Talk 3: “On the Classical Limit of Quantum Mechanics”

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Nelder Talk 3.

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Prologue

- This joint work with my Ph.D. student, Pun Wai Tong.
- The new results presented here are based on two papers in preparation, [Driver & Tong, 2014a, Driver & Tong, 2014b].
- Our original motivation came from trying to understand a paper by Rodnianski and Schlein [Rodnianski & Schlein, 2009] and many others.
- This lead us back to the pioneering paper of Hepp [Hepp, 1974].
- Although versions of the results to be presented are true in any dimension including infinite dimensions, we will be restricting our attention to $d = 1$.
- There are way too many papers relating to semi-classical analysis to list. To get a foothold into this literature the reader might start by looking at [Hagedorn, 1985, Zworski, 2012] and the references therein.

Hamiltonian Mechanics on $\mathbb{R}^2$

- Configuration $=$ (position space) $= \mathbb{R}$,
- State space $=$ (position,momentum)-space $= \mathbb{R}^2 \cong \mathbb{C}\ (T^*\mathbb{R})$.
- State = a point (a, b) in state space.
- Coordinates on states space are (q, p), i.e. $q(a, b) = a$ and $p(a, b) = b$.
- Observables are (real) functions, f, on state space.
- Observation is evaluation of an observable, f, on a state, (a, b) .
- A key observable should be the energy of the theory, $H : \mathbb{R}^2 \to \mathbb{R}$.
- The evolutions of states is by Hamilton's equations,

$$\dot{q} = \frac{\partial H}{\partial p}(q, p) \text{ and } \dot{p} = -\frac{\partial H}{\partial q}(q, p) .$$

More precisely, let

$$X^H(q, p) = \frac{\partial H}{\partial p}(q, p)\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}(q, p)\frac{\partial}{\partial p},$$

then $\alpha(t) \in \mathbb{R}^2$ satisfies Hamilton's equations of motion if

$$\dot{\alpha}(t) = X^H(\alpha(t)) \text{ with } \alpha(0) = \alpha_0 \in \mathbb{R}^2 \text{ given.} \tag{1}$$

Complex Notation

Let

$$z := \frac{1}{\sqrt{2}}(q + ip) \text{ and } \bar{z} = \frac{1}{\sqrt{2}}(q - ip) \text{ so that}$$

$$\partial := \frac{\partial}{\partial z} := \frac{1}{\sqrt{2}}(\partial_q - i\partial_p) \text{ and } \bar{\partial} := \frac{\partial}{\partial \bar{z}} := \frac{1}{\sqrt{2}}(\partial_q + i\partial_p). \tag{2}$$

Theorem 1. $\alpha(t) \in \mathbb{C} \cong \mathbb{R}^2$ *satisfies Hamilton's equations of motion iff*

$$i\dot{\alpha}(t) = \left(\frac{\partial}{\partial \bar{z}} H\right)(\alpha(t)) \text{ with } \alpha(0) = \alpha_0 \in \mathbb{C}. \tag{3}$$

Proof: If

$$\alpha(t) = \frac{1}{\sqrt{2}}(p(t) + iq(t)) \in \mathbb{C},$$

Eq. (3) states,

$$i\frac{1}{\sqrt{2}}(\dot{q}(t) + i\dot{p}(t)) = \frac{1}{\sqrt{2}}(\partial_q H + i\partial_p H)(\alpha(t))$$

which is equivalent to

$$\dot{q} = \partial_p H \text{ and } \dot{p} = -\partial_q H.$$

Q.E.D.

Examples H and their flows

1. **Translation generator.** If $w \in \mathbb{C}$, and
$$H_w(z,\bar{z}) = 2\,\mathrm{Im}(\bar{w}z) = i(w\bar{z} - \bar{w}z),$$
then
$$i\dot{\alpha}(t) = iw \text{ with } \alpha(0) = \alpha_0 \implies \alpha(t) = \alpha_0 + tw.$$
So H_w generates translation along w in phase space, $\mathbb{C}$.

2. **Newton's equations of motion.** If
$$H = \frac{p^2}{2} + V(q) = -\frac{1}{4}(z-\bar{z})^2 + V\left(\frac{z+\bar{z}}{\sqrt{2}}\right)$$
then Hamilton's equations are
$$\dot{q} = \partial_p H = p \text{ and } \dot{p} = -\partial_q H = -V'(q).$$

3. **Circular Motion.** If $V(q) = \frac{1}{2}q^2$, then
$$H = -\frac{1}{4}(z-\bar{z})^2 + \frac{1}{4}(z+\bar{z})^2 = z\bar{z}$$
and
$$i\dot{\alpha}(t) = \alpha(t) \implies \alpha(t) = e^{-it}\alpha_0.$$

Complex form of Poisson Brackets (Skip)

Proposition 2. *The Poisson bracket may be computed using*

$$\{f, g\} = i\left[\bar{\partial} f \cdot \partial g - \partial f \cdot \bar{\partial} g\right]. \tag{4}$$

Proof: We first solve Eq. (2) for ∂_q and ∂_p using

$$\partial + \bar{\partial} = \sqrt{2}\partial_q \text{ and } \partial - \bar{\partial} = -i\sqrt{2}\partial_p.$$

This then gives

$$\partial_q = \frac{1}{\sqrt{2}}\left(\partial + \bar{\partial}\right) \text{ and } \partial_p = \frac{i}{\sqrt{2}}\left(\partial - \bar{\partial}\right),$$

and hence

$$\begin{aligned}
\{f, g\} &= \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - (f \longleftrightarrow g) \\
&= \frac{i}{2}\left(\partial + \bar{\partial}\right) f \cdot \left(\partial - \bar{\partial}\right) g - (f \longleftrightarrow g) \\
&= \frac{i}{2}\left(-\partial f \cdot \bar{\partial} g + \bar{\partial} f \cdot \partial g\right) - (f \longleftrightarrow g) \\
&= i\left[\bar{\partial} f \cdot \partial g - \partial f \cdot \bar{\partial} g\right].
\end{aligned}$$

Q.E.D.

