Path integrals on Riemannian Manifolds

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Nelder Talk 1.

1pm-2:30pm, Wednesday 29th October, Room 340, Huxley

Imperial College, London

Newtonian Mechanics on $\mathbb{R}^d$

Given a potential energy function $V : \mathbb{R}^d \to \mathbb{R}$ we look to solve

$$m\ddot{q}(t) = -\nabla V(q(t)) \text{ for } q(t) \in \mathbb{R}^d$$

that is

Force = mass · acceleration

Recall that $p = m\dot{q}$ and

$$\begin{aligned} H(q,p) &= \frac{1}{2m} p \cdot p + V(q) \\ &= \text{ Conserved Energy} \\ &= E(q,\dot{q}) := \frac{1}{2} m |\dot{q}|^2 + V(q) \end{aligned}$$

Q.M. and Canonical Quantization on $\mathbb{R}^d$

We want to find

$$\psi(t,x) = \left(e^{\frac{t}{i\hbar}\hat{H}}\psi_0\right)(x)$$

i.e. solve the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi(t) \text{ for } \psi(t) \in L^2\left(\mathbb{R}^d\right)$$

$$\text{with } \psi(0,x) = \psi_0(x)$$

where by "Canonical Quantization,"

$$q \rightsquigarrow \hat{q} = M_q,\ p \rightsquigarrow \hat{p} = \frac{\hbar}{i}\nabla = \frac{\hbar}{i}\frac{\partial}{\partial q} \text{ and}$$

$$H(q,p) \rightsquigarrow H(\hat{q},\hat{p}) = -\frac{\hbar^2}{2m}\nabla^2 + M_{V(q)}.$$

Feynman Path Integral

Feynman explained that the solution to the Schrödinger equation should be given by

$$\left(e^{\frac{T}{i\hbar}\hat{H}}\psi_0\right)(x) = \frac{1}{Z(T)}\int_{W_{x,T}(\mathbb{R}^3)} e^{\frac{i}{\hbar}\int_0^T \text{(K.E. - P.E.)}(t)dt}\psi_0(\omega(T))\,d\,\mathrm{vol}(\omega) \qquad (1)$$

where $\psi_0(x)$ is the initial wave function,

$$\text{(K.E. - P.E.)}(t) = \frac{m}{2}\left|\dot{\omega}(t)\right|^2 - V(\omega(t)),$$

and

$$Z(T) = \int_{W_{x_0,T}(\mathbb{R}^3)} e^{\frac{i}{\hbar}\int_0^T \text{(K.E.)}(t)dt}\,d\,\mathrm{vol}(\omega).$$

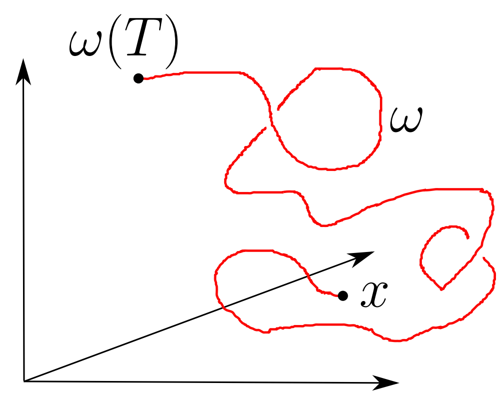


Figure 1: $W_{x,T}\left(\mathbb{R}^d\right) =$ the paths in $\mathbb{R}^d$ starting at x which are parametrized by $[0,T]$.

The Path Integral Prescription on $\mathbb{R}^d$

Theorem 1 (Meta-Theorem – Feynman (Kac) Quantization)**.** *Let* $V : \mathbb{R}^d \to \mathbb{R}$ *be a nice function and*

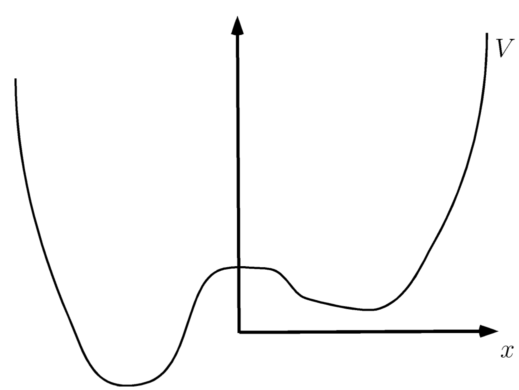
$$W\left(\mathbb{R}^d; x, T\right) := \left\{\omega \in C\left([0,T] \to \mathbb{R}^d\right) : \omega(0) = x\right\}.$$

Then

$$\left(e^{-T\hat{H}} f\right)(x) = \text{“}\frac{1}{Z_T} \int_{W\left(\mathbb{R}^d; x, T\right)} e^{-\int_0^T E(\omega(t), \dot{\omega}(t))dt} f(\omega(T)) \mathcal{D}\omega\text{”} \tag{2}$$

where $E(x,v) = \frac{1}{2} m |v|^2 + V(x)$ *is the classical energy and*

$$\text{“}Z_T := \int_{W\left(\mathbb{R}^d; x, T\right)} e^{-\frac{1}{2}\int_0^T |\dot{\omega}(t)|^2 dt} \mathcal{D}\omega\text{”.}$$



Proof of the Path Integral Prescription

Theorem 2 (Trotter Product Formula)**.** *Let A and B be $n \times n$ matrices. Then*

$$e^{(A+B)} = \lim_{n\to\infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}} \right)^n .$$

Proof: Since

$$\frac{d}{d\varepsilon}|_0 \log(e^{\varepsilon A} e^{\varepsilon B}) = A + B,$$
$$\log(e^{\varepsilon A} e^{\varepsilon B}) = \varepsilon\left(A + B\right) + O\left(\varepsilon^2\right),$$

i.e.

