HEAT KERNELS MEASURES AND INFINITE DIMENSIONAL ANALYSIS.

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Contents

1. Introduction

In these lecture notes we will generally be concerned with integral – differential analysis on infinite dimensional spaces equipped with measures related to heat kernels. As of yet, there is still no general theory within which to work. There have been attempts at a general structure, for example abstract Wiener–Riemann manifolds, but it has been hard to put interesting natural examples into this frame work. So these lectures will be a case study when the infinite dimensional manifold is either the paths or loops into a finite dimensional manifold and more specifically a Lie group.

In section 2, we will introduce the notion of the heat kernel measures on finite dimensional Riemannian manifolds. This notion will simply turn out to be the usual heat kernel function times the Riemannian volume form.

Section 3 is devoted to a description of the smoothness properties of positive measures on $\mathbb{R}^d$ without reference to Lebesgue measure. Although not technically needed for the rest of these notes, this section motivates some of our later considerations in the infinite dimensional setting.

Date: March 12, 2003 *File:notes.tex*.

This research was partially supported by NSF Grant and DMS 0202939.

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I.H.P. Lecture Notes, June 24, July 1, 2002.

Sections 4 – 6 are devoted to defining and proving existence of heat kernel measures associated to an infinite dimensional Hilbert space. The most important of these sections being Section 6 where classical Wiener measure on the space

$$W\left(\mathbb{R}^d\right) = \left\{\omega \in C\left([0,1], \mathbb{R}^d\right) : \omega(0) = 0\right\}$$

is considered as a heat kernel measure. Although the results in these sections are very classical (see for example Kuo [45] or Bogachev [5]), we still give the proofs in full detail. Our proofs will emphasizes the interpretation of Wiener measure as an infinite dimensional heat kernel measure.

Section 7 describes analogous results to those in Section 6 in the case $\mathbb{R}^d$ is replaced by a compact Lie group K. Results for the more complicated space of loops, $\mathcal{L}(K)$, on K are also described. The results of Section 7 rely on an analysis of Wiener measure on the path space of $\mathcal{L}(K)$. (Note this is a path space on a path space, i.e. maps from $[0,1]\times[0,T]$ to K.) Section 8 briefly outlines the results needed for Section 7 in the simpler setting where $\mathcal{L}(K)$ is replaced by a finite dimensional Riemannian manifold M.

Appendix 9 gives some motivations for these notes. Some readers may want to start here.

Acknowledgments: The author thanks the l'Institut Henri Poincaré for the opportunity to present the lectures which gave rise to these notes. The author is also greatly indebted to P. Auscher, G. Besson, T. Coulhon and A. Grigoryan who made possible the trimester programme on "Heat kernels, random walks and analysis to manifolds and graphs" held at IHP in Paris.

2. Finite Dimensional Heat Kernel Measures

Notation 2.1. Suppose (M^d, g) is a smooth d – dimensional manifold with Riemannian metric g. Let $C^k(M)$ denote the collection of k – times continuously differentiable functions $f : M \to \mathbb{R}$. As usual $C^k_c(M)$ will denote those $f \in C^k(M)$ with compact support. Similarly, let $BC^k(M)$ denote those $f \in C^k(M)$ such that $f,\ \nabla f,\ \dots, \nabla^k f$ are all bounded, where ∇ denotes the Levi-Civita covariant derivative of g. As usual Δ will be used to denote the **Riemannian Laplacian** associated to g. In local coordinates,

$$\Delta f = \operatorname{tr}(\nabla^2 f) = \sum_{i,j=1}^{d} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j}\right)$$

where $g = \sum_{i,j=1}^d g_{ij} dx^i \otimes dx^j$, $\left(g^{ij}\right)$ is the matrix inverse of (g_{ij}) and $\sqrt{g} = \det(g_{ij})$.

Notation 2.2. If μ is a probability measure on a measure space $(\Omega, \mathcal{F})$ and $f \in L^1(\mu) = L^1(\Omega, \mathcal{F}, \mu)$, we will often write $\mu(f)$ for the integral, $\int_\Omega f d\mu$.

Definition 2.3. Let (M, g) be a Riemannian manifold $o \in M$ be a fixed base point. A sequence $\{\nu_t\}_{t>0}$ of positive measures is called a **heat kernel sequence based** at $o \in M$ if:

(1) $\nu_t(M) \le 1$ for all $t > 0$.
(2) For all $f \in BC^2(M)$ the function $t \to \nu_t(f) := \int_M f d\nu_t$ is continuously differentiable,

$$\frac{d}{dt}\nu_t(f) = \frac{1}{2}\nu_t(\Delta f) \text{ and } \lim_{t\downarrow 0}\nu_t(f) = f(o). \tag{2.1}$$

Remark 2.4. If ν_t exists as in Definition 2.3, then necessarily $\nu_t(M) = 1$ for all t. This follows simply from the definition with $f \equiv 1$.

Proposition 2.5. *Suppose $M = \mathbb{R}^d$ with the standard flat metric, so that $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$. For each point $o \in \mathbb{R}^d$, there is exactly one sequence of positive measures $\{\nu_t\}_{t>0}$ with $\nu_t(\mathbb{R}^d) \le 1$ such that Eq. (2.1) holds for all $f \in C_c^\infty\left(\mathbb{R}^d\right)$. Moreover this sequence is given by*

$$\nu_t(dx) = p_t(o, x) dm(x) \tag{2.2}$$

where $p_t(x, y) := (2\pi t)^{-d/2} e^{-\frac{1}{2t}|x-y|^2}$ is the heat kernel and m is Lebesgue measure on $\mathbb{R}^d$.

Proof. Uniqueness. By assumption ν_t satisfies

$$\nu_t(f) = f(o) + \int_0^t \frac{1}{2}\nu_\tau(\Delta f) d\tau \text{ for all } f \in C_c^\infty\left(\mathbb{R}^d\right). \tag{2.3}$$

Now suppose $f \in C_c^2\left(\mathbb{R}^d\right)$ and $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \psi(x)dx = 1$. Letting $\psi_n(x) := n^d\psi(nx)$, we have $\psi_n * f \in C_c^\infty(\mathbb{R}^d)$ and

$$\psi_n * f \to f \text{ and } \psi_n * \Delta f \to \Delta f$$

boundedly as $n \to \infty$. Therefore passing to the limit, $n \to \infty$, in the equation,

$$\nu_t(\psi_n * f) = f(o) + \int_0^t \frac{1}{2}\nu_\tau(\psi_n * \Delta f) d\tau,$$

shows Eq. (2.3) holds for all $f \in C_c^2\left(\mathbb{R}^d\right)$.

Now suppose $f \in C^2\left(\mathbb{R}^d\right)$ such that $f, \nabla f$ and Δf are bounded and let $\phi \in C_c^\infty(B(0,1), [0,1])$ such that $\phi = 1$ in a neighborhood of 0 and set $\phi_n(x) := \phi(x/n)$. Then $f_n = \phi_n f$ is in $C_c^2(\mathbb{R}^d)$ and hence for large n,

$$\begin{aligned}\nu_t(\phi_n f) &= f(o) + \int_0^t \frac{1}{2}\nu_\tau(\Delta\phi_n f + 2\nabla\phi_n \cdot \nabla f + \phi_n \Delta f) d\tau \\ &= f(o) + \int_0^t \frac{1}{2}\nu_\tau(\frac{1}{n^2}\left(\Delta\phi\right)_n f + \frac{2}{n}\left(\nabla\phi\right)_n \cdot \nabla f + \phi_n\Delta f) d\tau.\end{aligned}$$

Using the dominated convergence theorem to pass to the limit in this equation allows us to conclude Eq. (2.3) holds for all $f \in C^2\left(\mathbb{R}^d\right)$ such that $f, \nabla f$ and Δf are bounded, i.e. $\{\nu_t\}_{t>0}$ is automatically heat kernel sequence based at $o \in \mathbb{R}^d$.

For $f \in C_c^\infty(\mathbb{R}^d)$ and $T > 0$ and $t \in [0, T)$, the function

$$F_t(x) = e^{(T-t)\Delta/2} f(x) = P_{(T-t)} f(x) := \int_{\mathbb{R}^d} p_{T-t}(x, y) f(y) dy$$

satisfies

$$|F_t| + |\nabla F_t| + |\Delta F_t| \le M := \sup_x \left[|f(x)| + |\nabla f(x)| + |\Delta f(x)|\right].$$

Claim: The function $\nu_t(F_s)$ is C^1 for $s, t \in (0, T)$ and

$$\frac{\partial}{\partial s}\nu_t(F_s) = -\frac{1}{2}\nu_t(\Delta F_s).$$

Indeed, we have

$$\begin{aligned}|\nu_t(F_s)-\nu_\tau(F_\sigma)| &\le |\nu_t(F_s-F_\sigma)|+|\nu_t(F_\sigma)-\nu_\tau(F_\sigma)|\\ &= |\nu_t(F_s-F_\sigma)|+\left|\frac{1}{2}\int_\tau^t \nu_r(\Delta F_\sigma)dr\right|\\ &\le |\nu_t(F_s-F_\sigma)|+\frac{1}{2}\|\Delta f\|_\infty |t-\tau| \to 0 \text{ as } (\sigma,\tau)\to(s,t) \quad (2.4)\end{aligned}$$

which shows $\nu_t(F_s)$ is continuous. Since

$$\frac{F_s-F_\sigma}{s-\sigma} = -\frac{1}{2(s-\sigma)}\int_\sigma^s \Delta F_r dr = -\frac{1}{2}\frac{1}{s-\sigma}\int_\sigma^s P_{T-r}\Delta f dr,$$

$\left|\frac{F_s-F_\sigma}{s-\sigma}\right|$ is bounded for s near σ and

$$\frac{F_s-F_\sigma}{s-\sigma} \to -\frac{1}{2}P_{T-\sigma}\Delta f = -\frac{1}{2}\Delta F_\sigma \text{ as } s\to\sigma.$$

Thus, by the dominated convergence theorem,

$$\frac{\nu_t(F_s)-\nu_t(F_\sigma)}{s-\sigma} = \nu_t\left(\frac{F_s-F_\sigma}{s-\sigma}\right) \to -\frac{1}{2}\nu_t\left(\Delta F_\sigma\right) \text{ as } s\to\sigma.$$

This shows that $\frac{\partial}{\partial s}\nu_t(F_s)$ exists and $\frac{\partial}{\partial s}\nu_t(F_s) = -\frac{1}{2}\nu_t(\Delta F_s)$. Since $\Delta F_s = P_{T-s}\Delta f$ and $\Delta f \in C_c^\infty(M)$, it follows from Eq. (2.4) with f replaced by Δf that

$$(s,t) \to \frac{\partial}{\partial s}\nu_t(F_s) = -\frac{1}{2}\nu_t(\Delta F_s) = -\frac{\partial}{\partial t}\nu_t(F_s)$$

is continuous proving the claim.

By the chain rule,

$$\frac{\partial}{\partial t}\nu_t(F_t) = \frac{1}{2}\nu_t(\Delta F_t) - \frac{1}{2}\nu_t(\Delta F_t) = 0$$

and therefore,

$$\nu_{T-\epsilon}(P_\epsilon f) = \nu_\epsilon(P_{T-\epsilon}f) \text{ for all } \epsilon>0. \quad (2.5)$$

Letting $\psi(\delta) := \sup\{|f(y)-f(x)| : |y-x|\le\delta\}$, we have

$$\begin{aligned}|P_\epsilon f(x)-f(x)| &= \left|\int_{\mathbb{R}^d} p_\epsilon(x,y)\left[f(y)-f(x)\right]dy\right|\\ &\le \int_{|y-x|\le\delta} p_\epsilon(x,y)\left|f(y)-f(x)\right|dy\\ &\quad + \int_{|y-x|>\delta} p_\epsilon(x,y)\left|f(y)-f(x)\right|dy\\ &\le \psi(\delta)+2\|f\|_\infty \int_{|y-x|>\delta} p_\epsilon(x,y)dy = \psi(\delta)+2\|f\|_\infty O(\epsilon), \quad (2.6)\end{aligned}$$

from which it follows $\lim_{\epsilon\downarrow 0}\|P_\epsilon f - f\|_\infty \le \psi(\delta) \to 0$ as $\delta\downarrow 0$. In particular this implies

$$|\nu_{T-\epsilon}(P_\epsilon f) - \nu_{T-\epsilon}(f)| \le \|P_\epsilon f - f\|_\infty \to 0 \text{ as } \epsilon\downarrow 0$$

and hence

$$\lim_{\epsilon\downarrow 0}\nu_{T-\epsilon}(P_\epsilon f) = \lim_{\epsilon\downarrow 0}\nu_{T-\epsilon}(f) = \nu_T(f). \quad (2.7)$$

Moreover,

$$\begin{aligned}|\nu_\epsilon(P_T f)-\nu_\epsilon(P_{T-\epsilon}f)| &= \left|\nu_\epsilon\left(\frac{1}{2}\int_{T-\epsilon}^{T}\Delta P_t f dt\right)\right| \\ &= \left|\nu_\epsilon\left(\frac{1}{2}\int_{T-\epsilon}^{T}P_t\Delta f dt\right)\right| \le \frac{1}{2}\epsilon\left\|\Delta f\right\|_\infty \to 0 \text{ as } \epsilon\downarrow 0.\end{aligned}$$

so that

$$\lim_{\epsilon\downarrow 0}\nu_\epsilon(P_{T-\epsilon}f)=\lim_{\epsilon\downarrow 0}\nu_\epsilon(P_T f)=P_T f(o)=\int_M p_T(o,y)f(y)dy. \tag{2.8}$$

The second equality in Eq. (2.8) requires a bit of explanation. By assumption $\lim_{\epsilon\downarrow 0}\nu_\epsilon(g)=g(o)$ for all $g\in C_c^\infty(\mathbb{R}^d)$. Let $\delta>0$ and B_δ be the ball of radius δ centered at $o\in\mathbb{R}^d$. Choosing $g\in C_c^\infty(B_\delta,[0,1])$ such that $g(o)=1$ implies $\lim_{\epsilon\downarrow 0}\nu_\epsilon(\mathbb{R}^d\setminus B_\delta)=0$. From this it follows that $\lim_{\epsilon\downarrow 0}\nu_\epsilon(g)=g(0)$ for all $g\in BC\left(\mathbb{R}^d\right)$.

Combining Eqs. (2.5) – (2.8) shows $\nu_T(f)=\int_{\mathbb{R}^d}p_T(o,y)f(y)dy$ for all $f\in C_c^\infty(\mathbb{R}^d)$. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $L^1(\nu_T+p_T(o,y)dy)$ and the latter space contains all bounded measurable functions, it follows that

$$d\nu_T(y)=p_T(o,y)dy,$$

i.e. Eq. (2.2) must hold.

Existence. This completes the proof, since it is now a simple matter to verify that ν_t defined as in Eq. (2.2) is a heat kernel sequence based at $o\in\mathbb{R}^d$. This fact will also follow from Theorem 2.6 below. ■

Recall (see for example Strichartz [56], Dodziuk [16] and Davies [14]) that if (M,g) is a complete Riemannian manifold, then $\Delta=\Delta_g$ acting on $C_c^\infty(M)$ is essentially self-adjoint, i.e. the closure $\bar\Delta$ of Δ is an unbounded self-adjoint operator on $L^2(M,dV)$. (Here $dV=\sqrt{g}dx^1\dots dx^n$ is being used to denote the Riemann volume measure on M.) Moreover the semi-group $P_t:=e^{t\bar\Delta/2}$ has a smooth integral kernel, $p_t(x,y)$, such that

$$p_t(x,y)\ge 0 \text{ for all } x,y\in M$$
$$\int_M p_t(x,y)dV(y)\le 1 \text{ for all } x\in M \text{ and}$$
$$P_t f(x):=\left(e^{t\bar\Delta/2}f\right)(x)=\int_M p_t(x,y)f(y)dV(y) \text{ for all } f\in L^2(M).$$

Theorem 2.6. *Let (M,g) be a complete Riemannian manifold with Ricci tensor bounded from below (i.e.* $\mathrm{Ric}\ge -Cg$ *for some $C\ge 0$) and $o\in M$ be a fixed point. Then there exists a unique heat kernel sequence $\{\nu_t\}_{t>0}$ based at $o\in M$. The measure ν_t is given by*

$$\nu_t(dx)=p_t(o,x)dV(x) \tag{2.9}$$

and satisfy

$$\nu_t(f):=\int_M f d\nu_t=:\left(e^{t\bar\Delta/2}f\right)(o) \text{ for all } f\in C_c^\infty(M). \tag{2.10}$$

Proof. Uniqueness. Suppose ν_t exists as described above. For $f \in C_c^2(M)$ and $T > 0$ let $F_t := e^{(T-t)\bar{\Delta}/2} f$. Then $\partial_t F_t = \frac{1}{2} e^{(T-t)\bar{\Delta}/2} \Delta f$ is a bounded function depending continuously on $t \in [0,T]$ and $x \in M$. Essentially the same argument as used in the proof of Proposition 2.5, shows if $\{\nu_t\}_{t>0}$ exits it must be given by Eq. (2.9). In doing this one should replace $\mathbb{R}^d$ by M and $|y-x|$ by $d(x,y)$ everywhere in the argument. The only other point is to note that the standard Gaussian heat kernel bounds along with volume growth estimates may be used in Eq. (2.6) to again conclude

$$|P_\epsilon f(x) - f(x)| \leq \psi(\delta) + 2\|f\|_\infty O(\epsilon).$$

Existence. According to Dodziuk [16][1], the kernel $p_t(x,y)$ may be written as the increasing limit of heat kernels $p_t^{\Omega_k}(x,y)$ with Dirichlet boundary conditions for relatively compact open subsets $\Omega_k \subset M$ with smooth boundary such that $\Omega_k \uparrow M$. Now for $f \in C^2(M)$ we then have, letting $\Omega = \Omega_k$ and $q = p^\Omega$,

$$\begin{aligned} \frac{d}{dt}\int_\Omega q_t(x,y) f(y)dy &= \frac{1}{2}\int_\Omega \Delta_y q_t(x,y) f(y)dy \\ &= -\frac{1}{2}\int_\Omega \nabla_y q_t(x,y)\cdot\nabla f(y)dy + \frac{1}{2}\int_{\partial\Omega} n(y)\cdot\nabla_y q_t(x,y) f(y)dy \\ &= \frac{1}{2}\int_\Omega q_t(x,y)\cdot\Delta f(y)dy + \frac{1}{2}\int_{\partial\Omega} n(y)\cdot\nabla_y q_t(x,y) f(y)d\sigma(y) \end{aligned}$$

where σ is the surface measure and n is the outward pointing unit normal on $\partial\Omega$ (the boundary of Ω). Integrating the previous equation on t gives

$$\begin{aligned} \int_{\Omega_k} p_t^{\Omega_k}(x,y) f(y)dy = f(x) &+ \frac{1}{2}\int_0^t d\tau \int_{\Omega_k} p_\tau^{\Omega_k}(x,y)\cdot\Delta f(y)dy \\ &+ \frac{1}{2}\int_0^t d\tau \int_{\partial\Omega_k} n(y)\cdot\nabla_y p_t^{\Omega_k}(x,y) f(y)d\sigma(y) \end{aligned}$$

and letting $k \to \infty$ in this equation implies

$$\int_M p_t(x,y) f(y)dy = f(x) + \frac{1}{2}\int_0^t d\tau \int_M p_\tau(x,y)\cdot\Delta f(y)dy + \lim_{k\to\infty} R_k(f)$$

where

$$R_k(f) := \frac{1}{2}\int_0^t d\tau \int_{\partial\Omega_k} n(y)\cdot\nabla_y p_\tau^{\Omega_k}(x,y) f(y)d\sigma(y).$$

We will now finish the proof by showing $\lim_{k\to\infty} R_k(f) = 0$.