Spectral Lines of Hydrogen

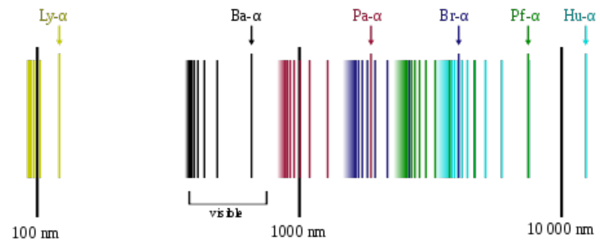
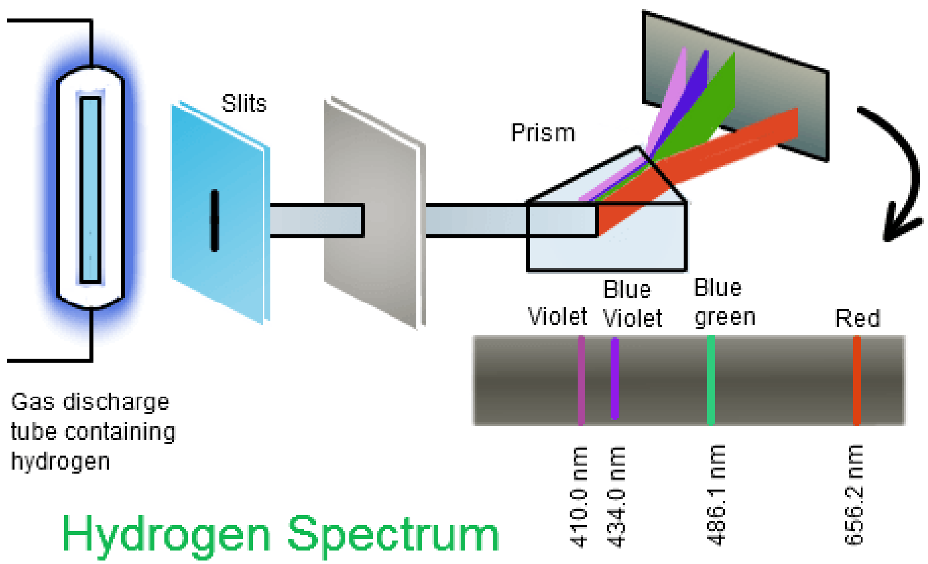


Figure 1: $\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$



Heisenberg Enters the Picture

Yeomen Warder (alias Beefeater at Tower of London):

"Of course Newton invented gravity in England."

That may be but nevertheless, the previous slide presents a problem for Newton. The next two slides are excerpts from the Wikipedia site;

http://en.wikipedia.org/wiki/Matrix_mechanics

In 1925 Werner Heisenberg was working in Göttingen on the problem of calculating the spectral lines of hydrogen. By May 1925 he began trying to describe atomic systems by observables only. On June 7, to escape the effects of a bad attack of hay fever, Heisenberg left for the pollen free North Sea island of Helgoland. While there, in between climbing and learning by heart poems from Goethe's West-östlicher Diwan,[1] he continued to ponder the spectral issue and eventually realized that adopting **non-commuting observables** might solve the problem, and he later wrote:[2]

Heisenberg: "It was about three o' clock at night when the final result of the calculation lay before me. At first I was deeply shaken. I was so excited that I could not think of sleep. So I left the house and awaited the sunrise on the top of a rock."

[1] West-östlicher Diwan ("West-Eastern Diwan", original title: West-östlicher Divan) is a diwan, or collection of lyrical poems, by the German poet Johann Wolfgang von Goethe. It was inspired by the Persian poet Hafez.

[2] W. Heisenberg, "Der Teil und das Ganze", Piper, Munich, (1969)The Birth of Quantum Mechanics.

Heisenberg and Born

After Heisenberg returned to Göttingen, he showed Wolfgang Pauli his calculations, commenting at one point:

"Everything is still vague and unclear to me, but it seems as if the electrons will no more move on orbits."

On July 9 Heisenberg gave the same paper of his calculations to Max Born, saying,

"...he had written a crazy paper and did not dare to send it in for publication, and that Born should read it and advise him on it..." prior to publication.

When Born read the paper, he recognized the formulation as one which could be transcribed and extended to the systematic language of matrices, which he had learned from his study under Jakob Rosanes at Breslau University.

Wiki: “Up until this time, matrices were seldom used by physicists; they were considered to belong to the realm of pure mathematics.”

Abstract Quantum Mechanics: Kinematics

- State space is a Hilbert space $\mathcal{K}$
- State is a unit vector, $\psi \in \mathcal{K}$
- Observables are self-adjoint operators, A, on $\mathcal{K}$

Definition 3. Given an operator A on $\mathcal{K}$ and a unit vector $\psi \in \mathcal{D}(A)$ let

$$\langle A \rangle_\psi := \langle A\psi, \psi \rangle$$

denote the **expectation** of A relative to the state ψ. The **variance** of A relative to the state $\psi \in \mathcal{D}\left(A^2\right)$ is then defined as

$$\operatorname{Var}_\psi(A) := \left\langle A^2 \right\rangle_\psi - \langle A \rangle_\psi^2 .$$

Remark 4. If $A = A^*$, then the spectral theorem guarantees there exists a unique probability measure μ on $\mathbb{R}$ such that

$$\langle f(A)\psi, \psi \rangle = \int_{\mathbb{R}} f(x)\, d\mu(x) .$$

This measure, μ, is called the **law of A relative to ψ** and is denoted by $\operatorname{Law}_\psi(A)$.

Abstract Quantum Mechanics: Dynamics

- The energy operator is, $\hat{H}$, self-adjoint operator bounded from below.
- The evolution of a state is governed by the **Schrödinger** equation,

$$i\hbar\frac{\partial}{\partial t}\psi(t) = \hat{H}\psi(t) \text{ with } \psi(0) = \psi_0 \in \mathcal{K}. \tag{5}$$

  As usual we denote the unique solution by $\psi(t) = e^{-i\frac{t}{\hbar}\hat{H}}\psi_0$.

- **Heisenberg picture:** if A is an observable (i.e. operator on $\mathcal{K}$), then

$$A(t) := e^{i\frac{t}{\hbar}\hat{H}} A e^{-i\frac{t}{\hbar}\hat{H}}. \tag{6}$$

- Note that

$$\langle A\rangle_{\psi(t)} = \langle A(t)\rangle_{\psi_0}.$$

Theorem 5. *Formally we have,*

$$\dot{A}(t) = i\frac{1}{\hbar}\left[\hat{H}, A(t)\right] = i\frac{1}{\hbar}\left[\hat{H}, A\right](t)$$

and therefore,

$$\frac{d}{dt}\langle A\rangle_{\psi(t)} = \frac{d}{dt}\langle A(t)\rangle_{\psi_0} = i\frac{1}{\hbar}\left\langle\left[\hat{H}, A\right](t)\right\rangle_{\psi_0} = \left\langle i\frac{1}{\hbar}\left[\hat{H}, A\right]\right\rangle_{\psi_0}.$$

Quantum observables for a $1\ D$ – particle

Definition 6. When we say we are considering the quantum mechanics of a particle in $\mathbb{R}^1$ we mean, given $\hbar > 0$, our Hilbert space $\mathcal{K}$ is equipped with a pair of self-adoint operators $\hat{q}_\hbar$ and $\hat{p}_\hbar$ such that

1. $\{\hat{q}_\hbar, \hat{p}_\hbar\}$ act irreducibly on $\mathcal{K}$, and
2. they satisfy the **canonical commutation** relations: $[\hat{q}_\hbar, \hat{p}_\hbar] = i\hbar I$.[3]

Remark 7. Morally speaking, the irreducibility assumption guarantees that all observables on $\mathcal{K}$ should be "functions" of $(\hat{q}_\hbar, \hat{p}_\hbar)$. Compare with Burnside's theorem.