$$e^{\varepsilon A} e^{\varepsilon B} = e^{\varepsilon(A+B)+O\left(\varepsilon^2\right)}$$

and therefore

$$\begin{aligned}(e^{n^{-1}A} e^{n^{-1}B})^n &= \left[e^{n^{-1}A+n^{-1}B+O\left(n^{-2}\right)} \right]^n \\ &= e^{A+B+O\left(n^{-1}\right)} \to e^{(A+B)} \text{ as } n \to \infty.\end{aligned}$$

- Let $A := \frac{1}{2}\Delta$;

$$\left(e^{t\Delta/2}f\right)(x) = \int_{\mathbb{R}^d} p_t(x,y)f(y)dy$$

where

$$p_t(x,y) = \left(\frac{1}{2\pi t}\right)^{d/2} \exp\left(\frac{1}{2t}|x-y|^2\right)$$

- Let $B = -M_V$ – multiplication by V; $e^{-tM_V} = M_{e^{-tV}}$

- By Trotter ($x_0 := x$),

$$\begin{aligned}
&\left(\left(e^{\frac{T}{n}\Delta/2}e^{-\frac{T}{n}V}\right)^n f\right)(x) \\
&= \int_{\left(\mathbb{R}^d\right)^n} p_{\frac{T}{n}}(x_0,x_1)e^{-\frac{T}{n}V(x_1)} \dots p_{\frac{T}{n}}(x_{n-1},x_n)e^{-\frac{T}{n}V(x_n)}f(x_n)dx_1\dots dx_n \\
&= \frac{1}{Z_n(T)}\int_{(\mathbb{R}^d)^n} e^{-\frac{n}{2T}\sum_{i=1}^{n}|x_i-x_{i-1}|^2-\frac{T}{n}\sum_{i=1}^{n}V(x_i)} f(x_n)dx_1\dots dx_n \\
&= \frac{1}{Z_n(T)}\int_{H_n} e^{-\int_0^T\left[\frac{1}{2}|\omega'(s)|^2+V(\omega(s_+))\right]ds} f(\omega(T))dm_{H_n}(\omega) \qquad (3)
\end{aligned}$$

where $Z_n(T) := (2\pi T/n)^{dn/2}$, $\mathcal{P}_n = \left\{\frac{k}{n}T\right\}_{k=0}^{n}$, and

$$H_n = \left\{\omega \in W\left(\mathbb{R}^d; x, T\right) : \omega''(s) = 0 \text{ for } s \notin \mathcal{P}_n\right\}.$$

Q.E.D.

Euclidean Path Integral Quantization on $\mathbb{R}^d$

Theorem 3 (Meta-Theorem – Path integral quantization)**.** *We can define $\hat{H}$ by;*

$$\left(e^{-T\hat{H}}\psi_0\right)(x)\text{“}=\text{”}\frac{1}{Z_T}\int_{\omega(0)=x} e^{-\int_0^T E(\omega(t),\dot{\omega}(t))dt}\psi_0(\omega(T))\mathcal{D}\omega \tag{4}$$

where

$$\text{“}Z_T := \int_{\omega(0)=0} e^{-\frac{1}{2}\int_0^T |\dot{\omega}(t)|^2 dt}\mathcal{D}\omega\text{”.}$$

and

$$\mathcal{D}\omega = \text{“Infinite Dimensional Lebesgue Measure.”}$$

- **Question:** what does this formula really mean?

1. Problems, $Z_T = \lim_{n\to\infty} Z_n(T) = 0$.

2. There is not Lebesgue measure in infinite dimensions.

3. The paths ω appearing in Eq. (4) are very rough and in fact nowhere differentiable.

Summary of Flat Results

- Let $\mathcal{P} := \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of $[0, T]$.
- Let $H_{\mathcal{P}}\left(\mathbb{R}^d\right) := \left\{\omega : [0, T] \to \mathbb{R}^d : \omega(0) = 0 \text{ and } \ddot{\omega}(t) = 0 \; \forall \, t \notin \mathcal{P}\right\}$
- $\lambda_{\mathcal{P}}$ be Lebesgue measure on $H_{\mathcal{P}}\left(\mathbb{R}^d\right)$
- $Z_{\mathcal{P}} := \int_{H_{\mathcal{P}}\left(\mathbb{R}^d\right)} \exp\left(-\frac{1}{2}\int_0^T |\dot{\omega}(t)|^2 \, dt\right) d\lambda_{\mathcal{P}}(\omega)$
- $d\mu_{\mathcal{P}} := \frac{1}{Z_{\mathcal{P}}} \exp\left(-\frac{1}{2}\int_0^T |\dot{\omega}(t)|^2 \, dt\right) d\lambda_{\mathcal{P}}(\omega)$

Theorem 4 (Wiener 1923)**.** *There exist a measure* μ *on* $W\left([0, T], \mathbb{R}^d\right)$ *such that* $\mu_{\mathcal{P}} \implies \mu$ *as* $|\mathcal{P}| \to 0$.

Theorem 5 (Feynman Kac)**.** *If* $E(x,v) = \frac{1}{2}|v|^2 + V(x)$ *where* V *is a nice potential, then*

$$\frac{1}{Z_{\mathcal{P}}} \exp\left(-\int_0^T E(\omega(t), \dot{\omega}(t))\, dt\right) d\lambda_{\mathcal{P}}(\omega) \implies e^{-\int_0^T V(\omega(s))ds} d\mu(\omega)$$

and morever,

$$\begin{aligned}\left(e^{-t\hat{H}} f\right)(0) &= \lim_{|\mathcal{P}|\to 0} \frac{1}{Z_{\mathcal{P}}} \int_{H_{\mathcal{P}}\left(\mathbb{R}^d\right)} \exp\left(-\int_0^T E(\omega(t), \dot{\omega}(t))\, dt\right) f(\omega(T))\, d\lambda_{\mathcal{P}}(\omega) \\ &= \int_{W\left([0,T],\mathbb{R}^d\right)} e^{-\int_0^T V(\omega(s))ds} f(\omega(T))\, d\mu(\omega).\end{aligned}$$

