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PROBABILITY MODELS IN MATHEMATICAL PHYSICS

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PROBABILITY MODELS IN MATHEMATICAL PHYSICS

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Preface

This volume communicates proceedings of the Conference on Probability Models in Mathematical Physics which met in Colorado Springs, Colorado, May 24–26, 1990.

The aim of the conference is to present some rigorous results of mathematical physics, especially probabilistic results motivated by modern physics. On the one hand, ideas and models born out of the study of statistical mechanics and quantum field theory stimulate the further development of probability theory. On the other hand, rigorous results in the areas of probability represented in this volume create a deeper understanding for physics.

We wish to thank the Department of Mathematics and the College of Engineering and Applied Science of the University of Colorado at Colorado Springs, and the National Science Foundation for both their human and financial supports. We also thank the conference staff of the Colorado College who accomodated the participants on their campus. We express our thanks to the participants for an especially lively conference.

Gregory J. Morrow and Wei–Shih Yang
Colorado Springs, June, 1990

12) L. Laanait, A. Messager, S. Miracle Solé, J. Ruis and S. Shlosman: *"Interfaces in the Potts model 1: Pirogov-Sinai theory of the Fortuin-Kasteleyn representation"*. CPT-prepint, Marseille, 1989.

13) S. N. Isakov, Commun. Math. Phys. **95** (1984), 427.

14) C. Borgs and J. Imbrie, Commun. Math. Phys. **123** (1989), 305.

15) A. M. Ferrenberg and R. H. Swendsen, Phys. Rev. Lett. **61** (1988), 2635.

16) A. M. Ferrenberg and R. H. Swendsen, Phys. Rev. Lett. **63** (1989), 1195.

17) U. Wolff, Nucl. Phys. **B280** (1987), 680.

18) C. M. Fortuin and P. W. Kasteleyn, Physica **57** (1972), 536.

19) D. Brydges: *"A short course on cluster expansions."* in K. Osterwalder, R. Stora (eds.) *"Critical phenomena, random systems, gauge theories."* (Les Houches 1984). Amsterdam: North Holland 1986.

20) A. Messager, S. Miracle Solé, C. Pfister, Commun. Math. Phys. **58** (1978), 19.

Two Dimensional Euclidean Quantized Yang-Mills Fields

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1. Introduction.

In this paper, I will first briefly explain how to gauge fix the "Yang-Mills" measure. It will then be seen, that choosing the "complete axial gauge" and restricting the space-time dimension (d) to be two, that the resulting gauge fixed measure is Gaussian. Once this measure is defined, the results in [D] pertaining to gauge invariant expectations will be summarized. To motivate the main Theorem (Theorem 3.2.), proofs of the theorems will be indicated under the further simplification that the "structure group" (G) is U(1).

The problem of mathematical existence of quantized Yang-Mills' fields is (on an informal level) equivalent to defining a certain probability measure on a space of connection forms. The informal description of this (Yang-Mills') measure is

$$d\mu(A) = Z^{-1}e^{-(F^A,F^A)/2g^2}DA, \qquad \text{(YM)}$$

where A runs over a space of connection forms ($\mathcal{A}$) on the trivial vector bundle $\mathbb{C}^N \times \mathbb{R}^d$, $F^A = dA+A\wedge A$ is the curvature of A, $(F,F) \equiv \int_{\mathbb{R}^d} \sum_{i<j} \mathrm{tr}(F_{ij}(x)^* F_{ij}(x))\, dx$,

$DA = \prod_{i=1}^{d} \prod_{x \in \mathbb{R}^d} d(A_i(x))$ is "infinite dimensional Lebesgue measure" on ($\mathcal{A}$), g^2 is a positive "coupling" constant (henceforth set equal to 1), and Z is a normalization constant which makes μ a probability measure. The connection forms are restricted to take their values in the Lie Algebra ($\mathcal{G}$) of the structure (or gauge) group (G), which is taken to be a closed subgroup of U(N). A rigorous definition and construction of the measure μ for $d \geq 3$ and general gauge group G is still a hard open problem. However, considerable success in understanding this problem has been made by Balaban [B] and Federbush [F] using renormalization group techniques on lattice or lattice like models.