Theorem 8 ([Burnside, 1905]). *If* $\{A_1, \ldots, A_k\} \subset \mathrm{End}\left(\mathbb{C}^d\right)$ *act irreducibly on* $\mathbb{C}^d$, *then every* $A \in \mathrm{End}\left(\mathbb{C}^d\right)$ *can be written as a non-commutative polynomial function of* $(A_1, \ldots, A_k)$.

Theorem 9 (Stone–von Neumann theorem (1931)). *Up to unitary equivalence, there is only one pair of self-adjoint operators satisfying Definition 6.*[4]

[3] The following formula needs more care since the operators involved are all unbounded!
[4] See the Wikipedia site on the "Stone–von Neumann theorem," for references.

Examples of $\left(\hat{q}_\hbar, \hat{p}_\hbar\right)$

We now take $\mathcal{K} = L^2(\mathbb{R}, m)$ where m is Lebesgue measure,

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x)\, \bar{g}(x)\, dm(x) \ \forall\, f, g \in L^2(m).$$

Example 1. Here are some choices for $(\hat{q}_\hbar, \hat{p}_\hbar)$

1. **Canonical Quantization I:**

$$\hat{q}_\hbar = M_x \text{ and } \hat{p}_\hbar = \frac{\hbar}{i}\frac{\partial}{\partial x}.$$

2. **Canonical Quantization II:**

$$\hat{q}_\hbar = \frac{\hbar}{i}\frac{\partial}{\partial x} \text{ and } \hat{p}_\hbar = -M_x.$$

   The unitary operator connecting this two is the Fourier transform.

3. **Hepp (egalitarian) quantization**,

$$\hat{q}_\hbar = \sqrt{\hbar} M_x \text{ and } \hat{p}_\hbar = \frac{\sqrt{\hbar}}{i}\frac{\partial}{\partial x}.$$

- All operators are taken to be their closures on $\mathcal{S} := \mathcal{S}(\mathbb{R}) \subset L^2(m)$ – the Schwartz test function space.

Lemma 10. *For $\hbar > 0$, let $S_\hbar : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the unitary map defined by*

$$(S_\hbar f)(x) := \hbar^{1/4} f\left(\sqrt{\hbar} x\right) \text{ for } x \in \mathbb{R}.$$

Then

$$M_x = S_\hbar^* \left(\sqrt{\hbar} M_x\right) S_\hbar \text{ and } \frac{\hbar}{i}\frac{\partial}{\partial x} = S_\hbar^* \left(\frac{\sqrt{\hbar}}{i}\frac{\partial}{\partial x}\right) S_\hbar.$$

Creation and annihilation operators

We now change to a complex basis.

Definition 11 (Creation and annihilation operators)**.** For $\hbar > 0$, let $a_\hbar$ and $a_\hbar^*$ be the creation and annihilation operators acting on $L^2(m)$ defined by

$$a_\hbar = \frac{1}{\sqrt{2}}\left(\hat{q}_\hbar + i\hat{p}_\hbar\right) = \sqrt{\frac{\hbar}{2}}\left(M_x + \partial_x\right)$$
$$a_\hbar^* = \frac{1}{\sqrt{2}}\left(\hat{q}_\hbar - i\hat{p}_\hbar\right) = \sqrt{\frac{\hbar}{2}}\left(M_x - \partial_x\right).$$

Notice: $a_\hbar = \sqrt{\hbar}a$ and $a_\hbar^* = \sqrt{\hbar}a^*$ where $a := a_1$.

Lemma 12. *The operators $a_\hbar$ and $a_\hbar^*$ satisfy the **canonical commutation relations***

$$[a_\hbar, a_\hbar^*] = \hbar I.$$

Notation 1. The class of **observables** we consider are of the form

$$\left\{P\left(a_\hbar, a_\hbar^*\right) : P \in \mathbb{C}\left\langle\theta, \theta^*\right\rangle\right\},$$

where or $\mathbb{C}\langle\theta, \theta^*\rangle$ is the space of polynomials in two non-commuting indeterminates, $\{\theta, \theta^*\}$.

Example Hamiltonian I

Example 2. Suppose that $w = \frac{1}{\sqrt{2}}(\xi + i\pi) \in \mathbb{C}$, and set

$$H(\theta, \theta^*) = i(w\theta^* - \bar{w}\theta).$$

Then

$$H(a_\hbar, a_\hbar^*) = i(wa_\hbar^* - \bar{w}a_\hbar) = \sqrt{\hbar}(i\pi M_x - \xi\partial_x).$$

We then set

$$U_\hbar(w) := e^{-\frac{1}{i\hbar}H_w\left(a_\hbar, a_\hbar^*\right)} = \exp\left(\frac{1}{\hbar}(w \cdot a_\hbar^* - \bar{w} \cdot a_\hbar)\right). \tag{7}$$

Proposition 13. *For* $\hbar > 0$ *and* $w = (\xi + i\pi)/\sqrt{2} \in \mathbb{C}$, *then*

$$(U_\hbar(w) f)(x) = \exp\left(i\frac{\pi}{\sqrt{\hbar}}\left(x - \frac{1}{2\sqrt{\hbar}}\xi\right)\right) f\left(x - \frac{1}{\sqrt{\hbar}}\xi\right) \tag{8}$$

and more importantly

$$U_\hbar(w)^* a_\hbar U_\hbar(w) = a_\hbar + w. \tag{9}$$

Proof: Use the method of characteristics to find Eq. (8) then prove Eq. (9) by direct computation. Another way to prove Eq. (9) is to integrate the identity,

$$\frac{d}{dt}U_\hbar(tw)^* a_\hbar U_\hbar(tw) = -U_\hbar(tw)^*\left[\frac{1}{\hbar}(w \cdot a_\hbar^* - w \cdot a_\hbar), a_\hbar\right] U_\hbar(tw) = w,$$

with respect to t. Q.E.D.

Example Hamiltonian II

Example 3. Suppose that $H(\theta, \theta^*) = \theta^* \theta$, then

$$\mathcal{N}_\hbar := H(a_\hbar, a_\hbar^*) := a_\hbar^* a_\hbar = \frac{\hbar}{2}(M_x - \partial_x)(M_x + \partial_x) = \frac{\hbar}{2}\left(-\partial_x^2 + M_x^2 - 1\right) \tag{10}$$

is the **Harmonic Oscillator Hamiltonian (**or **Number operator** when $\hbar = 1)$ number operator.

Fact. $\mathcal{N}_\hbar$ has a complete orthonormal basis of the form

$$\left\{H_n(x)\exp\left(-\frac{1}{2}x^2\right)\right\}_{n=0}^{\infty}$$

where $\{H_n\}_{n=0}^{\infty}$ are properly normalized Hermite polynomials.

Fact. $e^{-\frac{t}{\hbar}\mathcal{N}_\hbar}$ is given by integration against at Gaussian kernel function (Mehler Kernel).

