

r -Qsym is free over Sym

A proof

by

A. M. Garsia and N. Wallach

Notation

2

$$\mathbf{X}_n = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$$

$\mathbf{Q}[\mathbf{X}_n]$ denotes the algebra of polynomials in $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ with rational coefficients.

V a graded vector space

$\mathcal{H}_m(V)$ denotes the subspace of the homogeneous elements of degree m in V .

$$V = \mathcal{H}_0(V) \oplus \mathcal{H}_1(V) \oplus \mathcal{H}_2(V) \oplus \dots \oplus \mathcal{H}_m(V) \oplus \dots$$

The Hilbert series of V

$$\mathbf{F}_V(\mathbf{q}) = \sum_{m \geq 0} \dim(\mathcal{H}_m(V)) \mathbf{q}^m$$

Basics

Let \mathbf{A} be a Finitely Generated Graded Algebra.

$$\mathbf{A} = \bigoplus_{m \geq 0} \mathcal{H}_m(\mathbf{A})$$

Let

- (1) \mathbf{n}_1 = the order of “ $\mathbf{1}$ ” as a pole of the Hilbert series $\mathbf{F}_{\mathbf{A}}(\mathbf{q})$.
- (2) \mathbf{n}_2 = the maximum number of algebraically independent elements in \mathbf{A} .
- (3) \mathbf{n}_3 = the minimum number of elements $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n \in \mathbf{A}$ such that

$$\dim \mathbf{A}/(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n)_{\mathbf{A}} < \infty \quad (*)$$

Fact: $\mathbf{n}_1 = \mathbf{n}_2 = \mathbf{n}_3 = \mathbf{n}_{\mathbf{A}}$ = the “*Krull dimension*” of \mathbf{A}

If $(*)$ holds with $\mathbf{n} = \mathbf{n}_{\mathbf{A}}$ then $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ are called a “*System Of Parameters*”. “**S.O.P.**” in brief.

More Basics

Let $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n \in \mathbf{A}$ be homogeneous of degrees $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_n$ and suppose they constitute a **S.O.P** for \mathbf{A} . Let $\mathbf{B} = \{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_N\}$ be a basis for the quotient

$$\mathbf{A}/((\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n)_\mathbf{A}) < \infty$$

Then every $\mathbf{P} \in \mathbf{A}$ has an expansion of the form

$$\mathbf{P} = \sum_{i=1}^N \mathbf{f}_i \mathbf{Q}_i(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n)$$

with $\mathbf{Q}_1, \mathbf{Q}_2, \dots, \mathbf{Q}_N \in \mathbf{Q}[y_1, \dots, y_n]$. In other words the collection

$$\{\mathbf{f}_i \mathbf{q}_1^{\mathbf{p}_1}, \mathbf{q}_2^{\mathbf{p}_2}, \dots, \mathbf{q}_n^{\mathbf{p}_n}\}_{1 \leq i \leq N}$$

spans \mathbf{A} as a vector space. In particular it follows that

$$\mathbf{F}_\mathbf{A}(\mathbf{q}) \ll \frac{\sum_{i=1}^N \mathbf{q}^{\text{degree}(\mathbf{f}_i)}}{(\mathbf{1} - \mathbf{q}^{\mathbf{d}_1})(\mathbf{1} - \mathbf{q}^{\mathbf{d}_2}) \cdots (\mathbf{1} - \mathbf{q}^{\mathbf{d}_n})}$$

The Cohen Macaulay Property

If the expansion

$$\mathbf{P} = \sum_{i=1}^N \mathbf{f}_i \mathbf{Q}_i(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n)$$

is **unique** for every $\mathbf{P} \in \mathbf{A}$ then the collection

$$\{\mathbf{f}_i \mathbf{q}_1^{\mathbf{P}_1}, \mathbf{q}_2^{\mathbf{P}_2}, \dots, \mathbf{q}_n^{\mathbf{P}_n}\}_{1 \leq i \leq N}$$

is a **basis** for \mathbf{A} . This holds true **if and only if** we have the equality

$$\mathbf{F}_\mathbf{A}(\mathbf{q}) = \frac{\sum_{i=1}^N \mathbf{q}^{\text{degree}(\mathbf{f}_i)}}{(1 - \mathbf{q}^{\mathbf{d}_1})(1 - \mathbf{q}^{\mathbf{d}_2}) \cdots (1 - \mathbf{q}^{\mathbf{d}_n})}$$

Then \mathbf{A} is a free module over $\mathbf{Q}[\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n]$ of rank N and \mathbf{A} is said to be “Cohen-Macaulay”.

