The number of degree restricted rooted maps on the sphere

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Abstract

Let D be a set of positive integers. Let $m(n)$ be the number of n edged rooted maps on the sphere all of whose vertex degrees (or, dually, face degrees) lie in D. Using Brown's technique, we obtain the generating function for $m(n)$ implicitly. We use it to prove that, when $\gcd(D)$ is even,

$$m(n) \sim C(D) n^{-5/2} \gamma(D)^n.$$

It also yields known formulas for various special D.

Section 1: Introduction

Let D be a set of positive integers containing some element exceeding 2, let $M(x,y) = \sum_i M_i(x)y^i$ be the generating function by edges and root face degree for rooted maps on the sphere such that each nonroot face degree lies in D and let $m(n)$ be the number of n edged rooted maps all of whose face degrees lie in D. Define the coefficient operator with respect to y by

$$[y^k]\left(\sum_{i\ge 0} f_i(x)y^i\right) = f_k(x)$$

and define $[x^k]$ similarly. We will prove

Theorem 1. *There exist unique power series $R_1(x)$ and $R_2(x)$ such that*

$$R_1 = \frac{x}{2}\sum_{i\in D}[y^{i-1}](R^{-1/2}) \tag{1.1}$$

and

$$R_2 = \frac{x}{2}\sum_{i\in D}[y^{i}](R^{-1/2}) + x - 3R_1^2, \tag{1.2}$$

where $R = 1 - 4R_1y - 4R_2y^2$. We have

$$\begin{aligned} m(n) &= \frac{1}{n+1}\,[x^n]\big(M_2'(x)\big) \\ &= [x^n]\left(\frac{(R_2(x)+R_1(x)^2)(R_2(x)+9R_1(x)^2)}{(n+1)x^2}\right). \end{aligned} \tag{1.3}$$

Although the sums in (1.1) and (1.2) appear quite formidable, they can be simplified in some interesting cases. Two particularly simple situations are those in which $\gcd(D)$ is even and those related to arithmetic progressions and finite sets. We have

Corollary 1.1. *If, when viewed as a multiset, D differs from the union of a finite number of arithmetic progressions by a finite multiset, then $M(x,y)$ is algebraic.*

Corollary 1.2. (Tutte [6]). *The number of $2d$ regular, nd edge, rooted maps on the sphere is*

$$\frac{2(nd)!}{n!\,(nd-n+2)!}\binom{2d-1}{d}^n. \tag{1.4}$$

Corollary 1.3. (Liu [4]). *The number of n edge, rooted, bipartite (or, dually, eulerian) maps on the sphere is*

$$\frac{3}{2(n+1)(n+2)}\binom{2n}{n}2^n. \tag{1.5}$$

Theorem 2. *If $\gcd(D)=2d$, then*

$$m(n) \sim \frac{2d(\sigma\gamma)^{5/2}}{(\pi\lambda)^{1/2}} n^{-5/2}\gamma^n, \tag{1.6}$$

where n is a multiple of d, σ is the positive real root of

$$2=\sum_{2i\in D}(i-1)\binom{2i}{i}\sigma^i,$$

$$\lambda=\sum_{2i\in D} i(i-1)\binom{2i}{i}\sigma^i \quad \text{and} \quad \gamma=\tfrac{1}{2}\sum_{2i\in D} i\binom{2i}{i}\sigma^{i-1}.$$

We will use a Tutte type decomposition to obtain a quadratic equation for $M(x,y)$ when D is finite. Such equations are usually solved by the quadratic method [3]. That approach does not seem to work here. We must look more closely at what Brown's result [2] says about the discriminant of the quadratic. Having established Theorem 1 for finite D, we then pass to the limit. When $\gcd(D)$ is even, $R_1=0$. This results in considerable simplification of the equations in Theorem 1 which leads easily to

Corollaries 1.2 and 1.3. It also leads to equations which suggest that there may be an interesting bijection between various bipartite maps and pairs of some other objects. We have not been able to find the bijection. In Section 4, we use a result of Meir and Moon [5] to obtain Theorem 2 from Theorem 1.

