Equivalence of heat kernel measure and pinned Wiener measure on loop groups

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1 French Abridged Version

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We show that heat kernel measure and pinned Wiener measure on loop groups over simply connected compact Lie groups are equivalent.

Let K be a connected compact Lie group, $e \in K$ be the identity, $\mathfrak{k} \equiv T_eK$ be the Lie algebra of K, and $\langle \cdot, \cdot \rangle$ be an Ad_K-invariant inner product on $\mathfrak{k}$. Also let $\mathcal{L}$ be the loop group in Eq. (2.1), μ_t^0 be pinned Wiener measure on $\mathcal{L}$ with variance t and ν_t^0 be the time t heat kernel measure on $\mathcal{L}$, so that ν_t^0 is the law of the process $s \to \Sigma(t, s)$ described in Eq. (2.4). In [6], it was shown that the heat kernel measure ν_t^0 is absolute continuous relative to the pinned Wiener measure μ_t^0 and it was conjectured that these two measures are equivalent if K is simply connected. In this note, we will prove in Theorem 3.2 that this conjecture is true. The idea of the proof is as follows.

Let $Z_t(\gamma) = \frac{d\nu_t^0}{d\mu_t^0}(\gamma)$, $N := \{\gamma \in \mathcal{L} : Z_t(\gamma) = 0\}$, and $H^1(K)$ denotes the subgroup of finite energy loops in $\mathcal{L}$. Using the quasi-invariance of the pinned measure μ_t^0 proved in Malliavin and Malliavin [9] and of ν_t^0 proved in Driver [4] under left translation by $H^1(K)$, one shows that $h \cdot N = N$ modulo sets of μ_t^0 – measure zero for all $h \in H^1(K)$. Using Gross' ergodicity theorem, see [8] and Aida [1], it follows that either $\mu_t^0(N) = 0$ or that $\mu_t^0(\mathcal{L} \setminus N) = 0$. Since ν_t^0 is not the zero measure, we conclude that $\mu_t^0(N) = 0$, i.e. that $Z(\gamma) > 0$ on $\mathcal{L}$ for μ_t^0 - a.e. γ. That is to say that μ_t^0 and ν_t^0 are equivalent. A generalization of this result to non-simply connected groups K is given in Corollary 3.3.

2 Introduction

Let K be a connected compact Lie group, $e \in K$ be the identity, $\mathfrak{k} \equiv T_eK$ be the Lie algebra of K and $\langle \cdot, \cdot \rangle$ be an Ad_K-invariant inner product on $\mathfrak{k}$. (To simplify notation later we will, with out loss of generality, assume that K is a matrix group.) The loop group on K is defined by

$$\mathcal{L} = \mathcal{L}(K) \equiv \{\gamma \in C([0,1], K) : \ \gamma(0) = \gamma(1) = e\} \tag{2.1}$$

and when K is not simply connected, we let $\mathcal{L}_0$ denote those loops in $\mathcal{L}$ which are homotopic to the constant loop at $e \in K$. We view $\mathcal{L}$ as a topological space equipped with the topology of uniform convergence.

Let μ_t^0 denote *pinned Wiener measure* with variance t. This measure may be described as the law of the process $\{g_s : 0 \le s \le 1\}$ pinned so that $g_1 = e$, where g is the solution to the Stratonovich stochastic differential equation

$$dg_s = g_s \circ \sqrt{t} db_s \text{ with } g_0 = e. \tag{2.2}$$

Here b is a standard $\mathfrak{k}$ – valued Brownian motion with covariance determined by $\langle \cdot, \cdot \rangle$. As introduced in [5] and [10], the *heat kernel measure* ν_t^0 on $\mathcal{L}$ is the time t distribution of a certain $\mathcal{L}$ – valued Brownian motion which we now describe.

