# Heat equations on loop groups

## (AMS Sectional meeting at LSU)

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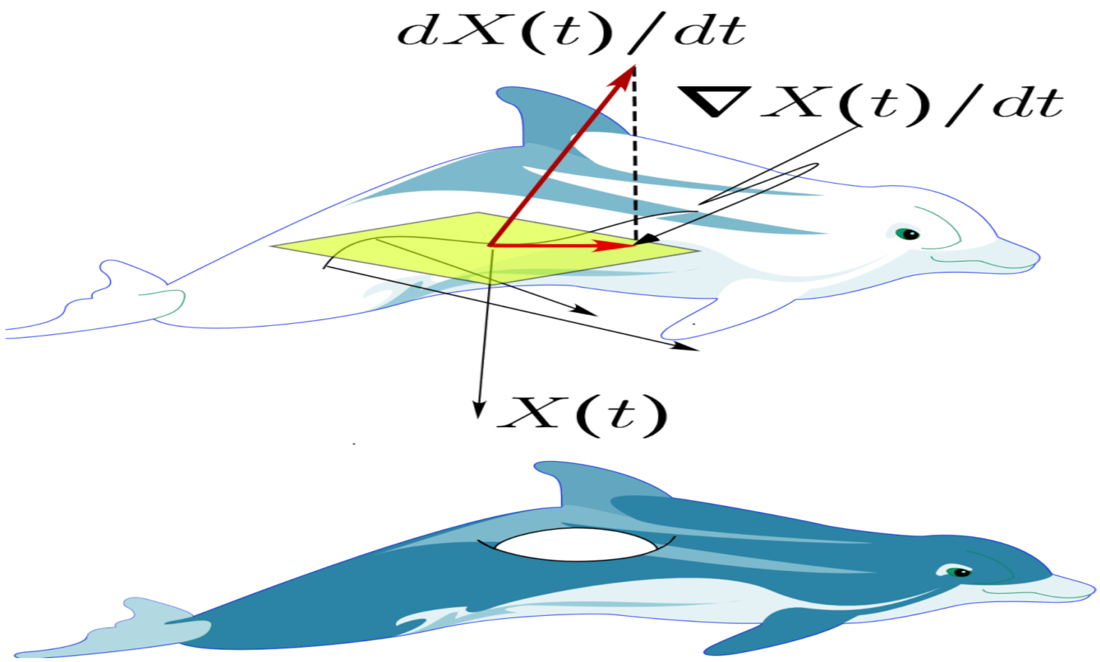
# 1 Heat Kernel Measures on Riemannian Manifolds

**Notation 1** *Suppose* $(M^d, g)$ *is a smooth* $d-$*dimensional manifold with Riemannian metric* $g$ *and* $\nabla$ *denotes the Levi-Civita covariant derivative of* $g$ *and* $\Delta$ *is the* ***Riemannian Laplacian.***

$$\Delta f = \mathrm{tr}(\nabla^2 f) = \sum_{i,j=1}^{d} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial f}{\partial x^j} \right)$$



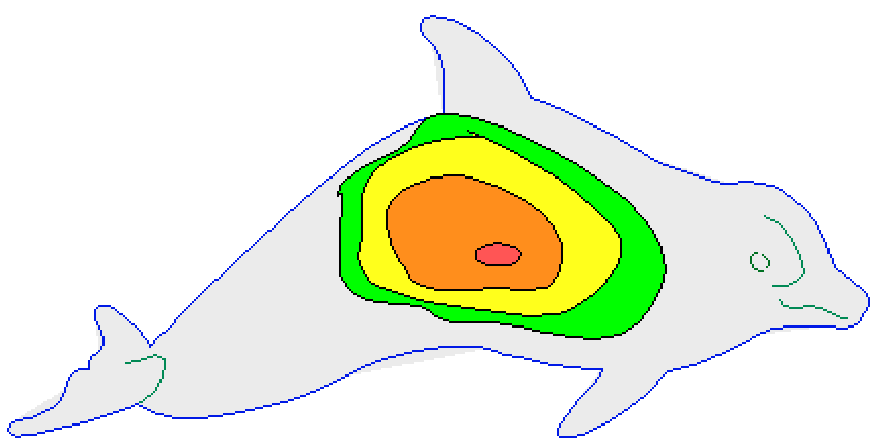
**Definition 2** *Let* $(M, g)$ *be a Riemannian manifold* $o \in M$ *be a fixed base point. A sequence* $\{\nu_t\}_{t>0}$ *of positive measures is called a* ***heat kernel sequence based*** *at* $o \in M$ *if:*

1. $\nu_t(M) \le 1$ for all $t > 0$.
2. For all $f \in BC^2(M)$ the function $t \to \nu_t(f) := \int_M f d\nu_t \in C^1(0, \infty)$ and

$$\frac{d}{dt}\nu_t(f) = \frac{1}{2}\nu_t(\Delta f)$$

and

$$\lim_{t \downarrow 0} \nu_t(f) = f(o).$$



**Remark 1** *If* $\nu_t$ *exists as in Definition 2, then necessarily* $\nu_t(M) = 1$ *for all* $t$*. Just take* $f \equiv 1$ *in the definition.*

# 2 The Case $M = \mathbb{R}^d$.

**Proposition 3** *Suppose*

$$M = \mathbb{R}^d, \ o = 0 \text{ and } \Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$$

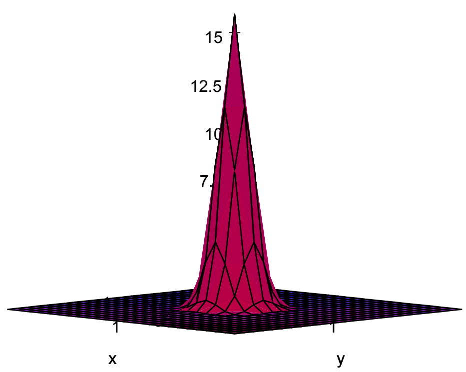
*Then there is exactly one sequence of positive measures* $\{\nu_t\}_{t>0}$ *. Moreover this sequence is given by*

$$\nu_t(dx) = p_t(x) dm(x) \tag{1}$$

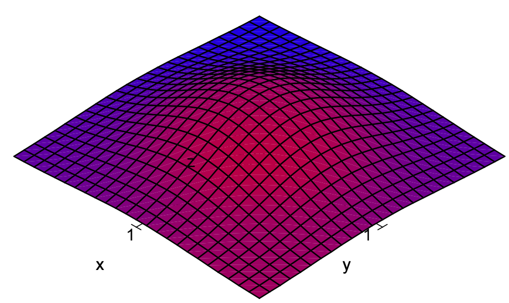
*where*

$$p_t(x) := (2\pi t)^{-d/2} \, e^{-\frac{1}{2t}|x|^2}$$

*is the heat kernel and* $m$ *is Lebesgue measure on* $\mathbb{R}^d$.