Norbert Wiener

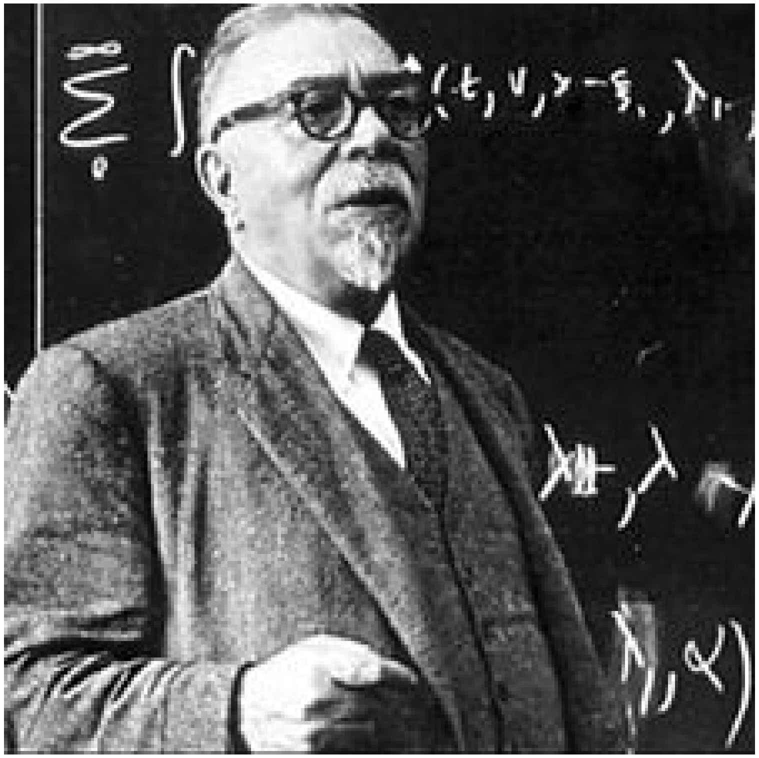
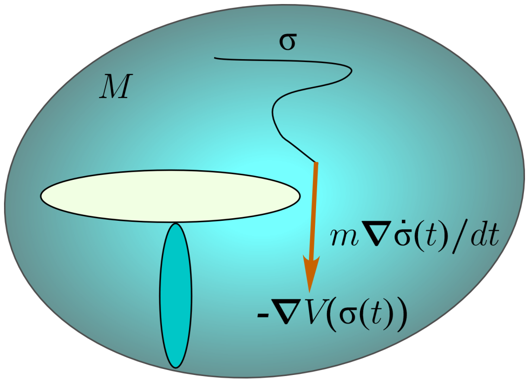


Figure 2: Norbert Wiener (November 26, 1894 – March 18, 1964). Graduated High School at 11, BA at Tufts College at the age of 14, and got his Ph.D. from Harvard at 18.

Classical Mechanics on a Manifold

- Let (M, g) be a Riemannian manifold.



- Newton's Equations of motion

$$m\frac{\nabla\dot{\sigma}\,(t)}{dt} = -\nabla V(q(t)), \tag{5}$$

i.e.

Force = mass · tangential acceleration

- In local coordinates $(q^1, \ldots, q^d)$;

$$H\,(q,p) = \frac{1}{2m}g^{ij}\,(q)\,p_i p_j + V\,(q) \text{ where}$$
$$ds^2 = g_{ij}\,(q)\,dq^i dq^j$$

(Not) Canonical Quantization on M

$$
\begin{aligned}
H(q,p) &= \frac{1}{2} g^{ij}(q) p_i p_j + V(q) \\
&= \frac{1}{2} \frac{1}{\sqrt{g}} p_i \sqrt{g} g^{ij}(q) p_j + V(q).
\end{aligned}
$$

- To quantize $H(q,p)$, let

$$
q_i \rightsquigarrow \hat{q}_i := M_{q^i}, \quad p_i \rightsquigarrow \hat{p}_i := \frac{1}{i} \frac{\partial}{\partial q^i}, \text{ and } H(q,p) \overset{?}{\rightsquigarrow} H(\hat{q}, \hat{p}).
$$

- Is

$$
\hat{H} = -\frac{1}{2} g^{ij}(q) \frac{\partial^2}{\partial q^i \partial q^j} + V(q)
$$

- or is it

$$
\hat{H} = -\frac{1}{2} \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^i} \sqrt{g} g^{ij}(q) \frac{\partial}{\partial q^j} + V(q) = -\frac{1}{2} \Delta_M + M_V,
$$

- or something else?

Path Integral Quantization of $\hat{H}$

The previous formulas on $\mathbb{R}^d$ suggest we can **define** $\hat{H}$ in the manifold setting by;

$$\left(e^{-T\hat{H}}\psi_0\right)(x_0) = \frac{1}{Z_T}\int_{\sigma(0)=x_0} e^{-\int_0^T E(\sigma(t),\dot{\sigma}(t))dt}\psi_0(\sigma(T))\mathcal{D}\sigma \tag{6}$$

where

$$E(x,v) = \frac{1}{2}g(v,v) + V(x)$$

is the classical energy.

- Formally, there no longer seems to be any ambiguity as there was with canonical quantization.
- On the other hand what does Eq. (6) actually mean?

Back to Curved Space Path Integrals

- Recall we now wish to mathematically interpret the expression;

$$d\nu(\sigma)\text{``}=\text{''}\frac{1}{Z\left(T\right)}e^{-\int_0^T\left[\frac{1}{2}|\dot{\sigma}(t)|^2+V(\sigma(t))\right]dt}\mathcal{D}\sigma.$$

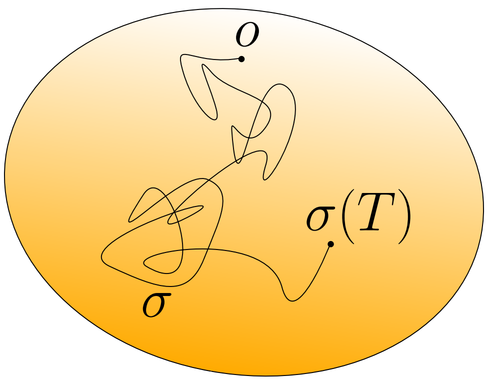


Figure 3: A path in $W_{o,T}\left(M\right)$.