A useful criterion

Theorem A

6

Let $q_1, q_2, \dots, q_n \in A$ be homogeneous of degrees d_1, d_2, \dots, d_n and an S.O.P. for A . Let

$$\dim A / (q_1, q_2, \dots, q_n)_A = M \leq N$$

with basis $B = \{f_1, f_2, \dots, f_M\}$. Then the condition

$$\lim_{\mathbf{q} \rightarrow \mathbf{1}} (1 - q^{d_1})(1 - q^{d_2}) \cdots (1 - q^{d_n}) F_A(\mathbf{q}) = N$$

forces $M = N$ and the equality

$$F_A(\mathbf{q}) = \frac{\sum_{i=1}^N q^{\text{degree}(f_i)}}{(1 - q^{d_1})(1 - q^{d_2}) \cdots (1 - q^{d_n})}$$

yielding that A is a free module over $\mathbb{Q}[q_1, q_2, \dots, q_n]$ of rank N and therefore A is a Cohen-Macaulay algebra.

r-Quasi-Symmetric

Recall the Florent Hivert “local” r-action of \mathbf{S}_n

$$s_i x_i^a x_{i+1}^b = \begin{cases} x_i^a x_{i+1}^b & \text{if } a, b \geq r \\ x_i^b x_{i+1}^a & \text{otherwise} \end{cases} \quad (*)$$

here $\mathbf{s}_i = \mathbf{s}_{i,i+1}$. Since we have

$$s_i^2 = \mathbf{1}, \quad \mathbf{s}_i \mathbf{s}_{i+1} \mathbf{s}_i = \mathbf{s}_{i+1} \mathbf{s}_i \mathbf{s}_{i+1}, \quad \mathbf{s}_i \mathbf{s}_j = \mathbf{s}_j \mathbf{s}_i \quad \forall |i - j| \geq 2$$

(*) defines an action of \mathbf{S}_n called the “r-action.” Florent Hivert defines

$$\mathbf{r-QSym} = \left\{ \mathbf{P}(\mathbf{x}) : \mathbf{P}(\mathbf{x}) \text{ is invariant under the r-action} \right\}$$

We have a descending chain of algebras

$$\mathbf{Q}[\mathbf{X}_n] \supset \mathbf{1-QSym}[\mathbf{X}_n] \supset \mathbf{2-QSym}[\mathbf{X}_n] \supset \cdots \supset \mathbf{r-QSym}[\mathbf{X}_n] \supset \cdots \supset \mathbf{Sym}[\mathbf{X}_n]$$

Note that $\mathbf{1-QSym}[\mathbf{X}_n]$ is the space of Gessel’s “quasi-symmetric functions”.

Preliminaries

For $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_k)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h)$ with $\mathbf{k} + \mathbf{h} \leq \mathbf{n}$ set

$$M_{\mathbf{p}, \lambda}[\mathbf{X}_n] = \sum_{\substack{\mathbf{T} \subseteq \{1, 2, \dots, n\} \\ |\mathbf{T}| = \mathbf{k}}} \mathbf{X}_{\mathbf{T}}^{\mathbf{p}} m_{\lambda}[\mathbf{X}_n - \mathbf{X}_{\mathbf{T}}] \quad \text{I.5}$$

where if $\mathbf{T} = \{\mathbf{i}_1 < \mathbf{i}_2 < \dots < \mathbf{i}_k\}$ then $\mathbf{X}_{\mathbf{T}}^{\mathbf{p}} = x_{\mathbf{i}_1}^{\mathbf{p}_1} x_{\mathbf{i}_2}^{\mathbf{p}_2} \dots x_{\mathbf{i}_k}^{\mathbf{p}_k}$

It was show by Florent Hivert that the collection

$$\left\{ M_{\mathbf{p}, \lambda}[\mathbf{X}_n] \right\}_{\lambda < \mathbf{r}, \mathbf{p} \geq \mathbf{r}}$$

is a basis for $r\text{QSym}[\mathbf{X}_n]$

First Breakthroughs

9

- (1) François and Nantel Bergeron discover that the collection of λ -free quasi-monomials

$$\left\{ M_{\mathbf{p}+\mathbf{r}}[\mathbf{X}_n] \right\}_{\mathbf{p}}$$

spans the quotient $\mathbf{rQSym}[\mathbf{X}_n]/(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{\mathbf{rQSym}[\mathbf{X}_n]}$. Here

$$M_{\mathbf{p}+\mathbf{r}}[\mathbf{X}_n] = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \mathbf{x}_{i_1}^{p_1+r} \mathbf{x}_{i_2}^{p_2+r} \dots \mathbf{x}_{i_k}^{p_k+r}$$

- (2) François and Nantel conjecture the existence of a collection C_n of semi-compositions such that the collection

$$\left\{ M_{\mathbf{p}+\mathbf{r}}[\mathbf{X}_n] \right\}_{\mathbf{p} \in C_n}$$

is a basis for this quotient for **all** $\mathbf{r} \geq \mathbf{1}$ (!!!!!)