Since $p_t^{\Omega_k}(x,y) \geq 0$ and vanishes for $y \in \partial\Omega$, $n(y)\cdot\nabla_y p_t^{\Omega_k}(x,y) \leq 0$ and hence

$$\begin{aligned} |R_k(f)| &\leq -\frac{1}{2}\|f\|_\infty \int_0^t d\tau \int_{\partial\Omega_k} n(y)\cdot\nabla_y p_\tau^{\Omega_k}(x,y)d\sigma(y) \\ &= -\|f\|_\infty \frac{1}{2}\int_0^t d\tau \int_{\Omega_k} \Delta_y p_\tau^{\Omega_k}(x,y)dy \\ &= -\|f\|_\infty \int_0^t d\tau \int_{\Omega_k} \frac{\partial}{\partial\tau} p_\tau^{\Omega_k}(x,y)dy = \|f\|_\infty\left[1 - \int_{\Omega_k} p_t^{\Omega_k}(x,y)dy\right]. \end{aligned}$$

[1] Dodziuk also proves, under the condition that (M,g) is complete and the Ricci curvature is bounded from below, that bounded solutions to the heat equation are uniquely determined by their initial values at $t = 0$.

Letting $k \to \infty$ in this expression then shows

$$\lim_{k\to\infty} |R_k(f)| \le \|f\|_\infty \left[1 - \int_M p_t(x,y)dy\right].$$

Thus $\lim_{k\to\infty} |R_k(f)| = 0$, since the lower bound on the Ricci curvature is sufficient to show $\int_M p_t(x,y)dy = 1$, see for example Theorem 5.2.6 in Davies [14]. ■

3. Describing Smooth Measures on $\mathbb{R}^d$ without reference to Lebesgue Measure

One of the main goals in these lectures is to give some examples of heat kernel sequences for infinite dimensional manifolds. Once we produce such a heat kernel sequence we will want to show the resulting measures $\{\nu_t\}_{t>0}$ are "smooth." However, in the infinite dimensional examples below there is no reasonable notion of Lebesgue measure or the Riemann volume measure. Hence it will not be possible to measure the smoothness of ν_t in terms of the smoothness of its density with respect to the Riemann volume measure. In this section, we will explain an intrinsic criteria for a finite measure on $\mathbb{R}^d$ to be smooth. This criteria will later be used as a definition in the infinite dimensional settings below.

Notation 3.1. For a measure μ on $\mathbb{R}^d$, let $L^{\infty-}(\mu) := \cap_{1\le p<\infty} L^p(\mu)$.

Definition 3.2. A Radon measure μ on $\mathbb{R}^d$ is said to be smooth if for all multi -indices $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ ($\mathbb{N} = \{1, 2 \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) there exists functions $g_\alpha \in C^\infty(\mathbb{R}^d) \cap L^{\infty-}(\mu)$ such that,

$$\int_{\mathbb{R}^d} (-D)^\alpha f d\mu = \int_{\mathbb{R}^d} f g_\alpha d\mu \text{ for all } f \in C_c^\infty(\mathbb{R}^d), \tag{3.1}$$

where $D^\alpha := \prod_{i=1}^d \left(\frac{\partial}{\partial x^i}\right)^{\alpha_i}$.

Theorem 3.3. *A measure μ on $\mathbb{R}^d$ is smooth iff there exists $\rho \in C^\infty(\mathbb{R}^d, (0,\infty))$ such that $d\mu = \rho dm$ where m is Lebesgue measure on $\mathbb{R}^d$.*

Proof. Let us begin by showing there are coefficients $c_\alpha(\beta) \in \mathbb{N}$ (in fact $c_\alpha(\beta) = \frac{\alpha!}{\beta!(\alpha-\beta)!}$) for $0 \le \beta \le \alpha$ such that for $f \in C_c^\infty(\mathbb{R}^d)$ and $h \in C^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} (-D)^\alpha f \cdot h d\mu = \sum_{\beta\le\alpha} c_\alpha(\beta) \int_{\mathbb{R}^d} f D^\beta h \cdot g_{\alpha-\beta} d\mu. \tag{3.2}$$

The proof of Eq. (3.2) will be by induction on $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Equation (3.1) with $\alpha = e_i$ and f being replaced by fh implies

$$\int_{\mathbb{R}^d} -\partial_i f \cdot h d\mu = \int_{\mathbb{R}^d} f \cdot (\partial_i h + g_i h)\, d\mu$$

which proves Eq. (3.2) for $|\alpha| = 1$.

Equation (3.1) with f being replaced by $-\partial_i f$ along with the previous identity shows

$$\begin{aligned}\int_{\mathbb{R}^d} f g_{\alpha+e_i} d\mu &= \int_{\mathbb{R}^d} (-D)^{\alpha+e_i} f d\mu = -\int_{\mathbb{R}^d} (-D)^\alpha \partial_i f d\mu \\ &= \int_{\mathbb{R}^d} -\partial_i f \cdot g_\alpha d\mu = \int_{\mathbb{R}^d} f \cdot [\partial_i g_\alpha + g_1 g_\alpha]\, d\mu.\end{aligned}$$

This equation being true for all $f \in C_c^\infty\left(\mathbb{R}^d\right)$ implies $\partial_i g_\alpha + g_1 g_\alpha = g_{\alpha+e_i}$, μ – a.e.

Now suppose Eq. (3.2) holds for all $|\alpha| \leq n$ with $n \geq 1$. Then

$$\begin{aligned}\int_{\mathbb{R}^d} (-D)^{\alpha+e_i} f \cdot h d\mu &= \int_{\mathbb{R}^d} (-D)^{\alpha} (-\partial_i f) \cdot h d\mu = \sum_{\beta \leq \alpha} c_\alpha(\beta) \int_{\mathbb{R}^d} (-\partial_i f) D^\beta h \cdot g_{\alpha-\beta} d\mu \\ &= \sum_{\beta \leq \alpha} c_\alpha(\beta) \int_{\mathbb{R}^d} f \left(\partial_i \left[D^\beta h \cdot g_{\alpha-\beta}\right] + D^\beta h \cdot g_i g_{\alpha-\beta}\right) d\mu \\ &= \sum_{\beta \leq \alpha} c_\alpha(\beta) \int_{\mathbb{R}^d} f \left(D^{\beta+e_i} h \cdot g_{\alpha-\beta} + D^\beta h \cdot [\partial_i g_{\alpha-\beta} + g_i g_{\alpha-\beta}]\right) d\mu \\ &= \int_{\mathbb{R}^d} \sum_{\beta \leq \alpha} c_\alpha(\beta) f \left(D^{\beta+e_i} h \cdot g_{\alpha-\beta} + D^\beta h \cdot g_{\alpha+e_i-\beta}\right) d\mu\end{aligned}$$

which finishes the induction argument.

For $\phi \in C_c^\infty(\mathbb{R}^d)$ let $l_\phi(f) := \int_{\mathbb{R}^d} \phi f d\mu$, then l_ϕ is a distribution on $\mathbb{R}^d$ with compact support. The Fourier transform of l_ϕ is given by

$$\hat{l}_\phi(k) = \int_{\mathbb{R}^d} e^{ik\cdot x} \phi(x) d\mu(x).$$

By Eq. (3.2),

$$\begin{aligned}k^\alpha \hat{l}_\phi(k) &= \int_{\mathbb{R}^d} k^\alpha e^{ik\cdot x} \phi(x) d\mu(x) = \int_{\mathbb{R}^d} \left(\frac{1}{i} D_x\right)^\alpha e^{ik\cdot x} \phi(x) d\mu(x) \\ &= \int_{\mathbb{R}^d} e^{ik\cdot x} \sum_{\beta \leq \alpha} c_\alpha(\beta) \left(-\frac{1}{i} D_x\right)^\beta \phi(x) \cdot g_{\alpha-\beta}(x) d\mu(x)\end{aligned}$$

from which we learn $\sup_k \left(1 + |k|^2\right)^N \left|\hat{l}_\phi(k)\right| < \infty$ for all N. Hence l_ϕ may be represented by a smooth function (still denoted by l_ϕ), i.e.

$$\int_{\mathbb{R}^d} \phi f d\mu = \int_{\mathbb{R}^d} l_\phi f dm \text{ for all } f \in C_c^\infty(\mathbb{R}^d).$$

Now choose $\phi \in C_c^\infty(\mathbb{R}^d, [0,1])$ such that $\phi = 1$ on $\overline{B(0,1)}$ and let $\phi_m(x) = \phi(x/m)$. Then one easily sees that $l_{\phi_m} = l_{\phi_n}$ on $B(0,n)$ for all $m \geq n$. Thus we may define $\rho(x) = l_{\phi_n}(x)$ for all $x \in B(0,n)$. Then ρ is a smooth function such that $d\mu = \rho dm$. Since $\mu \geq 0$ it follows that $\rho \geq 0$, so it only remains to prove that ρ is positive. By Eq. (3.1),

$$\int_{\mathbb{R}^d} f D^\alpha \rho dm = \int_{\mathbb{R}^d} (-D)^\alpha f \rho dm = \int_{\mathbb{R}^d} (-D)^\alpha f d\mu = \int_{\mathbb{R}^d} f g_\alpha d\mu = \int_{\mathbb{R}^d} f g_\alpha \rho dm$$

and hence $D^\alpha \rho = g_\alpha \rho$. Let $G = (g_1, g_2, \ldots, g_d)$ and fix a point $x_0 \in \mathbb{R}^d$ such that $\rho(x_0) > 0$. Then for any $y \in \mathbb{R}^d$,

$$\frac{d}{dt} \ln \rho(x_0+ty) = \frac{\nabla \rho(x_0+ty) \cdot y}{\rho(x_0+ty)} = \frac{\rho(x_0+ty) G(x_0+ty) \cdot y}{\rho(x_0+ty)} = G(x_0+ty) \cdot y \tag{3.3}$$

which is valid for all t such that $\rho(x_0+ty) > 0$. In particular this is valid for all t near zero. Integrating Eq. (3.3) on t implies

$$\rho(x_0+ty) = \rho(x_0) \exp\left(\int_0^t G(x_0+\tau y) \cdot y d\tau\right).$$

From this equation it follows that $\rho(x_0+ty)>0$ for all t, that is $\rho(x)>0$ for all x and then taking $x_0=0$ and $t=1$ that

$$\rho(y) \geq \rho(0)\exp\left(-|y|\int_0^1 |G(\tau y)|\,d\tau\right).$$

■

Corollary 3.4. *All smooth measures on $\mathbb{R}^d$ are mutually absolutely continuous relative to each other.*

Corollary 3.5. *If μ is a smooth measure on $\mathbb{R}^d$ and $\phi:\mathbb{R}^d\to\mathbb{R}^d$ is a diffeomorphism, then $\phi_*\mu$ is a smooth measure as well. In fact if $d\mu=\rho dm$, then*

$$d\left(\phi_*\mu\right)=\rho\circ\phi^{-1}\left|\left(\phi^{-1}\right)'\right|dm. \tag{3.4}$$

Proof. Let $f\in C_c(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} f d\phi_*\mu=\int_{\mathbb{R}^d} f\circ\phi d\mu=\int_{\mathbb{R}^d} f(\phi(x))\rho(x)dx.$$

So making the change of variables, $y=\phi(x)$ so that $x=\phi^{-1}(y)$, $dx=\left|\left(\phi^{-1}\right)'(y)\right|dy$ and hence

$$\int_{\mathbb{R}^d} f d\phi_*\mu=\int_{\mathbb{R}^d} f(y)\rho(\phi^{-1}(y))\left|\left(\phi^{-1}\right)'(y)\right|dy$$

which proves Eq. (3.4). ■

These finite dimensional results in Corollaries 3.4 and 3.5 are in stark contrast to what happens in infinite dimensional settings as we shall see below in Proposition 5.5 and Exercise 6.1. Also see Remark 6.22.

4. Infinite dimensional considerations

Let $(H,(\cdot,\cdot))$ be a separable Hilbert space, $|h|:=\sqrt{(h,h)}$ be the associate Hilbertian norm and $S\subset H$ be an orthonormal basis for H. As usual, for $f\in C^2(H)$, let

$$\Delta_H f(x)=\mathrm{tr}(D^2f(x))=\sum_{h\in S}\left(\partial_h^2 f\right)(x)$$

provided $D^2f(x)$ is trace class. Here $\partial_h f(x):=\frac{d}{dt}|_0 f(x+th)$, $Df(x)h:=\partial_h f(x)$ and $D^2f(x)(h,k):=\left(\partial_h\partial_k f\right)(x)$.

Example 4.1. Suppose $P:H\to H$ is a finite rank orthogonal projection and $F\in C^2(PH)$ and $f(x):=F(Px)$ for all $x\in H$. Then

$$\partial_h f(x)=\left(\partial_{Ph}F\right)(Px),$$
$$D^2f(x)(h,k)=D^2F(Px)\left(Ph,Pk\right)$$

and

$$\Delta_H f(x)=\left(\Delta_{PH}F\right)(x)$$

where Δ_{PH} represents the usual finite dimensional Laplacian acting on $C^2(PH)$.

Notation 4.2. A function of the form $f(x)=F(Px)$ with $F\in C^k(PH)$ and $P:H\to H$ is a finite rank orthogonal projection will be called a C^k – **cylinder function**. The collection of C^k – cylinder functions will be denoted by $\mathcal{F}C^k(H)$. Also let $\mathcal{F}C_c^k(H)$ $\left(\mathcal{F}BC^k(H)\right)$ denote those $f=F\circ P\in\mathcal{F}C^k(H)$ such that $F\in C_c^k(H)$ $\left(F\in BC^k(H)\right)$.

Proposition 4.3. *There does **not** exist a heat kernel sequence based at $0 \in H$. More explicitly there is no collection $\{\nu_t\}_{t>0}$ of positive measures on H such that*

1. $\nu_t(H) \leq 1$ *for all* $t > 0$ *and*
2. *For all* $f \in \mathcal{F}BC^2(H)$ *the function* $t \to \nu_t(f) := \int_H f d\nu_t$ *is continuously differentiable and*

$$\frac{d}{dt}\nu_t(f) = \frac{1}{2}\nu_t(\Delta_H f) \text{ and } \lim_{t\downarrow 0}\nu_t(f) = f(0).$$

The following basic Gaussian integration lemma will be needed for the proof of Proposition 4.3.