Before attacking the hard analytical problems in defining the measure (μ) one must first understand the problem of gauge invariance. It is well known that the heuristic expression of μ is invariant under the infinite dimensional group of gauge transformations $GT = \{g:\mathbb{R}^d \longrightarrow G: g \text{ is } C^\infty \text{ and } g(0) = id\}$. The gauge transformations act on $\mathcal{A}$ by $g \cdot A = gAg^{-1} + gdg^{-1}$, and under this action $F^{g \cdot A} = gF^A g^{-1}$. Therefore the exponent is invariant because of the properties of the trace function. Also, informally, the "infinite dimensional Lebesgue measure" is GT - invariant which may be seen as follows. Lebesgue measure should be invariant under translations, so we need only consider the transformation $A \longrightarrow gAg^{-1}$. This transformation does not mix the "components" $A_i(x)$ of A, so it suffices to show that the adjoint action of G on $\mathcal{G}$ leaves Lebesgue measure on $\mathcal{G}$ invariant. But $\det(Ad_g) = \pm 1$ which is a consequence of the fact that any real finite dimensional representation of a compact Lie group admits an inner product with respect to which the representation is orthogonal.

The gauge invariance makes it impossible to interpret μ as a probability measure on $\mathcal{A}$. As is well known, the correct interpretation is to consider μ as a measure on $\mathcal{A}/GT$.

2. Gauge Fixing.

In order to see how to interpret the informal expression for μ as a measure on the quotient space $\mathcal{A}/GT$ it is instructive to consider an analogous finite dimensional setting and then crib the results to the infinite dimensional setting. For a more detailed discussion of the material in this section the reader is referred to Z. Jaskolski [J] and the references therein. The notation for most of this section is as follows:

GT = a Lie group called the gauge transformations,
ω = a left invariant non-zero volume form on GT,
($\mathcal{A}, \mu$) = a finite dimensional smooth manifold with a volume form μ, and
S = a submanifold of $\mathcal{A}$.

We will assume:

i) GT acts smoothly to left on $\mathcal{A}$,
ii) the map $\Psi: GT \times S \longrightarrow \mathcal{A}$ given by $\Psi(g,s) = gs$ is a diffeomorphism, and
iii) μ is invariant under the action of GT, $L_g^* \mu = \mu$ for $g \in GT$, where L_g is the left action of g on M.

Note that condition ii) implies that S is a slice for the natural projection map $\pi: \mathcal{A} \longrightarrow \mathcal{A}/GT$, that is $\pi|_S$ is a bijection.

Proposition 2.1. Let $\xi_1, \cdots, \xi_p$ be a basis for the Lie algebra of GT (thought of as left invariant vector fields on GT) such that $\omega(\xi_1,\cdots,\xi_p) = 1$ and define a form (ν) on S by $\nu(v_1, \cdots, v_r) = \mu(\xi_1 s,\cdots,\xi_p s, v_1,\cdots v_r)$, where $\{v_i\}$ is a basis for $T_s S$ = the tangent space to S at s, $\xi_i s \equiv R_{s*}\xi_i$, and $R_s: GT \longrightarrow \mathcal{A}$ is given by $R_s(g) = gs$ for each s in $\mathcal{A}$. Then the pullback $\varphi^*\mu$ satisfies $\varphi^*\mu = \omega\wedge\nu$ or more precisely $\varphi^*\mu = pr_1{}^*\omega\wedge pr_2{}^*\nu$, where pr_1 and pr_2 are the projections of GT×S onto the first and second factor respectively.

Proof: First note, if φ_* denotes the differential of φ, then $\varphi_{*(g,s)}\xi_1(g) = \frac{d}{dt}\big|_0 g e^{t\xi_1(e)} s = L_{g*}(\xi_1(e)s)$, and that $\varphi_{*(g,s)} v_i = L_{g*} v_i$ for $v_i \in T_s S$. Therefore:

$$(\varphi^*\mu)_{(g,s)}\ (\xi_1,\cdots,\xi_p,v_1,\cdots,v_r)$$
$$= L_g^*\mu(\xi_1(e)s,\cdots,\xi_p(e)s,v_1,\cdots,v_r)$$
$$= \mu(\xi_1(e)s,\cdots,\xi_p(e)s,v_1,\cdots,v_r) \quad \text{(GT invariance)}$$
$$= \omega(\xi_1,\cdots,\xi_p)\nu(v_1,\cdots,v_r) \quad \text{(definition of } \nu \text{ and } \omega)$$
$$= \omega\wedge\nu(\xi_1,\cdots,\xi_p,v_1,\cdots,v_r).$$

Since $\varphi^*\mu$ and $\omega\wedge\nu$ are volume forms on GT×S, we conclude that $\varphi^*\mu = \omega\wedge\nu$. Q.E.D.

Definition 2.1. A function (f) on $\mathcal{A}$ is said to be gauge invariant if $f(gA) = f(A)$ for all gauge transformations g in GT.