- To simplify life (and w.o.l.o.g.) set $V=0, T=1$ so that we will now consider,

$$\frac{1}{Z}\int_{W_o(M)}e^{-\frac{1}{2}\int_0^1|\dot{\sigma}(t)|^2dt}\psi_0\left(\sigma\left(1\right)\right)\mathcal{D}\sigma.$$

- We need introduce (recall) six geometric ingredients.

I. Geometric Wiener Measure (ν) over M

Fact (Cartan's Rolling Map)**.** Relying on Itô to handle the technical (non-differentiability) difficulties, we may transfer Wiener's measure, μ, on $W_{0,T}\left(\mathbb{R}^d\right)$ to a measure, ν, on $W_{o,T}(M)$.

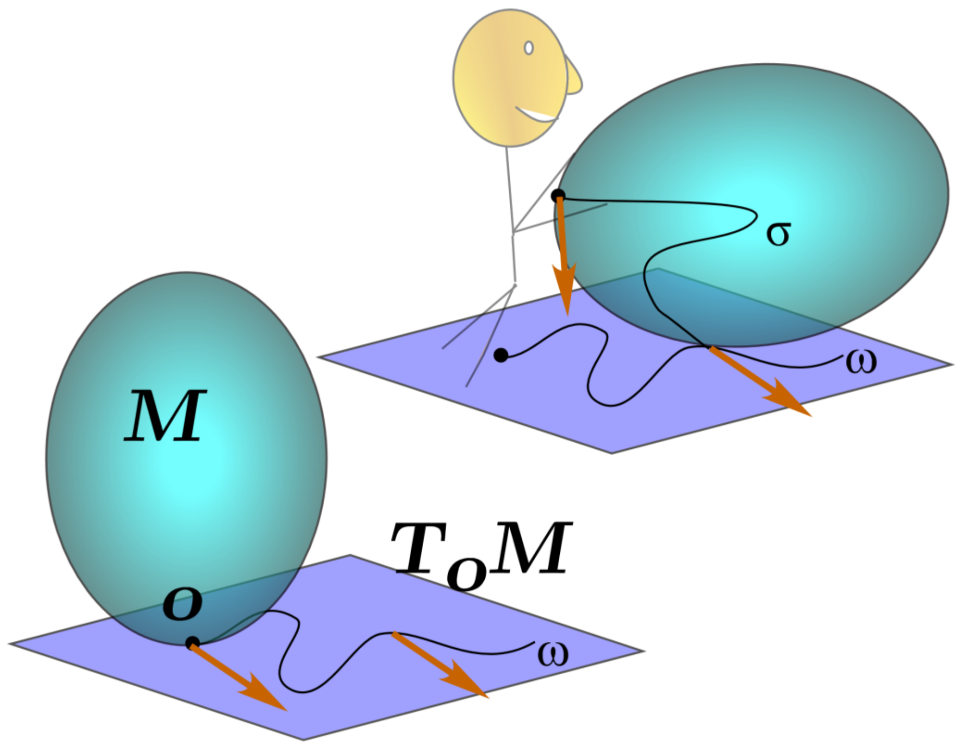


Figure 4: Cartan's rolling map gives a one to one correspondance between, $W_{0,T}\left(\mathbb{R}^d\right)$ and $W_{o,T}(M)$.

II. Riemannian Volume Measures

- On any finite dimensional Riemannian manifold (M, g) there is an associated **volume measure**,

$$d\,\mathrm{Vol}_g = \sqrt{\det\left(g\left(\frac{\partial \Sigma}{\partial t_i}, \frac{\partial \Sigma}{\partial t_j}\right)\right)} dt_1 \dots dt_n \qquad (7)$$

  where $\mathbb{R}^n \ni (t_1, \dots, t_n) \to \Sigma(t_1, \dots, t_n) \in M$ is a (local) parametrization of M.

Example 1. Suppose M is 2 dimensional surface, then we teach,

$$dS = \|\partial_{t_1}\Sigma(t_1, t_2) \times \partial_{t_2}\Sigma(t_1, t_2)\| \, dt_1 dt_2. \qquad (8)$$

Combining this with the identity,

$$\begin{aligned} \|a \times b\|^2 &= \|a\|^2 \|b\|^2 - (a \cdot b)^2 \\ &= \det \begin{bmatrix} a \cdot a & a \cdot b \\ a \cdot b & b \cdot b \end{bmatrix} \end{aligned}$$

shows,

$$dS = \sqrt{\det \begin{bmatrix} \partial_{t_1}\Sigma \cdot \partial_{t_1}\Sigma & \partial_{t_1}\Sigma \cdot \partial_{t_2}\Sigma \\ \partial_{t_1}\Sigma \cdot \partial_{t_2}\Sigma & \partial_{t_2}\Sigma \cdot \partial_{t_2}\Sigma \end{bmatrix}} dt_1 dt_2$$

that is Eq. (7) reduces to Eq. (8) for surfaces in $\mathbb{R}^3$.

III. Scalar Curvature

- On any *finite dimensional* Riemannian manifold (M, g) there is an associated function called **scalar curvatue**,

$$\mathrm{Scal} : M \to \mathbb{R}$$

such that at a point $m \in M$,

$$\mathrm{Vol}_g(B_\varepsilon(m)) = \left|B_\varepsilon^{\mathbb{R}^d}(0)\right| \left(1 - \frac{\varepsilon^2}{6(d+2)}\mathrm{Scal}(m) + O(\varepsilon^3)\right) \text{ for } \varepsilon \sim 0,$$

where $\left|B_\varepsilon^{\mathbb{R}^d}(0)\right|$ is the volume of a ε – Euclidean ball in $\mathbb{R}^d$.