Lemma 4.4. *For all* $\alpha > 0$ *and* $\beta \in \mathbb{C}$,

$$\int_{\mathbb{R}} e^{-\alpha x^2} e^{\beta x} dx = \sqrt{\frac{\pi}{\alpha}} e^{\frac{1}{4\alpha}\beta^2}. \tag{4.1}$$

More generally if $V \subset H$ *is a finite dimensional subspace,* $m := \dim(V)$, $\alpha > 0$ *and* $u, v \in V$, *then*

$$\int_V e^{-\alpha|y|^2} e^{(u,y)+i(v,y)} dy = \left(\frac{\pi}{\alpha}\right)^{m/2} e^{\frac{1}{4\alpha}(u+iv)^2} \tag{4.2}$$

where dy *denotes Lebesgue measure on* V *and* $(u+iv)^2 := |u|^2 - |v|^2 + 2i(u,v)$. *We also have, for any* $u \in V$,

$$\int_V e^{-\alpha|y|^2} (u,y)^2 dy = \left(\frac{\pi}{\alpha}\right)^{m/2} \frac{1}{2\alpha} |u|^2. \tag{4.3}$$

and any $p \in [1, \infty)$,

$$\left(\frac{\alpha}{\pi}\right)^{m/2} \int_V e^{-\alpha|y|^2} |y|^p dy = \frac{\Gamma(\frac{p+m}{2})}{\Gamma(\frac{m}{2})} \alpha^{-\frac{p}{2}}. \tag{4.4}$$

Proof. The proof of this lemma is standard. We leave the proof of Eq. (4.1) to the reader and note that Eq. (4.2) follows from Eq. (4.1) using Fubini's theorem after introducing an orthonormal basis on V. Equation (4.3) may be proved by differentiating Eq. (4.2) in λ to find

$$\begin{aligned}\int_V e^{-\alpha|y|^2} (u,y)^2 dy &= \frac{d^2}{d\lambda^2}|_{\lambda=0} \int_V e^{-\alpha|y|^2} e^{(\lambda u, y)} dy \\ &= \frac{d^2}{d\lambda^2}|_{\lambda=0} \left(\frac{\pi}{\alpha}\right)^{m/2} e^{\frac{\lambda^2}{4\alpha}|u|^2} = \left(\frac{\pi}{\alpha}\right)^{m/2} \frac{1}{2\alpha} |u|^2.\end{aligned}$$

Passing to polar coordinates, the left side of Eq. (4.4) satisfies

$$\left(\frac{\alpha}{\pi}\right)^{m/2} \int_V e^{-\alpha|y|^2} |y|^p dy = \sigma\left(S^{m-1}\right) \left(\frac{\alpha}{\pi}\right)^{m/2} \int_0^\infty e^{-\alpha r^2} r^p r^{m-1} dr,$$

where $\sigma\left(S^{m-1}\right)$ is the surface area of the unit sphere in $\mathbb{R}^m$. Letting $r=\sqrt{u/\alpha}$ in the last integral then shows

$$\begin{aligned}\left(\frac{\alpha}{\pi}\right)^{m/2}\int_V e^{-\alpha|y|^2}|y|^p\,dy &= \sigma\left(S^{m-1}\right)\left(\frac{\alpha}{\pi}\right)^{m/2}\int_0^\infty e^{-u}(u/\alpha)^{\frac{p+m-1}{2}}\frac{1}{2\alpha}(u/\alpha)^{-1/2}du\\ &= \frac{\sigma\left(S^{m-1}\right)}{2\pi^{m/2}}\alpha^{-\frac{p}{2}}\int_0^\infty u^{\frac{p+m-1}{2}-\frac{1}{2}}e^{-u}du\\ &= \frac{\sigma\left(S^{m-1}\right)}{2\pi^{m/2}}\alpha^{-\frac{p}{2}}\int_0^\infty u^{\frac{p+m}{2}}e^{-u}\frac{du}{u}\\ &= \frac{\sigma\left(S^{m-1}\right)}{2\pi^{m/2}}\alpha^{-\frac{p}{2}}\Gamma(\frac{p+m}{2}).\end{aligned}\tag{4.5}$$

Comparing this equation with $p=0$ and Eq. (4.2) with $u=v=0$, we find $1=\frac{\sigma\left(S^{m-1}\right)}{2\pi^{m/2}}\Gamma(\frac{m}{2})$ which put back into Eq. (4.5) proves Eq. (4.4). ■

Proof. Suppose $\{\nu_t\}_{t>0}$ were such a heat kernel sequence based at $0\in H$. Let $P:H\to H$ be a finite rank orthogonal projection and ν_t^P denote the measure on PH defined by

$$\int_{PH}F d\nu_t^P := \int_H F\circ P d\nu_t$$

for all $F:PH\to PH$ which are bounded and measurable. The hypothesis on ν_t now guarantees that $\left\{\nu_t^P\right\}_{t>0}$ is a heat kernel sequence based at $0\in PH$ and therefore by Proposition 2.5,

$$d\nu_t^P(y)=\left(\frac{1}{2\pi t}\right)^{\dim(PH)/2}e^{-\frac{1}{2t}|y|^2}dy$$

where dy denotes Lebesgue measure on PH. By Eq. (4.2) of Lemma 4.4, for any $\alpha>0$,

$$\int_{PH}e^{-\alpha|y|^2}d\nu_t^P(y)=\left(\frac{1}{2\pi t}\right)^{\dim(PH)/2}\left(\frac{\pi}{\alpha+\frac{1}{2t}}\right)^{\dim(PH)/2}=\left(\frac{1}{2t\alpha+1}\right)^{\dim(PH)/2}.$$

Let $P_n:H\to H$ be a sequence of increasing finite rank orthogonal projections such that $P_n\to I$ strongly as $n\to\infty$, then by the dominated convergence theorem,

$$\begin{aligned}\int_H e^{-\alpha|x|^2}d\nu_t(x) &= \lim_{n\to\infty}\int_H e^{-\alpha|P_nx|^2}d\nu_t(x)=\lim_{n\to\infty}\int_{P_nH}e^{-\alpha|y|^2}d\nu_t^{P_nH}(y)\\ &= \lim_{n\to\infty}\left(\frac{1}{2t\alpha+1}\right)^{\dim(P_nH)/2}=0.\end{aligned}$$

Since $e^{-\alpha|x|^2}$ is a positive function on H, it follows that ν_t must be the zero measure for all t, which clearly violates the initial condition: $\lim_{t\downarrow 0}\nu_t(f)=f(0)$. ■

Remark 4.5. Another way to "understand" Proposition 4.3 is that if ν_t were to exist as a measure on H it should be given by the formula

$$\text{“}\nu_t(dx)=\frac{1}{Z_t}e^{-\frac{1}{2t}|x|_H^2}dm_H(x),\text{”}\tag{4.6}$$

where m_H is "infinite dimensional Lebesgue measure," and

$$Z_t:=(2\pi t)^{\dim(H)/2}=\begin{cases}0 & \text{if}\quad t<1/2\pi\\ \infty & \text{if}\quad t>1/2\pi.\end{cases}$$

Clearly the expression in Eq. (4.6) has severe problems owing to the definition of Z_t. Moreover, it is well known that there is no reasonable notion of Lebesgue measure on an infinite dimensional Hilbert space as you are asked to show in Exercise 4.1 below.

Exercise 4.1. Suppose H is an infinite dimensional Hilbert space and m is a **countably additive** measure on $\mathcal{B}_H$ which is invariant under translations and satisfies, $m(B(0,\epsilon)) > 0$ for all $\epsilon > 0$. Show $m(V) = \infty$ for all non-empty open subsets $V \subset H$. **Hint:** Show $B(0,\epsilon)$ contains a infinite number of disjoint balls of radius $\delta = \epsilon/\sqrt{2}$.

L. Gross, in [38] and [39], describes how to characterize those "completions" of H to a Banach space W such that the heat kernel measures may be constructed on X. Rather than work in the full generality of Gross' abstract Wiener spaces, the discussion below will be restricted to two important special cases. The first is when $H = \ell^2$ and W is a certain Hilbertian extension of ℓ^2 and the second is in the context of "classical Wiener space."

5. Heat Kernel Measure associated to ℓ^2

When $H = \ell^2$, the expression in Eq. (4.6) may be informally re-written as

$$\nu_t(dx) = e^{-\frac{t}{2}\sum_{n=1}^{\infty} x_n^2} \prod_{n=1}^{\infty} \frac{dx_n}{\sqrt{2\pi t}} = \prod_{n=1}^{\infty} \left(e^{-\frac{t}{2}x_n^2} \frac{dx_n}{\sqrt{2\pi t}} \right) = \prod_{n=1}^{\infty} p_t(dx_n),$$

where $p_t(dx) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{t}{2}x^2} dx$.

Fact 5.1. Recall that Kolmogorov's existence theorem implies the existence of infinite products of probability measures. (See almost any graduate text book in probability theory.)

As a consequence, there exists a unique probability measure ν_t on $\mathbb{R}^{\mathbb{N}}$ such that

$$\int_{\mathbb{R}^{\mathbb{N}}} F(x_1, \dots, x_N) d\nu_t(x) = \int_{\mathbb{R}^N} F(x_1, \dots, x_N) \prod_{n=1}^{N} p_t(x_n) dx_n \tag{5.1}$$

holds for all $F : \mathbb{R}^N \to \mathbb{R}$ which are bounded and measurable and for all $N \in \mathbb{N}$. From Proposition 4.3, we expect that $\ell^2 \subset \mathbb{R}^{\mathbb{N}}$ is a set of ν_t – measure 0, i.e. $\nu_t(\ell^2) = 0$. This is verified in the following theorem.

Theorem 5.2. *For* $a = (a_1, a_2, \dots) \in (0,\infty)^{\mathbb{N}}$, *define*

$$X_a = \ell^2(a) = \{x \in \mathbb{R}^{\mathbb{N}} : \sqrt{\sum_{i=1}^{\infty} a_i x_i^2} =: \|x\|_a < \infty\},$$

then for any $t > 0$,

$$\nu_t(X_a) = \begin{cases} 1 & \text{if} \quad \sum_{i=1}^{\infty} a_i < \infty \\ 0 & \text{if} \quad \sum_{i=1}^{\infty} a_i = \infty. \end{cases} \tag{5.2}$$

In particular $\nu_t(\ell^2) = 0$.

Proof. The method of proof will be very similar to that of Proposition 4.3. Let $q(x) := \sum_{i=1}^{\infty} a_i x_i^2$ and for $N \in \mathbb{N}$ let $q_N(x) = \sum_{i=1}^{N} a_i x_i^2$. For any $\epsilon > 0$, using the monotone convergence theorem,

$$\begin{aligned} \int_{\mathbb{R}^{\mathbb{N}}} e^{-\epsilon q/2} d\nu_t &= \int_{\mathbb{R}^{\mathbb{N}}} \lim_{N\to\infty} e^{-\epsilon q_N/2} d\nu_t = \lim_{N\to\infty} \int_{\mathbb{R}^{\mathbb{N}}} e^{-\epsilon q_N/2} d\nu_t \\ &= \lim_{N\to\infty} \int_{\mathbb{R}^N} e^{-\frac{\epsilon}{2}\sum_1^N a_i x_i^2} \prod_{i=1}^N p_t(dx_i) \\ &= \lim_{N\to\infty} \prod_{i=1}^N \int_{\mathbb{R}} e^{-\frac{\epsilon}{2} a_i x^2} p_t(dx) = \lim_{N\to\infty} \prod_{i=1}^N \frac{\sqrt{\frac{\pi}{\frac{\epsilon}{2}a_i + \frac{1}{2t}}}}{\sqrt{2\pi t}} \\ &= \prod_1^\infty \frac{1}{\sqrt{1+\epsilon a_i}} = \left[\prod_1^\infty (1 + t\epsilon a_i)\right]^{-1/2}. \end{aligned} \tag{5.3}$$

Taking logarithms of Eq. (5.3) and then letting $\epsilon \downarrow 0$ implies

$$-\log\left(\int_{\mathbb{R}^{\mathbb{N}}} e^{-\epsilon q/2} d\nu_t\right) = \frac{1}{2}\sum_{i=1}^\infty \ln(1 + t\epsilon a_i) \overset{\epsilon\downarrow 0}{\to} \begin{cases} \infty & \text{if } \sum_{i=1}^\infty a_i = \infty \\ 0 & \text{if } \sum_{i=1}^\infty a_i < \infty \end{cases} \tag{5.4}$$

and hence

$$\lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^{\mathbb{N}}} e^{-\epsilon q/2} d\nu_t = \begin{cases} 0 & \text{if } \sum_{i=1}^\infty a_i = \infty \\ 1 & \text{if } \sum_{i=1}^\infty a_i < \infty. \end{cases}$$

Since $e^{-\epsilon q/2} \le 1$ and $\lim_{\epsilon\downarrow 0} e^{-\epsilon q/2} = 1_{X_a}$, this result along with the dominated convergence theorem proves Eq. (5.2). ∎

For the rest of this section, fix a linear subspace $W \subset \mathbb{R}^{\mathbb{N}}$ such that $\ell^2 \subset W$ and $\nu_t(W) = 1$ for all $t > 0$. (For example $W = X_a$ with $\sum_{i=1}^\infty a_i < \infty$.)

Notation 5.3. A function $f : W \to \mathbb{R}$ of the form $f(x) = F(x_1, \ldots, x_n)$ for some $F \in C^k(\mathbb{R}^n)$ will be called a **cylinder function** on W and the collection of such functions will be denoted by $\mathcal{F}C^k(W)$. As before, if $F \in C_c^k(\mathbb{R}^n)$ of $BC^k(\mathbb{R}^n)$, we will say $f \in \mathcal{F}C_c^k(W)$ or $f \in \mathcal{F}BC^k(W)$ respectively.

Proposition 5.4. *The measure $\{\nu_t\}_{t>0}$ form a heat kernel sequence based at $0 \in W$ in the sense that*

1. $\nu_t(W) = 1$ *for all* $t > 0$ *and*
2. *for all $f \in \mathcal{F}BC^2(W)$ the function $t \to \nu_t(f)$ is continuously differentiable,*

$$\frac{d}{dt}\nu_t(f) = \frac{1}{2}\nu_t(\Delta_H f) \text{ and } \lim_{t\downarrow 0}\nu_t(f) = f(0)$$

where

$$\Delta_H f(x) := \sum_{n=1}^\infty \partial_{e_n}^2 f(x) = (\Delta_{\mathbb{R}^n} F)(x_1, \ldots, x_n)$$

and $\{e_n\}_{n=1}^\infty$ is the standard orthonormal basis for ℓ^2, i.e. $e_n(i) = \delta_{ni}$.

Moreover, $\{\nu_t\}_{t>0}$ is the unique heat kernel sequence on W satisfying items 1. and 2. above.

Proof. The fact the ν_t satisfies items 1. and 2. above is a simple exercise left to the reader. For uniqueness, suppose $\{\nu_t\}_{t>0}$ is a heat kernel sequence based at $0 \in W$ and $n \in \mathbb{N}$, let ν_t^n be the measure on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} f(x_1, \dots, x_n) d\nu_t^n(x) = \int_W f(x_1, \dots, x_n) d\nu_t(x)$$

for all bounded measurable functions $f : \mathbb{R}^n \to \mathbb{R}$. Then one easily verifies $\{\nu_t^n\}_{t>0}$ is a heat kernel sequence based at $0 \in \mathbb{R}^n$ and hence by Proposition 2.5,

$$d\nu_t^n(x) = (2\pi t)^{-n/2} e^{-\frac{1}{2t}|x|^2_{\mathbb{R}^n}} dm(x)$$

which is equivalent to Eq. (5.1). ∎

Proposition 5.5. *If $s, t > 0$ and $s \neq t$ then $\nu_t \perp \nu_s$.*

Proof. For each $t > 0$, let

$$W_t := \left\{ x \in W : \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N x_i^2 = t \right\}.$$

Then the strong law of large numbers[2] asserts $\nu_t(W_t) = 1$ for all $t > 0$ and this proves the theorem since $W_t \cap W_s = \emptyset$ for all $s \neq t$. ∎

If $a \in \ell^2$ and $a_i = 0$ for $i \geq n$ for some n, then

$$\int_W (a \cdot x)^2 \, d\nu_t(x) = \int_{\mathbb{R}^n} \left(\sum_{i=1}^n a_i x_i \right)^2 d\nu_t^n(x) = t \, \|a\|_2^2 .$$

This simple computation along with a standard limiting argument leads to the following result.

Lemma 5.6. *For $a \in \ell^2$ and for $N < \infty$, let $a_i^N = a_i$ if $i \leq N$ and 0 if $i > N$. Then $\lim_{N\to\infty} a^N \cdot x$ exits in $L^2(\nu_t)$. By abuse of notation we will use $a \cdot x$ to denote this limit (even though the answer may depend on t). The limit $a \cdot x$ still satisfies*

$$\int_W (a \cdot x)^2 \, d\nu_t(x) = t \, \|a\|_2^2$$

and

$$\int_W e^{ia\cdot x} d\nu_t(x) = e^{-\frac{t}{2}\|a\|_2^2}.$$

The next proposition points out that even though ν_t is not supported on ℓ^2, its quasi-invariance (and hence differentiability properties) are still intimately connected with ℓ^2.

Proposition 5.7 (Cameron-Martin Type Theorem)**.** *For $a \in W$, let $\nu_t^a := \nu(\cdot - a)$, i.e.*

$$\int_W f(x) d\nu_t^a(x) = \int_W f(x) \nu_t (dx - a) := \int_W f(x + a) d\nu_t (x) . \tag{5.5}$$

Then $\nu_t^a := \nu_t(\cdot - a) \ll \nu_t$ iff $a \in \ell^2$ and if $a \in \ell^2$ then

$$\frac{d\nu_t^a(x)}{d\nu_t(x)} = e^{\frac{1}{t} a\cdot x - \frac{1}{2t}|a|^2}. \tag{5.6}$$

---

[2] Also see Exercise 6.1 which essentially sketches a proof of the law of large numbers in this context.

Proof. At an informal level, we have

$$\frac{d\nu_t^a(x)}{d\nu_t(x)} = \frac{e^{-\frac{1}{2t}|x-a|^2}}{e^{-\frac{1}{2t}|x|^2}} = e^{\frac{1}{t}a\cdot x-\frac{1}{2t}|a|^2}$$

which clearly only makes sense if $a \in \ell^2$.

For a rigorous proof, suppose first $a, b \in \ell^2$. Then

$$\int_W e^{ib\cdot x} d\nu_t^a(x) = \int_W e^{ib\cdot(x+a)} d\nu_t(x) = e^{-\frac{t}{2}|b|^2+ib\cdot a}$$

while

$$\int_W e^{ib\cdot x} e^{\frac{1}{t}a\cdot x-\frac{1}{2t}|a|^2} d\nu_t(x) = e^{\frac{t}{2}(t^{-1}a+ib)^2-\frac{1}{2t}|a|^2} = e^{-\frac{t}{2}|b|^2+ib\cdot a}.$$

and this suffices to prove Eq. (5.6).

Suppose that $a \in W \setminus \ell^2$ and let $\|\mu\|$ denotes the total variation norm of a measure μ. We will make use of the fact if $\|\nu_t^a - \nu_t\| = 2$ then that $\nu_t^a \perp \nu_t$. (The converse is true as well but is not needed here.) Indeed if $\|\nu_t^a - \nu_t\| = 2$ and $P \cup P^c$ is the Jordan decomposition of $\nu_t^a - \nu_t$, then

$$2 = \|\nu_t^a - \nu_t\| = (\nu_t^a - \nu_t)(P) - (\nu_t^a - \nu_t)(P^c) \le \nu_t^a(P) + \nu_t(P^c) \le 2$$

with equality iff $\nu_t^a(P) = 1$ and $\nu_t(P^c) = 1$. Therefore $\nu_t^a \perp \nu_t$.

