral, we find that

$$\lim_{|\mathscr{P}|\to 0}\left(\sum_{i=1}^{n} \langle k'(s_{i-1}+), \Delta_i B\rangle\right) = \int_0^1 \langle k', dB\rangle,$$

where the convergence takes place in $L^2(\mathrm{W}(\mathbb{R}^d), \mu)$. As in the proof of Theorem 4.17, $f(\phi(B_{\mathscr{P}}))$ converges to $F(u)$ in L^2 as well. Therefore we may pass to the limit in Eq. (7.36) to conclude that

$$\lim_{|\mathscr{P}|\to 0} \int_{\mathrm{H}_{\mathscr{P}}(M)} f\left(\sum_{i=1}^{n} \langle k'(s_{i-1}+), \Delta_i b\rangle\right) \nu_{\mathscr{P}}^1 = \mathbb{E}\left[F(u) \int_0^1 \langle k', dB\rangle\right].$$

Since (B, u) and $(\tilde{b}, \widetilde{//})$ have the same distribution,

$$\mathbb{E}\left[F(u) \int_0^1 \langle k', dB\rangle\right] = \int_{\mathrm{W}(M)} \left(\tilde{f} \int_0^1 \langle k', d\tilde{b}\rangle\right) d\nu.$$

The previous two displayed equations prove Eq. (7.35). ∎

Definition 7.15. A function $f\colon \mathrm{W}(M) \to \mathbb{R}$ is said to be *a smooth cylinder function* if f is of the form

$$f(\sigma) = F \circ \pi_{\mathscr{P}}(\sigma) = F(\sigma_{\mathscr{P}}) \tag{7.37}$$

for some partition $\mathscr{P}$ and some $F \in C^\infty(M^{\mathscr{P}})$.

We are now prepared for the main theorem of this section.

Theorem 7.16. *Let $k \in PC^1$, z be the solution to the differential equation* (7.18) *of Theorem* 7.11 *and f be a cylinder function on* $\mathrm{W}(M)$. *Then*

$$\int_{\mathrm{W}(M)} X^z f\, d\nu = \int_{\mathrm{W}(M)} f\left(\int_0^1 \langle k', d\tilde{b}\rangle\right) d\nu, \tag{7.38}$$

where

$$(X^z f)(\sigma) := \sum_{i=1}^{n} \langle (\nabla_i f)(\sigma), X^z_{s_i}(\sigma)\rangle = \sum_{i=1}^{n} \langle (\nabla_i f)(\sigma), \widetilde{//}_{s_i}(\sigma)\, z(s_i, \sigma)\rangle$$

and $(\nabla_i f)(\sigma)$ *denotes the gradient* F *in the* i*th variable evaluated at* $(\sigma(s_1), \sigma(s_2), ..., \sigma(s_n))$.

Proof. The proof is easily completed by passing to the limit $|\mathscr{P}| \to 0$ in Eq. (7.6) of Corollary 7.7 making use of Proposition 7.14, Theorems 7.11, 4.14, and Corollary 4.13. ∎

8. APPENDIX: BASIC ESTIMATES

8.1. *Determinant Estimates.*

LEMMA 8.1. *Let* U *be a* $d \times d$ *matrix such that* $|U| < 1$, *then*

$$\det(I - U) = \exp(-tr\, U + \Psi(U)), \tag{8.1}$$

where $\Psi(U) := -\sum_{n=2}^{\infty} \frac{1}{n} tr\, U^n$. *Moreover,* $\Psi(U)$ *satisfies the bound,*

$$|\Psi(U)| \leqslant \sum_{n=2}^{\infty} \frac{d}{n} |U|^n \leqslant d\, |U|^2 (1 - |U|)^{-1}. \tag{8.2}$$

Proof. Equation (8.1) is just a rewriting of the standard formula,

$$\log(\det(I - U)) = -\sum_{n=0}^{\infty} \frac{1}{n+1} \operatorname{tr}(U^{n+1}),$$

which is easily deduced by integrating the identity

$$\begin{aligned} \frac{d}{ds} \log(\det(I - sU)) &= -\operatorname{tr}((I - sU)^{-1} U) \\ &= -\operatorname{tr}\left(\sum_{n=0}^{\infty} s^n U^n U\right) = -\sum_{n=0}^{\infty} s^n \operatorname{tr}(U^{n+1}). \end{aligned}$$