Corollary 2.1. If f is a gauge invariant function on $\mathcal{A}$ then

$$\int_{\mathcal{A}} f\mu = \int_{GT\times S} \omega\wedge f|_S\nu = \int_{GT}\omega \cdot \int_S f|_S\nu \equiv \mathrm{Vol(GT)}\cdot\int_S f|_S\nu.$$

Furthermore if μ is a probability measure and $\bar{\mu} = \pi_*\mu$ is

the push-forward measure to $\mathcal{A}$/GT then

$$\int_{\mathcal{A}/GT} f\bar{\mu} = \frac{1}{Z}\int_S f|_S\nu,$$

where $Z = \int_S \nu$.

Definition 2.2. If the measure ν is a finite measure on S then define $\bar{\mu}$ to be the probability measure on $\mathcal{A}$/GT satisfying for all bounded gauge invariant functions (f):

$$\int_{\mathcal{A}/GT} f\bar{\mu} \equiv \frac{1}{Z}\int_S f|_S\nu,$$

where $Z = \int_S \nu$ is a normalization constant. (Note: μ is not assumed to be a finite measure now.)

Corollary 2.2. Assume that $h:\mathcal{A} \longrightarrow \mathbb{R}$ is a function such that $\int_{GT} h(gA)\nu(dg) \equiv K$, where K is a constant independent of $A\in\mathcal{A}$. If f is a gauge invariant function on $\mathcal{A}$, then

$$\int_{\mathcal{A}/GT} f\bar{\mu} = \frac{1}{Z}\int_{\mathcal{A}} f\cdot h\mu,$$

where $Z = \int h\mu$ is the normalization constant.

Proof:

$$\int_{\mathcal{A}} fh\mu = \int_{GT\times S} f(gs)h(gs)\omega(dg)\nu(ds) = \int_{GT\times S} f(s)h(gs)\omega(dg)\nu(ds)$$
$$= K\int_S f(s)\nu(ds)$$

from which the result easily follows. Q.E.D.

Example 2.1. Let $GT = (\mathbb{R}^+,\cdot)$ be the multiplicative group of positive real numbers acting on $\mathcal{A} = \mathbb{R}^2\setminus\{0\}$ by scalar multiplication, $S = S^1$ be the unit circle, $\mu = dx\wedge dy/r^2$ $(r^2 = x^2 + y^2)$, and $\omega = dt/t$. Using Proposition 2.1. one finds that $\nu_p = p_1 dx - p_2 dy = \text{"}d\theta\text{"}_p$ for all $p\in S$ and $\mu = \omega\wedge d\theta = dr\wedge d\theta/r$ which is equivalent to the well known fact that $dx\wedge dy = r dr\wedge d\theta$.

Example 2.2a. Let $\mathcal{A} = \mathbb{R}^n$, $\mu = \exp \frac{-1}{2}(Bx,x)\, dx$ where B is a positive semi-definite matrix, GT = Nul(B) acting on $\mathcal{A}$ by translation. Suppose that C is a positive semi-definite matrix such that A is the direct sum of the Nul(B) and the Nul(C). Then it is easy to check that $h(x) = \exp \frac{-1}{2}(Cx,x)$ satisfies the hypothesis of Corollary 2.2. so that for any function f on $\mathbb{R}^n$ invariant under translations by Nul(B),

$$\int_{A/\mathrm{Nul}(B)} f\bar{\mu} = \frac{1}{Z}\int_A f(x)\exp -\frac{1}{2}((B + C)x,x)dx,$$

where Z is again a normalization constant.

__Example 2.2b.__ (Free Euclidean Electro-Magnetic Field.) In this example we use the results of Example 2.2a. to interpret the expression (YM) as a Gaussian measure when the structure group G = U(1) and the Lie algebra $\mathcal{G} = i\mathbb{R}$. The expression (YM) reduces to

$$d\mu(A) = Z^{-1}\exp[-(dA,dA)/2]DA. \qquad \text{(EM1)}$$

Now every smooth gauge transformation $g:\mathbb{R}^d \longrightarrow U(1)$ is of the form $g = e^{-i\lambda}$, where $\lambda:\mathbb{R}^d \longrightarrow \mathbb{R}$ is a smooth function such that $\lambda(0) = 0$. This follows from Poincare's lemma and the fact that U(1) is abelian, so that $d(gdg^{-1}) = 0$. The function λ is unique since $\mathbb{R}^n$ is simply connected. We will identify the function λ with its corresponding gauge transformation $g = e^{-i\lambda}$, and let $\lambda A = gA = A + d\lambda$.