$\frac{d\nu_t}{dm}$ at $t = .01$




$\frac{d\nu_t}{dm}$ at $t = .25$

## 2.1 Proof.

**Existence.** Define $\nu_t$ as in Eq. (1), then

$$\begin{aligned}\frac{\partial}{\partial t}\nu_t(f) &= \int_{\mathbb{R}^d} f(x)\frac{\partial}{\partial t}p_t(x)dx = \frac{1}{2}\int_{\mathbb{R}^d} f(x)\Delta p_t(x)dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \Delta f(x)p_t(x)dx = \frac{1}{2}\nu_t(\Delta f).\end{aligned}$$

**Uniqueness.** By assumption $\nu_t$ satisfies

$$\nu_t(f) = f(0) + \int_0^t \frac{1}{2}\nu_\tau(\Delta f)d\tau \text{ for all } f \in C_c^\infty\left(\mathbb{R}^d\right). \tag{2}$$

Taking $T > 0$ and

$$F_t(x) = e^{(T-t)\Delta/2}f(x) = P_{(T-t)}f(x) := \int_{\mathbb{R}^d} p_{T-t}(x-y)f(y)dy$$

then

$$\frac{\partial}{\partial t}F_t = -\frac{1}{2}\Delta F_t \text{ with } F_T = f \text{ and } F_0 = P_T f.$$

So by the chain rule

$$\frac{\partial}{\partial t}\nu_t(F_t) = \frac{1}{2}\nu_t(\Delta F_t) - \frac{1}{2}\nu_t(\Delta F_t) = 0$$

and thus

$$\nu_T(f) = \nu_T(F_T) = \nu_0(F_0) \ = P_T f(0).$$

# 3 General finite dimensional Riemannian Manifolds $M$

**Theorem 4** *Let $o \in M$ be a fixed point and suppose*

- $(M, g)$ is a complete Riemannian manifold
- There exists $C \geq 0$ such that $\mathrm{Ricci} \geq -Cg$.

*Then there exists a unique heat kernel sequence $\{\nu_t\}_{t>0}$ based at $o \in M$. The measure $\nu_t$ is given by*

$$\nu_t(dx) = p_t(o, x)dV(x) \tag{3}$$

*and satisfy*

$$\nu_t(f) := \int_M f d\nu_t =: \left(e^{t\bar{\Delta}/2} f\right)(o) \text{ for all } f \in C_c^\infty(M) \tag{4}$$

*where*

$$p_t(o, x) := \left(e^{t\Delta/2}\delta_o\right)(x).$$

See Strichartz (1983) , Dodziuk (1983) and Davies (1990) for the existence of $p_t$.

# 4 Aside: Smooth Measure on $\mathbb{R}^n$

**Definition 5** *A Radon measure $\mu$ on $\mathbb{R}^d$ is said to be smooth if for all multi -indices $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ ($\mathbb{N} = \{1, 2 \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) there exists functions $g_\alpha \in C^\infty(\mathbb{R}^d) \cap L^{\infty-}(\mu)$ such that,*

$$\int_{\mathbb{R}^d} (-D)^\alpha f d\mu = \int_{\mathbb{R}^d} f g_\alpha d\mu \text{ for all } f \in C_c^\infty(\mathbb{R}^d), \quad (5)$$

*where $D^\alpha := \prod_{i=1}^d \left(\frac{\partial}{\partial x^i}\right)^{\alpha_i}$.*

**Theorem 6** *A measure $\mu$ on $\mathbb{R}^d$ is smooth iff there exists $\rho \in C^\infty(\mathbb{R}^d, (0, \infty))$ such that $d\mu = \rho dm$ where $m$ is Lebesgue measure on $\mathbb{R}^d$.*

**Corollary 7** *All smooth measures on $\mathbb{R}^d$ are mutually absolutely continuous relative to each other.*

**Corollary 8** *If $\mu$ is a smooth measure on $\mathbb{R}^d$ and $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism, then $\phi_*\mu$ is a smooth measure as well. In fact if $d\mu = \rho dm$, then*

$$d(\phi_*\mu) = \rho \circ \phi^{-1} \left| \left(\phi^{-1}\right)' \right| dm. \quad (6)$$

# 5 Infinite dimensional considerations

- Let $(H, (\cdot, \cdot))$ be a separable Hilbert space,
- $|h| := \sqrt{(h, h)}$ be the associate Hilbertian norm and
- $\partial_h f(x) := \frac{d}{dt}|_0 f(x + th)$, $Df(x)h := \partial_h f(x)$
- $D^2 f(x)(h, k) := (\partial_h \partial_k f)(x)$.
- $S \subset H$ be an orthonormal basis for $H$.
- As usual, for $f \in C^2(H)$ we will set
$$\Delta_H f(x) = \mathrm{tr}(D^2 f(x)) = \sum_{h \in S} \left(\partial_h^2 f\right)(x)$$
provided $D^2 f(x)$ is trace class.

**Proposition 9** *There does **not** exists a heat kernel sequence based at $0 \in H$, i.e. there is no collection $\{\nu_t\}_{t>0}$ of positive measures on $H$ such that*

- $\nu_t(H) \leq 1$ for all $t > 0$ and
- $\frac{d}{dt}\nu_t(f) = \frac{1}{2}\nu_t(\Delta_H f)$ with $\lim_{t \downarrow 0} \nu_t(f) = f(0)$.