Since for any $d \times d$ matrix $|\operatorname{tr} U| \leqslant d\, |U|$ and $|U^k| \leqslant |U|^k$, it follows that

$$|\operatorname{tr}(U^k)| \leqslant d\, |U|^k$$

and hence

$$|\Psi(U)| \leqslant \sum_{n=2}^{\infty} \frac{d}{n} |U|^n \leqslant d\, |U|^2 (1-|U|)^{-1}. \quad \blacksquare$$

8.2. *Ordinary Differential Equation Estimates.*

LEMMA 8.2. *Let $A(s)$ be a $d \times d$ matrix for all $s \in [0, 1]$ and let $Z(s)$ be either a $\mathbb{R}^d$ valued or $d \times d$ matrix valued solution to the second order differential equation*

$$Z''(s) = A(z)\, Z(s). \tag{8.3}$$

Then

$$|Z(s) - Z(0)| \leqslant |Z(0)|\, (\cosh \sqrt{K}\, s - 1) + |Z'(0)|\, \frac{\sinh \sqrt{K}\, s}{\sqrt{K}} \tag{8.4}$$

and

$$|Z(s) - Z(0)| \leqslant s\, |Z'(0)| + K \frac{s^2}{2} Z^*(s), \tag{8.5}$$

where $Z^*(s) := \max\{|Z(r)|: 0 \leqslant r \leqslant s\}$, $K := \sup_{s \in [0,1]} |A(s)|$ and $|A|$ denotes the operator norm of A.

Proof. By Taylor's theorem with integral remainder,

$$\begin{aligned} Z(s) &= Z(0) + sZ'(0) + \int_0^s Z''(u)(s-u)\, du \\ &= Z(0) + sZ'(0) + \int_0^s A(u)\, Z(u)(s-u)\, du \end{aligned} \tag{8.6}$$

and therefore

$$\begin{aligned} |Z(s) - Z(0)| &\leqslant s\, |Z'(0)| + K \int_0^s |Z(u)|\, (s-u)\, du \\ &\leqslant s\, |Z'(0)| + K \int_0^s |Z(u) - Z(0)|\, (s-u)\, du + \frac{s^2}{2} K\, |Z(0)| \\ &=: f(s). \end{aligned} \tag{8.7}$$

One may easily deduce Eq. (8.5) from the first inequality in this equation.
Note that $f(0)=0$,

$$f'(s)=|Z'(0)|+K\int_0^s |Z(u)-Z(0)|\,(s-u)\,du+sK\,|Z(0)|,$$

$f'(0)=|Z'(0)|$, and

$$f''(s)=K\,|Z(s)-Z(0)|+K\,|Z(0)|\leqslant Kf(s)+K\,|Z(0)|.$$

That is,

$$f''(s)=Kf(s)+\eta(s), \qquad f(0)=0, \qquad \text{and} \qquad f'(0)=|Z'(0)|, \tag{8.8}$$

where $\eta(s):=f''(s)-Kf(s)\leqslant K\,|Z(0)|$. Equation (8.8) may be solved by variation of parameters to find

$$\begin{aligned}
f(s)&=|Z'(0)|\,\frac{\sinh\sqrt{K}\,s}{\sqrt{K}}+\int_0^s \frac{\sinh\sqrt{K}\,(s-r)}{\sqrt{K}}\,\eta(r)\,dr\\
&\leqslant|Z'(0)|\,\frac{\sinh\sqrt{K}\,s}{\sqrt{K}}+|Z(0)|\int_0^s \sqrt{K}\sinh\sqrt{K}\,(s-r)\,dr\\
&=|Z'(0)|\,\frac{\sinh\sqrt{K}\,s}{\sqrt{K}}+|Z(0)|\,(\cosh\sqrt{K}\,s-1).
\end{aligned}$$