The Hilbert series of $rQsym$

10

Proposition

$$F_{rQsym[X_n]}(\mathbf{q}) = \frac{\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q D_k(\mathbf{q}) t^k}{(1 - \mathbf{q})(1 - \mathbf{q}^2) \cdots (1 - \mathbf{q}^n)}$$

with $t = \mathbf{q}^r$ and

$$D_n(\mathbf{q}) = [n]_{\mathbf{q}}! \sum_{s=0}^n \mathbf{q}^{\binom{s}{2}} \frac{(-1)^s}{[s]_{\mathbf{q}}!} \quad (\text{a } \mathbf{q}\text{-Derangement number!!})$$

Proof

$$\begin{aligned} F_{rQsym[X_n]}(\mathbf{q}) &= \sum_{k=0}^n \frac{\mathbf{q}^{r(n-k)} \begin{bmatrix} k+r-1 \\ r-1 \end{bmatrix}_q}{(1 - \mathbf{q})^{n-k}} \\ &= \sum_{k=0}^n \frac{\mathbf{q}^{r(n-k)} (1 - \mathbf{q}^r)(1 - \mathbf{q}^{r+1}) \cdots (1 - \mathbf{q}^{r+k-1})}{(1 - \mathbf{q})^{n-k} (1 - \mathbf{q})(1 - \mathbf{q}^2) \cdots (1 - \mathbf{q}^k)} \\ &= \sum_{k=0}^n \frac{t^{n-k} (1 - t)(1 - tq) \cdots (1 - tq^{k-1})}{(1 - \mathbf{q})^{n-k} (1 - \mathbf{q})(1 - \mathbf{q}^2) \cdots (1 - \mathbf{q}^k)} \end{aligned}$$

Use the q -binomial Theorem and rearrange terms.

The operator T_n

Let

$$T_n = (\mathbf{1}, \mathbf{2}, \dots, \mathbf{n}) + (\mathbf{2}, \mathbf{3}, \dots, \mathbf{n}) + \dots + (\mathbf{n} - \mathbf{1}, \mathbf{n}) + \mathbf{id}$$

Theorem(Phatarfod, Diaconis, etc)

The (right) action of T_n on the group algebra of S_n is semisimple with eigenvalues $(0, 1, 2, \dots, n-2, n)$ and the multiplicity of k is the number of permutations with k fixed points. In particular the multiplicity of 0 is the derangement number D_n . It follows from this that

$$\sum_{k=0}^n \binom{n}{k} D_k = n!$$

A basic Fact

Set

12

$$\mathbf{S}_{n\mathbf{k}} = \left\{ \mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_k) : \mathbf{0} \leq \mathbf{p}_i \leq n - \mathbf{i}, \text{ for } 1 \leq \mathbf{i} \leq k \right\}$$

We call the elements of $\mathbf{S}_{n\mathbf{k}}$ “ n -subtriangular semi-compositions of length \mathbf{k} ”

Let

$$\mathbf{I}_{n,\mathbf{k}} = (\mathbf{h}_{n+1-k}[\mathbf{X}_{\mathbf{k}}], \mathbf{h}_{n+2-k}[\mathbf{X}_{\mathbf{k}}], \dots, \mathbf{h}_n[\mathbf{X}_{\mathbf{k}}])_{\mathbf{Q}[\mathbf{X}_{\mathbf{k}}]}$$

Theorem

The quotient $\mathbf{R}_{n,\mathbf{k}} = \mathbf{Q}[\mathbf{X}_{\mathbf{k}}]/\mathbf{I}_{n\mathbf{k}}$ affords $\binom{n}{\mathbf{k}}$ copies of the regular representation of $\mathbf{S}_{\mathbf{k}}$, has basis

$$\mathbf{A}_{n,\mathbf{k}} = \left\{ \mathbf{x}_1^{\mathbf{a}_1} \mathbf{x}_2^{\mathbf{a}_2} \cdots \mathbf{x}_{\mathbf{k}}^{\mathbf{a}_{\mathbf{k}}} \right\}_{\mathbf{a} \in \mathbf{S}_{n\mathbf{k}}}$$

and Frobenius characteristic

$$\mathbf{Fch} \mathbf{R}_{n,\mathbf{k}} = \left[\begin{matrix} n \\ \mathbf{k} \end{matrix} \right]_{\mathbf{q}} \sum_{\lambda \vdash \mathbf{k}} \mathbf{S}_{\lambda}(\mathbf{x}) \sum_{\mathbf{T} \in \mathbf{ST}(\lambda)} \mathbf{q}^{\text{maj}(\mathbf{T})}$$

Third BreakThrough

13

For $\mathbf{f} = \sum_{\mathbf{p}} c_{\mathbf{p}} \mathbf{X}_{\mathbf{k}}^{\mathbf{p}}$ set $\Gamma_{\mathbf{n},\mathbf{k}}^{(r)} \mathbf{f} = \sum_{\mathbf{p}} c_{\mathbf{p}} \mathbf{M}_{\mathbf{p}+\mathbf{r}}[\mathbf{X}_{\mathbf{n}}]$.

