CONVERGENCE OF THE FOUR DIMENSIONAL U(1) LATTICE GAUGE THEORY TO ITS CONTINUUM LIMIT

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It is shown that in four space-time dimensions the compact U(1) lattice gauge theory with general energy function converges to a renormalized free electro-magnetic field on the current sector as the lattice spacing approaches zero, provided the coupling constant is sufficiently large. For the Wilson energy function, it is possible, by judicious choice of the Gibbs state, to get convergence for arbitrary coupling strengths. Furthermore, for all but a countable number of values of the coupling constant, the limit exists and is independent of the particular state chosen to define the lattice model.

Biographical Sketch

Bruce Driver was born February 27, 1960, in Minneapolis, Minnesota. He attended the University of Massachusetts at Amherst from 1977 to 1981 obtaining a B.S. in Physics. Since then he has been a graduate student at Cornell University in Ithaca, New York, earning an M.S. in Applied Mathematics in 1984 and then a Ph.D in 1986.

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Frequently Used Notation

$\mathcal{B}$	the set of oriented bonds on Z^d
$\mathcal{B}_+$	the set of positively oriented bonds on Z^d
$(\cdot,\cdot)$	the natural bilinear form on differential or lattice forms
C^k	denotes a function which is k-times continuously differentiable
$C^k(M)$	the set of C^k-functions on M
$C_c^k(M)$	the subset of functions of $C^k(M)$ with compact support
d	exterior derivative, exterior lattice derivative, or the dimension of the space
d^*	the adjoint of d with respect to the bilinear form $(\cdot,\cdot)$
$\mathcal{F}(\Lambda)$	the σ-algebra generated by the lattice variables over Λ
$\mathcal{F}_\Lambda$	the σ-algebra generated by the lattice variables outside of Λ
$\lvert\Lambda\rvert$	the number of elements in the set Λ
$\Lambda\subset\subset L$	$\Lambda\subset L$ and $\lvert\Lambda\rvert < \infty$
$\mu(f)$	$\int f d\mu$, the integral of f with respect to μ
$\mathcal{P}$	the set of oriented plaquettes on Z^d
$\mathcal{P}_+$	the set ot positively oriented plaquettes on Z^d
$S(\mathbb{R}^d)$	the space of C^∞-functions with rapid decrease
$S'(\mathbb{R}^d)$	the space of tempered distributions on $\mathbb{R}^d$

Chapter 1

INTRODUCTION

Section 1.1 The Yang-Mills' Measure

The "quantized" Yang-Mills' fields are widely taken as the basic mathematical model for the description of the four known forces of nature; strong, electro-magnetic, weak, and the gravitational force. The problem of mathematical existence of quantized Yang-Mills' fields is (on an informal level) equivalent to defining a certain probability measure on a space of connection forms. The informal description of this (Yang-Mills') measure is

$$d\mu(A) = Z^{-1}\exp\left[\frac{1}{2g^2}\int_{\mathbb{R}^d}\sum_{i<j}\text{trace}(F^A_{ij}(x)^2)dx\right]DA, \quad (1.1.1)$$

where A runs over a space of connection forms ($\underline{A}$) on the trivial vector bundle $\mathbb{C}^N \times \mathbb{R}^d$, $F^A = dA + A\wedge A$ is the curvature of A, $DA = \prod_{i=1}^{d}\prod_{x\in\mathbb{R}^d} d(A_i(x))$ is "infinite dimensional Lebesque measure" on ($\underline{A}$), g^2 is a positive "coupling" constant, and Z is a normalization constant which makes μ a probability measure. The connection forms are restricted to take their values in the Lie Algebra ($\underline{G}$) of the structure (or gauge) group (G), which is taken to be a closed subgroup of U(N). See Appendix B and especially equation (B.2.9) for motivation of (1.1.1).

The informal description of μ has a number of ailments.

First, the infinite dimensional Lebesque measure is not well defined. Second, the measure (μ) will certainly give zero measure to the space of differentiable or even continuous connection forms. Third, the normalization constant (Z) in similar situations has the interpretation of being either zero or infinite.

Besides these technical problems, the informal description of μ suffers from the algebraic property of "gauge invariance". The exponent of (1.1.1) is invariant under gauge transformations (Kobayashi and Nomizu [1]), that is under the mapping $A \longrightarrow A^g$. Here $g:\mathbb{R}^d \longrightarrow G$ is a C^∞ function called a gauge transformation, and $A^g(x) = g(x)^{-1}A(x)\, g(x) + g(x)^{-1}dg(x)$. (Geometrically, a gauge transformation is a change of global trivialization of the trivial vector bundle $\mathbb{C}^N \times \mathbb{R}^d$.) Due to this gauge invariance, if μ is to have a chance of being a probability measure, equation (1.1.1) must be interpreted as describing a measure on a space of gauge equivalence classes of connection forms. See L. Gross [4] and [5] and the references therein for more details.

Ignoring these details, it is expected (see Appendix B) that from a precise description of the measure (μ), one will be able to construct the quantum mechanical system corresponding to the relativistic Yang-Mills' equations — i.e. quantized Yang-Mills' fields. At this time it is impossible to formulate a precise theorem to this effect, since the definition and existence of the measure (μ) is still unknown for non-abelian gauge groups in space-time dimension $d \geq 3$ (but see Section 1.6). Hence the first step in quantizing Yang-Mills' fields is to make precise mathematical sense of the informal description of the measure μ in (1.1.1).

Section 1.2 Wilson's Lattice Approximation

As shown in Appendix B it is often possible to make sense out of informal expressions of the form (1.1.1) by "approximating" the measure with other measures that are better understood. These approximating measures should preserve as many symmetries of the theory as possible, so that the symmetries persist after taking limits. In particular for the Yang-Mills' measure it is desirable that the approximating measures are gauge invariant.

The lattice approximation of Ken Wilson [1] nicely preserves gauge invariance. The basic idea is to replace $\mathbb{R}^d$ by the lattice aZ^d $(a > 0)$, and the connection form A by its corresponding parallel translation operators. For motivation, let A be a C^∞ connection form with compact support. Recall (Kobayashi and Nomizu [1]) that

$$P^A(\partial p) = I + a^2 F^A_{ij}(x) + O(a^3), \qquad (1.2.1)$$

where $P^A(\partial p)$ denotes parallel translation with respect to the connection form A around the curve (∂p), and $O(a^3)/a^3$ remains bounded as $a \longrightarrow 0$. The curve (∂p) traverses the edge (in the correct direction) of an $a\times a$-dimensional planar square (p) parallel to the i'th and j'th coordinate directions which contains the point x. Such a square (p) is called a plaquette. Hence

$$\begin{aligned} F^A_{ij}(x)^2 &= -F^A_{ij}(x)F^A_{ij}(x)^* \\ &= -a^{-4}[P^A(\partial p) - I][P^A(\partial p) - I]^* + O(a) \\ &= 2a^{-4}[\mathrm{Re}P^A(\partial p) - I] + O(a), \end{aligned} \tag{1.2.2}$$

where $O(a)/a$ remains bounded as $a \longrightarrow 0$. The fact that G is a subgroup of U(N) (so $\underline{G}$ is a subspace of anti-hermetian matricies) has been used in the first and third equalities. With a little more care, it can be shown that

$$\int_{\mathbb{R}^d} \sum_{i<j} \mathrm{trace}(F^A_{ij}(x)^2)dx = a^{d-4} \sum_{p} \mathrm{trace}[\mathrm{Re}P^A(\partial p) - I] + O(a), \tag{1.2.3}$$

where the sum is over all "oriented" plaquettes (p) with corners in the lattice aZ^d.

To define the approximating measures we need the following notation which is covered in more detail in Chapter 2. Let $(\mathcal{B}_+)$ $\mathcal{B}$ denote the set of (positively) "oriented" bonds on Z^d, and $(\mathcal{P}_+)$ $\mathcal{P}$ the set of (positively) oriented plaquettes on Z^d. Let $\Omega \equiv \{\omega:\mathcal{B}_+ \longrightarrow G\}$, where $\omega(b)$ should be thought of as parallel translation $(P^A(b))$ along the oriented bond b. The configuration space (Ω) is identified with $\{\omega:\mathcal{B} \longrightarrow G$, such that $\omega(-b) = \omega(b)^{-1}\}$, where $-b$ denotes the bond b with the opposite orientation. If $p\in\mathcal{P}$ is a plaquette, we put $\mathrm{trace}[\omega(\partial p)] \equiv \mathrm{trace}[\omega(b_4)\omega(b_3)\omega(b_2)\omega(b_1)]$ (corresponding to $P^A(\partial p)$), see Figure (1.2.1). Because of the trace, $\mathrm{trace}[\omega(\partial p)]$ is independent of the corner from which we begin to traverse the edge of p.

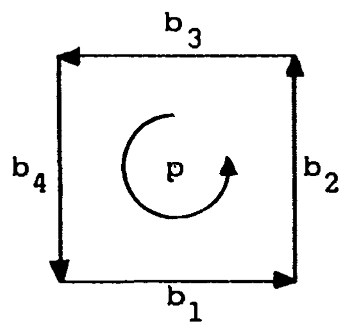

Figure 1.2.1

With this notation and the motivation of equation (1.2.3) it is reasonable to "approximate" (1.1.1) by,

$$d\mu_a(\omega) = Z_a^{-1}\exp\left\{a^{(d-4)}\frac{1}{2g^2}\sum_{p\in\mathcal{P}}\mathrm{trace}[\mathrm{Re}\,\omega(\partial p) -I]\right\}d\nu(\omega), \tag{1.2.4}$$

where $d\nu(\omega) = \prod_{b\in\mathcal{B}_+} d\lambda(\omega(b))$ with λ normalized Harr measure on G, and Z_a is the normalization constant to make μ_a a probability measure. The expression (1.2.4) is to be interpreted as a Gibbs state, see Appendix A, and Chapter 2. Also the lattice aZ^d has now been implicitly identified with the lattice Z^d. This identification leads to technical simplifications later on.

The ultimate goal is to show the measures μ_a converge in some sense to a measure μ with desirable properties, such as gauge invariance, Euclidean invariance, "reflection positivity", etc. This problem has proved to be very difficult in general. However, the special case where the gauge group is U(1) provides good testing grounds for techniques of proving convergence. The reason is that the measure μ of (1.1.1) may be understood as an infinite dimensional Gaussian measure. Before demonstrating this well known fact, we first specialize (1.2.4) to the case that G = U(1).

When G = U(1), it is convenient to identify U(1) with $[-\pi,\pi]$ with the end points identified. Under this identification ω is replaced by $e^{i\omega}$, where now $\omega:\mathcal{B}_+\longrightarrow \mathbb{R}$. The Harr measure on U(1) simply becomes normalized Lebesque measure (λ) on $[-\pi,\pi]$, and the measure in (1.2.4) then becomes

$$d\mu_a(\omega) = Z_a^{-1}\exp\left\{a^{(d-4)}\frac{1}{2g^2}\sum_{p\in\mathcal{P}}[\cos(\omega(\partial p)) - 1]\right\}d\nu(\omega), \tag{1.2.5}$$

where $d\nu(\omega) = \prod_{b\in\mathcal{B}_+} d\lambda(\omega(b))$.

Section 1.3 U(1)-Model as a Gaussian Measure

We now specialize (1.1.1) to the case where the vector bundle is $\mathbb{C}\times\mathbb{R}^d$ and the gauge group is U(1). We also make a slight change of conventions by replacing A by iA. This makes A real, since the Lie algebra of U(1) is $i\mathbb{R}$. Expression (1.1.1) then becomes

$$d\mu(A) = Z^{-1}\exp[\frac{-1}{2g^2}(dA,dA)]DA, \tag{1.3.1}$$

where $(\cdot,\cdot)$ denotes the natural bilinear form (see equation (2.2.3)) on the compactly supported differential forms on $\mathbb{R}^d$. (Equation (1.3.1) is the same as (B.2.9) of Appendix B.) Hence the connection form (A) only enters in the exponent quadratically, so that the (1.3.1) would be interpretable as an infinite dimensional Gaussian measure (as in Section B.2) if it were not for the gauge problem mentioned in Section 1.1.

Every smooth gauge transformation $g:\mathbb{R}^d \longrightarrow U(1)$ is of the form $g = e^{i\lambda}$, where $\lambda:\mathbb{R}^d \longrightarrow \mathbb{R}$ is a smooth function. This follows by Poincare's lemma (Spivak [1]) and the easily verified fact that $d(g^{-1}dg) = 0$, since U(1) is abelian. (The function λ is unique up to an integer multiple of 2π.) We will identify the function λ with its corresponding gauge transformation $g = e^{i\lambda}$, and write A^λ for A^g. In this case $A^\lambda = A + d\lambda$. Hence, $[A] = \{A + d\lambda\colon \lambda:\mathbb{R}^d \longrightarrow \mathbb{R}\}$ is the gauge equivalence class associated to the connection form A. As has already been noted, it is such a space of [A]'s on which the measure μ should be defined.

In order to see how to proceed, I will consider the analogous finite dimensional situation. Let B be a k by k positive semi-definite matrix with a non-trivial kernel (N). The analogue of (1.3.1) is

$$d\mu(x) = Z^{-1}\exp[\frac{-1}{2}(Bx,x)_{\mathbb{R}^k}]dx, \tag{1.3.2}$$

to be interpreted as a probability measure on the space $\mathbb{R}^k/N$. Since $\mathbb{R}^k$ is the orthogonal direct sum of N and $M \equiv N^{\perp}$, $\mathbb{R}^k/N$ is naturally isomorphic to M. Hence, equation (1.3.2) is naturally interpreted as the probability measure

$$d\mu(m) = Z^{-1}\exp[\frac{-1}{2}(Bm,m)_{\mathbb{R}^k}]dm \tag{1.3.3}$$

on M with dm denoting Lebesque measure on M. In effect the "extra factor" of Lebesque measure (dn) on N has been removed from the measure (dx) in (1.3.2).

Now choose a positive semi-definite matrix (C) such that $C|_M \equiv 0$, and $C|_N : N \longrightarrow \mathbb{R}^k$ is one to one. For such a C, put

$$d\mu_C(x) = Z_C^{-1}\exp[\frac{-1}{2}((B + C)x,x)_{\mathbb{R}^k}]dx, \tag{1.3.4}$$

where Z_C is chosen such that μ_C is a probability measure on $\mathbb{R}^k$. Each function $F:M \longrightarrow \mathbb{R}$ may be considered to be a function on $\mathbb{R}^k$ by putting $F(m + n) \equiv F(m)$ if $m \in M$ and $n \in N$. With this convention and definitions (1.3.3) and (1.3.4) it is easy to check that

$$\mu(F) = \mu_C(F), \tag{1.3.5}$$

for all measurable μ-integrable functions F on M. Hence, μ and μ_C have the same effect on all functions (F) on $\mathbb{R}^k$ which are invariant under the action of N on $\mathbb{R}^k$; $(x \longrightarrow x + n):\mathbb{R}^k \longrightarrow \mathbb{R}^k$ for each $n \in N$.

With this finite dimensional example in mind, we are led to replace (1.3.1) by

$$d\mu_\alpha(A) = Z^{-1}\exp[\frac{1}{2g^2}((d^*d + \alpha dd^*)A,A)]DA, \qquad (1.3.6)$$

where d^* is the adjoint of d, and α is any positive number. One should think of B as d^*d and C as αdd^*. The expression (1.3.6) should capture the meaning of (1.3.1) provided only gauge invariant functions are integrated. Furthermore, this "gauge fixing" process has rendered the exponent of (1.3.6) non-degenerate, so μ_α may be interpreted as an infinite dimensional Gaussian measure. (Compare with Section B.2, specifically equation (B.2.5).) According to the discussion above, the measures μ_α should be independent of $\alpha > 0$ when acting on gauge invariant functions.

I will introduce some notation before giving the precise definition of the free Euclidean measure. Let $\mathrm{Re}S'(\mathbb{R}^d)\otimes\mathbb{R}^d$ [$S'(\mathbb{R}^d)$ is the space of tempered distributions] be the space of generalized 1-forms on $\mathbb{R}^d$ under the identification, $A(j) = \sum_{i=1}^{d} A_i(j_i)$, where $A = \sum_{i=1}^{d} A_i e_i$ in $\mathrm{Re}S'(\mathbb{R}^d)\otimes\mathbb{R}^d$ and $j = \sum_{i=1}^{d} j_i dx^i$ is a test (C^∞ with compact support) 1-form on $\mathbb{R}^d$. Let $\mathcal{F}$ be the smallest

σ-algebra on $\mathrm{Re}S'(\mathbb{R}^d)\otimes\mathbb{R}^d$ such that the maps $(A \longrightarrow A(j)):\mathrm{Re}S'(\mathbb{R}^d)\otimes\mathbb{R}^d \longrightarrow \mathbb{C}$ are measurable for all test 1-forms (j) with $d^*j = 0$. The σ-algebra $(\mathcal{F})$ is called the gauge invariant σ-algebra, since $(A + d\lambda)(j) \equiv A(j) + \lambda(d^*j) = A(j)$ for all $\lambda \in \mathrm{Re}S'(\mathbb{R}^d)$.

<u>Definition 1.3.1</u>: The free Euclidean measure is the unique probability measure (μ) on the gauge invariant σ-algebra $(\mathcal{F})$ such that for all complex test 1-forms (j) on $\mathbb{R}^d$ with $d^*j = 0$,

$$\int e^{A(j)}\mu(dA) = \exp[\frac{-g^2}{2}(\Delta^{-1}j,j)], \tag{1.3.7}$$

where $\Delta \equiv -(d^*d + dd^*)$ is the Laplacian on forms, and $(\cdot,\cdot)$ is the natural complex bilinear form on test forms, see equation (2.2.3).

<u>Remark 1.3.1</u> The existence of the measure μ is guaranteed by a slight variation of Minlo's Theorem, see Simon [2].

<u>Remark 1.3.2</u>: The right hand side of (1.3.7) is found by using the informal expression (1.3.6) with $\alpha = 1$ along with a completion of the squares argument to compute (informally) the left hand side of (1.3.7). Furthermore, any different value of $\alpha > 0$ will give the same measure on $\mathcal{F}$, since if $\Delta_\alpha = -(d^*d + \alpha dd^*)$ then $\Delta_\alpha^{-1} = \Delta^{-1} - (1-\alpha)\Delta_\alpha^{-1}\Delta^{-1}dd^*$. Hence, $\Delta_\alpha^{-1}j = \Delta^{-1}j$ if $d^*j = 0$.

<u>Remark 1.3.3</u>: If φ is a test 2-form, then the function $A \longrightarrow F^A(\varphi) \equiv A(d^*\varphi)$ is a gauge invariant function of A,

since $d^*d^*\varphi = 0$. Hence, the integral of functions of $F^A(\varphi)$ are defined independent of the method of gauge fixing which we used, i.e. the choice of $C = dd^*$.

Remark 1.3.4: Suppose $\varphi = dj$, where j is a test 1-form, then

$$\int e^{F^A(\varphi)}\mu(dA) = \exp[\frac{g^2}{2}(\varphi,\varphi)]. \tag{1.3.8}$$

Section 1.4: Convergence Results for d = 3

Let φ be a test 2-form on $\mathbb{R}^d$ and $a > 0$ (the lattice spacing parameter), then put $\varphi_a(p) \equiv \int_{ap} \varphi \cong \pm a^2\varphi_{ij}(ax)$, (Definition 2.2.1), where $p \in \mathcal{P}$ is a plaquette, x is a corner of p, and p is parallel to the i'th and j'th coordinate directions. The function φ_a is called a lattice approximation to the form φ. Also let $F(p)(\omega) \equiv \sin(\omega(\partial p))$, where $\omega : \mathcal{B}_+ \longrightarrow \mathbb{R}$ — compare with the imaginary part ($G = U(1)$) of equation (1.2.1) and see Definition 2.3.2. Finally, let $(F,\varphi_a)(\omega) \equiv \frac{1}{2}\sum_{p\in\mathcal{P}} F(p)(\omega)\varphi_a(p)$. The following theorem is due to L. Gross [3].

Theorem 1.4.1: Let d=3 and G=U(1). Let $\varphi = dj$ with j a real valued test 1-form on $\mathbb{R}^3$. Then

$$\lim_{a\downarrow 0} \mu_a(e^{ia^{-1}(F,\varphi_a)}) = \exp[\frac{-g^2}{2}(\varphi,\varphi)], \tag{1.4.1}$$

where μ_a is a translation and 90^o-rotation invariant Gibbs state corresponding to equation (1.2.5).

<u>Remark 1.4.1</u>: As pointed out by L. Gross, the requirement that φ be exact is related to the fact that the lattice version of dF is not zero as it is for the free Euclidean field (Definition 1.3.1). It is shown in Gross [3], that if the cosine interaction in (1.2.5) is replaced by the Villain action (Example 2.3.3), then

$$\lim_{a \downarrow 0} \mu_a(|(F,(d^*\varphi)_a)|^r) = 0, \qquad (1.4.2)$$

for all real test 3-forms on $\mathbb{R}^3$ and $r \geq 1$. So in this case the Bianci identity ($dF = 0$) holds in the limit, i.e. there are no magnetic monopoles in the limit.

<u>Remark 1.4.2</u>: The statement of Theorem 1.4.1 is also shown to hold (Gross [3]) for certain more general interactions other than the cosine interaction of (1.2.5).

<u>Section 1.5</u> Convergence Results for $d > 3$

When $d > 3$, the heuristics leading up to the lattice approximating measures of equation (1.2.5) are not so convincing. The key assumption was that $\omega(\partial p)$ was "close" to zero. When $d = 3$ the factor of a in the exponent of (1.2.5) forces the measures μ_a to concentrate on configurations for which $\omega(\partial p)$ is close to zero, and in fact this is a main ingredient of the proof of Theorem 1.4.1. On the other hand when $d > 3$, the measures μ_a of (1.2.5) are no longer concentrated on configurations with $\omega(\partial p)$ small. Nevertheless, similar results to Theorem 1.4.1 still hold in four dimensions. In this case, the key fact is the clustering properties of extreme or unique Gibbs states, see Theorems A.3.1 and A.3.2.