Combining this equation with Eq. (8.7) proves Eq. (8.4). ∎

LEMMA 8.3. *Suppose that Z is a $d\times d$-matrix valued solution to Eq. (8.3) with $Z(0)=0$ and $Z'(0)=I$. Let $K>0$, $K_1>0$ be constants so that* $\sup_{s\in[0,1]}|A(s)|\leqslant K$ *and* $\sup_{s\in[0,1]}|A'(s)|\leqslant K_1$. *Then*

$$Z(s)=sI+\frac{s^3}{6}A(0)+s\mathscr{E}(s), \tag{8.9}$$

where

$$|\mathscr{E}(s)|\leqslant\tfrac{1}{6}(2K_1s^3+\tfrac{1}{2}K^2s^4)\cosh(\sqrt{K}\,s). \tag{8.10}$$

Proof. Using the definition of Z in Eq. (8.3) we have that $Z(0)=Z''(0)=0$, $Z'(0)=I$,

$$Z^{(3)}(s):=\frac{d^3}{ds^3}Z(s)=A'(s)\,Z(s)+A(s)\,Z'(s),$$

and hence $Z^{(3)}(0)=A(0)$. By Taylor's theorem with integral remainder

$$\begin{aligned} Z(s) &= sI+\frac{1}{2}\int_0^s Z^{(3)}(\xi)(s-\xi)^2\,d\xi \\ &= sI+\frac{s^3}{6}A(0)+\frac{1}{2}\int_0^s (Z^{(3)}(\xi)-A(0))(s-\xi)^2\,d\xi. \end{aligned}$$

Now using Lemma 8.2 with $Z(0)=0$, we find

$$\begin{aligned} |Z^{(3)}(\xi)-A(0)| &= \left|A'(\xi)\,Z(\xi)+A(\xi)\left(I+\int_0^\xi A(r)\,Z(r)\,dr\right)-A(0)\right| && (8.11)\\ &\leqslant \frac{\sinh(\sqrt{K}\,\xi)}{\sqrt{K}}+K(\cosh(\sqrt{K}\,\xi)-1)+K_1\xi && (8.12)\\ &\leqslant K_1\xi(\cosh(\sqrt{K}\,\xi)+1)+\frac{1}{2}K^2\xi^2\cosh(\sqrt{K}\,\xi) && (8.13)\\ &\leqslant \left(2K_1 s+\frac{1}{2}K^2s^2\right)\cosh(\sqrt{K}\,s), && (8.14) \end{aligned}$$

where we used the elementary inequalities $\sinh(a)/a\leqslant\cosh(a)$ and $\cosh(a)-1\leqslant\frac{1}{2}a^2\cosh(a)$ valid for all $a\in\mathbb{R}$. Using $Z^{(3)}(0)=A(0)$ and the definition of $\mathscr{E}$ completes the proof. ∎

8.3. *Gaussian Bounds.* In this subsection, $B(s)$ will always denote the standard $\mathbb{R}^d$-valued Brownian motion defined in Notation 1.2.

LEMMA 8.4 (Fernique). *For $\gamma\in(0,1/2)$ let K_γ, be the random variable,*

$$K_\gamma:=\sup\left\{\frac{|B(s)-B(s)|}{|s-r|^\gamma}:0\leqslant s<r\leqslant 1\right\}. \tag{8.15}$$

Then there exists an $\varepsilon=\varepsilon(\gamma)>0$ such that $\mathbb{E}[e^{\varepsilon K_\gamma^2}]<\infty$

Proof. Since K_γ as a functional of B is a "measurable" semi-norm, Eq. (8.15) is a direct consequence of Fernique's theorem [67, Theorem 3.2]. ∎

LEMMA 8.5. *For $p\in[1,\infty)$,*

$$\mathbb{E}e^{(p/2)\,C\sum_{j=1}^n|\Delta_jB|^2}=\prod_{j=1}^n(1-pC\,\Delta_js)^{-d/2} \tag{8.16}$$

provided that $pC\,\Delta_j s<1$ *for all* j. *Furthermore,*

$$\lim_{|\mathcal{P}|\to 0} \mathbb{E}e^{(p/2)\,C\sum_{j=1}^{n}|\Delta_j B|^2}=e^{dpC/2}. \tag{8.17}$$

