Submap Density and Asymmetry Results for Two Parameter Map Families

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1. Introduction

Throughout this paper, all limits are understood to be taken through those values of n (and k) for which the numbers involved are nonzero.

Definition (map). Let S be a compact surface without boundary. A *map* is a graph G (loops and multiple edges allowed) which has been embedded in S so that S minus the embedding is a set of discs—called the *faces* of the map. A map on the sphere is called *planar*. A map is *rooted* if an edge, a direction along the edge and a side of the embedded edge are distinguished. The face lying on that side of the edge is called the *root face*.

The requirement that the faces of a map be discs implies that G is connected and that the Euler relation holds for the map. Rooted planar maps are often drawn on the plane so that the root face is unbounded. Since all our results will deal with exponentially small fractions and since there are at most $4n$ rootings of an n-edged map, the results do not depend on whether or not the map is rooted.

Definition (submap). Let M be a map on the surface S and let C be a cycle formed by a subset of the edges of M. Imagine that the edges of M have nonzero width so that we can cut the surface by running a cut along C through the middle of its edges. If this separates the surface, identifying the cut in one piece with the boundary of a disc produces a map on some surface. This map is called a *submap* of M. It is rooted by choosing and directing some edge e in C and making the disc that was used to close the cut the root face. Thus, a cycle C in M yields either two submaps of M or, if the cut does not disconnect the surface, no submaps of M.

Let $\mathcal{M}(S)$ be the set of rooted maps in some class (e.g. 2-connected) on S. Let $\mathcal{M}_n(S)$ be those with n edges and $\mathcal{M}_{n,k}(S)$ those with n edges and k vertices. It was shown in [3,4,5,6], that almost all maps in $\mathcal{M}_n(S)$ contain many copies of any reasonable planar submap. Various applications were given in those papers and Richmond and Wormald [13] used the result to show that almost all of the maps in $\mathcal{M}_n(S)$ are asymmetric. In this paper, we extend these ideas to $\mathcal{M}_{n,k}(S)$. Since the Euler relation $v - e + f = \chi(S)$ relates the number of vertices, edges and faces, n and k can equally well stand for any two of these three numbers.

Definition (positively dense submaps). Let $m_{n,k}(S) = |\mathcal{M}_{n,k}(S)|$. Let $S = 0$ denote the sphere and let a (resp. A) be the lim inf (resp. lim sup) as $n \to \infty$ of those k/n for which $m_{n,k}(0) \neq 0$. Let P be a planar map. We say that P has *positive edge-vertex density* in a class of maps if, for every closed interval $I \subset (a, A)$, there are constants $\epsilon > 0$, $c < 1$ and N depending on I, P, S and the class of maps such that, for all $n > N$ and all integers $k \in nI$, all but a fraction c^{-n} of the maps in $\mathcal{M}_{n,k}(S)$ contain at least ϵn copies of P as a submap.

Note that this definition is *uniform* in n and k; that is, ϵ, c and N do not depend on how n and k approach infinity so long as $k/n \in I$. Let α and β be any two of "edge," "vertex" and "face." It should be clear how to define "positive α-β density." The following lemma shows that we can drop the qualifier and simply say "positive density."

Lemma. *For any choice of α and β in the previous discussion, P has positive α-β density in a class of maps if and only if it has positive edge-vertex density.*

Proof. All cases are similar. We give a proof for the vertex-face case. If a and A are the values for the edge-vertex case, it is easily seen from Euler's relation that

$$a' = \frac{1-A}{A} \quad \text{and} \quad A' = \frac{1-a}{a}$$

are the values for the vertex-face case, with the usual arithmetic for the reals with infinity. More generally, an interval $[b, B]$ corresponds to the interval $[b', B']$ where $b' = B^{-1} - 1$ and $B' = b^{-1} - 1$. This allows us to transform one problem into another. ■

It is possible to prove denseness for a variety of classes of maps. In order to avoid excessive length, we have selected some classes that illustrate various techniques and contain maps of interest in their own right:

Theorem 1. *Let D be a set of positive even integers with $1 < |D| < \infty$. For all surfaces S, the following planar submaps have positive density in the indicated classes:*

(a) P is any planar map and $\mathcal{M}(S)$ is all maps;

(b) P is any 2-connected planar map and $\mathcal{M}(S)$ is all 2-connected maps;

(c) P is any 3-connected planar map and $\mathcal{M}(S)$ is all 3-connected maps;

(d) P is any planar map with face degrees in D and $\mathcal{M}(S)$ is all maps with face degrees in D.

