THE HAGLUND MACDONALD STATISTICS

AND

THE PROBLEMS THAT IT CREATES

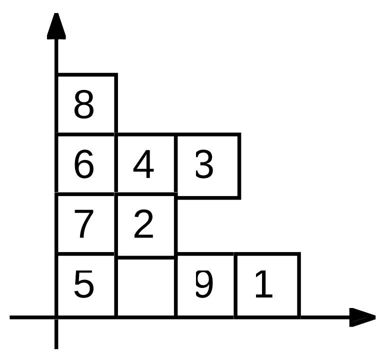
THE HAGLUND STATISTICS

$$\mathbf{inv}(\mathbf{T}) \;=\; \sum_{\mathbf{u}\in\mathbf{T}}\sum_{\mathbf{v}\in\mathbf{T}}\sum_{\mathbf{w}\in\mathbf{T}} \chi\begin{bmatrix}\mathbf{u} & \mathbf{v}\\ \mathbf{w} & \end{bmatrix} \;+\; \sum_{\mathbf{u}\in\mathbf{T_{bottom}}}\sum_{\mathbf{v}\in\mathbf{T}} \chi(\mathbf{u}\to\mathbf{v})\chi(\mathbf{u}>\mathbf{v})$$

with

$$\chi\begin{bmatrix}\mathbf{u} & \mathbf{v}\\ \mathbf{w} & \end{bmatrix} \;=\; \chi(\mathbf{u}<\mathbf{w}<\mathbf{v}) \;+\; \chi(\mathbf{v}<\mathbf{u}<\mathbf{w}) \;+\; \chi(\mathbf{w}<\mathbf{v}<\mathbf{u})$$

$$\mathbf{maj}(\mathbf{T}) = \sum_{\mathbf{c}\in\mathbf{U}} \chi\Big(\begin{bmatrix}\mathbf{c}\\ \mathbf{c}'\end{bmatrix}\in\mathbf{T}\Big)\,\chi\big(\mathbf{T}(\mathbf{c})>\mathbf{T}(\mathbf{c}')\big)\big(\mathbf{1}+\mathbf{leg}(\mathbf{c})\big)$$



$$\mathbf{W}(\mathbf{T}) \;=\; \mathbf{q}^{\mathbf{inv}(\mathbf{T})}\mathbf{t}^{\mathbf{maj}(\mathbf{T})} \;=\; \mathbf{q}^{\mathbf{3+2}}\mathbf{t}^{\mathbf{1+3+1}}$$

THE HAGLUND POLYNOMIALS

$$\mathbf{P_U(x;q,t)} = \sum_{\lambda(\mathbf{T})=\mathbf{U}} \mathbf{q^{inv(T)}t^{maj(T)}x(T)}$$

$$\mathbf{x(T)} = \mathbf{x_1^{m_1(T)}x_2^{m_2(T)}\cdots x_n^{m_n(T)}}$$

Theorem (Haglund)

$\mathbf{P_U(x;q,t)}$ **is always a Symmetric function**

Conjecture (Haglund)

$\mathbf{P_U(x;q,t)}$ **is Schur-positive**

Theorem (Haglund-Haiman-Loehr)

$$\mathbf{P}_\mu(\mathbf{x;q,t}) = \tilde{\mathbf{H}}_\mu(\mathbf{x;q,t}) \left(= \sum_{\lambda\vdash \mathbf{n}} \mathbf{S}_\lambda(\mathbf{x})\tilde{\mathbf{K}}_{\lambda,\mu}(\mathbf{q,t})\right)$$

Corollary (Haglund-Haiman-Loehr)

An extremely simple proof of the Lascoux-Schüzenberger Charge interpretation of Hall-Littlewood polynomials

Enter Quasi-Symmetric Functions

$$\mathbf{F_p(x)} \;=\; \sum_{\substack{\mathbf{a_1 \le a_2 \le \cdots \le a_n} \\ \mathbf{i \in S \to a_i < a_{i+1}}}} \mathbf{x_{a_1} x_{a_2} \cdots x_{a_n}} \;=\; \mathbf{F_S(x)}$$

$$\mathbf{p} = (\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_{k+1}}) \quad \longrightarrow \quad \mathbf{S} = \mathbf{S(p)} = \big\{\mathbf{p_1}, \mathbf{p_1} + \mathbf{p_2}, \ldots, \mathbf{p_1} + \mathbf{p_2} + \cdots + \mathbf{p_k}\big\}$$

