Li-Yau inequalities for positive harmonic functions by Stochastic Analysis

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Abstract

We prove Li-Yau type inequalities for positive harmonic functions on Riemannian manifolds by using methods of Stochastic Analysis. Rather than evaluating an exact Bismut formula for the differential of a harmonic function, our method relies on a Bismut type inequality which is derived by an elementary integration by parts argument from an underlying submartingale. It is the monotonicity inherited in this submartingale which allows to establish the pointwise estimates.

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1 Introduction

The effect of curvature on the behaviour of harmonic functions is a classical problem: it manifests itself most directly in gradient estimates and Harnack inequalities with constants depending only on a lower bound of the Ricci curvature of the manifold, the dimension, and the radius of the ball on which the function is defined. Such estimates in global form, i.e., for positive harmonic functions on Riemannian manifolds, are due to S.T. Yau [9]; local versions have been established by Cheng and Yau [3].

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The classical proof of gradient estimates relies on two ingredients: the use of comparison theorems for the Laplacian of the Riemannian distance, which allow to bound the mean curvature of geodesic spheres from above in terms of the lower bound of the Ricci curvature, and Bochner's formula which yields a lower bound for $\Delta|\operatorname{grad} u|^2$ for a harmonic function u in terms the lower Ricci bound. The estimate itself then relies on a clever use of the maximum principle, see [6].

In this paper we show that gradient estimates for positive harmonic functions can be derived from elementary submartingales by means of stochastic analysis. In particular, we give a stochastic proof of the following gradient estimate due to Cheng and Yau [3]. It should be noted that our approach gives in addition an explicit value for the constant.

Theorem 1.1 *Let M be a complete Riemannian manifold of dimension $n \geq 2$ and let $D \subset M$ be a relatively compact domain. Let $u\colon D \to \mathbb{R}$ be a harmonic function which is strictly positive. Put $r(x) = \operatorname{dist}(x, \partial D)$. Then:*

$$\frac{|du|}{u}(x) \leq c(n)\left[k + \frac{1}{r(x)}\right] \tag{1.1}$$

if $\operatorname{Ric} \geq -(n-1)\,k^2$ *where* $k \geq 0$.

Our method of proof is inspired by the stochastic approach to gradient estimates used in [8], [1] where one represents the differential of a harmonic map by a Bismut type mean value formula which can be evaluated in explicit terms. As in [8], explicit estimates depend on a reasonable choice of a finite-energy process which is used for integration by parts on path space. See [7] for background on Bismut formulas.

2 Some elementary calculations

Let M be a (not necessarily complete) Riemannian manifold of dimension $n \geq 2$, and $u\colon M \to \mathbb{R}$ be a harmonic function. For $x \in M$ let $\varphi(x) = |\operatorname{grad} u|(x)$. For $x \in M$ with $\varphi(x) \neq 0$, let $\mathfrak{n}(x) = \varphi(x)^{-1}\operatorname{grad} u(x)$.

If $f\colon M \to \mathbb{R}$ is a smooth function, then we have the well-known formula

$$\Box \operatorname{grad} f = \operatorname{grad} \Delta f + \operatorname{Ric}^\sharp \operatorname{grad} f, \tag{2.1}$$

where $\Box = \operatorname{trace} \nabla^2$ denotes the rough Laplacian on $\Gamma(TM)$, and where by definition,

$$\langle \operatorname{Ric}^\sharp X, Y\rangle = \operatorname{Ric}(X, Y), \quad X, Y \in \Gamma(TM).$$

Eq. (2.1) applied to u gives

$$\Box \operatorname{grad} u = \operatorname{Ric}^\sharp \operatorname{grad} u. \tag{2.2}$$

In the following Lemma we calculate $\operatorname{grad}\varphi$ and $\Delta\varphi$ in terms of $\mathfrak{n}$.

Lemma 2.1 *Let $u\colon M \to \mathbb{R}$ be a harmonic function, $\varphi = |\operatorname{grad} u|$, and $\mathfrak{n} = (\operatorname{grad} u)/\varphi$ where it is defined. Then on M, where φ does not vanish,*

$$\operatorname{grad}\log\varphi = \nabla_{\mathfrak{n}}\mathfrak{n} - (\operatorname{div}\mathfrak{n})\mathfrak{n}, \tag{2.3}$$

and

$$\Delta\varphi = \varphi\left[\operatorname{Ric}(\mathfrak{n},\mathfrak{n}) + |\nabla\mathfrak{n}|^2_{\text{H.S.}}\right]. \tag{2.4}$$

In particular,

$$|\operatorname{grad}\log\varphi|^2 \le (n-1)\ |\nabla\mathfrak{n}|^2_{\text{H.S.}}\,. \tag{2.5}$$

Proof (i) We start by proving Eq. (2.4). To this end, fix $x \in M$ such that $\varphi(x) \neq 0$ and choose an orthonormal frame $(e_i)_{1\le i\le n}$ at x such that $(\nabla_{e_i}e_j)(x) = 0$ for all i, j. Then, we have at x,

$$\Box(\varphi\mathfrak{n}) = (\Delta\varphi)\,\mathfrak{n} + 2\,\langle\nabla_{e_i}\operatorname{grad} u, \mathfrak{n}\rangle\,\nabla_{e_i}\mathfrak{n} + \varphi\,\Box\mathfrak{n}.$$

Taking scalar product with $\mathfrak{n}$ makes the second term of the r.h.s. to vanish and yields

$$\Delta\varphi = \langle\Box(\varphi\mathfrak{n}), \mathfrak{n}\rangle - \varphi\,\langle\Box\mathfrak{n}, \mathfrak{n}\rangle\,. \tag{2.6}$$

It is easily seen that

$$\langle\Box\mathfrak{n}, \mathfrak{n}\rangle = -\,|\nabla\mathfrak{n}|^2_{\text{H.S.}}\ ,$$

so that with Eq. (2.2) the claimed equality for $\Delta\varphi$ follows.

(ii) To establish (2.3), note that $\Delta u = 0$ writes as

$$0 = \operatorname{div}(\varphi\mathfrak{n}) = d\varphi\,(\mathfrak{n}) + \varphi\operatorname{div}\mathfrak{n}.$$

Let $\mathfrak{n}^\flat = \langle\mathfrak{n}, \cdot\rangle = \varphi^{-1}\,\langle\operatorname{grad} u, \cdot\rangle = \varphi^{-1}du$. Then on one hand,

$$\begin{aligned}
\iota_{\mathfrak{n}}d\mathfrak{n}^\flat &= \iota_{\mathfrak{n}}\left(-\varphi^{-2}d\varphi\wedge du\right) = -\varphi^{-2}\left(d\varphi\,(\mathfrak{n})\,du - du\,(\mathfrak{n})\,d\varphi\right)\\
&= -\varphi^{-2}\left(-\varphi\operatorname{div}\mathfrak{n}\,du - \varphi\,|\mathfrak{n}|^2\,d\varphi\right)\\
&= \operatorname{div}\mathfrak{n}\,\langle\mathfrak{n}, \cdot\rangle + \langle\operatorname{grad}\log\varphi, \cdot\rangle\,,
\end{aligned}$$

while on the other hand,

$$\iota_{\mathfrak{n}}d\mathfrak{n}^\flat = \langle\nabla_{\mathfrak{n}}\mathfrak{n}, \cdot\rangle - \langle\nabla_{\cdot}\mathfrak{n}, \mathfrak{n}\rangle = \langle\nabla_{\mathfrak{n}}\mathfrak{n}, \cdot\rangle\,.$$

Comparing these last two equations proves Eq. (2.3).