Let $\{\chi(t,s)\}_{0 \le s \le 1, 0 \le t < \infty}$ be a $\mathfrak{k}$ – valued *Brownian bridge sheet,* i.e. χ is a $\mathfrak{k}$ – valued centered continuous Gaussian process such that

$$\mathbb{E}[\langle A, \chi(t,s)\rangle \langle B, \chi(\tau,\sigma)\rangle] = \langle A, B\rangle \min(t,\tau) \left(\min(s,\sigma) - \sigma s\right) \tag{2.3}$$

for all $s, \sigma \in [0,1]$, $t, \tau \in [0, \infty)$ and $A, B \in \mathfrak{k}$. Let $\Sigma(t,s)$ for $0 \le t < \infty$ and $0 \le s \le 1$ denote the solution to the Stratonovich stochastic differential equation in t (with s as a parameter),

$$\Sigma(dt, s) = \Sigma(t,s) \circ \chi(dt, s) \text{ with } \Sigma(0,s) = e \quad \forall s \in [0,1]. \tag{2.4}$$

By Malliavin [10] (see also Baxendale [3] or Driver [4]), we may and do choose $\Sigma(t,s)$ to be jointly continuous in (t,s). The *heat kernel measure* ν_t^0 at time $t > 0$ is the law of the process $s \to \Sigma(t,s) \in K$. Notice by construction that $\nu_t^0(\mathcal{L}_0) = 1$. For further details and motivations for these constructions the reader is referred to [4, 5, 6].

In [6], it was shown that ν_t^0 is absolute continuous relative to μ_t^0 and it was conjectured that these two measures are equivalent if K is simply connected. In this note, we will prove in Theorem 3.2 that this conjecture is true. The proof is based on an elementary fact (Lemma 3.1 below) on the equivalence of two quasi-invariant measures.

3 An elementary lemma and the equivalence of ν_t^0 and μ_t^0

Let X be a topological group and Y be a subgroup. For a finite Borel measure ν and $h \in Y$, define the new measure ν^h on X by $\nu^h(A) = \nu(h \cdot A)$, where A is a Borel measurable subset and $h \cdot A = \{hx \mid x \in A\}$.

We recall the basic notion of measures on groups. We say that ν is quasi-invariant relative to Y if ν^h is equivalent to ν for all $h \in Y$. For a quasi-invariant measure ν relative to Y, ν is said to be ergodic relative to the left action of Y if a Borel measurable subset A satisfying $\nu((A \setminus h \cdot A) \cup (h \cdot A \setminus A)) = 0$ for any $h \in Y$, implies that either $\nu(X \setminus A) = 0$ or $\nu(A) = 0$.

Lemma 3.1 *Let ν_1 and ν_2 be quasi-invariant finite non-zero measures on X relative to a subgroup Y. Let $\alpha_i^h(x) = \frac{d\nu_i^h}{d\nu_i}(x)$ $(i = 1, 2)$. Suppose that*

(1) *ν_1 is ergodic relative to Y and that*

(2) *ν_2 is absolute continuous relative to ν_1.*

Then ν_1 and ν_2 are equivalent.

Proof. Let $Z(x) = \frac{d\nu_2}{d\nu_1}(x)$. We need only prove that $Z(x) > 0$ for ν_1-a.e. x. We will do this by showing that $\nu_1(N) = \nu_1(\{x \in X \mid Z(x) = 0\}) = 0$.

By definition of $\alpha_i^h(x)$,

$$\int_X f(h^{-1}x)d\nu_i(x) = \int_X f(x)\alpha_i^h(x)d\nu_i(x) \quad (i = 1, 2) \tag{3.1}$$

for all bounded measurable functions f on X and $h \in Y$. Using Eq. (3.1) and the definition of Z we have

$$\int_X f(h^{-1}x)d\nu_2(x) = \int_X f(h^{-1}x)Z(x)d\nu_1(x) = \int_X f(x)Z(hx)\alpha_1^h(x)d\nu_1(x) \tag{3.2}$$

and

$$\int_X f(h^{-1}x)d\nu_2(x) = \int_X f(x)\alpha_2^h(x)d\nu_2(x) = \int_X f(x)\alpha_2^h(x)Z(x)d\nu_1(x). \tag{3.3}$$