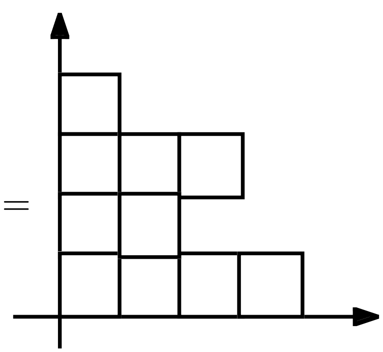
Theorem

$$\mathbf{P_U(x; q, t)} \;=\; \sum_{\sigma \in \mathbf{S_n}} \mathbf{q}^{\mathbf{inv_U}(\sigma)} \mathbf{t}^{\mathbf{maj_U}(\sigma)} \; \mathbf{F}_{\mathbf{ID}(\sigma)}(\mathbf{x})$$

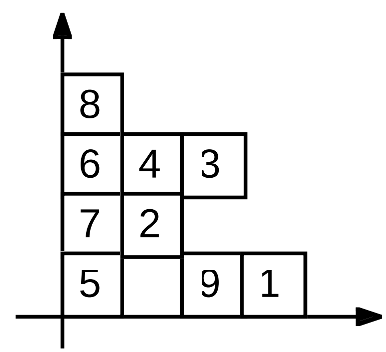
$$\mathbf{inv_U}(\sigma) = \mathbf{inv}\big(\mathbf{T}(\sigma)\big)\,, \qquad \mathbf{maj_U}(\sigma) = \mathbf{maj}\big(\mathbf{T}(\sigma)\big)$$

Example

$\sigma \;=\; \mathbf{864372591} \quad \mathbf{U} \;=$



$\longrightarrow \qquad \mathbf{T}(\sigma) =$



Another q-analogue of the Hook formula

Theorem

$$\sum_{\sigma\in\mathbf{S_n}} \mathbf{q}^{\mathbf{inv_U}(\sigma)\ +\ \mathbf{n}(\mu)-\mathbf{maj_U}(\sigma)} \ = \ \mathbf{f}_\mu \ \prod_{\mathbf{c}\in\mu}[\mathbf{h}(\mathbf{c})]_\mathbf{q}$$

Proof

$$\mathbf{LHS} \ = \ \mathbf{P}_\mu(\mathbf{x};\mathbf{q},\mathbf{t})\Big|_{\mathbf{x_1 x_2 \cdots x_n}} \ = \ \tilde{\mathbf{H}}_\mu(\mathbf{x};\mathbf{q},\mathbf{t})\Big|_{\mathbf{x_1 x_2 \cdots x_n}} \ = \ \mathbf{F}_\mu(\mathbf{q},\mathbf{t})$$

where

$$\mathbf{F}_\mu(\mathbf{q},\mathbf{t}) \ = \ \textbf{Hilbert series of the Garsia-Haiman modules}$$

and we proved that

$$\frac{\mathbf{q}^{\mathbf{n}(\mu)}\mathbf{F}_\mu(\mathbf{q},\mathbf{q}^{-1})}{\prod_{\mathbf{c}\in\mu}[\mathbf{h}(\mathbf{c})]_\mathbf{q}} \ = \ \sum_{\nu\to\mu} \frac{\mathbf{q}^{\mathbf{n}(\mu)}\mathbf{F}_\nu(\mathbf{q},\mathbf{q}^{-1})}{\prod_{\mathbf{c}\in\nu}[\mathbf{h}(\mathbf{c})]_\mathbf{q}}$$

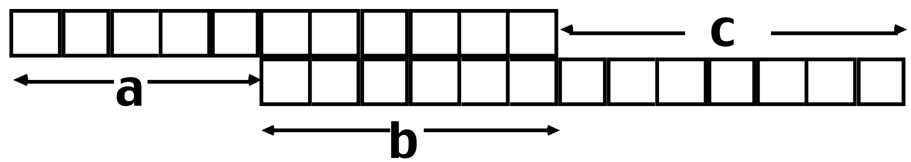
with appropriate initial conditions

Open problem:

Combinatorial Proof ?