Positive density of a submap implies a "1 law" for that submap. (See [3] for a definition.) Perhaps more importantly, if we have submaps with positive density, it follows from [13] that, for every closed interval $I \subset (a, A)$ there is a constant c so that, for all sufficiently large n and each $k \in nI$, the fraction of maps in $\mathcal{M}_{n,k}(S)$ which have symmetries is less than c^{-n} for all suffciently large n.

The results in Theorem 1 are probably not best possible since they do not let us approach the boundaries of (a, A). For example, it is known that for 3-connected maps almost all maps in $\mathcal{M}_{n,k}(0)$ are asymmetric for *all* sufficiently large n and k [9].

The proof of Theorem 1 involves some particular considerations for each case as well as a rather technical general theorem. The theorem is stated and proved in the next section. The remaining sections contain the particular considerations needed to prove Theorem 1.

2. A Technical Theorem

Let $M_S(x, y) = \sum m_{n,k}(S)x^n y^k$. For a generating function $F(x, y)$, let $r(F, t)$ denote the radius of convergence of $F(x, t)$.

Definition (typical behaviour), Let the interval (a, A) be as in the definition of positive density. If the following conditions hold for each closed interval $I \subset (a, A)$, we say that the class of maps being studied *behaves typically.*

T1. For all $t > 0$, $\lim_{n\to\infty} \left(\sum_k m_{n,k}(0)t^k\right)^{-1/n} = r(M_0, t)$. (As always, the limit is through those n for which the sum is nonzero.)

T2. There is a continuous function $t(\alpha) > 0$ on I, a function $k(n, \alpha)$ and an increasing function $f(n) = o(n)$ such that for each n and each $\alpha \in I$

$$|k(n, \alpha) - \alpha n| < f(n) \quad \text{and} \quad \sum_i m_{n,i}(0)t(\alpha)^i \Big/ m_{n,k}(0)t(\alpha)^k$$

is bounded by a polynomial in n.

T3. There is a $g(n) = o(n)$ with $f(n) = o(g(n))$ such that $m_{n,k}(S) \geq m_{\nu,\kappa}(0)$ whenever (a) n is sufficiently large, (b) $k/n \in I$, (c) in the degree restricted case, the number of edges is a multiple of $\gcd(D)/2$, (d) $\nu = n - g(n)$ and (e) $|\kappa - \nu k/n| < f(n)$.

T4. For all S and all $t > 0$, $r(M_S, t) \geq r(M_0, t)$.

When $t = 1$, T1 reduces to what was termed "smoothness" in [3, p. 111]. For general t, one can usually verify T1 in the same way one verifies smoothness. T2 is related to "mean shifting" ideas connected with some combinatorial applications of central and local limit theorems. One can usually set $f(n) = n^{1/2}$. Although T3 looks messy, it can usually be proved by a relatively simple construction akin to the methods used for condition E that is described next. Although we do not need it, we remark in passing that T3 and T4 easily imply that $r(M_S, t) = r(M_0, t)$ and this implies that T1 can be extended to $m_{n,k}(S)$.

We need an assumption about being able to embed the submap P uniquely in maps of the class being considered. The following is taken from the assumption on page 111 of [3].

Definition (embeddable submap). We call P *embeddable* in a class of maps if P is a submap of a rooted map Q and we can attach in some manner copies of Q to maps in $\mathcal{M}(S)$ so that

E1 the number of possible attachments is a positive fraction of the number of edges in the map to which copies are being attached,

E2 only maps in $\mathcal{M}_n(S)$ are produced by such attachment,

E3 for any map so produced, we can identify the copies of Q that may have been added and they are all pairwise disjoint, and

E4 given the copies that have been added, the original map and the places of attachment can be uniquely identified.