(iii) Finally, to establish (2.5), note first that, as a consequence of (2.3),

$$|\mathrm{grad}\log\varphi|^2 = (\mathrm{div}\,\mathfrak{n})^2 + |\nabla_{\mathfrak{n}}\mathfrak{n}|^2 . \tag{2.7}$$

Next, fix $x \in M$ such that $\varphi(x) \neq 0$ and choose an orthonormal frame $(e_i')_{1\le i\le n}$ at x such that $e_n' = \mathfrak{n}$. Then

$$\begin{aligned}
|\mathrm{grad}\log\varphi|^2 &= \left(\sum_{j=1}^{n-1}\left\langle \nabla_{e_j'}\mathfrak{n}, e_j'\right\rangle\right)^2 + |\nabla_{\mathfrak{n}}\mathfrak{n}|^2 \\
&\le (n-1)\sum_{j=1}^{n-1}\left\langle \nabla_{e_j'}\mathfrak{n}, e_j'\right\rangle^2 + \left|\nabla_{e_n'}e_n'\right|^2 \\
&\le (n-1)\sum_{j=1}^{n}\left|\nabla_{e_j'}\mathfrak{n}\right|^2 = (n-1)\,|\nabla\mathfrak{n}|^2_{\mathrm{H.S.}} . \qquad \square
\end{aligned}$$

3 Gradient estimates for positive harmonic functions

The following theorem gives the submartingale property which will be crucial for our estimates, see Bakry [2] for related analytic results.

Theorem 3.1 *Let M be a (not necessarily complete) Riemannian manifold of dimension $n \ge 2$. Let X be a Brownian motion on M and let $u\colon M \to \mathbb{R}$ be a harmonic function. For any $\alpha \ge \frac{n-2}{n-1}$, the process*

$$Y_t := |\,\mathrm{grad}\,u|^\alpha(X_t)\,\exp\left\{-\frac{\alpha}{2}\int_0^t \mathrm{Ric}(\mathfrak{n},\mathfrak{n})(X_r)\,dr\right\} \tag{3.1}$$

is a local submartingale (with the convention that $\mathrm{Ric}(\mathfrak{n},\mathfrak{n})(x) = 0$ *at points x where* $\mathrm{grad}\,u(x)$ *vanishes).*

Proof First assume that $\mathrm{grad}\,u$ does not vanish on M. Then, making use of Eq. (2.4), we have

$$\begin{aligned}
\Delta\varphi^\alpha &= \alpha\varphi^{\alpha-1}\Delta\varphi + \alpha(\alpha-1)\varphi^{\alpha-2}\,|\mathrm{grad}\,\varphi|^2 \\
&= \alpha\varphi^\alpha\left(\mathrm{Ric}(\mathfrak{n},\mathfrak{n}) + |\nabla\mathfrak{n}|^2_{\mathrm{H.S.}} + (\alpha-1)\,|\mathrm{grad}\log\varphi|^2\right).
\end{aligned}$$

From the assumption on α, along with estimate (2.5), we obtain

$$|\nabla\mathfrak{n}|^2_{\mathrm{H.S.}} + (\alpha-1)\,|\mathrm{grad}\log\varphi|^2 \ge |\nabla\mathfrak{n}|^2_{\mathrm{H.S.}} - \frac{1}{n-1}\,|\mathrm{grad}\log\varphi|^2 \ge 0,$$

and hence

$$\Delta\varphi^\alpha \ge \alpha\varphi^\alpha\,\mathrm{Ric}(\mathfrak{n},\mathfrak{n}). \tag{3.2}$$

An application of Itô's lemma now shows, $Y_t = N_t + A_t$ where

$$dN_t = \exp\left\{-\frac{\alpha}{2}\int_0^t \mathrm{Ric}(\mathfrak{n},\mathfrak{n})(X_r)\,dr\right\}\left\langle //_t^{-1}\operatorname{grad}\varphi^\alpha(X_t), db_t\right\rangle$$

and

$$dA_t = \frac{1}{2}\left(\frac{\Delta\varphi^\alpha}{\varphi^\alpha}(X_t) - \alpha\,\mathrm{Ric}(\mathfrak{n},\mathfrak{n})(X_t)\right)Y_t\,dt.$$

By the inequality (3.2), $dA_t \geq 0$ and therefore, Y_t is a local submartingale, which completes the proof under the assumption that φ never vanishes on M. This assumption however is easily removed by letting $\mathrm{Ric}(\mathfrak{n},\mathfrak{n})(x) = 0$ in (3.1) at points x where $\operatorname{grad} u(x) = 0$.

Indeed, let $[0,\zeta[$ be the maximal interval on which our Brownian motion X is defined. Fixing $\varepsilon > 0$, we consider the partition

$$0 = \tau_0 \leq \sigma_1 \leq \tau_1 \leq \sigma_2 \leq \tau_2 \leq \dots$$

of $[0,\zeta[$ defined by

$$\sigma_i = \inf\{t \geq \tau_{i-1} : Y_t \leq \varepsilon/2\} \quad\text{and}\quad \tau_i = \inf\{t \geq \sigma_i : Y_t \geq \varepsilon\}, \quad i \geq 1.$$

Now consider $Y_t^\varepsilon := Y_t \vee \varepsilon$ which is seen to be a local submartingale on each of the sub-intervals of our partition. Indeed, on $[\sigma_i,\tau_i[$ the process is constant, $Y^\varepsilon \equiv \varepsilon$, while on $[\tau_{i-1},\sigma_i[$ it is a local submartingale, since there Y itself is a local submartingale by Itô's formula, as shown above, using the fact that $X|[\tau_{i-1},\sigma_i[$ takes its values in $\{x \in M\colon \operatorname{grad} u(x) \neq 0\}$. Now since each Y^ε is a local submartingale, $Y_t = \lim_{\varepsilon\downarrow 0} Y_t^\varepsilon$ itself is a local submartingale. □

Remark 1 In (3.1) we adopted the convention $\mathrm{Ric}(\mathfrak{n},\mathfrak{n})(x) = 0$ at points x where $\operatorname{grad} u(x)$ vanishes. It should be noted that any other convention gives a local submartingale as well.