PISTOL POLYNOMIALS

A pistol shape $\mathbf{\Pi[a,b,c]}$



The pistol polynomials may be defined as the solutions of the recursion

$$\mathbf{\Pi[a,b,c]} = \frac{\mathbf{t-q^{a-1}}}{\mathbf{t-q^c}}\ \mathbf{P[a-1,b,c+1]} + \frac{\mathbf{1-q^c}}{\mathbf{t-q^c}}\ \mathbf{\Pi[a-1,b+1,c-1]} + \frac{\mathbf{q^{a-1}-1}}{\mathbf{t-q^c}}\ \mathbf{\Pi[a-2,b+1,c]}\ ,$$

with the initial conditions

$$(1) \quad \mathbf{\Pi[-1,b,c]} = \mathbf{0}$$

$$(2) \quad \mathbf{\Pi[0,b,c]} = \mathbf{\tilde{H}_{(b,b+c)}(x;q,t)}$$

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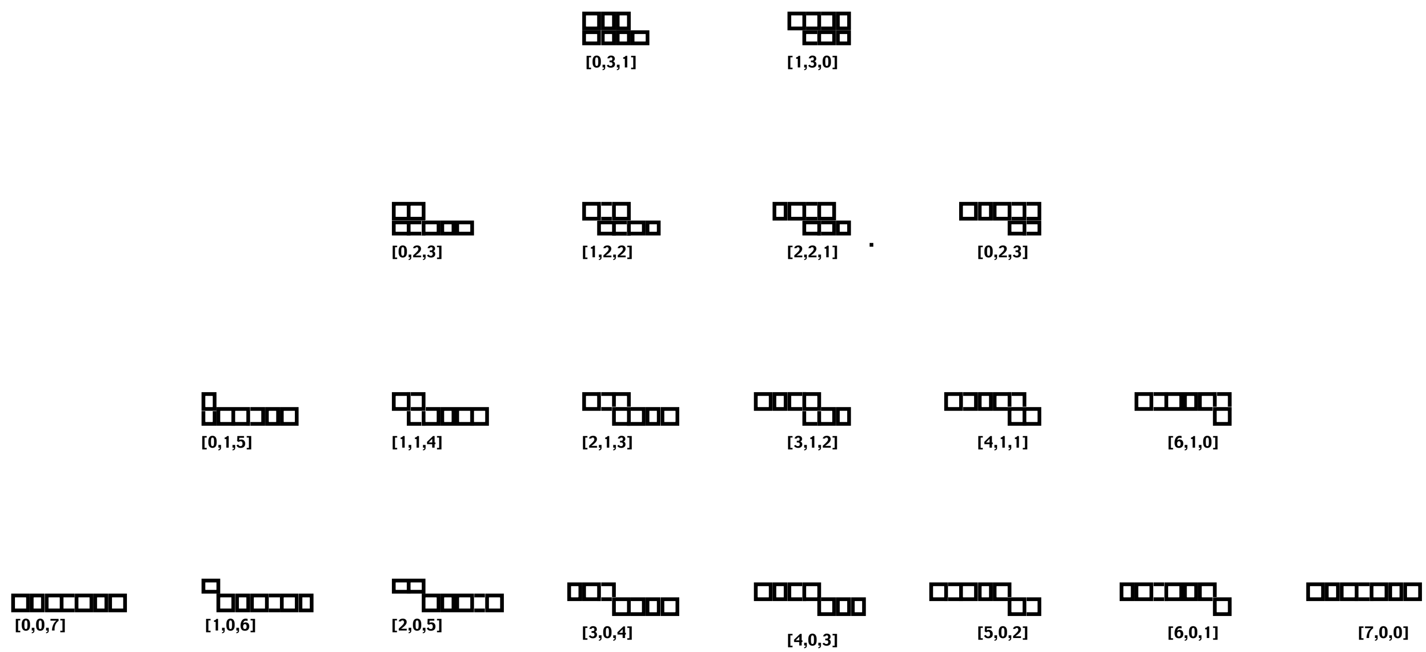
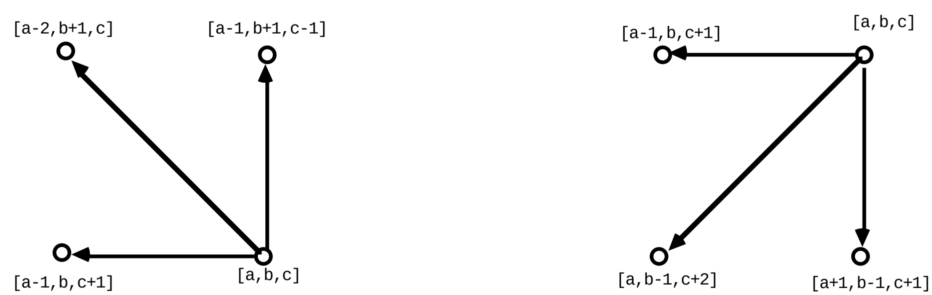
RECURSIONS


[0,3,1]
[1,3,0]
[0,2,3]
[1,2,2]
[2,2,1]
[0,2,3]
[0,1,5]
[1,1,4]
[2,1,3]
[3,1,2]
[4,1,1]
[6,1,0]
[0,0,7]
[1,0,6]
[2,0,5]
[3,0,4]
[4,0,3]
[5,0,2]
[6,0,1]
[7,0,0]