IV. Tangent Vectors in Path Spaces

- The space

$$H(M) = \left\{ \sigma \in W_o(M) : E(\sigma) := \int_0^1 |\dot{\sigma}(t)|^2 \, dt < \infty \right\}$$

is an infinite dimensional Hilbert manifold.

- The tangent space to $\sigma \in H(M)$ is

$$T_\sigma H(M) = \left\{ \begin{array}{c} X : [0,1] \to TM : X(t) \in T_{\sigma(t)} M \text{ and} \\ G^1(X, X) := \int_0^1 g\left(\frac{\nabla X(t)}{dt}, \frac{\nabla X(t)}{dt}\right) dt < \infty \end{array} \right\}.$$

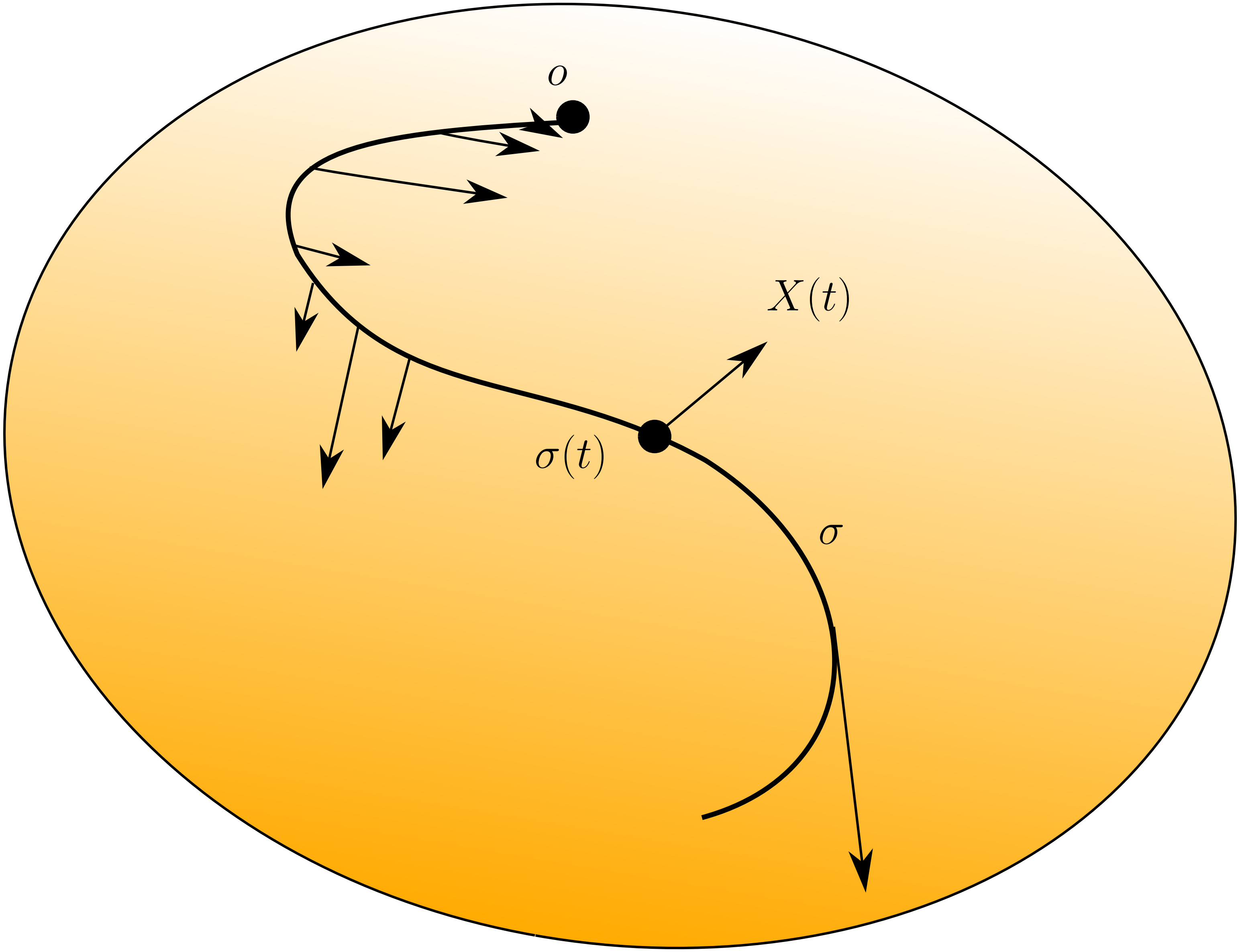


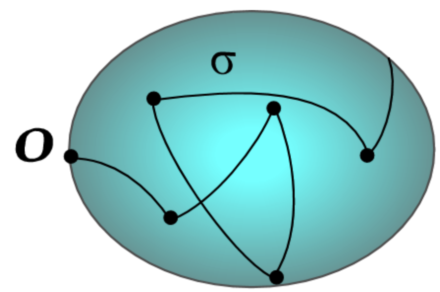
Figure 5: A tangent vector at $\sigma \in H(M)$.

V. Piecewise Geodesics Approximations

- Given a partition $\mathcal{P}$ of $[0,1]$ the space

$$H_{\mathcal{P}}(M) = \left\{\sigma \in W_o(M) : \frac{\nabla}{dt}\dot{\sigma}(t) = 0 \text{ for } t \notin \mathcal{P}\right\}$$

is a smooth finite dimensional embedded sub-manifold of $H(M)$.