As in Gross [3], we restrict our attention to studying the lattice current ($J = d^{*}F$) rather than the field strengths (F). This allows us to avoid the Dirac monopoles (breakdown of the Bianchi identity) which are inadvertently introduced into the lattice theory. Avoidance of the monopoles seems reasonable in the abelian theory because a similar mechanism for avoiding them in the non-abelian theory is now available, Gross [4].

The results of this thesis pertain to a more general statistical mechanical model than that indicated in (1.2.5). We will be concerned with Gibbs states given informally by

$$d\mu_a(\omega) = Z_a^{-1}\exp\left\{-a^{(d-4)}\frac{1}{2g^2}\sum_{p\in\mathcal{P}} h(\omega(\partial p))\right\}d\nu(\omega), \quad (1.5.1)$$

where $h:\mathbb{R} \longrightarrow \mathbb{R}$ is an (energy) function which is C^2, even, and 2π-periodic (See Section 2.3). The collection of Gibbs states associated to (1.5.1) will be denoted by $G(g^{-2}a^{(d-4)}h)$. The lattice version of the field strength tensor (F) for this model is $F(p)(\omega) \equiv h'(\omega(\partial p))$, this agrees with the previous definition when $h(x) = \cos(x) - 1$. (See Chapter 2 and Appendix A for more precise definitions.)

The main results of this thesis will now be summarized. The energy function h is assumed to have the properties described above.

<u>**Theorem 1.5.1**</u> Let $d > 4$, and φ be a closed ($d\varphi = 0$) complex valued test 2-form on $\mathbb{R}^d$. For each $a > 0$, let $\mu_a \in G(g^{-2}a^{(d-4)}h)$, then

$$\lim_{a \downarrow 0} \mu_a(\exp(a^{(d-4)}(F,\varphi_a))) = 1, \qquad (1.5.2)$$

where $(F,\varphi_a)(\omega) \equiv \frac{1}{2}\sum_{p\in\mathcal{P}} F(p)(\omega)\varphi_a(p)$ as before.

<u>Remark 1.5.1</u> The fact that φ is closed is equivalent to the existence of a test 1-form (j) such that $\varphi = dj$ (provided $d > 2$), as follows from standard compact De Rham cohomology theory (Bott and Tu [1]). Hence the theorem is a statement about the lattice current d^*F, since $(F,\varphi) = (d^*F,j)$.

<u>Proof</u>: This is Theorem 3.3.1 after a rescaling of h.

In the following, we will assume that $d = 4$. This is an exceptional case, since the interaction in (1.5.1) no longer depends on the lattice spacing. The influence of the lattice spacing is only felt in the lattice approximations φ_a to the test 2-form φ..

<u>**Theorem 1.5.2**</u> ($d = 4$) Suppose that $G(g^{-2}h) = \{\mu\}$, is a one element set. Then for any closed complex test 2-form (φ) on $\mathbb{R}^4$,

$$\lim_{a \downarrow 0} \mu(e^{(F,\varphi_a)}) = \exp(\tfrac{\alpha}{2} g^2(\varphi,\varphi)), \qquad (1.5.3)$$

where $\alpha \equiv \mu(h''(\omega(\partial p))) \geq 0$ (independent of $p\in\mathcal{P}$). Furthermore, $\alpha = 0$ if and only if h is a constant function.

<u>Proof</u>: This is Theorem 3.2.2 and Lemma 3.2.1.

<u>Remark 1.5.3</u> The theorem says that the lattice Laplace transforms restricted to closed 2-forms converge to the corresponding Laplace transforms of the free Euclidean field provided the coupling constant (g^2) is renormalized by the factor α.

<u>Corollary 1.5.1</u> Let h be a given energy function, and let the coupling constant (g^2) be chosen sufficiently large such that $[\sup(h) - \inf(h)] < 2g^2/9$. Then $|G(g^{-2}h)| = 1$, and hence the conclusion of Theorem 1.5.2 holds.

<u>Proof</u>: An application of Dobrushin's uniqueness theorem, see Lemma 3.2.1.

<u>Theorem 1.5.3</u> (d = 4) Let φ be a real valued exact test 2-form on $\mathbb{R}^4$. Suppose that $\mu \in G(g^{-2}h)$ is an extreme Gibbs state which is also translation and 90°-rotation invariant. Then

$$\lim_{a \downarrow 0} \mu(e^{i(F,\varphi_a)}) = \exp(\frac{-\alpha}{2} g^2(\varphi,\varphi)), \qquad (1.5.4)$$

where α is the constant defined in Theorem 1.5.1.

<u>Proof</u>: This is Theorem 3.2.2.

<u>Remark 1.5.4</u> Theorem 1.5.3 is a special case of Theorem 1.5.2 if $|G(g^{-2}h)| = 1$.

<u>Theorem 1.5.4</u> (d = 4) Suppose h is a Wilson-like energy function, that is $h(x) = b - \sum_{k=1}^{N} b_k \cos(kx)$ with $b_k \geq 0$ (Definition 5.1.1). Let $\mu_g^o \in G(g^{-2}h)$ be the Gibbs state constructed by taking the thermodynamic limit with zero

boundary conditions, see equation (5.1.2). Then μ_g^o satisfies the hypothesis of Theorem 1.5.3, and hence also the conclusion.

Proof: This is Theorem 5.1.1 and Corollary 5.1.1.

Theorem 1.5.5: (d = 4) Let h be a Wilson-like energy function, and φ a closed real valued test 2-form on $\mathbb{R}^4$. Let $\mu_g \in G(g^{-2}h)$ denote any translation and 90^o-rotation invariant Gibbs state. Then for all but at most a countable number of $g > 0$ (independent of φ),

$$\lim_{a \downarrow 0} \mu_g(e^{i(F,\varphi_a)}) = \exp(\frac{-\alpha}{2} g^2(\varphi,\varphi)), \tag{1.5.5}$$

where $\alpha \equiv \mu_g^o(h''(\omega(\partial p)))$ independent of the particular choice of the Gibbs state (μ_g).

Proof: This is Theorem 5.2.1.

It should be pointed out that Theorems 1.5.3 - 1.5.5 are only needed when there is a first order phase transition for the model, that is when $|G(g^{-2}h)| > 1$. Otherwise the stronger result of Theorem 1.5.2 would be applicable. The existence of a first order phase transition is still an open question for these $U(1)_4$-gauge models. Recall that Guth's theorem (Guth [1]) asserts that in d = 4, the $U(1)_4$ lattice gauge theory with the Villain action does have a phase transition. However, this phase transition is characterized by the decay properties of "Wilson loop variables" as the size of the loop increases to infinity. It is not clear how a phase transition of this type is related to the more standard phase transitions which are characterized by smoothness properties of the thermodynamic potentials.

Section 1.6 Related Work and Discussion

T. Balaban [1-14] and P. Federbush [1-7] have some of the most promising results towards constructing the non-abelian Yang-Mills' measure (1.1.1). Balaban's idea is to use a renormalization group transformation (modeled on the transformation introduced in K. G. Wilson [2]) to construct the measure in three and four space-time dimensions. Balaban has found a number of "stability" estimates on the renormalized lattice actions and partition functions independent of the lattice spacing (a). P. Federbush has some similar results for four space-time dimensions using methods based, in part, on the "averaging" techniques of Balaban.

C. King [1-2] has used the results and idea's of Balaban [1-3] to prove the existence of the continuum U(1)-Higgs (abelian) model in two and three space-time dimensions. The existence is in the sense of convergence of Laplace transforms of the field strength variables ($F(x)$) and the gauge invariant renormalized Higgs field variables ($:|\phi|^2:(x)$). However, the appoximating measures that King uses are non-compact versions of the Wilson lattice approximation. The generalization of these non-compact approximations to the non-abelian setting is as yet unknown.

Part of the motivation of the work of Gross [3] and of this thesis was to use an abelian model which did have a non-abelian analogue. As already mentioned, the abelian case is considered to be a testing ground for techniques to be applied in the non-abelian case. With this criterion in mind, one might object to using the smoothed field strengths ($F(\varphi)$) as the basic variables of the theory, since they do not immediately generalize to gauge invariant functions in the non-abelian case. However, there are non-abelian analogues of the field strengths ($F(x)$) (called lassos) which are path dependent functions, see Gross [4] for their definition and properties. It is hoped that these lassos (which are essentially the field strengths when the gauge group is abelian) will prove to be useful in defining the Yang-Mills' measure in the non-abelian case.

Another collection of gauge invariant functions which may be useful for defining the Euclidean Yang-Mills' fields are the ("renormalized") Wilson-loop variables. (The Wilson-loop variables are the maps ($A \longrightarrow P^A(\sigma)$) where $P^A(\sigma)$ denotes parallel translation around a <u>closed</u> curve (σ) with respect to the connection form A.). The Wilson-loop variables have the disadvantage of being extremely singular. In fact, it seems that the Wilson-loop variables are too singular to be "smoothed" into a genuine measurable function. (The free Euclidean electo-magnetic field, Definition 1.3.1, is a testing ground for these statements.) For this reason the field strength variables seem more desirable than the Wilson-loop variables. However, see Chapter 8 of Seiler [1] for a tentative set of axioms that the Euclidean Yang-Mills' fields might satisfy in terms of "expectations" involving the Wilson-loop variables.

Chapter 2

PRELIMINARIES

Section 2.1 Lattice Complexes and Forms

Definition 2.1.1 Let $\{e_i\}_{i=1}^d$ be the standard unit vectors in $\mathbb{R}^d$, and k be a non-negative integer less than d. The positively oriented k-cells based at $x \in Z^d$, are the formal symbols: $(e_{i_1} \wedge e_{i_2} \wedge \cdots\cdots \wedge e_{i_k})_x$ where $1 \le i_1 < i_2 < \cdots\cdots < i_k \le i_d$, provided $k \ge 1$. The positive oriented 0-cells are the symbols $(+)_x$, where $x \in Z^d$. When there is no confusion $(+)_x$ will be abbreviated simply by x. We also define negatively oriented k-cells based at x to be the formal symbols $-(e_{i_1} \wedge e_{i_2} \wedge \cdots\cdots \wedge e_{i_k})_x$ if $k \ge 1$, and $(-)_x$ or $-x$ for the 0-cells. Let $(Z^d)^{(k)}$ denote the set of k-cells of both orientations.

The basic k-cell $(e_{i_1} \wedge e_{i_2} \wedge \cdots\cdots \wedge e_{i_k})_x$ should be thought of as a k-dimensional oriented cube with x as its "lower left hand" corner. The edges of the cube consist of translates of the unit vectors $\{e_{i_j}\}_{j=1}^k$.

Definition 2.1.2 A k-chain (c) is the formal sum of a finite number of k-cells with integer coefficients:

$$c = a_1 c_1 + \cdots\cdots + a_n c_n \tag{2.1.1}$$

where the c_i's are k-cells and the a_i's are integers. If c is a k-cell, identify $-1c$ with $-c$, where $-c$ denotes c with the opposite orientation.

__Remark 2.1.1__ Another way to state Definition 2.1.2 is: the set of k-chains is the free Z-module generated by the k-cells modulo the relation -1c = -c.

__Definition 2.1.3__ Let $k \geq 2$. The boundary operator (∂) applied to a k-cell ($c = (e_{i_1} \wedge e_{i_2} \wedge \cdots\cdots \wedge e_{i_k})_x$) is the k-1 chain:

$$\partial c = \sum_{\varepsilon=0}^{1} \sum_{j=1}^{k} (-1)^{\varepsilon+j} (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge \hat{e}_{i_j} \cdots \wedge e_{i_k})_{x+\varepsilon e_{i_j}} \qquad (2.1.2)$$

where the basis vector under the circumflex is to be omitted. If k=1, and $c=(e_i)_x$, put

$$\partial c = (+)_{x+e_i} - (+)_x = (+)_{x+e_i} + (-)_x. \qquad (2.1.3)$$

If c is a 0-cell, put $\partial c=0$.

__Remark 2.1.2__ The definition of the boundary operator conforms to the usual notion of the induced orientation of a face determined by the "outward pointing normal".

__Remark 2.1.3__ An easy computation shows that $\partial^2=\partial\circ\partial \equiv 0$.

__Definition 2.1.4__ A (lattice) k-form is a homomorphism on the Z module of k-chains to the complex numbers. So if Ψ is a k-form and c is the k-chain given in equation (2.1.1), then

$$\Psi(c) = a_1\Psi(c_1)+\cdots\cdots\cdots+a_n\Psi(c_n). \qquad (2.1.4)$$

In particular, $\Psi(-c) = -\Psi(c)$.

__Definition 2.1.5__ A k-form is said to have compact support if it is identically zero on all k-cells based sufficiently far from 0.

There is a natural complex bi-linear form on the collection of compactly supported k-forms. Namely, if φ and ψ are two k-forms with compact support, set

$$(\varphi,\psi) = \frac{1}{2}\sum_{c\in(Z^d)^{(k)}} \varphi(c)\psi(c). \tag{2.1.5}$$

It is for later convenience that we do not conjugate one of the factors in (2.1.5).

<u>Definition 2.1.6</u> The differential ($d\psi$) of k-form (ψ) is the k+1 form determined on (k + 1)-cells (c) by

$$d\psi(c) = \psi(\partial c) \equiv \sum_{p\in\partial c} \psi(p) \tag{2.1.6}$$

A k-cell p is said to be in ∂c if p is one of the summands (including the correct orientation) of equation (2.1.2). If ψ is a d-chain, then $d\psi$ is to be interpreted as zero.

<u>Remark 2.1.4</u> Since $\partial^2 \equiv 0$, we have also $d^2 \equiv 0$.

<u>Definition 2.1.7</u> The co-differential ($d^*\psi$) of (k+1)-form (ψ) is the k form determined on k-cells (p) by

$$d^*\psi(p) = \sum_{c:p\in\partial c} \psi(c). \tag{2.1.7}$$

If ψ is a 0-chain, then $d^*\psi$ is to be interpreted as zero.

<u>Remark 2.1.5</u> It is easily checked that d^* and d are adjoint to each other with respect to the bilinear form (2.1.5) on the space of compactly supported forms.

<u>Section 2.2</u> Approximation of Differential Forms

This section deals with approximating differential forms on $\mathbb{R}^d$ by lattice forms on Z^d. (A differential form is always assumed to be C^∞.) A "small" parameter (a) will be introduced into the definitions which should be thought of as the "lattice spacing" of Z^d. That is for conceptual purposes it will often be clearer to think of the lattice Z^d as the lattice aZ^d.

<u>Definition 2.2.1</u> If ψ is a differential k-form on $\mathbb{R}^d$, and $a>0$, then put

$$\psi_a(c) = \int_{ac} \psi$$

$$\equiv \int_{[0,a]^k} \psi_{i_1 i_2 \cdots i_k}(ax+s_1 e_{i_1}+\cdots+s_k e_{i_k})ds_1\cdots ds_k \quad (2.2.1)$$

where $c = (e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k})_x$ and

$\psi = \sum \psi_{i_1 i_2\cdots i_k} dx^{i_1}\wedge\cdots\wedge dx^{i_k}$. The sum in the last expression is over all increasing sequences $i_1<i_2<\cdots<i_k$, and the dx^i's are the standard basis forms on $\mathbb{R}^d$. The k-form (ψ_a) is called a lattice approximation of the differential k-form (ψ).

<u>Remark 2.2.1</u> By a change of variables, equation (2.2.1) is the same as

$$\psi_a(c)$$

$$= a^k \int_{[0,1]^k} \psi_{i_1 i_2\cdots i_k}(a[x+s_1 e_{i_1}+\cdots+s_k e_{i_k}])\, ds_1\cdots ds_k \quad (2.2.2)$$

Thus $\psi_a(c) \simeq a^k \psi_{i_1 i_2\cdots i_k}(ax)$ for a close to zero.

If ψ is a differential form on $\mathbb{R}^d$, let $d\psi$ denote its exterior differential. The symbol (d) now has two uses.

However, they are essentially the same, and after the next lemma no distiction will be made between the two versions of (d).

<u>Lemma 2.2.1</u> Let ψ be a differential k-form on $\mathbb{R}^d$ and ψ_a be a lattice approximation. Then $(d\psi)_a$ and $d(\psi_a)$ are the same k-forms on Z^d, where the first d is the exterior differential and the second d is the lattice differential.

<u>Proof</u>: $$(d\psi)_a(c) = \int_{ac} d\psi = \int_{\partial(ac)} \psi = \int_{a(\partial c)} \psi = \psi_a(\partial c)$$
$$= (d\psi_a)(c)$$

The second equality is Stokes' theorem, see Spivak [1]. The rest is only a matter of unwinding definitions. Q.E.D.

Recall the "usual" complex bilinear form on the space of compactly supported differential k-forms is

$$(\psi,\varphi) = \sum \int_{\mathbb{R}^d} \psi_{i_1 i_2 \cdots i_k}(x)\, \varphi_{i_1 i_2 \cdots i_k}(x)\, dx, \qquad (2.2.3)$$

where the sum is over increasing subsequences of length k of $\{1,2,3,\cdots\cdots,d\}$, and $\psi_{i_1 i_2 \cdots i_k}(x)$ and $\varphi_{i_1 i_2 \cdots i_k}(x)$ are the components of the differential k-forms ψ and φ respectively. Again the same notation is used for the inner products on lattice forms as well as on differential forms. This should cause no confusion since the type of form in the inner product indicates which definition is in force.

<u>Lemma 2.2.2</u> Let ψ and φ be compactly supported differential k-forms on $\mathbb{R}^d$, then

$$(\Psi,\varphi) = \lim_{a\downarrow 0} a^{(d-2k)} (\Psi_a,\varphi_a). \tag{2.2.4}$$

Proof: Let $f = \Psi_{i_1 i_2 \cdots i_k}$ be a fixed component of Ψ, and $c = (e_{i_1} \wedge e_{i_2} \wedge \cdots\cdots \wedge e_{i_k})_x$. By the mean value theorem and the definition of $\Psi_a(c)$ it follows that,

$$|a^k f(ax) - \Psi_a(c)| \le K\cdot a^{(k+1)} \sup |\nabla f(y)|, \tag{2.2.5}$$

where the supremum is over $\{y \in \mathbb{R}^d : |ax - y| \le Ka\}$, and K is a dimension dependent constant. Let N be any number larger than the dimension (d). Since the support of f is compact, there exists a constant (B) such that $|\nabla f(y)| \le B(1+|y|)^{-N}$ for all $y \in \mathbb{R}^d$. So for a sufficiently small

$$|a^k f(ax) - \Psi_a(c)| \le 2^N K B a^{(k+1)} (1+a|x|)^{-N} \tag{2.2.6}$$

holds for all $x \in Z^d$. Furthermore, the estimate

$$\sum_{x\in Z^d} (1+a|x|)^{-N} \le C a^{-d} \tag{2.2.7}$$

holds for a small, where C is a constant ($C < \infty$ since $N > d$). Equation (2.2.7) is easily verified by comparing the sum with an integral.

Using the estimates of the form (2.2.4 – 2.2.7) and standard Riemann integral techniques, we find that

$$a^{(d-2k)} (\Psi_a,\varphi_a) = a^d \sum_{x\in Z^d} \sum \Psi_{i_1 i_2 \cdots i_k}(ax)\, \varphi_{i_1 i_2 \cdots i_k}(ax) + O(a), \tag{2.2.8}$$

where $O(a) \longrightarrow 0$ as $a \longrightarrow 0$. The lemma now follows by Riemann integral techniques and equation (2.2.8).

Q.E.D.

Section 2.3 Definition of the Model

Let $h:\mathbb{R} \longrightarrow \mathbb{R}$ be a real twice continuously differentiable even periodic function with period 2π. Any such function (h) will be called an energy function. The main examples of interest are given below.

Example 2.3.1 Wilson action: $h(x)=1-\cos(x)$.

Example 2.3.2 Generalized Wilson action: $h(x) = 1-\cos(mx)$, where m is any integer.

Example 2.3.3 Villain action: For each $\beta > 0$, define h_β by

$$\exp(-\beta h_\beta(x)) = c_\beta \sum_{n=-\infty}^{\infty} \exp[-\beta(x-2\pi n)^2/2], \qquad (2.3.1)$$

where c_β is a constant chosen such that the right hand side is one at $x=0$.

The model will now be defined, following the notation in the appendix. Let $(\mathcal{B}_+)$ $\mathcal{B}$ be the set of (positively oriented) one cells on Z^d. Let $(\mathcal{P}_+)$ $\mathcal{P}$ be the set of (positively oriented) two cells. The 1-cells will also be referred to as bonds and the 2-cells as plaquettes. The lattice (L) for the model is the set of positively oriented bonds $\mathcal{B}_+$. The state space for the model is $S = S^1$, the unit circle. The unit circle will be identified with the interval $[-\pi,\pi]$ with the end points identified. The apriori measure is normalized Lebesque measure (λ) on $[-\pi,\pi]$. The configuration space (Ω) is then $(S^1)^{\mathcal{B}_+}$.

It is convenient to embed configuration space (Ω) in $(S^1)^{\mathbf{B}}$, by defining $\omega(-b) \equiv -\omega(b)$ where $b \in \mathbf{B}_+$ and $\omega \in \Omega$. (Remember that the unit circle has been identified with $[-\pi,\pi]$.) With this convention each configuration may be considered as a 1-form on Z^d.