## 5.1 Heuristic Proof.

Formally,

$$“\nu_t(dx) = \frac{1}{Z_t} e^{-\frac{1}{2t}|x|_H^2} dm_H(x),” \tag{7}$$

where $m_H$ is “infinite dimensional Lebesgue measure,”

and

$$Z_t := (2\pi t)^{\dim(H)/2} = \begin{cases} 0 & \text{if } t < 1/2\pi \\ \infty & \text{if } t > 1/2\pi. \end{cases}$$

## 5.2 Rigorous Proof.

Let $\alpha > 0$, then one shows

$$\begin{aligned} \int_H e^{-\alpha|x|^2} d\nu_t(x) &= \lim_{n\to\infty} \int_H e^{-\alpha|P_n x|^2} d\nu_t(x) \\ &= \lim_{n\to\infty} \int_{P_n H} e^{-\alpha|y|^2} d\nu_t^{P_n H}(y) \\ &= \lim_{n\to\infty} \left(\frac{1}{2t\alpha+1}\right)^{\dim(P_n H)/2} = 0. \end{aligned}$$

Since $e^{-\alpha|x|^2} > 0$ for all $x \in H$, this implies that

$\nu_t \equiv 0$. This clearly **contradicts** the assumption that

$$\lim_{t\downarrow 0} \nu_t(f) = f(0).$$

# 6 Path Spaces

**Notation 10 (Path Spaces)** *Given a pointed Riemannian manifold* $(M, g, o)$, *let*

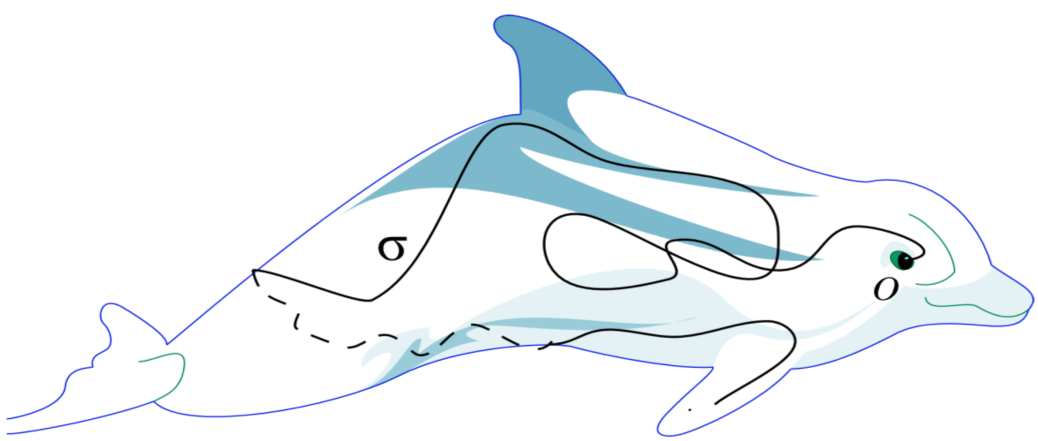
$$W(M) = \{\sigma \in C([0,1] \to M) \,|\, \sigma(0) = o\}.$$

*For those* $\sigma \in W(M)$ *which are absolutely continuous, let*

$$E_M(\sigma) := \int_0^1 |\sigma'(s)|_g^2 \, ds$$

*denote the* ***energy of*** $\sigma$. *The space of* ***finite energy paths*** $H(M)$ *is given by*

$$H(M) := \left\{ \begin{array}{c} \sigma \in W(M) | \sigma \text{ is absolutely continuous} \\ \text{and } E_M(\sigma) < \infty \end{array} \right\}.$$

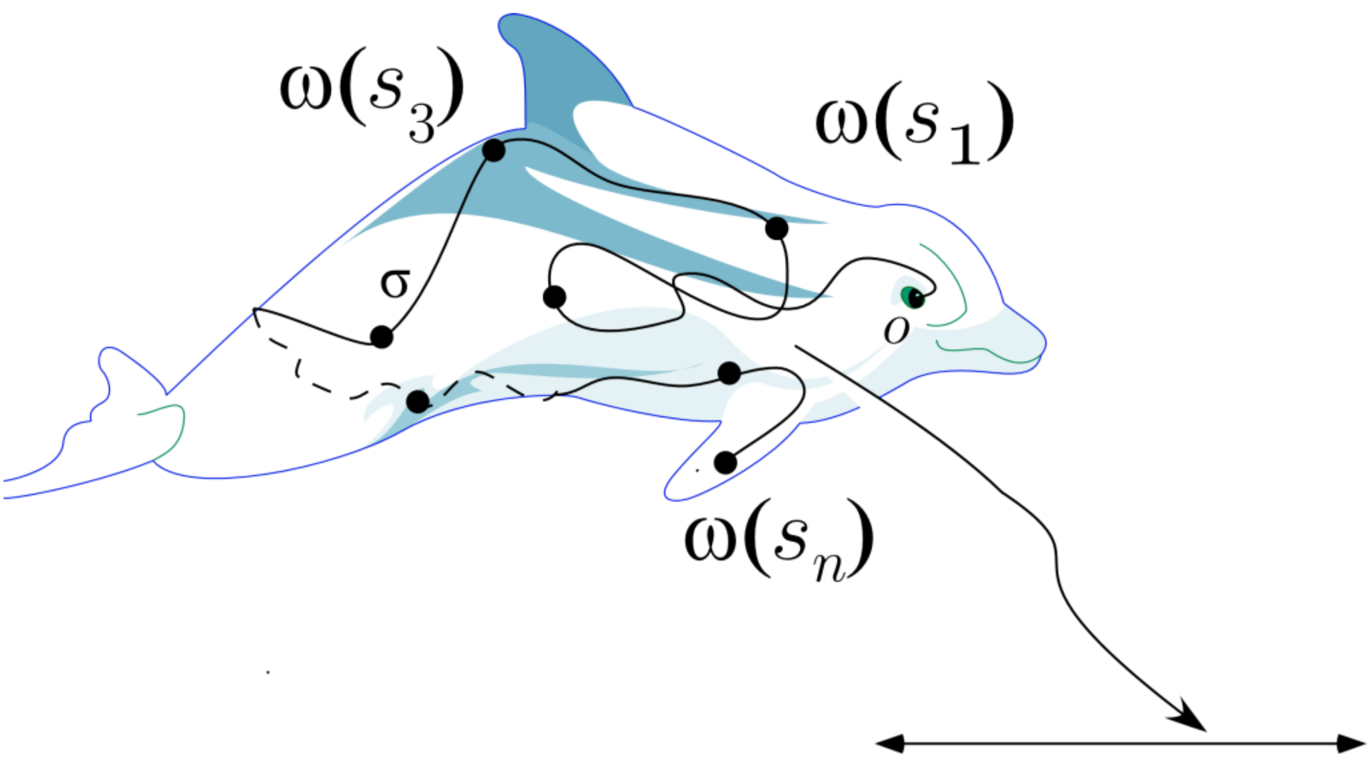


**Definition 11** *A function* $f : W(M) \to \mathbb{C}$ *is a* $C^k-$***cylinder function*** $(f \in \mathcal{F}C^k(W))$ *provided there exists a partition*

$$\pi := \{0 = s_0 < s_1 < \cdots < s_n = 1\}$$

*of* $[0, 1]$ *and a smooth function* $F \in C^k(M^n)$ *such that*

$$f(\sigma) = F(\sigma(s_1), \ldots, \sigma(s_n)) = F(\sigma|_\pi). \qquad (8)$$



# 7 Classical Wiener Measure as a Heat Kernel Measure

**Theorem 12 (Wiener 1923)** *There exits a unique heat kernel sequence*[1] $\{\nu_t\}_{t>0}$ *based at* $0 \in W = W(\mathbb{R}^d)$ *satisfying*