VI. Four Riemannian Metrics on $H_{\mathcal{P}}(M)$

Let $\sigma \in \mathrm{H}_{\mathcal{P}}(M)$, and $X, Y \in T_\sigma \mathrm{H}_{\mathcal{P}}(M)$. **Metrics:**

- H^0–**Metric on** $H(M)$

$$G^0(X,X) := \int_0^1 \langle X(s), X(s)\rangle \, ds,$$

- H^1–**Metric on** $H(M)$

$$G^1(X,X) := \int_0^1 \left\langle \frac{\nabla X(s)}{ds}, \frac{\nabla X(s)}{ds} \right\rangle ds,$$

- H^1–**Metric on** $H_{\mathcal{P}}(M)$ (Riemannian Sum Approximation)

$$G^1_{\mathcal{P}}(X,Y) := \sum_{i=1}^n \langle \frac{\nabla X(s_{i-1}+)}{ds}, \frac{\nabla Y(s_{i-1}+)}{ds} \rangle \Delta_i s,$$

- H^0–"**Metric**" **on** $H_{\mathcal{P}}(M)$ (Riemannian Sum Approximation)

$$G^0_{\mathcal{P}}(X,Y) := \sum_{i=1}^n \langle X(s_i), Y(s_i)\rangle \Delta_i s.$$

Riemann Sum Metric Results

Theorem 6 (Andersson and D. JFA 1999.). *Suppose that* $f : W(M) \to \mathbb{R}$ *is a bounded and continuous and*

$$d\nu_{\mathcal{P}}^{*}(\sigma) = \frac{1}{Z_{\mathcal{P}}} e^{-\frac{1}{2}\int_0^1 |\dot{\sigma}(t)|^2 dt} d\,\mathrm{vol}_{G_{\mathcal{P}}^{*}}(\sigma) \text{ for } * \in \{0,1\}.$$

Then

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma) d\nu_{\mathcal{P}}^{1}(\sigma) = \int_{\mathrm{W}(M)} f(\sigma) d\nu(\sigma)$$

$$\implies \hat{H} = -\frac{1}{2}\Delta_M = -\frac{1}{2}\Delta_M + \frac{1}{\infty}\mathrm{Scal}.$$

and

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma) d\nu_{\mathcal{P}}^{0}(\sigma)$$

$$= \int_{\mathrm{W}(M)} f(\sigma) e^{-\frac{1}{6}\int_0^1 \mathrm{Scal}(\sigma(s))ds} d\nu(\sigma)$$

$$\implies \hat{H} = -\frac{1}{2}\Delta_M + \frac{1}{6}\mathrm{Scal}.$$

Some Other (Markovian) Results

If $\hat{H}$ is "defined" by

$$e^{-T\hat{H}} f(x_0) = \frac{1}{Z_T} \int_{\sigma(0)=x_0} e^{-\int_0^T E(\sigma(t),\dot{\sigma}(t))dt} f(\sigma(T)) \mathcal{D}\sigma \tag{9}$$

then

$$\hat{H} = -\frac{1}{2}\Delta + \frac{1}{\kappa}\mathrm{S}$$

where

- S is the scalar curvature of M, and
- $\kappa \in \{6, 8, 12, \infty\}$.
- $\kappa = 6$ Cheng 72.
- $\kappa = 12$, De Witt 1957, Um 73, Atsuchi & Maeda 85, and Darling 85. Geometric Quantization. (AIDA says to check these names: Atsuchi & Maeda as at least one is a given name rather than the family name.)
- $\kappa = 8$ Marinov 1980 and De Witt 1992.

- Inahama (2005) Osaka J. Math.
- Semi-group proofs and extensions of AD1999;
  - Butko (2006)
  - O. G. Smolyanov, Weizsäcker, Wittich, Potential Anal. 26 (2007).
  - Bär and Frank Pfäffle, Crelle 2008.
- Fine and Sawin CMP (2008) – supersymmetic version.
- In the real Feynman case see for example S. Albeverio and R. Hoegh-Krohn (1976), Lapidus and Johnson, etc. etc.

Continuum H^1 – Metric Result

Now let

$$d\nu^1_{\mathcal{P}}(\sigma) = \frac{1}{Z_{\mathcal{P}}} e^{-\frac{1}{2}\int_0^1 |\dot{\sigma}(t)|^2 dt} d\,\mathrm{vol}_{G^1|_{H_{\mathcal{P}}(M)}}(\sigma)\,.$$

Theorem 7 (Adrian Lim 2006). *(Reviews in Mathematical Physics 19 (2007), no. 9, 967–1044.) Assume* (M, g) *satisfies,*

$$0 \le \text{\textit{Sectional-Curvatures}} \le \frac{1}{2d}.$$

If $f : W(M) \to \mathbb{R}$ *is a bounded and continuous function, then*

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma)\, d\nu^1_{\mathcal{P}}(\sigma)$$
$$= \int_{W(M)} f(\sigma) e^{-\frac{1}{6}\int_0^1 \mathrm{Scal}(\sigma(s))\ ds} \sqrt{\det\left(I + \frac{1}{12}K_\sigma\right)}\, d\nu(\sigma).$$

where, for $\sigma \in H(M)$, K_σ *is a certain integral operator acting on* $L^2([0,1];\mathbb{R}^d)$.

- K_σ is defined by
$$(K_\sigma f)(s) = \int_0^1 (s \wedge t)\, \Gamma_{\sigma(t)} f(t)\, dt$$
where
$$\Gamma_m = \sum_{i,j=1}^{d} \left(\begin{array}{c} R_m\left(e_i, R_m(e_i,\cdot)e_j\right) e_j + R_m\left(e_i, R_m(e_j,\cdot)e_i\right) e_j \\ +R_m\left(e_i, R_m(e_j,\cdot)e_j\right) e_i \end{array} \right).$$
Here R_m is the curvature tensor at $m \in M$ and $\{e_i\}_{i=1,2,\ldots,d}$ is any orthonormal basis in $T_m(M)$.

- Adrian Lim's limiting measure has lost the Markov property and no nice $\hat{H}$ in this case. See "Fredholm Determinant of an Integral Operator driven by a Diffusion Process," Journal of Applied Mathematics and Stochastic Analysis, Vol. 2008, Article ID 130940.