Proof. By the independence of increments and scaling properties of B we have

$$\mathbb{E}[e^{(p/2)\,C\sum_{j=1}^{n}|\Delta_j B|^2}]=\prod_{j=1}^{n}\mathbb{E}[e^{pC\,|\Delta_j B|^2/2}]=\prod_{j=1}^{n}(\mathbb{E}[e^{pC\,\Delta_j sN^2/2}])^d,$$

where N is an standard normal random variable. This proves Eq. (8.16), since an elementary Gaussian integration gives

$$\mathbb{E}[e^{pC\,\Delta_j sN^2/2}]=(1-pC\,\Delta_j s)^{-1/2}$$

provided that $pC\,\Delta_j s<1$. Equation (8.17) is an elementary consequence of (8.16). ∎

LEMMA 8.6 (Gaussian Bound). *For every* $k\geqslant 0$ *there is a constant* $C=C(k,d)$ *which is increasing in* k *such that*

$$\mathbb{E}[e^{k\,|B(1)|}\colon |B(1)|\geqslant\rho]\leqslant Ce^{-(1/4)\,\rho^2}/\rho^2 \qquad \textit{for all} \quad \rho\geqslant 1. \tag{8.18}$$

Proof. A compactness argument shows that there is a constant $\tilde{C}(k,d)$ such that $r^{d-1}e^{kr}e^{-(1/2)\,r^2}\leqslant\tilde{C}(k,d)\,e^{-(3/8)\,r^2}$ for all $r\geqslant 0$. Passing to polar coordinates and using this inequality shows that

$$\begin{aligned}\mathbb{E}[e^{k\,|B(1)|}\colon |B(1)|\geqslant\rho]&=\omega_{d-1}(2\pi)^{d/2}\int_\rho^\infty r^{d-1}e^{kr}e^{-(1/2)\,r^2}\,dr\\&\leqslant\omega_{d-1}(2\pi)^{d/2}\,\tilde{C}(k,d)\int_\rho^\infty\frac{r}{\rho}\,e^{-(3/8)\,r^2}\,dr\\&=\omega_{d-1}(2\pi)^{d/2}\,\tilde{C}(k,d)\,\frac{4}{3\rho}\,e^{-(3/8)\,\rho^2}\\&\leqslant Ce^{-(1/4)\,\rho^2}/\rho^2,\end{aligned}$$

where ω_{d-1} is the volume of the $d-1$ sphere in $\mathbb{R}^d$. ∎

LEMMA 8.7. *Fix* $\varepsilon>0$ *and* $K\geqslant 0$. *Let* $\chi_\varepsilon(r)=1_{r\geqslant\varepsilon}$, *let* B *be a standard* $\mathbb{R}^d$*-valued Brownian motion and let* $\mathcal{P}=\{0=s_0<s_1<\ldots<s_n=1\}$ *be a partition of* $[0,1]$.

Define the function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *by*

$$\psi(u) := \mathbb{E}\left[\left(\frac{\sinh(\sqrt{K}\,|B(u^2)|)}{\sqrt{K}\,|B(u^2)|}\right)^{d-1}\right]$$
$$= \mathbb{E}\left[\left(\frac{\sinh(\sqrt{K}\,u\,|B(1)|)}{\sqrt{K}\,u\,|B(1)|}\right)^{d-1}\right]. \tag{8.19}$$

Then there is a constant $C = C(K, d) < \infty$ *such that*

$$\sum_{i=1}^{n} \mathbb{E}\left[\chi_\varepsilon(|\Delta_i B|)\left(\frac{\sinh(\sqrt{K}\,|\Delta_i B|)}{\sqrt{K}\,|\Delta_i B|}\right)^{d-1}\right] \prod_{j\neq i} \psi(\sqrt{\Delta_j s})$$
$$\leqslant C\varepsilon^{-2} \exp\left(-\frac{\varepsilon^2}{4\,|\mathcal{P}|}\right). \tag{8.20}$$