1. $\nu_t(W) = 1$ for all $t > 0$ and
2. for all $f \in \mathcal{F}BC^2(W)$, the function $t \to \nu_t(f)$ is continuously differentiable,
$$\frac{d}{dt}\nu_t(f) = \frac{1}{2}\nu_t(\triangle_{H(\mathbb{R}^d)} f) \text{ and } \lim_{t \downarrow 0} \nu_t(f) = f(0)$$
where
$$\triangle_{H(\mathbb{R}^d)} f := \sum_{h \in S} \partial_h^2 f$$
and $S$ is an orthonormal basis for $H\left(\mathbb{R}^d\right)$.

[1] Wiener did not state the theorem this way, but the results are equivalent.

**Notation 13** *To each partition*

$$\pi := \{0 = s_0 < s_1 < \cdots < s_n = 1\}$$

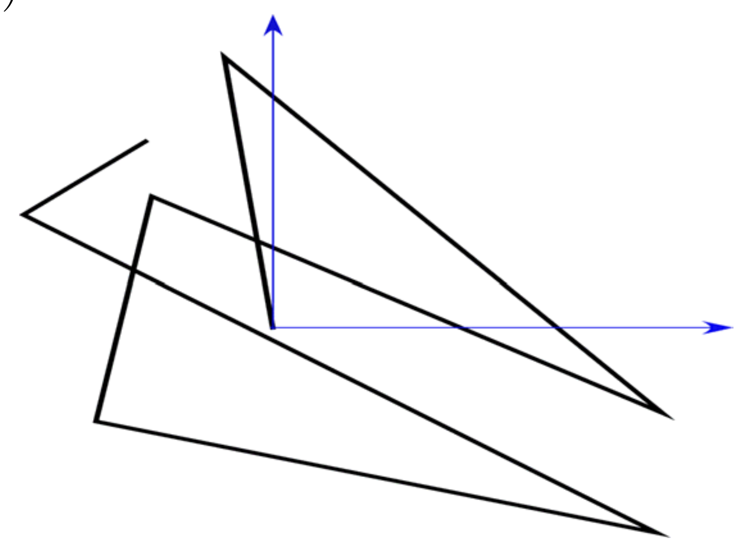
*let*

$$H_\pi(\mathbb{R}^d) := \left\{\omega \in H(\mathbb{R}^d) : \omega''(s) = 0 \text{ if } s \notin \pi\right\}$$

*and* $\{\nu_t^\pi\}_{t>0}$ *denote the heat kernel sequence on* $H_\pi(\mathbb{R}^d)$,

$$d\nu_t^\pi(h) := \frac{1}{Z_\pi(t)} e^{-\frac{1}{2t}E(h)} dm_\pi(h)$$

*where* $Z_\pi(t)$ *is a normalization constant.*



**Proposition 14 (Path Integral intepretation)** *Suppose* $\{\nu_t\}_{t>0}$ *is the heat kernel sequence based at* $0 \in W = W(\mathbb{R}^d)$. *For each* $f : W \to \mathbb{R}$ *which is bounded and continuous,*

$$\int_{W(\mathbb{R}^d)} f(\omega) d\nu_t(\omega) = \lim_{|\pi|\to 0} \frac{1}{Z_\pi(t)} \int_{H_\pi(\mathbb{R}^d)} f(h) e^{-\frac{1}{2t}E(h)} dm_\pi(h). \tag{9}$$

# 8 Cameron – Martin Theorem and Integration by Parts Formula

**Definition 15** *For $h \in W$, let $\nu_t^h := \nu_t(\cdot - h)$, i.e. $\nu_t^h$ is such that*

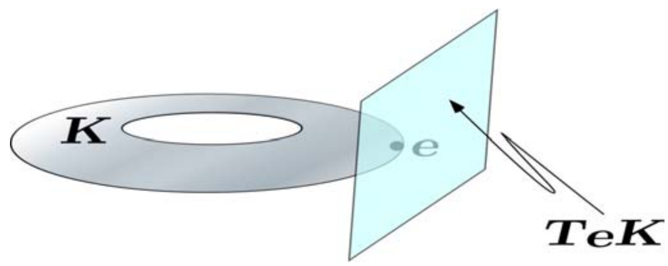
$$\int_W f(\omega) d\nu_t^h(\omega) = \int_W f(\omega + h) d\nu_t(\omega).$$

**Theorem 16** *(Cameron-Martin 44, Cameron-Martin 49 and Cameron 51.)*

1. If $h \in W \setminus H$ then $\nu_t^h \perp \nu_t$.
2. If $h \in H$ then $\nu_t^h \ll \nu_t$ and
$$\frac{d\nu_t^h}{d\nu_t}(\omega) = e^{\frac{1}{t}(h,\omega) - \frac{1}{2t}|h|^2} = \exp\left(\frac{1}{t}\int_0^1 h'(s) d\omega(s) - \frac{1}{2t}|h|^2\right).$$
3. For all $h \in H$, $\partial_h^* = \left(-\partial_h + \frac{1}{t}(h,\omega)\right)$, i.e.
$$\int_W \partial_h f(\omega) \cdot g(\omega) d\nu_t(\omega) = \int_W f(\omega)\left(-\partial_h + \frac{1}{t}(h,\omega)\right) g(\omega) d\nu_t(\omega).$$
In particular $\nu_t$ is a smooth measure.