[a-2,b+1,c]
[a-1,b+1,c-1]
[a-1,b,c+1]
[a,b,c]
[a-1,b,c+1]
[a,b,c]
[a,b-1,c+2]
[a+1,b-1,c+1]

Remarkable Computer data

Gives

$$\mathbf{\Pi[a,b,c](x;q,t) \quad = \quad P_{\Pi[a,b,c]}(x;q,t)}$$

Problem

Prove it or find a counterexample

This brings us to the paper

Lattice Diagram Polynomials

and
Extended Pieri Rules

F. Bergeron, N. Bergeron, A. M. Garsia, M. Haiman and G. Tesler

LATTICE POLYNOMIALS

(2,0)	(2,1)	(2,2)	(2,3)
(1,0)	(1,1)	(1,2)	(1,3)
(0,0)	(0,1)	(0,2)	(0,3)

$$\mathbf{L} \;=\; \{\, (\mathbf{0},\mathbf{2})\,,\; (\mathbf{0},\mathbf{3})\,,\; (\mathbf{1},\mathbf{1})\,,\; (\mathbf{1},\mathbf{2})\,,\; (\mathbf{2},\mathbf{3})\,\}$$

$$\mathbf{\Delta_L} \;=\; \mathbf{det} \begin{pmatrix} y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 \\ y_1^3 & y_2^3 & y_3^3 & y_4^3 & y_5^3 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_4y_5 \\ x_1y_1^2 & x_2y_2^2 & x_3y_3^2 & x_4y_4^2 & x_4y_5^2 \\ x_1^2y_1^3 & x_2^2y_2^3 & x_3^2y_3^3 & x_4^2y_4^3 & x_5^2y_5^3 \end{pmatrix}$$

More generally if

$$\mathbf{L} \;=\; \{\, (\mathbf{p_1},\mathbf{q_1})\,,\; (\mathbf{p_2},\mathbf{q_2})\,,\; \ldots\;,\; (\mathbf{p_n},\mathbf{q_n})\,\}$$

$$\mathbf{\Delta_L} \;=\; \mathbf{det}\, \|\mathbf{x_i^{p_j} y_i^{q_j}}\, \|_{\mathbf{i,j=1}}^{\mathbf{n}}$$

ACTION OF THE OPERATORS

$$\mathbf{D_{h,k}} \;=\; \sum_{\mathbf{i=1}}^{\mathbf{n}} \partial_{\mathbf{x_i}}^{\mathbf{h}} \partial_{\mathbf{y_i}}^{\mathbf{k}}$$

Proposition 1

Let $L = \{(p_1, q_1)\ ,\ (p_2, q_2)\ ,\ \ldots\ ,\ (p_n, q_n)\ \}$ *be a lattice diagram. Then for any integers* $h, k \geq 0$ *(with* $h + k \geq 1$*) we have*

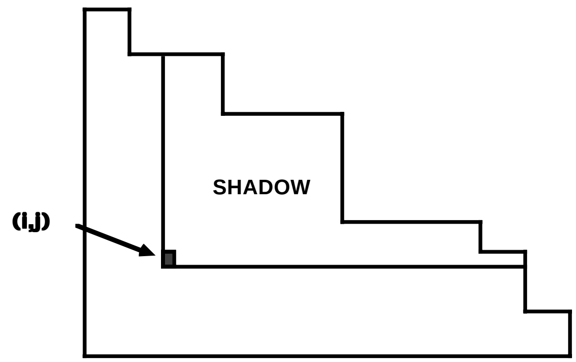
$$\sum_{\mathbf{i=1}}^{\mathbf{n}} \partial_{\mathbf{x_i}}^{\mathbf{h}}\ \partial_{\mathbf{y_i}}^{\mathbf{k}}\ \mathbf{\Delta_L(x; y)} \;=\; \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{c(L \downarrow_{hk}^{i})\ \Delta_{L\downarrow_{hk}^{i}}(x; y)}$$