Proof. It is easily checked that ψ is an even smooth (in fact analytic) function and that $\psi(u) = 1 + (d(d-1)/6)\,u^2 + O(u^4)$ and hence there is a constant $C < \infty$ such that $\psi(u) \leqslant e^{Cu^2}$ for $0 \leqslant u \leqslant 1$. Thus

$$\prod_{j\neq i} \psi(\sqrt{\Delta_j s}) \leqslant e^{C \sum_{j\neq i} \Delta_j s} \leqslant e^{C}.$$

Recall the elementary inequalities $\sinh(a)/a \leqslant \cosh(a) \leqslant e^{|a|}$ which are valid for all $a \in \mathbb{R}$. Using these inequalities and the scaling properties of B and Lemma 8.6,

$$\mathbb{E}\left[\chi_\varepsilon(|\Delta_i B|)\left(\frac{\sinh(\sqrt{K}\,|\Delta_i B|)}{\sqrt{|K|}\,|\Delta_i B|}\right)^{d-1}\right]$$
$$= \mathbb{E}\left[\chi_\varepsilon(\sqrt{\Delta_i s}\,|B(1)|)\left(\frac{\sinh(\sqrt{K\,\Delta_i s}\,|B(1)|)}{\sqrt{K\,\Delta_i s}\,|B(1)|}\right)^{d-1}\right]$$
$$\leqslant \mathbb{E}[\chi_{\varepsilon \Delta_i s^{-1/2}}(|B(1)|)\exp((d-1)\sqrt{K\,\Delta_i s}\,|B(1)|)]$$
$$\leqslant C(K\,|\mathcal{P}|, d)\,\frac{\Delta_i s}{\varepsilon^2} \exp\left(-\frac{\varepsilon^2}{4\Delta_i s}\right)$$
$$\leqslant C(K\,|\mathcal{P}|, d)\,\frac{\Delta_i s}{\varepsilon^2} \exp\left(-\frac{\varepsilon^2}{4\,|\mathcal{P}|}\right).$$

Combining the above estimates completes the proof of Lemma 8.7. ∎

Proposition 8.8. *Let* B *be the* $\mathbb{R}^d$*-valued Brownian motion defined on* $(\mathrm{W}(\mathbb{R}^d), \mu)$ *as in Notation* 1.2 *above and let* $\mathbf{R}_i$ *for* $i = 0, 1, ..., n$ *be random* *symmetric* $d\times d$ *matrices which are* $\sigma(B_s\colon s\leqslant s_{i-1})$*-measurable for each* i. *Note that* $\mathbf{R}_0$ *is non-random. Further assume there is a non-random constant* $K<\infty$ *such that* $|\mathbf{R}_i|\leqslant K$ *for all* i. *Then for all* $p\in\mathbb{R}$ *there is an* $\varepsilon=\varepsilon(K,d,p)>0$

$$1\leqslant\mathbb{E}[e^{p\sum_{i=1}^{n}(\langle\mathbf{R}_i\Delta_iB,\Delta_iB\rangle-tr(\mathbf{R}_i)\Delta_is)}]\leqslant e^{dp^2K^2|\mathcal{P}|} \tag{8.21}$$

whenever $|\mathcal{P}|\leqslant\varepsilon$.

Proof. By Itô's Lemma,

$$\langle\mathbf{R}_i\Delta_iB,\Delta_iB\rangle-\mathrm{tr}(\mathbf{R}_i)\Delta_is=2\int_{s_{i-1}}^{s_i}\langle\mathbf{R}_i(B(s)-B(\underline{s})),dB(s)\rangle,$$

and hence $\sum_{i=1}^{n}(\langle\mathbf{R}_i\Delta_iB,\Delta_iB\rangle-\mathrm{tr}(\mathbf{R}_i)\Delta_is)=M_1$, where M_t is the continuous square integrable martingale

$$M_t:=2\int_0^t\langle\mathbf{R}_s(B(s)-B(\underline{s})),dB(s)\rangle$$

and $\mathbf{R}_s:=\mathbf{R}_i$ if $s\in(s_{i-1},s_i]$. The quadratic variation of this martingale is

$$\langle M\rangle_t=4\int_0^t|\mathbf{R}_s(B(s)-B(\underline{s}))|^2\,ds\leqslant 4K^2\int_0^t|B(s)-B(\underline{s})|^2\,ds.$$