Remark 2 In Appendix A we provide a generalization of Theorem 3.1, along with a unified proof of the submartingale property of (3.1), which directly takes care of the possible vanishing of the gradient of u and does not require the case distinction made in the proof of Theorem 3.1.

Let ℓ_t be a real-valued non-negative adapted continuous process with C^1 paths. Since

$$d\left(Y_t\ell_t - \int_0^t Y_s\dot{\ell}_s\,ds\right) = \ell_t\,dY_t,$$

the process

$$Z_t := Y_t\ell_t - \int_0^t Y_s\dot{\ell}_s\,ds = Y_0\ell_0 + \int_0^t \ell_s dY_s \tag{3.3}$$

is a local submartingale. Fix $x \in M$ and assume $X_0 = x$. Further let τ be a stopping time such that (3.3) stopped at τ is a (true) submartingale. (Note that this can always be achieved by choosing τ to be the first exit time of X from some relatively compact neighbourhood of x.) Then, if $\ell_0 = 1$ and $\ell_\tau = 0$, the inequality

$$Z_0 \le \mathbb{E}\left[Z_\tau\right]$$

yields

$$Y_0 \le -\mathbb{E}\left[\int_0^\tau Y_s \dot{\ell}_s\, ds\right]$$

which gives in our situation

$$\varphi^\alpha(x) \le -\mathbb{E}\left[\int_0^\tau \varphi^\alpha(X_s)\, \exp\left\{-\frac{\alpha}{2}\int_0^s \mathrm{Ric}(\mathfrak{n},\mathfrak{n})(X_r)\, dr\right\} \dot{\ell}_s\, ds\right]. \tag{3.4}$$

Assuming $\alpha < 2$, by Hölder's inequality, we may estimate

$$\begin{aligned}\varphi(x) \le \mathbb{E}\Bigg[&\left(\int_0^\tau \varphi^2(X_s)\, ds\right)^{\alpha/2} \\ &\left(\int_0^\tau \exp\left\{-\frac{\alpha}{2-\alpha}\int_0^s \mathrm{Ric}(\mathfrak{n},\mathfrak{n})(X_r)\, dr\right\} \left|\dot{\ell}_s\right|^{2/(2-\alpha)}\, ds\right)^{(2-\alpha)/2}\Bigg]^{1/\alpha}.\end{aligned} \tag{3.5}$$

Theorem 3.2 *Let* $\alpha \in [\frac{n-2}{n-1}, 2[$ *and let* $p > 1$, $q > 1$ *such that* $p^{-1} + q^{-1} = 1$. *Then*

$$|\operatorname{grad} u|(x) \le I_1(\alpha,p)\cdot I_2(\alpha,p) \tag{3.6}$$

where

$$I_1(\alpha,p) = \mathbb{E}\left[\left(\int_0^\tau |\operatorname{grad} u|^2(X_s)\, ds\right)^{\alpha p/2}\right]^{1/\alpha p}$$

$$I_2(\alpha,p) = \mathbb{E}\left[\left(\int_0^\tau \exp\left\{\frac{-\alpha}{2-\alpha}\int_0^s \mathrm{Ric}(\mathfrak{n},\mathfrak{n})(X_r)\, dr\right\} \left|\dot{\ell}_s\right|^{2/(2-\alpha)}\, ds\right)^{(2-\alpha)q/2}\right]^{1/\alpha q}$$

and where ℓ_t *is a real-valued non-negative adapted continuous process with* C^1 *paths such that* $\ell_0 = 1$ *and* $\ell_\tau = 0$.

Proof We apply one more time Hölder's inequality to Eq. (3.5). □

To estimate $I_1(\alpha,p)$ we use the following Lemma.

Lemma 3.3 *Let* $\beta \in]0,1[$ *and*

$$C_\beta = 2^{-1/2}\left(\frac{\Gamma\left(\frac{1-\beta}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^{1/\beta}. \tag{3.7}$$

For every positive local martingale Y *with infinite lifetime and deterministic starting point* $Y_0 = y$,

$$\mathbb{E}\left[\langle Y,Y\rangle_\infty^{\beta/2}\right]^{1/\beta} \le C_\beta\, y. \tag{3.8}$$

Proof Without loss of generality we may assume that $y = 1$. Moreover, applying the Dambis, Dubins-Schwarz Theorem (cf. [4] or [5]), by "enriching" the filtered probability space if necessary, we may assume there exists a Brownian motion B started at 0 such that

$$Y = 1 + B_{\langle Y,Y\rangle}.$$

Let $T := \inf\{t \geq 0,\ 1 + B_t = 0\}$. By the reflection principle which yields

$$\mathbb{P}\{B_t \leq -1\} = \frac{1}{2}\,\mathbb{P}\{T \leq t\} \quad \text{for all } t \geq 0,$$

and the scaling property of Brownian motion, we conclude that T has the same law as $1/B_1^2$. Consequently,

$$\mathbb{E}\left[T^{\beta/2}\right]^{1/\beta} = \mathbb{E}\left[|B_1|^{-\beta}\right]^{1/\beta} = C_\beta.$$

Moreover, we have

$$\langle Y, Y\rangle_\infty \leq T \quad \text{a.s.},$$

so that

$$\mathbb{E}\left[\langle Y, Y\rangle_\infty^{\beta/2}\right]^{1/\beta} \leq \mathbb{E}\left[T^{\beta/2}\right]^{1/\beta} = C_\beta. \qquad \square$$

To exploit estimate (3.6) we now choose $\alpha \in \left[\frac{n-2}{n-1}, 1\right[$ if $n \geq 3$, $\alpha \in \left]0, 1\right[$ if $n = 2$, and $p > 1$ such that $\beta := \alpha p < 1$.

Proposition 3.4 (Gradient estimate; general form) *Let M be a Riemannian manifold. Assume that $u\colon M \to \mathbb{R}$ is a positive harmonic function and let $\mathfrak{n} = \operatorname{grad} u/|\operatorname{grad} u|$ where it is defined. Further, let $\alpha \in \left[\frac{n-2}{n-1}, 1\right[$ and $q > 1$ such that $\beta := \frac{q}{q-1}\,\alpha < 1$. Then, for each $x \in M$, the following estimate holds:*

$$\begin{aligned} &|\operatorname{grad}\log u|(x) \qquad\qquad (3.9)\\ &\leq C_\beta\,\mathbb{E}\left[\left(\int_0^\tau \exp\left\{-\frac{\alpha}{2-\alpha}\int_0^s \operatorname{Ric}(\mathfrak{n},\mathfrak{n})(X_r)\,dr\right\}\left|\dot{\ell}_s\right|^{2/(2-\alpha)}\,ds\right)^{(2-\alpha)q/2}\right]^{1/\alpha q} \end{aligned}$$

where C_β is given by (3.7) and the process ℓ_s is chosen as in Theorem 3.2.