Harmonic Oscillator Evolution

Proposition 14. *If* $a_\hbar(t) := e^{\frac{i}{\hbar}t\mathcal{N}_\hbar} a_\hbar e^{-\frac{i}{\hbar}t\mathcal{N}_\hbar}$, *then*

$$\dot{a}_\hbar(t) = -i a_\hbar(t) \implies a_\hbar(t) = e^{-it} a_\hbar$$

just as in the Classical Mechanics case.

Proof: We need only make use of the evolution equation for the Heisenberg picture and the commutation relations;

$$\begin{aligned}\dot{a}_\hbar(t) &= \frac{i}{\hbar} e^{\frac{i}{\hbar}t\mathcal{N}_\hbar} \left[\mathcal{N}_\hbar, a_\hbar\right] e^{-\frac{i}{\hbar}t\mathcal{N}_\hbar} = \frac{i}{\hbar} e^{\frac{i}{\hbar}t\mathcal{N}_\hbar} \left[a_\hbar^* a_\hbar, a_\hbar\right] e^{-\frac{i}{\hbar}t\mathcal{N}_\hbar} \\ &= -\frac{i}{\hbar} e^{\frac{i}{\hbar}t\mathcal{N}_\hbar}_\hbar a_\hbar e^{-\frac{i}{\hbar}t\mathcal{N}_\hbar} = -i a_\hbar(t).\end{aligned}$$

Q.E.D.

Operator Evolution Facts

Definition 15. The symbol (or **classical** residue) of $P \in \mathbb{C}\langle\theta, \theta^*\rangle$ is the function $P^{\mathrm{cl}} : \mathbb{C} \to \mathbb{C}$ defined by $P^{\mathrm{cl}}(\alpha) := P(\alpha, \bar{\alpha})$ where we view $\mathbb{C}$ as a commutative algebra with involution given by complex conjugation.

- **Fact:** If $H(\theta, \theta^*)$ has degree 2 or less, then

$$a_\hbar(t) := e^{\frac{i}{\hbar}tH\left(a_\hbar, a_\hbar^*\right)} a_\hbar e^{-\frac{i}{\hbar}tH\left(a_\hbar, a_\hbar^*\right)}$$

  $\left(a_\hbar(t), a_\hbar(t)^*\right)$ is related to $\left(a_\hbar, a_\hbar^*\right)$ by the same affine transformation that is determined by the flow of $X^{H^{cl}}$ where $H^{cl}(z) := H(z, \bar{z})$.

- **Fact:** in general $\hat{q}_\hbar(t)$ satisfies Newton's equations of motion when $H = \frac{p^2}{2} + V(q)$.

Algebra of $\mathbb{C}\left\langle \theta, \theta^* \right\rangle$

Notation 2 (Taylor Expansion)**.** If $\alpha \in \mathbb{C}$ and $H(\theta, \theta^*) \in \mathbb{C}\langle \theta, \theta^* \rangle$ with $d = \deg H$, then

$$H(\theta + \alpha, \theta^* + \bar{\alpha}) = \sum_{k=0}^{d} H_k(\alpha : \theta, \theta^*) \tag{11}$$

where

$$H_k(\alpha : \theta, \theta^*) = \frac{1}{k!}\left(\frac{d}{dt}\right)^k |_{t=0} H(t\theta + \alpha, t\theta^* + \bar{\alpha}). \tag{12}$$

is homogeneous of degree k in $\{\theta, \theta^*\}$.

Theorem 16. *If $H(\theta, \theta^*)$ is symmetric, $\{H_k\}$ are as in Eq. (11), then*

$$\begin{aligned} H_0(\alpha : \theta, \theta^*) &= H^{\text{cl}}(\alpha) \\ H_1(\alpha : \theta, \theta^*) &= \left[\left(\partial H^{\text{cl}}\right)(\alpha)\,\theta + \left(\bar{\partial} H^{\text{cl}}\right)(\alpha)\,\theta^*\right] \text{ and} \\ H_2(\alpha : \theta, \theta^*) &= \frac{1}{2}\partial^2 H^{\text{cl}}(\alpha)\,\theta^2 + \frac{1}{2}\bar{\partial}^2 H^{\text{cl}}(\alpha)\,\theta^{*2} \\ &\quad + u(\alpha)\,\theta^*\theta + v(\alpha)\,\theta\theta^* \end{aligned}$$

where $u, v \in \mathbb{R}[\alpha]$ are polynomials (depending on H) such that

$$u(\alpha) + v(\alpha) = \partial\bar{\partial} H^{\text{cl}}(\alpha).$$

Example 4. Suppose that $H(\theta, \theta^*) = \theta^2\theta^*\theta + \theta^*\theta\theta^{*2}$ so that

$$H^{cl}(\alpha) = \bar{\alpha}\alpha^3 + \bar{\alpha}^3\alpha,$$

$$\begin{aligned} H(\theta + \alpha, \theta^* + \bar{\alpha}) &= (\theta + \alpha)^2 (\theta^* + \bar{\alpha})(\theta + \alpha) \\ &\quad + (\theta^* + \bar{\alpha})(\theta + \alpha)(\theta^* + \bar{\alpha})^2 \end{aligned}$$

and

$$H_2(\alpha; \theta, \theta^*) = \left[\alpha^2 + \bar{\alpha}^2\right] \theta^*\theta + 2\left[\alpha^2 + \bar{\alpha}^2\right] \theta\theta^*.$$

Notice that

$$\partial\bar{\partial}H^{cl}(\alpha) = 3\left(\alpha^2 + \bar{\alpha}^2\right) = \left[\alpha^2 + \bar{\alpha}^2\right] + 2\left[\alpha^2 + \bar{\alpha}^2\right].$$

Corollary 17. *For all* $H \in \mathbb{C}\langle\theta, \theta^*\rangle$ *symmetric and* $\hbar > 0$,

$$\begin{aligned} H_2(\alpha : a_\hbar, a_\hbar^*) &= \frac{1}{2}\partial^2 H^{cl}(\alpha) \cdot a_\hbar^2 + \frac{1}{2}\bar{\partial}^2 H^{cl}(\alpha) \cdot a_\hbar^{*2} \\ &\quad + \partial\bar{\partial}H^{cl}(\alpha) \cdot a_\hbar^* a_\hbar + \hbar \cdot v_H(\alpha) I. \end{aligned}$$

Squeezing States

Corollary 18 (Concentrated states)**.** *Let* $P(\theta, \theta^*) \in \mathbb{C}\langle \theta, \theta^* \rangle$, $\psi \in \mathcal{S}$, $\hbar > 0$, *and* $\alpha \in \mathbb{C}$, *then*

$$\langle P(a_\hbar, a_\hbar^*) \rangle_{U_\hbar(\alpha)\psi} = P(\alpha, \bar{\alpha}) + O\left(\sqrt{\hbar}\right) \tag{13}$$

$$\mathrm{Var}_{U_\hbar(\alpha)\psi}\left(P(a_\hbar, a_\hbar^*)\right) = O\left(\sqrt{\hbar}\right), \tag{14}$$

and

$$\lim_{\hbar \downarrow 0} \left\langle P\left(\frac{a_\hbar - \alpha}{\sqrt{\hbar}}, \frac{a_\hbar^* - \bar{\alpha}}{\sqrt{\hbar}}\right) \right\rangle_{U_\hbar(\alpha)\psi} = \langle P(a, a^*) \rangle_\psi . \tag{15}$$

[In fact, the equality in the last equation holds before taking the limit as $\hbar \to 0$.]