Theorem

For each $1 \leq \mathbf{k} \leq \mathbf{n}$ we have

$$\Gamma_{\mathbf{n},\mathbf{k}}^{(r)} \mathbf{I}_{\mathbf{n},\mathbf{k}} \subseteq (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{\mathbf{n}})_{\mathbf{r}-\mathbf{Qsym}}[\mathbf{X}_{\mathbf{n}}] \cdot$$

In particular $\Gamma_{\mathbf{n},\mathbf{k}}^{(r)}$ is well defined as an operator from $\mathbf{R}_{\mathbf{n},\mathbf{k}}$ into

$$\mathbf{rQsym}[\mathbf{X}_{\mathbf{n}}] / (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{\mathbf{n}})_{\mathbf{rQsym}}[\mathbf{X}_{\mathbf{n}}] \cdot$$

Likewise if $\mathcal{B}_{\mathbf{n},\mathbf{k}} \subseteq \mathbf{Q}[\mathbf{X}_{\mathbf{k}}]$ yields a basis for $\mathbf{R}_{\mathbf{n},\mathbf{k}}$, for $1 \leq \mathbf{k} \leq \mathbf{n}$, then the collection

$$\{\mathbf{1}\} \cup \bigcup_{\mathbf{k}=1}^{\mathbf{n}} \Gamma_{\mathbf{n},\mathbf{k}}^{(r)} \mathcal{B}_{\mathbf{n},\mathbf{k}}$$

spans $\mathbf{rQsym}[\mathbf{X}_{\mathbf{n}}]$ as a $\Lambda_{\mathbf{n}}$ module.

A simple consequence

14

Let $\mathbf{E}_{n,k}$ be a basis for the eigenvectors of \mathbf{T}_k on the quotient $\mathbf{R}_{n,k}$. From the previous theorem it follows that the collection

$$\{\mathbf{1}\} \cup \bigcup_{k=1}^n \Gamma_{n,k}^{(r)} \mathbf{E}_{n,k}$$

spans the quotient

$$\mathbf{rQsym}[\mathbf{X}_n] / (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{\mathbf{rQsym}[\mathbf{X}_n]}$$

However since $\mathbf{R}_{n,k}$ consists of $\binom{n}{k}$ copies of the left regular representation of \mathbf{S}_k the cardinality of this collection is

$$1 + \sum_{k=1}^n \binom{n}{k} k!$$

Too many!!!! What we have to throw away???

The crucial Happening

Since $\mathbf{R}_{n,k}$ is $\binom{n}{k}$ copies of the left regular representation, the subset $\mathbf{E}_{nk}^{\circ} \subset \mathbf{E}_{nk}$ of zero eigenvectors has cardinality

$$|\mathbf{E}_{nk}^{\circ}| = \binom{n}{k} D_k$$

and thus the subcollection

$$\{\mathbf{1}\} \cup \bigcup_{k=1}^n \Gamma_{n,k}^{(r)} \mathbf{E}_{n,k}^{\circ}$$

has cardinality

$$1 + \sum_{k=1}^n \binom{n}{k} D_k = n!$$

COULD IT BE!!!???

The crucial Happening

Since $\mathbf{R}_{n,k}$ is $\binom{n}{k}$ copies of the left regular representation, the subset $\mathbf{E}_{nk}^{\circ} \subset \mathbf{E}_{nk}$ of zero eigenvectors has cardinality

$$|\mathbf{E}_{nk}^{\circ}| = \binom{n}{k} D_k$$

and thus the subcollection

$$\{\mathbf{1}\} \cup \bigcup_{k=1}^n \Gamma_{n,k}^{(r)} \mathbf{E}_{n,k}^{\circ}$$

has cardinality

$$1 + \sum_{k=1}^n \binom{n}{k} D_k = n!$$

YES!!! YES!!!

IN FACT

$$\Gamma_{n,k}^{(r)}(\mathbf{E}_{nk} - \mathbf{E}_{nk}^o) \subset \mathbf{L} \left[\bigcup_{s=1}^{k-1} \Gamma_{ns}^{(r)} \mathbf{E}_{n,s}^o \right]$$

IN FACT

$$\Gamma_{nk}^{(r)}(\mathbf{E}_{nk} - \mathbf{E}_{nk}^o) \subset \mathbf{L} \left[\bigcup_{s=1}^{k-1} \Gamma_{ns}^{(r)} \mathbf{E}_{n,s}^o \right]$$

Why?

IN FACT

19

$$\mathbf{G}_{nk}(\mathbf{E}_{nk} - \mathbf{E}_{nk}^{\circ}) \subset \mathbf{L} \left[\bigcup_{s=1}^{k-1} \mathbf{G}_{ns} \mathbf{E}_{n,s}^{\circ} \right]$$

Multiplication by power symmetric functions does it!