<u>Definition 2.3.1</u> Let h be a given energy function, then the associated interaction potential ($\varphi^h = \{\varphi^h_B\}_{B\subset\subset\mathbf{B}_+}$) is given by

$$\varphi^h_B(\omega) = \begin{cases} h(d\omega(p)) & \text{if } \overline{B} = \overline{\partial p} \text{ for some } p \in \mathcal{P}_+ \\ 0 & \text{otherwise} \end{cases}, \qquad (2.3.2)$$

where $\overline{B}$ denotes the set of bonds in B disregarding orientation. (The notation $B\subset\subset\mathbf{B}_+$ means $B\subset\mathbf{B}_+$ and $|B| < \infty$.)

The energy ($H^h_B(\omega|\omega')$) of a configuraton (ω) given the boundary conditions (ω') over $B\subset\subset\mathbf{B}_+$ is

$$H^h_B(\omega|\omega') = \sum_{p\in\mathcal{P}_+:\overline{p}\cap\overline{B} \neq \phi} h(d[\omega_B \times \omega'_{\mathbf{B}_+\setminus B}](p)). \qquad (2.3.3)$$

The corresponding specification (Π^{φ^h}) is denoted by $\Pi^h = \{\Pi^h_B\}_{B\subset\subset\mathbf{B}_+}$. If (f) is a continuous function on Ω then,

$$\Pi^h_B(\omega,f) = Z^h_B(\omega)^{-1}\int_{\Omega(B)} e^{-H^h_B(\omega'|\omega)} f(\omega'_B \times \omega_{\mathbf{B}_+\setminus B})\, d\lambda^B(\omega'_B), \qquad (2.3.4)$$

where $Z^h_B(\omega)$ is the normalization constant. The set of (extreme) Gibbs states will be denoted by ($G_e(h)$) $G(h)$.

<u>Definition 2.3.2</u> The lattice version of the field strength tensor is

$$F(p)(\omega) = h'(d\omega(p)), \qquad (2.3.5)$$

where $p \in \mathcal{P}$, and $\omega \in \Omega$. (Remember that each ω is extended to be a 1-form on Z^d.)

Remark 2.3.1 Since the energy function (h) is assumed to be even, h' is then odd. So for each configuration (ω), $F(\cdot)(\omega)$ may be considered a 2-form on Z^d.

Definition 2.3.3 The "current" associated to F is

$$J(b)(\omega) = d^* F(b)(\omega) = \sum_{P \in \mathcal{P}:\ b \in \partial P} F(P)(\omega), \tag{2.3.6}$$

for all $b \in \mathcal{B}$, and $\omega \in \Omega$.

Remark 2.3.2 The factor $a^{(d-4)} g^{-2}$ in (1.5.1) has been absorbed into the energy function (h). This factor is easily re-inserted if one notes that F and J depend linearly on h.

Section 2.4 Pressure

The purpose of this section is to generalize the notion of pressure and its properties to the case where the underlying lattice is $\mathcal{B}_+$. In this section the interaction potential (φ) is allowed to be arbitrary except for the restriction of finite range. The results of this section will only be used in the proof of Theorem 5.2.1 of Chapter 5.

The group Z^d naturally acts on the lattice $\mathcal{B}_+$ via

$$T_x[(e_i)_y] \equiv (e_i)_{(x+y)}. \tag{2.4.1}$$

So T_x acts on a bond simply by translating the base point. As in the appendix, the action on Z^d naturally induces actions on Ω, and the functions on Ω,

$$T_x(\omega) = \omega \circ T_{-x} \tag{2.4.2}$$

$$T_x(f)(\omega) = f(\omega \circ T_x). \tag{2.4.3}$$

We will only consider interaction potentials (φ) which are translation invariant,

$$\varphi_{T_x(B)}(\omega \circ T_x) = \varphi_B(\omega). \tag{2.4.4}$$

<u>Definition 2.4.1</u> Let $\Lambda \subset Z^d$, the set of positively oriented bonds associated to Λ is

$$\Lambda^{(1)} = \{ \ (e_i)_x \ | \ x \in \Lambda^d \text{ and } i = 1, 2, \cdots, d \ \}. \tag{2.4.5}$$

<u>Definition 2.4.2</u> The pressure ($P_\Lambda(\omega,\varphi)$) of a translation invariant, finite range interaction potential (φ) on Ω, given a configuration (ω) and $\Lambda \subset\subset Z^d$, is

$$P_\Lambda(\omega,\varphi) = |\Lambda|^{-1} \ln[Z^{\varphi}_{\Lambda^{(1)}}(\omega)], \tag{2.4.6}$$

where $Z^{\varphi}_{\Lambda^{(1)}}(\omega)$ is the normalization constant in the definition of the specification, see Definition A.2.4.

<u>Definition 2.4.3</u> If $B \subset\subset \mathcal{B}_+$, put $B^o = \{x \in Z^d | \exists$ a bond $b \in B$ with x as its base point$\}$, the base of B.

<u>Theorem 2.4.1</u> The infinite volume pressure (P) defined by the limit

$$P(\varphi) = \lim_{\Lambda \uparrow Z^d} P_\Lambda(\omega,\varphi) \tag{2.4.7}$$

exists and is independent of the boundary conditions (ω), as Λ increases to Z^d through cubes. Furthermore, as in Theorem A.8.1, there is a one to one correspondence between tangent functionals to the pressure at φ and translation invariant

Gibbs states of the interaction potential φ. If μ is a translation invariant Gibbs state, the associated tangent functional is

$$\alpha(\psi) = -\mu\left[\sum_{B:\ 0\in B^o} |B^o|^{-1}\ \psi_B \right]. \tag{2.4.8}$$

If α is a tangent functional, the associated translation invariant Gibbs state is determined by

$$\mu(f) = -\ \alpha(\psi^f) \tag{2.4.9}$$

where f is a continuous $\mathcal{F}(A)$-measurable function for some $A\subset\subset\mathbf{B}_+$, and

$$\psi_B^f(\omega) = \begin{cases} f(\omega\circ T_x) & \text{if } B = T_x(A) \text{ for some } x\in Z^d \\ 0 & \text{otherwise} \end{cases}. \tag{2.4.10}$$

<u>Proof</u>: We will reduce this theorem to Theorem A.8.1 of Appendix A. The procedure is to map a lattice system over $\mathbf{B}_+$ to a lattice system over Z^d. The state space of the new system is $T = S^d$, where S is the unit circle as before. The mapping between the two configuration spaces is K: $S^{\mathbf{B}_+} \longrightarrow T^{Z^d}$, given by $K(\omega)(x) = \{\omega(\ (e_i)_x)\}_{i=1}^d$. Each interaction potential (φ) over $\mathbf{B}_+$ maps to an interaction potential (φ^*) over Z^d,

$$\varphi^*_\Lambda(K(\omega)) \equiv \sum_{B\in\mathbf{B}_+:B^o=\Lambda} \varphi_B(\omega), \tag{2.4.11}$$

where $\Lambda\subset\subset L$. By an easy computation we find

$$H_\Lambda^{\varphi^*}(K(\omega)|K(\omega')) = H^{\varphi}_{\Lambda^{(1)}}(\omega|\omega'), \tag{2.4.12}$$

for all $\Lambda\subset\subset Z^d$. Hence, the corresponding specifications obey

$$\Pi^{\varphi^*}_{\Lambda}(K(\omega), f\circ K^{-1}) = \Pi^{\varphi}_{\Lambda(1)}(\omega, f), \tag{2.4.13}$$

where f is any continuous function of $\Omega = S^{\mathbb{B}_+}$. Using Remark A.2.4 of Appendix A and equation (2.4.13), K induces an affine bijection

$$(\mu \longrightarrow \mu K^{-1}) : G(\Pi^{\varphi}) \longrightarrow G(\Pi^{\varphi^*}) \tag{2.4.14}$$

which preserves the subset of translation invariant states.

The finite volume pressures of the two systems behave in a similar fashion,

$$P^*_{\Lambda}(K(\omega), \varphi^*) = P_{\Lambda(1)}(\omega, \varphi), \tag{2.4.15}$$

where P^* is the pressure in the system over Z^d. Passing to the limit in equation (2.4.15) yields

$$P^*(\varphi^*) = P(\varphi). \tag{2.4.16}$$

Hence the two systems are essentially the same. The theorem follows by simply transcribing the known results about the system over Z^d to the desired results for the system over $\mathbb{B}_+$. Q.E.D.

Chapter 3

CONVERGENCE RESULTS

General criteria are studied for the convergence of the lattice Laplace and Fourier transforms (on the current algebra) to the respective transforms of Euclidean free electromagnetic field. The case of most interest will be when the dimension (d) is four. The results for d=3 is the subject of Gross [3]. If $d>4$, the Laplace transform converges but to the trivial limit (the function 1). Recall from the introduction, that d=4 has the distinguished feature that the lattice measure does not depend on the lattice spacing parameter (a).

The main theorems of this chapter conclude that lattice Fourier (Laplace) transforms restricted to the "current algebra" converge to the (renormalized) Fourier (Laplace) transforms of the free Euclidean electromagnetic field provided the lattice measure is an extreme (unique) Gibbs state. Furthermore, if d=4, the resulting transforms are not "trivial" provided the energy function (Section 2.3) is not constant.

Section 3.1 Schwinger – Dyson Equations

Lemma 3.1.1 (Schwinger-Dyson Equations, see Gross [3]) Let h be an energy function, $\mu \in G(h)$ be an arbitrary Gibbs state. Let f be a differentiable periodic function depending on only finitely many bond variables (i.e. f is $\mathcal{F}(B)$-measurable for some $B \subset\subset \mathbf{B}_+$). Then for any bond b

$$\mu(\partial f/\partial\omega(b)) = \mu(J(b)\cdot f), \qquad (3.1.1)$$

where J(b) is the lattice current in Definition 2.3.3. (Recall that the constants $a^{(d-4)}g^{-2}$ have been absorbed into the energy function (h), see Remark 2.3.2.)

<u>Proof</u>: Choose $B \subset\subset \mathbf{B}_+$ such that f is $\mathcal{F}(B)$-measurable and such that b and the bonds of any neighboring plaquettes are contained in B. Then by finite dimensional integration by parts, noting that the boundary terms are zero by periodicity,

$$\Pi^h_B(\omega, \partial f/\partial\omega(b)) = \Pi^h_B(\omega, J(b)\cdot f), \qquad (3.1.2)$$

for all configurations ω. See equations (2.3.3), (2.3.4) and the definition of J(b) (Definition 2.3.3) and the following computation.

$$\begin{aligned}\partial H^h_B(\omega'|\omega)/\partial\omega'(b) &= \sum_{p\in\mathcal{P}_+ : \bar{p}\cap\bar{B}\neq\phi} h'(d\omega'(p))\cdot\partial d\omega'(p)/\partial\omega'(b) \\ &= \sum_{p\in\mathcal{P} : b\in\partial p} h'(d\omega'(p)) = J(b)(\omega').\end{aligned}$$

In the second equality we used the assumption that h' is odd, the convention of ω' being extended to a 1-chain, and the assumption that b was in the "interior" of B.

The theorem now follows by integrating both sides of equations (3.1.2) with respect to the Gibbs state (μ) and using the D.L.R. equations (A.2.3). Q.E.D.

We will see that this form of integration by parts is a very powerful tool for studying the properties of U(1) lattice gauge models. This technique was put to good use by Gross [3] in studying the case d=3.

Section 3.2 Convergence of Transforms in Dimension Four

Although the main results in this section are for d=4, many of the results are dimension independent. Unless explicitly stated, it is assumed that the dimension is d.

Through out this section h will denote an energy function as in Section 2.3, and j will be a test 1-form on $\mathbb{R}^d$. A test 1-form is a C^∞ differential 1-form with compact support on $\mathbb{R}^d$.

Definition 3.2.1 A Gibbs state, $\mu \in G(h)$, is said to be invariant if μ is invariant under both translations and 90^o-rotations.

Theorem 3.2.1 (d = 4) Let j be a complex valued test 1-form on $\mathbb{R}^4$. Assume that there is only one Gibbs state (μ) associated to the energy function (h); $G(h)=\{\mu\}$. Then

$$\lim_{a \downarrow 0} \mu(e^{(J,j_a)}) = \exp(\frac{\alpha}{2}(dj,dj)) \qquad (3.2.1)$$

where $\alpha=\mu(h''(d\omega(p)))$ — p is any plaquette. Note: the bilinear form in the left hand exponent of (3.2.1) is on lattice forms, whereas in the right hand exponent it is on differential forms.

Remark 3.2.1 The Gibbs state μ must be invariant (translation and 90°-rotation invariant), since such a Gibbs state always exists (by Remark A.8.1) if the interaction potential is invariant. Since μ is unique, it must be the invariant state. Thus the constant (α) is well defined.

Remark 3.2.2 The constant (α) must be nonnegative otherwise the right hand side of (3.2.1) would be larger than one when evaluated on purely imaginary test 1-forms, which is impossible. This fact will also be shown explicitly in Lemma 3.2.2, where it is also shown that $\alpha > 0$ if h is not a constant.

Lemma 3.2.1 If $[\sup(h) - \inf(h)] < 2/(3[d-1])$, then $|G(h)| = 1$.

Proof: We first note that $G(h - c) = G(h)$ for any real constant (c), since $\pi_B^{(h-c)} = \pi_B^h$ for all real c and $B \subset\subset \mathbf{B}_+$. Thus we may assume that h is normalized such that $\|h\|_\infty = \frac{1}{2}[\sup(h) - \inf(h)]$.

An easy computation using the Definition 2.3.2 shows

$$\sup_{b \in \mathbf{B}_+} \sum_{B: b \in B} (|B|-1)\|\varphi_B^h\|_\infty = 3(d-1)\|h\|_\infty \tag{3.2.2}$$

in d-dimensions. The factor 3 is from the fact that $|\partial p| = 4$ if p is a plaquette. The factor d-1 counts the number of positively oriented plaquettes with b in the boundary. Thus the lemma is a consequence of Dobrushin's uniqueness theorem (Theorem A.4.1), and equation (3.2.2). Q.E.D.

As a consequence of Lemma 3.2.1 and Theorem 3.2.1, we have the immediate corollary.

<u>Corollary 3.2.1</u> If $d=4$ and $[\sup(h) - \inf(h)] < 2/9$, then the conclusion of Theorem 3.2.1 holds.

<u>Theorem 3.2.2</u> Suppose $d=4$, and $\mu \in G(h)$ ($G(h)$ not necessarily a one element set) such that μ is <u>both</u> extreme (among all Gibbs states not just the invariant states) and invariant. Then

$$\lim_{a \downarrow 0} \mu(e^{i(J,j_a)}) = \exp\left(\frac{-\alpha}{2}(dj,dj)\right), \tag{3.2.3}$$

where j is a <u>real</u> test 1-form, and α as above.

<u>Remark 3.2.3</u> The collection of Gibbs states ($G(h)$) is not assumed to contain only one element in Theorem 3.2.2. In fact, Theorem 3.2.2 would be a consequence of Theorem 3.2.1 if $|G(h)|=1$.

<u>Remark 3.2.4</u> It will be shown in Chapter 5 that for Wilson "like" energy functions there always exists an extreme and translation invariant Gibbs state. In fact, this state may be found by taking the thermodynamic limit resulting from zero boundary conditions.

The proofs of the two main theorems (Theorems 3.2.1 and 3.2.2) will be postponed. First we will prove a number of lemmas which will lead to the main results.

<u>Lemma 3.2.2</u> Let μ be an invariant Gibbs state, then

$$\alpha \equiv \mu(h''(d\omega(p))) = (2(d-1))^{-1} \cdot \mu(J(b)^2), \tag{3.2.4}$$

where b is any bond, and d is the dimension. Furthermore, $\alpha > 0$ if h is not a constant.

Proof:

$$\mu(J(b)^2) = \mu(\partial J(b)/\partial\omega(b)) = \mu\Big(\frac{\partial}{\partial\omega(b)}\Big\{ \sum_{p:b\in\partial p} F(p)\Big\}\Big)$$

$$= \sum_{p:b\in\partial p} \mu(\ h''(d\omega(p))) = 2(d-1)\alpha.$$

The Schwinger-Dyson equations were used in the first equality, the definition of J in the second, the definition of F in the third, and the fact that 2(d-1) is the number of plaquettes with a given bond in their boundaries. So the validity of equation (3.2.4) has been shown.

Suppose that J(b) were identically zero. Let $p\in\mathbf{B}_+$ with $b\in\partial p$, and choose $b'\in\partial p$ such that $b'\neq b$. Then $\partial J(b)/\partial\omega(b')\equiv 0$ implies that $h''\equiv 0$. Hence, h' is a constant. Since h' is odd, the constant must be 0. So J(b) is not identically zero if h is not a constant.

Fix a bond $b\in\mathbf{B}_+$, and let $B\subset\subset\mathbf{B}_+$ be such that $b\in B$ and J(b) is $\mathcal{F}(B)$-measurable. If h is not constant there is a neighborhood in Ω with $J(b)^2>0$, using the continuity of J(b). It then follows that $\Pi^h_B(\omega,J(b)^2)>0$ for <u>all</u> configurations ω, since finite dimensional Lebesque measure charges open sets. Since $\mu(J(b)^2) = \mu\Pi^h_B(\cdot,J(b)^2)$ by the D.L.R. equations, we conclude that $\mu(J(b)^2)>0$, and thus $\alpha>0$.

Q.E.D.

<u>Lemma 3.2.3</u> (See Gross [3].) Let j be a lattice 1-form with finite support, $\mu\in G(h)$, and $u(s) = \mu(\ e^{s(J,j)})$ for s real. Then

$$u'(s) = s\sum_{p\in\mathcal{P}_+} dj(p)^2\mu(h''(d\omega(p)\cdot e^{s(J,j)}). \qquad (3.2.5)$$

<u>Proof</u>: $u'(s) = s\mu(\ (J,j)e^{s(J,j)})$

$$= (1/2)\Sigma j(b)\mu(J(b)e^{s(J,j)})$$

$$= (1/2)\Sigma j(b)\mu(\frac{\partial}{\partial\omega(b)}e^{s(J,j)}))$$

$$= (s/2)\Sigma j(b)\mu(\frac{\partial\ (J,j)}{\partial\omega(b)}\cdot e^{s(J,j)}), \qquad (3.2.6)$$

where the sums are over all bonds. The Schwinger-Dyson equations were used in the third equality.

Set φ=dj, so by Definitions 2.3.2 and 2.3.3, (J,j) = (F,φ). We now compute the derivative in (3.2.6).

$$\sum_b \frac{\partial(F,\varphi)}{\partial\omega(b)}j(b) =$$

$$=(1/2)\ \sum_b\sum_p j(b)\varphi(p)\{h''(d\omega(p))[1_{\partial p}(b) - 1_{\partial p}(-b)]\}$$

$$=(1/2)\ \sum_p \varphi(p)h''(d\omega(p))[dj(p)-dj(-p)]$$

$$=\sum_p \varphi(p)^2h''(d\omega(p)), \qquad (3.2.7)$$

where $$1_{\partial p}(b)\equiv \begin{cases} 1 \text{ if } b\in\partial p \\ 0 \text{ otherwise}\end{cases}. \qquad (3.2.8)$$

Plug (3.2.7) into (3.2.6) to finish the proof, noting the extra factor of two in (3.2.5) arises from the restriction of the sum to $\mathcal{P}_+$. Q.E.D.

<u>Proposition 3.2.1</u> (d=4) Suppose that j is a complex valued test 1-form on $\mathbb{R}^4$ and $\mu\in G(h)$ is an invariant Gibbs state. Define

$$c(a)\equiv\sup\{\,|cov_\mu(h''(d\omega(p),e^{s(J,j_a)})|:p\in\mathcal{P}_+,\ s\in[0,1]\}, \qquad (3.2.9)$$

where $cov_\mu(f,g) \equiv \mu(fg) - \mu(f)\mu(g)$. Suppose that

$\lim_{a\downarrow 0} c(a) = 0$, then

$$\lim_{a\downarrow 0} \mu(e^{(J,j_a)}) = \exp(\frac{\alpha}{2}(dj,dj)), \qquad (3.2.10)$$

with α as in Theorem 3.2.1.

<u>Proof</u>: Let $k_a = \alpha(dj_a,dj_a)$, $u_a(s)=\mu(e^{s(J,j_a)})$, and $v_a(s) = \exp(-k_a s^2/2)\cdot u_a(s)$. Then by Lemma 3.2.3

$$\begin{aligned} v_a'(s) &= \exp(-k_a s^2/2)\{-k_a s u_a(s) + u_a'(s)\} \\ &= \exp(-k_a s^2/2)\{(1/2)s \\ &\quad \times \sum_p dj_a(p)^2 \mathrm{cov}_\mu(h''(d\omega(p)),e^{s(J,j_a)})\} \end{aligned} \qquad (3.2.11)$$

So

$$|v_a'(s)| \leq K\cdot c(a)\cdot \| dj_a\|^2 \qquad (3.2.12)$$

where

$$K = \sup \{(s/2)\exp(-k_a s^2/2) \mid 0<a<1 , 0\leq s\leq 1\}, \qquad (3.2.13)$$

and

$$\|dj_a\|^2 \equiv (dj_a,\overline{dj_a}). \qquad (3.2.14)$$

By Lemma 2.2.2 $k_a \longrightarrow k \equiv \alpha(dj,dj)$ as $a \longrightarrow 0$, so that $K<\infty$. Since by assumption $c(a) \longrightarrow 0$ as $a \longrightarrow 0$, $v_a' \longrightarrow 0$ as $a \longrightarrow 0$ uniformly in $s\in[0,1]$. Since $v_a(0) = 1$ ($u_a(0)= 1$), it follows that v_a converges to one uniformly in s. Putting s=1, we conclude that $u_a(1) \longrightarrow e^{k/2}$ as $a \rightarrow 0$.

Q.E.D.