Proof Lemma 3.3 applied to $Y_t := u(X_t^\tau)$ gives

$$\mathbb{E}\left[\left(\int_0^\tau \varphi^2(X_s)\,ds\right)^{\beta/2}\right]^{1/\beta} \leq C_\beta\,u(x). \qquad (3.10)$$

Bounding the term $I_1(\alpha, p)$ in estimate (3.6) by means of (3.10), and dividing by $u(x)$, the claimed inequality follows from (3.6). $\square$

Corollary 3.5 *We keep the assumptions of Proposition* 3.4 *and assume that* $\mathrm{Ric} \geq -K$ *for some constant* $K \geq 0$*. Then*

$$|\operatorname{grad} \log u|(x) \leq C_\beta \, \mathbb{E}\left[\left(\int_0^\tau \exp\left\{\frac{\alpha K s}{2-\alpha}\right\} \left|\dot{\ell}_s\right|^{2/(2-\alpha)} \, ds\right)^{(2-\alpha)q/2}\right]^{1/\alpha q}. \quad (3.11)$$

4 Explicit constants

For methods of estimating the right hand sides in Eqs. (3.9) and (3.11) we refer to [8] and see especially [8, Corollary 5.1]. The remaining question then is to optimize the choices of α and p.

Fix $x \in M$ and let $D \subset M$ be a relatively compact open neighbourhood of x in M such that ∂D is smooth. Let $f \in C^2(\bar{D})$ be a positive function on D which is bounded by 1 and vanishing on ∂D. Define

$$T(t) := \int_0^t f^{-2}(X_s(x)) \, ds \quad \text{and} \quad \rho(t) := \inf\{s \geq 0 : T(s) \geq t\}.$$

We have $\rho(t) \leq t$. The process $X'_t := X_{\rho(t)}$ is a diffusion with generator $L' := \frac{1}{2} f^2 \Delta$ and infinite lifetime, see [8, Proposition 2.5]. We fix $t > 0$ and let

$$h_0(s) := \int_0^s f^{-2}(X_r(x)) \, 1_{\{r<\rho(t)\}} \, dr.$$

Hence $h_0(s) = h_0(\rho(t)) = T(\rho(t)) = t$ for $s \geq \rho(t)$. Let $h_1 \in C^1([0,t], \mathbb{R})$ be a function with non-positive derivative such that $h_1(0) = 1$ and $h_1(t) = 0$, and define $\ell_s := (h_1 \circ h_0)(s)$. Since ℓ_s has non-positive derivative, $|\dot{\ell}_s| \, ds$ is a probability measure on $[0, \rho(t)]$.

We want to estimate $I^{1/\alpha q}$ where

$$I := \mathbb{E}\left[\left(\int_0^{\rho(t)} \exp\left\{\frac{\alpha K s}{2-\alpha}\right\} \left|\dot{\ell}_s\right|^{2/(2-\alpha)} \, ds\right)^{(2-\alpha)q/2}\right].$$

Assume that $(2-\alpha)q/2 > 1$ (note this is automatically satisfied if $n \geq 3$ and $\alpha = \frac{n-2}{n-1}$). By means of Jensen's inequality, we get

$$\begin{aligned} I &= \mathbb{E}\left[\left(\int_0^{\rho(t)} \exp\left\{\frac{\alpha K}{2-\alpha} s\right\} \left|\dot{\ell}_s\right|^{\alpha/(2-\alpha)} \left|\dot{\ell}_s\right| \, ds\right)^{(2-\alpha)q/2}\right] \\ &\leq \mathbb{E}\left[\int_0^{\rho(t)} \exp\left\{\frac{\alpha K q}{2} s\right\} \left|\dot{\ell}_s\right|^{q\alpha/2} \left|\dot{\ell}_s\right| \, ds\right] \\ &= \mathbb{E}\left[\int_0^{\rho(t)} \exp\left\{\frac{\alpha K q}{2} s\right\} \left|\dot{\ell}_s\right|^{(q\alpha+2)/2} \, ds\right] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\int_0^{\rho(t)} \exp\left\{\frac{\alpha K q}{2}\, s\right\} \left|\dot h_1(h_0(s))\right|^{(q\alpha+2)/2} \left|\dot h_0(s)\right|^{(q\alpha+2)/2} ds\right] \\
&= \mathbb{E}\left[\int_0^{\rho(t)} \exp\left\{\frac{\alpha K q}{2}\, s\right\} \left|\dot h_1(h_0(s))\right|^{(q\alpha+2)/2} f^{-q\alpha-2}(X_s)\, ds\right] \\
&= \mathbb{E}\left[\int_0^{t} \exp\left\{\frac{\alpha K q}{2}\, \rho(r)\right\} \left|\dot h_1(r)\right|^{(q\alpha+2)/2} f^{-q\alpha}(X'_r)\, dr\right] \\
&= \int_0^{t} \left|\dot h_1(r)\right|^{(q\alpha+2)/2} \mathbb{E}\left[\exp\left\{\frac{\alpha K q}{2}\, \rho(r)\right\} f^{-q\alpha}(X'_r)\right] dr.
\end{aligned}
$$

Let $Z_s = e^{\alpha K q \rho(s)/2}\, f^{-\alpha q}(X'_s)$. We have

$$
\begin{aligned}
dZ_s &\overset{\mathrm{m}}{=} \frac{1}{2} Z_s \left(\alpha K q f^2(X'_s) - \alpha q (f\Delta f)(X'_s) + \alpha q(\alpha q+1) \left|\operatorname{grad} f\right|^2 (X'_s)\right) ds \\
&\le C(\alpha, K, q, f) Z_s\, ds
\end{aligned}
$$

where

$$
C(\alpha, K, q, f) = \frac{1}{2} \sup_{x\in D} \left\{\alpha K q f^2(x) - \alpha q (f\Delta f)(x) + \alpha q(\alpha q+1) \left|\operatorname{grad} f\right|^2 (x)\right\}.
$$

Let $C := C(\alpha, K, q, f)$. Then

$$
\mathbb{E}[Z_s] \le Z_0 e^{Cs} = f^{-\alpha q}(x) e^{Cs}
$$

which yields

$$
\begin{aligned}
I^{1/\alpha q} &\le \left(\int_0^t f^{-\alpha q}(x)\, e^{Cs} \left|\dot h_1(s)\right|^{(q\alpha+2)/2} ds\right)^{1/\alpha q} \\
&= f^{-1}(x) \left(\int_0^t e^{Cs} \left|\dot h_1(s)\right|^{(q\alpha+2)/2} ds\right)^{1/\alpha q}.
\end{aligned}
$$