Using Example 2.2a. as our guide we are led to replace (EM1) by

$$d\mu(A) = Z^{-1}\exp(-((d^*d + dd^*)A,A)/2]DA, \qquad \text{(EM2)}$$

where d^* is the adjoint of d. One should think of B as d^*d and C as dd^*. (If one ignores domain questions it is easy to see that A is the direct sum of $\mathrm{Nul}(d^*) = \mathrm{Nul}(dd^*)$ and $\mathrm{Nul}(d) = \mathrm{Nul}\ (d^*d)$.) The expression in (EM2) should capture the meaning of (YM) provided only gauge invariant functions are integrated. The exponent in (EM2) is now a non-degenerate bi-linear form. Therefore, by Minlo's Theorem, see Simon [S], the expression (EM2) may be interpreted as a mean zero Gaussian measure on the space A of generalized imaginary-valued 1-forms with characteristic functional

$$\int_A e^{iA(j)}\mu(dA) = e^{-\frac{1}{2}(\Delta^{-1}j,j)},$$

where $\Delta = dd^* + d^*d$, $j = \Sigma j_i dx^i$ is a smooth compactly supported $i\mathbb{R}$ - valued 1-form on $\mathbb{R}^d$, and $A(j) = \Sigma_{i=1}^d A_i(j_i)$ when $A = \Sigma_{i=1}^d A_i dx^i$ with the A_i's being $i\mathbb{R}$ - valued distributions.

__Remark 2.1.__ A necessary and sufficient condition for the function $A \longrightarrow A(j)$ to be gauge invariant is that $d^*j = 0$, since $(A + d\lambda)(j) \equiv A(j) + \lambda(d^*j)$ must be equal to A(j). For example; $A \longrightarrow F^A(\varphi) \equiv dA(\varphi) \equiv A(d^*\varphi)$ is gauge invariant for all test 2-forms (φ), since $d^*d^*\varphi = 0$. If φ is closed ($d\varphi = 0$) then

$$\int e^{F^A(\varphi)}\mu(dA) = \exp[(\varphi,\varphi)/2],$$

because $(\Delta^{-1}d^*\varphi,d^*\varphi) = (\Delta^{-1}dd^*\varphi,\varphi) = (\Delta^{-1}\Delta\varphi,\varphi) = (\varphi,\varphi)$. In particular, if d = 2, then φ is always closed and we see that $A \longrightarrow iF^A$ is a real valued white noise. (See section 3.)

__Example 2.3a.__ Suppose that $A = \mathbb{R}^n$, GT is a Lie group, μ is Lebesgue measure on A, and S is a linear subspace of A. Further assume that the action of GT on A has the form $g\cdot A = \rho(g)A + C(g)$, where $C:GT \longrightarrow A$, and $\rho:GT \longrightarrow \mathrm{Aut}(A)$ is a representation of GT for which the $|\det(\rho)| \equiv 1$ and S is an invariant subspace. Clearly Lebesgue measure is invariant under this action. We will now show that the measure ν on S given by Proposition 2.1. is a Lebesgue measure. Let $\{\xi_i\}$ be a basis for the Lie algebra of GT (thought of as the tangent space to the identity) and let $\{v_j\}$ be a basis for S which we identify with a basis of the tangent space to s at S. Since $\xi\cdot s = \rho_*(\xi)s + C_*(\xi)$ and $\rho_*(\xi)s$ in S for all ξ, it follows that

$$\nu_s(v_1,\cdots,v_r) = \mu(\xi_1 s,\cdots,\xi_p s,v_1,\cdots v_r)$$
$$= \mu(C_*\xi_1,\cdots,C_*\xi_p,v_1,\cdots,v_r)$$

which is independent of $s\in S$. This shows that ν is a Lebesgue measure on S. Q.E.D.

Example 3.3b. (Complete Axial Gauge Fixing of (YM)) Let $\mathcal{A}$ be the space of $\mathcal{G}$-valued connection 1-forms on $\mathbb{R}^d$, S be the linear subspace of $\mathcal{A}$ containing the elements $A = \Sigma_{i=1}^d A_i dx^i$ with $A_i(x_1,\cdots,x_i,0,\cdots,0) \equiv 0$ for all $(x_1,\cdots,x_i)$ in $\mathbb{R}^i$, μ be the informal expression given by (YM), GT be the space of gauge transformations acting on $\mathcal{A}$ as described in the introduction. An element A of S is said to be in the complete axial gauge.

Proposition 2.2. The natural map φ:GT×S $\longrightarrow \mathcal{A}$ is a bijection.