IN FACT

$$\Gamma_{\mathbf{nk}}^{(r)}(\mathbf{E}_{\mathbf{nk}} - \mathbf{E}_{\mathbf{nk}}^{\circ}) \subset \mathbf{L} \left[\bigcup_{s=1}^{k-1} \Gamma_{\mathbf{ns}}^{(r)} \mathbf{E}_{\mathbf{n},s}^{\circ} \right]$$

20

Proof

Note that for any semi-composition \mathbf{b} of length $k-1$ and any $\mathbf{1} \leq \mathbf{j} \leq \mathbf{n}$ we have

$$\mathbf{p}_{\mathbf{j}+\mathbf{r}}[\mathbf{X}_{\mathbf{n}}] \mathbf{M}_{\mathbf{b}+\mathbf{r}}[\mathbf{X}_{\mathbf{n}}] = \sum_{\mathbf{q} \in \mathbf{b} \cup \mathbf{j}} \mathbf{M}_{\mathbf{q}+\mathbf{r}}[\mathbf{X}_{\mathbf{n}}] + \sum_{s=1}^{k-1} \mathbf{M}_{\mathbf{b}+\mathbf{r}+\mathbf{e}_s(\mathbf{r}+\mathbf{j})}[\mathbf{X}_{\mathbf{n}}]$$

with \mathbf{e}_s the coordinate $k-1$ -vector with $\mathbf{1}$ in position \mathbf{s} .

IN FACT

$$\Gamma_{\mathbf{nk}}^{(r)}(\mathbf{E}_{\mathbf{nk}} - \mathbf{E}_{\mathbf{nk}}^{\circ}) \subset \mathbf{L} \left[\bigcup_{s=1}^{k-1} \Gamma_{\mathbf{ns}}^{(r)} \mathbf{E}_{\mathbf{n},s}^{\circ} \right]$$

21

Proof

Note that for any semi-composition \mathbf{b} of length $k-1$ and any $1 \leq \mathbf{j} \leq \mathbf{n}$ we have

$$\mathbf{p}_{\mathbf{j}+\mathbf{r}}[\mathbf{X}_{\mathbf{n}}] \mathbf{M}_{\mathbf{b}+\mathbf{r}}[\mathbf{X}_{\mathbf{n}}] = \sum_{\mathbf{q} \in \mathbf{b} \cup \mathbf{j}} \mathbf{M}_{\mathbf{q}+\mathbf{r}}[\mathbf{X}_{\mathbf{n}}] + \sum_{s=1}^{k-1} \mathbf{M}_{\mathbf{b}+\mathbf{r}+\mathbf{e}_s(\mathbf{r}+\mathbf{j})}[\mathbf{X}_{\mathbf{n}}]$$

with \mathbf{e}_s the coordinate $k-1$ -vector with 1 in position s . This gives

$$\sum_{\mathbf{q} \in \mathbf{b} \cup \mathbf{j}} \mathbf{M}_{\mathbf{q}+\mathbf{r}}[\mathbf{X}_{\mathbf{n}}] \equiv - \sum_{s=1}^{k-1} \mathbf{M}_{\mathbf{b}+\mathbf{r}+\mathbf{e}_s(\mathbf{r}+\mathbf{j})}[\mathbf{X}_{\mathbf{n}}] \pmod{(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{\mathbf{r}-\mathbf{Q}_{\text{sym}}[\mathbf{X}_{\mathbf{n}}]}}$$

IN FACT

$$\Gamma_{nk}^{(r)}(\mathbf{E}_{nk} - \mathbf{E}_{nk}^o) \subset \mathbf{L} \left[\bigcup_{s=1}^{k-1} \Gamma_{ns}^{(r)} \mathbf{E}_{n,s}^o \right]$$

22

Proof

Note that for any semi-composition \mathbf{b} of length $k-1$ and any $1 \leq \mathbf{j} \leq \mathbf{n}$ we have

$$\mathbf{p}_{\mathbf{j}+\mathbf{r}}[\mathbf{X}_n] \mathbf{M}_{\mathbf{b}+\mathbf{r}}[\mathbf{X}_n] = \sum_{\mathbf{q} \in \mathbf{b} \cup \mathbf{j}} \mathbf{M}_{\mathbf{q}+\mathbf{r}}[\mathbf{X}_n] + \sum_{s=1}^{k-1} \mathbf{M}_{\mathbf{b}+\mathbf{r}+\mathbf{e}_s(\mathbf{r}+\mathbf{j})}[\mathbf{X}_n]$$

with \mathbf{e}_s the coordinate $k-1$ -vector with 1 in position s . This gives

$$\sum_{\mathbf{q} \in \mathbf{b} \cup \mathbf{j}} \mathbf{M}_{\mathbf{q}+\mathbf{r}}[\mathbf{X}_n] \equiv - \sum_{s=1}^{k-1} \mathbf{M}_{\mathbf{b}+\mathbf{r}+\mathbf{e}_s(\mathbf{r}+\mathbf{j})}[\mathbf{X}_n] \quad \left(\text{mod } (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{\mathbf{r}-\mathbf{Q}_{\text{sym}}[\mathbf{X}_n]} \right)$$