In view of this proposition, the proofs of the main theorems are reduced to showing in each case that $c(a) \longrightarrow 0$ as $a \longrightarrow 0$. The remainder of this section will be devoted to this goal.

<u>Lemma 3.2.4</u> (Dimension = d.) Let j be a complex lattice 1-form with finite support, $\mu \in G(h)$ a translation invariant Gibbs state, and $u(s) = \mu(e^{s(J,j)})$. Then

$$|u(s)| \leq \exp\{(s^2/2)\|dj\|^2 \cdot \| h''\|_\infty\}, \tag{3.2.15}$$

where $\|dj\|^2$ is defined in equation (3.2.14).

<u>Proof</u>: Without loss of generality, it may be assumed that j is real since $|u(s)| \leq \mu(|e^{s(J,j)}|) = \mu(e^{s(J,\mathrm{Re} j)})$, and $\|d\mathrm{Re}(j)\|^2 \leq \|dj\|^2$. So assume j is real.

By equation (3.2.5),

$$|u'(s)| \leq s\|dj\|^2\|h''\|_\infty u(s). \tag{3.2.16}$$

Hence

$$|\ln(u(s)/u(0))| \leq \int_{t\in[0,s]} |d\ln(u(t))/dt|dt \leq \|dj\|^2\|h''\|_\infty s^2/2.$$

The lemma follows by exponentiating this last inequality using $u(0) = 1$. Q.E.D.

For notational ease, let $K(\cdot,\cdot,\cdots,\cdot)$ denote a generic function which is increasing in each of its variables. From lemma to lemma and even line to line there may be many such functions K, which will all be denoted by the same letter.

<u>Lemma 3.2.5</u> (Dimension = d) Let $\mu \in G(h)$ be an invariant Gibbs state, and j be a complex lattice 1-form on Z^d, with finite support. Then

$$\mu(|e^{(J,j)} - 1|^2) \leq K(\|h''\|_\infty \cdot \|dj\|_2)\|h''\|_\infty \cdot \|dj\|_2^2. \tag{3.2.17}$$

<u>Proof</u>: Let

$$v(s) = \mu(|e^{s(J,j)} - 1|^2)$$

$$= \mu(e^{2sRe(J,j)} + 1 - e^{s(J,\bar{j})} - e^{s(J,\bar{j})}). \qquad (3.2.18)$$

Differentiating (3.2.18) using equation (3.2.5) one easily finds the estimate

$$|v'(s)| \le s\|h''\|_\infty \sum_p |dj(p)^2|[\mu(|e^{2s(J,j)}|) + \mu(|e^{s(J,j)}|)].$$

By using the estimate in equation (3.2.15) and this last equation we conclude that

$$|v'(s)| \le K(\|h''\|_\infty \cdot \|dj\|_2)\|h''\|_\infty \cdot \|dj\|_2^2 \qquad (3.2.19)$$

for $s \in [0,1]$. Since $v(0) = 0$, the same estimate holds for v.

Q.E.D.

The idea behind the proofs of the main theorems is to use the cluster properties of unique or extreme Gibbs states to conclude that $c(a) \longrightarrow 0$ as $a \longrightarrow 0$. To carry this procedure out we must divide the test 1-form into "near" and "far" pieces. The cluster properties will be applied to the far pieces. The following lemma enables us to control this splitting of the test 1-form.

For the remainder of this section let g be a real infinitely differentiable function with compact support on $\mathbb{R}^d$. Furthermore assume g is radial, $0 \le g \le 1$, and

$$g(x) = \begin{cases} 0 \text{ if } |x| \ge 2 \\ 1 \text{ if } |x| \le 1 \end{cases}. \qquad (3.2.20)$$

Put $g^r(x) = g(r^{-1}x)$ for all $r > 0$.

<u>Lemma 3.2.6</u> Let j be a complex test 1-form on $\mathbb{R}^d$ and g^r be as above. Define $j^r \equiv g^r \cdot j$ for all $r > 0$. Then

$$a^{d-4}\|(dj^r)_a\|_2^2 \le K \cdot [\|j\|_\infty^2 \cdot r^{(d-2)} + \|dj\|_\infty^2 r^d], \qquad (3.2.21)$$

for all positive r and a. The sup-norms are supremums over all the components of the forms and over all of $\mathbb{R}^d$. K is a constant which only depends on the dimension (d), and the function (g).

<u>Proof</u>:

$$\|(dj^r)_a\|_2^2 \le [\|(dg^r \wedge j)_a\|_2 + \|(g^r \cdot dj)_a\|_2]^2$$
$$\le 2[\|(dg^r \wedge j)_a\|_2^2 + \|(g^r \cdot dj)_a\|_2^2] \qquad (3.2.22)$$

We estimate the two terms of (3.2.22) separately. Starting with the first term observe,

$$|(dg^r \wedge j)_a(p)| \le \int_{ap} |dg^r \wedge j| \le \|j\|_\infty \cdot \int_{ap} |dg^r|, \qquad (3.2.23)$$

where the absolute value of a form denotes the maximum over the absolute value of the components By the definition of g^r, there exists a dimension dependent constant (c) such that

$$\int_{ap} |dg^r| = 0 \text{ if } \mathrm{dist}(ap,0) \ge cr. \qquad (3.2.24)$$

Using the estimate,

$$|dg^r| = |r^{-1}\nabla g(r^{-1}x) \cdot dx| \le r^{-1}\|\nabla g\|_\infty \qquad (3.2.25)$$

and (3.2.24) we conclude that

$$\int_{ap} |dg^r| \le \|\nabla g\|_\infty r^{-1} a^2 \chi_{(cr/a)}(p) \qquad (3.2.26)$$

where

$$\chi_s(p) \equiv \begin{cases} 1 \text{ if } \mathrm{dist}(p,0) \le 1 \\ 0 \text{ otherwise} \end{cases}. \qquad (3.2.27)$$

Combine the estimates (3.2.23) into (3.2.26) to get

$$|(dg^r \wedge j)_a(p)| \le K\|j\|_\infty r^{-1} a^2 \chi_{cr/a}(p). \qquad (3.2.28)$$

Square and then sum this last estimate over all plaquettes to obtain

$$a^{(d-4)}\|(dg^r\wedge j)_a\|_2^2 \le K\cdot\|j\|_\infty^2\cdot r^{(d-2)}. \tag{3.2.29}$$

To obtain this last inequality I have used

$$|\{p:\ \mathrm{dist}(p,0) \le cr/a\}| \le K\cdot(r/a)^d, \tag{3.2.30}$$

where K denotes a constant only depending on d.

By a similar (easier) argument it follows

$$a^{(d-4)}\|(g^r dj)_a\|_2^2 \le K\cdot\|dj\|_\infty^2\cdot r^d, \tag{3.2.31}$$

where K is a constant only depending on d.

The theorem now follows from the estimates (3.2.22), (3.2.29) and (3.2.31). Q.E.D.

<u>**Theorem 3.2.3**</u> (d=4) Let h be an energy function, $\mu\in G(h)$, and j be a complex test 1-form on $\mathbb{R}^4$. Fix a plaquette (p_o) based at $0\in Z^4$. Then

a) if $|G(h)| = 1$, then

$$|\mathrm{cov}_\mu(h''(d\omega(p_o),e^{(J,j_a)})| \le K(\|j\|_\infty,\|dj\|_\infty,\|dj_a\|_2)\cdot O_{p_o}(a) \tag{3.2.32}$$

b) if μ is extreme $(\mu\in G_e(h))$ and j is purely imaginary, then the estimate (3.2.32) still holds.

Where in (3.2.32), $O_{p_o}(a)$ denotes a function which is independent of the test 1-form (j), and tends to zero as a tends to zero.

<u>Proof</u>: Let r be a positive number less that one, j^r be as above, and put $k^r\equiv j - j^r$. For notational ease put $j_a^r = (j^r)_a$, $k_a^r = (k^r)_a$, and $f(\omega) = h''(d\omega(p_o))$. The parameter (r)

will eventually be chosen to be a function of a which converges to zero as a goes to zero.

First split $\mathrm{cov}_\mu(f, e^{(J,j_a)})$ into two parts,

$$\mathrm{cov}_\mu(f, e^{(J,j_a)}) = \mathrm{cov}_\mu(f, e^{(J,j_a^r)}\, e^{(J,k_a^r)})$$
$$= \mathrm{cov}_\mu(f, [e^{(J,j_a^r)}-1]e^{(J,k_a^r)})$$
$$+ \mathrm{cov}_\mu(f, e^{(J,k_a^r)}). \qquad (3.2.33)$$

Call the first term in (3.2.33) A and the second term B. We now estimate $|A|$,

$$|A| \le 2\|f\|_\infty \mu(|[e^{(J,j_a^r)}-1]e^{(J,k_a^r)}|)$$
$$\le 2\|f\|_\infty \|[e^{(J,j_a^r)}-1]\|_{L^2(\mu)} \|e^{(J,k_a^r)}\|_{L^2(\mu)}$$
$$\le K(\|dj_a^r\|_2, \|dk_a^r\|_2)\cdot\|dj_a^r\|_2$$
$$\le K(\|dj\|_\infty, \|j\|_\infty, \|dj_a\|_2)\cdot r, \qquad (3.2.34)$$

where $K(\cdot,\cdot,\cdot)$ denotes a function (depending on h) which is increasing in its arguments. The Cauchy-Schwartz inequality was used in the second inequality, Lemma 3.2.4 and Lemma 3.2.5 in the third, and $\|dk_r^a\|_2 \le \|dj_a\|_2 + \|dj_a^r\|_2$, Lemma 3.2.6 and $r \le 1$ in the last inequality.

To estimate $|B|$ we will have to divide the proof into two cases corresponding to the two cases of the theorem. However, first note that $(J,k_a^r)(\omega)$ depends only on the bond variables outside a ball of radius (cr/a), where c is a positive constant. At this time choose $r = a^{(1/2)}$, hence $r/a \longrightarrow \infty$ as $a \longrightarrow 0$.

Case (a) $|G(h)| = 1$: By the strong cluster property

(Theorem A.3.2), there exists a function $O_{p_o}(a)$ as in the statement of the theorem such that

$$|B| = |\mathrm{cov}_\mu(f, e^{(J,k_a^r)})| \le O_{p_o}(a)\cdot\mu(|e^{(J,k_a^r)}|). \qquad (3.2.35)$$

Using Lemma 3.2.4 , Lemma 3.2.6 and equation (3.2.35) we conclude that

$$|\mathrm{cov}_\mu(f, e^{(J,k_a^r)})| \le O_{p_o}(a)\cdot K(\|j\|_\infty, \|dj\|_\infty, \|dj_a\|_2). \qquad (3.2.36)$$

Case (b) $\mu\in G(h)$ is extreme and j is purely imaginary: The observation that $|e^{(J,k_a^r)}| = 1$ and the cluster property of Theorem A.3.1 asserts again an estimate of type (3.2.36) holds (with K independent of j in this case).

The estimate (3.2.32) follows from combining the estimates (3.2.34), (3.2.36), and (3.2.33). Q.E.D.

<u>Lemma 3.2.7</u> Suppose that $\mu\in G(h)$ is an invariant (translation and 90° rotation invariant) Gibbs state satisfying an estimate of the form (3.2.32), then

$$|\mathrm{cov}_\mu(h''(d\omega(p)), e^{(J,j_a)})| \le K(\|j\|_\infty, \|dj\|_\infty, \|dj_a\|_2)\cdot O_{p_o}(a) \qquad (3.2.37)$$

holds for all $p\in\mathcal{P}$.

<u>Proof</u>: Let T_x denote the natural translation operators on differential forms and on lattice forms. Then if j is a test 1-form

$$(T_{ax}j)_a = T_x j_a. \qquad (3.2.38)$$

So using (3.2.38) and the fact that both sides of the estimate (3.2.32) are invariant under translations of j, allows us to conclude that inequality (3.2.37) is valid if the plaquette (p_o) in the left hand side of (3.2.37) is replaced by any of its translates. Using a similar argument for rotations we conclude that the plaquette (p_o) on the left hand side of the estimate (3.2.37) may be replaced by any plaquette (p). Q.E.D.

Proof of Theorems 3.2.1 and 3.2.2: Let $c(a)$ be as in equation (3.2.9),

$$c(a) \equiv \sup\{\mathrm{cov}_\mu(h''(d\omega(p),e^{s(J,j_a)})\,|\,p\in\mathcal{P}_+,\ s\in[0,1]\}.$$

By hypothesis of Theorem 3.2.1, Theorem 3.2.3 and Lemma 3.2.7 may be used to conclude that $c(a) \longrightarrow 0$ as $a \longrightarrow 0$ if j is a test 1-form. Similarly, under the hypothesis of Theorem 3.2.2, $c(a) \longrightarrow 0$ as $a \longrightarrow 0$ if j is purely imaginary test 1-form. Both theorems now follow by applying Proposition 3.2.1. Q.E.D.

Section 3.3 Convergence for $d > 4$

In this section it is shown that the Laplace transform of the lattice measure (on the current algebra) converges to the function one, which is the Laplace transform of the Free Euclidean measure with infinite coupling constant. Unfortunately, the limiting value is rather uninteresting.

Theorem 3.3.1 Assume the dimension (d) is larger than four. Let h be an energy function, $a > 0$ the lattice spacing parameter, and $\mu_a \in G(a^{(d-4)}h)$ be any Gibbs state of the energy function $a^{(d-4)}h$. Then for each test 1-form (j) on $\mathbb{R}^d$,

$$\lim_{a \downarrow 0} \mu_a(\exp[a^{(d-4)}(J,j_a)]) = 1, \tag{3.3.1}$$

where J is the current associated with the energy function (h), see Definition 2.3.3.

Remark 3.3.1 By Dobrushin's uniqueness theorem (see Lemma 3.2.1) $|G(a^{(d-4)}h)| = 1$, if a is sufficiently small. So as in Remark 3.2.4 the measures (μ_a) in Theorem 3.3.1 are invariant for small a.

Proof: For notational ease, let $\varphi \equiv dj$, $\pi_B^a(\omega',\cdot) \equiv \pi_B^{a^{(d-4)}h}(\omega,\cdot)$, and $u_a(s) = \mu_a(e^{sa^{(d-4)}(J,j_a)})$ for $s\in[0,1]$. Then by Lemma 3.2.3 with h replaced by $a^{(d-4)}h$,

$$u_a'(s) = (s/2)a^{(d-4)} \sum_{p\in\mathcal{P}} \varphi_a(p)^2 \mu_a(h''(d\omega(p))e^{sa^{(d-4)}(J,j_a)}). \tag{3.3.2}$$

Since $(s/2)a^{(d-4)} \sum_{p\in\mathcal{P}} \varphi_a(p)^2 \longrightarrow s(\varphi,\varphi)$ as $a \to 0$, it suffices to show that $\mu_a(h''(d\omega(p))e^{sa^{(d-4)}(J,j_a)}) \longrightarrow 0$ uniformly in p as $a \to 0$, because $u_a(0) = 1$.

Let p be a plaquette based at zero. Choose a subset $B \subset\subset \mathcal{B}_+$ which contains all the bonds (disregarding orientation) of any plaquettes having a bond in common with p. By the finite volume Schwinger-Dyson equations (see equation (3.1.2)) with h replaced by $a^{(d-4)}h$,

$$\Pi_B^a(\omega',a^{(d-4)}h''(d\omega(p)) = \Pi_B^a(\omega',a^{(d-4)}\frac{\partial}{\partial\omega(b)}[h'(d\omega(p)])$$

$$= a^{2(d-4)}\Pi_B^a(\omega',J(b)F(p)), \qquad (3.3.3)$$

where b is any bond in the ∂p_o, and $\omega'\in\Omega$ is any configuration. Dividing equation (3.3.3) by $a^{(d-4)}$, using the fact that J and F are uniformly bounded, we find

$$|\Pi_B^a(\omega',h''d\omega(p))| \le Ka^{(d-4)}, \qquad (3.3.4)$$

where K is a constant depending on $\|h\|_\infty$.

As in Theorem 3.2.3, split the 1-form j_a into its "near" and "far" pieces,

$$j_a^n(b) = \begin{cases} j_a(b) & \text{if b or -b is in B} \\ 0 & \text{otherwise} \end{cases} \qquad (3.3.5)$$

and

$$j_a^f = j_a - j_a^n \,. \qquad (3.3.6)$$

We do not have to be so careful in this case.

Now

$$|\mu_a(h''(d\omega(p))e^{sa^{(d-4)}(J,j_a)})|$$

$$= |\mu_a(h''(d\omega(p))e^{sa^{(d-4)}(J,j_a^n)}\cdot e^{sa^{(d-4)}(J,j_a^f)})|$$

$$= |\mu_a\left\{\Pi_B^a[\cdot,h''(d\omega(p))e^{sa^{(d-4)}(J,j_a^n)}]\cdot e^{sa^{(d-4)}(J,j_a^f)}\right\}|$$

$$\le e^{K\|j\|_\infty a^{(d-3)}}\cdot O(a^{(d-4)})\cdot\mu_a(|e^{sa^{(d-4)}(J,j_a^f)}|), \qquad (3.3.7)$$

where K is a constant depending on $|B|$ and $\|h\|_\infty$. The D.L.R. equations were used in the second equality along with the fact that (J,j_a^f) is $\mathcal{F}_B$-measurable (i.e. only depends on the bond variables outside of B). The inequality is a consequence of equation (3.3.4) and the easy estimate

$$\|e^{sa^{(d-4)}(J,j_a^n)}\|_\infty \le e^{Ka^{(d-3)}\cdot\|j\|_\infty}, \tag{3.3.8}$$

where K only depends on $|B|$. But by Lemma 3.2.4 with h replaced by $a^{(d-4)}h$,

$$\begin{aligned}\mu_a(|e^{sa^{(d-4)}(J,j_a^f)}|) &\le \exp\{a^{(d-4)}\|dj_a^f\|_2^2\|h\|_\infty/2\}\\ &\le \exp\{a^{(d-4)}\|dj_a\|_2^2\|h\|_\infty/2\},\end{aligned} \tag{3.3.9}$$

Combine the estimates (3.3.9) and (3.3.7) to get

$$|\mu_a(h''(d\omega(p))e^{sa^{(d-4)}(J,j_a)})| \le K(\|j\|_\infty, a^{(d-4)}\|dj_a\|_2)\cdot a^{(d-4)}, \tag{3.3.10}$$

where K is an increasing function in its arguments. As in Lemma 3.2.7 the estimate (3.3.10) remains valid when (p) is any plaquette, since for small a the Gibbs state μ_a is invariant. Hence we have shown that

$\mu_a(h''(d\omega(p))e^{sa^{(d-4)}(J,j_a)}) \longrightarrow 0$ uniformly in p as $a \longrightarrow 0$,

so the theorem is proved. Q.E.D.

Chapter 4

CORRELATION INEQUALITIES

This chapter reviews some basic correlation inequalities for statistical mechanical models with the circle as state space and "cosine" type interactions. These correlation inequalities along with their consequences will be used in Chapter 5 to prove (among other results) for "Wilson-like" actions the existence of an extreme invariant Gibbs state.

I will follow closely the treatment of Messager, Miracle-Sole, and Pfister [1], and Pfister [1]. Also see Frohlich and Pfister [1] and [2]. For related material see Bricmont, Fontaine and Landau [1].

Section 4.1 Notation

Let $S = S^1$ be the unit circle which is to be identified with $[-\pi,\pi]$ with end points identified as before. Let N be a positive integer. If $\omega \in \mathbb{R}^N$ and $m \in Z^N$, then let $m\cdot\omega$ denote the usual inner product on $\mathbb{R}^N$, $m\cdot\omega = \sum_{i=1}^{N} m_i\omega_i$.

Definition 4.1.1 Let J and ϕ be real valued functions on Z^N with finite support. The associated probability measure (μ_ϕ^J) on S^N is

$$d\mu_\phi^J(\omega) = Z(\phi)^{-1}\exp\left\{ \sum_{m\in Z^N} J(m)\cos(m\cdot\omega - \phi(m)) \right\} d\omega, \quad (4.1.1)$$

where $\omega \in [-\pi,\pi]^N$, $Z(\phi)$ is the normalization constant, and $d\omega$ is N-dimensional Lebesque measure. If ϕ is the zero function, we write μ_0^J for μ_ϕ^J.

Section 4.2 Correlation Inequalities

The following lemma is basic to all correlation inequalities involving measures of the form (4.1.1), see Ginibre [1].

Lemma 4.2.1 Let $F:\mathbb{R}^N \longrightarrow \mathbb{R}$ be a bounded function which is 2π-periodic in each coordinate when the other coordinates are fixed. Furthermore, it is assumed that

$$G(\omega,\omega') \equiv F([\omega+\omega']/2)F([\omega-\omega']/2) \tag{4.2.1}$$

is 2π-periodic in each coordinate of ω and ω', where $\omega,\omega' \in \mathbb{R}^N$. Then

$$\int G(\omega,\omega')d\omega d\omega' = 4^{-N}\left[\int F(\omega)d\omega\right]^2, \tag{4.2.2}$$

where each integral in the above iterated integrals is over any interval of length 2π (which interval is not important due to the periodicity assumptions).