Proof: Given $A\in\mathcal{A}$ and $x\in\mathbb{R}^d$, let $g(x) = P_1^A(\sigma_x) \equiv p(1)$ where $p(t) = P_t^A(\sigma_x)$ is the G - valued solution to the ordinary differential equation for parallel translation:

$$\dot{p}(t) + A(\dot{\sigma}_x(t))p(t) = 0$$

with initial condition p(0) = Id in G. (We write $A(v) = \Sigma_i A_i(p)v^i$ is $v\in T_p\mathbb{R}^d$.) Here σ_x is the path in $\mathbb{R}^d$ going from 0 to x by traversing the polygonal path: $0 = (0,\cdots,0) \longrightarrow (x_1,0,\cdots,0) \longrightarrow (x_1,x_2,0,\cdots,0) \longrightarrow \cdots \longrightarrow (x_1,\cdots,x_{n-1},0) \longrightarrow (x_1,\cdots,x_n) = x$. Now given $h\in$GT, a connection 1-form A, and a curve σ in $\mathbb{R}^d$, it is easy to check that $t \longrightarrow h(\sigma(t))P_t^A(\sigma)h(\sigma(0))^{-1}$ satisfies the same differential equation as $P_t^{hA}(\sigma)$ so that

$$P_t^{hA}(\sigma) = h(\sigma(t))P_t^A(\sigma)h(\sigma(0))^{-1}.$$

Using this last equation (with $h = g^{-1}$ and $\sigma = \sigma_x$) and the fact that parallel translation is parametrization

independent one finds that $P_t^{g^{-1}A}(\sigma_x) = id$. Differentiating this last equality with respect to t and use the defining equation for parallel translation we conclude that $(g^{-1}A)(\dot{\sigma}_x) = 0$ for all x in $\mathbb{R}^d$ from which it easily follows that $g^{-1}A$ is in the complete axial gauge. This shows that φ is surjective. To see that the map is injective suppose gA = hB where A and B are in S and that g, and h are in GT. Now for A in S, $A(\dot{\sigma}_x) \equiv 0$, and thus $P_t^A(\sigma_x) = id$ for all x in $\mathbb{R}^d$. Therefore, $P_1^{gA}(\sigma_x) = g(x)g(0)^{-1} = g(x)$, since g(0) = id by definition of GT. Similarly, $P_1^{hB}(\sigma_x) = h(x)$, which implies that $g(x) = P_1^{gA}(\sigma_x) = P_1^{hB}(\sigma_x) = h(x)$ and in turn this implies that $A = g^{-1}gA = h^{-1}hB = B$. Hence φ is injective. Q.E.D.

So informally Example 2.3b. is the infinite dimensional analogue Example 2.3a. This motivates replacing the informal expression (YM) for μ by

$$d\nu(A) = Z^{-1}e^{-(F^A,F^A)/2}DA, \qquad \text{(YMG)}$$

the same expression as before but A is now restricted to be in the complete axial gauge. Of course this expression is still informal, however when G = U(1) or d = 2 the curvature F^A is equal to dA and the factor in the exponent is quadratic in A. The expression for ν is then easily given meaning as an infinite dimensional Gaussian measure. The case d = 2 will be the topic of the next section.

3. d = 2 YM-Measure and Expectations.

In this section, for d = 2, I will explain how to interpret the expression (YMG) as a Gaussian measure and then how to compute gauge invariant expectations. For a more detailed analysis of this measure see [D] and [GKS].

For a construction of the Yang-Mills measure on the two sphere see Sengupta [Se].

For the rest of this paper the space-time dimension (d) will be fixed at two. Because d = 2, for $A \in S$ (S = the space of connections in the complete axial gauge as in Example 2.3b) we see that $F = dA = -\frac{\partial a}{\partial y} dx \wedge dy$, where $A = a\,dx$ and (x,y) are the usual cartesian coordinate functions on $\mathbb{R}^2$. To simplify notation identify F with $\frac{\partial a}{\partial y}$. After making the linear change of variables $F = \frac{\partial a}{\partial y}$ in (YMG) we find an informal Gaussian expression for the distribution of the F-variables :

$$d\nu(F) = Z^{-1}\exp\left\{\frac{-1}{2}(F,F)\right\}DF.$$

(Formally the Jacobian of the transformation is a constant which is canceled by the normalization constant.) This last expression suggests the interpretation of F as a mean-zero generaliled Gaussian random process indexed by $L^2(\mathbb{R}^2)$ with covariance $E(F^a(f)F^b(g)) = (f,g)_{L^2(\mathbb{R}^2)}\delta_{ab}$, where $f,g \in L^2$, $F^a = \text{trace}(T^aF)$, and $(T^a)_{a=1}^{\dim(\mathfrak{G})}$ is a basis for $\mathfrak{G}$ such that $\text{trace}(T^aT^b) = -\delta_{ab}$. In other words, the F^a's are independent $\mathbb{R}$-valued white noises on $\mathbb{R}^2$. Now given F we may recover the process (a) by the formula: $a(x,y) = \int_0^y F(x,y)dy$ which is to be interpreted in the sense of generalized functions.