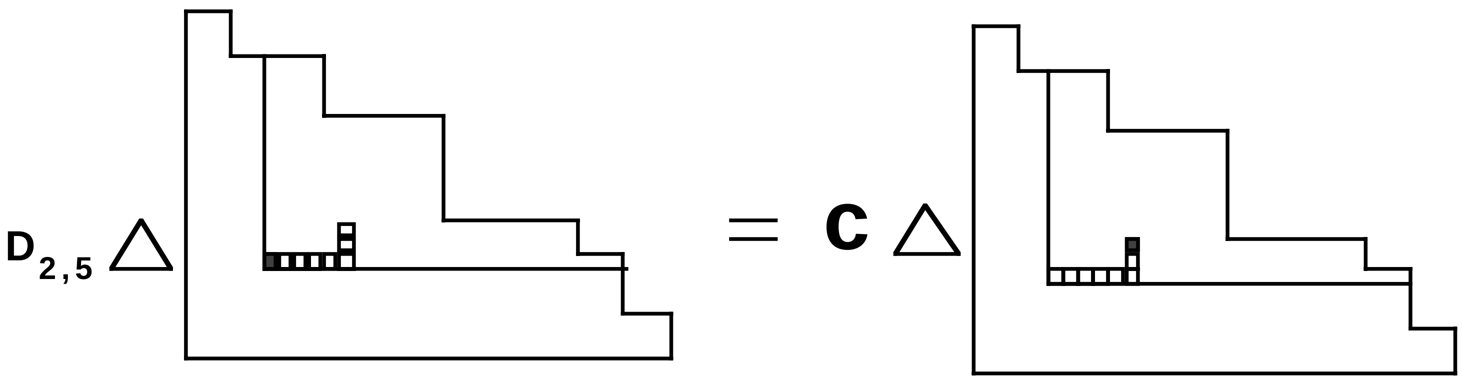
where

$$\mathbf{L \downarrow_{hk}^{i} = \{(p_1, q_1)\ ,\ \ldots (p_i - h, q_i - k)\ ,\ \ldots\ ,\ (p_n, q_n)\ \}}$$

and the coefficient $\mathbf{c(L \downarrow_{hk}^{i})}$ *is different from zero only if* $(p_i - h, q_i - k)$ *in the positive quadrant and* $L \downarrow_{hk}^{i}$ *consist of* n *distinct cells.*

PIERCED FERRERS DIAGRAMS





LATTICE MODULES

$$\mathbf{M_L} \;=\; \mathcal{L}\left[\, \partial_{\mathbf{x}}^{\mathbf{p}} \partial_{\mathbf{y}}^{\mathbf{q}}\, \mathbf{\Delta_L} \,\right]$$

Conjecture 1(*now false → interesting problem)*

For any Lattice diagram L with n cells, the module $\mathbf{M}_L$ decomposes into a direct sum of left regular representations of S_n.

RECALL

n! Conjecture (Now Theorem)

For any Ferrers diagram μ with $\mathbf{n}$ cells, the module

$$\mathbf{M}_\mu \;=\; \mathbf{L}\left[\, \partial_{\mathbf{x}}^{\mathbf{p}} \partial_{\mathbf{y}}^{\mathbf{q}}\, \mathbf{\Delta}_\mu \,\right]$$

gives a bigraded version of the left regular representations of $\mathbf{S_n}$ and

$$\tilde{\mathbf{H}}_\mu(\mathbf{x};\mathbf{q},\mathbf{t}) \;=\; \textbf{bigraded Frobenius characteristic of } \mathbf{M}_\mu$$