Proof: Let I be the left hand side of (4.2.2). By the periodicity assumptions on G,

$$I = 2^{-2N}\int_{[-2\pi,2\pi]^{2N}} G(\omega,\omega')d\omega d\omega'. \tag{4.2.3}$$

Now make the change of variables $\alpha = [\omega+\omega']/2$ and $\alpha'=[\omega-\omega']/2$. Then $d\alpha d\alpha'=2^{-N}d\omega d\omega'$, and the new domain of integration is the N fold product of the domain $D\subset\mathbb{R}^2$ which is the union of regions 1-4 in Figure 4.2.1. So,

$$I = 2^{-N} \int_{D^N} F(\alpha)F(\alpha')d\alpha d\alpha' . \qquad (4.2.4)$$

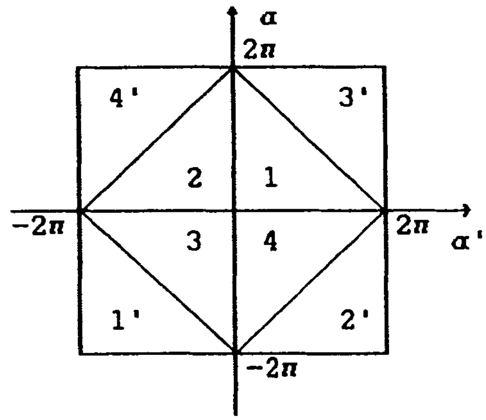

Figure 4.3.1

Using the periodicity assumptions on F, it is easy to check that for each component (α_i, α_i') the integral over any of the regions 1-4 may be replaced by an integral over 1'-4' respectively. For instance, the change of variables $(\alpha_i, \alpha_i') \longrightarrow (\alpha_i - 2\pi, \alpha_i' - 2\pi)$ takes region 1 to region 1' without changing the integrand in equation (4.2.4). Hence we may integrate over $[-2\pi, 2\pi]^{2N}$ in (4.2.4) provided we divide the result by 2^N. But this is precisely the statement of the lemma. Q.E.D.

<u>Remark 4.2.1</u> The key result used from this lemma is not the explicit expression on the right hand side of equation (4.2.4), but the fact that this quantity is non-negative.

<u>Proposition 4.2.1</u> Let J, J' and ϕ be real valued functions on Z^N with finite support. Furthermore, assume that $J(m) \geq |J'(m)|$ for all m. Let $\langle\cdot\rangle$ and $\langle\cdot\rangle'$ denote the expectations with respect to the measures μ_0^J and $\mu_\phi^{J'}$ respectively. Then for any $m,n \in Z^N$ and real function ψ on Z^N,

$$\langle\cos(m\cdot\omega)\cos(n\cdot\omega)\rangle - \langle\cos(m\cdot\omega - \psi(m))\cos(n\cdot\omega - \psi(n))\rangle'$$
$$\geq |\langle\cos(m\cdot\omega)\rangle\cdot\langle\cos(n\cdot\omega - \psi(n))\rangle' - \langle\cos(m\cdot\omega - \psi(m))\rangle'\cdot\langle\cos(n\cdot\omega)\rangle|. \quad (4.2.5)$$

<u>Remark 4.2.2</u> The special case when J = J', and both ψ and ϕ are zero may be found in Ginibre [1]. The proposition in this form is Proposition 1 of Messager et al. [1].

<u>Proof</u>: The trick in correlation inequalities is to introduce auxilliary variables, and then reformulate the inequalities in such a fashion that the normalization constants of the measures play no role.

The inequality (4.2.5) is equivalent to showing the quantities

$$I_\pm \equiv \langle\cos(m\cdot\omega)\cos(n\cdot\omega)\rangle - \langle\cos(m\cdot\omega - \psi(m))\cos(n\cdot\omega - \psi(n))\rangle'$$
$$\pm [\langle\cos(m\cdot\omega)\rangle\cdot\langle\cos(n\cdot\omega - \psi(n))\rangle' - \langle\cos(m\cdot\omega - \psi(m))\rangle'\cdot\langle\cos(n\cdot\omega)\rangle] \quad (4.2.6)$$

are greater than or equal to zero. But $I_\pm$ may be written in terms of auxilliary variables as

$$I_\pm = \frac{1}{2}\iint \mu(d\omega)\mu'(d\omega')[\cos(m\cdot\omega) - (\pm)\cos(m\cdot\omega' - \psi(m))]$$
$$\times[\cos(n\cdot\omega) \pm \cos(n\cdot\omega' - \psi(n))]. \quad (4.2.7)$$

where μ and μ' denote μ_0^J and $\mu_\phi^{J'}$ respectively. So the lemma is equivalent to showing this last expression is greater than or equal to zero; a statement that is independent of the normalization constants (the normalization constants are positive).

The rest of the proof is a matter of using trigonometric identities to get the expressions in (4.2.7) into a form for which Lemma 4.2.1 is applicable. The relevant identities are:

$$\cos(m\cdot\omega) + \cos(m\cdot\omega' - \psi(m))$$
$$= 2\cos(m\cdot\frac{\omega' + \omega}{2} - \psi(m)/2)\cos(m\cdot\frac{\omega' - \omega}{2} - \psi(m)/2) \qquad (4.2.8)$$

and

$$\cos(m\cdot\omega) - \cos(m\cdot\omega' - \psi(m))$$
$$= 2\sin(m\cdot\frac{\omega' + \omega}{2} - \psi(m)/2)\sin(m\cdot\frac{\omega' - \omega}{2} - \psi(m)/2). \qquad (4.2.9)$$

The main consequence of these last two equations is that arbitrary products with factors of the form $[\cos(m\cdot\omega) \pm \cos(m\cdot\omega' - \psi(m))]$ may be written as $F([\omega+\omega']/2)F([\omega-\omega']/2)$ with F obeying the hypothesis of Lemma 4.2.1. Hence the integral of such products is non-negative.

Now the measure $\mu(d\omega)\mu(d\omega')$ is proportional to

$$\exp\{\sum_m [J(m)\cos(m\cdot\omega) + J'(m)\cos(m\cdot\omega' - \psi(m))]\}d\omega d\omega'. \quad (4.2.10)$$

But the summand in the exponent may be written,

$$[J(m)\cos(m\cdot\omega) + J'(m)\cos(m\cdot\omega' - \psi(m))]$$
$$= (1/2)\ [J(m) + J'(m)][\cos(m\cdot\omega) + \cos(m\cdot\omega' - \psi(m))]$$
$$+ (1/2)\ [J(m) - J'(m)][\cos(m\cdot\omega) - \cos(m\cdot\omega' - \psi(m))]. \qquad (4.2.11)$$

By assumption, the coefficients $[J(m) \pm J'(m)]$ are greater than or equal to zero. Thus upon expanding out the exponent in (4.2.10) using the factorization (4.2.11) and the expression (4.2.7) for $I_\pm$, we find that $I_\pm$ may be written in the form

$$I_\pm = \sum_k a_k \iint G_k^\pm(\omega,\omega')\, d\omega d\omega', \qquad (4.2.12)$$

with $G_k^\pm$ a function of the form in Lemma 4.2.1, and $a_k \geq 0$. So by Lemma 4.2.1, each term in the sum (4.2.12) is non-negative which shows that $I_\pm \geq 0$. Q.E.D.

<u>Corollary 4.2.1</u> With the same notation as above,

$$\langle\cos(m\cdot\omega)\rangle \geq \left|\langle\cos(m\cdot\omega - \psi(m))\rangle'\right|, \qquad (4.2.13)$$

where $\psi(m)$ is an arbitrary real number.

<u>Proof</u>: The case $m = 0$ is trivial, so assume $m \neq 0$. Choose $n = 0$ and apply Proposition 4.2.1 with $\psi(0) = 0$ and π, noting that the right hand side of (4.2.5) is larger than or equal to zero. Q.E.D.

<u>Corollary 4.2.2</u> (Messager et al. [1]) Suppose that $\langle\cos(m\cdot\omega)\rangle = \langle\cos(m\cdot\omega)\rangle'$ for some $m\in Z^N$, then $\langle\sin(m\cdot\omega)\rangle' = 0$.

<u>Proof</u>: Again, if $m = 0$ the result is trivial, so assume $m \neq 0$. Apply Proposition 4.2.1 to the case $n = 0$, $\psi(m) = 0$, and $\psi(0) = \Psi$ (Ψ an arbitrary real number) to find

$$\begin{aligned}\langle\cos(m\cdot\omega)\rangle &\geq \langle\cos(m\cdot\omega - \Psi)\rangle' \\ &= \cos(\Psi)\langle\cos(m\cdot\omega)\rangle' + \sin(\Psi)\langle\sin(m\cdot\omega)\rangle'. \qquad (4.2.14)\end{aligned}$$

Using the assumption that $\langle\cos(m\cdot\omega)\rangle = \langle\cos(m\cdot\omega)\rangle'$ in equation (4.2.14) yields

$$[1 - \cos(\Psi)]\langle\cos(m\cdot\omega)\rangle \geq \sin(\Psi)\langle\sin(m\cdot\omega)\rangle'. \qquad (4.2.15)$$

Divide both sides of (4.2.15) by $|\Psi|$ and then take the limits as Ψ tends to zero from above and below to prove the corollary. Q.E.D.

<u>Corollary 4.2.3</u> Let $m,n\in Z^N$ and J be a non-negative function on Z^N with finite support. Then

$$\frac{\partial}{\partial J(n)}\,\mu_0^J(\cos(m\cdot\omega)) \geq 0. \qquad (4.2.16)$$

<u>Proof</u>: An easy computation shows that

$$\frac{\partial}{\partial J(n)}\,\mu_0^J(\cos(m\cdot\omega)) = \mathrm{cov}_{\mu_0^J}(\cos(m\cdot\omega),\cos(n\cdot\omega)). \qquad (4.2.17)$$

This last expression is non-negative by Proposition 4.2.1 with $\psi = \phi \equiv 0$, and $J = J' \geq 0$. Q.E.D.

<u>Section 4.3</u> Extreme States

The above correlation inequalities are useful for producing extreme Gibbs states in "ferromagnetic" models with cosine interaction terms, see Messager et al. [1]. This will be demonstrated in an abstract setting. The application to lattice gauge models will be given in Chapter 5.

Following the notation in the appendix, the lattice (L) will be a countable set, the state space (S) will be the unit circle, and the configuration space (Ω) will be the collection of functions from L to S, $\Omega \equiv S^L$. Let J be a function on Z^L satisfying the following assumptions:

A1) $J \geq 0$ (ferromagnetic)

A2) There exists a positive integer M, such that $J(m) = 0$ if $|\mathrm{supp}(m)| > M$, where $\mathrm{supp}(m) \equiv \{x \in L: m(x) \neq 0\}$.

A3) For each $x \in L$, $|\{m: J(m) > 0, \text{ and } x \in \mathrm{supp}(m)\}| < \infty$.

<u>Definition 4.3.1</u> The interaction potential $\varphi = \varphi^J$ associated to a function (J) satisfying A1-A3 is given by

$$\varphi^J_\Lambda(\omega) = -\sum_{m:\mathrm{supp}(m) = \Lambda} J(m)\cos(m\cdot\omega), \tag{4.3.1}$$

where $m\cdot\omega \equiv \sum_{x\in L} m(x)\omega(x)$. The sum in (4.3.1) is finite by assumptions A2 and A3.

As in the appendix, the specification (Π^J) associated to the interaction potential φ^J is determined by,

$$\Pi^J_\Lambda(\omega, f) = Z^J_\Lambda(\omega)^{-1}\int_{\Omega(\Lambda)} f(\omega'_\Lambda \times \omega_{L\setminus\Lambda})\exp\Big\{\sum_{m:\mathrm{supp}(m)\cap\Lambda\neq\phi} J(m)\cos(m\cdot\omega'_\Lambda \times \omega_{L\setminus\Lambda})\Big\}d\omega'_\Lambda, \tag{4.3.2}$$

where f is any continuous function on Ω, $Z^J_\Lambda(\omega)$ is the normalization constant, and $d\omega'_\Lambda$ denotes Lebesque measure on $\Omega(\Lambda) = S^\Lambda$.

<u>Lemma 4.3.1</u> Let $\Lambda \subset\subset L$ ($\Lambda \subset L$ $|\Lambda| < \infty$) and $m \in Z^L$ with $\mathrm{supp}(m) \subset \Lambda$, then

$$|\Pi^J_\Lambda(\omega, \cos(m\cdot\omega'))| \leq \Pi^J_\Lambda(0, \cos(m\cdot\omega')), \tag{4.3.3}$$

where 0 denotes the zero configuration.

<u>Proof</u>: This is an immediate consequence of Corollary 4.2.2, and the definition of the specification in equation (4.3.2). Q.E.D.

<u>Lemma 4.3.2</u> Let f be a continuous function on Ω, and

$$\Pi^{+}_{\Lambda}(f) \equiv \max_{\omega\in\Omega} \Pi^{J}_{\Lambda}(\omega,f). \tag{4.3.4}$$

Then $\Pi^{+}_{\Lambda}(f)$ decreases as Λ increases.

<u>Proof</u>: Let $\Lambda\subset\Lambda'\subset\subset L$. Then by Lemma A.2.1

$$\Pi^{J}_{\Lambda'}(\omega,f) = \Pi^{J}_{\Lambda'}(\omega,\Pi^{J}_{\Lambda}(\cdot,f)) \le \Pi^{J}_{\Lambda'}(\omega,\Pi^{+}_{\Lambda}(f)) = \Pi^{+}_{\Lambda}(f) \tag{4.3.5}$$

for all $\omega\in\Omega$. Hence, $\Pi^{+}_{\Lambda'}(f) \le \Pi^{+}_{\Lambda}(f)$. Q.E.D.

As a result of this lemma, $\mu^{+}(f) \equiv \lim_{\Lambda\uparrow L} \Pi^{+}_{\Lambda}(f)$ exists for each continuous function f.

<u>Remark 4.3.1</u> Lemma 4.3.2 is a standard result, see for example Preston [1].

<u>Lemma 4.3.3</u> For each continuous function (f),

$$\mu^{o}(f) \equiv \lim_{\Lambda\uparrow L} \Pi^{J}_{\Lambda}(0,f) \tag{4.3.6}$$

exists. Furthermore,

$$\mu^{o}(e^{im\cdot\omega}) = \mu^{o}(\cos(m\cdot\omega)) = \mu^{+}(\cos(m\cdot\omega)) \tag{4.3.7}$$

for any $m\in Z^{L}$ with finite support.

<u>Proof</u>: By the Stone-Weierstrass theorem, linear combinations of functions of the form $e^{im\cdot\omega}$ are dense (in the sup-norm) in the set of continuous functions on Ω. So by an $\epsilon/3$ argument it suffices to show that (4.3.7) is valid. But $\Pi^{J}_{\Lambda}(0,e^{im\cdot\omega}) = \Pi^{J}_{\Lambda}(0,\cos(m\cdot\omega))$, since by reflection invariance $\Pi^{J}_{\Lambda}(0,\sin(m\cdot\omega)) = 0$. So by Lemmas 4.3.1 and 4.3.2, it follows that

$$\begin{aligned}\mu^{o}(e^{im\cdot\omega}) &= \lim_{\Lambda\uparrow L} \Pi^{J}_{\Lambda}(0,e^{im\cdot\omega}) = \lim_{\Lambda\uparrow L} \Pi^{J}_{\Lambda}(0,\cos(m\cdot\omega)) \\ &= \lim_{\Lambda\uparrow L} \Pi^{+}_{\Lambda}(\cos(m\cdot\omega)) = \mu^{+}(\cos(m\cdot\omega)).\end{aligned} \tag{4.3.8}$$

Q.E.D.

<u>Lemma 4.3.4</u> The correlation inequality (4.2.5) still holds with $\langle\cdot\rangle$ replaced by $\mu^{o}(\cdot)$ (of equation (4.3.6)) and $\langle\cdot\rangle'$ replace by $\mu(\cdot)$, where μ is any Gibbs state for the interaction φ^{J}. Consequently, the inequality

$$\mu^{o}(\cos(m\cdot\omega)) \geq \mu(\cos(m\cdot\omega)) \qquad (4.3.9)$$

and the result that if $\mu^{o}(\cos(m\cdot\omega)) = \mu(\cos(m\cdot\omega))$ then $\mu^{o}(\sin(m\cdot\omega)) = \mu(\sin(m\cdot\omega))$ are still valid.

<u>Proof</u>: Let m and n be integer valued functions on Z with finite support and $\Lambda \subset\subset L$ be such that Λ contains the supports of both m and n. By Proposition 4.2.1, inequality (4.2.5) holds with $\langle\cdot\rangle$ replaced by $\Pi^{J}_{\Lambda}(0,\cdot)$ and $\langle\cdot\rangle'$ replaced by $\Pi^{J}_{\Lambda}(\omega,\cdot)$. Replace the absolute value in (4.2.5) by $(\pm)$, and then integrate the equation with respect to the Gibbs state μ. After using the D.L.R. equations we get

$$\begin{aligned}&\Pi^{J}_{\Lambda}(0,\cos(m\cdot\omega)\cos(n\cdot\omega)) - \mu(\cos(m\cdot\omega - \psi(m))\cos(n\cdot\omega - \psi(n)))\\ &\geq \pm[\Pi^{J}_{\Lambda}(0,\cos(m\cdot\omega))\mu(\cos(n\cdot\omega - \psi(n))\\ &\qquad - \mu(\cos(m\cdot\omega - \psi(m))\Pi^{J}_{\Lambda}(0,\cos(n\cdot\omega))],\end{aligned} \qquad (4.3.9)$$

where $\psi(m)$ and $\psi(n)$ are arbitrary real numbers. The desired inequality follows by letting $\Lambda\uparrow L$ in equation (4.3.9). The rest of the Lemma follows by the same arguments as in the proofs of Corollaries 4.2.1 and 4.2.2. Q.E.D.

<u>Proposition 4.3.1</u> The thermodynamic limit $\mu^{o}(\cdot)$ (see (4.3.6)) is an extreme Gibbs state. (See Theorem 1 of Messager et al. [1].)

Proof: Suppose that $\mu^o = s\mu_1 + (1-s)\mu_2$ with $s\in(0,1)$. I will show that $\mu_1 = \mu_2 = \mu^o$, so that μ^o must be extreme.

Let $m\in Z^L$ be a function of finite support. By Lemma 4.3.4, $\mu^o(\cos(m\cdot\omega)) \geq \mu_i(\cos(m\cdot\omega))$, so $\mu^o(\cos(m\cdot\omega)) = \mu_i(\cos(m\cdot\omega))$ for i = 1 or 2. Hence, again by Lemma 4.3.4, $\mu^o(\sin(m\cdot\omega)) = \mu_i(\sin(m\cdot\omega))$ for i = 1 or 2. Since linear combinations of such cosines and sine functions are dense in the continuous functions with the sup-norm topology (Stone-Weierstrass theorem), it follows that $\mu^o(f) = \mu_1(f) = \mu_2(f)$ for all continuous f. So the probability (necessarily Borel) measures μ^o, μ_1, and μ_2 are all equal. Q.E.D.

Section 4.4 Invariance Properties of μ^o

Let $T:L\longrightarrow L$ be a bijection of the lattice (L). The map (T) induces actions on Ω and functions on Ω,

$$T(\omega) = \omega\circ T^{-1} \tag{4.4.1}$$

and

$$T(f)(\omega) = f(\omega\circ T) \tag{4.4.2}$$

respectively.

Proposition 4.4.1 Let J be an integer valued function on Z^L satisfying the assumptions A1 - A3 in Section 4.3 and which is also invariant under the action of T, that is $J(m\circ T) = J(m)$ for all $m\in Z^L$. Then the Gibbs state μ^o (see (4.3.6)) is invariant under the action of T, $\mu^o T^{-1} = \mu^o$.

Before proving Proposition 4.4.1, we first observe the following easy lemma.

<u>**Lemma 4.4.1**</u> Let J satisfy the hypothesis of Proposition 4.4.1, then the associated interaction potential (φ^J), energy ($H^J \equiv H^{\varphi^J}$), and specification ($\Pi^J \equiv \Pi^{\varphi^J}$) have the transformation properties:

$$\text{a)} \quad \varphi^J_\Lambda(\omega\circ T) = \varphi^J_{T(\Lambda)}(\omega) \tag{4.4.3}$$

$$\text{b)} \quad H^J_\Lambda(\omega\circ T|\omega'\circ T) = H^J_{T(\Lambda)}(\omega|\omega') \tag{4.4.4}$$

$$\text{c)} \quad \Pi^J_\Lambda(\omega\circ T, f\circ T) = \Pi^J_{T(\Lambda)}(\omega, f), \tag{4.4.5}$$

where ω, $\omega'\in\Omega$, $\Lambda\subset\subset L$, and f is a bounded function on Ω.

<u>Proof</u>: a)

$$\varphi^J_\Lambda(\omega\circ T) = \sum_{m:\mathrm{supp}(m) = \Lambda} J(m)\cos(m\cdot\omega\circ T) = \sum_{m:\mathrm{supp}(m) = \Lambda} J(m)\cos(m\circ T^{-1}\cdot\omega)$$

$$= \sum_{m:\mathrm{supp}(m) = \Lambda} J(m\circ T^{-1})\cos(m\circ T^{-1}\cdot\omega) = \sum_{m:\mathrm{supp}(m) = T(\Lambda)} J(m)\cos(m\cdot\omega),$$

$$= \varphi^J_{T(\Lambda)}(\omega),$$

where the invariance of J was used in the third inequality.

The proofs of (b), and (c) are also easy computations of the same spirit as the proof of (a). Q.E.D.

<u>Proof of Proposition 4.4.1</u>: Let f be a function on Ω, then

$$\mu^o T^{-1}(f) = \mu^o(f\circ T) = \lim_{\Lambda\uparrow L} \Pi^J_\Lambda(0, f\circ T)$$

$$= \lim_{\Lambda\uparrow L} \Pi^J_{T(\Lambda)}(0\circ T^{-1}, f) = \lim_{\Lambda\uparrow L} \Pi^J_{T(\Lambda)}(0, f)$$

$$= \mu^o(f),$$

where equation (4.4.5) was used in the third equality, the fact that the zero configuration is invariant under the action (T) in the fourth, and the observation that $T(\Lambda)\uparrow L$ as $\Lambda\uparrow L$ in the last equality. Q.E.D.