Theorem 2. *If P is embeddable in a class of maps which behaves typically, then P has positive density in the class.*

The remainder of this section is devoted to a proof of the theorem. The uniformity in the definition of density will follow from the compactness of I, so we need not prove it. The proof consists of carrying out the following three steps:

1. Let $h_{n,k}(S)$ be those maps containing less than ϵn copies of of P. Fix $t > 0$. For sufficiently small ϵ, $r(H_S, t) > r(M_S, t)$ whenever the latter is nonzero.
2. $h_{n,k}(S)$ is exponentially smaller than $m_{\nu,\kappa}(0)$ for some κ with $|\kappa - (k/n)\nu| < f(n)$.
3. $h_{n,k}(S)$ is exponentially smaller than $m_{n,k}(S)$.

Let $f_{n,k} \ll g_{n,k}$ denote the fact that $f_{n,k}$ is exponentially smaller than $g_{n,k}$; that is, for some $C > 1$ and all sufficiently large n we have $C^n f_{n,k} \leq g_{n,k}$. Let $f_{n,k} \approx g_{n,k}$ denote the fact that $\lim_{n\to\infty} (f_{n,k}/g_{n,k})^{1/n} = 1$. Of course, $k = k(n) \in nI$.

Step 1 is nothing more than a simple adaptation of the proof of Theorem 1 of [3, p. 111-112], which dealt with the case $t = 1$. All one needs to do is change the notation slightly, replace the equation $G(x) = H(x + x^{e(P)-1})$ in that paper with $G(x,t) = H(x + xe(Q) - 1t^{v(Q)-2}), t)$, and introduce t in obvious places. Details are left to the diligent reader.

From T4 it follows that $r(H_S, t) > r(M_0, t)$. From the smoothness condition T1 and the fact that $\nu = n - o(n)$, it follows that $\sum h_{n,i}(S)t^i \ll \sum m_{\nu,i}t^i$. With $t = t(k/n)$, it follows from T2 and the above that

$$h_{n,k}(S)t^k \ll \sum m_{\nu,i}(0)t^i \approx m_{\nu,\kappa}(0)t^\kappa$$

for some $\kappa = \kappa(n,t)$ with $|\kappa - (k/n)\nu| < f(\nu) \leq f(n)$. Since $k - \kappa = o(n)$, it follows that $h_{n,k}(S) \ll m_{\nu,\kappa}(0)$. This completes Step 2.

Step 3 follows immediately from Step 2 and T3. This completes the proof of Theorem 2.

3. The Planar Conditions

In this section, we determine (a, A) and establish T1 and T2 for the maps in Theorem 1.

An obvious approach is by means of explicit formulas. Brown and Tutte [11] showed that the number of 2-connected maps on the sphere having $i + 1$ vertices and $j + 1$ faces is given by

$$f_{i,j} = \frac{(2i + j - 2)!\,(2j + i - 2)!}{i!\,j!\,(2i - 1)!\,(2j - 1)!}. \tag{1}$$

From Euler's formula, there are $n = i + j$ vertices. It follows easily that $(a, A) = (0, 1)$. Using (1), it follows that the maximum term of $\sum m_{n,k}(0)t^k$ occurs near the solution to $1 \approx f_{k+1,n-k-1}t^{k+1}/f_{k,n-k}t^k$; that is, near the real number $0 < k < n$ for which

$$1 = \frac{(n+k)(n-k)^3 t}{(2n-k)k^3}.$$

Since the value of the sum lies between the maximum term and n times the maximum term, T1 and T2 follow after a bit of computation.

Instead of exact formulas, one can use asymptotic information. It is known [8] that $p_{i,j}$, the number of 3-connected maps on the sphere with $i + 1$ vertices and $j + 1$ faces satisfies

$$p_{i,j} \sim \frac{1}{3^5 ij} \binom{2i}{j+3} \binom{2j}{i+3}$$

uniformly as $\max(i, j) \to \infty$. The ideas in the previous paragraph can be applied to establish T1 and T2 and the fact that the formula is uniform over a wide range allows us to deduce that $(a, A) = (1/2, 2)$ for the face/vertex case. (This transforms to $(1/3, 2/3)$ for the vertex/edge case.)

For all maps, the asymptotic behaviour of $m_{n,k}(S)$ is known for all S [2]. Conditions T1 and T2 again follow easily from this result and condition T4 is immediate. In this case, $(a, A) = (0, 1)$ since a map can consist solely of loops and a connected graph can have at most one more vertex than edges.