Since these equations hold for all bounded measurable functions, it follows

$$Z(hx)\alpha_1^h(x) = Z(x)\alpha_2^h(x) \text{ for } \nu_1 - \text{a.e. } x \tag{3.4}$$

for all $h \in Y$. Similar calculations shows that $\alpha_i^{h^{-1}}(hx)\alpha_i^h(x) = 1$ for ν_i – a.e. x, $i = 1, 2$ and $h \in Y$. In particular

$$\alpha_i^h(x) > 0 \text{ for } \nu_i - \text{a.e. } x. \tag{3.5}$$

Thus we may write Eq. (3.4) as

$$Z(hx) = Z(x)\frac{\alpha_2^h(x)}{\alpha_1^h(x)}. \tag{3.6}$$

By Eqs. (3.5) and (3.6) and the fact that $\nu_2(N) = 0$,

$$h \cdot N = N \quad \text{up to a } \nu_1\text{-null set} \tag{3.7}$$

for all $h \in Y$. Hence by the ergodicity of ν_1, either $\nu_1(X \setminus N) = 0$ or $\nu_1(N) = 0$. Since ν_2 is not zero we know $\nu_1(X \setminus N) > 0$ and therefore, $\nu_1(N) = 0$ which completes the proof. ■

Now we are going to prove the equivalence of heat kernel measure and pinned Wiener measure. Let $\tilde{\mu}_t^0$ be the restriction of the pinned Wiener measure μ_t^0 to $\mathcal{L}_0$. The absolute continuity of ν_t^0 with μ_t^0 was proved in [6]. Let $Z_t(\gamma) = \frac{d\nu_t^0}{d\mu_t^0}(\gamma)$.

Theorem 3.2 *The Radon – Nikodým derivative $Z_t(\gamma)$ is positive for $\tilde{\mu}_t^0$-a.e. γ. Namely μ_t^0 and ν_t^0 are equivalent on $\mathcal{L}_0$*

Proof. Let $H^1(K)$ and $H^1(K)_0$ denote the subgroups of finite energy loops in $\mathcal{L}$ and $\mathcal{L}_0$ respectively. The quasi-invariance of the **un**-pinned measure relative to $H^1(K)$ is standard (see [2], [7], [11]). However the quasi-invariance of pinned Wiener measure relative to $H^1(K)$ is much more delicate and is due to M.-P. Malliavin and P. Malliavin [9]. The quasi-invariance of heat kernel measure ν_t^0 relative to $H^1(K)_0$ was proved in [4]. Furthermore the ergodicity of $\tilde{\mu}_t^0$ relative to $H^1(K)_0$ was proved by Gross [8], see also Aida [1] for this result and its generalizations. Hence the theorem is a consequence of Lemma 3.1 after setting $X = \mathcal{L}_0,\ Y = H^1(K)_0,\ \nu_1 = \tilde{\mu}_t^0$ and $\nu_2 = \nu_t^0$. ■

Corollary 3.3 *For each $h \in H^1(K)$, let $\nu_t^0(h, A) := \nu_t^0(h^{-1}A)$ – heat kernel measure on $\mathcal{L}$ starting at h. Let $\Pi \subset H^1(K)$ be chosen so that to each homotopy class in $\mathcal{L}$, there is a unique representative in Π. Then μ_t^0 and $\sum_{h\in\Pi} \nu_t^0(h, \cdot)$ are equivalent measures on $\mathcal{L}$.*

Proof. For $h \in H^1(K)$, let $\mathcal{L}_h$ denote those loops in $\mathcal{L}$ which are homotopic to h. By Theorem 3.2, $\nu_t^0(h, \cdot)$ is equivalent to $\tilde{\mu}_t^0(h^{-1}(\cdot))$ on $\mathcal{L}_{h^{-1}}$. By the quasi-invariance of μ_t^0 relative to $H^1(K)$, it follows that $\tilde{\mu}_t^0(h^{-1}(\cdot))$ is equivalent to $\mu_t^0|_{\mathcal{L}_{h^{-1}}}$ on $\mathcal{L}_{h^{-1}}$. Therefore $\nu_t^0(h, \cdot)$ is equivalent to $\mu_t^0|_{\mathcal{L}_{h^{-1}}}$ on $\mathcal{L}_{h^{-1}}$ and hence $\sum_{h\in\Pi} \nu_t^0(h, \cdot)$ is equivalent to $\sum_{h\in\Pi} \mu_t^0|_{\mathcal{L}_{h^{-1}}} = \mu_t^0$. ■

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