We now compute $\|\nu_t^a - \nu_t\|$ formally. For this let $z := e^{\frac{1}{2t}a\cdot x-\frac{1}{4t}|a|^2}$ then

$$\begin{aligned}\|\nu_t^a - \nu_t\| &= \int_W \left|e^{\frac{1}{t}a\cdot x-\frac{1}{2t}|a|^2} - 1\right| d\nu = \int_W |z-1|\,|z+1| d\nu_t \\ &\ge \int_W |z-1|^2 d\nu_t = \int_W (z^2-2z+1) d\nu_t = 2(1-\int_W z d\nu_t).\end{aligned}$$

Now

$$\int_W z d\nu_t = \int_W e^{\frac{1}{2t}a\cdot x-\frac{1}{4t}|a|^2} d\nu_t = e^{\frac{t}{8}|a|^2-\frac{1}{4t}|a|^2} = e^{-\frac{1}{8t}|a|^2}$$

from which it follows that

$$\|\nu_t^a - \nu_t\| \ge 2(1-e^{-\frac{1}{8}|a|^2}). \tag{5.7}$$

This proof is of course not rigorous. However the idea is right and in fact the same type of computations show

$$\begin{aligned}\|\nu_t^a - \nu_t\| &= \sup\{(\nu_t^a - \nu_t)(f) : f \text{ bounded and measurable}\} \\ &\ge \sup\{(\nu_t^a - \nu_t)(f) : f(x) = F(x_1, \dots, x_N) \text{ bounded and measurable}\} \\ &= \int_W \left|e^{\frac{1}{t}a^N\cdot x-\frac{1}{2t}|a^N|^2} - 1\right| d\nu_t(x) \ge 2(1-e^{-\frac{1}{8t}|a^N|^2}).\end{aligned}$$

Letting $N \to \infty$ in this estimate shows that Eq. (5.7) is indeed valid and in particular if $a \notin \ell^2$ we have $\|\nu_t^a - \nu_t\| = 2$. ■

Corollary 5.8 (A Cameron type integration by parts formula)**.** *For $h_1, \dots, h_n \in \ell^2$ and $f, g \in \mathcal{F}C_c^\infty(W)$,*

$$\nu_t((\partial_{h_1} \dots \partial_{h_n} f) \cdot g) = \nu_t\left(f \cdot \partial_{h_n}^* \dots \partial_{h_1}^* g\right)$$

where $\partial_h^* = -\partial_h + t^{-1}M_{h\cdot x}$ and $\partial_h(k \cdot x)$ is to be interpreted as $k \cdot h$.

Proof. (Sketch.) From Eq. (5.5) and (5.6),

$$\int_W f(x+sh)g(x+sh)d\nu_t(x) = \int_W f(x)g(x)e^{\frac{s}{t}h\cdot x - \frac{s^2}{2t}|h|^2} d\nu_t(x).$$

Differentiating this equation in s and evaluating at $s=0$ shows

$$\int_W \left[\partial_h f(x)g(x) + f(x)\partial_h g(x)\right] d\nu_t(x) = \int_W \frac{1}{t}(h\cdot x) f(x)g(x)d\nu_t(x),$$

i.e. $\partial_h^* = -\partial_h + t^{-1}M_{h\cdot x}$. The analytic details are left to the reader or see, for example, [22]. ■

6. Classical Wiener Measure

Notation 6.1 (Path Spaces)**.** Given a pointed Riemannian manifold (M,g,o), let

$$W(M) = \{\sigma \in C([0,1]\to M) \,|\, \sigma(0) = o\}. \tag{6.1}$$

For those $\sigma \in W(M)$ which are absolutely continuous, let

$$E_M(\sigma) := \int_0^1 |\sigma'(s)|_g^2 \, ds \tag{6.2}$$

denote the **energy of** σ. The space of **finite energy paths** $H(M)$ is given by

$$H(M) := \{\sigma \in W(M) | \sigma \text{ is absolutely continuous and } E_M(\sigma) < \infty\}. \tag{6.3}$$

Notation 6.2. If M is an inner product space we will always take $o = 0 \in M$ and g to be the Riemannian metric associated to the inner product on M. The supremum norm,

$$\|\omega\| = \max_{s\in[0,1]} |\omega(s)|,$$

makes the **Wiener space** $W(M)$ into a Banach space. The **Cameron − Martin space** $H(M)$ becomes a Hilbert space when equipped with the inner product

$$(h,k) = (h,k)_{H(M)} := \int_0^1 (h'(s), k'(s))_M \, ds \text{ for all } h,k \in H(M).$$

The associated Hilbertian norm $h \to \sqrt{(h,h)}$ on $H(M)$ will be denoted by $|h|$.

Definition 6.3. A function $f : W(M) \to \mathbb{C}$ is a C^k − cylinder function ($f \in \mathcal{F}C^k(W)$) provided there exists a partition

$$\pi := \{0 = s_0 < s_1 < \cdots < s_n = 1\} \tag{6.4}$$

of $[0,1]$ and a smooth function $F \in C^k(M^n)$ such that

$$f(\sigma) = F(\sigma(s_1), \ldots, \sigma(s_n)) = F(\sigma|_\pi). \tag{6.5}$$

As usual we will say $f \in \mathcal{F}C_c^k(W(M))$ or $f \in \mathcal{F}BC^k(W(M))$ if $F \in C_c^k(M^n)$ or $F \in BC^k(M^n)$ respectively.

For the rest of this section we are going to take $M = \mathbb{R}^d$. (The case where M is a more general manifold will be considered in Sections 7 and 8 below.)

Definition 6.4 (Differential Operators)**.** For $f \in C^2(W(\mathbb{R}^d))$ and $h \in H(\mathbb{R}^d)$ let $\partial_h f(\omega) := \frac{d}{dt}|_0 f(\omega + th)$ and $\operatorname{grad} f(\omega) \in H(\mathbb{R}^d)$ for the unique element in $H(\mathbb{R}^d)$ such that $\partial_h f(\omega) = (\operatorname{grad} f(\omega), h)$ for all $h \in H(\mathbb{R}^d)$. We also let S be an orthonormal basis for $H(\mathbb{R}^n)$ and define $\triangle_{H(\mathbb{R}^d)} f := \sum_{h\in S} \partial_h^2 f$ whenever the sums converge.

See Proposition 6.11 below for an explicit description of grad$f$ and $\triangle_{H(\mathbb{R}^d)}f$ when f is a cylinder function. The existence (and the hard) part of the following theorem is due to N. Wiener [58].

Theorem 6.5 (Wiener 1923). *There exits a unique heat kernel sequence*[3] $\{\nu_t\}_{t>0}$ *based at* $0 \in W = W(\mathbb{R}^d)$ *satisfying*

(1) $\nu_t(W) = 1$ *for all* $t > 0$ *and*
(2) *for all* $f \in \mathcal{F}BC^2(W)$, *the function* $t \to \nu_t(f)$ *is continuously differentiable,*

$$\frac{d}{dt}\nu_t(f) = \frac{1}{2}\nu_t(\triangle_{H(\mathbb{R}^d)}f) \text{ and } \lim_{t\downarrow 0}\nu_t(f) = f(0).$$

Remark 6.6. The existence proof in subsection 6.3 below will show that ν_t is concentrated on $\alpha -$ Hölder continuous paths for any $\alpha < 1/2$. It is also well known that $\nu_t -$ lives on the set of nowhere differentiable paths. It is not our aim here to study the sample path properties of ν_t in any detail. The reader interested in such matters is referred to the very nice survey article of Y. Peres' [54].

Before going into the proof of Theorem 6.5 we need to pause to develop the differential calculus on $H(\mathbb{R}^d)$. The uniqueness assertion will be proved in subsection 6.2 and the existence assertion will be proved in subsection 6.3 below.

6.1. Differential Calculus on H.

In what follows, for notational simplicity, we will often state and/or prove results in the special case, $d = 1$ in which case we write $W = W(\mathbb{R}^1)$ and $H = H(\mathbb{R}^1)$. The reader is invited to fill in the details for $d > 1$ which are omitted.

Proposition 6.7. *Let* $G(s,t) = \min(s,t) = s \wedge t$. *Then* G *is the* ***reproducing kernel*** *for* H, *i.e.* $(G(s,\cdot), h) = h(s)$ *for all* $s \in [0,1]$ *and* $h \in H$.

Proof. For $h \in H$,

$$h(t) = \int_0^t h'(s)ds = \int_0^1 1_{s\le t}h'(s)ds = (G(s,\cdot),h)$$

where

$$\partial G(s,t)/\partial s = 1_{s\le t}$$

and therefore $G(s,t) = \int_0^s 1_{r\le t}dr = s \wedge t$. ■

Remark 6.8. $G(s,t)$ is the Green's function for $-d^2/ds^2$ with Dirichlet boundary conditions at 0 and Neumann boundary conditions at 1.

Corollary 6.9 (A simple Sobolev embedding Theorem). *The inclusion map* $i : H \to W$ *is continuous and in fact*

$$\|h\|_W \le \|h\|_H \text{ for all } h \in H.$$

Proof. By Proposition 6.7, for $s \in [0,1]$,

$$|h(s)| = |(G(s,\cdot),h)_H| \le \|G(s,\cdot)\|_H \|h\|_H .$$

This proves the Proposition since

$$\|G(s,\cdot)\|_H^2 = \int_0^1 (1_{t\le s})^2 dt = s \le 1.$$

■

[3]Wiener did not state the theorem this way, but the results are equivalent.

Corollary 6.10. *Let $S \subset H$ be* ***any*** *Orthonormal basis for H. Then*

$$\sum_{h \in S} h(s)h(t) = G(s,t).$$

Proof. The proof is simply Parsavel's equality along with the reproducing kernel properties of G,

$$\sum_{h \in S} h(s)h(t) = \sum_{h \in S} (G(s,\cdot),h)(G(t,\cdot),h) = (G(s,\cdot),G(t,\cdot)) = G(s,t).$$

■

Proposition 6.11. *Suppose that $f \in \mathcal{F}C^2(W)$, then*

$$\mathrm{grad} f(\omega) = \sum_{i=1}^{n} \partial_i F(\omega(s_1),\dots,\omega(s_n))G(s_i,\cdot) \tag{6.6}$$

and

$$\Delta_H f(\omega) = \sum_{i,j=1}^{n} G(s_i,s_j)\partial_i\partial_j F(\omega(s_1),\dots,\omega(s_n)) =: \Delta_\pi F(\omega|_\pi). \tag{6.7}$$

If f is expressed as

$$f(\omega) = F(\delta_1\omega,\dots,\delta_n\omega) \tag{6.8}$$

where $\delta_i\omega = \omega(s_i) - \omega(s_{i-1})$ for $i = 1,2,\dots,n$, then (with $\delta_i := s_i - s_{i-1}$)

$$\Delta_H f(\omega) = \sum_{i=1}^{n} \delta_i \left(\partial_i^2 F\right)(\delta_1\omega,\dots,\delta_n\omega). \tag{6.9}$$

Proof. By definition,

$$\partial_h f(\omega) = \sum_{i=1}^{n} h(s_i)\partial_i F(\omega(s_1),\dots,\omega(s_n)) = \sum_{i=1}^{n} \partial_i F(\omega(s_1),\dots,\omega(s_n))\,(G(s_i,\cdot),h)$$

and

$$\begin{aligned}\Delta_H f(\omega) &= \sum_{h \in S} \partial_h^2 f(\omega) = \sum_{h \in S}\sum_{i,j=1}^{n} h(s_i)h(s_j)\partial_i\partial_j F(\omega(s_1),\dots,\omega(s_n)) \\ &= \sum_{i,j=1}^{n} G(s_i,s_j)\partial_i\partial_j F(\omega(s_1),\dots,\omega(s_n))\end{aligned}$$

which proves Eqs. (6.6) and (6.7). Similarly, if f is given as in Eq. (6.8), then

$$\begin{aligned}\partial_h f(\omega) &= \sum_{i=1}^{n} \left(h(s_i) - h(s_{i-1})\right)\partial_i F(\delta_1\omega,\dots,\delta_n\omega) \\ \mathrm{grad} f(\omega) &= \sum_{i=1}^{n} \partial_i F(\delta_1\omega,\dots,\delta_n\omega)\left(G(s_i,\cdot) - G(s_i,\cdot)\right)\end{aligned}$$

and

$$\begin{aligned}\Delta_H f(\omega) &= \sum_{h\in S} \partial_h^2 f(\omega) = \sum_{h\in S}\sum_{i,j=1}^{n} \left(h(s_i) - h(s_{i-1})\right)\left(h(s_j) - h(s_{j-1})\right)\partial_i\partial_j F(\delta_1\omega,\dots,\delta_n\omega) \\ &= \sum_{i,j=1}^{n} \left(s_i \wedge s_j - s_i \wedge s_{j-1} - s_{i-1}\wedge s_j + s_{i-1}\wedge s_{j-1}\right)\partial_i\partial_j F(\delta_1\omega,\dots,\delta_n\omega) \\ &= \sum_{i=1}^{n} \delta_i \partial_i^2 F(\delta_1\omega,\dots,\delta_n\omega).\end{aligned}$$

■

Remark 6.12. The operators Δ_π are all elliptic. Indeed, if $\vec{\xi} = (\xi_i)_{i=1}^n \in \mathbb{R}^n$ then

$$\sum_{i,j=1}^{n} G(s_i,s_j)\xi_i\xi_j = \sum_{i,j=1}^{n} (G(s_i,\cdot),G(s_i,\cdot))\xi_i\xi_j = \left\|\sum_{i=1}^{n} G(s_i,\cdot)\xi_i\right\|^2 \ge 0$$

with equality iff $h(s) = \sum_{i=1}^n G(s_i,s)\xi_i$ is zero. This would imply

$$0 = h'(s) = \sum_{i=1}^{n} 1_{s\le s_i}\xi_i \text{ for } s \notin \pi$$

from which it easily follows that $\xi = 0$.

Proposition 6.13. *Let π be a partition of $[0,1]$ as in Eq. (6.4), $F \in C_c^2(\mathbb{R}^n)$, then*

$$e^{t\Delta_\pi/2}F(0) = \int_{\mathbb{R}^n} F(x_1,\dots,x_n)\left[\prod_{i=1}^{n} p_{t(s_i-s_{i-1})}(x_{i-1},x_i)\right] dx_1\cdots dx_n \tag{6.10}$$

where p is the heat kernel on $\mathbb{R}^d$ as in Proposition 2.5 and Δ_π is defined in Eq. (6.7).

Proof. If $F(x_1,\dots,x_n) = G(x_1, x_2 - x_1 \dots, x_n - x_{n-1})$ then

$$\Delta_\pi F(x_1,\dots,x_n) = \sum_{i=1}^{n} (s_i - s_{i-1})\left(\partial_i^2 G\right)(x_1, x_2 - x_1\dots, x_n - x_{n-1}). \tag{6.11}$$

This may be deduced from Proposition 6.11 or proved directly as follows. By the chain rule

$$\partial_i F(x_1,\dots,x_n) = \left[\left(\partial_i - \partial_{i+1}\right)G\right](x_1, x_2 - x_1\dots, x_n - x_{n-1})$$

where by convention $\partial_{n+1} = 0$. Hence, with $x = (x_1,\dots,x_n)$ and $y = (x_1, x_2 - x_1\dots, x_n - x_{n-1})$

$$\begin{aligned}\Delta_\pi F(x) &= \sum_{i,j} s_i \wedge s_j \left[\left(\partial_i - \partial_{i+1}\right)\left(\partial_j - \partial_{j+1}\right)G\right](y) \\ &= 2\sum_{i<j} s_i \left[\left(\partial_i - \partial_{i+1}\right)\left(\partial_j - \partial_{j+1}\right)G\right](y) + \sum_i s_i\left[\left(\partial_i - \partial_{i+1}\right)^2 G\right](y) \\ &= 2\sum_i s_i\left[\left(\partial_i - \partial_{i+1}\right)\partial_{i+1}G\right](y) + \sum_i s_i\left[\left(\partial_i - \partial_{i+1}\right)^2 G\right](y)\end{aligned}$$

where we have used a telescoping series to compute the sum on j. Elementary algebra now shows

$$\Delta_\pi F(x) = \sum_i s_i \left[(\partial_i - \partial_{i+1})(\partial_i + \partial_{i+1}) G\right](y) = \sum_i s_i \left[\left(\partial_i^2 - \partial_{i+1}^2\right) G\right](y)$$

which is equivalent to Eq. (6.11) after re-indexing the second term in the last sum.

A consequence of Eq. (6.11) is

$$\begin{aligned}\left(e^{t\Delta_\pi/2}F\right)(0) &= \left(e^{\frac{t}{2}\sum_{i=1}^n (s_i - s_{i-1})\partial_i^2} G\right)(0) \\ &= \int_{(\mathbb{R}^d)^n} G(y_1, \dots, y_n) \prod_{i=1}^k p_{(s_i - s_{i-1})t}(y_i) dy_1 \cdots dy_n. \\ &= \int_{(\mathbb{R}^d)^n} F(y_1, y_1 + y_2, \dots, x_n) \prod_{i=1}^k p_{(s_i - s_{i-1})t}(y_i) dy_1 \cdots dy_n.\end{aligned}$$

Making the change of variable $y = (x_1, x_2 - x_1 \dots, x_n - x_{n-1})$ in the previous integral gives

$$\begin{aligned}\left(e^{t\Delta_\pi/2}F\right)(0) &= \int_{(\mathbb{R}^d)^n} G(x_1, x_2 - x_1 \dots, x_n - x_{n-1}) \prod_{i=1}^k p_{(s_i - s_{i-1})t}(x_i - x_{i-1}) dx_1 \cdots dx_n \\ &= \int_{(\mathbb{R}^d)^n} F(x_1, x_2 - x_1 \dots, x_n - x_{n-1}) \prod_{i=1}^k p_{(s_i - s_{i-1})t}(x_i - x_{i-1}) dx_1 \cdots dx_n\end{aligned}$$

as desired. ■

6.2. **Properties of $\{\nu_t\}_{t>0}$.** In this section we will develop some of the basic properties of the heat kernel sequence $\{\nu_t\}_{t>0}$ in Theorem 6.5.