At this time we have no general asymptotic result when $\gcd(D)$ is odd. We suspect that a result of the form (1.6) will hold. This can be verified on a case by case basis with lengthy calculations. For example, with D the set of odd positive integers, we have used Maple to prove that

$$m(n) \sim Cn^{-5/2} x_0^{-n}$$

where

$$x_0 = 0.10519\ldots, \qquad C = 3\tau^{1/2}/4\pi^{1/2} = 0.71772\ldots$$

and both x_0 and τ are algebraic of degree 6.

Section 2: Proof of Theorem 1

For all of this section except the last paragraph, we assume that D is a finite set with largest element t.

Note that (1.1) tells us that R_1 has no constant term. If we specifiy R_1 and R_2 through terms of degree k in x and substitute them in the right sides of (1.1) and (1.2), the left sides give us R_1 and R_2 through degree $k+1$. Thus the power series R_1 and R_2 are uniquely determined by (1.1) and (1.2) and have no constant terms.

Let $\mathbf{C}$ denote the complex numbers, let $\mathcal{R}[[x]]$ denote formal power series over the commutative ring $\mathcal{R}$ and let $\mathcal{R}[y]$ denote polynomials. Let

$$\theta_k(y) = \sum_{\substack{i\in D\\ i\geq k}} y^{t-i}. \tag{2.1}$$

Note that $\theta_k(y) \equiv \theta_0(y) \pmod{y^{t-k+1}}$.

We use a standard construction [7] to obtain a functional equation for $M(x,y)$. Either a map consists of just one vertex, with generating function 1, or it has a root edge. The generating function for maps for which the removal of the root edge leaves two components is given by $xy^2M(x,y)^2$. There is one more case, namely removing the root edge does not disconnect the map. Reversing these removals gives a recursive construction for the maps. If the root face has degree j and we wish to add a new root that reverses the last case, we can create a nonroot face of degree k and a root face of degree $j+2-k$. This construction leads to

$$M(x,y) = 1 + xy^2M(x,y)^2 + x\sum_{j\geq 0} M_j(x) \sum_{\substack{k\in D\\ k\leq j+1}} y^{j+2-k}.$$

After some algebra,

$$\begin{aligned} M(x,y) &= 1 + xy^2M(x,y)^2 + xy^{2-t}\theta_0(y)M(x,y) \\ &\quad - x\sum_{j=0}^{t-2} \theta_{j+2}(y)y^{j+2-t}M_j(x). \end{aligned} \tag{2.2}$$

It is important to note that the recursive nature of this construction guarantees that *there is a unique power series solution to* (2.2).

Regarding (2.2) as a quadratic in $M(x,y)$, we obtain

$$2xy^tM(x,y) = y^{t-2} - x\theta_0(y) \pm B(x,y)^{1/2} \tag{2.3}$$

where

$$B(x,y) = \left(x\theta_0(y) - y^{t-2}\right)^2 - 4xy^{2t-2} + 4x^2y^t\sum_{j=0}^{t-2}\theta_{j+2}(y)M_j(x)y^j. \tag{2.4}$$

We show that (2.3) is true algebraically modulo y^{2t-1} for any $M_j(x) \in \mathbf{C}[[x]]$ provided that $M_0(x) = 1$ and the proper sign is chosen for the square root. (By "algebraically", we mean that no map information is needed.) To see this, first rearrange (2.3) as

$$\pm B(x,y)^{1/2} \equiv x\theta_0(y) - y^{t-2} + 2xy^t M(x,y) \pmod{y^{2t-1}}$$

and note that the right side contains a nonzero term in y^0. Thus this congruence holds if and only if

$$\begin{aligned} B(x,y) &\equiv \big(x\theta_0(y) - y^{t-2} + 2xy^t M(x,y)\big)^2 \pmod{y^{2t-1}} \\ &\equiv (x\theta_0(y) - y^{t-2})^2 + 4x^2y^t\theta_0(y)M(x,y) - 4xy^{2t-2}M_0(x) \pmod{y^{2t-1}}, \end{aligned}$$

which is easily verified using (2.4).

