A Multivariate Lagrange Inversion Formula
for Asymptotic Calculations

Edward A. Bender
Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112, USA
ebender@ucsd.edu

L. Bruce Richmond
Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario N2L 3G1, Canada
lbrichmond@watdragon.uwaterloo.ca

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Abstract

The determinant that is present in traditional formulations of multivariate Lagrange inversion causes difficulties when one attempts $(d+1)^{d-1}$ terms in contrast to the $d!$ terms of the determinantal form. Thus it is likely to prove useful only for asymptotic purposes.

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1. Introduction

Many researchers have studied the Lagrange inversion formula, obtaining a variety of proofs and extensions. Gessel [4] has collected an extensive set of references. For more recent results see Haiman and Schmitt [6], Goulden and Kulkarni [5], and Section 3.1 of Bergeron, Labelle, and Leroux [3].

Let boldface letters denote vectors and let a vector to a vector power be the product of componentwise exponentiation as in $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_d^{n_d}$. Let $[\mathbf{x}^{\mathbf{n}}]\, h(\mathbf{x})$ denote the coefficient of $\mathbf{x}^{\mathbf{n}}$ in $h(\mathbf{x})$. Let $\|a_{i,j}\|$ denote the determinant of the $d \times d$ matrix with entries $a_{i,j}$. A traditional formulation of multivariate Lagrange inversion is

Theorem 1. *Suppose that $g(\mathbf{x}),\ f_1(\mathbf{x}), \cdots, f_d(\mathbf{x})$ are formal power series in $\mathbf{x}$ such that $f_i(\mathbf{0}) \neq 0$ for $1 \leq i \leq d$. Then the set of equations $w_i = t_i f_i(\mathbf{w})$ for $1 \leq i \leq d$ uniquely determine the w_i as formal power series in $\mathbf{t}$ and*

$$[\mathbf{t}^{\mathbf{n}}]\, g(\mathbf{w}(\mathbf{t})) = [\mathbf{x}^{\mathbf{n}}] \left\{ g(\mathbf{x})\, \mathbf{f}(\mathbf{x})^{\mathbf{n}} \left\| \delta_{i,j} - \frac{x_i}{f_j(\mathbf{x})} \frac{\partial f_j(\mathbf{x})}{\partial x_i} \right\| \right\}, \tag{1}$$

where $\delta_{i,j}$ is the Kronecker delta.

If one attempts to use this formula to estimate $[\mathbf{t}^{\mathbf{n}}]\, g(\mathbf{w}(\mathbf{t}))$ by steepest descent or stationary phase, one finds that the determinant vanishes near the point where the integrand is maximized, and this can lead to difficulties as $\min(n_i) \to \infty$. We derive an alternate formulation of (1) which avoids this problem. In [2], we apply the result to asymptotic problems.

Let $\mathcal{D}$ be a directed graph with vertex set V and edge set E. Let the vectors $\mathbf{x}$ and $\mathbf{f}(\mathbf{x})$ be indexed by V. Define

$$\frac{\partial \mathbf{f}}{\partial \mathcal{D}} = \prod_{j \in V} \left\{ \left(\prod_{(i,j) \in E} \frac{\partial}{\partial x_i} \right) f_j(\mathbf{x}) \right\}.$$

We prove

Theorem 2. *Suppose that $g(\mathbf{x}),\ f_1(\mathbf{x}), \cdots, f_d(\mathbf{x})$ are formal power series in $\mathbf{x}$ such that $f_i(\mathbf{0}) \neq 0$ for $1 \leq i \leq d$. Then the set of equations $w_i = t_i f_i(\mathbf{w})$ for $1 \leq i \leq d$ uniquely determine the w_i as formal power series in $\mathbf{t}$ and*

$$[\mathbf{t}^{\mathbf{n}}]\, g(\mathbf{w}(\mathbf{t})) = \frac{1}{\prod n_i}\, [\mathbf{x}^{\mathbf{n}-\mathbf{1}}] \sum_{\mathcal{T}} \frac{\partial(g, f_1^{n_1}, \ldots, f_d^{n_d})}{\partial \mathcal{T}}, \tag{2}$$

where $\mathbf{1} = (1, \ldots, 1)$, the sum is over all trees $\mathcal{T}$ with $V = \{0, 1, \ldots, d\}$ and edges directed toward 0, and the vector in $\partial/\partial\mathcal{T}$ is indexed from 0 to d.

When $d = 1$, this reduces to the classical formula

$$[t^n]\, g(w(t)) = \frac{[x^{n-1}]\, g'(t) f(t)^n}{n}.$$

Derivatives with respect to trees have also appeared in Bass, Connell, and Wright [1].

2. Proof of Theorem 2

Expand the determinant $\|\delta_{i,j} - a_{i,j}\|$. For each subset S of $\{1, \ldots, d\}$ and each permutation π on S, select the entries $-a_{i,\pi(i)}$ for $i \in S$ and $\delta_{i,i}$ for $i \notin S$. The sign of the resulting term will be $(-1)^{|S|}$ times the sign of π. Since (i) the sign of π is -1 to the number of even cycles in π and (ii) $|S|$ has the same parity as the number of odd cycles in π, it follows that

$$\|\delta_{i,j} - a_{i,j}\| = \sum_{S,\pi} (-1)^{c(\pi)} \prod_{i\in S} a_{i,\pi(i)}, \tag{3}$$

where $c(\pi)$ is the number of cycles of π and the sum is over all S and π as described above. (When $S = \emptyset$, the product is 1 and $c(\pi) = 0$.)