# 9 Replacing $W\left(\mathbb{R}^d\right)$ by $W(K)$ and $\mathcal{L}(K)$

**Notation 17** • $K$ be a connected compact Lie group (which we take to be a matrix group.)



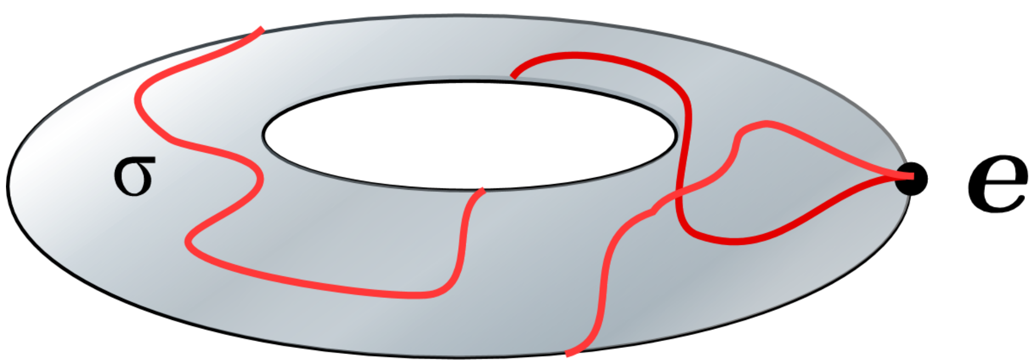

- $\mathfrak{k} := T_eK$ be the Lie algebra of $K$
- $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ be an $Ad_K$-invariant inner product on $\mathfrak{k}$
- $g$ denote the unique bi-invariant Riemannian metric on $K$.
- For $A \in \mathfrak{k}$ and $f \in C^\infty(K)$ let
$$Af(x) = \frac{d}{dt}|_0 f(xe^{tA}).$$

**Example 1** *$K = SO(3)$ be the group of $3\times3$ real orthogonal matrices with determinant $1$. The Lie algebra of $K$ is $\mathfrak{k} = so(3)$, the set of $3 \times 3$ real skew symmetric matrices, and the inner product $\langle A, B\rangle_{\mathfrak{k}} := -\mathrm{tr}(AB)$.*

**Notation 18** *For a compact Lie group $K$ let*

- $W(K) := \{\sigma \in C([0,1] \to K) \,|\, \sigma(0) = e\}$
- $\mathcal{L}(K) := \{\sigma \in W(K) \,|\, \sigma(1) = e\}$
- $\mathbf{e} \in \mathcal{L}(K) \subset W(K)$ denote the constant path at $e \in K$.
- $H(K)$ and $H_0(K)$ are the finite energy paths in $W(K)$ and $\mathcal{L}(K)$ respectively.
- $E_K(\sigma)$ is the energy of $\sigma$.

$$E_K(\sigma) := \int_0^1 \left| [\sigma(s)]^{-1} \sigma'(s) \right|_{\mathfrak{k}}^2 ds = \int_0^1 \left| \sigma'(s)\sigma(s)^{-1} \right|_{\mathfrak{k}}^2 ds, \tag{10}$$

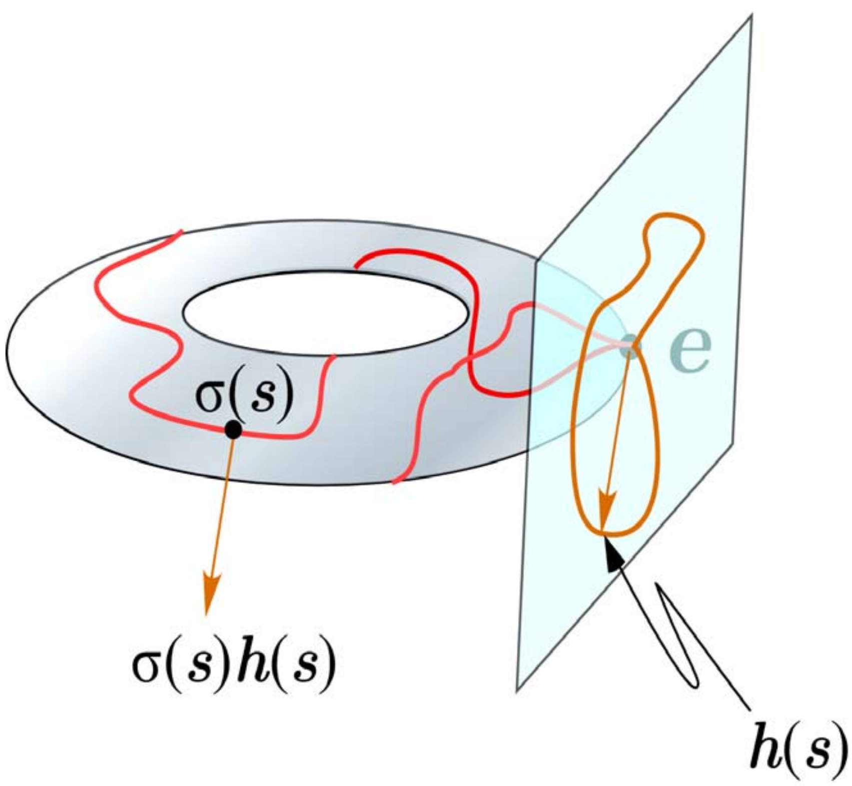


**Definition 19 (Differential Operators)** *For* $h \in H(\mathfrak{k})$ *and* $f : W(K) \to \mathbb{R}$ *let*

$$\tilde{h} f(\sigma) = \frac{d}{dt}\bigg|_0 f\left(\sigma e^{th}\right) \text{ for } \sigma \in W(K)$$

*where* $\left(\sigma e^{th}\right)(s) = \sigma(s) e^{th(s)}$ *for all* $s \in [0,1]$. *Also let*