Continuum H^0 – Metric Result

Theorem 8 (Tom Laetsch: JFA 2013)**.** *If*

$$d\nu^0_{\mathcal{P}}(\sigma) = \frac{1}{Z_{\mathcal{P}}} e^{-\frac{1}{2}\int_0^1 |\dot{\sigma}(t)|^2 dt} d\,\mathrm{vol}_{G^0|_{H_{\mathcal{P}}(M)}}(\sigma)\,,$$

then

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma)\, d\nu^0_{\mathcal{P}}(\sigma) = \int_{W(M)} f(\sigma) e^{-\frac{2+\sqrt{3}}{20\sqrt{3}}\int_0^1 \mathrm{Scal}(\sigma(s))\, ds} d\nu(\sigma).$$

- The quantization implication of this result is that we should take

$$\hat{H} = -\frac{1}{2}\Delta_M + \frac{2+\sqrt{3}}{20\sqrt{3}}\mathrm{Scal}.$$

Summary: Quantization of Free Hamiltonian

$$\hat{H} = -\frac{1}{2}\Delta_M + \frac{1}{\kappa}\mathrm{Scal}.$$

- $\kappa \in \{8, 12\} \cup \{\infty, 6, \emptyset, 10\}$.

Non Intrinsic Considerations

- Sidorova, Smolyanov, Weizsäcker, and Olaf Wittich, JFA2004, consider squeezing a ambient Brownian motion onto an embedded submanifold. This then result in

$$\hat{H} = -\frac{1}{2}\Delta_M - \frac{1}{4}S + V_{\mathsf{SF}}$$

  where V_{SF} is a potential depending on the embedding through the second fundamental form.

Applications

Corollary 9 (Trotter Product Formula for $e^{t\Delta/2}$). *For $s > 0$ let Q_s be the symmetric integral operator on $L^2(M, dx)$ defined by the kernel*

$$Q_s(x, y) = (2\pi s)^{-d/2} \exp\left(-\frac{1}{2s}d^2(x, y) + \frac{s}{12}\mathrm{S}(x) + \frac{s}{12}\mathrm{S}(y)\right)$$

for all $x, y \in M$. Then for all continuous functions $F : M \to \mathbb{R}$ and $x \in M$,

$$(e^{\frac{s}{2}\Delta}F)(x) = \lim_{n\to\infty}(Q^n_{s/n}F)(x).$$

See also Chorin, McCracken, Huges, Marsden (78) and Wu (98).

Proof. This is a special case of the L^2 – limit theorem. The main points are:

- $\nu^0_{\mathcal{P}}$ is essentially product measure on M^n.
- From this one shows that

$$(Q^n_{s/n}F)(x) \cong \int_{H_{\mathcal{P}}(M)} e^{\frac{1}{6}\int_0^1 \mathrm{S}(\sigma(s))ds} F\left(\sigma\left(s\right)\right) d\nu^0_{\mathcal{P}}(\sigma)$$

Corollary 2: Integration by Parts for ν on $W(M)$

See Bismut, Driver, Enchev, Elworthy, Hsu, Li, Lyons, Norris, Stroock, Taniguchi, ...

Let $k \in PC^1$, and z solve:

$$z'(s) + \frac{1}{2}\mathrm{Ric}_{\tilde{//}_s(\sigma)} z(s) = k'(s), \qquad z(0) = 0.$$

and f be a cylinder function on $\mathrm{W}(M)$. Then

$$\int_{\mathrm{W}(M)} X^z f \, d\nu = \int_{\mathrm{W}(M)} f \int_0^1 \langle k', d\tilde{b}\rangle \, d\nu, \text{ where}$$

$$(X^z f)(\sigma) = \sum_{i=1}^{n} \langle (\nabla_i f)(\sigma), X^z_{s_i}(\sigma)\rangle$$
$$= \sum_{i=1}^{n} \langle (\nabla_i f)(\sigma), \tilde{//}_{s_i}(\sigma) z(s_i, \sigma)\rangle$$

and $(\nabla_i f)(\sigma)$ denotes the gradient F in the ith variable evaluated at $(\sigma(s_1), \sigma(s_2), \ldots, \sigma(s_n))$. **Proof.** Integrate by parts on $H_{\mathcal{P}}(M)$ and then pass to the limit as $|\mathcal{P}| \to 0$.

More Detailed Proof

Proof. Given $k \in C^1 \cap H(T_oM)$, let $X^{\mathcal{P}}_{\cdot}(\sigma) \in T_\sigma H_{\mathcal{P}}(M)$ such that

$$\frac{\nabla X^{\mathcal{P}}_s(\sigma)}{ds}|_{s=s_i+} = //_{s_i}(\sigma)k'(s_i+).$$

1. $X^{\mathcal{P}}(\sigma)$ is a certain projection of $//.(\sigma)k(\cdot)$ into $T_\sigma H_{\mathcal{P}}(M)$.

2.

$$\begin{aligned} dE(X^{\mathcal{P}}) &= 2\int_0^1 \langle \sigma'(s), \frac{\nabla X^{\mathcal{P}}_s}{ds}\rangle ds \\ &= 2\sum_{i=1}^n \langle \Delta_i b, k'(s_{i-1}+)\rangle \end{aligned}$$

3. $L_{X^{k_{\mathcal{P}}}} \mathrm{Vol}_{G^1_{\mathcal{P}}} = 0.$

4. 1 & 2 imply that

$$L_{X^{k_{\mathcal{P}}}} \nu^1_{\mathcal{P}} = -\sum_{i=1}^n \langle \Delta_i b, k'(s_{i-1}+)\rangle \nu^1_{\mathcal{P}}.$$

Equivalently:

$$\int_{\mathrm{H}_{\mathcal{P}}(M)} \left(X^{k_{\mathcal{P}}} f\right) \nu_{\mathcal{P}}^1 = \int_{\mathrm{H}_{\mathcal{P}}(M)} \sum_{i=1}^{n} \langle k'(s_{i-1}+), \Delta_i b\rangle \, f \, \nu_{\mathcal{P}}^1.$$