¿From the previous paragraph, it follows that *any* choice for $M_1(x), \ldots, M_{t-2}(x) \in \mathbf{C}[[x]]$ which leads to a power series for $B(x,y)^{1/2}$ will be a solution to (2.3) and, hence, the unique solution.

Since $B(x,y) \in \mathbf{C}[[x]][y]$ has degree $2t-2$ in y, Brown's theorem [2] guarantees that $B(x,y) = Q(x,y)^2R(x,y)$, for some Q, $R \in \mathbf{C}[[x]][y]$ with $R(x,0) = 1$. We will show that one solution can be found with

$$Q(x,y) = x + \sum_{i=1}^{t-2} Q_i(x)y^i \quad \text{and} \quad R(x,y) = 1 - 4R_1(x)y - 4R_2(x)y^2. \tag{2.5}$$

To begin with, Q^2R has the same degree as B with respect to y, namely $2t-2$. Since (2.3) is true algebraically modulo y^{2t-1}, it suffices to determine the t unknown functions R_1, R_2, $Q_1, \ldots, Q_{t-2}$ by looking at the coefficients of y, y^2 up to y^t in

$$2xy^tM(x,y) = y^{t-2} - x\theta_0(y) + Q(x,y)R(x,y)^{1/2} \tag{2.6}$$

and then showing that $M_1(x), \ldots, M_{t-2}(x)$ are in fact power series.

It follows from (2.6) that

$$Q(x,y) \equiv R(x,y)^{-1/2}\big(x\theta_0(y) - y^{t-2} + 2xy^t\big) \pmod{y^{t+1}}. \tag{2.7}$$

Since we defined Q to have degree $t-2$ in y, reduction modulo y^{t-1} determines Q in terms of R_1 and R_2 while the coefficients of y^{t-1} and y^t in (2.7) give two polynomial equations in the two unknowns R_1 and R_2. These equations are (1.1) and (1.2). We have shown that $Q_i(x) \in \mathbf{C}[[x]]$ and $R_j \in \mathbf{C}[[x]]$ can be found. Since $R_1(x)$ and $R_2(x)$ have no constant terms and $Q(x,y)$ has degree $t-2$ in y, it follows from (2.6) that $xM_j(x)$ has no constant term and so $M_j(x)$ is a power series. This completes the proof that (2.5) and (2.7) determine the unique power series solution $M(x,y)$ to (2.3).

We now prove (1.3). Using G' to denote $\partial G/\partial x$, we have

$$(QR^{1/2})' = \tfrac{1}{2}R^{-1/2}(2Q'R + QR') \tag{2.8}$$

and, from (2.7),

$$(QR^{1/2})' \equiv \theta_0(y) + 2y^t \pmod{y^{t+1}}. \tag{2.9}$$

Thus

$$2Q'R + QR' \equiv 2R^{1/2}\big(\theta_0(y) + 2y^t\big) \pmod{y^{t+1}}. \tag{2.10}$$

Define the truncation operator with respect to y by

$$\mathcal{T}_k\left(\textstyle\sum_{i\ge 0} f_i(x)y^i\right) = \sum_{i=0}^{k} f_i(x)y^i.$$