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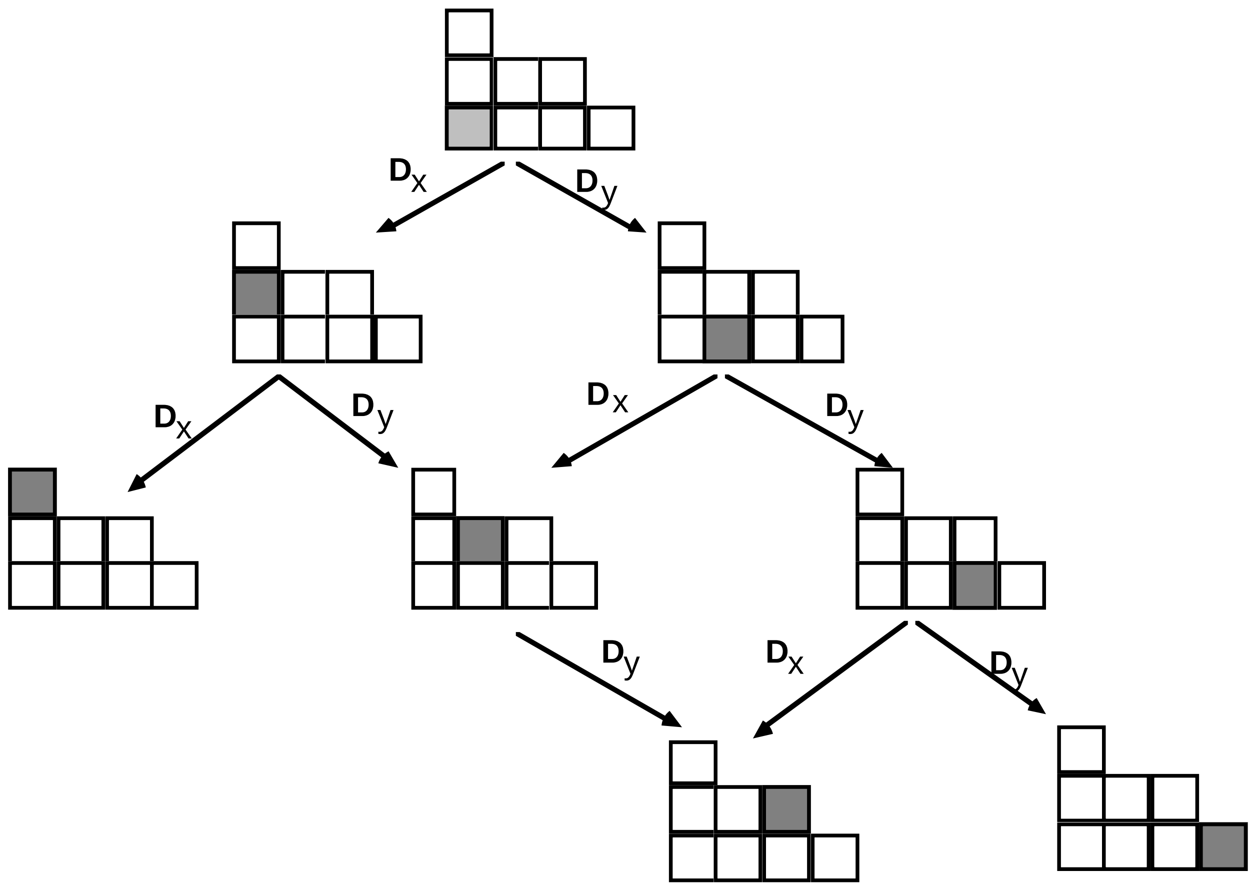
THE CASE OF (4,3,1)



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THE DIMENSION OF pierced-FERRERS MODULES

Proposition 3

The collection of polynomials

$$\{\ \boldsymbol{\Delta}_{\mu/\mathbf{i'j'}}(\mathbf{x};\mathbf{y})\ :\ (\mathbf{i'},\mathbf{j'})\in\mu\ \textbf{and}\ (\mathbf{i'},\mathbf{j'})\geq(\mathbf{i},\mathbf{j})\ \}$$

forms a basis for the submodule of alternants of $\mathbf{M}_{\mu/ij}$.

Conjecture*(Would follow from Conjecture 1)*

For any $\mu\vdash\mathbf{n+1}$ *and any* $(\mathbf{i},\mathbf{j})\in\mu$ *the* S_n*-module* $\mathbf{M}_{\mu/ij}$ *decomposes into the direct sum of* $\mathbf{m}$ *left regular representations of* $\mathbf{S_n}$ *where* $\mathbf{m}$ *gives the number of cells in the shadow of* $(\mathbf{i},\mathbf{j})$.

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BIGRADED FROBENIUS CHARACTERISTICS

For

$$\mathbf{L} \;=\; \{\, (\mathbf{p_1},\mathbf{q_1})\,,\ (\mathbf{p_2},\mathbf{q_2})\,,\ \ldots\ ,\ (\mathbf{p_n},\mathbf{q_n})\,\}$$

$$\mathbf{M_L} \;=\; \bigoplus_{\mathbf{r=0}}^{|\mathbf{p}|} \bigoplus_{\mathbf{s=0}}^{|\mathbf{q}|} \mathcal{H}_{\mathbf{r,s}}[\mathbf{M_L}]$$

$$|\mathbf{p}| \;=\; \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{p_i} \quad , \qquad |\mathbf{q}| \;=\; \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{q_i}$$

$$\mathbf{C_L}(\mathbf{x};\mathbf{q},\mathbf{t}) \;=\; \sum_{\mathbf{r=0}}^{|\mathbf{p}|} \sum_{\mathbf{s=0}}^{|\mathbf{q}|} \mathbf{t^r}\ \mathbf{q^s}\ \mathbf{F\ ch}\ \ \mathcal{H}_{\mathbf{r,s}}[\mathbf{M_L}]$$

.