It should be noted for non-abelian G that the curvature F is not a gauge invariant function on $\mathcal{A}$. In order to construct measurable gauge invariant functions it is necessary to construct parallel translation with respect to the random connection form A = a dx. For this we first consider "horizontal" curves $\sigma(x) = (x,\underline{\sigma}(x))$ where $\underline{\sigma}$: $[a,b] \longrightarrow \mathbb{R}$ is a continuous function. Given A in the complete axial gauge and horizontal curve σ, we have $A(\dot{\sigma}(t)) = \frac{d}{dt}\int_a^t dx \int_0^{\underline{\sigma}(\tau)} dy\, F(x,y) = \frac{d}{dt}F(f_t^\sigma)$, where f_t^σ is the function on $\mathbb{R}^2$ which is 1 (-1) on the region above (below) the x-axis and bounded by the vertical lines x = a and x = t. Unfortunately the process $t \longrightarrow F(f_t^\sigma)$ is not differentiable so the above computation is only formal. However, $t \longrightarrow F(f_t^\sigma)$ is a martingale with a continuous version and so we may interpret the differential equation for parallel translation as the stochastic differential equation:

$$dP_t(\sigma) + dF(f_t^\sigma)\circ P_t(\sigma) = 0$$

with initial condition $P_a(\sigma) = id$, where "$\circ$" denotes the Stratonovich multiplication of differentials which is necessary to insure that P_t remains in the structure group G. We now set $P(\sigma) = P_b(\sigma)$, which defines parallel translation along left to right moving horizontal curves as a G-valued random process. If σ is a horizontal curve which is oriented from right to left, set $P(\sigma) = P(\bar{\sigma})^{-1}$ where $\bar{\sigma}$ denotes the curve σ with the opposite orientation. For paths (σ) which are vertical line segments (ie. whose x-components remain constant) we set $P(\sigma) = id$ -- recall that we are in the complete axial gauge so $A_2 \equiv 0$. Now call a path (σ) admissible if σ can be broken into pieces $\sigma_1,\cdots,\sigma_n$ in such a way that each piece is either a vertical line segment or a horizontal path. Write $\sigma = \sigma_1\cdots\sigma_n$ if σ is the path constructed by traversing σ_n then σ_{n-1} , $\cdots$,and then σ_1. (We assume that the final point of σ_{i+1} matches the initial point of

σ_i.) For an admissible path σ decomposed as above, we define $P(\sigma) = P(\sigma_1)\cdots P(\sigma_n)$. This defines parallel translation as a G-valued random process index by the class admissible curves.

For simplicity, I will specialize G to be U(1). In this case there is only one T^a which is taken to be $T^1 = i$ and the process iF is a real valued white noise on $\mathbb{R}^2$. If σ is a horizontal curve lying above the x-axis, it is easy to check that the explicit solution to the Stratonovich stochastic differential equation for parallel translation is $P_t(\sigma) = e^{-F(f_t^\sigma)} = e^{i(iF(f_t^\sigma))}$. As an easy consequence, for an admissible simple closed curve (σ) one finds $P(\sigma) = e^{\varepsilon F(R)}$, where R is the bounded component of $\mathbb{R}^2\backslash$(Image of σ), ε is (+1) if σ is traversed counter-clockwise and (-1) otherwise, and $F(R) \equiv F(1_R)$.

Theorem 3.1. Suppose that G = U(1), d = 2, $\Sigma = \{\sigma_i\}_{i=1}^k$ is a collection of simple closed curves in the plane, and $f:U(1)^k: \longrightarrow \mathbb{R}$ is a bounded measurable function. Let $\hat{\Sigma}$ be the collection of bounded components (R) of $\mathbb{R}^2\backslash$(images of the σ_i's), $\hat{\sigma}_i = \{R\in\hat{\Sigma} | R$ is contained inside $\sigma_i\}$, and

$$Q_R(e^{i\theta}) = \Sigma_{n=-\infty}^{\infty} (2\pi/|R|)^{1/2}\exp -(\theta - 2\pi n)^2/2|R|,$$

where $|R|$ denotes the area of R. Then

$$Ef(P(\sigma_1),\cdots,P(\sigma_k)) = \int_{U(1)^{\hat{\Sigma}}} f(\Pi_{R\in\hat{\sigma}_1} z_R,\cdots,\Pi_{R\in\hat{\sigma}_k} z_R)\Pi_{R\in\hat{\Sigma}} Q_R(z_R)dz_R,$$

where $dz = "d\theta/2\pi"$ is normalized Haar measure on U(1).

Proof: The proof of this theorem is quite straight forward so it will be omitted -- but see Example 3.1. below. Let me just say that the appearance of the function Q_R is a direct result of the following easily verified equality:

$$\int_{\mathbb{R}} f(e^{ix})\cdot(2\pi|R|)^{-1/2}e^{-x^2/2|R|}dx = \int_{U(1)} f(z)Q_R(z)\,dz,$$

where f is any bounded function on U(1). Q.E.D.