We fix $a > 0$ and let $h_1(s) = 1 - \dfrac{1-e^{-as}}{1-e^{-at}}$ for $s \in [0, t]$. Then, in terms of $I \equiv I(t, a)$, we have

$$
|\operatorname{grad} \log u|(x) \le C_\beta \inf_{a>0} \inf_{t>0} I^{1/\alpha q}(t, a).
$$

But, if $a > 2C/(q\alpha + 2)$, then

$$
\begin{aligned}
I^{1/\alpha q}(t, a) &\le f^{-1}(x) \left(\frac{a}{1-e^{-at}}\right)^{(q\alpha+2)/(2\alpha q)} \left(\int_0^t e^{(C-(q\alpha+2)a/2)s}\, ds\right)^{1/\alpha q} \\
&= f^{-1}(x) \left(\frac{a}{1-e^{-at}}\right)^{(q\alpha+2)/(2\alpha q)} \left(\frac{1-e^{(C-(q\alpha+2)a/2)t}}{(q\alpha+2)a/2 - C}\right)^{1/\alpha q}
\end{aligned}
$$

and

$$
\inf_{t>0} I^{1/\alpha q}(t, a) \le \lim_{t\to\infty} I^{1/\alpha q}(t, a) = f^{-1}(x) \left(\frac{a^{(q\alpha+2)/2}}{(q\alpha+2)a/2 - C}\right)^{1/\alpha q}.
$$

The minimum in $a > 2C/(q\alpha+2)$ of the last expression is attained for $a = 2C/\alpha q$, and is equal to $(2C/\alpha q)^{1/2}$. Consequently

$$|\operatorname{grad}\log u|(x) \le f^{-1}(x)\, C_\beta \left(\frac{2C}{\alpha q}\right)^{1/2}.$$

Recall that

$$C_\beta = C_{\alpha p} = 2^{-1/2}\left(\frac{\Gamma\left(\frac{1-\alpha p}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^{1/\alpha p}.$$

Combining our results, we get

Theorem 4.1 (Gradient estimate; specific form) *Let M be a Riemannian manifold such that* $\mathrm{Ric} \ge -K$ *for some constant $K \ge 0$. Let u be a positive harmonic function defined on some relatively compact open domain $D \subset M$ with smooth boundary ∂D. Further, let $f \in C^2(\bar{D})$ be strictly positive on D, bounded by 1 and vanishing on ∂D. Then, for any $x \in D$,*

$$|\operatorname{grad}\log u|(x) \le f^{-1}(x)\left(\frac{\Gamma\left(\frac{1-\alpha p}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^{1/\alpha p}\sqrt{C_1(\alpha,K,q,f)} \tag{4.1}$$

where

$$C_1(\alpha,K,q,f) = \frac{1}{2}\sup_{x\in D}\left\{Kf^2(x) - (f\Delta f)(x) + (\alpha q+1)\,|\operatorname{grad} f|^2(x)\right\}. \tag{4.2}$$

Here we have $\alpha \in \left[\frac{n-2}{n-1}, 1\right[$ and $p > 1$, $q > 1$ such that $p^{-1} + q^{-1} = 1$ and $p\alpha < 1$.

Note that $C_1(\alpha,K,q,f)$ differs from the constant $c_1(f)/2$ in [8], Eq. (4.2) only by the term $\alpha q + 1$. In the following remark we specify estimate (4.1) by a particular choice of the function f.

Remark 3 Let M be a complete Riemannian manifold and $D \subset M$ be a relatively compact open domain with smooth boundary. Fixing $x \in D$ and using the function

$$f := \cos\left(\frac{\pi}{2}\,\frac{d(x,\cdot)}{\operatorname{dist}(x,\partial D)}\right),$$

defined on the open ball $B(x, \operatorname{dist}(x,\partial D))$ about x of radius $\operatorname{dist}(x,\partial D)$, we may follow the method applied in [8], Sect. 5 to obtain for strictly positive harmonic functions u on D the estimate

$$|\operatorname{grad}\log u|(x) \le \left(\frac{\Gamma\left(\frac{1-\alpha p}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^{1/\alpha p}\sqrt{C(\operatorname{dist}(x,\partial D))} \tag{4.3}$$

where

$$C(r) := \frac{\pi^2}{4}\,\frac{n+\alpha q+1}{r^2} + \frac{\pi}{2}\,\frac{\sqrt{K\,(n-1)}}{r} + K. \tag{4.4}$$

Here $-K \leq 0$ is again a lower bound for Ricci on M.

It remains to minimize the r.h.s. of (4.1) in α and p (resp. q). Letting for instance $m = \dim M \vee 3$, $\alpha = \frac{m-2}{m-1}$, $p = \frac{2m-3}{2m-4}$ so that $\beta = \alpha p = \frac{2m-3}{2m-2} < 1$, $q = 2m-3$ the conjugate exponent of p, the estimate depends only on the dimension of M, the lower bound K for the Ricci curvature, and on the the distance $\operatorname{dist}(x, \partial D)$ to the boundary. This establishes estimate (1.1) claimed in the Introduction.

Theorem 4.2 (Gradient estimate; Cheng-Yau [3]) *Let M be a complete Riemannian manifold and let $D \subset M$ be a relatively compact domain. Let $u\colon D \to \mathbb{R}$ be a harmonic function which is strictly positive. Put $r(x) = \operatorname{dist}(x, \partial D)$. Then:*

$$\frac{|du|}{u}(x) \leq c(n)\left[k + \frac{1}{r(x)}\right]$$

if $\operatorname{Ric} \geq -(n-1)\,k^2$ *where* $k \geq 0$.

Corollary 4.3 (Gradient estimate on geodesic balls) *Let M be a complete Riemannian manifold with* $\operatorname{Ric} \geq -(n-1)k^2$, $k \geq 0$. *If u is a positive harmonic function on a geodesic ball $B_r(x) \subset M$, then*

$$\sup_{B_{r/2}(x)} \frac{|du|}{u} \leq c(n)\left(\frac{1+k\,r}{r}\right). \tag{4.5}$$

In particular, if $\operatorname{Ric} \geq 0$ *then any positive harmonic function u on M is constant.*

Corollary 4.4 (Elliptic Harnack inequality) *Let M be a complete Riemannian manifold with* $\operatorname{Ric} \geq -(n-1)k^2$. *Suppose that u is a positive harmonic function on a geodesic ball $B_r(x) \subset M$. Then*

$$\sup_{B_{r/2}(x)} u \leq c(n,r,k) \inf_{B_{r/2}(x)} u. \tag{4.6}$$

Proof By the gradient estimate, we have $\sup_{B_{r/2}(x)} |du|/u \leq c(n,r,k)$. Let $x_1, x_2 \in \bar{B}_{r/2}(x)$ be such that $\sup_{B_{r/2}(x)} u = u(x_1)$ and $\inf_{B_{r/2}(x)} u = u(x_2)$. Let γ be a minimal geodesic joining x_1 and x_2. Then

$$\log\frac{u(x_1)}{u(x_2)} = \left|\int \frac{d\log u(\gamma(s))}{ds}\,ds\right| \leq \int_\gamma \frac{|du|}{u} \leq c(n,r,k)\int_\gamma 1 \leq c(n,r,k)\,2r.$$

Therefore, $u(x_1) \leq \exp\left(2r\,c(n,r,k)\right)\,u(x_2)$. $\square$

A Expansion on the submartingale proof

We start by generalizing inequality (3.2).