THE CHARACTERISTIC OF P-FERRERS MODULES

Conjecture 2

For any $(\mathbf{i},\mathbf{j})\in\mu$ *we have*

$$\mathbf{C}_{\mu/\mathbf{ij}}(\mathbf{x};\mathbf{q},\mathbf{t}) \;=\; \sum_{\beta\to\alpha} \mathbf{c}_{\alpha\beta}(\mathbf{q},\mathbf{t})\;\tilde{\mathbf{H}}_{\mu-\alpha+\beta}\;(\mathbf{x};\mathbf{q},\mathbf{t}) \qquad (*)$$

where α *denotes the partition corresponding to the shadow of* $(\mathbf{i},\mathbf{j})$ *and the symbol "*$\mu-\alpha+\beta$*" is to represent replacing* α *by* β *in the shadow of* $(\mathbf{i},\mathbf{j})$*. Where the* $\mathbf{c}_{\alpha\beta}(\mathbf{q},\mathbf{t})$ *are the same as those giving the identity*

$$\partial_{\mathbf{p_1}}\tilde{\mathbf{H}}_\alpha \;=\; \sum_{\beta\to\alpha}\mathbf{c}_{\alpha\beta}(\mathbf{q},\mathbf{t})\tilde{\mathbf{H}}_\rho$$

Theorem 2

Equation (*) above is equivalent to the four term recursion

$$\mathbf{C}_{\mu/\mathbf{ij}} \;=\; \frac{\mathbf{t^l}-\mathbf{q^{a+1}}}{\mathbf{t^l}-\mathbf{q^a}}\,\mathbf{C}_{\mu/\mathbf{i},\mathbf{j+1}} + \frac{\mathbf{t^{l+1}}-\mathbf{q^{a}}}{\mathbf{t^l}-\mathbf{q^a}}\,\mathbf{C}_{\mu/\mathbf{i+1},\mathbf{j}} - \frac{\mathbf{t^{l+1}}-\mathbf{q^{a+1}}}{\mathbf{t^l}-\mathbf{q^a}}\,\mathbf{C}_{\mu/\mathbf{i+1},\mathbf{j+1}}\;,$$

with $\mathbf{l}$ *and* $\mathbf{a}$ *the leg an arm of* $(\mathbf{i},\mathbf{j})$ *in* μ*.*

KERNELS AND ATOMS

Let $\mathbf{K}^x_{ij}$ *denote the kernel of the operator* $\mathbf{D_x}$ *as a map of* $\mathbf{M}_{ij}$ *onto* $\mathbf{M}_{i+1,j}$. *Similarly, let* $\mathbf{K}^y_{ij}$ *be the kernel of* $\mathbf{D_y}$ *as a map of* $\mathbf{M}_{ij}$ *onto* $\mathbf{M}_{i,j+1}$. *We have*

$$\mathbf{K}^x_{i,j+1} \subseteq \mathbf{K}^x_{ij} \qquad \textit{as well as} \qquad \mathbf{K}^y_{i+1,j} \subseteq \mathbf{K}^y_{ij}$$

Set

$$\mathbf{A}^x_{ij} = \mathbf{K}^x_{ij} / \mathbf{K}^x_{i,j+1} \qquad \textit{and} \qquad \mathbf{A}^y_{ij} = \mathbf{K}^y_{ij} / \mathbf{K}^y_{i+1,j}$$

and let A^x_{ij} *and* A^y_{ij} *denote their respective Frobenius characteristics.*

Proposition I.6

$$\mathbf{K^x_{ij}} = \mathbf{C}_{\mu/\mathbf{ij}} - \mathbf{t}\, \mathbf{C}_{\mu/\mathbf{i+1j}} \qquad \textbf{and} \qquad \mathbf{K^y_{ij}} = \mathbf{C}_{\mu/\mathbf{ij}} - \mathbf{q}\, \mathbf{C}_{\mu/\mathbf{ij+1}}$$

and

$$\mathbf{A^x_{ij}} = \mathbf{K^x_{ij}} - \mathbf{K^x_{ij+1}} \qquad \textbf{and} \qquad \mathbf{A^y_{ij}} = \mathbf{K^y_{ij}} - \mathbf{K^y_{i+1j}}$$

Then the four term recursion reduces to

$$\mathbf{t^l}\, \mathbf{A^x_{ij}} = \mathbf{q^a}\, \mathbf{A^y_{ij}}$$

$$\mathbf{\Xi}_{\mu/\mathbf{ij}} = \mathbf{q^{-a}} \mathbf{A^x_{ij}} = \mathbf{t^{-l}} \mathbf{A^y_{ij}}$$

Remarkable Computer data

(1) *In a variety of cases we get*

$$\mathbf{\Xi}_{\mu/\mathbf{ij}} \;=\; \mathbf{P}_{\mu/\mathbf{ij}}(\mathbf{x};\mathbf{q},\mathbf{t})$$

But not in all cases

WHY?