Proposition 6.14 (Uniqueness of Heat Kernel Measures on W). *Suppose $\{\nu_t\}_{t>0}$ is a heat kernel sequence based at $0 \in W = W(\mathbb{R}^d)$ as in Theorem 6.5 and $f \in B\mathcal{F}C^2(W)$ is a cylinder function as in Eq. (6.5) then*

$$\nu_t(f) = \int_{(\mathbb{R}^d)^n} F(x_1, \dots, x_n) \left[\prod_{i=1}^n p_{t(s_i - s_{i-1})}(x_{i-1}, x_i)\right] dx_1 \cdots dx_n. \tag{6.12}$$

In particular if $\{\nu_t\}_{t>0}$ exists then it is uniquely determined by Eq. (6.12).

Proof. The proof follows in the same manner as the proof of uniqueness in Proposition 5.4 and making use of Proposition 6.13. ■

Let $E(x) := \int_0^1 |x'(s)|^2 ds$ be the energy of a path $x \in H(\mathbb{R}^d)$, then (as in Remark 4.5) we have informally

$$\text{“}\nu_t(dx) = \frac{1}{Z_t} e^{-\frac{1}{2t}E(x)} dm_H(x).\text{”} \tag{6.13}$$

Proposition 6.17 below makes this formula precise.

Notation 6.15. To each partition π of $[0,1]$ let

$$H_\pi(\mathbb{R}^d) := \left\{\omega \in H(\mathbb{R}^d) : \omega''(s) = 0 \text{ if } s \notin \pi\right\}$$

and for $\omega \in W\left(\mathbb{R}^d\right)$ let $\omega_\pi \in H_\pi(\mathbb{R}^d)$ denote the unique element of $H_\pi(\mathbb{R}^d)$ such that $\omega_\pi(s) = \omega(s)$ for all $s \in \pi$.

Lemma 6.16. *The mapping $h \in H\left(\mathbb{R}^d\right) \to h_\pi \in H_\pi(\mathbb{R}^d) \subset H(\mathbb{R}^d)$ is orthogonal projection onto $H_\pi(\mathbb{R}^d)$.*

Proof. Since it is clear that $h_\pi = h$ for $h \in H_\pi\left(\mathbb{R}^d\right)$, we need only prove $(h_\pi, k) = (h, k_\pi)$ for all $h, k \in H\left(\mathbb{R}^d\right)$. If π is a partition as in Eq. (6.4), then

$$h'_\pi = \sum_{i=0}^{n-1} \frac{h(s_{i+1}) - h(s_i)}{s_{i+1} - s_i} 1_{(s_i, s_{i+1}]}$$

and hence

$$(h_\pi, k) = \sum_{i=0}^{n-1} \frac{h(s_{i+1}) - h(s_i)}{s_{i+1} - s_i} \cdot (k(s_{i+1}) - k(s_i))$$

which is clearly symmetric in h and k. ■

Proposition 6.17. *Suppose $\{\nu_t\}_{t>0}$ is a heat kernel sequence based at $0 \in W = W(\mathbb{R}^d)$, π is a partition of $[0,1]$ and f is a cylinder function written as $f(\omega) = F(\omega_\pi)$ with ω_π as in Notation 6.15. Let m_π denote a Lebesgue measure on $H_\pi(\mathbb{R}^d)$ (i.e. any non-trivial translation invariant measure on $H_\pi(\mathbb{R}^d)$) then*

$$\int_{W(\mathbb{R}^d)} f(\omega) d\nu_t(\omega) = \frac{1}{Z_\pi(t)} \int_{H_\pi(\mathbb{R}^d)} f(h) e^{-\frac{1}{2t}E(h)} dm_\pi(h) \tag{6.14}$$

where $Z_\pi(t)$ is a normalization constant chosen so that

$$d\nu_t^\pi(h) := \frac{1}{Z_\pi(t)} e^{-\frac{1}{2t}E(h)} dm_\pi(h) \tag{6.15}$$

is a probability measure.

Proof. First notice that the measure ν_t^π is independent of the possible choices of m_π since translation invariant measures are unique up to a multiplicative constant and this ambiguity of the constant is cancelled by the normalization constant $Z_\pi(t)$. For each $x \in \left(\mathbb{R}^d\right)^n$ let h_x denote the unique element of $H(\mathbb{R}^d)$ such that $h_x(s_i) = x_i$ for $i = 1, 2, \ldots n$. The mapping $x \in \left(\mathbb{R}^d\right)^n \to h_x \in H_\pi(\mathbb{R}^d)$ is a vector space isomorphism with the property that (with $\delta_i := s_i - s_{i-1}$ and $\delta_i x := x_i - x_{i-1}$)

$$E(h_x) = \sum_{i=1}^{n} \frac{\left|x_i - x_{i-1}\right|^2}{\delta_i^2} \delta_i = \sum_{i=1}^{n} \frac{\left|\delta_i x\right|^2}{\delta_i}$$

and hence

$$\begin{aligned}
\prod_{i=1}^{n} p_{t(s_i - s_{i-1})}(x_{i-1}, x_i) &= \prod_{i=0}^{n-1} \frac{1}{\left(2\pi t \delta_i\right)^{d/2}} e^{-\frac{1}{2t\Delta_i}|\delta_i x|^2} \\
&= \prod_{i=0}^{n-1} \frac{1}{\left(2\pi t \delta_i\right)^{d/2}} \cdot e^{-\frac{1}{2t}E(h_x)}.
\end{aligned}$$

So if we now fix m_π by requiring m_π to be the push forward of Lebesgue measure on $\left(\mathbb{R}^d\right)^n$ under the map $x \to h_x$ we have shown

$$\begin{aligned}
\int_{W(\mathbb{R}^d)} f(\omega) d\nu_t(\omega) &= \int_{(\mathbb{R}^d)^n} F(x_1, \ldots, x_n) \prod_{i=1}^{n} p_{t(s_i - s_{i-1})}(x_{i-1}, x_i) dx_1 \cdots dx_n \\
&= \int_{H_\pi(\mathbb{R}^d)} f(h) \prod_{i=0}^{n-1} \frac{1}{\left(2\pi t \delta_i\right)^{d/2}} \cdot e^{-\frac{1}{2t}E(h_x)} dm_\pi(h).
\end{aligned}$$

■

Corollary 6.18. *Let f be a bounded and continuous function on $W\left(\mathbb{R}^d\right)$ relative to the sup-norm topology, then*

$$\int_{W(\mathbb{R}^d)} f(\omega)d\nu_t(\omega) = \lim_{|\pi|\to 0}\int_{H_\pi(\mathbb{R}^d)} f(h)d\nu_t^\pi(h).$$

Proof. For each partition π and $\omega \in W(\mathbb{R}^d)$ let $\omega_\pi \in H_\pi(\mathbb{R}^d)$ be as in Notation 6.15. Then by uniform continuity, $\omega_\pi(s) \to \omega(s)$ uniformly in s as $|\pi| \to 0$ and so by the dominated convergence theorem,

$$\int_{W(\mathbb{R}^d)} f(\omega)d\nu_t(\omega) = \lim_{|\pi|\to 0}\int_{W(\mathbb{R}^d)} f(\omega_\pi)d\nu_t(\omega) = \lim_{|\pi|\to 0}\int_{H_\pi(\mathbb{R}^d)} f(h)d\nu_t^\pi(h)$$

wherein we have used Proposition 6.17 for the second equality. ■

Exercise 6.1. Use the following outline to show $\nu_t \perp \nu_s$ if $s \neq t$. To simplify notation assume $d = 1$. For $n \in \mathbb{N}$ let $\pi_n := \{k2^{-n} : k = 0, 1, 2\dots, 2^n\}$ and define

$$\xi_k(\omega) = \left|\omega\left((k+1)\,2^{-n}\right) - \omega\left(k2^{-n}\right)\right|^2 - 2^{-n}t$$

and

$$S_n(\omega) := \sum_{k=0}^{2^n-1}\left|\omega\left((k+1)\cdot 2^{-n}\right) - \omega\left(k\cdot 2^{-n}\right)\right|^2.$$

(1) Show $\nu_t(\xi_k) = 0$, $\nu_t(\xi_k\xi_j) = 0$ if $k \neq j$ and $\nu_t(\xi_k^2) = 3\cdot 2^{-2n}t^2$.
(2) Use 1. to conclude for any $\epsilon > 0$,

$$\nu_t\left(|S_n - t| > \epsilon\right) \le \epsilon^{-2}\int_W |S_n(\omega) - t|^2\, d\nu_t(\omega) = 3\cdot 2^{-n}t^2\epsilon^{-2}.$$

(3) Use 2. to conclude for any $\epsilon > 0$ that

$$\sum_{n=1}^{\infty} 1_{|S_n(\omega)-t|>\epsilon} < \infty \text{ for } \nu_t - \text{a.e. } \omega.$$

(4) Use 3. to conclude that $\nu_t(W_t) = 1$ where

$$W_t := \left\{\omega \in W(\mathbb{R}) : \lim_{n\to\infty}\sum_{k=0}^{2^n-1}\left|\omega\left((k+1)\,2^{-n}\right) - \omega\left(k2^{-n}\right)\right|^2 = t\right\}.$$

(5) Observe that $W_t \cap W_s = \emptyset$ if $s \neq t$.

Proposition 6.19 (Itô integral)**.** *Suppose $h \in H(\mathbb{R}^d)$, π is a partition of $[0,1]$ as in Eq. (6.4) and for $\omega \in W\left(\mathbb{R}^d\right)$ let $\omega_\pi \in H_\pi\left(\mathbb{R}^d\right)$ be as in Notation 6.15. Then for each $t > 0$, the limit of the function, $\omega \to (h, \omega_\pi)$, exist in $L^2(\nu_t)$ as $|\pi| \to 0$. By abuse of notation we will write this $L^2(\nu_t)$ limit as*

$$L^2(\nu_t) - \lim_{|\pi|\to 0}(h, \omega_\pi) =: (h, \omega) = \int_0^1 h'(s)\cdot d\omega(s).$$

*(**Warning**: As in Lemma 5.6, the limit will in general depend on $t > 0$.) Furthermore, for all $h \in H\left(\mathbb{R}^d\right)$ and bounded measurable functions $F : \mathbb{R} \to \mathbb{R}$,*

$$\int_{W(\mathbb{R}^d)} F\left((h,\omega)\right)d\nu_t(\omega) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} F\left(\sqrt{t}\,|h|\,x\right)e^{-x^2/2}dx. \tag{6.16}$$

Proof. Before starting the proof let us notice that

$$(h,\omega_\pi)=\int_0^1 h'(s)\cdot\omega_\pi'(s)ds=\sum_{i=0}^{n-1}\left[\frac{h(s_{i+1})-h(s_i)}{s_{i+1}-s_i}\right]\cdot\left[\omega(s_{i+1})-\omega(s_i)\right]. \tag{6.17}$$

For $h\in\cup_\pi H_\pi\left(\mathbb{R}^d\right)$ (the union being over all partitions of $[0,1]$) with

$$h'(s)=\sum_{i=0}^{n-1}a_i 1_{(s_i,s_{i+1}]}(s) \text{ for a.e. } s \tag{6.18}$$

let

$$n_h(\omega):=\sum_{i=0}^{n-1}a_i\cdot\left[\omega(s_{i+1})-\omega(s_i)\right].$$

The reader is invited to check that $n_h(\omega)$ is well defined independent of how h' is written in the form given in Eq. (6.18). Making use of Propositions 6.13 and 6.14, we have for $a,b\in\mathbb{R}^d$,

$$\int_{W(\mathbb{R}^d)} a\cdot\left[\omega(s_{i+1})-\omega(s_i)\right]b\cdot\left[\omega(s_{j+1})-\omega(s_j)\right]d\nu_t(\omega)=t\delta_{ij}a\cdot b\left(s_{i+1}-s_i\right).$$

and therefore

$$\int_{W(\mathbb{R}^d)}\left[n_h(\omega)\right]^2 d\nu_t(\omega)=t\sum_{i=0}^{n-1}\left|a_i\right|^2\left(s_{i+1}-s_i\right)=t\left|h\right|^2.$$

This shows that the map

$$h\in\cup_\pi H_\pi\left(\mathbb{R}^d\right)\to n_h\in L^2(\nu_t)$$

is a bounded linear map and hence extends uniquely to $H\left(\mathbb{R}^d\right)=\overline{\cup_\pi H_\pi\left(\mathbb{R}^d\right)}$. If $h\in H\left(\mathbb{R}^d\right)$ is chosen so that $h_\pi\in H_\pi\left(\mathbb{R}^d\right)$ with $h_\pi(s)=h(s)$ for all $s\in\pi$, then Eq. (6.17) shows $n_{h_\pi}(\omega)=(h,\omega_\pi)$. So to finish the proof it suffices to prove, for all $h\in H$, $h_\pi\to h$ in H as $|\pi|\to 0$.

When $k\in C^1\left([0,1],\mathbb{R}^d\right)\cap H\left(\mathbb{R}^d\right)$, $k_\pi'\to k'$ uniformly and therefore $k_\pi\to k$ in H as $|\pi|\to 0$. For general $h\in H\left(\mathbb{R}^d\right)$ and $k\in C^1\left([0,1],\mathbb{R}^d\right)\cap H\left(\mathbb{R}^d\right)$ we have

$$\begin{aligned}\lim\sup_{|\pi|\to\infty}|h-h_\pi| &\le \lim\sup_{|\pi|\to\infty}\left(|h-k|+|k-k_\pi|+\left|(k-h)_\pi\right|\right)\\ &\le |h-k|+\lim\sup_{|\pi|\to\infty}\left|(k-h)_\pi\right|\le 2\left|h-k\right|\end{aligned} \tag{6.19}$$

wherein the last equality we have used Lemma 6.16 to conclude $\left|(k-h)_\pi\right|\le|h-k|$. Letting $k\to h$ in Eq. (6.19) completes the proof of existence of the $L^2(\nu_t)-$ limit.

Since probability measures on $\mathbb{R}$ are uniquely characterized by their Fourier transform, it suffices to to prove Eq. (6.16) in the case that $F(x)=e^{i\lambda x}$ for some $\lambda\in\mathbb{R}$. Now choose a sequence of partitions π_n such that $|\pi_n|\to 0$ and $\lim_{n\to\infty}(h,\omega_{\pi_n})=(h,\omega)$ for ν_t- a.e. ω. Then using Lemma 4.4 and Corollary 6.18,

$$\begin{aligned}\int_{W(\mathbb{R}^d)} e^{i\lambda(h,\omega)}d\nu_t(\omega)&=\lim_{n\to\infty}\int_{W(\mathbb{R}^d)} e^{i\lambda(h,\omega_{\pi_n})}d\nu_t(\omega)=\lim_{n\to\infty}\int_{H_{\pi_n}(\mathbb{R}^d)} e^{i\lambda(h,k)}d\nu_t^{\pi_n}(k)\\ &=\lim_{n\to\infty}e^{-\frac{t}{2}\lambda^2|h_{\pi_n}|^2}=e^{-\frac{t}{2}\lambda^2|h|^2}=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\lambda\left(\sqrt{t}|h|x\right)}e^{-x^2/2}dx.\end{aligned}$$

Items 2. and 3. of the following theorem may be found in [7, 8, 9]. ■

Theorem 6.20 (Cameron – Martin Theorem and Integration by Parts Formula)**.** *For $h \in W\left(\mathbb{R}^d\right)$, let $\nu_t^h := \nu_t(\cdot - h)$, i.e.*

$$\int_{W(\mathbb{R}^d)} f(\omega) d\nu_t^h(\omega) = \int_{W(\mathbb{R}^d)} f(\omega) \nu_t\left(d\omega - h\right) := \int_{W(\mathbb{R}^d)} f(\omega + h) d\nu_t\left(\omega\right).$$

(1) *If $h \in W\left(\mathbb{R}^d\right) \setminus H\left(\mathbb{R}^d\right)$ then $\nu_t^h \perp \nu_t$.*
(2) *If $h \in H\left(\mathbb{R}^d\right)$ then $\nu_t^h \ll \nu_t$ and*

$$\frac{d\nu_t^h}{d\nu_t}(\omega) = e^{\frac{1}{t}(h,\omega) - \frac{1}{2t}|h|^2} = \exp\left(\frac{1}{t}\int_0^1 h'(s) \cdot d\omega(s) - \frac{1}{2t}|h|^2\right).$$

(3) *For all $h \in H\left(\mathbb{R}^d\right)$, $\partial_h^* = \left(-\partial_h + \frac{1}{t}(h,\omega)\right)$, i.e.*

$$\int_{W(\mathbb{R}^d)} \partial_h f(\omega) \cdot g(\omega) d\nu_t(\omega) = \int_{W(\mathbb{R}^d)} f(\omega)\left(-\partial_h + \frac{1}{t}(h,\omega)\right) g(\omega) d\nu_t(\omega).$$

In particular ν_t is a smooth measure.