Remark 3.1. The significance of the function Q_R lies in the fact that it is the convolution heat kernel for the Laplacian on the Lie group U(1).

By considering a few simple examples for $\Sigma = \{\sigma_1,\cdots,\sigma_k\}$ the reader should be able to convince him/herself that Theorem 3.1. can be expressed in the following form (again see Example 3.1.):

Corollary 3.1. Let G = U(1), d = 2, $\Sigma = \{\sigma_1,\cdots,\sigma_k\}$ as above, and $\mathcal{B} = \mathcal{B}\Sigma$ denote the directed planer graph with vertices $V = V\Sigma$ given by the intersection points of the curves in Σ and bonds consisting of portions of the curves in Σ which join any two vertices. We will identify each curve σ_i with the directed path of bonds in $\mathcal{B}\Sigma$ in to which σ_i decomposes. Also let $U(1)^{\mathcal{B}} = \{z:\mathcal{B} \longrightarrow U(1) | z(b) \equiv z(\bar{b})^{-1}$ for all $b\in\mathcal{B}\}$ where if b is a bond in $\mathcal{B}$ then $\bar{b}$ denotes the same bond with the opposite orientation. Then

$$Ef(P(\sigma_1),\cdots,P(\sigma_k)) = \int_{U(1)^{\mathcal{B}}} f(z(\sigma_1),\cdots,z(\sigma_k))\Pi_{R\in\hat{\Sigma}} Q_R(z(\partial R))\cdot\Pi_{b\in\mathcal{B}'} dz(b),$$

where , $z(\sigma) \equiv z(b_1)\cdots z(b_m)$ if $\sigma = b_1\cdots b_m$ is a directed path of bonds in $\mathcal{B}$, ∂R denotes any path of bonds around the boundary of the region $R\in\hat{\Sigma}$, dz(b) is normalized Haar measure on U(1), and $\mathcal{B}'$ is a subset of $\mathcal{B}$ such that for each $b\in\mathcal{B}$ either (but not both) b or $\bar{b}$ is in $\mathcal{B}'$.

Example 3.1. Let $\Sigma = \{\sigma_1,\sigma_2\}$ be two concentric counter clockwise oriented curves with σ_1 inside of σ_2. Let R be the region inside σ_1 and S be the region between σ_2 and σ_1. Then $P(\sigma_1) = e^{F(R)}$, and $P(\sigma_2) = e^{F(R\cup S)} = e^{F(R)+F(S)} = e^{F(R)}e^{F(S)}$. Now the variables F(R) and F(S) are independent mean - zero Gaussian random variables with covariances $|R|$ and $|S|$ respectively. For $R\subset\mathbb{R}^2$ with $|R|<\infty$, set $p_R(x) = (2\pi|R|)e^{-x^2/2|R|}$ to be the Gaussian density with variance $|R|$, then

$$
\begin{aligned}
Ef(P(\sigma_1),P(\sigma_2)) &= \int_{\mathbb{R}^2} f(e^{i\theta}, e^{i\theta}e^{i\alpha})p_R(\theta)p_S(\alpha)\,d\theta\, d\alpha \\
&= \int_{U(1)^2} f(z_R, z_R z_S)Q_R(z_R)Q_S(z_S)\,dz_R\, dz_S \\
&= \int_{U(1)^2} f(z_1, z_2)Q_R(z_1)Q_S(z_1^{-1}z_2)\,dz_1\, dz_2,
\end{aligned}
$$

where the third equality is the result of renaming $z_R = z_1$, $z_S = z_2$, and using the invariance of Haar measure to make the change of variables $z_2 \longrightarrow z_1^{-1}z_2$. Observe that the second line is expressed in the form given by Theorem 3.1. and the third line is expressed in the form given by Corollary 3.1. after identifying z_i with $z(\sigma_i)$. (Note: $z(\partial S) = z_1^{-1}z_2$, and $z(\partial R) = z(\sigma_1) = z_1$.)

It turns out that the expression for the expectation given in Corollary 3.1. correctly generalizes to the case of non-abelian structure groups (G). Before stating the theorem we need some notation. Let $\mathcal{B}$ be a directed planar graph with vertices V, for example $\mathcal{B} = \mathcal{B}\mathcal{L}$ as above. Set $G^{\mathcal{B}} = \{z:\mathcal{B} \longrightarrow G \mid z(b) = z(\bar{b})^{-1}\}$, and $G^V = \{g:V \longrightarrow G\}$ which acts on $G^{\mathcal{B}}$ by $(g\cdot z)(b) = g(b^f)z(b)g(b^i)^{-1}$, where b^f and b^i are the initial and final points respectively of the bond b in $\mathcal{B}$.

