Montecarlo Combinatorics

or better

How to transform Probabilistic Existence Proofs

into Constructions

Problem 1

Given an $n \times n$ matrix

$$egin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \ a_{21} & a_{22} & \cdots & a_{2n} \ dots & dots & dots & dots & dots \ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad a_{ij} = \pm 1.$$

Construct two ± 1 vectors

$$X = (x_1, x_2, \dots, x_n), \qquad Y = (y_1, y_2, \dots, y_n)$$

Such that

$$\sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} y_j a_{i,j} \right) \ge \sqrt{\frac{2}{\pi}} \times n^{3/2} \qquad (1)$$

Algorithm:

- **Step 1** Produce a random Y, with $P[y_i = \pm 1] = 1/2$
- **Step 2** Compute for each *i* the sum $S_i = \sum_{j=1}^n y_j a_{i,j}$
- **Step 3** Set $x_i = sign S_i$
- Step 4 Test if (1) is satisfied.

Repeat until it is

Why does this work?

This gives

$$\sum_{i=1}^{n} x_i \Big(\sum_{j=1}^{n} y_j a_{i,j} \Big) = \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} y_j a_{i,j} \Big|$$

Since for each i the random variables

$$a_{i,1}y_1 \ , \ a_{i,2}y_1 \ , \ \ldots \ , \ a_{i,n}y_n$$

are independent identically distributed and mean zero, the Central Limit Theorem gives

$$E\Big[\Big|\sum_{j=1}^n y_j a_{i,j}\Big|\Big] \approx \sqrt{\frac{2}{\pi}} \times n^{1/2} \qquad (2)$$

This suggests that if we repeat steps 1,2,3,4 we are bound to find the desired X and Y.

How good is the approximation in (2)? This is easily verified by computing

$$E\Big[\Big|\sum_{j=1}^n y_j a_{i,j}\Big|\Big] = \sum_{i=0}^n \binom{n}{i} |2i-n| imes rac{1}{2^n}$$

next

Problem 2

Given a set of n integers

 ${\bf S} \ = \ \{ {\bf r_1} \ , \ {\bf r_2} \ , \ \ldots \ , \ {\bf r_n} \}$

construct a "sum free" subset T of cardinality n/3.

(Erdos proved that such a subset exists)

Algorithm:

Step 1 Produce a Random subset $\mathbf{T} \subseteq \mathbf{S}$

Step 2 Check if T is sum free.

Repeat until you find it.

Note: It is good to test if S itself is sum free. If so we are done.

Computer experimentation with this algorithm reveals two interesting facts:

- a) The overwhelming majority of cardinality n/3 subsets are sum free.
- b) If the integers are all positive then any subset of the largest n/2 elements of S

Note : Define for a given S

is very likely to be sum free.

 $\mathbf{i_0} = \max{\{\mathbf{i} : \text{there exists } k > j > i \text{ such that } r_i + r_j = r_k }$

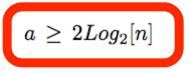
By choosing at random n distinct positive integers we could not find an example for which $i_0 > n/2$

On 2-Colorings of the edges of Kn

Theorem

For each n > 1 there exists a 2-coloring of the edges of $\mathbf{K_n}$ with no monochromatic a-cliques,

provided



Proof

Let ${\mathcal C}$ be a random 2-coloring of the edges of ${\mathbf K}_{\mathbf n}$ and set

$$N(\mathcal{C}) = \sum_{T \subseteq [1,n] \& |T| = a} \chi(K_T ext{ is monochromatic })$$

Under a random 2-coloring C the probability that the clique with vertices in T is monochromatic is

$$2 imes rac{1}{2^{\binom{|T|}{2}}}$$

Thus the expected number of monochromatic a-cliques is

$$E[N(\mathcal{C})] \;\;=\;\; {n \choose a} imes rac{2}{2^{{a \choose 2}}}$$

In particular it follows that

$$P[N(\mathcal{C}) \geq 1] \leq \binom{n}{a} \times \frac{2}{2^{\binom{a}{2}}}$$

Now it can be shown that $\binom{n}{a} \times \frac{2}{2\binom{a}{2}} < 1$ when $a > 2Log_2[n]$. For such choices of a we have $1 - P[N(\mathcal{C}) \ge 1] = P[N(\mathcal{C}) = 0] > 0$ Q.E.D.

Problem 3

For a given n construct the 2-coloring of the edges of \mathbf{K}_n guaranteed by the previous theorem **Algorithm:**

Step 1 Produce a Random 2-coloring of the edges of K_n

Step 2 Count the number of Monochromatic *a*-cliques.

Repeat until you get a zero count

The previous theorem guarantees that you will find the desired coloring when $a \ge 2Log_2[n]$ Note: In the following table, under each $10 \le n \le 30$, I placed the smallest a such that

$$\binom{n}{a}2\times\frac{1}{2^{\binom{a}{2}}}<1$$

and under it I placed the ceiling of $2Log_2[n]$:

In particular we see that in a random 2-coloring of the edges of $\mathbf{K_{20}}$

we should expect to find about 30 monochromatic 5-cliques.