Proof. We will not give the proof of item 1. here which is similar to the proof of the corresponding result in Proposition 5.7. For item 2., first suppose $f \in BC(W\left(\mathbb{R}^d\right))$, $h \in H_\pi\left(\mathbb{R}^d\right)$ for some partition π of $[0,1]$ and let π_n be a sequence of partitions containing π such that $|\pi_n| \downarrow 0$. Since $H_\pi\left(\mathbb{R}^d\right) \subset H_{\pi_n}\left(\mathbb{R}^d\right)$, by Corollary 6.18 and the translation invariance of finite dimensional Lebesgue measure,

$$\begin{aligned}
\int_{W(\mathbb{R}^d)} f(\omega + h) d\nu_t\left(\omega\right) &= \lim_{n\to\infty} \int_{H_{\pi_n}(\mathbb{R}^d)} f(k+h) d\nu_t^{\pi_n}\left(k\right) \\
&= \lim_{n\to\infty} \int_{H_{\pi_n}(\mathbb{R}^d)} f(k+h) \frac{1}{Z_{\pi_n}(t)} e^{-\frac{1}{2t}E(k)} dm_{\pi_n}(k) \\
&= \lim_{n\to\infty} \int_{H_{\pi_n}(\mathbb{R}^d)} f(k) \frac{1}{Z_{\pi_n}(t)} e^{-\frac{1}{2t}E(k-h)} dm_{\pi_n}(k) \\
&= \lim_{n\to\infty} \int_{H_{\pi_n}(\mathbb{R}^d)} f(k) e^{-\frac{1}{2t}\left[-2(h,k)+|h|^2\right]} d\nu_t^{\pi_n}\left(k\right) \\
&= \int_{W(\mathbb{R}^d)} f(\omega) e^{-\frac{1}{2t}\left[-2(h,\omega)+|h|^2\right]} d\nu_t\left(\omega\right).
\end{aligned}$$

For general $h \in H\left(\mathbb{R}^d\right)$, the previous result proves

$$\int_{W(\mathbb{R}^d)} f(\omega + h_\pi) d\nu_t\left(\omega\right) = \int_{W(\mathbb{R}^d)} f(\omega) e^{\frac{1}{t}(h_\pi,\omega) - \frac{1}{2t}|h_\pi|^2} d\nu_t\left(\omega\right). \tag{6.20}$$

By the dominated convergence theorem,

$$\lim_{|\pi|\to 0} \int_{W(\mathbb{R}^d)} f(\omega + h_\pi) d\nu_t\left(\omega\right) = \int_{W(\mathbb{R}^d)} f(\omega + h) d\nu_t\left(\omega\right) \tag{6.21}$$

while for any $p \geq 1$,

$$\begin{aligned}
\epsilon_\pi &= \int_{W(\mathbb{R}^d)} \left| e^{-\frac{1}{2t}\left[-2(h,\omega)+|h|^2\right]} - e^{-\frac{1}{2t}\left[-2(h_\pi,\omega)+|h_\pi|^2\right]} \right|^p d\nu_t\left(\omega\right) \\
&= \int_{W(\mathbb{R}^d)} e^{\frac{p}{t}(h,\omega) - \frac{p}{2t}|h|^2} \left| 1 - e^{\frac{p}{t}(h_\pi - h,\omega) - \frac{p}{2t}\left[|h_\pi|^2 - |h|^2\right]} \right|^p d\nu_t\left(\omega\right).
\end{aligned}$$

Hence by the Cauchy Schwarz inequality and Eq. (6.16), $\epsilon_\pi^2 \le AB_\pi$ where

$$A = \int_{W(\mathbb{R}^d)} e^{\frac{2p}{t}(h,\omega)-\frac{2p}{2t}|h|^2} d\nu_t(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{2p}{t}\sqrt{t}|h|x-\frac{2p}{2t}|h|^2} e^{-x^2/2} dx < \infty$$

and

$$\begin{aligned} B_\pi &= \int_{W(\mathbb{R}^d)} \left|1 - e^{\frac{p}{t}(h_\pi - h,\omega)-\frac{p}{2t}\left[|h_\pi|^2-|h|^2\right]}\right|^{2p} d\nu_t(\omega) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left|1 - e^{\frac{p}{t}\sqrt{t}|h_\pi - h|x-\frac{p}{2t}\left[|h_\pi|^2-|h|^2\right]}\right|^{2} e^{-x^2/2} dx. \end{aligned}$$

The dominated convergence theorem now shows $\lim_{|\pi|\to 0} B_\pi = 0$ and hence $\epsilon_\pi \to 0$ so that
(6.22)

$$\lim_{|\pi|\to 0} \int_{W(\mathbb{R}^d)} f(\omega) e^{\frac{1}{t}(h_\pi,\omega)-\frac{1}{2t}|h_\pi|^2} d\nu_t(\omega) = \int_{W(\mathbb{R}^d)} f(\omega) e^{\frac{1}{t}(h,\omega)-\frac{1}{2t}|h|^2} d\nu_t(\omega).$$

Combining Eqs. (6.20) – (6.22) shows

$$\int_{W(\mathbb{R}^d)} f(\omega + h) d\nu_t(\omega) = \int_{W(\mathbb{R}^d)} f(\omega) e^{\frac{1}{t}(h,\omega)-\frac{1}{2t}|h|^2} d\nu_t(\omega) \tag{6.23}$$

for all $h \in H(\mathbb{R}^d)$ and $f \in BC(W(\mathbb{R}^d))$. By general measure theoretic arguments it now follows that Eq. (6.23) holds for all bounded measurable functions f on $W(\mathbb{R}^d)$ and this proves item 2. Lastly item 3. is proved similarly to the proof of Corollary 5.8. ■

Notation 6.21. It is customary to call the measure $\mu := \nu_1$ – **Wiener measure** on $W(\mathbb{R}^d)$.

Remark 6.22. From Proposition 6.17, it is easily shown that $d\nu_t$ is the measure on $W(\mathbb{R}^d)$ determined by

$$\int_{W(\mathbb{R}^d)} f(\omega) d\nu_t(\omega) = \int_{W(\mathbb{R}^d)} f(\sqrt{t}\omega) d\mu(\omega)$$

and this equation clearly shows that

$$\lim_{t\downarrow 0} \nu_t(f) = \lim_{t\downarrow 0} \int_{W(\mathbb{R}^d)} f(\sqrt{t}\omega) d\mu(\omega) = f(0)$$

for all $f \in BC(W(\mathbb{R}^d))$. It is also interesting to note that we can deduce from Theorem 6.20 that $\partial_t \nu_t(f) = \nu_t(\frac{1}{2}\Delta_H f)$. Indeed let $f \in \mathcal{F}BC^2(W(\mathbb{R}^d))$ be a cylinder function and S be an orthonormal basis for $H(\mathbb{R}^d)$, then

$$\begin{aligned} \partial_t \nu_t(f) &= \int_{W(\mathbb{R}^d)} \frac{1}{2\sqrt{t}} (\nabla f(\sqrt{t}\omega), \omega) d\mu(\omega) = \sum_{h\in S} \int_{W(\mathbb{R}^d)} \frac{1}{2\sqrt{t}} (\nabla f(\sqrt{t}\omega), h)(h,\omega) d\mu(\omega) \\ &= \sum_{h\in S} \int_{W(\mathbb{R}^d)} \frac{1}{2\sqrt{t}} (\partial_h f)(\sqrt{t}\omega) \partial_h^* 1 d\mu(\omega) = \sum_{h\in S} \int_{W(\mathbb{R}^d)} \frac{1}{2\sqrt{t}} \sqrt{t} \left(\partial_h^2 f\right)(\sqrt{t}\omega) d\mu(\omega) \\ &= \nu_t(\frac{1}{2}\Delta_H f). \end{aligned}$$

6.3. **Construction of $\{\nu_t\}_{t>0}$ on $W\left(\mathbb{R}^d\right)$.** Our construction of ν_t will be based on the ideas in Proposition 6.17 and Corollary 6.18.

Notation 6.23. For each $n \in \mathbb{N}_0$ let

$$\pi_n := \left\{k2^{-n} : k = 0, 1, 2 \dots, 2^n\right\}$$

and let $V_0 := H_{\pi_0}\left(\mathbb{R}^d\right) = \left\{h \in H(\mathbb{R}^d) : h'' = 0\right\}$ and for $n \geq 1$ let V_n denote the orthogonal complement of $H_{\pi_{n-1}}(\mathbb{R}^d)$ in $H_{\pi_n}(\mathbb{R}^d)$.

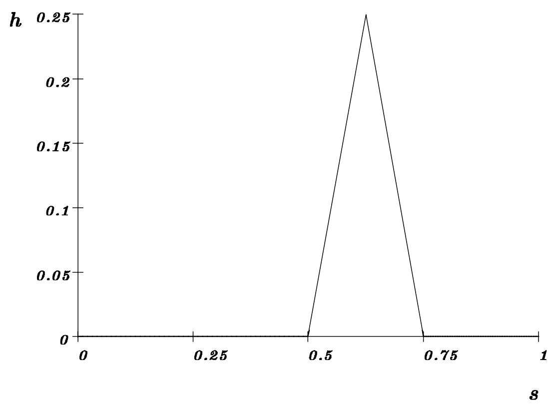

FIGURE 1. This function is a typical element of V_3. It is the function h_3 as described in Lemma 6.24 below with $n = 3$.

Lemma 6.24. *Using the notation above:*

(1) *Suppose π is a partition of $[0,1]$ then $h \in H_\pi(\mathbb{R}^d)^\perp$ iff $h|_\pi = 0$.*
(2) *For $n \geq 1$, $V_n = \left\{h \in H_{\pi_n}(\mathbb{R}^d) : h|_{\pi_{n-1}} = 0\right\}$.*
(3) *$H\left(\mathbb{R}^d\right) = \oplus_{n=0}^\infty V_n$ with the sum being the Hilbert space orthogonal direct sum.*
(4) *Given $n \geq 1$ and $0 \leq k < 2^{n-1}$, let $h_k \in V_n(\mathbb{R})$ be the unique "tent" function (see Figure 1) such that*

$$h_k|_{[0,k2^{-n+1}]\cup[(k+1)2^{-n+1},1]} = 0 \text{ and } h_k((k+\frac{1}{2})2^{-n+1}) = 2^{-\frac{n+1}{2}}.$$

Then $\left\{h_k e_j : 0 \leq k < 2^{n-1}, j = 1, \dots, d\right\}$ is an orthonormal basis for $V_n\left(\mathbb{R}^d\right)$.

Proof. Items 1. and 2. Suppose $h \in H_\pi(\mathbb{R}^d)^\perp$, $s \in \pi$ and $a \in \mathbb{R}^d$. Let $k(t) = G(s,t)a$, then $k \in H_\pi(\mathbb{R}^d)$ and hence $0 = (h,k) = h(s) \cdot a$ from which it follows $h|_\pi = 0$. Any easy computation using the fundamental theorem of calculus shows that if $h|_\pi = 0$ and $k \in H_\pi(\mathbb{R}^d)$, then $(h,k) = 0$.

Item 3. If $m \neq n$, V_n and V_m are orthogonal subspaces and $H_{\pi_n}\left(\mathbb{R}^d\right) = \oplus_{k=0}^n V_k$ by construction. If $h \in H\left(\mathbb{R}^d\right)$ and $h \perp V_n$ for all n, then $h \perp H_{\pi_n}\left(\mathbb{R}^d\right)$ for all n. So by item 1., $h(s) = 0$ on all dyadic rationals in $[0,1]$. Since the later set is dense in $[0,1]$ and h is continuous, $h \equiv 0$ and this completes the proof of item 3.

Item 4. is a simple verification left to the reader. ■

Lemma 6.25. *Let $\{\nu_t^n\}_{t>0}$ denote the heat kernel sequence on $V_n := V_n(\mathbb{R})$ at 0 and for $q \in [1,\infty)$ let*

$$C(q) := \left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |x|^q e^{-\frac{1}{2}x^2}dx\right)^{1/q} = \sqrt{2}\left[\pi^{-\frac{1}{2}}\Gamma\left(\frac{q+1}{2}\right)\right]^{1/q} < \infty. \tag{6.24}$$

Then for any $\rho, p \in [1,\infty)$,

$$\left(\int_{V_n} \|h\|_\infty^p \, d\nu_t^n(h)\right)^{1/p} \le C(p\rho)t^{1/2}2^{-n\left(\frac{1}{2}-\frac{1}{\rho p}\right)}. \tag{6.25}$$

Moreover for $\alpha \in (0,1)$ let

$$\|h\|_\alpha := \sup\left\{\frac{|h(t)-h(s)|}{|t-s|^\alpha} : t,s \in [0,1] \text{ with } t \neq s\right\}$$

then we have

$$\left(\int_{V_n} \|h\|_\alpha^p \, d\nu_t^n(h)\right)^{1/p} \le C(p\rho)t^{1/2}2^{-(n-1)\left(\frac{1}{2}-\alpha-\frac{1}{p\rho}\right)}. \tag{6.26}$$

Proof. Let $\{h_k : k < 2^{n-1}\}$ be as in Lemma 6.24 and set $\xi_k(h) := (h_k, h)$. Then $h = \sum_{k<2^{n-1}} \xi_k(h)\, h_k$ for all $h \in V_n$. Since the sets $\{h_k \neq 0\}_{k=0}^{2^{n-1}}$ are disjoint and $\|h_k\|_\infty = 2^{-\frac{n+1}{2}}$, it follows that $\|h\|_\infty = 2^{-\frac{n+1}{2}}M_n(h)$ where

$$M_n(h) := \max\left\{|\xi_k(h)| : k < 2^{n-1}\right\}.$$

For any $q \ge 1$,

$$\begin{aligned}\left(\int_{V_n} M_n^q(h) d\nu_t^n(h)\right)^{1/q} &\le \left(\int_{V_n} \sum_{k<2^{n-1}} |\xi_k(h)|^q \, d\nu_t^n(h)\right)^{1/q}\\ &= \left(2^{n-1}\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} |x|^q e^{-\frac{1}{2t}x^2}dx\right)^{1/q} = C(q)t^{1/2}2^{(n-1)/q} \qquad (6.27)\end{aligned}$$

where $C(q)$ is defined in Eq. (6.24). (The second equality in Eq. (6.24)is a consequence of the integrand being even and the change of variables $u = x^2/2$.)

Therefore for any $\rho, p \ge 1$,

$$\begin{aligned}\left(\int_{V_n} \|h\|_\infty^p \, d\nu_t^n(h)\right)^{1/p} &\le \left(\int_{V_n} \|h\|_\infty^{p\rho} \, d\nu_t^n(h)\right)^{1/p\rho} \le 2^{-\frac{n+1}{2}}\left(\int_{V_n} M_n^{p\rho}(h) d\nu_t^n(h)\right)^{1/p\rho}\\ &\le C(p\rho)t^{1/2}2^{\frac{n-1}{p\rho}}2^{-\frac{n+1}{2}} \le C(p\rho)t^{1/2}2^{\frac{n}{p\rho}}2^{-\frac{n}{2}}\end{aligned}$$

which proves Eq. (6.25).

Since $\left\|\dot{h}_k\right\|_\infty = 2^{-\frac{n+1}{2}}/2^{-n} = 2^{\frac{n-1}{2}}$, $\dot{h} = \sum_{k<2^{n-1}} \xi_k(h)\, \dot{h}_k$, and $\{\dot{h}_k\}$ have essentially disjoint supports, it follows that $\left\|\dot{h}\right\|_\infty = 2^{\frac{n-1}{2}}M_n(h)$. By the mean value theorem,

$$\frac{|h(t)-h(s)|}{|t-s|^\alpha} \le \frac{\left\|\dot{h}\right\|_\infty |t-s|}{|t-s|^\alpha} = 2^{\frac{n-1}{2}}M_n(h)\,|t-s|^{1-\alpha}. \tag{6.28}$$

From Eq. (6.28), if $|t-s| \le 2^{-(n-1)}$,

$$\frac{|h(t)-h(s)|}{|t-s|^\alpha} \le 2^{\frac{n-1}{2}}M_n(h)2^{-(1-\alpha)(n-1)} = M_n(h)2^{-(n-1)\left(\frac{1}{2}-\alpha\right)}$$

and if $|t-s| \geq 2^{-(n-1)}$,

$$\frac{|h(t)-h(s)|}{|t-s|^{\alpha}} \leq \frac{2\|h\|_{\infty}}{2^{-\alpha(n-1)}} \leq 2\frac{2^{-\frac{n+1}{2}}M_n(h)}{2^{-\alpha(n-1)}} = M_n(h)2^{-(n-1)\left(\frac{1}{2}-\alpha\right)}.$$

The previous two equations imply

$$\|h\|_{\alpha} \leq M_n(h)2^{-(n-1)\left(\frac{1}{2}-\alpha\right)}.$$

Therefore for any $\rho \geq 1$ (working as above)

$$\begin{aligned}\left(\int_{V_n}\|h\|_{\alpha}^{p}\,d\nu_t^n(h)\right)^{1/p} &\leq 2^{-(n-1)\left(\frac{1}{2}-\alpha\right)}\left(\int_{V_n}M_n^{p\rho}(h)d\nu_t^n(h)\right)^{1/p\rho}\\ &\leq 2^{-(n-1)\left(\frac{1}{2}-\alpha\right)}C(p\rho)t^{1/2}2^{\frac{n-1}{p\rho}}\end{aligned}$$

which proves Eq. (6.26). ■

6.3.1. *Existence proof of* $\{\nu_t\}_{t>0}$ *in Theorem 6.5.* Again for simplicity of notation we carry out the proof when $d=1$. Let $\Omega := \prod_{n=0}^{\infty} V_n$, $P_t := \prod_{n=0}^{\infty}\nu_t^n$ (see Fact 5.1) and for $n \in \mathbb{N}$ let $S_n : \Omega \to V_n$ be the natural projection onto V_n. Then by construction, $\{S_n\}_{n=0}^{\infty}$ are all mutually independent and $(S_n)_* P_t = \nu_t^n$ for each $n \in \mathbb{N}$. If $\alpha < \frac{1}{2}$ and $p \in [1,\infty)$ we may choose $\rho \in [1,\infty)$ such that $\frac{1}{2}-\alpha-\frac{1}{p\rho} > 0$. Then by Lemma 6.25,

$$\left\|\sum_{n=0}^{\infty}\|S_n\|_{\alpha}\right\|_{L^p(P_t)} \leq \sum_{n=0}^{\infty}\left\|\|S_n\|_{\alpha}\right\|_{L^p(P_t)} \leq C(p\rho)t^{1/2}\sum_{n=0}^{\infty}2^{-(n-1)\left(\frac{1}{2}-\alpha-\frac{1}{p\rho}\right)} < \infty.$$

This shows: 1) $\sum_{n=0}^{\infty}\|S_n\|_{\alpha} < \infty$ for P_t – a.e. and hence $S := \sum_{n=0}^{\infty} S_n$ exists in $C^{0,\alpha}([0,1])$ off a P_t – null set and 2) $S := \sum_{n=0}^{\infty} S_n$ converges in $L^p(\Omega, P_t, C^{0,\alpha}([0,1]))$ for all $p \in [1,\infty)$.