Remark 19 (Moral)**.** Consequently, $U_\hbar(\alpha)\psi$ is a state which is concentrated in phase space near the α and are therefore reasonable quantum mechanical approximations of the classical state α.

Proof of Squeezing

Proof: Since

$$U_\hbar(w)^* a_\hbar U_\hbar(w) = a_\hbar + w = \sqrt{\hbar}a + w,$$

we conclude,

$$U_\hbar(\alpha)^* P(a_\hbar, a_\hbar^*) U_\hbar(\alpha) = P(a_\hbar + \alpha, a_\hbar^* + \bar{\alpha}).$$

Therefore,

$$\begin{aligned}\langle P(a_\hbar, a_\hbar^*)\rangle_{U_\hbar(\alpha)\psi} &= \langle U_\hbar(\alpha)^* P(a_\hbar, a_\hbar^*) U_\hbar(\alpha)\rangle_\psi = \langle P(a_\hbar + \alpha, a_\hbar^* + \bar{\alpha})\rangle_\psi \\ &= \left\langle \sum_{k=0}^d P_k(\alpha; a_\hbar, a_\hbar^*) \right\rangle_\psi = P_0(\alpha) + \sum_{k=0}^d \hbar^{k/2} \langle P_k(\alpha; a, a^*)\rangle_\psi \\ &= P_0(\alpha) + O\left(\sqrt{\hbar}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\left\langle P^2\left(a_\hbar, a_\hbar^*\right)\right\rangle_{U_\hbar(\alpha)\psi} &= \left(P^2\right)_0(\alpha) + \sum_{k=1}^{2d} \hbar^{k/2}\left\langle \left(P^2\right)_k(\alpha; a, a^*)\right\rangle_\psi \\ &= \left(P_0^2\right)(\alpha) + \sum_{k=1}^{2d} \hbar^{k/2}\left\langle \left(P^2\right)_k(\alpha; a, a^*)\right\rangle_\psi \\ &= \left(P_0^2\right)(\alpha) + O\left(\sqrt{\hbar}\right).\end{aligned}$$

This then implies

$$\mathrm{Var}_{U_\hbar(\alpha)\psi}\left(P\left(a_\hbar, a_\hbar^*\right)\right) := \left\langle P^2\left(a_\hbar, a_\hbar^*\right)\right\rangle_{U_\hbar(\alpha)\psi} - \left\langle P\left(a_\hbar, a_\hbar^*\right)\right\rangle_{U_\hbar(\alpha)\psi}^2 = O\left(\sqrt{\hbar}\right).$$

Equation (15) is even simpler,

$$\begin{aligned}\left\langle P\left(\frac{a_\hbar - \alpha}{\sqrt{\hbar}}, \frac{a_\hbar^* - \bar{\alpha}}{\sqrt{\hbar}}\right)\right\rangle_{U_\hbar(\alpha)\psi} &= \left\langle P\left(\frac{a_\hbar + \alpha - \alpha}{\sqrt{\hbar}}, \frac{a_\hbar^* + \bar{\alpha} - \bar{\alpha}}{\sqrt{\hbar}}\right)\right\rangle_\psi \\ &= \left\langle P\left(a, a^*\right)\right\rangle_\psi.\end{aligned}$$

Q.E.D.

Assumptions

Assumption 1. We assume $H(\theta, \theta^*) \in \mathbb{R}\langle \theta, \theta^* \rangle$ be a non-commutative polynomial with real coefficients satisfying;

1. $H(\theta, \theta^*) \in \mathbb{R}\langle \theta, \theta^* \rangle$ is symmetric and $d = \deg H$ is even. [5]

2. There exists $\eta > 0$ such that for all $\hbar \in (0, \eta)$ and $n \in \mathbb{N}$,

   (a) $H^n(a_\hbar, a_\hbar^*) = \left[H\left(a_\hbar, a_\hbar^*\right)\right]^n$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R})$, and

   (b) $\exists\, C_n > 0$ such that

$$\langle \psi, \mathcal{N}_\hbar^n \psi \rangle \le C_n \langle \psi, (H(a_\hbar, a_\hbar^*) + I)^n \psi \rangle \;\forall\, \psi \in \mathcal{S}(\mathbb{R}). \tag{16}$$

   where $\mathcal{N}_\hbar = a_\hbar^* a_\hbar$.

[5] See [Chernoff, 1973, Kato, 1973] for a large class of example of potentials for which the standard Schrödinger operator satisfies item 1 of the assumption.

Theorem 20 (Example Hamiltonians [Driver & Tong, 2014a]). *Assumption 1 holds if;*

1. *For all* $m > 0$ *and* $V \in \mathbb{R}[x]$ *such that* $\deg V \in 2\mathbb{N}$ *such that* $\lim_{x\to\infty} V(x) = \infty$, *the non-commutative polynomial,*

$$H(\theta, \theta^*) = m(\theta - \theta^*)^2 + V\left(\frac{1}{\sqrt{2}}(\theta + \theta^*)\right), \tag{17}$$

   satisfies Assumption 1.

2. *More generally we can take*

$$H(\theta, \theta^*) = \sum_{l=0}^{m} (-2)^l (\theta - \theta^*)^l b_l\left(\frac{1}{\sqrt{2}}(\theta + \theta^*)\right)(\theta - \theta^*)^l \tag{18}$$

   where $b_l \in \mathbb{R}[x]$ *are polynomials satisfying;*

   (a) *each* b_l *is an even polynomial with positive leading order coefficient,* $b_1, b_m > 0$,

   (b) $\deg b_0 \geq 2$ *and* $\deg(b_l) \leq \deg(b_{l-1})$ *for* $1 \leq l \leq m$.