5. After some work one shows

$$\lim_{|\mathcal{P}|\to 0} \int_{\mathrm{H}_{\mathcal{P}}(M)} \left(X^{k_{\mathcal{P}}} f\right) \nu_{\mathcal{P}}^1 = \int_{W(M)} X^z f \, d\nu$$

and

6.

$$\lim_{|\mathcal{P}|\to 0} \int_{\mathrm{H}_{\mathcal{P}}(M)} \sum_{i=1}^{n} \langle k'(s_{i-1}+), \Delta_i b\rangle f d\nu_{\mathcal{P}}^1 = \int_{\mathrm{W}(M)} X^z f d\nu$$

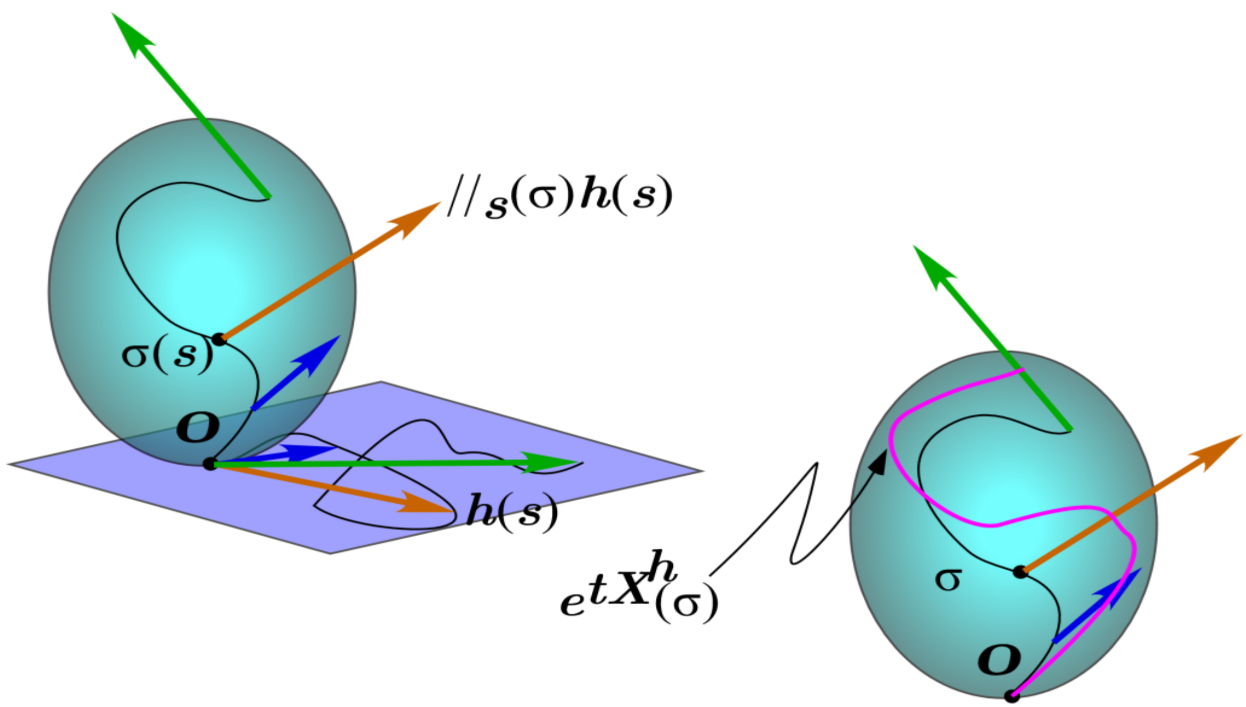
7. The previous three equations and the limit theorem imply the IBP result.

Quasi-Invariance Theorem for $\nu_{W(M)}$

Theorem 10 (D. 92, Hsu 95)**.** *Let $h \in H(T_oM)$ and X^h be the $\nu_{W(M)}$ – a.e. well defined vector field on $W(M)$ given by*

$$X_s^h(\sigma) = //_s(\sigma)h(s) \text{ for } s \in [0,1]. \tag{10}$$

*Then X^h admits a flow e^{tX^h} on $W(M)$ and this flow leaves $\nu_{W(M)}$ quasi-invariant. (**Ref:** D. 92, Hsu 95, Enchev-Strook 95, Lyons 96, Norris 95, ...)*



A word from our sponsor: Quantized Yang-Mills Fields

- A $1,000,000 question, `http://www.claymath.org/millennium-problems`
- "... Quantum Yang-Mills theory is now the foundation of most of elementary particle theory, and its predictions have been tested at many experimental laboratories, but its mathematical foundation is still unclear. ..."
- Roughly speaking one needs to make sense out of the path integral expressions above when $[0,T]$ is replaced by $\mathbb{R}^4 = \mathbb{R}\times\mathbb{R}^3$:

$$d\mu(A)\text{``}=\text{''}\frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^3}\left|F^A\right|^2 dt\,dx\right)\mathcal{D}A, \tag{11}$$

More Motivation: Physics proof of the Atiyah–Singer Index Theorem

Physics proof of the Atiyah–Singer Index Theorem (Alvarez-Gaumé, Friedan & Windey, Witten)

$$\text{index}(D)\text{“} = \text{”} \lim_{T\to 0} \int_{L(M)} e^{-\int_0^T \left[|\sigma'(s)|^2 - \psi(s)\cdot\frac{\nabla\psi(s)}{ds}\right]ds} \mathcal{D}\sigma\mathcal{D}\psi$$

$$\vdots$$

(Laplace Asymptotics)

$$\vdots$$

$$= C^{2n} \int_M \hat{A}(R).$$

- Toy Model for Constructive Field Theory,
- Intuitive understanding of smoothness properties of ν.
- Heuristic path integral methods have lead to many interesting conjectures and theorems.

End