Lemma A.1 *Let $\varepsilon \geq 0$, $u\colon M \to \mathbb{R}$ be a harmonic function,*

$$M' = \{x \in M : \operatorname{grad} u\,(x) \neq 0\}\,, \quad \varphi_\varepsilon = \sqrt{|\operatorname{grad} u|^2 + \varepsilon^2},$$

and

$$\mathfrak{n}_\varepsilon\,(x) := \varphi_\varepsilon^{-1} \operatorname{grad} u = \Big(|\operatorname{grad} u|^2 + \varepsilon^2\Big)^{-1/2} \operatorname{grad} u,$$

with the convention that $\mathfrak{n}_0\,(x) := 0$ if $x \notin M'$. Then for $\alpha \in \left[\frac{n-2}{n-1}, 1\right]$,

$$\Delta \varphi_\varepsilon^\alpha \geq \alpha \varphi_\varepsilon^\alpha \operatorname{Ric}(\mathfrak{n}_\varepsilon, \mathfrak{n}_\varepsilon) \tag{A.1}$$

where (A.1) *holds for all $x \in M$ if $\varepsilon > 0$ and for all $x \in M'$ if $\varepsilon = 0$.*

Proof Suppose either $\varepsilon > 0$ or $\varepsilon = 0$ and $\operatorname{grad} u\,(x) \neq 0$. We begin by observing that

$$\begin{aligned}\Delta \varphi_\varepsilon^\alpha &= \alpha \varphi_\varepsilon^{\alpha-1} \Delta \varphi_\varepsilon + \alpha\,(\alpha - 1)\,\varphi_\varepsilon^{\alpha-2}\,|\operatorname{grad} \varphi_\varepsilon|^2 \\ &= \alpha \varphi_\varepsilon^\alpha \left\{\frac{\Delta \varphi_\varepsilon}{\varphi_\varepsilon} + (\alpha - 1)\,|\operatorname{grad} \log \varphi_\varepsilon|^2\right\}.\end{aligned} \tag{A.2}$$

Set $f_\varepsilon\,(s) = (\varepsilon^2 + s)^{1/2}$, so that $\varphi_\varepsilon\,(x) = f_\varepsilon\Big(|\operatorname{grad} u\,(x)|^2\Big)$. Then

$$f_\varepsilon'\,(s) = \frac{1}{2}\left(\varepsilon^2 + s\right)^{-1/2} \text{ and } f_\varepsilon''\,(s) = -\frac{1}{4}\left(\varepsilon^2 + s\right)^{-3/2}$$

and for $v \in TM$,

$$\begin{aligned}v\varphi_\varepsilon &= 2 f_\varepsilon'\left(|\operatorname{grad} u|^2\right)\langle \nabla_v \operatorname{grad} u, \operatorname{grad} u\rangle \\ &= \varphi_\varepsilon^{-1}\,\langle \nabla_v \operatorname{grad} u, \operatorname{grad} u\rangle \\ &= \langle \nabla_v \operatorname{grad} u, \mathfrak{n}_\varepsilon\rangle = \langle \nabla_{\mathfrak{n}_\varepsilon} \operatorname{grad} u, v\rangle\end{aligned} \tag{A.3}$$

where in the last equality we have used the fact that ∇ has zero torsion. From this equation it follows that

$$\operatorname{grad} \varphi_\varepsilon = \nabla_{\mathfrak{n}_\varepsilon} \operatorname{grad} u = \nabla_{\mathfrak{n}_\varepsilon}(\varphi_\varepsilon \mathfrak{n}_\varepsilon) = \mathfrak{n}_\varepsilon \varphi_\varepsilon \cdot \mathfrak{n}_\varepsilon + \varphi_\varepsilon \nabla_{\mathfrak{n}_\varepsilon} \mathfrak{n}_\varepsilon, \tag{A.4}$$

and in particular that

$$\operatorname{grad} \log \varphi_\varepsilon = \mathfrak{n}_\varepsilon \log \varphi_\varepsilon \cdot \mathfrak{n}_\varepsilon + \nabla_{\mathfrak{n}_\varepsilon} \mathfrak{n}_\varepsilon. \tag{A.5}$$

Since

$$\operatorname{div} \mathfrak{n}_\varepsilon = \operatorname{div}\left(\varphi_\varepsilon^{-1} \operatorname{grad} u\right) = -\varphi_\varepsilon^{-2}\,\langle \operatorname{grad} \varphi_\varepsilon, \operatorname{grad} u\rangle + \varphi_\varepsilon^{-1} \Delta u$$

$$= -\langle \operatorname{grad} \log \varphi_\varepsilon, \mathfrak{n}_\varepsilon \rangle = -\mathfrak{n}_\varepsilon \log \varphi_\varepsilon,$$

Eq. (A.5) may be written as

$$\operatorname{grad} \log \varphi_\varepsilon = \nabla_{\mathfrak{n}_\varepsilon} \mathfrak{n}_\varepsilon - (\operatorname{div} \mathfrak{n}_\varepsilon)\, \mathfrak{n}_\varepsilon. \tag{A.6}$$