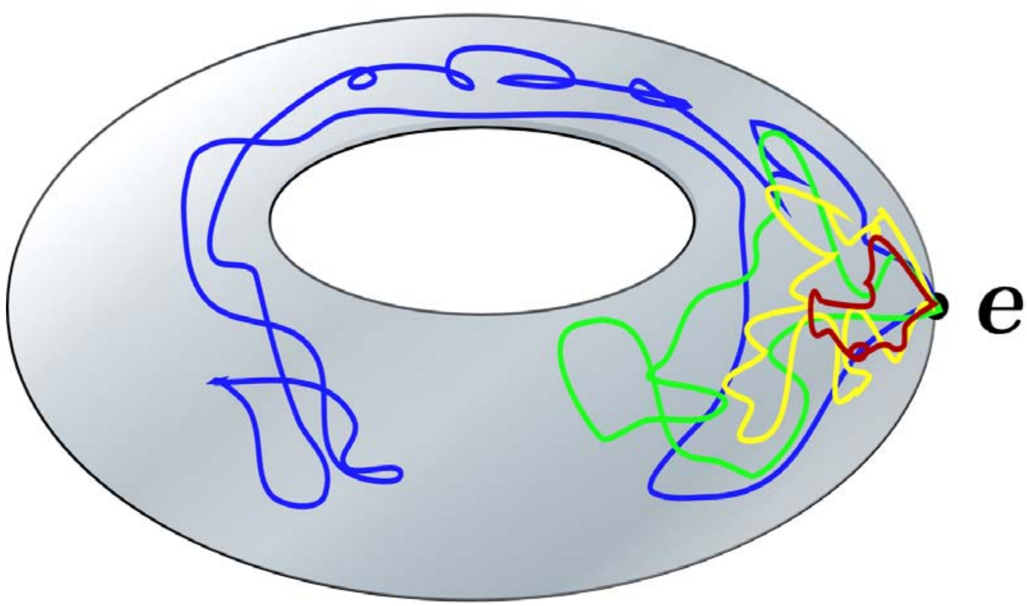
- $S_0$ be an orthonormal basis for $H_0(\mathfrak{k})$
- $\triangle_{H_0(K)} f := \sum_{h \in S_0} \tilde{h}^2 f.$
- $\|\text{grad}_0 f\|^2 := \sum_{h \in S_0} \left(\tilde{h} f\right)^2$



## 9.1 Heat Kernel Measure

**Theorem 20 (Malliavin 90, Driver and Lohrenz 96, Driver 97)** *There exists unique heat kernel sequences* $\{\nu_t\}_{t>0}$ *and* $\{\nu_t^{\mathbf{e}}\}_{t>0}$ *based at* $\mathbf{e}$ *on* $W(K)$ *and* $\mathcal{L}(K)$ *respectively, i.e. for all* $f \in \mathcal{F}C^2(W(K))$,
$\partial_t \nu_t(f) = \frac{1}{2}\nu_t\left(\triangle_{H(K)} f\right)$ *and* $\partial_t \nu_t^{\mathbf{e}}(f) = \frac{1}{2}\nu_t^{\mathbf{e}}\left(\triangle_{H_0(K)} f\right)$
*and*

$$\lim_{t\downarrow 0} \nu_t(f) = f(\mathbf{e}) = \lim_{t\downarrow 0} \nu_t^{\mathbf{e}}(f).$$



## 9.2 Heat Kernel Logarithmic Sobolev Theorem

**Theorem 21 (D. Lohrenz 1996, Carson 97 & 99, Fang 99)** *There is a constant* $C < \infty$ *such that*

$$\int_{\mathcal{L}(K)} f^2 \log \frac{f^2}{\nu_t^{\mathbf{e}}(f^2)} d\nu_t^{\mathbf{e}} \leq C \int_{\mathcal{L}(K)} \|grad_0 f\|^2 \, d\nu_t^{\mathbf{e}} \quad (11)$$

*for all smooth cylinder functions* $f : \mathcal{L}(K) \to \mathbb{R}$.

- Eq. (11) when $K = \mathbb{R}^d$ is Gross' original Logarithmic Sobolev inequality.
- This is analogous to Gross' Logarithmic Sobolev inequality for "Pinned Wiener Measure.
- Eq. (11) is an analogue of a similar result by Bakry, Ledoux, Emery for Riemannian manifolds $M$ with Ricci curvature bounded below.
- The Ricci curvature was computed by D. Freed ≅ (86) in this example.

## 9.3 Quasi-invariance for heat kernel measure

**Notation 22** *For each $k \in H_0(K)$ let $\nu_t^k$ denote the unique measure on $\mathcal{L}(K)$ such that*

$$\int_{\mathcal{L}(K)} f(\sigma) d\nu_t^k(\sigma) = \int_{\mathcal{L}(K)} f(k\sigma) d\nu_t^{\mathbf{e}}(\sigma)$$

**Theorem 23 (D. 97, Fang 99)** *For each $k \in H_0(G)$ which is null homotopic, $\nu_t^{\mathbf{e}}$ quasi-invariant under the right and left translations by $k$, i.e. there is a function $Z^k(\sigma)$ such that*

$$\int_{\mathcal{L}(K)} f(k\sigma) d\nu_t^{\mathbf{e}}(\sigma) = \int_{\mathcal{L}(K)} f(\sigma) Z^k(\sigma) d\nu_t^{\mathbf{e}}(\sigma).$$

- This is proved by proving theorems about the $\mathcal{L}(K)$ – valued Brownian motion.
- The free loop space version of these results was carried out by Trevor Carson 97 & 99 and also see Inahama 2001 for generalizations to include "$H^s$ – metrics" on $\mathcal{L}(K)$ for $s > 1/2$.

# 10 Wiener Measure on $W(M)$

**Notation 24** *Let $M$ be a Riemannian manifold with base point $o \in M$.*

**Theorem 25 (Wiener measure)** *There exists a unique probability measure $\mu_{W(M)}$ on $W(M)$ such that*

$$\int_{W(M)} F(\sigma(s_1), \ldots, \sigma(s_n)) d\mu_{W(M)}(\sigma)$$
$$= \int_{M^n} F(x_1, \ldots, x_n) \prod_{i=0}^{n-1} p_{(s_{i+1}-s_i)}(x_i, x_{i+1}) dx_1 \cdots dx_n.$$

*where $x_0 = o$ and $dx$ denotes the volume measure on $M$.* ***Moreover****, $\mu_{W(M)}$ is the pushforward of $\mu_{W(T_oM)}$ under stochastic version of Cartan's Rolling map.*