Since the left side of (2.10) equation has degree t in y,

$$2Q'R + QR' = \mathcal{T}_t\Big(2R^{1/2}\big(\theta_0(y) + 2y^t\big)\Big).$$

Thus

$$\begin{aligned}
x(QR^{1/2})'R^{1/2} &= x\mathcal{T}_t\Big(R^{1/2}\big(\theta_0(y)+2y^t\big)\Big) \\
&= \mathcal{T}_t\Big(R^{1/2}\big(x\theta_0(y)-y^{t-2}+2xy^t\big)\Big)+\mathcal{T}_t\big(R^{1/2}y^{t-2}\big) \\
&= \mathcal{T}_t\Big(R\,R^{-1/2}\big(x\theta_0(y)-y^{t-2}+2xy^t\big)\Big)+y^{t-2}\mathcal{T}_2\big(R^{1/2}\big) \\
&= RQ+y^{t-2}\mathcal{T}_2(R^{1/2}),
\end{aligned}$$

by (2.7). Combining this with (2.3), we have

$$\begin{aligned}
2x(xM_j(x))' &= [y^{t+j}]\Big(x(QR^{1/2})'\Big) \\
&= [y^{t+j}](QR^{1/2})+[y^{t+j}]\big(R^{-1/2}y^{t-2}\mathcal{T}_2(R^{1/2})\big) \\
&= 2xM_j(x)+[y^{j+2}]\big(R^{-1/2}\mathcal{T}_2(R^{1/2})\big).
\end{aligned}$$

Rearranging the two ends of this equation gives us

$$M_j'(x)=\frac{1}{2x^2}\,[y^{j+2}]\big(R^{-1/2}\mathcal{T}_2(R^{1/2})\big). \tag{2.11}$$

Set $m(0)=1$ and let $M(x)$ be the generating function for $m(n)$. By removing the root edge from a map counted by $M_2(x)$, it is easily seen that

$$M(x)=M_2(x)/x. \tag{2.12}$$

This combined with a bit of algebra on (2.11) gives us (1.3). The proof of Theorem 1 for finite D is complete.

Let D be arbitrary and define $D(t)$ to be those elements of D which do not exceed t. We may apply Theorem 1 to $D(t)$. If we replace t by $t'>t$, then it is a simple matter to check that the terms of degree less than $t/2$ do not change in the formulas for R_1 and R_2. Thus, we may simply let $t\to\infty$.

Section 3: Simple applications of Theorem 1

The reader may be amused to carry out the calculations in Theorem 1 when D is the positive integers, thereby rederiving the generating function for all maps: In this case, (1.1) and (1.2) become

$$\begin{aligned} R_1 &= \frac{x}{2}(1-4R_1-4R_2)^{-1/2} \\ R_2 &= \frac{x}{2}\Big((1-4R_1-4R_2)^{-1/2}-1\Big)+x-3R_1^2. \end{aligned}$$

Eliminating R_2 leads to the quartic equation

$$0=(12R_1^2-2R_1+x)(4R_1^2-2R_1-x).$$

Since R_1 has no constant term and since the generating function for maps must have a positive real singularity, the correct solution is

$$R_1=\frac{1-\sqrt{1-12x}}{12}.$$

The equation for $M(x)$ follows easily from (1.3) and (2.12).

To prove Corollary 1.1, we observe that (1.1) and (1.2) can be summed for i in an arithmetic progression by using multisection of the series $R^{-1/2}$ with respect to y and then setting $y=1$. Thus the equations for R_1 and R_2 are algebraic. Now use (2.6) and the value of Q from (2.7).

When $\gcd(D)$ is even, $R_1=0$. One can see this either by noting that $M(x,y)$ cannot have any odd degree terms in y or by noting that the assumption $R_1=0$ leads to a solution and so must be the unique solution. Since $R_1=0$, we have

$$[y^{2i}](R^{-1/2})=[y^{2i}]\big((1-4R_2y^2)^{-1/2}\big)=\binom{2i}{i}R_2^i. \tag{3.1}$$

Thus, (1.1) becomes $0 = 0$, (1.2) becomes

$$R_2 = \frac{x}{2} \sum_{2i \in D} \binom{2i}{i} R_2^i + x \tag{3.2}$$

and (1.3) becomes

$$m(n) = \tfrac{1}{n+1}[x^n]\big(M_2'(x)\big) = \tfrac{1}{n+1}[x^n]\big(x^{-2}R_2(x)^2\big) \tag{3.3}$$

Since $\binom{2i}{i}$ is even, it follows from (3.1) that $R_2(x)$ has nonnegative integer coefficients. This combined with $M_2'(x) = \big(R_2(x)/x\big)^2$ from (3.3) suggests that there is probably an interesting bijection between rooted maps with a distinguished edge and pairs of combinatorial objects when $\gcd(D)$ is even.