For degree restricted maps, less is available. Let f and F be the maximum and minimum of D, respectively. Since each face contains between f and F edges and each side of an edge appears in a face, it follows that the ratio of faces to edges lies between $2/F$ and $2/f$. Hence

$$(a, A) = (1 - 2/F,\ 1 - 2/f).$$

Theorem 3. *Let D be a set of even integers with $1 < |D| < \infty$. Let I be a closed subinterval of (a, A). There are continuous functions s, t and u such that*

$$m_{n,k}(0) \sim s(k/n) n^{-3} t(k/n)^n u(k/n)^k$$

uniformly for $k \in nI$ and n a multiple of $\gcd(D)/2$.

As will be shown, conditions T1 and T2 follow from this theorem. The remainder of this section is devoted to proving the theorem and deriving the conditions.

It is a simple matter to include a parameter for the number of vertices in the derivation of generating functions in [1]. When $\gcd(D)$ is even, we obtain

$$\frac{\partial\big(xM_0(x,y)\big)}{\partial x} = \big(R_2(x,y)/x\big)^2 \quad \text{where} \quad R_2(x,y) = xy + \frac{x}{2}\sum_{2i\in D}\binom{2i}{i}R_2(x,y)^i \tag{2}$$

uniquely defines the generating function R_2.

We begin by locating the singularity of $R_2(x,t)$ nearest the origin for real positive t. Let $w(x) = R_2(x,t)/x$ and $z = x^{1/d}$ where $d = \gcd(D)/2$. Then (2) becomes

$$w = t + \frac{1}{2}\sum_{2jd\in D}\binom{2jd}{jd}(zw^d)^j.$$

By Meir and Moon's Theorem 1 in [12], this has a singularity at $(z,w) = (\rho,\tau)$ and there is no other singularity with $|z| \le \rho$. By considering derivatives or studying their proof, one can show that the singularity is a branch point due to a square root and then that $R_2(x,t)^2$ has the same singularity. Since this singularity is isolated and algebraic, Darboux's theorem can be used to show that the coefficient of x^n in $R_2(x,t)^2$ is asymptotic to $Cn^{-3/2}r^{-n}$, where C and r depend on t. (See [14, p. 205] for a statement of Darboux's theorem.) By (2), the ratio of the coefficients of R_2^2 and M_0 is n. This establishes T1.

It is an easy matter to show that the conditions of Theorem 1 of [7] are satisfied. Thus $m_{n,k}(0)$ satisfies a local limit theorem with variance proportional to n and mean equal to $n\mu = -n\, d\log r(t)/d\log t$ where $x = r(t)$ is the singularity. To complete the proof, it suffices to verify that μ can assume all values in (a,A). The value of $r(t)$ is given by the simultaneous solution for $r(t)$ and R_2 of (2) and the partial of (2) with respect to R_2:

$$R_2 = r(t)t + \frac{r(t)}{2}\sum_{2i\in D}\binom{2i}{i}R_2^i \quad \text{and} \quad 1 = \frac{r(t)}{2}\sum_{2i\in D} i\binom{2i}{i}R_2^{i-1}.$$

Thus

$$r(t) = 2/\sigma' \quad \text{and} \quad t = (R_2\sigma' - \sigma)/2, \quad \text{where} \quad \sigma = \sum_{2i\in D}\binom{2i}{i}R_2^i. \tag{3}$$

Treating r and t as functions of R_2, we have

$$\mu = \frac{-d\log r}{d\log t} = \frac{-d\log r/dR_2}{d\log t/dR_2} = \frac{\sigma''}{\sigma'}\,\frac{R_2\sigma' - \sigma}{R_2\sigma''} = 1 - \frac{\sigma}{R_2\sigma'}.$$

It is easily seen from (3) that R_2 and $-r(t)$ are continuous strictly increasing functions of t. Furthermore, R_2 ranges from 0 to ∞. As this happens, the last displayed equation ranges from $1-2/F$ to $1-2/f$. This proves the theorem.