Chapter 5

CONVERGENCE FOR WILSON ACTION

In this chapter we will find a class of interaction functions (h) and associated Gibbs states which verify the hypothesis of Theorem 3.2.2. Furthermore, for this class of energy functions it will be shown (for all but a countable number of values of the coupling constant) that the limit in Theorem 3.2.2 exists and is independent of the invariant Gibbs state chosen.

Section 5.1 Extreme States for Wilson-like Actions

Definition 5.1.1 The functions $h:\mathbb{R} \longrightarrow \mathbb{R}$ of the form

$$h(x) = b - \sum_{k=1}^{N} b_k \cos(k \cdot x), \qquad (5.1.1)$$

where $b_k \geq 0$, and $b \in \mathbb{R}$ are called Wilson-like energy functions.

Remark 5.1.1 A Wilson-like energy function is an energy function as defined in Section 2.3. The Wilson and Generalized Wilson actions of Examples 2.3.1 and 2.3.2 are examples of Wilson-like functions.

Remark 5.1.2 The constant b is irrelevant since the associated specifications are independent b, see equation (2.3.4).

<u>**Theorem 5.1.1**</u> Let h be a Wilson-like function and Π^h its associated specification, see equation (2.3.4). Then the thermodynamic limit

$$\mu^o \equiv \text{weak-}\lim_{B \uparrow \mathcal{B}_+} \Pi^h_B(0,\cdot), \tag{5.1.2}$$

exists (independent of how $B \uparrow \mathcal{B}_+$), and furthermore μ^o is an extreme (translation and 90^o-rotation) invariant Gibbs state.

<u>Proof</u>: To each positively oriented plaquette (p), let

$$m_p(b) = 1_{\partial p}(b) - 1_{-\partial p}(b), \tag{5.1.3}$$

where $b \in \mathcal{B}_+$ (so $m_p \in Z^{\mathcal{B}_+}$). Define an integer valued function (J) on $Z^{\mathcal{B}_+}$,

$$J(m) = \begin{cases} b_k & \text{if } m = km_p \text{ for some } p \in \mathcal{P}_+ \text{ and } k = 1,\cdots,N \\ 0 & \text{otherwise} \end{cases}, \tag{5.1.4}$$

with the b_k's as in (5.1.1). With this definition of J it is easy to check that the specification (Π^{φ^J}) as defined in equation 4.3.2 is the same as the specification (Π^h) as defined in equation (2.3.4). Also the invariance of J with respect to translations and 90^o-rotations is obvious. Hence, the existence of the limit in (5.1.2) and the fact that μ^o is extreme follows from Lemma 4.3.3 and Proposition 4.3.1 respectively. The invariance of the measure follows by applying Proposition 4.4.1 a number of times with T being a translation in the various coordinate directions or a 90^o-rotation. Q.E.D.

<u>Corollary 5.1.1</u> (d=4) Let h be a Wilson-like action, μ^o be the Gibbs state as in Theorem 5.5.1, and j be a real valued test 1-form on $\mathbb{R}^4$. Then

$$\lim_{a \downarrow 0} \mu^o(e^{i(J,j_a)}) = \exp(\frac{-\alpha}{2}(dj,dj)), \qquad (5.1.5)$$

where J is the lattice current associated to h, j_a is the lattice approximation to j, (Section 2.3), and $\alpha = \mu^o(h''(d\omega(p)))$ where p is any plaquette.

<u>Proof</u>: The Corollary is a direct consequence of Theorems 5.1.1 and 3.2.2. Q.E.D.

<u>Seciton 5.2</u> Independence of the Limit

<u>Theorem 5.2.1</u> (d=4) Let h be a Wilson-like energy function, j a real valued test 1-form on $\mathbb{R}^4$, and $\beta > 0$ ($\beta = g^{-2}$, where g is the coupling constant). Let $\mu_\beta \in G(\beta h)$ be any invariant Gibbs state. Then for all but at most a countable number of β (independent of j),

$$\lim_{a \downarrow 0} \mu_\beta(e^{i(J,j_a)}) = \exp(\frac{-\alpha}{2\beta}(dj,dj)), \qquad (5.2.1)$$

where $\alpha = \mu_\beta^o(h''(d\omega(p))$ and μ_β^o is the Gibbs state for the interaction βh as defined in equation (5.1.2).

The proof of Theorem 5.2.1 will be postponed. Our first goal will be to find a criteria on a Gibbs state which insures that the limit in (5.2.1) exists and is the desired value. The following theorem is closely related to Theorem 3.2.3, and will take its place in this setting.

<u>Theorem 5.2.2</u> Let h be a Wilson-like energy function, p_o be a fixed plaquette based at 0, and $\mu \in G(h)$ be any Gibbs state such that $\mu^o(\cos(d\omega(p_o)) = \mu(\cos(d\omega(p_o)))$. Then

$$|\mathrm{cov}_\mu(h''(d\omega(p_o)), e^{i(J,j_a)})| \le K(\|j\|_\infty, \|dj\|_\infty, \|dj_a\|_2) \cdot O_{p_o}(a), \tag{5.2.2}$$

where K is an increasing function in its arguments, $O_{p_o}(a) \longrightarrow 0$ as $a \longrightarrow 0$, and j is any real test 1-form on $\mathbb{R}^4$ (i.e. the estimate (3.2.32) is still valid).

<u>Proof</u>: For ease of notation set $f_o = h''(d\omega(p_o))$. By Theorem A.5.1 we may decompose μ into its extreme states;

$$\mu(f) = \int_{G_e(h)} \nu(f) P(d\nu), \tag{5.2.3}$$

where P is a probability measure on $G_e(h)$ and f is any continuous function on Ω. Set $f = f_o$ in (5.2.3) and use $\mu(f_o) \ge \mu^o(f_o) \ge \nu(f_o)$ for all $\nu \in G(h)$ (Lemma 4.3.4) to conclude that

$$P\{\nu \in G_e(h): \nu(f_o) = \mu^o(f_o)\} = 1. \tag{5.2.4}$$

If f is any continuous function on Ω, let $\hat{f}: G_e(h) \longrightarrow \mathbb{R}$ be the function $\hat{f}(\nu) = \nu(f)$. By definition, the function $\hat{f}$ is measurable on $G_e(h)$, see Theorem A.5.1. With this notation we may restate (5.2.4) as $\hat{f}_o = \mu^o(f_o)$ P-almost surely.

Now if f and g are two continuous functions on Ω and μ is given by equation (5.2.3), an easy computation shows that

$$\mathrm{cov}_\mu(f,g) = \int_{G_e(h)} \mathrm{cov}_\nu(f,g) P(d\nu) + \mathrm{cov}_P(\hat{f},\hat{g}). \tag{5.2.5}$$

In particular if $f = f_o$, then

$$\operatorname{cov}_\mu(f_o, g) = \int_{G_e(h)} \operatorname{cov}_\nu(f_o, g) P(d\nu), \tag{5.2.6}$$

since $\hat{f}_o$ is a constant P-almost surely. By Theorem 3.2.3, for each $\nu \in G_e(h)$ there exist functions K and O_ν such that

$$|\operatorname{cov}_\nu(f_o, e^{i(J,j_a)})| \le K(\|j\|_\infty, \|dj\|_\infty, \|dj_a\|_2) \cdot O_\nu(a), \tag{5.2.7}$$

where K is increasing in its arguments, $O_\nu(a) \longrightarrow 0$ as $a \longrightarrow 0$, and j is any real test 1-form. Furthermore, looking at the proof of Theorem 3.2.3, the function K may be chosen to be continuous and independent of the Gibbs state ν. For later convenience the function K is also chosen to be larger than one. It will be shown below (Lemma 5.2.3) that the functions $O_\nu(a)$ may be chosen so that for each fixed a the map $\nu \longrightarrow O_\nu(a)$ is measurable on $G_e(h)$ and the map is bounded by one. So use the estimate (5.2.7) in equation (5.2.6) with $g = e^{i(J,j_a)}$ to get

$$|\operatorname{cov}_\mu(f_o, e^{i(J,j_a)})| \le K(\|j\|_\infty, \|dj\|_\infty, \|dj_a\|_2) \cdot O_{P_o}(a), \tag{5.2.8}$$

where

$$O_{P_o}(a) = \int_{G_e(h)} O_\nu(a) \, P(d\nu). \tag{5.2.9}$$

An application of the dominated convergence theorem shows that $O_{P_o}(a) \longrightarrow 0$ as $a \longrightarrow 0$. Q.E.D.

<u>Corollary 5.2.1</u> Let h be a Wilson-like energy function and $\mu \in G(h)$ be an invariant Gibbs state such that $\mu(h''(d\omega(p))) = \mu^o(h''(d\omega(p)))$ for some plaquette p (and hence

all p's). Then for each real test 1-form on $\mathbb{R}^4$,

$$\lim_{a\downarrow 0} \mu(e^{i(J,j_a)}) = \exp(\frac{-\alpha}{2}(dj,dj)), \tag{5.2.10}$$

where $\alpha = \mu^{o}(h''(d\omega(p_o)))$.

Proof: The proof is the same as the proof of Theorem 3.2.2 after using Theorem 5.2.2 in place of Theorem 3.2.3 part (b). Q.E.D.

This corollary is the desired convergence criteria that we were seeking. Its proof will be complete once we dispense with the technical detail of measurability in the proof of Theorem 5.2.2.

Lemma 5.2.1 The collection of real continuous functions on $\mathbb{R}^4$ with compact support ($C_c(\mathbb{R}^4)$) is separable in the sup-norm topology. Furthermore, a countable dense set $D\subset C_c(\mathbb{R}^4)$ may be chosen to have the following property. If $f\in C_c(\mathbb{R}^4)$ and n is sufficiently large such that the $\mathrm{supp}(f)\subset B(0,n) \equiv \{x\in\mathbb{R}^4: |x| \le n\}$, then there exists $g\in D$ arbitrarily close to f with the $\mathrm{supp}(g)\subset B(0,n+1)$.

Proof: Let D' be the collection of continuous functions formed by taking polynomials with rational coefficients of the functions $x \longrightarrow |x - y| :\mathbb{R}^4 \longrightarrow \mathbb{R}$, where $y\in\mathbb{R}^4$ with rational components. The collection D' is a countable set. By the Stone-Weierstrass theorem the collection D' when restricted to any compact set $K\subset\mathbb{R}^4$ is dense in C(K).

For each positive integer n, choose $g_n \in C_c(\mathbb{R}^4)$ such that $\mathrm{supp}(g_n) \subset B(0,n+1)$, $g_n \equiv 1$ on $B(0,n)$, and $0 \le g_n \le 1$. We now define the countable collection of functions (D) as $D = \{fg_n : f \in D' \text{ and } n \text{ a positive integer}\}$. Then if $f \in C_c(\mathbb{R}^4)$ with $\mathrm{supp}(f) \subset B(0,n)$ there exists $h_k \in D'$ such that $\|f - h_k\|_{L^\infty(B(0,n+1))} \longrightarrow 0$ as $k \longrightarrow \infty$. Hence $\|f - g_n h_k\|_\infty \longrightarrow 0$ as $k \longrightarrow \infty$, since $\|f - g_n h_k\|_\infty \le \|f - h_k\|_{L^\infty(B(0,n+1))}$.

Q.E.D.

<u>Definition 5.2.1</u> A continuous k-form (φ) on $\mathbb{R}^d$ is a continuous function from $\mathbb{R}^d$ to the degree-k exterior algebra over $\mathbb{R}^d$. In other words, φ is a differential k-form except that the standard coefficients are only required to be continuous rather than smooth.

<u>**Lemma 5.2.2**</u> Let $a > 0$ be fixed. Let X denote the space of pairs $\langle j,\varphi\rangle$, where j is a continuous 1-form and φ is a continuous 2-form on $\mathbb{R}^4$ both with compact support. The space X is given the norm

$$\|\langle j,\varphi\rangle\| \equiv \|j\|_\infty + \|\varphi\|_\infty + \|\varphi_a\|_2. \tag{5.2.11}$$

With the above notation, the space $(X, \|\cdot\|)$ is a separable space.

<u>Proof</u>: Put $\|\langle j,\varphi\rangle\|_\infty = \|j\|_\infty + \|\varphi_a\|_\infty$. Let $D_o \subset X$ be the collection of pairs which have all of their components in the set D of Lemma 5.2.1. The countable set D_o is clearly dense in the space $(X, \|\langle\cdot,\cdot\rangle\|_\infty)$. So let $\epsilon > 0$, and $\langle j,\varphi\rangle \in X$ be given, and suppose that $\mathrm{supp}(\langle j,\varphi\rangle) \subset B(0,n)$. Then by Lemma 5.2.1, there exist $\langle j_k,\varphi_k\rangle \in D_o$ supported in $B(0,n+1)$

converging to $\langle j,\varphi\rangle$ in the sup-norm. It follows by the easy estimate

$$\|(\varphi - \varphi_k)_a\|_2 \le K[n + 1 + 2a]^4\|\varphi - \varphi_k\|_\infty, \tag{5.2.12}$$

that $\|\langle j_k,\varphi_k\rangle - \langle j,\varphi\rangle\| \longrightarrow 0$ as $n \longrightarrow \infty$, where K is the volume of the unit sphere in 4-dimensions. Q.E.D.

__Lemma 5.2.3__ Let K be a continuous function, increasing in its arguments, $K \ge 1$, and such that an estimate of form (3.2.32) of Theorem 3.2.3 is valid for all extreme Gibbs states ν. Then for each $a\in(0,1)$, the function

$$O_\nu(a) \equiv \sup\left\{\frac{|\mathrm{cov}_\nu(f_o, e^{i(J,j_a)})|}{K(\|j\|_\infty, \|dj\|_\infty, \|dj_a\|_2)} : j \text{ is a test 1-form.}\right\} \tag{5.2.13}$$

is measurable as a function of $\nu\in G_e(h)$. Furthermore $O_\nu(a)$ is uniformly bounded by $2\|h''\|_\infty$, and $O_\nu(a) \longrightarrow 0$ as $a \longrightarrow 0$.

__Proof__: Each real test 1-form (j) may be identified with the element $\langle j,dj\rangle\in X$, where X is the space defined in Lemma 5.2.2. The space of test 1-forms given the norm

$$\|j\| \equiv \|\langle j,dj\rangle\| \tag{5.2.14}$$

is a subspace of the separable normed space X, and hence is separable. The expression in the bracket of equation (5.2.13) is easily seen to be continuous in the $\|\cdot\|$ - topology. So it suffices to take the supremum in (5.2.13) over a countable set. But the expression in the braces of (5.2.13) when considered as a function of ν is measurable (by definition), and hence so is $\nu \longrightarrow O_\nu(a)$.

The estimate that $O_\nu(a) \leq 2\|h''\|_\infty$ is trivial. The fact that $O_\nu(a) \longrightarrow 0$ as $a \longrightarrow 0$ is Theorem 3.2.3. Q.E.D.

The next objective is the proof of Theorem 5.2.1. In view of Corollary 5.2.1, it is enough to show the following proposition.

<u>Proposition 5.2.1</u> (Dimension = d.) Let h be a Wilson-like energy function, then each translation invariant measure $\mu \in G(\beta h)$ satisfies $\mu(h''(d\omega(p))) = \mu_\beta^0(h''(d\omega(p)))$ (p is any plaquette), for all but at most a countable number of $\beta > 0$.

<u>Remark 5.2.1</u> This result is modeled on Corollary 4.3 of Pfister [1], and Proposition 3.5 of Frohlich and Pfister [1].

<u>Proof</u>: As noted in Remark 5.1.2 there is no loss of generality in assuming that b = 0 in equation (5.1.1). Under this assumption

$$h''(x) = -h(x) = \sum_{k=1}^{N} b_k \cos(kx). \tag{5.2.15}$$

Let

$$P(\beta) \equiv P(\varphi^{\beta h}) \tag{5.2.16}$$

where $P(\varphi^{\beta h})$ is defined in Theorem 2.4.1. By Theorem 2.4.1, the function $P(\beta)$ is a convex continuous function. So by standard facts about convex functions, $P'(\beta)$ exists for all but a countable number of $\beta > 0$. By Theorem 2.4.1, Theorem A.8.1, and $\dfrac{d\varphi^{\beta h}}{d\beta} = \varphi^h$;

$$P'(\beta) = -\beta\mu\Big(\sum_{B\subset\mathcal{B}_+:0\in B^o} |B^o|^{-1}\cdot\varphi_B^h\Big) = -\beta\mu\Big(\sum_{p\subset\mathcal{P}_+:0\in\partial p^o} h(d\omega(p))/4\Big)$$
$$= -\beta\mu\Big(\sum_{p\subset\mathcal{P}_+:0\in\text{base}(p)} h(d\omega(p))\Big) = \beta\mu\Big(\sum_{p\subset\mathcal{P}_+:0\in\text{base}(p)} h''(d\omega(p))\Big), \tag{5.2.17}$$

where $\mu\in G(\beta h)$ is any translation invariant Gibbs state and β is a point where $P'(\beta)$ exists. By equation (5.2.15), h'' is a sum of cosine terms with positive coefficients, and hence by Lemma 4.3.4

$$\mu_\beta^o(h''(d\omega(p))) \geq \mu(h''(d\omega(p))) \tag{5.2.18}$$

for all plaquettes p and $\mu\in G(\beta h)$. In view of equation (5.2.17) and (5.2.18) we conclude that

$$\mu_\beta^o(h''(d\omega(p))) = \mu(h''(d\omega(p))) \tag{5.2.19}$$

for all plaquettes p and translation invariant $\mu\in G(\beta h)$. Equation (5.2.19) is valid for all β for which $P'(\beta)$ exists, that is for all but a countable number of β's. Q.E.D.

<u>Proof of Theorem 5.2.1</u>: As already noted, Theorem 5.2.1 is a direct consequence of Corollary 5.2.1 and Proposition 5.2.1. Q.E.D.

Appendix

COMPACT LATTICE BASICS

Section A.1 General Notation

This appendix is devoted to some of the basic facts about compact lattice statistical mechanical models which are used in this thesis. In particular the notion of Gibbs states and their basic properties will be reviewed. The proofs of the results stated here may be found in the manuscripts of Israel [1], Preston [1], and Ruelle [1].

The following notation will be fixed in this appendix. Let S (state space) be a compact metric space, L an arbitrary countable set, and ν a given probability measure on the Borel sets of S. The configuration space (Ω) for the system is defined to be S^L. That is, Ω is the collection of functions from the lattice (L) to the state space (S). More generally, the configurations over Λ ($\Omega(\Lambda)$) is $\Omega(\Lambda) \equiv S^{\Lambda}$, where $\Lambda \subset L$. Since Λ is a countable set, $\Omega(\Lambda)$ may be considered as a compact metric space.

The sets $\Omega(\Lambda)$ may also be considered as measurable spaces when endowed with the Borel σ-algebra. The Borel σ-algebras on $\Omega(\Lambda)$ may be pulled back to σ-algebras on Ω. Namely, let $\mathcal{F}(\Lambda)$ be the smallest σ-algebra on Ω for which the projection maps of Ω onto $\Omega(\Lambda)$ are measurable. The projection maps Ω onto $\Omega(\Lambda)$ will be denoted by $\omega \longrightarrow \omega_{\Lambda}$, where ω_Λ is ω restricted to the set Λ. (A function (f) is $\mathcal{F}(\Lambda)$-mesurable iff there is a measurable function (g) on $\Omega(\Lambda)$ with $f(\omega) = g(\omega_\Lambda)$.) It will also be convenient to set $\mathcal{F}_\Lambda \equiv \mathcal{F}(L\backslash\Lambda)$, $\mathcal{F}_\infty \equiv \bigcap_{\Lambda\subset\subset L} \mathcal{F}_\Lambda$, and $\mathcal{F}$ the smallest σ-algebra containing $\mathcal{F}_\Lambda$ for all $\Lambda\subset\subset L$. (A function is $\mathcal{F}_\Lambda$ measurable if it only depends on the configurations outside of Λ.) For $\Lambda\subset L$, $\nu^\Lambda \equiv \prod_{i\in\Lambda} \nu$ will denote the product measure on $\Omega(\Lambda)$.

Finally if $\Lambda\subset L$, it is natural to write Ω as a product of two factors, $\Omega \cong \Omega(\Lambda) \times \Omega(L\backslash\Lambda)$. Corresponding to this decomposition we write $\omega = \omega_\Lambda \times \omega_{L\backslash\Lambda}$.

Section A.2 Definition of Gibbs States

Definition A.2.1 An interaction potential (φ) is a collection $\{\varphi_\Lambda\}_{\Lambda\subset\subset L}$ of real functions on configuration space (Ω), such that φ_Λ is $\mathcal{F}(\Lambda)$ measurable. The notation $\Lambda\subset\subset L$ denotes $\Lambda\subset L$ and $|\Lambda|<\infty$, where $|\Lambda|$ is the cardinality of Λ. The interaction potential $\varphi = \{\varphi_\Lambda\}_{\Lambda\subset\subset L}$ is said to be of finite range if there exists a positive integer n, such that $\varphi_\Lambda \equiv 0$ if $|\Lambda|>n$.

Remark A.2.1 The space of (finite range) interaction potentials naturally forms a real vector space. For our purposes it is sufficient to restrict attention to the vector space of finite range interaction potentials. This restriction will be assumed below without further mention, even though the results hold with more general hypotheses.