Thus the measure $\nu_t := S_* P_t$ is a probability measure on W which is supported on $C^{0,\alpha}([0,1])$ for any $\alpha < 1/2$ and satisfies

$$\int_W \|\omega\|_{\alpha}^{p}\,d\nu_t(\omega) < \infty \text{ for all } p \in [1,\infty).$$

It now only remains to show $\{\nu_t\}_{t>0}$ is the desired heat kernel sequence. To this end, suppose that $f \in \mathcal{F}C_c^2(W)$ with $f(\omega) = F(\omega_{\pi})$ for some partition π of $[0,1]$. For each n, let $p_n : W \to H_{\pi_n}(\mathbb{R})$ be defined by $p_n(\omega) = \omega_{\pi_n}$ as defined in Notation 6.15. and set

$$g_n(t) = \int_W f \circ p_n d\nu_t = \int_W f \circ p_n(\sqrt{t}\omega)d\nu_1(\omega).$$

Then one shows using Proposition 6.14 and the semigroup $p_t * p_s = p_{t+s}$ that

$$\begin{aligned} g_n(t) &\to g(t) = \int_W f d\nu_t = \nu_t(f) \text{ and}\\ \dot{g}_n(t) &= \frac{1}{2}\int_W \Delta_H\left(f \circ p_n\right)d\nu_t \to \frac{1}{2}\int_W \Delta_H f d\nu_t = \frac{1}{2}\nu_t(\Delta_H f)\end{aligned}$$

uniformly for t in compact subsets of $(0,\infty)$. Hence g is differentiable and $\frac{d}{dt}\nu_t(f) = \frac{1}{2}\nu_t(\Delta_H f)$ as desired.

7. Path and Loop Group Extensions

In this section we will discuss the analogues of the results in Section 6 when $W\left(\mathbb{R}^d\right)$ is replaced by the path $W(K)$ or the loop space $\mathcal{L}(K)$ on a compact Lie group K. Our description of the results in this section will be rather brief compared to the previous sections. This is because to understand these heat kernel measures on $W(K)$ and $\mathcal{L}(K)$ one must understand "Wiener measure" on the path space of $W(K)$ and $\mathcal{L}(K)$ respectively. Section 8, describes these type of results in the simpler setting where $W(K)$ and $\mathcal{L}(K)$ are replaced by a finite dimensional Riemannian manifold M.

Notation 7.1. Let K be a connected compact Lie group, $\mathfrak{k} := T_eK$ be the Lie algebra of K, $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ be an Ad_K-invariant inner product on $\mathfrak{k}$ and let $g := g_{\mathfrak{k}}$ denote the unique bi-invariant Riemannian metric on K which agrees with $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ on $\mathfrak{k} := T_eK$. To simplify notation later we will assume that K is a matrix group in which case $\mathfrak{k}$ may also be viewed as a matrix Lie algebra. (Since K is compact, this is no restriction, see for example Theorem 4.1 on p. 136 in [6].) Elements $A \in \mathfrak{k}$ will be identified with the unique left invariant vector field on K agreeing with A at the identity in K, i.e. if $f \in C^\infty(K)$ then

$$Af(x) = \frac{d}{dt}|_0 f(xe^{tA}).$$

Example 7.2. As an example, let $K = SO(3)$ be the group of 3×3 real orthogonal matrices with determinant 1. The Lie algebra of K is $\mathfrak{k} = so(3)$, the set of 3×3 real skew symmetric matrices, and the inner product $\langle A, B\rangle_{\mathfrak{k}} := -\mathrm{tr}(AB)$ is an example of an Ad_K – invariant inner product on $\mathfrak{k}$.

Our main interest here is the path and loop spaces built on K. In this section, let $M = K$ and $o = e$ ($e \in K$ is the identity element) in Notation 6.1.

Notation 7.3. For a compact Lie group K let

$$W(K) := \{\sigma \in C([0,1] \to K) \,|\, \sigma(0) = e\}, \tag{7.1}$$

$$\mathcal{L}(K) := \{\sigma \in W(K) \,|\, \sigma(1) = e\} \tag{7.2}$$

and $\mathbf{e} \in \mathcal{L}(K) \subset W(K)$ denote the constant path at $e \in K$. As in Notation 6.1, $H(K)$ and $H_0(K)$ are the finite energy paths in $W(K)$ and $\mathcal{L}(K)$ respectively. In this case the energy E_K on $H(K)$ is given explicitly by

$$E_K(\sigma) := \int_0^1 \left|[\sigma(s)]^{-1}\sigma'(s)\right|_{\mathfrak{k}}^2 ds = \int_0^1 \left|\sigma'(s)\sigma(s)^{-1}\right|_{\mathfrak{k}}^2 ds, \tag{7.3}$$

wherein the last equality is a consequence of the Ad_K – invariance of $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$.

As usual we will refer to $H(\mathfrak{k})$ equipped with the Hilbertian inner product,

$$(h,k) := \int_0^1 \langle h'(s), k'(s)\rangle \, ds, \tag{7.4}$$

as the Cameron – Martin Hilbert space.

Remark 7.4. It is well known that $H(K)$ is a Hilbert Lie group under pointwise multiplication and that the map

$$(x,h) \in H(K) \times H(\mathfrak{k}) \to L_{x*}h \in T(H(K))$$

is a trivialization of the tangent bundle of $H(K)$. (We are using $L_x : H(K) \to H(K)$ to denote left multiplication by x.) This trivialization induces a left-invariant Riemannian metric $(\cdot,\cdot)$ on $H(K)$ given explicitly by

$$(L_{x*}h, L_{x*}h) = \int_0^1 \langle h'(s), h'(s)\rangle_{\mathfrak{k}}\, ds \quad \forall\, x \in H(K) \text{ and } h \in H(\mathfrak{k}). \tag{7.5}$$

See Appendix A in [20] and the references therein for more details.

Definition 7.5 (Differential Operators)**.** For $h \in H(\mathfrak{k})$ ($h \in H_0(\mathfrak{k})$), let $\tilde{h}$ denote the left invariant vector field on $W(K)$ ($\mathcal{L}(K)$) such that $\tilde{h}(\mathbf{e}) = h$, i.e. if $f \in C^1(W(K))$ $\left(f \in C^1(\mathcal{L}(K))\right)$ and $x \in W(K)$ ($x \in \mathcal{L}(K)$) then

$$\tilde{h}f(x) = \frac{d}{dt}\Big|_0 f\left(xe^{th}\right),$$

where $\left(xe^{th}\right)(s) = x(s)e^{th(s)}$ for all $s \in [0,1]$. For those $f \in C^2(W(K))$ for which the following sums converge (for example smooth cylinder functions), let

$$\|\mathrm{grad} f\|^2 := \sum_{h\in S}\left(\tilde{h}f\right)^2 \text{ and } \triangle_{H(K)}f := \sum_{h\in S}\tilde{h}^2 f.$$

and

$$\|\mathrm{grad}_0 f\|^2 := \sum_{h\in S_0}\left(\tilde{h}f\right)^2 \text{ and } \triangle_{H_0(K)}f := \sum_{h\in S_0}\tilde{h}^2 f.$$

Here S and S_0 are orthonormal bases for $H(\mathfrak{k})$ and $H_0(\mathfrak{k})$ respectively.

Theorem 7.6 (Heat Kernel Measure)**.** *There exists unique heat kernel sequences $\{\nu_t\}_{t>0}$ and $\{\nu_t^{\mathbf{e}}\}_{t>0}$ based at $\mathbf{e}$ on $W(K)$ and $\mathcal{L}(K)$ respectively, i.e. for all $f \in \mathcal{F}C^2(W(K))$,*

$$\partial_t \nu_t(f) = \frac{1}{2}\nu_t\left(\triangle_{H(K)}f\right) \text{ and } \partial_t \nu_t^{\mathbf{e}}(f) = \frac{1}{2}\nu_t^{\mathbf{e}}\left(\triangle_{H_0(K)}f\right)$$

and

$$\lim_{t\downarrow 0}\nu_t(f) = f(\mathbf{e}) = \lim_{t\downarrow 0}\nu_t^{\mathbf{e}}(f).$$

The reader is referred to Malliavin [52], Driver and Lohrenz [21], and Driver and [18] for the existence of ν_t and $\nu_t^{\mathbf{e}}$.

Theorem 7.7 (Quasi-invariance for heat kernel measure)**.** *For each $k \in H_0(G)$ which is null homotopic, $\nu_t^{\mathbf{e}}$ is quasi-invariant under the right and left translations by k.*

Proof. See Driver [18, 19] and Fang [32, 33]. The free loop space version of these results was carried out by Trevor Carson in [10, 11]. The reader should also see Inahama [43] for generalizations of Theorem 7.8 and Corollary 7.7 to include "H^s – metrics" on $\mathcal{L}(K)$ for $s > 1/2$. ■

Theorem 7.8 (Heat Kernel Logarithmic Sobolev Theorem, [21])**.** *There is a constant $C < \infty$ such that*

$$\int_{\mathcal{L}(K)} f^2 \log\frac{f^2}{\nu_t^{\mathbf{e}}(f^2)}d\nu_t^{\mathbf{e}} \le C\int_{\mathcal{L}(K)} \|grad_0 f\|^2\, d\nu_t^{\mathbf{e}} \tag{7.6}$$

for all smooth cylinder functions $f : \mathcal{L}(K) \to \mathbb{R}$. (Eq. (7.6) when $K = \mathbb{R}^d$ is Gross' original Logarithmic Sobolev inequality.)

Proof. See Driver and Lohrenz [21], Carson [10, 11] and Fang [33]. ■

Remark 7.9. Similar results hold for ν_t and they are much easier to prove.

8. Wiener measure on $W(M)$ and its properties

The proofs of the results in Section 7 rely on properties of "Wiener measure" on $W(\mathcal{L}(K))$ and $W(W(K))$ to deduce properties about the heat kernel measures ν_t and $\nu_t^{\mathbf{e}}$ respectively. This section will describe some of the relevant results needed in the simpler setting where $\mathcal{L}(K)$ $(W(K))$ is replaced by a finite dimensional Riemannian manifold M with a fixed base point $o \in M$. We will continue to use the notation and results from Section 2. In particular $p_t(x,y)$ denotes the heat kernel on M as described just before Theorem 2.6. To simplify the exposition, let us assume M is compact. (Most of the results are valid under the weaker assumption that M is complete and the Ricci curvature is bounded from below.)

Notation 8.1. To each $\sigma \in H(M)$ and $s \in [0,1]$ let $//_s(\sigma) : T_oM \to T_{\sigma(s)}M$ denote **parallel translation** along $\sigma|_{[0,s]}$ relative to the Levi-Civita covariant derivative ∇, i.e. $//_s(\sigma)$ is the unique solution to the ordinary differential equation

$$\frac{\nabla}{ds}//_s(\sigma) = 0 \text{ with } //_0(\sigma) = Id_{T_oM}.$$

Also let $\varphi^\nabla : H(T_oM) \longrightarrow H(M)$ denote **Cartan's rolling map**, defined by $\sigma = \phi^\nabla(\omega)$ where σ is the unique solution to the functional differential equation

$$\sigma'(s) = //_s(\sigma)\omega'(s) \quad \text{with } \sigma(0) = o. \tag{8.1}$$

Remark 8.2. Suppose M is the boundary of a smooth convex region in $\mathbb{R}^3$ equipped with the metric inherited from $\mathbb{R}^3$. Then the curve σ in (8.1) has the interpretation of being the curve on M found by *rolling* M along the curve ω in T_oM. The reader is invited to try this by rolling a balloon along a curve, ω, drawn on a chalk board.

Theorem 8.3 (Wiener measure)**.** *There exists a unique probability measure $\mu_{W(M)}$ on $W(M)$ such that for all cylinder functions $f \in \mathcal{F}C(W(M))$, as described in Definition 6.3,*

$$\int_{W(M)} f(\sigma)d\mu_{W(M)}(\sigma) = \int_{M^n} F(x_1,\dots,x_n)\prod_{i=0}^{n-1} p_{(s_{i+1}-s_i)}(x_i,x_{i+1})dx_1\cdots dx_n. \tag{8.2}$$

where $x_0 = o$ and dx denotes the volume measure on M.

Remark 8.4 (Warning). Comparing Eq. (8.2) with Eq. (6.12) with $t = 1$, the reader may be lead to think that $\mu_{W(M)}$ is a heat kernel measure on $W(M)$. This is however not the case for general Riemannian manifolds M. Of course $\mu_{W(M)}$ is intimately connected to the heat Kernel measures ν_t on M based at $o \in M$ by the formula

$$\nu_t(F) = \int_{W(M)} F(\sigma(t))d\mu_M(\sigma) \text{ for all } F \in C(M). \tag{8.3}$$

It is this relationship which is exploited to prove the results in Section 7.

It turns out that there is another (often more useful) way to construct the measure $\mu_{W(M)}$ which involves solving a "stochastic differential" equation. We will hide this stochastic differential equation in the formulation given in the next theorem.

Theorem 8.5 (Eelles & Elworthy stochastic rolling construction of $\mu_{W(M)}$)**.** *Let $\mu_{W(\mathbb{R}^d)}$ be Wiener measure on $W\left(\mathbb{R}^d\right)$ as in Notation 6.21 and for $\omega \in W\left(\mathbb{R}^d\right)$ and a partition π of $[0,1]$ let $\omega_\pi \in H_\pi\left(\mathbb{R}^d\right)$ be as defined in Notation 6.15. Then $\tilde{\phi}(\omega) := \lim_{|\pi|\to 0}\phi(\omega_\pi)$ exists for $\mu_{W(\mathbb{R}^d)}$ – a.e. ω and moreover $\mu_{W(M)} = \tilde{\phi}_*\mu_{W(\mathbb{R}^d)} := \mu_{W(\mathbb{R}^d)} \circ \tilde{\phi}^{-1}$. In words, $\mu_{W(M)}$ is the push – forward of Wiener measure $\mu_{W(\mathbb{R}^d)}$ on $W\left(\mathbb{R}^d\right)$ by the "stochastic" extension $\tilde{\phi}$ of Cartan's rolling map.*

Proof. The fact that $\tilde{\phi}$ has a "stochastic extension" seems to have first been observed by Eells and Elworthy [23] who used ideas of Gangolli [36]. The relationship of the stochastic development map to stochastic differential equations on the orthogonal frame bundle $O(M)$ of M is pointed out in Elworthy [24, 25, 26]. The frame bundle point of view has also been developed by Malliavin, see for example [50, 49, 51]. For a more detailed history of the stochastic development map, see pp. 156–157 in Elworthy [26]. ■

Proposition 8.6 (Stochastic parallel translation)**.** *There exists a continuous process, $(\sigma, s) \in W(M) \times [0,1] \to \widetilde{//}_s(\sigma) \in End\,(T_oM, TM)$, such that $\widetilde{//}_s(\sigma) \in End\left(T_oM, T_{\sigma(s)}M\right)$ for all σ and s and*

$$\widetilde{//}_s(\tilde{\phi}(\omega)) = \mu_{W(\mathbb{R}^d)}\text{-}\lim_{|\pi|\to 0} //_s(\phi\,(\omega_\pi)),$$

where the limit is taken in the sense of $\mu_{W(\mathbb{R}^d)}$ – measure.

Theorem 8.7 (Cameron-Martin Theorem for M)**.** *Let $h \in H(T_oM)$ and X^h be the $\mu_{W(M)}$ – a.e. well defined vector field on $W(M)$ given by*

$$X^h_s(\sigma) = //_s(\sigma)h(s) \quad \text{for } s \in [0,1]. \tag{8.4}$$

Then X^h admits a flow e^{tX^h} on $W(M)$ and this flow leaves $\mu_{W(M)}$ quasi-invariant.

This theorem first appeared in Driver [17] when $h \in H\,(T_oM) \cap C^1([0,1], T_oM)$ and was soon extended to all $h \in H\,(T_oM)$ by E. Hsu [40, 41]. Other proofs may also be found in [30, 48, 53].

Corollary 8.8 (Integration by Parts for $\mu_{W(M)}$)**.** *For $h \in H(T_oM)$ and $f \in \mathcal{F}C^1(W(M))$ be as in Eq. (6.5), let*

$$(X^hf)(\sigma) = \frac{d}{dt}|_0 f(e^{tX^h}(\sigma)) = \sum_{i=1}^n (\nabla_i f(\sigma), X^h_{s_i}(\sigma))_g = \sum_{i=1}^n (\nabla_i f)(\sigma), \widetilde{//}_{s_i}(\sigma)h(s_i))_g.$$

Then

$$\int_{W(M)} X^hf\,d\mu_{W(M)} = \int_{W(\mathbb{R}^d)} f\,(\tilde{\phi}(\omega))z^h(\omega)\,d\mu_{W(\mathbb{R}^d)}(\omega)$$

where

$$z^h(\omega) := \int_0^1 \langle h'(s) + \frac{1}{2}\mathrm{Ric}_{\tilde{//}_s(\tilde{\phi}(\omega))}h'(s), d\omega(s)\rangle$$

and $\mathrm{Ric}_{\tilde{//}_s(\sigma)} := \tilde{//}_s(\sigma)^{-1}\mathrm{Ric}_{\sigma(s)}\tilde{//}_s(\sigma) \in End(T_oM)$ and Ric is the Ricci tensor on TM.