Remark 21. If H is as in Eq. (18), then

$$H(a_\hbar, {a_\hbar}^*) = \sum_{l=0}^{m} (-1)^l \hbar^l D^l M_{b_l\left(\sqrt{\hbar}(\cdot)\right)} D^l.$$

Classical Equations

Lemma 22. *Letting C_1 denote the constant appearing in Eq. (16), then $H^{cl}(\alpha) := H(\alpha, \bar{\alpha})$ satisfies*

$$|\alpha|^2 \le C_1 \left(H^{cl}(\alpha) + 1\right) \text{ for all } \alpha \in \mathbb{C}. \tag{19}$$

Proof: By Assumption 1 with $n = 1$ and ψ replaced by $U_\hbar(\alpha)\psi$, we find

$$\langle a_\hbar^* a_\hbar \rangle_{U_\hbar(\alpha)\psi} \le C_1 \left[\langle H(a_\hbar, a_\hbar^*) \rangle_{U_\hbar(\alpha)\psi} + 1 \right]$$

which is equivalent to

$$\langle (a_\hbar^* + \bar{\alpha})(a_\hbar + \alpha) \rangle_\psi \le C_1 \left[\langle H((a_\hbar + \alpha), (a_\hbar^* + \bar{\alpha})) \rangle_\psi + 1 \right].$$

The result follows by letting $\hbar \downarrow 0$. Q.E.D.

Corollary 23. *For all $\alpha_0 \in \mathbb{C}$, there exists a unique global solution, $\alpha(t) = \alpha(t; \alpha_0)$ to Hamilton's ODE,*

$$i\dot{\alpha}(t) = \frac{\partial H^{cl}}{\partial \bar{z}}(\alpha(t)) \text{ with } \alpha(0) = \alpha_0 \in \mathbb{C}. \tag{20}$$

Notation 3. For $z \in \mathbb{C}$, let $\gamma(t)$ and $\delta(t)$ be the unique $\mathbb{C}$ – valued functions such that

$$\alpha'(t, \alpha_0) z := \frac{d}{ds} \alpha(t, \alpha_0 + sz) = \gamma(t) z + \delta(t) \bar{z}.$$

[To $\gamma(t)$ and $\delta(t)$ carry the information about the linearization of Eq. (20).]

A Semi-Classical Limit Theorem

Theorem 24. *Suppose $H(\theta, \theta^*)$ is a non-commutative polynomial in two indeterminates which satisfies Assumptions 1. For $\alpha_0 \in \mathbb{C}$ and a $L^2(m)$ – normalized state $\psi \in \mathcal{S}$ let;*

1. *$\alpha(t) \in \mathbb{C}$ be the solution to Hamilton's classical equations of motion (20),*
2. *$a(t) \in Ops\left(L^2(m)\right)$ be defined by*
$$a(t) = \gamma(t) a_1 + \delta(t) a_1^*,$$
3. *for $\hbar > 0$ let*
$$\psi_\hbar(t) := e^{-i\frac{t}{\hbar}H\left(a_\hbar^*, a_\hbar\right)} U_\hbar(\alpha_0) \psi \tag{21}$$
be the Shrödinger evolution of the state $U_\hbar(\alpha_0)\psi$ (which is concentrated near α_0).

Then for all $t \in \mathbb{R}$ the following weak limits (in the sense of non-commutative probability) hold;

$$\mathrm{Law}_{\psi_\hbar(t)}[a_\hbar] \to \alpha(t) \text{ as } \hbar \downarrow 0 \tag{22}$$

and

$$\mathrm{Law}_{\psi_\hbar(t)}\left[\frac{a_\hbar - \alpha(t)}{\sqrt{\hbar}}\right] \to \mathrm{Law}_\psi[a(t)] \text{ as } \hbar \downarrow 0. \tag{23}$$

Intuitive Meaning

The meaning of the limits in Eqs. (22) and (23) are as follows; for all $P(\theta,\theta^*)\in\mathbb{C}\langle\theta,\theta^*\rangle$,

$$\lim_{\hbar\downarrow 0}\langle P(a_\hbar,a_\hbar^*)\rangle_{\psi_\hbar(t)}=P(\alpha(t),\bar{\alpha}(t)) \tag{24}$$

and

$$\lim_{\hbar\downarrow 0}\left\langle P\left(\frac{a_\hbar-\alpha(t)}{\sqrt{\hbar}},\frac{a_\hbar^*-\bar{\alpha}(t)}{\sqrt{\hbar}}\right)\right\rangle_{\psi_\hbar(t)}=\langle P(a(t),a^*(t))\rangle_\psi . \tag{25}$$

respectively.

Remark 25. More intuitively,

$$\mathrm{Law}_{\psi_\hbar(t)}[a_\hbar]\cong\mathrm{Law}_\psi\left[\alpha(t)+\sqrt{\hbar}a(t)\right]\text{ for } 0<\hbar\ll 1, \tag{26}$$

i.e. for all $P\in\mathbb{C}\langle\theta,\theta^*\rangle$

$$\langle P(a_\hbar,a_\hbar^*)\rangle_{\psi_\hbar(t)}=\left\langle P\left(\alpha(t)+\sqrt{\hbar}a_1(t),\bar{\alpha}(t)+\sqrt{\hbar}a^*(t)\right)\right\rangle_\psi+o\left(\sqrt{\hbar}\right). \tag{27}$$

Motivations for the Proof

- Let $H_\hbar := H\left(a_\hbar^*, a_\hbar\right)$.
- $U_\hbar(\alpha_0)\,\psi$ is a state in $L^2(m)$ "concentrated" near $\alpha_0 \in \mathbb{C}$.
- The hope is that $\psi_\hbar(t) := e^{-iH_\hbar t/\hbar} U_\hbar(\alpha_0)\,\psi$ is a state concentrated near $\alpha(t) \in \mathbb{C}$.
- For any unitary operator $W_0(t)$, $U_\hbar(\alpha(t))\,W_0(t)\,\psi$ is a state concentrated near $\alpha(t) \in \mathbb{C}$.
- Consequently we might hope that if we choose $W_0(t)$ properly, then

$$e^{-iH_\hbar t/\hbar} U_\hbar(\alpha_0)\,\psi \sim U_\hbar(\alpha(t))\,W_0(t)\,\psi.$$

- This motivates us to consider

$$V_\hbar(t) := U_\hbar(-\alpha(t))\,e^{-iH_\hbar t/\hbar} U_\hbar(\alpha_0) = U_\hbar(\alpha(t))^*\,e^{-iH_\hbar t/\hbar} U_\hbar(\alpha_0). \tag{28}$$

Computing some Derivatives

Lemma 26. *If* $\alpha \in C^1(\mathbb{R} \to \mathbb{C})$ *and*

$$V_\hbar(t) = U_\hbar(\alpha(t))^* e^{-iH_\hbar t/\hbar} U_\hbar(\alpha_0),$$

then

$$\partial_t V_\hbar(t)\psi = \Gamma_\hbar(t) V_\hbar(t)\psi$$

where

$$\Gamma_\hbar(t) := \frac{1}{\hbar}\left(\overline{\dot{\alpha}(t)} a_\hbar - \dot{\alpha}(t) a_\hbar^* + i\,\mathrm{Im}\left(\alpha(t)\overline{\dot{\alpha}(t)}\right) - iH\left(a_\hbar + \alpha(t), a_\hbar^* + \bar{\alpha}(t)\right)\right). \tag{29}$$

Proof: Rather direct computation shows,

$$\begin{aligned}
\partial_t V_\hbar(t) =& \partial_t U_\hbar(-\alpha(t)) \cdot e^{-iH_\hbar t/\hbar} U_\hbar(\alpha_0) - \frac{i}{\hbar} U_\hbar(-\alpha(t)) H_\hbar e^{-iH_\hbar t/\hbar} U_\hbar(\alpha_0) \\
=& \left(-\frac{\dot{\alpha}(t)}{\sqrt{\hbar}} a^* + \frac{\overline{\dot{\alpha}(t)}}{\sqrt{\hbar}} a + \frac{i}{\hbar}\,\mathrm{Im}\left(\alpha(t)\overline{\dot{\alpha}(t)}\right)\right) V_\hbar(t) \\
& - \frac{i}{\hbar} U_\hbar(-\alpha(t)) H_\hbar U_\hbar(\alpha(t)) V_\hbar(t) \\
=& \Gamma_\hbar(t) V_\hbar(t).
\end{aligned}$$

Q.E.D.