Thus

$$\sum_{\mathbf{q} \in \mathbf{b} \cup \mathbf{j}} \mathbf{M}_{\mathbf{q}+\mathbf{r}}[\mathbf{X}_n] \in \mathbf{L} \left[\Gamma_{n,k-1}^{(r)} \mathbf{E}_{n,k-1} \right] \quad \left(\text{mod } (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{\mathbf{r}-\mathbf{Q}_{\text{sym}}[\mathbf{X}_n]} \right)$$

The role of T_n

Let $\mathbf{b}[\mathbf{X}_k] \in \mathbf{E}_{nk} - \mathbf{E}_{nk}^\circ$ be of eigenvalue $\lambda \neq 0$ with

$$\mathbf{b}[\mathbf{X}_k] = \sum_{\alpha \in S_{nk}} \mathbf{c}_\alpha \mathbf{x}_1^{a_1} \mathbf{x}_2^{a_2} \cdots \mathbf{x}_k^{a_k}$$

The role of T_n

Let $\mathbf{b}[\mathbf{X}_k] \in \mathbf{E}_{nk} - \mathbf{E}_{nk}^o$ be of eigenvalue $\lambda \neq 0$ with

24

$$\mathbf{b}[\mathbf{X}_k] = \sum_{\alpha \in S_{nk}} \mathbf{c}_\alpha \mathbf{x}_1^{a_1} \mathbf{x}_2^{a_2} \cdots \mathbf{x}_k^{a_k}$$

Now setting $\mathbf{a}' = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{k-1})$ we have

$$\mathbf{b}[\mathbf{X}_k] = \frac{1}{\lambda} \sum_{\mathbf{a} \in S_{nk}} \mathbf{c}_a \mathbf{T}_k \mathbf{x}_1^{a_1} \mathbf{x}_2^{a_2} \cdots \mathbf{x}_k^{a_k} = \frac{1}{\lambda} \sum_{\mathbf{a} \in S_{nk}} \mathbf{c}_a \sum_{\mathbf{q} \in \mathbf{a}_k \cup \mathbf{a}'} \mathbf{X}_k^{\mathbf{q}}$$

The role of \mathbf{T}_n

Let $\mathbf{b}[\mathbf{X}_k] \in \mathbf{E}_{n_k} - \mathbf{E}_{n_k}^\circ$ be of eigenvalue $\lambda \neq 0$ with

25

$$\mathbf{b}[\mathbf{X}_k] = \sum_{\alpha \in \mathbf{S}_{n_k}} \mathbf{c}_\alpha \mathbf{x}_1^{a_1} \mathbf{x}_2^{a_2} \cdots \mathbf{x}_k^{a_k}$$

Now setting $\mathbf{a}' = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{k-1})$ we have

$$\mathbf{b}[\mathbf{X}_k] = \frac{1}{\lambda} \sum_{\mathbf{a} \in \mathbf{S}_{n_k}} \mathbf{c}_\mathbf{a} \mathbf{T}_k \mathbf{x}_1^{a_1} \mathbf{x}_2^{a_2} \cdots \mathbf{x}_k^{a_k} = \frac{1}{\lambda} \sum_{\mathbf{a} \in \mathbf{S}_{n_k}} \mathbf{c}_\mathbf{a} \sum_{\mathbf{q} \in \mathbf{a}_k \cup \mathbf{a}'} \mathbf{X}_k^{\mathbf{q}}$$

from which it follows that

$$\mathbf{\Gamma}_{n,k}^{(r)} \mathbf{b}[\mathbf{X}_k] = \frac{1}{\lambda} \sum_{\mathbf{a} \in \mathbf{S}_{n_k}} \mathbf{c}_\mathbf{a} \sum_{\mathbf{q} \in \mathbf{a}_k \cup \mathbf{a}'} \mathbf{M}_{\mathbf{q}+r}[\mathbf{X}_n].$$

The role of \mathbf{T}_n

Let $\mathbf{b}[\mathbf{X}_k] \in \mathbf{E}_{nk} - \mathbf{E}_{nk}^o$ be of eigenvalue $\lambda \neq 0$ with

26

$$\mathbf{b}[\mathbf{X}_k] = \sum_{\alpha \in \mathbf{S}_{nk}} \mathbf{c}_\alpha \mathbf{x}_1^{a_1} \mathbf{x}_2^{a_2} \cdots \mathbf{x}_k^{a_k}$$