The central limit theorem implies that

$$\sum_i m_{n,i}(0)t^i \Big/ \sum_{|i-n\mu|<n^{1/2}} m_{n,i}(0)t^i$$

is bounded by a constant. By choosing k to be value of i with $|i-n\mu|<n^{1/2}$ that maximizes $m_{n,i}(0)t^i$, we obtain T2.

4. Constructions for Conditions T3 and E

Select a map $G \in \mathcal{M}(S)$. Using G, one can establish T3 by means of two constructions

1. Show that there exist constants e' and v' such that, for all sufficiently large n', we can combine a map in $\mathcal{M}_{n',k'}(0) \neq \emptyset$ with G to produce a map in $\mathcal{M}_{n'+e',k'+v'}(S)$.
2. Show that there exist constants e and v such that any map H of the sort produced in the previous construction can be combined with any map in $\mathcal{M}_{\nu,\kappa}(0)$ to produce a map in $\mathcal{M}_{\nu+n'+e,\kappa+k'+v}(S)$. Furthermore, show that this construction is injective over $\mathcal{M}(0)$ for fixed H.

It is a simple matter to see that these constructions suffice to establish T3 provided the values needed in the first construction are such that $\mathcal{M}_{n',k'}(0) \neq \emptyset$. We have $\nu+n'+e=n$ and $\kappa+k'+v=k$ and so

$$\frac{k'}{n'} = \frac{k-\kappa-v}{n-\nu-e} = \frac{k-\nu k/n + o(g(n))}{n-\nu-e} = \frac{k}{n}\,\frac{g(n)+o(g(n)n/k)}{g(n)-e} \sim \frac{k}{n}$$

since n/k is bounded for $k/n \in I$. This shows that, for all $\epsilon > 0$ and all sufficiently large n, the distance from k'/n' to I is less than ϵ. It is a relatively simple matter to show that this implies that $\mathcal{M}_{n',k'}(0) \neq \emptyset$ when one has either exact or asymptotic formulas for $m_{n',k'}(0)$.

The constructions needed in 1 and 2 above can be easily obtained by adapting the constructions used for proving embedability. Hence we will just describe the latter. The arguments in [3] suffice to establish embedability for k-connected maps. For degree restricted maps, a construction in [3] can be modified slightly: Let $d \in D$ exceed 2. In Figure 2 of [3], replace each of the two straight edges of positive slope with a path of length $d-2$. The reasoning in that paper applies equally well to the modified figure.

5. Condition T4

For all maps, condition T4 follows immediately from [2].

For degree restricted maps with no constraint on D, we can adapt Lemma 1 of [6] to include another parameter. That lemma gives a construction for transforming degree restricted maps on S having n edges and k vertices to degree restricted maps on the sphere having n edges and $k+2-\chi(S)$ vertices. (The last subscript in Lemma 1 should be n, not $n+2g$.) This transformation is many to one, but it is shown that the multiplicity is at most $O(n^{4-2\chi(S)})$. It follows from this that $r(M_S,t) \geq r(M_0,t)$.

Let A_S, B_S and C_S denote the generating functions for all rooted maps, all 2-connected rooted maps and all 3-connected rooted maps, respectively, on S. All maps on the sphere can be obtained by attaching arbitrary maps in the "corners" of 2-connected maps. Since the number of corners is twice the number of edges, this leads to the functional equation

$$A_0(x,y) = B_0\big(xU(x,y),\ y\big) \quad \text{where} \quad U(x,y) = \big(1 + A_0(x,y)/y\big)^2.$$

The locations of the singularities of B_0 are thus completely determined by the locations of the singularities of A_0. Furthermore, they are algebraic since the singularities of A_0 are. We can define B_S^* by

$$A_S(x,y) = B_S^*\big(xU(x,y),\ y\big).$$

Since resulting function $B_S^*(x,y)$ includes more than the 2-connected maps, $r(B_S^*,t) \leq r(B_S,t)$. (In fact, B_S^* includes more than the "unchoppable" maps of [10].) The locations of the singularities of B_S^* are determined by the locations of the singularities of A_0 and A_S. By [2], A_S has singularities just where A_0 does. Thus $r(B_S^*,t) = r(B_0,t)$.

In Section 2 of [5], it was shown that $r(C_S,1) = r(C_0,1)$. That argument is easily extended to general t, thus proving T4 for 3-connected maps.

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