From Eq. (A.3) we also have

$$\begin{aligned} \nabla^2_{v\otimes v} \varphi_\varepsilon &= \left\langle \nabla^2_{v\otimes v} \operatorname{grad} u, \mathfrak{n}_\varepsilon \right\rangle + \langle \nabla_v \operatorname{grad} u, \nabla_v \mathfrak{n}_\varepsilon \rangle \\ &= \left\langle \nabla^2_{v\otimes v} \operatorname{grad} u, \mathfrak{n}_\varepsilon \right\rangle + \langle \nabla_v (\varphi_\varepsilon \mathfrak{n}_\varepsilon), \nabla_v \mathfrak{n}_\varepsilon \rangle \\ &= \left\langle \nabla^2_{v\otimes v} \operatorname{grad} u, \mathfrak{n}_\varepsilon \right\rangle + v\varphi_\varepsilon \cdot \langle \mathfrak{n}_\varepsilon, \nabla_v \mathfrak{n}_\varepsilon \rangle + \varphi_\varepsilon \langle \nabla_v \mathfrak{n}_\varepsilon, \nabla_v \mathfrak{n}_\varepsilon \rangle \end{aligned}$$

which upon summing on v running through an orthonormal frame shows

$$\begin{aligned} \Delta\varphi_\varepsilon &= \langle \square \operatorname{grad} u, \mathfrak{n}_\varepsilon \rangle + \langle \mathfrak{n}_\varepsilon, \nabla_{\operatorname{grad} \varphi_\varepsilon} \mathfrak{n}_\varepsilon \rangle + \varphi_\varepsilon |\nabla \mathfrak{n}_\varepsilon|^2 \\ &= \left\langle \operatorname{grad} \Delta u + \operatorname{Ric}^\# \operatorname{grad} u, \mathfrak{n}_\varepsilon \right\rangle + \langle \mathfrak{n}_\varepsilon, \nabla_{\operatorname{grad} \varphi_\varepsilon} \mathfrak{n}_\varepsilon \rangle + \varphi_\varepsilon |\nabla \mathfrak{n}_\varepsilon|^2 \\ &= \left\langle \operatorname{Ric}^\# \nabla u, \mathfrak{n}_\varepsilon \right\rangle + \langle \mathfrak{n}_\varepsilon, \nabla_{\operatorname{grad} \varphi_\varepsilon} \mathfrak{n}_\varepsilon \rangle + \varphi_\varepsilon |\nabla \mathfrak{n}_\varepsilon|^2 \\ &= \varphi_\varepsilon \left\{ \operatorname{Ric}(\mathfrak{n}_\varepsilon, \mathfrak{n}_\varepsilon) + \langle \mathfrak{n}_\varepsilon, \nabla_{\operatorname{grad} \log \varphi_\varepsilon} \mathfrak{n}_\varepsilon \rangle + |\nabla \mathfrak{n}_\varepsilon|^2 \right\}. \end{aligned} \tag{A.7}$$

When $\varepsilon \neq 0$ and $\operatorname{grad} u(x) = 0$, we see from Eq. (A.7) that

$$\Delta\varphi_\varepsilon(x) = \varphi_\varepsilon(x)\, |\nabla \mathfrak{n}_\varepsilon|^2 (x)$$

and from Eq. (A.4) that $(\operatorname{grad} \log \varphi_\varepsilon)(x) = 0$. Combining these identities with Eq. (A.2) gives

$$(\Delta\varphi_\varepsilon^\alpha)(x) = \alpha\varphi_\varepsilon^\alpha(x)\, |\nabla \mathfrak{n}_\varepsilon|^2 (x) \geq 0 = \alpha\varphi_\varepsilon^\alpha(x) \operatorname{Ric}(\mathfrak{n}_\varepsilon, \mathfrak{n}_\varepsilon)(x). \tag{A.8}$$

This shows that Eq. (A.1) is valid for $x \notin M'$. To finish the proof it suffices to show that Eq. (A.1) is valid for all $x \in M'$.

So for the rest of the proof we will assume that $x \in M'$. Since $\langle \mathfrak{n}_0, \mathfrak{n}_0 \rangle = 1$ on M',

$$0 = v1 = 2\langle \nabla_v \mathfrak{n}_0, \mathfrak{n}_0 \rangle \text{ for all } v \in T_x M \text{ and } x \in M'. \tag{A.9}$$

Taking $\varepsilon = 0$ in Eq. (A.7) gives

$$\Delta\varphi_0 = \varphi_0 \left(\operatorname{Ric}(\mathfrak{n}_0, \mathfrak{n}_0) + |\nabla \mathfrak{n}_0|^2 \right) \tag{A.10}$$

and from Eqs. (A.6) and (A.9) we find

$$|\operatorname{grad} \log \varphi_0|^2 = |\nabla_{\mathfrak{n}_0} \mathfrak{n}_0|^2 + (\operatorname{div} \mathfrak{n}_0)^2 \tag{A.11}$$

on M'. Let $F_\varepsilon(s) := (s^2 + \varepsilon^2)^{\alpha/2}$ so that $\varphi_\varepsilon^\alpha = F_\varepsilon(\varphi_0)$ and by elementary calculus,

$$F'_\varepsilon(s) = \alpha s \left(s^2 + \varepsilon^2\right)^{-1} F_\varepsilon(s) \text{ and}$$

$$F''_\varepsilon(s) = \alpha \left(s^2+\varepsilon^2\right)^{-2} \left(\varepsilon^2 - (1-\alpha) s^2\right) F_\varepsilon(s).$$

Therefore we find

$$\begin{aligned}
\Delta\varphi_\varepsilon^\alpha &= \Delta F_\varepsilon(\varphi_0) = \operatorname{div}\left(F'_\varepsilon(\varphi_0)\operatorname{grad}\varphi_0\right) = F'_\varepsilon(\varphi_0)\Delta\varphi_0 + F''_\varepsilon(\varphi_0)\left|\operatorname{grad}\varphi_0\right|^2 \\
&= \alpha\varphi_0\left(\varphi_0^2+\varepsilon^2\right)^{-1} F_\varepsilon(\varphi_0)\Delta\varphi_0 \\
&\quad + \alpha\left(\varphi_0^2+\varepsilon^2\right)^{-2}\left(\varepsilon^2-(1-\alpha)\varphi_0^2\right) F_\varepsilon(\varphi_0)\left|\operatorname{grad}\varphi_0\right|^2 \\
&\geq \alpha\varphi_\varepsilon^\alpha\left\{\varphi_0\left(\varphi_0^2+\varepsilon^2\right)^{-1}\Delta\varphi_0 - \left(\varphi_0^2+\varepsilon^2\right)^{-2}(1-\alpha)\varphi_0^2\left|\operatorname{grad}\varphi_0\right|^2\right\} \\
&= \alpha\varphi_\varepsilon^\alpha\frac{\varphi_0^2}{\varphi_0^2+\varepsilon^2}\left\{\frac{\Delta\varphi_0}{\varphi_0} - (1-\alpha)\frac{\varphi_0^2}{\varphi_0^2+\varepsilon^2}\left|\operatorname{grad}\log\varphi_0\right|^2\right\} \\
&\geq \alpha\varphi_\varepsilon^\alpha\frac{\varphi_0^2}{\varphi_0^2+\varepsilon^2}\left\{\frac{\Delta\varphi_0}{\varphi_0} - (1-\alpha)\left|\operatorname{grad}\log\varphi_0\right|^2\right\} \\
&= \alpha\varphi_\varepsilon^\alpha\frac{\varphi_0^2}{\varphi_0^2+\varepsilon^2}\left\{\operatorname{Ric}(\mathfrak{n}_0,\mathfrak{n}_0) + \left|\nabla\mathfrak{n}_0\right|^2 - (1-\alpha)\left|\operatorname{grad}\log\varphi_0\right|^2\right\} \qquad \text{(A.12)}
\end{aligned}$$

where in the last equality we have used Eq. (A.10).