<u>Definition A.2.2</u> $\|\varphi\| \equiv \sup_{x\in L} \sum_{\Lambda : x\in\Lambda\subset\subset L} |\Lambda|^{-1} \| \varphi_\Lambda \|_\infty$

<u>Definition A.2.3</u> Let $\Lambda\subset\subset L$. The energy $(H_\Lambda^\varphi(\omega|\omega'))$ of a configuration $\omega\in\Omega$ ("in" Λ) given a configuration $\omega'\in\Omega$ ("outside" of Λ) is defined by:

$$H_\Lambda^\varphi(\omega|\omega') = \Sigma_{\Lambda' : \Lambda'\cap\Lambda\neq\phi} \, \varphi_{\Lambda'}(\omega_\Lambda x \omega'_{L\setminus\Lambda}). \tag{A.2.1}$$

<u>Definition A.2.4</u> The specification $\Pi^\varphi = \{\Pi_\Lambda^\varphi\}_{\Lambda\subset\subset L}$ assosciated to the interaction potential (φ) is the collection of maps $\Pi_\Lambda^\varphi : \Omega \times \mathcal{F} \longrightarrow [0,1]$, defined by:

$$\Pi_\Lambda^\varphi(\omega,A) \equiv Z_\Lambda^\varphi(\omega)^{-1} \int_A e^{-H_\Lambda^\varphi(\omega'|\omega)} \, \nu^\Lambda(d\omega'_\Lambda) \, \delta_{\omega_{L\setminus\Lambda}}(d\omega'_{L\setminus\Lambda}) \tag{A.2.2}$$

where $\delta_{\omega_{L\setminus\Lambda}}$ denotes the point mass at $\omega_{L\setminus\Lambda}$ on $\Omega(L\setminus\Lambda)$ and $Z_\Lambda^\varphi(\omega)$ is chosen so that $\Pi_\Lambda^\varphi(\omega,\Omega) = 1$.

<u>Lemma A.2.1</u> The specifications Π_Λ^φ's have the following properties:

a) $\Pi_\Lambda^\varphi(\cdot,A)$ is a continuous $\mathcal{F}_\Lambda$-measurable function on Ω for all $\Lambda\subset\subset L$ and $A\in\mathcal{F}$.

b) $\Pi_\Lambda^\varphi(\omega,\cdot)$ is a probability measure on $(\Omega,\mathcal{F})$.

c) If $\Lambda \subset \Lambda'\subset\subset L$, then
$$\Pi_\Lambda^\varphi(\omega,\Pi_{\Lambda'}^\varphi(\cdot,A)) = \Pi_{\Lambda'}^\varphi(\omega,\Pi_\Lambda^\varphi(\cdot,A)) = \Pi_{\Lambda'}^\varphi(\omega,A).$$

d) If f is $\mathcal{F}_\Lambda$ measurable, then $\Pi_\Lambda^\varphi(\cdot,f) = f$.

<u>Remark A.2.2</u> Lemma A.2.1 follows by inspection and straight forward computations. Property (c) and (d) lead to

the interpretation of $\Pi_{\Lambda}^{\varphi}(\omega,\cdot)$ as conditional probabilities. The probability measures $\Pi_{\Lambda}^{\varphi}(\omega,\cdot)$ are said to be a finite volume Gibbs state over Λ with boundary conditions ω.

<u>Definition A.2.5</u> An infinite volume Gibbs state (μ) for the interaction potential (φ) is a probability measure on $(\Omega,\mathcal{F})$ such that for each $\Lambda\subset\subset L$ and $A\in\mathcal{F}$:

$$\mu\Pi_{\Lambda}^{\varphi}(\cdot,A)=\mu(A), \tag{A.2.3}$$

where $\mu\Pi_{\Lambda}^{\varphi}(\cdot,A) = \mu(\Pi_{\Lambda}^{\varphi}(\cdot,A))$

<u>Remark A.2.3</u> The equations (A.2.3) are know as the D.L.R. equations (Dobrushin, Lanford, and Ruelle). Probabilisticly, the D.L.R. equations express the fact that μ is a Gibbs state iff $\Pi_{\Lambda}^{\varphi} = \mu(\cdot|\mathcal{F}_{\Lambda})$ μ-almost surely for all $\Lambda\subset\subset L$, where $\mu(\cdot|\mathcal{F}_{\Lambda})$ is the conditional expectation of μ given $\mathcal{F}_{\Lambda}$. Let $G(\varphi)$ denote the collection of Gibbs states associated with the interaction potential (φ).

<u>Remark A.2.4</u> By property (c) of Lemma A.2.1, to determine if a probability measure is a Gibbs state, it suffices to check the D.L.R. equations for any fixed sequence $\Lambda_n\subset\subset L$ with $\Lambda_n\uparrow L$.

Let $\mu_n= \Pi_{\Lambda_n}(\omega_n,\cdot)$, with ω_n an arbitrary sequence in Ω, and $\Lambda_n\subset\subset L$ such that $\Lambda_n\uparrow L$ in the sense that Λ_n eventually encloses any finite subset of L. By compactness we may pass to a subsequence (again denoted by μ_n) for which μ_n converges weakly ($\mu_n(f) \longrightarrow \mu(f)$ for all continuous f) to a probability measure μ on Ω.

Definition A.2.6 Any probability measure (μ) constructed above is called a thermodynamic limit.

Theorem A.2.1 The set of Gibbs states $G(\Phi)$ is a nonempty convex set of probability measures on Ω. The set $G(\Phi)$ contains all thermodynamic limits. In fact $G(\Phi)$ may be constructed by taking convex combinations of the thermodynamic limits.

Section A.3 Cluster Properties of Gibbs States

Definition A.3.1 A Gibbs state μ is said to be extreme if it can not be written as a convex combination of two distinct Gibbs states. Let $G_e(\Phi)$ denote the set of extreme Gibbs states.

Definition A.3.2 A Gibbs state (μ) is said to have the cluster property (C) if for all $f \in C(\Omega)$ there exists $\Lambda \subset\subset L$ such that if $g \in C(\Omega)$ and is $\mathcal{F}_\Lambda$ measurable then

$$|\mathrm{cov}_\mu(f,g)| \leq \| g \|_\infty, \tag{A.3.1}$$

where $\mathrm{cov}_\mu(f,g) \equiv \mu(fg)-\mu(f)\mu(g)$. and $\|\cdot\|_\infty$ is the sup-norm.

Theorem A.3.1 (Ruelle [1] or Preston [1]) A Gibbs state μ is extreme if and only if μ has the cluster property (C).

Corollary A.3.1 $G(\Phi)$ contains only one element iff each Gibbs state satisfies the cluster property (C).

The corollary is a direct consequence of the theorem, since $G(\Phi)$ has more than one element iff there exists a non-extreme Gibbs state.

Definition A.3.3 A Gibbs state (μ) is said to satisfy the strong cluster property (SC) if for all $f \in C(\Omega)$ there exists $\Lambda \subset\subset L$ such that if $g \in C(\Omega)$ and is $\mathcal{F}_{\Lambda}$ measurable then

$$|\mathrm{cov}_{\mu}(f,g)| \leq \mu(|g|). \qquad (A.3.2)$$

Theorem A.3.2 (Ruelle [1]) If $G(\varphi)$ consists of only one element μ, then μ has the strong cluster property.

Section A.4 Dobrushin's Uniqueness Theorem

This section contains a version of Dobrusin's uniqueness theorem which is a combination of Dobrushin's results and an estimate of B. Simon [3]. See Dobrushin [1] for Dobrushin's original work, or Gross [1] or Follmer [1] for another proof. For related material on decay of correlations see Gross [2], and Kunsch [1] and the references therein.

Theorem A.4.1 Suppose $\sup_{x \in L} \sum_{\Lambda : x \in \Lambda} (|\Lambda|-1) \, \| \varphi_{\Lambda} \|_{\infty} < 1$, then $G(\varphi)$ has only one element.

Section A.5 Simplex Properties of $G(\varphi)$

This section contains a version of the well known fact that every Gibbs state is a unique convex decomposition of extreme Gibbs states. The usual version appeals to theorems about Choquet simplexes, for example see Ruelle [1]. Rather than taking this route, we will follow the treatment of Dynkin [1] and Preston [1], and apply a theorem of Dynkin's on "Sufficient Statistics".

Let $\Lambda_n \subset\subset L$ be a sequence increasing to L, and

$$\Omega_o \equiv \{\omega\in\Omega : \lim_{n\to\infty} \Pi^{\varphi}_{\Lambda_n}(\omega,f) \text{ exists for all } f\in C(\Omega)\}.$$

Fix any $\mu_o \in G(\varphi)$ and put

$$\Pi^{\varphi}(\omega,f) \equiv \begin{cases} \lim_{n\to\infty} \Pi^{\varphi}_{\Lambda_n}(\omega,f) & \text{if } \omega\in\Omega_o \\ \mu_o(f) & \text{otherwise} \end{cases} \tag{A.5.1}$$

where f is any continuous function. It is easy to check that $\Pi^{\varphi}(\omega,\cdot)$ is a positive linear functional with $\Pi^{\varphi}(\omega,1) = 1$. Hence, by the Riesz representation theorem $\Pi^{\varphi}(\omega,\cdot)$ may be considered as a probability measure on Ω. In fact, if $\omega\in\Omega_o$ then $\Pi^{\varphi}(\omega,\cdot)$ is a thermodynamic limit, and thus is also a Gibbs state.

<u>**Lemma A.5.1**</u> Let $\Pi^{\varphi}(\cdot,\cdot)$ be as above and μ be any Gibbs state, then:

a) $\Pi^{\varphi}(\cdot,A)$ is $\mathcal{F}_\infty$ measurable for all $A\in\mathcal{F}$.

b) $\mu(\Omega_o) = 1$

c) the conditional probability $\mu(A|\mathcal{F}_\infty) = \Pi^{\varphi}(\cdot,A)$ μ almost surely.

d) $\Pi^{\varphi}(\omega,\cdot)\in G(\varphi)$ for all $\omega\in\Omega$.

<u>**Proof**</u>: Assertion (a) is clear by inspection, and (d) has already been noted. Let $\{f_n\}$ be a countable dense subset (in the sup-norm) for $C(\Omega)$. By standard $\varepsilon/3$ argument one finds that $\Omega_o = \{\omega\in\Omega : \lim_{n\to\infty} \Pi^{\varphi}_{\Lambda_n}(\omega,f_m)$ exists for all m.$\}$. But for any fixed m: $X_n \equiv \Pi^{\varphi}_{\Lambda_n}(\cdot,f_m)$ is a reverse martingale with respect to the measure μ and the filtration $\mathcal{F}_{\Lambda_n}$.

Hence, by standard martingale convergence theorems (see Ikeda and Watanabe [1]) the set $\{\omega\in\Omega : \lim_{n\to\infty} \Pi^{\varphi}_{\Lambda_n}(\omega,f_m) \text{ exists }\}$ has μ-measure 1 for each fixed m. Since $\{f_n\}$ is a countable set, it follows that Ω_o also has μ-measure one, which proves (b). It is easy to show that the conditional expectation $\mu(f|\mathcal{F}_{\infty}) = \Pi^{\varphi}(\cdot,f)$ (μ-almost surely) if f is a continuous function. By the monotone class theorem one may extend the result to all bounded measurable functions. This proves (c). Q.E.D.

<u>Theorem A.5.1</u> For all $\mu\in G(\varphi)$ there exists a unique probability measure (P) on $G_e(\varphi)$ such that :

$$\mu(f) = \int_{G_e(\varphi)} \nu(f)\, dP(\nu) \qquad (A.5.2)$$

for all bounded $\mathcal{F}$-measurable functions (f). The σ-algebra on $G_e(\varphi)$ is taken to be the smallest σ-algebra such that the maps $\nu\longrightarrow \nu(f):G_e(\varphi) \longrightarrow \mathbb{R}$ are measurable for all bounded $\mathcal{F}$-measurable functions (f). Furthermore, the measure (P) is explicitly given by

$$P(M) = \mu(\{\omega: \Pi^{\varphi}(\omega,\cdot)\in M \}) \qquad (A.5.3)$$

where M is a measurable subset of $G_e(\varphi)$.

<u>Remark A.5.1</u> Theorem A.5.1 is the content of Theorems 3.1 and 5.1 in Dynkin [1], specialized to compact lattice models. See also Chapter 2 of Preston [1].

<u>Corollary A.5.1</u> To construct $G(\varphi)$ it suffices to consider only convex combinations of thermodynamic limits of

the form $\lim_{n\to\infty} \Pi^{\varphi}_{\Lambda_n}(\omega,\cdot)$ where Λ_n is any fixed sequence of subsets of L converging to L, and $\omega\in\Omega$ are points for which the limit exists (weak limit).

Section A.6 Definition of Pressure

For the remainder of this appendix we specialize the lattice (L) to be Z^d where Z is the set of integers and d is the dimension — a positive integer. Hence, Ω is now S^{Z^d}. Let $T_x : Z^d \longrightarrow Z^d$, be translation by the element $x\in Z^d$. The action of T_x naturally induces actions on Ω by $T_x(\omega)\equiv\omega\circ T_{-x}$, and on functions (f) on Ω by $T_x(f)\equiv f\circ T_{-x}$, that is $T_x(f)(\omega)=f(\omega\circ T_x)$. Note the abuse of notation of using the same symbol T_x in all cases.

All interaction potentials $\varphi = \{\varphi_\Lambda\}_{\Lambda\subset\subset L}$ are now assumed invariant under translations by elements of Z^d. Explicitly, this means that

$$\varphi_{\Lambda+x}(\omega)=\varphi_\Lambda(\omega\circ T_x), \qquad (A.6.1)$$

for all $x\in Z^d$, $\Lambda\subset\subset L$, and $\omega\in\Omega$.

Definition A.6.1 The pressure of a translation invariant interaction potential (φ), over a finite subset $\Lambda\subset\subset Z^d$ with boundary conditions $\omega\in\Omega$ is the function

$$P_\Lambda(\omega,\varphi) \equiv |\Lambda|^{-1} \ln Z^{\varphi}_{\Lambda}(\omega), \qquad (A.6.2)$$

where $Z^{\varphi}_{\Lambda}(\omega)$ is the normalization constant in Definition A.2.4.

<u>**Theorem A.6.1**</u> Let Λ_n be a sequence of cubes in Z^d with $\Lambda_n \uparrow Z^d$. Then $P(\varphi) \equiv \lim_{n\to\infty} P_{\Lambda_n}(\omega,\varphi)$ exists for all $\omega \in \Omega$, and is in fact independent of the boundary condition ω, and the sequence Λ_n. Furthermore, $P(\cdot)$ is a convex function of the translation invariant interaction potentials and satisfies the following estimate:

$$|P(\varphi) - P(\psi)| \le \| \varphi - \psi \| \tag{A.6.3}$$

where φ, and ψ are any finite range interaction potentials. ($\|\cdot\|$ was defined in Definition A.2.2.)

<u>Section A.7</u> Convex Functions

Let $(X, \| \ \|)$ be a real normed linear vector space.

<u>**Definition A.7.1**</u> A function $P: X \longrightarrow \mathbb{R}$ is said to be convex if for all $x,y \in X$ and $s \in [0,1]$

$$P(sx+(1-s)y) \le sP(x)+(1-s)P(y). \tag{A.7.1}$$

<u>**Definition A.7.2**</u> Let P be a continuous convex function on X, and x a fixed point in X. A tangent functional to P at x is a continuous linear functional (α) on X such that

$$P(y)-P(x) \ge \alpha(y-x) \tag{A.7.2}$$

for all $y \in X$.

<u>**Theorem A.7.1**</u> If P is a continuous convex function, then

$$\partial_y^+ P(x) \equiv \lim_{h\downarrow 0} \frac{P(x+hy) - P(x)}{h} \tag{A.7.3}$$

exists for all $x,y \in X$, and furthermore there exists a tangent functional α at x to P with $\alpha(y) = \partial_y^+ P(x)$.

<u>Proof</u>: The fact that the limit in equation (A.7.3) exists follows from the basic fact that convex functions of one real variable have one sided derivatives everywhere. For the rest of the proof we may assume without loss of generality that x=0 and P(x)=0. Let Y be the subspace of X which is the span of the vector y. Define a linear functional α_o on Y by $\alpha_o(sy) \equiv s\partial_y^+ P(0)$ for all $s \in \mathbb{R}$. Then $\alpha_o \le P$ on the subspace Y. So by the Hahn-Banach theorem (see Reed and Simon [1]) there exists a continuous linear functional (α) on X which is an extension of α_o on Y satisfying $\alpha \le P$. This α is the desired tangent functional.

Q.E.D.

<u>Corollary A.7.1</u> The continuous convex function (P) has at least one tangent functional at each point x of X.

<u>Section A.8</u> Equilibrium States

<u>Definition A.8.1</u> An Equilibrium state is a translation invariant Gibbs state. That is $\mu \in G(\Psi)$ is an equilibrium state if $\mu \circ T_x = \mu$ for all translations T_x.

<u>Definition A.8.2</u> If Ψ is a translation invariant interaction potential, put

$$A\Psi \equiv -\sum_{\Lambda : 0 \in \Lambda \subset\subset L} |\Lambda|^{-1} \Psi_\Lambda \qquad (A.8.1)$$

<u>Definition A.8.3</u> If f is a continuous $\mathcal{F}(\Lambda_o)$-measurable function, $(\Lambda_o \subset\subset L)$, then let Ψ^f be the interaction potential

$$\Psi^f_\Lambda(\omega) \equiv \begin{cases} f(\omega \circ T_x) & \text{if } \Lambda = \Lambda_o + x \text{ for some } x \in Z^d \\ 0 & \text{otherwise} \end{cases} . \qquad (A.8.2)$$

<u>Theorem A.8.1</u> There is a one to one correspondence between equilibrium states and tangent functionals to the pressure. Specifically, if Φ is a finite range translation invariant interaction potential, and μ an equilibrium state for Φ then the corresponding tangent functional α is

$$\alpha(\Psi) = \mu(A\Psi) \qquad (A.8.3)$$

where Ψ is any other such interaction potential. On the other hand, if α is a tangent functional to P at Φ, the corresponding Gibbs state (μ) is determined by

$$\mu(f) = -\alpha(\Psi^f), \qquad (A.8.4)$$

where f and Ψ^f are as in definition (A.8.3).

<u>Corollary A.8.1</u> If Φ is a translation invariant finite range interaction potential then $G(\Phi)$ contains an equilibrium state.

<u>Proof</u>: This follows from Corollary A.7.1 and Theorem A.8.1. Q.E.D.

<u>Remark A.8.1</u> The corollary may also be proved by taking any Gibbs state and then averaging over translations by elements of finite subsets of Z^d. By translation invariance of the potential and convexity of $G(\Phi)$, the averages will still be Gibbs states. By compactness, there will be a subsequence of these averages which converges to a Gibbs state which is necessarily translation invariant.

Corollary A.8.2: Let φ and ψ be translation invariant interaction potentials, then $\frac{d}{ds}|_0 P(\varphi + s\psi)$ exists if and only if $\mu(A\psi) = C$ (C a constant) independent of all $\mu \in G(\varphi)$ which are translation invariant. Furthermore, when the derivative exists it is given by the constant C.

Proof: By Theorem A.7.1 and Theorem A.8.1, if $\mu(A\psi) =$ C independent of the translation invariant μ, then $\partial_\psi^+ P(\varphi) =$ C. Similarly, the left handed derivative $\partial_\psi^- P(\varphi)$ must also be the constant C. Hence, $\frac{d}{ds}|_0 P(\varphi + s\psi)$ exists and is equal to C.

On the other hand, it is easy to check that

$$\partial_\psi^- P(\varphi) \leq \alpha(\varphi) \leq \partial_\psi^+ P(\varphi) \qquad \text{(A.8.5)}$$

for all tangent functionals (α) to the pressure (P). So if $\frac{d}{ds}|_0 P(\varphi + s\psi)$ exists, $\alpha(\psi) = \partial_\psi^+ P(\varphi) = \partial_\psi^- P(\varphi) = \frac{d}{ds}|_0 P(\varphi + s\psi)$ independent of the tangent functional α. But by Theorem A.8.1 the tangent functionals to the pressure are in one to one correspondence with the translation invariant Gibbs states. Q.E.D.

Appendix B

BASIC IDEAS OF CONSTRUCTIVE FIELD THEORY

<u>Section B.1</u> Quantum Mechanics

This appendix discusses the basic ideas of constructive quantum field theory, with the goal of motivating the informal expression for the Euclidean Yang-Mills measure. I will concentrate on the probabilistic approach first introduced by Symanzik [1] and Nelson [1]. The computational aspects will not be emphasized in an effort not to obscure the main ideas. For more detailed accounts see Glimm and Jaffe [1], Simon [1], and Simon [2] For a more physical review, see Kogut [1], and Seiler [1].

Before jumping into field theory it is wise to first look at the probabilistic approach applied to quantum mechanics. For this purpose consider the classical mechanical system described by Newton's equations of motion

$$x''(t) = -\nabla V(x(t)), \tag{B.1.1}$$

where $x:\mathbb{R} \longrightarrow \mathbb{R}^3$, and $V:\mathbb{R}^3 \longrightarrow \mathbb{R}$. The corresponding quantum mechanical time evolution is given by the Schordinger equation

$$i\psi'(t) = H\psi(t), \tag{B.1.2}$$

where $\psi:\mathbb{R} \longrightarrow L^2(\mathbb{R}^3,d\lambda)$ (λ is Lebesque measure), and H is the Hamiltonian,

$$H = -(1/2)\Delta + M_V. \tag{B.1.3}$$

The symbol M_V denotes multiplication by the function V. The domain of H is chosen such that H is self-adjoint. Hence, by the spectral theorem the solution to (B.1.2) is given by the unitary time evolution operator

$$U(t) = e^{-itH} . \tag{B.1.4}$$

In order to incorporate probability theory into the discussion in the relevant form it is necessary to evaluate the time evolution operator at imaginary times

$$S(t) \equiv U(-it) = e^{-tH}, \tag{B.1.5}$$

where $t \geq 0$. The operators $\{S(t)\}_{t\geq 0}$ form a bounded semigroup provided that $\lambda_o = \inf \operatorname{spec}(H) > -\infty$. (This condition on the spectrum of H is physically reasonable, since it prevents the system from releasing an infinite amount of energy under small perturbations.) We now replace H by the physically equivalent Hamiltonian $\hat{H} \equiv H - \lambda_o I$ and replace S(t) by $\hat{S}(t) \equiv e^{-t\hat{H}}$, then $\hat{S}$ is a contraction semigroup of positivity preserving operators. Thus $\hat{S}$ may be used to specify transition probabilities for a Markov process (X_t). We will see below that the path space measure (μ) for the process (X_t) (started with the invariant measure if it exists) is well characterized by the informal expression

$$d\mu(x) = Z^{-1}\exp\left[\int_{\mathbb{R}} [-(1/2)|x'(t)|^2 + V(x(t))]dt\right]Dx. \tag{B.1.6}$$

Where $x \in \Omega \equiv \{x: x:\mathbb{R} \longrightarrow \mathbb{R}^3$ and x is continuous$\}$,

$Dx = \prod_{i=1}^{3} \prod_{t\in\mathbb{R}} dx^i(t)$ is "infinite dimensional Lebesque measure", and Z is a normalization constant so that μ is a probability measure on Ω. Before giving a more precise meaning to (B.1.6), it will be convenient to introduce some notation.