Let $p\in(1,\infty)$. Then by the independent increment property of the Brownian motion B, it follows that

$$\begin{aligned}\mathbb{E}[e^{p^2\langle M\rangle_1}]&\leqslant\mathbb{E}\left[\exp\left(4p^2K^2\int_0^1|B(s)-B(\underline{s})|^2\,ds\right)\right]\\&=\prod_{i=1}^{n}\mathbb{E}\left[\exp\left(4p^2K^2\int_{s_{i-1}}^{s_i}|B(s)-B(\underline{s})|^2\,ds\right)\right]\\&=\prod_{i=1}^{n}\mathbb{E}\left[\exp\left(4p^2K^2\,\Delta_is^2\int_0^1|B(s)|^2\,ds\right)\right],\end{aligned}\tag{8.22}$$

wherein the last equality we have used scaling and independence properties of B to conclude that $\int_{s_{i-1}}^{s_i}|B(s)-B(\underline{s})|^2\,ds$, $\int_0^{\Delta_is}|B(s)|^2\,ds$ and $\int_0^{\Delta_is}\Delta_is\,|B(s/\Delta_is)|^2\,ds=\Delta_is^2\int_0^1|B(s)|^2\,ds$ all have the same distribution.

Fernique's theorem [67, Theorem 3.2] implies that

$$\psi(\lambda):=\mathbb{E}\left[\exp\left(\frac{\lambda}{2}\int_0^1|B(s)|^2\,ds\right)\right]$$

is a well defined analytic function of λ in a neighborhood of 0. Because $\psi(0)=1$ and

$$\psi'(0)=\frac{1}{2}\,\mathbb{E}\int_0^1 |B(s)|^2\,ds=\frac{d}{4}$$

it follows that $\psi(\lambda)\leqslant e^{d\lambda/2}$ for all positive λ sufficiently near 0. Using this fact in Eq. (8.22) gives the bound

$$\begin{aligned}\mathbb{E}[e^{p^2\langle M\rangle_1}]&\leqslant\prod_{i=1}^n \exp(4\,dp^2K^2\,\Delta_i s^2)=\exp\left(4\,dK^2p^2\sum_{i=1}^n\Delta_i s^2\right)\\&\leqslant\exp(4\,dK^2p^2\,|\mathcal{P}|)<\infty,\end{aligned}\tag{8.23}$$

which is valid when the mesh of $\mathcal{P}$ is sufficiently small.

By Itô's Lemma,

$$Z_t^{(p)}=\exp\left(pM_t-\frac{p^2}{2}\langle M\rangle_t\right)$$

is a positive local martingale. Because of the bound in Eq. (8.23), Novikov's criterion [87, Proposition 1.15, p. 308] implies that $Z_t^{(p)}$ is in fact a martingale and hence in particular $\mathbb{E}[Z_1^{(p)}]=1$. Therefore,

$$\mathbb{E}[e^{pM_1}]=\mathbb{E}[e^{pM_1-(p^2/2)\langle M\rangle_1}e^{(p^2/2)\langle M\rangle_1}]\geqslant\mathbb{E}[e^{pM_1-(p^2/2)\langle M\rangle_1}]=1$$

and

$$\begin{aligned}\mathbb{E}[e^{pM_1}]&=\mathbb{E}[\exp(pM_1-p^2\langle M\rangle_1)\exp(p^2\langle M\rangle_1)]\\&\leqslant\sqrt{\mathbb{E}[\exp(2pM_1-2p^2\langle M\rangle_1)]}\,\sqrt{\mathbb{E}[\exp(p^2\langle M\rangle_1)]}\\&=\sqrt{\mathbb{E}Z_t^{(2p)}}\,\sqrt{\mathbb{E}[\exp(p^2\langle M\rangle_1)]}=\sqrt{\mathbb{E}[\exp(p^2\langle M\rangle_1)]}\\&\leqslant\exp(4\,dK^2p^2\,|\mathcal{P}|).\end{aligned}$$

This completes the proof of Proposition 8.8. ∎

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