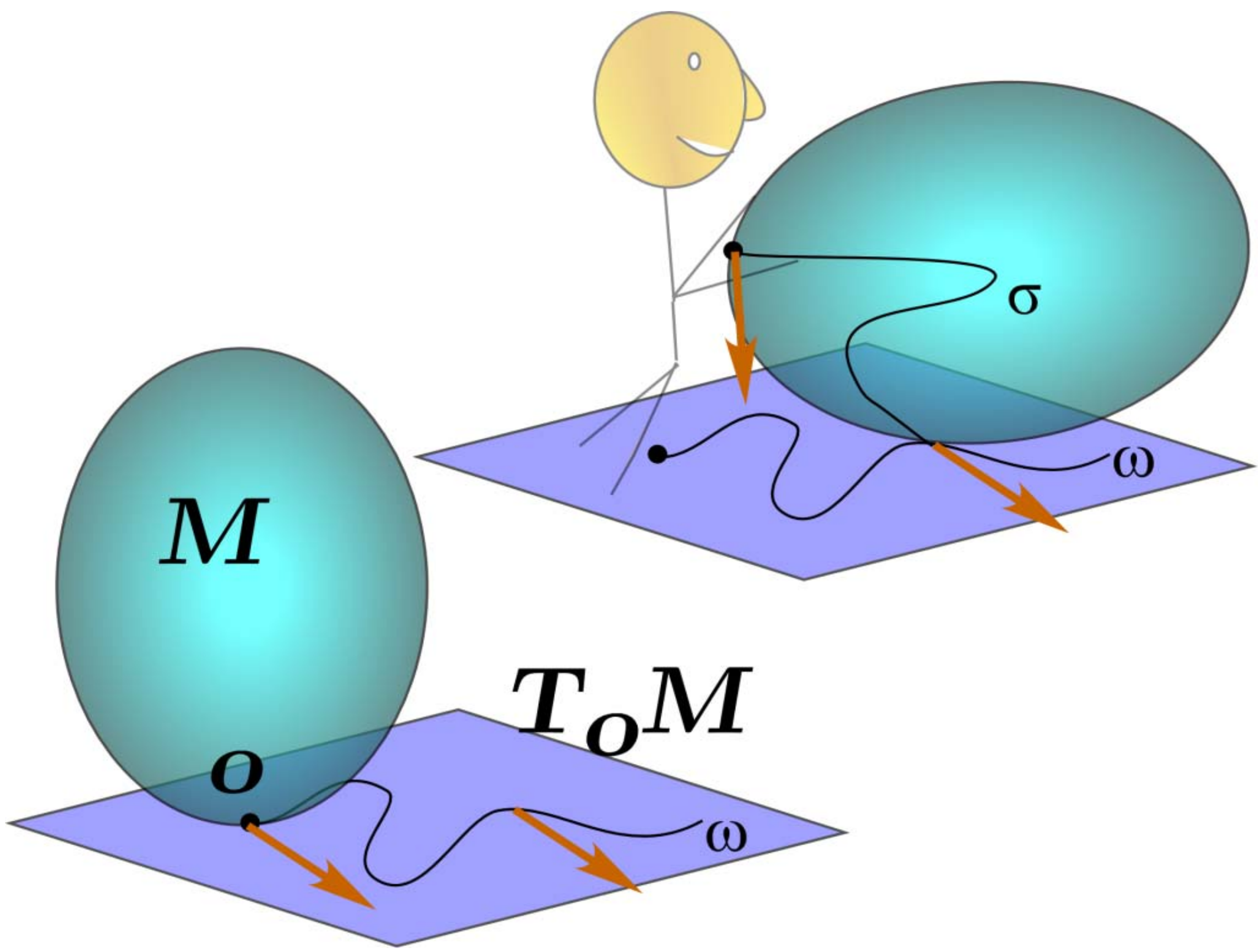
**Notation 26** *To each $\sigma \in H(M)$ and $s \in [0,1]$ let*

- **Parallel translation:** $//_s(\sigma) : T_oM \to T_{\sigma(s)}M$

$$\frac{\nabla}{ds}//_s(\sigma) = 0 \text{ with } //_0(\sigma) = Id_{T_oM}.$$

- **Cartan's rolling map:** $\varphi^{\nabla} : H(T_oM) \longrightarrow H(M)$ given by $\sigma = \phi^{\nabla}(\omega)$ where

$$\sigma'(s) = //_s(\sigma)\omega'(s) \text{ with } \sigma(0) = o. \tag{12}$$

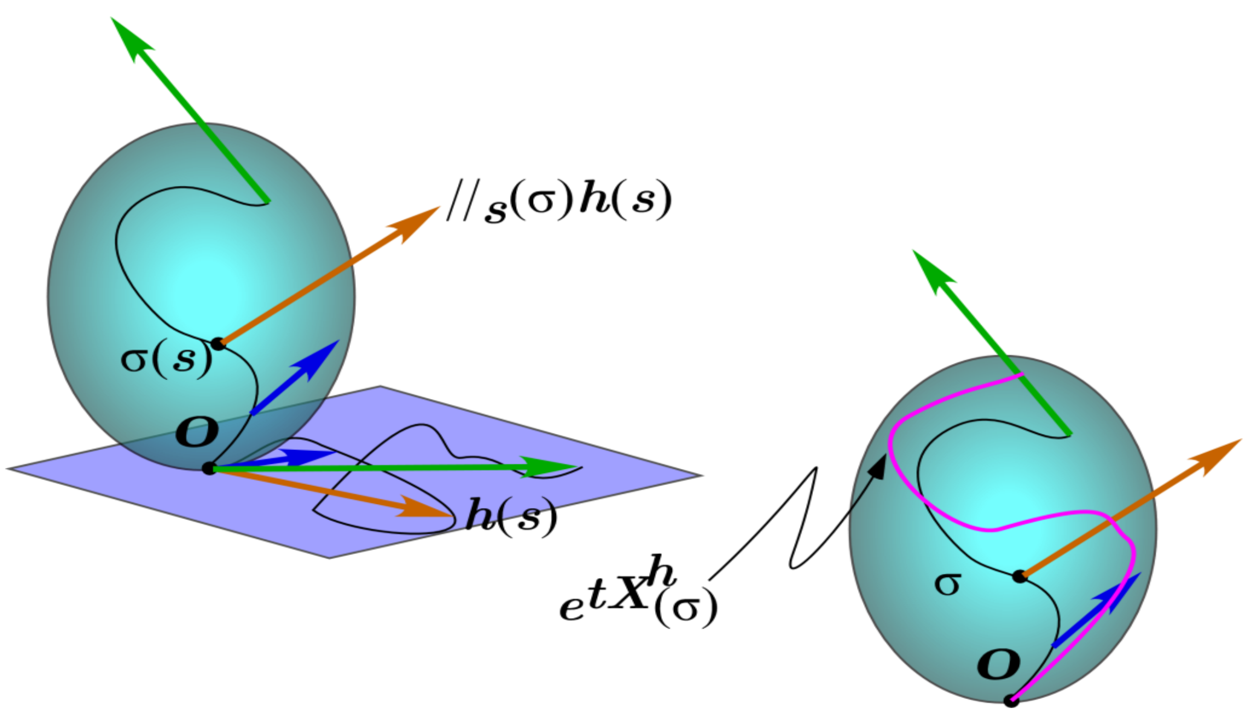


# 11 Quasi-Invariance Theorem for $\mu_W(M)$

**Theorem 27 (D. 92, Hsu 95)** *Let $h \in H(T_oM)$ and $X^h$ be the $\mu_{W(M)}$ – a.e. well defined vector field on $W(M)$ given by*

$$X_s^h(\sigma) = //_s(\sigma)h(s) \text{ for } s \in [0,1]. \tag{13}$$

*Then $X^h$ admits a flow $e^{tX^h}$ on $W(M)$ and this flow leaves $\mu_{W(M)}$ quasi-invariant.* ***(Ref:** D. 92, Hsu 95, Enchev-Strook 95, Lyons 96, Norris 95, ...)*