Now setting $\mathbf{a}' = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{k-1})$ we have

$$\mathbf{b}[\mathbf{X}_k] = \frac{1}{\lambda} \sum_{\mathbf{a} \in \mathbf{S}_{nk}} \mathbf{c}_\mathbf{a} \mathbf{T}_k \mathbf{x}_1^{a_1} \mathbf{x}_2^{a_2} \cdots \mathbf{x}_k^{a_k} = \frac{1}{\lambda} \sum_{\mathbf{a} \in \mathbf{S}_{nk}} \mathbf{c}_\mathbf{a} \sum_{\mathbf{q} \in \mathbf{a}_k \cup \mathbf{a}'} \mathbf{X}_k^{\mathbf{q}}$$

from which it follows that

$$\Gamma_{n,k}^{(r)} \mathbf{b}[\mathbf{X}_k] = \frac{1}{\lambda} \sum_{\mathbf{a} \in \mathbf{S}_{nk}} \mathbf{c}_\mathbf{a} \sum_{\mathbf{q} \in \mathbf{a}_k \cup \mathbf{a}'} \mathbf{M}_{\mathbf{q}+r}[\mathbf{X}_n].$$

and the previous result gives

$$\Gamma_{n,k}^{(r)} \mathbf{b}[\mathbf{X}_k] \in \mathbf{L} \left[\Gamma_{n,k-1}^{(r)} \mathbf{E}_{n,k-1} \right] \pmod{(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{r-Q_{\text{sym}}[\mathbf{X}_n]}}$$

Completion of the proof

27

We have established that

$$\Gamma_{n,k}^{(r)}(\mathbf{E}_{n,k} - \mathbf{E}_{n,k}^0) \subseteq \mathbf{L} \left[\Gamma_{n,k-1}^{(r)} \mathbf{E}_{n,k-1} \right] \left(\text{mod } (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{r-\mathbf{Qsym}[\mathbf{X}_n]} \right) \quad (*)$$

Completion of the proof

28

We have established that

$$\Gamma_{n,k}^{(r)} (\mathbf{E}_{n,k} - \mathbf{E}_{n,k}^0) \subseteq \mathbf{L} \left[\Gamma_{n,k-1}^{(r)} \mathbf{E}_{n,k-1} \right] \left(\text{mod } (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{r-\mathbf{Qsym}[X_n]} \right) \quad (*)$$

For $\mathbf{k} = \mathbf{1}$

$$\Gamma_{n,1}^{(r)} \mathbf{R}_{n,1} \in \mathbf{L}(\{\mathbf{p}_{1+r}, \mathbf{p}_{2+r}, \dots, \mathbf{p}_{n+r}\}) \subseteq (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{r-\mathbf{Qsym}[X_n]}$$

Completion of the proof

29

We have established that

$$\Gamma_{n,k}^{(r)} (\mathbf{E}_{n,k} - \mathbf{E}_{n,k}^{\circ}) \subseteq \mathbf{L} \left[\Gamma_{n,k-1}^{(r)} \mathbf{E}_{n,k-1} \right] \left(\text{mod } (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{r-\mathbf{Qsym}[\mathbf{X}_n]} \right) \quad (*)$$

For $\mathbf{k} = \mathbf{1}$

$$\Gamma_{n,1}^{(r)} \mathbf{R}_{n,1} \in \mathbf{L}(\{\mathbf{p}_{1+r}, \mathbf{p}_{2+r}, \dots, \mathbf{p}_{n+r}\}) \subseteq (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{r-\mathbf{Qsym}[\mathbf{X}_n]}$$

So assuming by induction that (modulo the ideal $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{r-\mathbf{Qsym}[\mathbf{X}_n]}$)

$$\Gamma_{n,k-1}^{(r)} \mathbf{E}_{n,k-1} \subseteq \mathbf{L} \left[\bigcup_{s=1}^{k-1} \Gamma_{n,s}^{(r)} \mathbf{E}_{n,s}^{\circ} \right]$$

Completion of the proof

30

We have established that

$$\Gamma_{n,k}^{(r)}(\mathbf{E}_{n,k} - \mathbf{E}_{n,k}^{\circ}) \subseteq \mathbf{L} \left[\Gamma_{n,k-1}^{(r)} \mathbf{E}_{n,k-1} \right] \left(\text{mod } (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{r-\mathbf{Qsym}[\mathbf{X}_n]} \right) \quad (*)$$

For $\mathbf{k} = \mathbf{1}$

$$\Gamma_{n,1}^{(r)} \mathbf{R}_{n,1} \in \mathbf{L}(\{\mathbf{p}_{1+r}, \mathbf{p}_{2+r}, \dots, \mathbf{p}_{n+r}\}) \subseteq (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{r-\mathbf{Qsym}[\mathbf{X}_n]}$$