(2) *In a variety of cases we get for* $\mu \vdash \mathbf{n+1}$ *with minor modification of the statistics*

$$\mathbf{C}_{\mu/\mathbf{ij}}(\mathbf{x};\mathbf{q},\mathbf{t}) \;=\; \sum_{\substack{\sigma\in\mathbf{S_{n+1}}\\ \mathbf{n+1}\ \textbf{in the shadow of ij}}} \mathbf{q}^{\mathbf{inv}_{\mu/\mathbf{ij}}(\sigma)}\mathbf{t}^{\mathbf{inv}_{\mu/\mathbf{ij}}(\sigma)}\mathbf{F}_{\mathbf{ID}(\sigma)}(\mathbf{x})$$

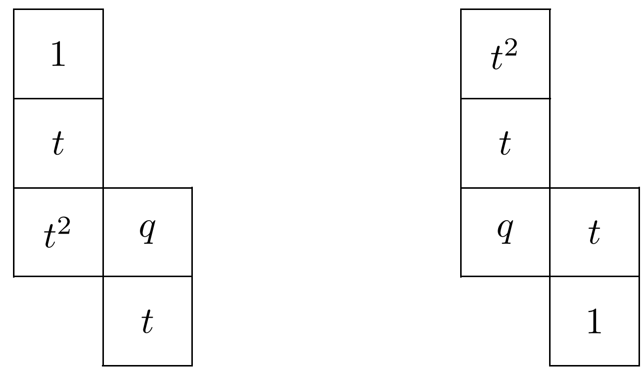
But not in all cases

WHY?

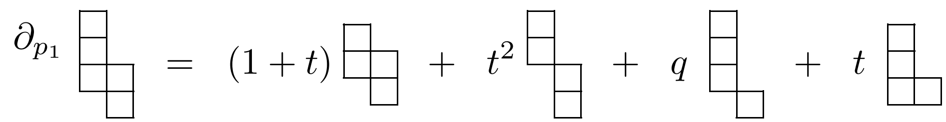
Problem:

Make it work in all cases

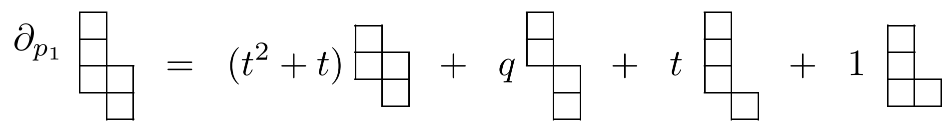
The Gistol polynomials



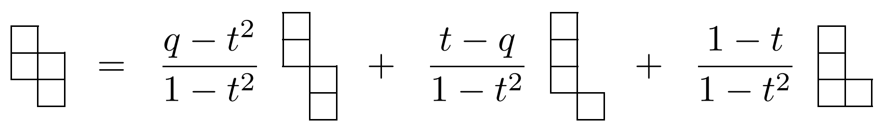
gives



and



so by subtraction



Some Pieri Rules

$$\ydiagram{1,1,1+1,1+1} = \frac{(1-t)(1-t^2)}{(q-t)(q-t^2)}\ \ydiagram{2,2} + \frac{(1-t)(q-1)(1+t)^2}{(q-t)(q-t^3)}\ \ydiagram{1,1,2} + \frac{(q-t)(q-1)}{(q-t^2)(q-t^3)}\ \ydiagram{1,1,1,1} \qquad (A)$$

and

$$\ydiagram{1,1,1,1+1} = \frac{t^3-1}{t^3-q}\ \ydiagram{1,1,2} + \frac{1-q}{t^3-q}\ \ydiagram{1,1,1,1}\,. \qquad (B)$$

Substituting (A) and (B) in

$$\ydiagram{1,2,1+1} = \frac{q-t^2}{1-t^2}\ \ydiagram{1,1,1+1,1+1} + \frac{t-q}{1-t^2}\ \ydiagram{1,1,1,1+1} + \frac{1-t}{1-t^2}\ \ydiagram{1,1,2}$$

gives

$$\ydiagram{1,2,1+1} = \frac{1-t}{q-t}\ \ydiagram{2,2} + \frac{q-1}{q-t}\ \ydiagram{1,1,2}\,,$$

Remarkable computer data

Many Gistol polynomials are identical to the corresponding Haglund polynomials

Problem

WHY?