Applying (3) to (1) with $h_0 = g$, $h_1 = f_1^{n_1}, \ldots, h_d = f_d^{n_d}$ and understanding that $S \subseteq \{1, \ldots, d\}$, we obtain

$$\begin{aligned}
&\left(\textstyle\prod n_i\right)[\mathbf{x}^{\mathbf{n}}]\, g(\mathbf{w}(\mathbf{t})) \\
&\quad = [\mathbf{x}^{\mathbf{n}}] \sum_{S,\pi} (-1)^{c(\pi)} \left\{ \prod_{\substack{i\notin S\\ i\neq 0}} n_i \times \prod_{i\notin S} h_i(\mathbf{x}) \times \prod_{i\in S} x_i n_i\, f_{\pi(i)}(\mathbf{x})^{n_i - 1} \frac{\partial f_{\pi(i)}(\mathbf{x})}{\partial x_i} \right\} \\
&\quad = [\mathbf{x}^{\mathbf{n-1}}] \sum_{S,\pi} (-1)^{c(\pi)} \left\{ \prod_{\substack{i\notin S\\ i\neq 0}} \frac{n_i}{x_i} \times \prod_{i\notin S} h_i(\mathbf{x}) \times \prod_{i\in S} \frac{\partial h_{\pi(i)}(\mathbf{x})}{\partial x_i} \right\} \\
&\quad = [\mathbf{x}^{\mathbf{n-1}}] \sum_{S,\pi} (-1)^{c(\pi)} \left\{ \left(\prod_{\substack{i\notin S\\ i\neq 0}} \frac{\partial}{\partial x_i} \right) \left(\prod_{i\notin S} h_i(\mathbf{x}) \times \prod_{i\in S} \frac{\partial h_{\pi(i)}(\mathbf{x})}{\partial x_i} \right) \right\},
\end{aligned} \tag{4}$$

where, in the last line, the $\partial/\partial x_i$ operators replaced n_i/x_i because we are extracting the coefficient of $x_i^{n_i-1}$.

If we expand a particular S, π term in (4) by distributing the partial derivative operators, we obtain a sum of terms of the form

$$\prod_{j\in V} \left\{ \left(\prod_{(i,j)\in E} \frac{\partial}{\partial x_i} \right) h_j(\mathbf{x}) \right\},$$

where $V = \{0, 1, \ldots, d\}$ and $E \subset V \times V$. Since each $\partial/\partial x_i$ appears exactly once per term, all vertices in the directed graph $\mathcal{D} = (V, E)$ have outdegree one, except for vertex 0 which has outdegree zero. Thus adding the edge $(0, 0)$ to $\mathcal{D}$ gives a functional digraph. The cycles of π are among the cycles of $\mathcal{D}$, and, since the $\partial/\partial x_i$ for $i \notin S$ can be applied to any factor, the remaining edges are arbitrary. Hence

$$\left(\prod_{\substack{i\notin S\\ i\neq 0}} \frac{\partial}{\partial x_i} \right) \left(\prod_{i\notin S} h_i(\mathbf{x}) \times \prod_{i\in S} \frac{\partial h_{\pi(i)}(\mathbf{x})}{\partial x_i} \right) = \sum_{\mathcal{D}} \frac{\partial \mathbf{h}}{\partial \mathcal{D}},$$

where the sum ranges over all directed graphs $\mathcal{D}$ on $V = \{0, 1, \ldots, d\}$ such that (i) adjoining $(0, 0)$ produces a functional digraph and (ii) the cycles of $\mathcal{D}$ include π. Denote condition (ii) by $\pi \subseteq \mathcal{D}$. We have shown that

$$\left(\prod n_i\right) [\mathbf{x}^{\mathbf{n}}]\, g(\mathbf{w}(\mathbf{t})) = [\mathbf{x}^{\mathbf{n}-\mathbf{1}}] \sum_{S,\pi} (-1)^{c(\pi)} \sum_{\mathcal{D}:\pi\subseteq\mathcal{D}} \frac{\partial \mathbf{h}}{\partial \mathcal{D}}$$
$$= [\mathbf{x}^{\mathbf{n}-\mathbf{1}}] \sum_{\mathcal{D}} \sum_{\pi:\pi\subseteq\mathcal{D}} (-1)^{c(\pi)} \frac{\partial \mathbf{h}}{\partial \mathcal{D}}.$$

Since $\sum_{\pi\subseteq\mathcal{D}} (-1)^{c(\pi)} = 0$ when $\mathcal{D}$ has cyclic points and is 1 otherwise, the sum reduces to a sum over acyclic directed graphs $\mathcal{D}$ such that adjoining $(0, 0)$ gives a functional digraph. Since these are precisely the trees with edges directed toward 0, the proof is complete.

References

[1] H. Bass, E. H. Connell, and D. Wright, The Jacobian conjecture: Reduction of degree and formal expansion of the inverse, *Bull. Amer. Math. Soc. (N.S.)* **7** (1982) 287–330.

[2] E. A. Bender and L. B. Richmond, Asymptotics for multivariate Lagrange inversion, in preparation.

[3] F. Bergeron, G. Labelle, and P. Leroux (trans. by M. Readdy), *Combinatorial Species and Tree-Like Structures*, Encylopedia of Math. and Its Appl. Vol 67, Cambridge Univ. Press, 1998.

[4] I. M. Gessel, A combinatorial proof of the multivariate Lagrange inversion formula, *J. Combin. Theory Ser. A* **45** (1987) 178–195.

[5] I. P. Goulden and D. M. Kulkarni, Multivariable Lagrange invers, Gessel-Viennot cancellation and the Matrix Tree Theorem, *J. Combin. Theory Ser. A* **80** (1997) 295–308.

[6] M. Haiman and W. Schmitt, Incidence algebra antipodes and Lagrange inversion in one and several variables, *J. Combin. Theory Ser. A* **50** (1989) 172–185.