Proof. A special case of this type of theorem for $f(\sigma) = F(\sigma(s))$ for some $F \in C^\infty(M)$ first appeared in Bismut [4]. The result stated here was proved in [17]. Other proofs of this corollary may be found in [1, 2, 18, 28, 29, 27, 30, 31, 40, 41, 46, 47, 48, 53]

■

Example 8.9. When $M = \mathbb{R}^d$ then $//_s(\sigma)v_o = v_{\sigma(s)}$ for all $v \in \mathbb{R}^d$ and $\sigma \in W(\mathbb{R}^d)$. Thus $X^h_s(\sigma) = (h(s))_{\sigma(s)}$ and $e^{tX^h}(\sigma) = \sigma + th$ and so Theorem 8.7 becomes the classical Cameron-Martin Theorem, see Theorem 6.20 with $t = 1$.

8.1. **Path Integral Interpretation.** In this subsection we will state a couple of analogues of Proposition 6.17.

Notation 8.10. Given a partition π of $[0, 1]$, let

$$H_\pi(M) = \{\sigma \in H(M) \cap C^2(I \setminus \pi) : \nabla\sigma'(s)/ds = 0 \text{ for } s \notin \pi\}$$

be the piecewise geodesics paths in $H(M)$ which change directions only at the partition points.

It is possible to check that $H_\pi(M)$ is a finite dimensional submanifold of $H(M)$ which is in fact diffeomorphic to $\left(\mathbb{R}^d\right)^n$. For $\sigma \in H_\pi(M)$, the tangent space $T_\sigma H_\pi(M)$ can be identified with elements $X \in T_\sigma H_\pi(M)$ satisfying the Jacobi equations on $I \setminus \pi$.

Definition 8.11 (The π–Metrics)**.** For each partition π of $[0, 1]$ as in Eq. (6.4) let G^1_π be the metric on $TH_\pi(M)$ given by

$$G^1_\pi(X, Y) := \sum_{i=1}^{n} \left(\frac{\nabla X(s_{i-1}+)}{ds}, \frac{\nabla Y(s_{i-1}+)}{ds}\right)_g (s_i - s_{i-1}) \tag{8.5}$$

for all $X, Y \in T_\sigma H_\pi(M)$ and $\sigma \in H_\pi(M)$. (We are writing $\frac{\nabla X(s_{i-1}+)}{ds}$ as a shorthand for $\lim_{s\downarrow s_{i-1}} \frac{\nabla X(s)}{ds}$.) Similarly, let G^0_π be the *degenerate* metric on $H_\pi(M)$ given by

$$G^0_\pi(X, Y) := \sum_{i=1}^{n} (X(s_i), Y(s_i))_g (s_i - s_{i-1}), \tag{8.6}$$

for all $X, Y \in T_\sigma H_\pi(M)$ and $\sigma \in H_\pi(M)$.

Remark 8.12. Notice that G^1_π and G^0_π are the Riemann sum approximations to the metrics,

$$G^1(X, Y) := \int_0^1 \left(\frac{\nabla X(s)}{ds}, \frac{\nabla Y(s)}{ds}\right)_g ds \text{ and } G^0(X, Y) := \int_0^1 (X(s), Y(s))_g\, ds.$$

If N^p is an oriented manifold equipped with a possibly degenerate Riemannian metric G, let Vol_G denote the p–form on N determined by

$$\mathrm{Vol}_G(v_1, v_2, \ldots, v_p) := \sqrt{\det\left(\{G(v_i, v_j)\}^p_{i,j=1}\right)}, \tag{8.7}$$

where $\{v_1, v_2, \ldots, v_p\} \subset T_nN$ is an oriented basis and $n \in N$. We will identify a p–form on N with the Radon measure induced by the linear functional $f \in C_c(N) \to \int_N f\mathrm{Vol}_G$.

Definition 8.13 (π – Volume Forms)**.** Let $\mathrm{Vol}_{G^0_\pi}$ and $\mathrm{Vol}_{G^1_\pi}$ denote the volume forms on $H_\pi(M)$ determined by G^0_π and G^1_π in accordance with equation (8.7).

Given the above definitions, there are now two natural finite dimensional "approximations" to $\mu_{W(M)}$ in equation (7.4) given in the following definition.

Definition 8.14 (Approximates to Wiener Measure)**.** For each partition $\pi = \{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$ of $[0,1]$, let μ_π^0 and μ_π^1 denote measures on $H_\pi(M)$ defined by

$$\mu_\pi^0 := \frac{1}{Z_\pi^0} e^{-\frac{1}{2}E_M} \mathrm{Vol}_{G_\pi^0}$$

and

$$\mu_\pi^1 = \frac{1}{Z_\pi^1} e^{-\frac{1}{2}E_M} \mathrm{Vol}_{G_\pi^1},$$

where $E_M : H(M) \to [0,\infty)$ is the energy functional defined in equation (6.2) and Z_π^0 and Z_π^1 are normalization constants given by

$$Z_\pi^0 := \prod_{i=1}^n (\sqrt{2\pi}\,(s_i - s_{i-1}))^d \text{ and } Z_\pi^1 := (2\pi)^{dn/2}. \tag{8.8}$$

Theorem 8.15 (Path Integral interpretation of $\mu_{W(M)}$)**.** *Suppose that* $f : W(M) \to \mathbb{R}$ *is bounded and continuous, then*

$$\lim_{|\pi|\to 0} \int_{H_\pi(M)} f(\sigma) d\mu_\pi^1(\sigma) = \int_{\mathrm{W}(M)} f(\sigma) d\mu_{\mathrm{W}(M)}(\sigma) \tag{8.9}$$

and

$$\lim_{|\pi|\to 0} \int_{H_\pi(M)} f(\sigma) d\mu_\pi^0(\sigma) = \int_{\mathrm{W}(M)} f(\sigma) e^{-\frac{1}{6}\int_0^1 \mathrm{Scal}(\sigma(s))ds} d\mu_{\mathrm{W}(M)}(\sigma), \tag{8.10}$$

where Scal *is the scalar curvature of* (M, g).

There is a large literature pertaining to results of the type in Theorem 8.15, see for example [12, 57, 55, 35, 13, 44, 42, 59]. The version given here is contained in Andersson and Driver [3].

9. Motivations

9.1. **Malliavin's Method.**

Malliavin's idea is to embed questions about heat kernels on finite dimensional manifolds into questions about Wiener measure on $W(\mathbb{R}^d)$. In the elliptic (i.e. Riemannian geometry) setting, Equation (8.3) along with Corollary 8.8 may be used as a basis for this method. Malliavin's idea also extends to certain hypoelliptic settings as well. Although this is a strong motivation, I am more motivated by problems related to quantum mechanics and quantum field theories to be described next.

9.2. **Canonical Quantization & Path Integral Quantization.**

Let $q(t) \in \mathbb{R}^d$ describe the motion of a particle of mass m in the force due to a potential function $V(q)$. Then q satisfies Newton's equations of motion,

$$m\ddot{q}(t) = -\nabla V(q(t)).$$

The Lagrangian density associated to this equation is $L(q,v) := \frac{1}{2}m|v|^2 - V(q)$, the momentum p conjugate to v is given by $p = \frac{\partial L(q,v)}{\partial v} = mv$ and the associated Hamiltonian is given by

$$H(q,p) = p \cdot v - L(q,v) \text{ where } p = mv,$$

i.e.

$$H(q,p) = \frac{1}{m}p^2 - \frac{1}{2m}p^2 + V(q) = \frac{1}{2m}p^2 + V(q) = E(q,v) \tag{9.1}$$

where $E(q,v) = \frac{1}{2}m|v|^2 + V(q)$ is the energy of the "state," (q,v). To quantize this system, we should take $K = L^2(\mathbb{R}^d, dm)$ for the quantum Hilbert space and replace q by $Q = M_q$ and p by $P = -i\hbar\nabla_q$. These are the usual "canonical quantization" rules one learns in a quantum mechanics class. Let us summarize the usual story in the following table.

CONCEPT	**CLASSICAL**	**QUANTUM**
CONFIGURATION SPACE	$\mathbb{R}^d$	No analogue
STATE SPACE	$T^*\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d$ $(p,q) =$ Position $\times$ Momentum	$K = PL^2(\mathbb{R}^d, dm)$, i.e. $\psi \in L^2(\mathbb{R}^d, dm) \ni \|\psi\|_K = 1$ and $\psi \sim e^{i\theta}\psi$.
OBSERVABLES **Examples**	Functions $f : T^*\mathbb{R}^d \longrightarrow \mathbb{R}$ p_k q_k $H(q,p) = \frac{1}{2m}p^2 + V(q)$	Self adjoint operators θ on K $\hat{p}_k = \frac{\hbar}{i}\frac{\partial}{\partial q_k}$ $\hat{q}_k = M_{q_k}$ $\hat{H} = -\frac{\hbar^2}{2m}\Delta + V(q)$
DYNAMICS	Newtons Equations of Motion $\ddot{q}(t) = -\nabla V(q(t))$, $q(t) \in \mathbb{R}^d$	Shrödinger Equation $i\hbar\dot{\psi}(t) = H\psi(t)$ $\psi(t) \in K$
MEASUREMENTS	Evaluation of an observable on a state, i.e. $f(q,p)$	$\langle \psi, \theta\psi\rangle$ – expected value of θ in the state ψ.

The **formal** "path integral quantization" of the system described by H in Eq. (9.1) is given by

$$e^{-T\hat{H}}f(x) \text{``}=\text{''} \frac{1}{Z_T}\int_{\omega(0)=x} e^{-\int_0^T E(\omega(t),\dot{\omega}(t))dt} f(\omega(T))\mathcal{D}\omega$$
$$= \frac{1}{Z_T}\int_{\omega(0)=0} e^{-\int_0^T E(x+\omega(t),\dot{\omega}(t))dt} f(x+\omega(T))\mathcal{D}\omega \tag{9.2}$$

where

$$\text{``}Z_T := \int_{\omega(0)=0} e^{-\frac{m}{2}\int_0^T |\dot{\omega}(t)|^2 dt}\mathcal{D}\omega\text{''}$$

is the "normalization constant" chosen so that

$$d\mu(\omega) \text{``}=\text{''} \frac{1}{Z_T} e^{-\frac{m}{2}\int_0^T |\omega(t)|^2 dt}\mathcal{D}\omega \tag{9.3}$$

is a probability "measure". With this notation Eq. (9.2) states

$$e^{-T\hat{H}}f(x) = \int_{\omega(0)=x} f(x+\omega(T))e^{-\int_0^T V(x+\omega(t))dt} d\mu(\omega)$$

which is the Feynman Kac formula. This last formula is in fact rigorous provided one interprets μ as Wiener measure with variance $m^{-1/2}$ on $W(\mathbb{R}^d)$ and some mild restrictions are put on the potential V.

The use of "path integrals" in physics including heuristic expressions like those in equations (9.2) started with Feynman in [34] with very early beginnings being traced back to Dirac [15]. See Section 6 for the correct interpretation of Eq. (9.3).

9.3. **Quantization on Riemannian Manifolds.** Now suppose M is a Riemannian manifold with metric g and $q(t) \in M^d$ describe the motion of a particle in M subject to the force due to a potential function $V(q)$. Then q satisfies Newton's equations of motion,

$$\frac{\nabla \dot{q}(t)}{dt} = -\nabla V(q(t)). \tag{9.4}$$

As before, the Lagrangian density associated to this equations is given by

$$L(q,v) := \frac{1}{2}|v|^2 - V(q) = \frac{1}{2}g_{ij}(q)v^i v^j - V(q)$$

where $v^i = dx^i(v)$ in local coordinates. The corresponding Hamiltonian is given by the Legendre transform,

$$H(q,p) = p_i v^i - L(q,v), \text{ where } p_i = \frac{\partial L(q,v)}{\partial v^i} = g_{ij}(q)v^j$$

and p_i is the conjugate momentum to v^i. So $v^i = g^{ij}(x)p_j$ and hence

$$\begin{aligned} H(q,p) &= p_i v^i - L(q,v) = p_i v^i - \left(\frac{1}{2}g_{ij}(q)v^i v^j - V(q)\right) \\ &= p_i v^i - \left(\frac{1}{2}p_i v^i - V(q)\right) = \frac{1}{2}p_i v^i + V(q) \\ &= \frac{1}{2}g^{ij}(x)p_i p_j + V(q). \end{aligned} \tag{9.5}$$

If $q(t)$ solves Eq. (9.4) and $q^i(t) := x^i(q(t))$ and $p_i(t) := g_{ij}(q(t))\dot{q}^j(t)$ then

$$\dot{q}^j(t) = \frac{\partial H(q,p)}{\partial p_j} \text{ and } \dot{p}_i(t) = -\frac{\partial H(q,p)}{\partial q^i}.$$

We now want to quantize $H(q,p)$ by replacing:

$$p_i \to P_i := \frac{1}{i}\frac{\partial}{\partial x^i} \text{ and } q_i \to Q_i := M_{x^i}$$

where Q_i is multiplication by q_i. Working formally from Eq. (9.5) we conclude

$$\hat{H} = -\frac{1}{2}g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + V(q).$$

This is not a very good answer since it is coordinate dependent. To remedy this, notice at the classical level we could also write

$$H(q,p) = \frac{1}{2}\frac{1}{\sqrt{g}}p_i\sqrt{g}g^{ij}(x)p_j + V(q)$$

which when quantized gives the operator,

$$\hat{H} = -\frac{1}{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\sqrt{g}g^{ij}(x)\frac{\partial}{\partial x^j} + V(q) = -\frac{1}{2}\Delta_M + M_V.$$

The latter expression has the virtue of at least being coordinate independent.

The formal path integral quantization of the above system is given by

$$e^{-T\hat{H}}f(x_0) = \frac{1}{Z_T}\int_{\sigma(0)=x_0} e^{-\int_0^T E(\sigma(t),\dot{\sigma}(t))dt} f(\sigma(T))\mathcal{D}\sigma \tag{9.6}$$

where $E(x,v) = \frac{1}{2}g(v,v) + V(x)$ is the energy. Possible rigorous interpretations of the right side of Eq. (9.6) and its relationship to $e^{-T\hat{H}}$ when $V = 0$ are discussed in Theorem 8.15 above.

9.4. **Quantization of infinite dimensional classical systems.** Quantization of infinite dimensional classical systems leads to infinite dimensional Shrödinger Equations. The simplest of which are standard type heat equations.

9.4.1. *Klein-Gordon Equations.* A non-linear Klein-Gordon equation is a non-linear wave equation of the form,

$$\phi_{tt} + (-\Delta + m^2)\phi + \phi^3 = 0$$

for some function $\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. This may be phrased as $\ddot{\phi} = -\nabla V(\phi)$ where

$$V(\phi) = \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla\phi|^2 + \frac{m^2}{2}\phi^2 + \frac{1}{4}\phi^4 \right) dx$$

The quantization of this equation leads one to consider the partial differential equation in infinitely many variable,

$$\partial_t u(t,\phi) = \frac{1}{2}\Delta_{L^2(\mathbb{R}^d)} u(t,\phi) - V(\phi)u(t,\phi).$$

The formal path integral quantization of this system is given by

$$e^{t\left(\frac{1}{2}\Delta_{L^2(\mathbb{R}^d)} - V\right)} f(\phi_o) = \frac{1}{Z_T} \int_{\phi(0)=\phi_0} e^{-\int_0^T \left[\frac{1}{2}\left\|\dot{\phi}(t)\right\|_{L^2(\mathbb{R}^d)} + V(\phi(t))\right] dt} f(\phi(T)) \mathcal{D}\phi.$$

See Glimm and Jaffe [37] and the references therein for more information about this expression.

9.4.2. *Yang – Mills Equations.* The Yang – Mills equations are the Euler Lagrange equations for

$$I(A) = \int_{\mathbb{R}\times\mathbb{R}^d} \langle F^A \rangle_L^2 dt dx$$

where $F^A = dA + A \wedge A$ and $A : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \otimes \mathfrak{g}$ is a connection one form and $\langle \cdot \rangle_L^2$ is a non-degenerate quadratic form determined by the Lorhenzian metric on $\mathbb{R}^{d+1}$ and an inner product on $\mathfrak{g} = Lie(G)$ and G is a compact Lie group. The corresponding path integral quantization measure is given informally by

$$d\mu(A) = \frac{1}{Z} e^{-\frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^d} |F^A|^2 dt dx} \mathcal{D}A. \tag{9.7}$$

Because of "gauge invariance" of the problem, this measure is really to be defined on the non-linear space of connections modulo gauge transformations, $\mathcal{M}/\mathcal{G}$. Making sense out of Eq. (9.7) is a part of the million dollar Clay Mathematics prize pertaining to quantization of Yang–Mills fields.[4]

When $d = 1$ and $\mathbb{R}^d = \mathbb{R}^1$ is replace by S^1 the space $\mathcal{M}/\mathcal{G}_0$ simply becomes G itself and the path integral in (9.7) reduces to the one like that in Eq. (9.6) with $M = G$ and $V = 0$. See the Driver and Hall [20] for more on this point and the relation to symplectic reduction.

[4] More information about this problem may be found at
http://www.claymath.org/Millennium_Prize_Problems/Yang-Mills_Theory/.

9.5. **Loop spaces.** The loop spaces $\mathcal{L}(K)$ considered in Section 7 is a model of the configuration space in "string theory." The action used in physics is the relativistic area swept out by the string which leads to considering the so called non-linear $\sigma-$ models in the path integral formulation. In Section 7 we considered a more tractable action which leads to reasonable heat equation on $\mathcal{L}(K)$. The heat "kernels" for this heat equation may be thought of as a replacements for the non-existent Lebesgue measure on $\mathcal{L}(K)$. As such one would eventually like to understand the relationship between the analysis on $\mathcal{L}(K)$ and the topology of $\mathcal{L}(K)$, i.e. something like a Hodge deRham theory and index theory for loop spaces.

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