**Corollary 28 (Integration by Parts for $\mu_{W(M)}$)** *For $h \in H(T_oM)$ and $f \in \mathcal{F}C^1(W(M))$ be as in Eq. (8), let*

$$\begin{aligned}(X^h f)(\sigma) &= \frac{d}{dt}|_0 f(e^{tX^h}(\sigma)) = \sum_{i=1}^{n} (\nabla_i f(\sigma), X^h_{s_i}(\sigma))_g \\ &= \sum_{i=1}^{n} (\nabla_i f)(\sigma), \widetilde{//}_{s_i}(\sigma) h(s_i))_g.\end{aligned}$$

*Then*

$$\int_{\mathrm{W}(M)} X^h f \, d\mu_{W(M)} = \int_{\mathrm{W}(\mathbb{R}^d)} f \, z^h \, d\mu_{W(M)}$$

*where*

$$z^h(\sigma) := \int_0^1 \langle h'(s) + \frac{1}{2}\mathrm{Ric}_{//_s(\sigma)} h'(s), d\sigma(s)\rangle$$

*and*

$$\mathrm{Ric}_{//_s(\sigma)} := //_s(\sigma)^{-1} \mathrm{Ric}_{\sigma(s)} //_s(\sigma) \in End(T_oM).$$

# 12 Path Integral Representations

**Definition 29 (The $\pi$–Metrics)** *For each partition* $\pi = \{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$ *of* $[0,1]$ *let*

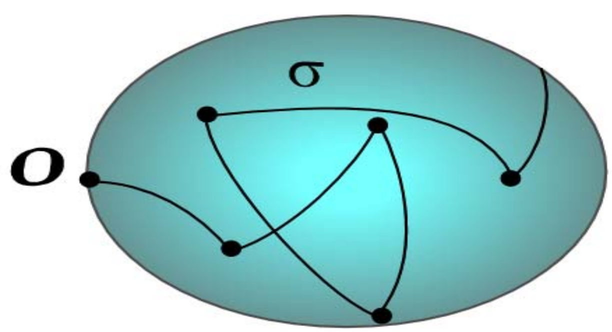
- $H_\pi(M) = \left\{ \sigma \in H(M) : \frac{\nabla \sigma'(s)}{ds} = 0 \text{ for } s \notin \pi \right\}$
- For $X, Y \in TH_\pi(M)$ let

$$G^1_\pi(X,Y) := \sum_{i=1}^n \langle \frac{\nabla X(s_{i-1}+)}{ds}, \frac{\nabla Y(s_{i-1}+)}{ds} \rangle \, (s_i - s_{i-1})$$

- $G^0_\pi(X,Y) := \sum_{i=1}^n \langle X(s_i), Y(s_i) \rangle \, (s_i - s_{i-1})$.
- $G^1_\pi$ and $G^0_\pi$ are the Riemann sum approximations to

$$G^1(X,Y) := \int_0^1 \langle \frac{\nabla X(s)}{ds}, \frac{\nabla Y(s)}{ds} \rangle ds \text{ and}$$

$$G^0(X,Y) := \int_0^1 \langle X(s), Y(s) \rangle ds$$



**Definition 30 ($\pi$ – Volume Forms)** *Let* $\mathrm{Vol}_{G_\pi^0}$ *and* $\mathrm{Vol}_{G_\pi^1}$ *denote the volume forms on* $H_\pi(M)$ *determined by* $G_\pi^0$ *and* $G_\pi^1$.

**Definition 31 (Approximates to Wiener Measure to $\mu_{W(M)}$)** *For each partition* $\pi = \{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$ *of* $[0,1]$, *let* $\mu_\pi^0$ *and* $\mu_\pi^1$ *denote measures on* $H_\pi(M)$ *defined by*

$$\mu_\pi^0 := \frac{1}{Z_\pi^0} e^{-\frac{1}{2}E_M} \mathrm{Vol}_{G_\pi^0}$$

*and*

$$\mu_\pi^1 = \frac{1}{Z_\pi^1} e^{-\frac{1}{2}E_M} \mathrm{Vol}_{G_\pi^1},$$

*where* $E_M : H(M) \to [0,\infty)$ *is the energy functional*

$$E_M(\sigma) := \int_0^1 |\sigma'(s)|_g^2 \, ds$$

*and* $Z_\pi^0$ *and* $Z_\pi^1$ *are normalization constants given by*

$$Z_\pi^0 := \prod_{i=1}^{n} (\sqrt{2\pi}\,(s_i - s_{i-1}))^d \text{ and } Z_\pi^1 := (2\pi)^{dn/2}. \quad (14)$$

**Theorem 32 (Andersson and D. 1999.)** *Suppose that* $f : W(M) \to \mathbb{R}$ *is a bounded and continuous, then*

$$\lim_{|\pi|\to 0} \int_{H_\pi(M)} f(\sigma) d\mu_\pi^1(\sigma) = \int_{\mathrm{W}(M)} f(\sigma) d\mu_{\mathrm{W}(M)}(\sigma) \tag{15}$$

*and*

$$\lim_{|\pi|\to 0} \int_{H_\pi(M)} f(\sigma) d\mu_\pi^0(\sigma) = \int_{\mathrm{W}(M)} f(\sigma) e^{-\frac{1}{6}\int_0^1 \mathrm{Scal}(\sigma(s))ds} d\mu_{\mathrm{W}(M)}(\sigma), \tag{16}$$

*where* $\mathrm{Scal}$ *is the scalar curvature of* $(M, g)$.

**Proof.** There is a large literature pertaining to results of the type in Theorem 32, see for example Cheng72, Um74, Pinsky78, Fujiwara 80, Darling84, A. Inoue and Y. Maeda 85, W. Ichinose 97 and Jyh-Yang Wu 98. The version given here is contained in Andersson and Driver 98.