<u>Definition</u> <u>3.1.</u> A function $f:G^{\mathcal{B}} \longrightarrow \mathbb{R}$ is gauge invariant if $f(gz) = f(z)$ for all $(g,z)\in G^V \times G^{\mathcal{B}}$.

<u>Remark</u> <u>3.2.</u> If A is a smooth connection 1-form on $\mathbb{R}^2$, then $z = P^A|_{\mathcal{B}}$ is an element of $G^{\mathcal{B}}$ and Definition 3.2. is precisely what is required on the function f to guarantee that $\hat{f}(A) \equiv f(P^A|_{\mathcal{B}})$ is a gauge invariant function on $\mathcal{A}$. Finally, if $f(z) = f(z(\sigma_1),\cdots,z(\sigma_k))$ (abuse of notation) as in Theorem 3.1. and Corollary 3.1., and if $G = U(1)$, then the function f(z) is automatically gauge invariant.

We now come to the main result which is the non-abelian version of Corollary 3.1.

<u>Theorem</u> <u>3.2.</u> Let $\mathcal{B}$ be a directed planer graph on $\mathbb{R}^2$ with vertices V and suppose that $f:G^{\mathcal{B}} \longrightarrow \mathbb{R}$ is a gauge invariant function, then

$$
Ef(P|_{\mathcal{B}}) = \int_{G^{\mathcal{B}}} f(z)\Pi_R\, Q_R(z(\partial R))\cdot \Pi_{b\in\mathcal{B}'} dz(b),
$$

where the product Π_R is over all bounded connected regions in $\mathbb{R}^2\backslash$(the union of the bonds in $\mathcal{B}$), dz(b) is Haar measure on G, and Q_R is the convolution semi-group heat kernel on G at time $t \doteq |R|$.

The proof of this theorem is considerably more involved than the abelian version. For the details of the proof and the proof of convergence of the corresponding lattice gauge models the reader is referred to [D] and [GKS].

<u>References</u>

[B] Balaban, T.: Renormalization group approach to lattice gauge fields theories. I. Generation of effective actions in a small fields approximation and a coupling constant renormalization in four dimensions. Commun. Math. Phys. 109, 249 - 301 (1987) and see the references therein.

[D] Driver, B. K.: YM_2:Continuum expectations, lattice convergence, and lassos. Commun in Math. Phys. 123, 575-616 (1989)

[F] Federbush, P.: A phase cell approach to Yang-Mills theory, IV. The choice of variables. Commun. Math. Phys. 114, 317-343 (1988) Also see the references therein and P. Federbush's contribution to these proceedings.

[GKS] Gross L., King, C., Sen Gupta, A.: Two dimensional Yang-Mills via stochastic differential equation, Ann. Phys. 194, 65 - 112 (1989)

[J] Jaskolski, Z.: The integration of G-invariant functions and the geometry of the Faddeev-Popov procedure. Commun. in Math. Phys. 111, 439-468 (1987)

[Se] Sengupta, A.: The Yang-Mills measure for the two sphere. (Cornell University Thesis 1990)

[S] Simon, B.: Functional integration and quantum physics. New York: Academic Press 1979.

Construction of the Gross-Neveu Model in Three Dimensions[1]

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The work we present below has been done in collaboration with C. de Calan, J. Magnen and R. Sénéor from the Ecole Polytechnique (Palaiseau/France) and will appear soon[1].

Let us begin by making a series of heuristic arguments before stating our result. We consider the massive Gross-Neveu model, which is formally defined by the Lagrangian

$$L = \overline{\psi}(x)\,(i\zeta\partial\!\!\!/ + m)\,\psi(x) + \lambda\,(:\overline{\psi}(x)\,\psi(x):)^2/2N \qquad (1)$$

where, for x being a point in $\mathbb{R}^3$ and $N \in \mathbb{N}$, $\overline{\psi}(x)$ and $\psi(x)$ are N-flavor component fermionic fields (a set of Grassman random variables), and m, ζ and $\lambda \in \mathbb{R}_+$ are the fermion mass, the field strength (re-)normalization and the coupling constant respectively. Finally, : : denotes a Wick ordering with respect to the free-fermion gaussian measure to be referred to below, and $\partial\!\!\!/ = \partial_\alpha\gamma_\alpha$ (α=0,1,2 and the γ_α are chosen among the 4x4 Dirac matrices).

The fact that, for dimension d>2, λ has a positive dimension in mass implies that the usual perturbation expansion in λ for the model is non-renormalizable and the divergences appearing in the high-momentum or

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