So assuming by induction that (modulo the ideal $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{r-\mathbf{Qsym}[\mathbf{X}_n]}$)

$$\Gamma_{n,k-1}^{(r)} \mathbf{E}_{n,k-1} \subseteq \mathbf{L} \left[\bigcup_{s=1}^{k-1} \Gamma_{n,s}^{(r)} \mathbf{E}_{n,s}^{\circ} \right]$$

then $(*)$ gives

$$\Gamma_{n,k}^{(r)}(\mathbf{E}_{n,k}) \subset \mathbf{L} \left[\bigcup_{s=1}^k \Gamma_{n,s}^{(r)} \mathbf{E}_{n,s}^{\circ} \right]$$

completing the induction

Wrapping thing up

We have proved that the collection

$$\{\mathbf{1}\} \cup \bigcup_{k=1}^n \Gamma_{n,k}^{(r)} \mathbf{E}_{n,k}^o$$

of cardinality

$$1 + \sum_{k=1}^n \binom{n}{k} D_k = n! < \infty$$

spans the quotient

$$\mathbf{rQsym}[X_n] / (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{\mathbf{rQsym}[X_n]}$$

Thus $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are a homogeneous S.O.P for $\mathbf{A} = \mathbf{rQsym}$ and

$$\dim \mathbf{rQsym}[X_n] / (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{\mathbf{rQsym}[X_n]} \leq n!$$

Recall the useful criterion

Theorem A

Let $q_1, q_2, \dots, q_n \in A$ be homogeneous of degrees d_1, d_2, \dots, d_n and an S.O.P. for A . Let

$$\dim A / (q_1, q_2, \dots, q_n)_A = M \leq N$$

with basis $B = \{f_1, f_2, \dots, f_M\}$. Then the condition

$$\lim_{q \rightarrow 1} (1 - q^{d_1})(1 - q^{d_2}) \cdots (1 - q^{d_n}) F_A(q) = N$$

forces $M = N$ and the equality

$$F_A(q) = \frac{\sum_{i=1}^N q^{\text{degree}(f_i)}}{(1 - q^{d_1})(1 - q^{d_2}) \cdots (1 - q^{d_n})}$$

yielding that A is a free module over $\mathbb{Q}[q_1, q_2, \dots, q_n]$ of rank N and therefore A is a Cohen-Macaulay algebra.

What we have in our case

The elementary symmetric functions $e_1, e_2, \dots, e_n \in rQ_{\text{sym}}$ are homogeneous of degrees $1, 2, \dots, n$ and an S.O.P. for A . We have shown that

$$\dim rQ_{\text{sym}} / (q_1, q_2, \dots, q_n)_{rQ_{\text{sym}}} \leq n!$$

with spanning set $B = \{\mathbf{1}\} \cup \bigcup_{k=1}^n \Gamma_{n,k}^{(r)} E_{n,k}^o$. Moreover we have

$$\lim_{q \rightarrow 1} (1 - q)(1 - q^2) \cdots (1 - q^n) F_{rQ_{\text{sym}}}(q) = \lim_{q \rightarrow 1} \sum_{k=0}^n \binom{n}{k}_q D_k(q) q^{rk} = n!$$

thus B is a basis and we must also have

$$1 + \sum_{k=1}^n \binom{n}{k}_q D_k(q) q^{rk} = 1 + \sum_{k=1}^n \sum_{b \in E_{n,k}^o} q^{\text{degree}(b)} q^{rk}$$

yielding that rQ_{sym} is a free module over $Q[e_1, e_2, \dots, e_n]$ of rank $n!$ and therefore rQ_{sym} is a Cohen-Macaulay algebra.

Some interesting byproducts of the proof

34

$$[k]_q \mathbf{D}_k(\mathbf{q}) = \sum_{\mathbf{b} \in \mathbf{E}_{n,k}^o} \mathbf{q}^{\text{degree}(\mathbf{b})}$$

In particular the q -derangement polynomial

$$\mathbf{D}_n(\mathbf{q}) = [n]_q! \sum_{s=0}^n \frac{\mathbf{q}^{\binom{s}{2}} (-1)^s}{[s]_q!}$$

gives the Hilbert series of the kernel of \mathbf{T}_n in the polynomial ring $\mathbf{Q}[\mathbf{X}_n]$

The Frobenius characteristic of the Kernel of \mathbf{T}_n in the Group algebra of \mathbf{S}_n is the symmetric polynomial

$$\sum_{\mathbf{i}=1}^n (-1)^i \mathbf{e}_i \mathbf{e}_1^{n-i}$$

A Remarkable Corollary

Given a standard tableau T let us set

$$\text{exit}(\mathbf{T}) = \min\{j : j \text{ is in the first column of } T \text{ and } j + 1 \text{ is not } \}$$

This given, Theorem 3.1 yields

Corollary 3.1 (Reiner-Wachs)

The multiplicity $m_\lambda(\mathbf{k})$ of Young's irreducible representation indexed by λ in the \mathbf{k}^{th} eigenspace of \mathbb{T}_n is equal to the number of standard tableaux \mathbf{T} of shape λ with $1, 2, \dots, \mathbf{k}$ in the first row, $\mathbf{k} + 2$ in the first column and such that the difference $\text{exit}(\mathbf{T}) - \mathbf{k}$ is even.