Letting $\{e_i\}_{i=1}^n$ be an orthonormal frame such that $e_n = \mathfrak{n}_0$ shows

$$\begin{aligned}
(\operatorname{div}\mathfrak{n}_0)^2 &= \left(\sum_{i=1}^{n}\langle\nabla_{e_i}\mathfrak{n}_0, e_i\rangle\right)^2 = \left(\sum_{i=1}^{n-1}\langle\nabla_{e_i}\mathfrak{n}_0, e_i\rangle\right)^2 \\
&\leq (n-1)\sum_{i=1}^{n-1}\langle\nabla_{e_i}\mathfrak{n}_0, e_i\rangle^2 \leq (n-1)\sum_{i=1}^{n-1}\left|\nabla_{e_i}\mathfrak{n}_0\right|^2
\end{aligned}$$

and therefore, using Eq. (A.9),

$$\begin{aligned}
\left|\operatorname{grad}\log\varphi_0\right|^2 &= \left|\nabla_{\mathfrak{n}_0}\mathfrak{n}_0\right|^2 + (\operatorname{div}\mathfrak{n}_0)^2 \\
&\leq (n-1)\sum_{i=1}^{n-1}\left|\nabla_{e_i}\mathfrak{n}_0\right|^2 + \left|\nabla_{e_n}\mathfrak{n}_0\right|^2 \leq (n-1)\left|\nabla\mathfrak{n}_0\right|^2.
\end{aligned}$$

Combining this estimate with Eq. (A.12) shows

$$\Delta\varphi_\varepsilon^\alpha \geq \alpha\varphi_\varepsilon^\alpha\frac{\varphi_0^2}{\varphi_0^2+\varepsilon^2}\left\{\operatorname{Ric}(\mathfrak{n}_0,\mathfrak{n}_0) + \Big(1-(1-\alpha)(n-1)\Big)\left|\nabla\mathfrak{n}_0\right|^2\right\}. \qquad \text{(A.13)}$$

Taking $\alpha \geq \frac{n-2}{n-1}$ (which is equivalent to $1-(1-\alpha)(n-1)\geq 0$) in Eq. (A.13) implies

$$\Delta\varphi_\varepsilon^\alpha \geq \alpha\varphi_\varepsilon^\alpha\frac{\varphi_0^2}{\varphi_0^2+\varepsilon^2}\operatorname{Ric}(\mathfrak{n}_0,\mathfrak{n}_0) = \alpha\varphi_\varepsilon^\alpha\operatorname{Ric}(\mathfrak{n}_\varepsilon,\mathfrak{n}_\varepsilon). \qquad \text{(A.14)}$$

Combining this estimate with that in Eq. (A.8) proves inequality (A.1). $\square$

Theorem A.2 *We keep the notation and the assumptions of Theorem* (3.1). *For all* $\alpha \in \left[\frac{n-2}{n-1}, 1\right]$,

$$Y_t := |\operatorname{grad} u(X_t)|^\alpha \exp\left\{-\frac{\alpha}{2}\int_0^t \operatorname{Ric}(\mathfrak{n},\mathfrak{n})(X_s)\, ds\right\} \tag{A.15}$$

is a local submartingale, where

$$\mathfrak{n}(x) := \mathfrak{n}_0(x) = \begin{cases} |\operatorname{grad} u(x)|^{-1} \operatorname{grad} u(x) & \text{if } \operatorname{grad} u(x) \neq 0 \\ 0 & \text{if } \operatorname{grad} u(x) = 0. \end{cases}$$

Proof Let $\varepsilon > 0$, then by Itô's formula along with Lemma A.1,

$$\begin{aligned} d\varphi_\varepsilon^\alpha(X_t) &= \left\langle (\operatorname{grad}\varphi_\varepsilon^\alpha)(X_t), //_t\, db_t\right\rangle + \frac{1}{2}(\Delta\varphi_\varepsilon^\alpha)(X_t)\, dt \\ &= dM_t^\varepsilon + \frac{\alpha}{2}\varphi_\varepsilon^\alpha(X_t)\operatorname{Ric}(\mathfrak{n}_\varepsilon,\mathfrak{n}_\varepsilon)(X_t)\, dt + \rho_t^\varepsilon\, dt \end{aligned}$$

where ρ_t^ε is a non-negative process. In particular this implies that

$$\begin{aligned} &d\left(\exp\left\{-\frac{\alpha}{2}\int_0^t \operatorname{Ric}(\mathfrak{n}_\varepsilon,\mathfrak{n}_\varepsilon)(X_s)\, ds\right\}\varphi_\varepsilon^\alpha(X_t)\right) \\ &\quad= \exp\left\{-\frac{\alpha}{2}\int_0^t \operatorname{Ric}(\mathfrak{n}_\varepsilon,\mathfrak{n}_\varepsilon)(X_s)\, ds\right\} dM_t \\ &\qquad+ \exp\left\{-\frac{\alpha}{2}\int_0^t \operatorname{Ric}(\mathfrak{n}_\varepsilon,\mathfrak{n}_\varepsilon)(X_s)\, ds\right\}\rho_t^\varepsilon\, dt. \end{aligned}$$

So if $\varepsilon > 0$ and $\alpha \in \left[\frac{n-2}{n-1}, 1\right]$, then

$$Y_t(\varepsilon) := \left(|\operatorname{grad} u(X_t)|^2 + \varepsilon^2\right)^{\alpha/2} \exp\left\{-\frac{\alpha}{2}\int_0^t \operatorname{Ric}(\mathfrak{n}_\varepsilon,\mathfrak{n}_\varepsilon)(X_s)\, ds\right\}$$

is a local submartingale. If τ is the first exit time of X_t from a precompact open subset of M, $Y_t^\tau(\varepsilon)$ is an honest submartingale. If G is a bounded non-negative $\mathscr{F}_s$-measurable function, then

$$\mathbb{E}\Big[G\Big(Y_t^\tau(\varepsilon) - Y_s^\tau(\varepsilon)\Big)\Big] = \mathbb{E}\left[G\int_s^t \exp\left\{-\frac{\alpha}{2}\int_0^r \operatorname{Ric}(\mathfrak{n}_\varepsilon,\mathfrak{n}_\varepsilon)(X_s)\, ds\right\}\rho_r^\varepsilon\, dr\right] \geq 0.$$

Using the dominated convergence theorem, we may let $\varepsilon \downarrow 0$ in the above inequality to conclude,

$$\mathbb{E}\left[G\left(Y_t^\tau - Y_s^\tau\right)\right] \geq 0$$

which completes the proof. □

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