Removing the $\hbar^{-1/2}$ – terms

Remark 27. Recall that

$$
\begin{aligned}
&H\left(a_\hbar + \alpha(t), a_\hbar^* + \bar{\alpha}(t)\right) \\
&\quad = H^{\mathrm{cl}}(\alpha(t)) + \left(\partial H^{\mathrm{cl}}\right)(\alpha(t))\, a_\hbar + \left(\bar{\partial} H^{\mathrm{cl}}\right)(\alpha(t))\, a_\hbar^* \\
&\quad + H_2\left(\alpha(t) : a_\hbar, a_\hbar^*\right) + H_{\geq 3}\left(\alpha(t) : a_\hbar, a_\hbar^*\right). \qquad (30)
\end{aligned}
$$

so that

$$
\begin{aligned}
\hbar \Gamma_\hbar(t) &:= i\left[\operatorname{Im}\left(\alpha(t)\,\overline{\dot{\alpha}(t)}\right) - H^{\mathrm{cl}}(\alpha(t))\right] \\
&+ \left[\overline{\dot{\alpha}(t)} - i\left(\partial H^{\mathrm{cl}}\right)(\alpha(t))\right] a_\hbar + \left[\dot{\alpha}(t) + i\left(\bar{\partial} H^{\mathrm{cl}}\right)(\alpha(t))\right] a_\hbar^* \\
&- iH_2\left(\alpha(t) : a_\hbar, a_\hbar^*\right) - iH_{\geq 3}\left(\alpha(t) : a_\hbar, a_\hbar^*\right).
\end{aligned}
$$

Key point: if $\alpha(t)$ solves Hamilton's equations of motion,

$$
i\dot{\alpha}(t) = \left(\bar{\partial} H^{\mathrm{cl}}\right)(\alpha(t)),
$$

then

$$
\begin{aligned}
\Gamma_\hbar(t) :=& \frac{i}{\hbar}\left[\operatorname{Im}\left(\alpha(t)\,\overline{\dot{\alpha}(t)}\right) - H^{\mathrm{cl}}(\alpha(t))\right] \\
&- iH_2\left(\alpha(t) : a, a^*\right) - \frac{i}{\hbar} H_{\geq 3}\left(\alpha(t) : a_\hbar, a_\hbar^*\right).
\end{aligned}
$$

Remark 28. The highly oscillatory phase factor in

$$\frac{i}{\hbar}\left[\operatorname{Im}\left(\alpha\left(t\right)\overline{\dot{\alpha}\left(t\right)}\right)-H^{\mathsf{cl}}\left(\alpha\left(t\right)\right)\right]$$

is inessential and is easily removed.

Hepp's Method

Corollary 29. *If*

$$W_\hbar(t) = e^{\frac{i}{\hbar}f(t)}V_\hbar(t) = e^{\frac{i}{\hbar}f(t)}U_\hbar(-\alpha(t))\, e^{-iH_\hbar t/\hbar}U_\hbar(\alpha_0)$$

where

$$f(t) := \int_0^t \left(H^{cl}(\alpha(\tau)) - \operatorname{Im}\left(\alpha(\tau)\overline{\dot{\alpha}(\tau)}\right)\right) d\tau, \tag{31}$$

then

$$i\partial_t W_\hbar(t) = \left[H_2(\alpha(t) : a, a^*) + \frac{1}{\hbar}H_{\geq 3}(\alpha(t) : a_\hbar, a_\hbar^*)\right] W_\hbar(t) \text{ with } W_\hbar(0) = I. \tag{32}$$

Remark 30 (The heart of [Hepp, 1974]'s method.)**.** Observe that

$$\frac{i}{\hbar}H_{\geq 3}(\alpha(t) : a_\hbar, a_\hbar^*) = i\sqrt{\hbar}\sum_{l\geq 3}\hbar^{(l-3)/2}H_l(\alpha(t), a, a^*)$$

and so

$$\frac{i}{\hbar}H_{\geq 3}(\alpha(t) : a_\hbar, a_\hbar^*)\,\psi = \text{“}O\left(\sqrt{\hbar}\right) \to 0\text{” as } \hbar \downarrow 0.$$

Formally letting $\hbar \downarrow 0$ in Eq. (32) should imply that $W_\hbar(t) \to W_0(t)$ where

$$i\frac{\partial}{\partial t}W_0(t) = H_2(\alpha(t) : a, a^*)\, W_0(t) \text{ with } W_0(0) = I. \tag{33}$$

The Key Limiting Result

Theorem 31. *For any continuous function* $\alpha : \mathbb{R} \to \mathbb{C}$ *there exists a unique one parameter strongly continuous family of unitary operators* $\{W_0(t)\}_{t\in\mathbb{R}}$ *satisfying Eq. (33).*

Theorem 32 (Strong(er) Convergence Theorem)**.** *Continuing the previous notation,*

$$W_\hbar(t) \stackrel{s}{\to} W_0(t) \text{ as } \hbar \downarrow 0.$$

Moreover we have the following stronger convergence, if $n \in \mathbb{N}_0$ *there exists* $K = K_n < \infty$ *such that*

$$\left\|\mathcal{N}^n \left(W_0(t) - W_\hbar(t)\right)\psi\right\| \le K\sqrt{\hbar}\cdot \left\|(I+\mathcal{N})^{d(2n+1)}\psi\right\| \;\forall\, \psi \in \mathcal{D}\left(\mathcal{N}^{d(2n+1)}\right). \quad (34)$$

Proof: The fact that $W_\hbar(t) \stackrel{s}{\to} W_0(t)$ as $\hbar \downarrow 0$ is quite plausible. Proving this statement along with Eq. (34) is fairly technical and the interested reader is referred to [Driver & Tong, 2014b]. Q.E.D.