When $D = \{2d\}$, the sum in (3.2) has only one term and Corollary 1.2 follows easily by Lagrange inversion.

To prove Corollary 1.3, note that (3.2) becomes

$$R_2 = \frac{x}{2}\Big((1-4R_2)^{-1/2} - 1\Big) + x.$$

After a bit of algebra, one obtains

$$R_2(x) = \frac{4x + 1 - \sqrt{1-8x}}{8}, \tag{3.4}$$

where the minus sign was chosen on the square root because $R_2(0) = 0$. Thus

$$\frac{R_2(x)^2}{x^2} = \frac{8x^2+1}{32x^2} + \left(\frac{(1-8x)^{3/2}}{32x}\right)'.$$

By (2.12) and (3.3),

$$M(x) = M_2(x)/x = \frac{1}{4} - \frac{1}{32x^2} + \frac{(1-8x)^{3/2}}{32x^2} + \frac{3}{8x}.$$

Corollary 1.3 follows easily.

Section 4: Proof of Theorem 2

Suppose that $2d = \gcd(D)$. It follows from (3.2) that $R_2(x) = x + xw(x^d)$ where the power series $w(z)$ is determined by

$$w = F(z,w) = \tfrac{1}{2} \sum_{2id \in D} \binom{2id}{id} z^i (w+1)^{id}. \tag{4.1}$$

Let (ρ, τ) be a positive real solution, if any, of the simultaneous equations (4.1) and $1 = F_w(z,w)$. With a bit of algebra, $\rho(\tau+1)^d$ is the positive real root of

$$2 = \sum_{2id \in D} (id-1) \binom{2id}{id} \bigl(\rho(\tau+1)^d\bigr)^i \tag{4.2}$$

and $\tau > 0$ is then determined by (4.1).

We wish to apply Meir and Moon's [5, Thm. 1] and thereby obtain asymptotics by using their correction to [1, Thm. 5]. If D is finite, $F(z,w)$ is analytic for all z and w and the conditions for [5, Thm. 1] are satsified. Suppose that D is infinite. Then $F(z,w)$ is analytic for $4\bigl|z(w+1)^d\bigr| < 1$ since $\binom{2n}{n} \sim 4^n/(\pi n)^{1/2}$ and (4.2) has a solution with $4\rho(\tau+1) < 1$ since $(id-1)\binom{2id}{id}$ is unbounded. Again, [5, Thm. 1] applies.

Combining this with [1, (7.1)], we find that

$$[z^n](w+1) \sim \Bigl((\tau+1)/2\pi F_{ww}(\rho,\tau)d\Bigr)^{1/2} n^{-3/2} \rho^{-n}$$

and w behaves like $\tau + 1 + C(1 - z/\rho)^{1/2}$ near $z = \rho$. Thus

$$[z^n]\bigl((w+1)^2\bigr) \sim 2(\tau+1)\Bigl((\tau+1)/2\pi F_{ww}(\rho,\tau)d\Bigr)^{1/2} n^{-3/2} \rho^{-n}. \tag{4.3}$$

In terms of the notation in Theorem 2, one finds with a bit of algebra that $\rho = \gamma^{-d}$, $\tau + 1 = \sigma\gamma$ and $F_{ww}(\rho,\tau) = \lambda/2(\sigma\gamma)^2$. Theorem 2 now follows easily from (3.3), $R_2/x = w + 1$ and (4.3).

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