For each $T > 0$, let $\Omega_T = \{x: x:[-T,T] \longrightarrow \mathbb{R}^3$, x is continuous$\}$. The space Ω_T is a Banach space when given the sup-norm. Let ν_T denote Wiener measure on the Borel σ-algebra of Ω_T with the starting distribution of Lebesque measure. Explicitly, ν_T is the unique measure on Ω_T such that

$$\int_{\Omega_T} f(x(t_0),x(t_2),\cdots,x(t_n))\nu_T(dx) =$$

$$\int_{\mathbb{R}^{3(n+1)}} f(x_0,x_2,\cdots,x_n)[p_{t_1-t_0}(x_0,x_1)p_{t_2-t_1}(x_1,x_2)\cdots$$

$$\times p_{t_n-t_{n-1}}(x_{n-1},x_n)]dx_0dx_2\cdots dx_n, \qquad (B.1.7)$$

where f is a bounded measurable function on $\mathbb{R}^{3(n+1)}$, $-T = t_0 < t_1 < t_2 < \cdots < t_n = T$, and $p_t(x,y) = [\text{kernel of } e^{t\Delta/2}](x,y) = (2\pi t)^{-3/2}\, e^{-(1/2t)|x-y|^2}$. It should be noted that ν_T is not a probability measure, in fact x(t) is distributed by Lebesque measure for each $t\in[-T,T]$.

I will now describe a class of probability measures which is closely related to the informal expression (B.1.6) To each pair of non-negative functions $f,g\in L^2(\mathbb{R}^3,d\lambda)$, we assign a probability measure $(\mu_{f,g,T})$ on Ω_T. The measure μ_T

$= \mu_{f,g,T}$ is uniquely determined by

$$\int_{\Omega_T} \prod_{i=0}^{n} f_i(x(t_i))\mu_T(dx) =$$

$$= \frac{(ff_0, \hat{S}(t_1-t_0)M_{f_1}\hat{S}(t_2-t_1)M_{f_2}\cdots\hat{S}(t_{n-1}-t_{n-2})M_{f_{n-1}}S(t_n-t_{n-1})f_n g)}{(f,\hat{S}(2T)g)},$$

(B.1.8)

where f_i's are bounded measurable functions on $\mathbb{R}^3$, $(\cdot,\cdot)$ denotes $L^2(\mathbb{R}^3,d\lambda)$ inner product, and $-T = t_0 < t_1 < \cdots < t_n = T$. The existence of such a measure is guaranteed by the Feynman-Kac formula if V is sufficiently nice, see the proof of Theorem B.1.1 below.

Finally, to make the connection between the measures $\mu_{f,g,T}$ with the expression (B.1.6), it is useful to approximate $x \in \Omega_T$ by a piecewise-linear curve. If $x \in \Omega_T$ and P is a partition of $[-T,T]$ ($P = \{t_0,t_1,\cdots,t_n\}$ with $-T=t_0<t_1<\cdots<t_n=T$), let x_P be the piecewise linear approximation to x which agrees with x on the partition P. It should be noted that in this notation equation (B.1.7) may be written as

$$\int_{\Omega_T} f(x(t_0),x(t_2),\cdots,x(t_n))\nu_T(dx) =$$

$$Z_P^{-1}\int_{\mathbb{R}^{3(n+1)}} \Big\{f(x(t_0),x(t_1),\cdots,x(t_n))$$

$$\times\exp\left[\frac{-1}{2}\int_{-T}^{T} |x_P'(t)|^2dt\right]\Big\}dx(t_0)dx(t_1)\cdots dx(t_n).,$$

(B.1.9)

where Z_P is a normalization constant. The constant Z_P is defined by

$$Z_P = \int_{\mathbb{R}^{3n}} \exp\left[\frac{-1}{2}\int_{-T}^{T} |x_P'(t)|^2 dt\right] dx(t_1)dx(t_2)\cdots dx(t_n), \quad \text{(B.1.10)}$$

where $x(t_0)$ is held fixed at some point $x \in \mathbb{R}^3$. (It is easily checked that Z_P is independent of which x is chosen.) To make sense out of these last expressions one should interpret $dx(t_0)dx(t_1)\cdots dx(t_n)$ as the measure on Ω_T which is induced by Lebesque measure on the "coordinates" $\{x(t_i)\}_{i=0}^{n}$. Similarly, $dx(t_1)dx(t_2)\cdots dx(t_n)$ is the measure on Ω_T induced by Lebesque measure on the coordinates $\{x(t_i)\}_{i=1}^{n}$, which is concentrated on a set of paths for which $x(t_0) = x \in \mathbb{R}^3$.

<u>Theorem B.1.1</u> Let $V:\mathbb{R}^3 \longrightarrow \mathbb{R}$ be continuous, and bounded from below. ($H = -(1/2)\Delta + V$ is then essentially self-adjoint on the set of infinitely differentiable functions with rapid decrease.) Let λ_o, $\hat{H}$, $\hat{S}$, and $\mu_T = \mu_{f,g,T}$ be as above and $F(x)$ be a bounded continuous function on Ω_T. Then

$$\int_{\Omega_T} F(x)\mu_T(dx) =$$

$$= \lim_{|P|\to 0} Z_P^{-1}\int_{\mathbb{R}^{3(n+1)}} f(x(-T))g(x(T))F(x_P)\exp\left\{-\int_{-T}^{T}[\frac{1}{2}|x_P'(t)|^2 + V(x_P(t))]dt\right\}dx(t_0)dx(t_2)\cdots dx(t_n), \quad \text{(B.1.11)}$$

where $P = \{-T=t_0 < t_1 < t_2 < \cdots < t_n = T\}$ is a partition $[-T,T]$, Z_P is a normalization constant such that for each P the right hand side of (B.1.11) is 1 if $F \equiv 1$, and $|P|$ denotes the mesh size of the partition P.

<u>Proof</u>: By replacing V by $V-\lambda_o$ we may assume that λ_o is zero, since this change does not affect the right hand side of (B.1.11). The extra factor in the numerator is canceled by the same factor in the normalization constant of (B.1.11).

By the Feynaman-Kac formula and the Markov property of the Wiener measure it is well known (Glimm and Jaffe [1] or Simon [1 or 2]) that

$$\mu_T(F) \equiv \mu_{f,g,T}(F) = \frac{[f_T(x(-T))g_T(x(T))F(x)]_T}{[f_T(x(-T))g_T(x(T))]_T}, \tag{B.1.12}$$

where

$$[G]_T \equiv \upsilon_T\left(G\cdot\exp\left\{-\int_{-T}^{T} V(x(s))ds\right\}\right). \tag{B.1.13}$$

Equation (B.1.12) is first proved for F of the form $\prod_{i=0}^{n} f_i(x(t_i))$ where the t_i's form a partition for $[-T,T]$, and then extended to more general F by standard measure theoretic arguments. An application of the dominated convergence theorem implies

$$\mu_T(F) = \lim_{|P|\to 0} \frac{\upsilon_T(f_T(x_P(-T))g_T(x_P(T))F(x_P)e^{-\int_{-T}^{T} V(x_P(s))ds})}{\upsilon_T(f_T(x(-T))g_T(x(T))e^{-\int_{-T}^{T} V(x(s))ds})}.$$

(B.1.14)

The theorem now follows after using equation (B.1.9) in (B.1.14) with

$$f(x(t_0),x(t_1),\cdots,x(t_n)) \equiv G(x_p)g(x(T))\exp\left\{-\int_{-T}^{T} V(x_p(t))dt\right\},$$

with $G \equiv F$ in the numerator of (B.1.14) and $G \equiv 1$ in the denominator. Q.E.D.

<u>Remark B.1.1</u> Informally, Theorem 1.1.1 states that

$$\int_{\Omega_T} F(x)\mu_T(dx) = Z_T^{-1}\int_{\Omega_T} f(x(-T))g(x(T))F(x) \times\exp\left\{-\int_{-T}^{T}[\frac{1}{2}|x'(t)|^2 + V(x(t))]dt\right\}Dx. \tag{B.1.15}$$

It is instructive to view the informal expression (B.1.6) as describing a measure (μ) in the Gibbs state formalism of statistical mechanics. Remark B.1.1 suggests that the measures $\mu_{f,g,T}$ should be considered as a "finite volume" Gibbs states with "boundary conditions" determined by the two functions f and g. It is then natural to take the "thermodynamic limit", that is the limit as T tends to infinity. This is the subject of the next theorem. Before stating the theorem we require some more assumptions on the potential V.

Assume the potential V is continuous, bounded from below, and chosen such that λ_o is an eigenvalue for the operator H with eigenfunction h_o. It is well known under these assumptions (see Corollary 3.3.4 of Glimm and Jaffe [1]) that the multiplicity of λ_o is one and the eigenfunction (h_o) may be chosen to be strictly positive. Furthermore, by the spectral and dominated convergence theorems, $\hat{S}(t)$ converges strongly as $t \longrightarrow \infty$ to the orthogonal projection onto the λ_o eigenspace.

<u>**Theorem B.1.2**</u> Assume V satisfies the assumptions above, f,g are fixed non-negative and non-zero $L^2(\mathbb{R}^3,d\lambda)$-functions, and the measures $\mu_T = \mu_{f,g,T}$ are defined as before. Then there exists a probability measure (μ) on Ω which is independent of the functions f and g and satisfies

$$\lim_{T\longrightarrow\infty} \mu_T(F) = \mu(F), \tag{B.1.16}$$

for all bounded measurable functions F(x) of the form

$$F(x) = G(x|_{[-T_o,T_o]}), \tag{B.1.17}$$

where $0 < T_o < \infty$, and G is a measurable function on Ω_{T_o}. (Under this assumption the expression $\mu_T(F)$ makes sense for $T > T_o$.) Furthermore, the value of $\mu(F)$ is given by

$$\mu(F) = \mu_{h_o,h_o,T}(F), \tag{B.1.18}$$

where h_o is the ground state for H and T is any number larger than T_o.

<u>**Proof**</u>: For $T > T_o$, and $f \in L^2(\mathbb{R}^3,d\lambda)$, put $f_T \equiv \hat{S}(T-T_o)f$. Using the definition of $\mu_{f,g,T}$, it is easily seen that

$$\mu_{f,g,T}(F) = \mu_{f_T,g_T,T_o}(F). \tag{B.1.19}$$

(Again first check the validity of (B.1.19) on functions of the form $\prod_{i=0}^{n} f_i(x(t_i))$ where the t_i's form a partition for $[-T,T]$, and then extended to more general F by standard measure theoretic arguments.) In particular (B.1.19) implies that $\mu_{h_o,h_o,T}(F) = \mu_{h_o,h_o,T_o}(F)$, since $\hat{S}(t)h_o = h_o$ for all $t \geq 0$. Hence the measures $\mu_{h_o,h_o,T}$ are consistently defined, and thus define a unique measure on Ω, see 1.3.5 Theorem of Stroock and Varadhan [1].

By equations (B.1.19), (B.1.12), and (B.1.13) we get

$$\mu_{f,g,T}(F) = \frac{[f_T(x(-T_o))g_T(x(T_o))F(x)]_{T_o}}{[f_T(x(-T_o))g_T(x(T_o))]_{T_o}}. \tag{B.1.20}$$

Since $f_T \longrightarrow (f,h_o)h_o$ in $L^2(\mathbb{R}^3,d\lambda)$, it follows that $f_T(x(-T_o)) \longrightarrow (f,h_o)h_o(x(-T_o))$ in $L^2(\Omega_T,\nu_{T_o})$ ($x(t)$ is distributed as Lebesque measure for each $t\in[-T_o,T_o]$). A similar statement holds for the functions g_T. Hence after letting $T \longrightarrow \infty$ in equation (B.1.20) and canceling out the common factor of $(f,h_o)(g,h_o)$ in both the numerator and the denominator, one finds that

$$\lim_{T\longrightarrow\infty} \mu_T(F) = \frac{[h_o(x(-T_o))h_o(x(T_o))F(x)]_{T_o}}{[h_o(x(-T_o))h_o(x(T_o))]_{T_o}}$$

$$= \mu_{h_o,h_o,T_o}(F) \equiv \mu(F). \tag{B.1.21}$$

Q.E.D.

<u>Remark B.1.2</u> The combination of Theorems B.1.1 and B.1.2 show that the measure (μ) defined in Theorem B.1.2 may

be described by approximations which are suggested by the informal description of the measure (μ) given in equation (B.1.6).

Let us now take stock in what has been accomplished.

On one hand there is a natural measure (μ) associated to each quantum mechanical system associated to the classical system described by (B.1.1). On the other hand Theorems B.1.1 and B.1.2 may be used to construct the measure (μ) without reference to any quantum mechanics (i.e. operator theory.) The informal description (B.1.6) of the measure (μ) suggests the method of construction given in the two theorems. Once we have constructed the measure μ, we may reconstruct the quantum mechanical Hilbert space, up to a natural unitary equivalence. Indeed, $\{F \in L^2(\Omega, d\mu): F(x) = f(x(0))\}$ is equivalent to $L^2(\mathbb{R}^3, h_o^2 d\lambda)$ by the unitary operator $(f(x(0)) \longrightarrow f)$, where h_o is the normalized ground state for H. Since $h_o > 0$, $L^2(\mathbb{R}^3, d\lambda)$ is naturally equivalent to $L^2(\mathbb{R}^3, h_o^2 d\lambda)$ by the unitary map $(f \longrightarrow f/h_o)$. Furthermore, from the measure μ, we may reconstruct the renormalized quantum mechanical Hamiltonian $\hat{H}$. To accomplish this, we first note that $\hat{S}$ may be determined by

$$(fh_o, \hat{S}(t) gh_o)_{L^2(\mathbb{R}^3, d\lambda)} = \mu(f(x(0))g(x(t))) \quad \text{(B.1.22)}$$

for all bounded f, and g on $\mathbb{R}^3$. The operator $\hat{H}$ is then determined by differentiating $\hat{S}$,

$$\hat{H}f = \lim_{h \downarrow 0} \frac{\hat{S}(h) - I}{h} \cdot f, \quad \text{(B.1.23)}$$

with the domain of H given by the set of $f \in L^2(\mathbb{R}^3, d\lambda)$ for which the limit exists.

So we have found that making sense out of the informal expression (B.1.6) is essentially equivalent to "quantizing" the classical mechanical system described by (B.1.1).

Section B.2 Quantum Field Theory

We now want to use the above ideas to quantize certain partial differential equations, rather than the O.D.E. (B.1.1). This amounts to replacing the "3" in $\mathbb{R}^3$ by "∞" in some appropriate sense.

The simplest example to consider is the free Klein Gordon equation

$$\phi_{tt} + (-\Delta_3 + m^2)\phi = 0, \qquad \text{(B.2.1)}$$

where $\phi = \phi(t,x) \in \mathbb{R}$, $(t,x) \in \mathbb{R} \times \mathbb{R}^3$, Δ_3 is the Laplacian only acting on the x-variables, and m is a positive constant called the mass. To make the connection of (B.2.1) with (B.1.1) we consider ϕ as a map from $\mathbb{R}$ to $\mathrm{Re}L^2(\mathbb{R}^3, d\lambda)$. Then (B.2.1) may be written as

$$\phi''(t) = -\mathrm{grad}V(\phi(t)), \qquad \text{(B.2.2)}$$

where $V(\phi(t)) \equiv \frac{1}{2}(\phi(t), (-\Delta_3 + m^2)\phi(t))_{L^2(\mathbb{R}^3, d\lambda)}$, and grad denotes the functional gradient. In analogy with (B.1.6) the informal description of the path space measure associated to the equation (B.2.2) is

$$d\mu(\phi) = Z^{-1}\exp\int\Big\{(-1/2)\|\phi'(t)\|^2_{L^2(\mathbb{R}^3,d\lambda)} + \frac{1}{2}(\phi(t),(-\Delta_3 + m^2)\phi(t))_{L^2(\mathbb{R}^3,d\lambda)}\Big\}dt\cdot D\phi$$

$$= Z^{-1}\exp\Big[(-1/2)(\phi,(-\Delta_4 + m^2)\phi)_{L^2(\mathbb{R}^4,d\lambda)}\Big]D\phi, \qquad \text{(B.2.3)}$$

where $D\phi$ is an "infinite dimensional Lebesque measure", Z is the normalization constant, and Δ_4 denotes the four dimensional Laplacian acting on both t and x.

It is now possible to give precise meaning to the measure μ by following procedures analogous to Theorems B.1.1 and B.1.2, see Simon [1]. Rather than going this route we will instead exploit the fact that the "measure" (μ) is "Gaussian". Recall the finite dimensional Gaussian Fourier transform formula

$$\left\{\frac{\det(A)}{(2\pi)^k}\right\}^{\frac{1}{2}}\int_{\mathbb{R}^k} e^{i(x,y)}\exp(\frac{-1}{2}(Ax,x))dx = \exp(\frac{-1}{2}(A^{-1}y,y)),$$

(B.2.4)

which is valid if A is a positive definite matrix on $\mathbb{R}^k$. In analogy to (B.2.4), we define μ to be the unique probability measure on $\mathrm{ReS}'(\mathbb{R}^4)$ whose Fourier transform is

$$\int_{\mathrm{ReS}'(\mathbb{R}^4)} e^{i\phi(f)}d\mu(\phi) = \exp[\frac{-1}{2}(f,(-\Delta_4 + m^2)^{-1}f)_{L^2(\mathbb{R}^4,d\lambda)}],$$

(B.2.5)

where f is in $S(\mathbb{R}^4)$. The existence and uniqueness of the measure μ is guaranteed by Minlos' theorem, see Simon [2].

In analogy to the discussion at the end of Section B.1, the measure μ may be used to construct a quantum mechanical system associated to the partial differential equation (B.2.1), see Simon [1] and Glimm and Jaffe [1] for more details. The generalization to the case where the term $m^2\phi$ (B.2.1) is replaced by a polynomial in ϕ is also covered in Simon [1] and Glimm and Jaffe [1] (in the case where the spatial dimension is one rather than three).

As a final example, we will consider the free Maxwell's equations. Let $E(t,x)\in\mathbb{R}^3$ denote the electic field and $B(t,x)\in\mathbb{R}^3$ denote the magnetic field $((t,x)\in\mathbb{R}\times\mathbb{R}^3)$. Put,

$$\begin{aligned} F &= E\cdot dx\wedge dt + B\cdot d\sigma \\ &\equiv E_1dx_1\wedge dt + E_2dx_2\wedge dt + E_3dx_3\wedge dt \\ &\quad + B_1dx_2\wedge dx_3 - B_2dx_1\wedge dx_3 + B_3dx_1\wedge dx_2, \end{aligned} \tag{B.2.6}$$

a differential 2-form on $\mathbb{R}^4$ called the field-strength tensor. With this notation the free Maxwell's equations may be written

$$dF = 0 \text{ and } d^*F = 0, \tag{B.2.7}$$

where d^* is the adjoint of d computed with respect to the flat Minikowski metric (g) with signature (1,-1,-1,-1).

The Maxwell's equations (B.2.7) are first order in time, and hence do not seem to be analogous to "Newtonian-like" equations of motion. This can be remedied by noting that the equation $dF = 0$ implies, by Poincare's Lemma (Spivak [1]), that there is a 1-form (A) such that $F = dA$. In terms of this potential A, the Maxwells equations become,

$$d^* dA = 0. \tag{B.2.8}$$

Equation (B.2.8) written in components yields a system of four coupled partial differential equations, three of which are second order in time. The other equation is a "constraint" equation. The reason for this extra complication of a constraint is due to the fact that the potential A is not uniquely determined by the field strength tensor F. This is the so called "gauge" problem. For the purposes of finding the informal expression for the path space measure, we will ignore this gauge problem (but see Chapter 1 and especially Section 1.3).

In analogy to the previous examples, the informal expression for the path space measure is

$$d\mu(A) = Z^{-1}\exp[\frac{-1}{2}\int_{\mathbb{R}^4}\sum_{i<j} dA_{ij}(x)^2 dx]DA \tag{B.2.9}$$

where Z is a normalization constant, and DA is an "infinite dimensional Lebesque measure" on a space of 1-forms. Note that $\sum_{i<j} dA_{ij}(x)^2 = \sum_{i<j} F_{ij}(x)^2 = |E(x)|^2 + |B(x)|^2$, which is the classical energy of the electro-magnetic fields. (In the examples above, it has always been the energy that went into the exponents of the path space measure.) See Section 1.3 for a precise description of the informal expression (B.2.9).

The informal description of the Yang Mills' measure (equation (1.1.1)) is a natural generalization of (B.2.9).

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