Proof of the Classical Limit Theorem

Let $P \in \mathbb{C}\langle\theta,\theta^*\rangle$ and recall that

$$\psi_\hbar(t) := e^{-iH\left(a_\hbar^*,a_\hbar\right)t/\hbar}U_\hbar(\alpha_0)\,\psi \text{ and}$$
$$W_\hbar(t) = e^{\frac{i}{\hbar}f(t)}U_\hbar(-\alpha(t))\,e^{-iH_\hbar t/\hbar}U_\hbar(\alpha_0).$$

Therefore, **given Theorem 32**,

$$\begin{aligned}
\langle P(a_\hbar,a_\hbar^*)\rangle_{\psi_\hbar(t)} &= \left\langle U_\hbar(\alpha_0)^*\,e^{iH_\hbar t/\hbar}P(a_\hbar,a_\hbar^*)\,e^{-iH_\hbar t/\hbar}U_\hbar(\alpha_0)\right\rangle_\psi \\
&= \langle W_\hbar(t)^*\,U_\hbar(\alpha(t))^*\,P(a_\hbar,a_\hbar^*)\,U_\hbar(\alpha(t))\,W_\hbar(t)\rangle_\psi \\
&= \langle W_\hbar(t)^*\,P(a_\hbar+\alpha(t),a_\hbar^*+\bar\alpha(t))\,W_\hbar(t)\rangle_\psi \\
&= P(\alpha(t),\bar\alpha(t)) + \sum_{k=1}^n \hbar^{k/2}\langle W_\hbar(t)^*\,P_k(\alpha(t);a,a^*)\,W_\hbar(t)\rangle_\psi \\
&= P(\alpha(t),\bar\alpha(t)) + O\left(\sqrt{\hbar}\right). \qquad (35)
\end{aligned}$$

Remark 33. So we have now shown that the quantum expectations, $\left\langle P\left(a_\hbar,a_\hbar^*\right)\right\rangle_{\psi_\hbar(t)}$, closely track their classical counterparts, $P(\alpha(t),\bar\alpha(t))$.

Proof of the Quantum Fluctuation Eq. (23)

From Eq. (35) with

$$P(\theta, \theta^*) \rightsquigarrow P\left(\frac{1}{\sqrt{\hbar}}(\theta - \alpha(t)), \frac{1}{\sqrt{\hbar}}\left(\theta^* - \overline{\alpha(t)}\right)\right)$$

we find

$$\begin{aligned}\left\langle P\left(\frac{a_\hbar - \alpha(t)}{\sqrt{\hbar}}, \frac{a_\hbar^* - \overline{\alpha(t)}}{\sqrt{\hbar}}\right)\right\rangle_{\psi_\hbar(t)} &= \left\langle W_\hbar(t)^* P\left(\frac{a_\hbar}{\sqrt{\hbar}}, \frac{a_\hbar^*}{\sqrt{\hbar}}\right) W_\hbar(t)\right\rangle_\psi \\ &= \langle W_\hbar(t)^* P(a, a^*) W_\hbar(t)\rangle_\psi \\ &\to \langle W_0(t)^* P(a, a^*) W_0(t)\rangle_\psi \\ &= \left\langle P\left(a(t), a(t)^*\right)\right\rangle_\psi\end{aligned}$$

wherein we have used the next lemma for the last equality.

Remark 34. This result gives the next order quantum corrections to the classical theory.

Interpreting $W_0(t)^* aW_0(t)$

Lemma 35. *Keeping the notation as above,*

$$W_0(t)^* aW_0(t) = a(t) = \gamma(t) a_1 + \delta(t) a_1^*.$$

Proof: If $a(t) := W_0(t)^* aW_0(t)$, then from the definition of $W_0(t)$ (see Theorem 31),

$$i\dot{a}(t) = i\frac{d}{dt}\left(W_0(t)^* aW_0(t)\right) = -W_0(t)^* \left[H_2(\alpha(t) : a, a^*), a\right] W_0(t). \tag{36}$$

From Corollary 17 above,

$$\begin{aligned} H_2(\alpha : a_\hbar, a_\hbar^*) = {} & \frac{1}{2}\partial^2 H^{\text{cl}}(\alpha) \cdot a_\hbar^2 + \frac{1}{2}\bar{\partial}^2 H^{\text{cl}}(\alpha) \cdot a_\hbar^{*2} \\ & + \partial\bar{\partial} H^{\text{cl}}(\alpha) \cdot a_\hbar^* a_\hbar + \hbar \cdot v_H(\alpha) I \end{aligned}$$

and hence using the commutation relations,

$$-\left[H_2(\alpha(t) : a, a^*), a\right] = \left(\bar{\partial}^2 H^{\text{cl}}\right)(\alpha(t))\, a^* + \left(\partial\bar{\partial} H^{\text{cl}}\right)(\alpha(t)).$$

Using this in Eq. (36) then shows

$$i\dot{a}(t) = \left(\bar{\partial}^2 H^{\text{cl}}\right)(\alpha(t))\, a(t)^* + \left(\partial\bar{\partial} H^{\text{cl}}\right)(\alpha(t))\, a(t)$$

which is precisely the linearization of Hamilton's equations. Q.E.D.

In Summary

For $\alpha_0 \in \mathbb{C}$, let $\alpha(t) = \alpha(t, \alpha_0)$ be the solution to Hamilton's equations,

$$i\dot{\alpha}(t) = \left(\frac{\partial}{\partial \bar{z}} H^{\text{cl}}\right)(\alpha(t)) \text{ with } \alpha(0) = \alpha_0 \in \mathbb{C}.$$

Let $\gamma(\alpha_0, t)$ and $\delta(\alpha_0, t)$ be determined by

$$\alpha'(t, \alpha_0) z := \frac{d}{ds}\alpha(t, \alpha_0 + sz) = \gamma(\alpha_0, t) z + \delta(\alpha_0, t) \bar{z}.$$

Let $W_0(\alpha_0, t) = W_0(t)$ be the one parameter family of unitary operators satisfying,

$$i\frac{\partial}{\partial t} W_0(t) = H_2(\alpha(t) : a, a^*) W_0(t) \text{ with } W_0(0) = I.$$

Further let $A_\hbar(t)$ be the evolution of $a_\hbar$ in the **Heisenberg picture**, i.e.

$$A_\hbar(t) := e^{iH\left(a_\hbar^*, a_\hbar\right)t/\hbar} a_\hbar e^{-iH\left(a_\hbar^*, a_\hbar\right)t/\hbar}.$$

Theorem 36 (Summary)**.** *If $H(\theta, \theta^*)$ satisfies Assumption 1, $\alpha_0 \in \mathbb{C}$, and $\hbar$ small, then*

$$U_\hbar(\alpha_0)^* A_\hbar(t) U_\hbar(\alpha_0) = \alpha(t) + \sqrt{\hbar}\, W_\hbar(\alpha_0, t)^* a W_\hbar(\alpha_0, t) \quad (37)$$

$$\cong \alpha(t) + \sqrt{\hbar}\, W_0(\alpha_0, t)^* a W_0(\alpha_0, t) \quad (38)$$

$$= \alpha(t) + \sqrt{\hbar}\, a(\alpha_0, t). \quad (39)$$

End

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