A New Recursion

In

The Theory of Macdonald Polynomials

Joint work
with
Jim Haglund

next

Some Basic Ingredients

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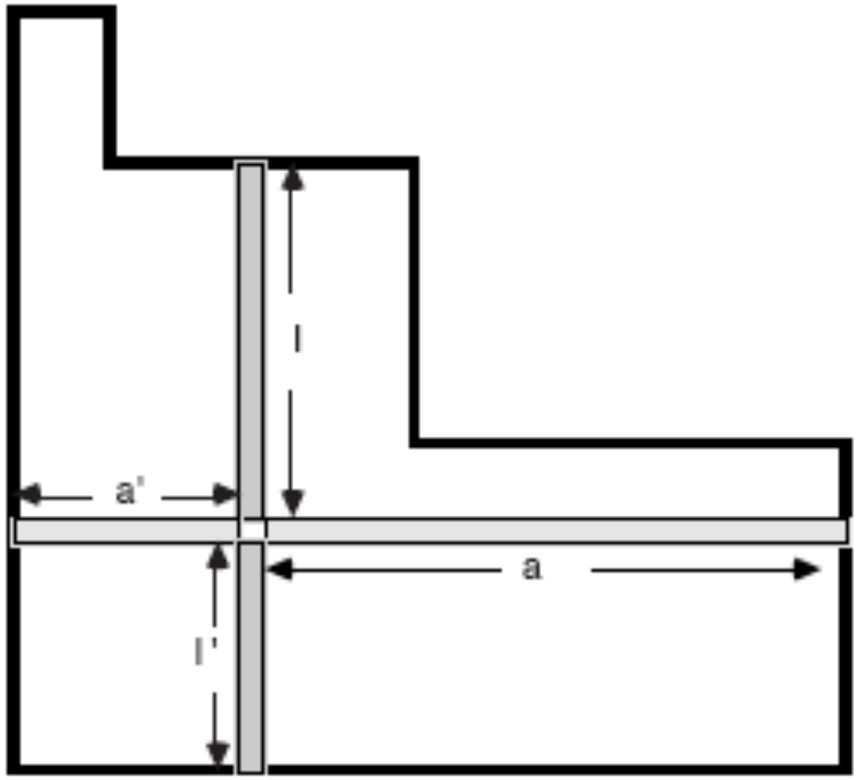

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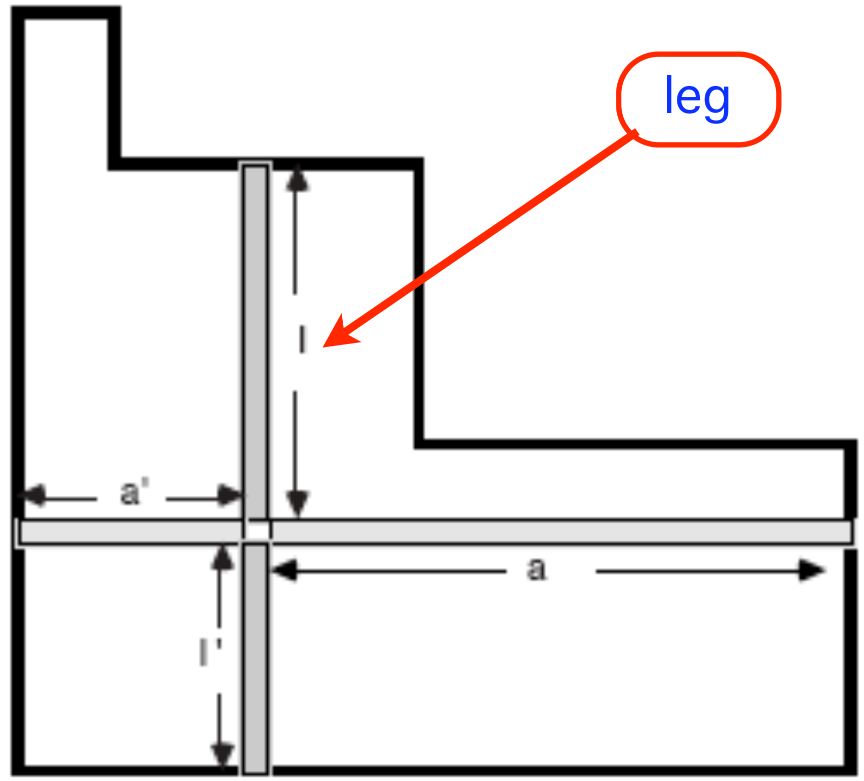

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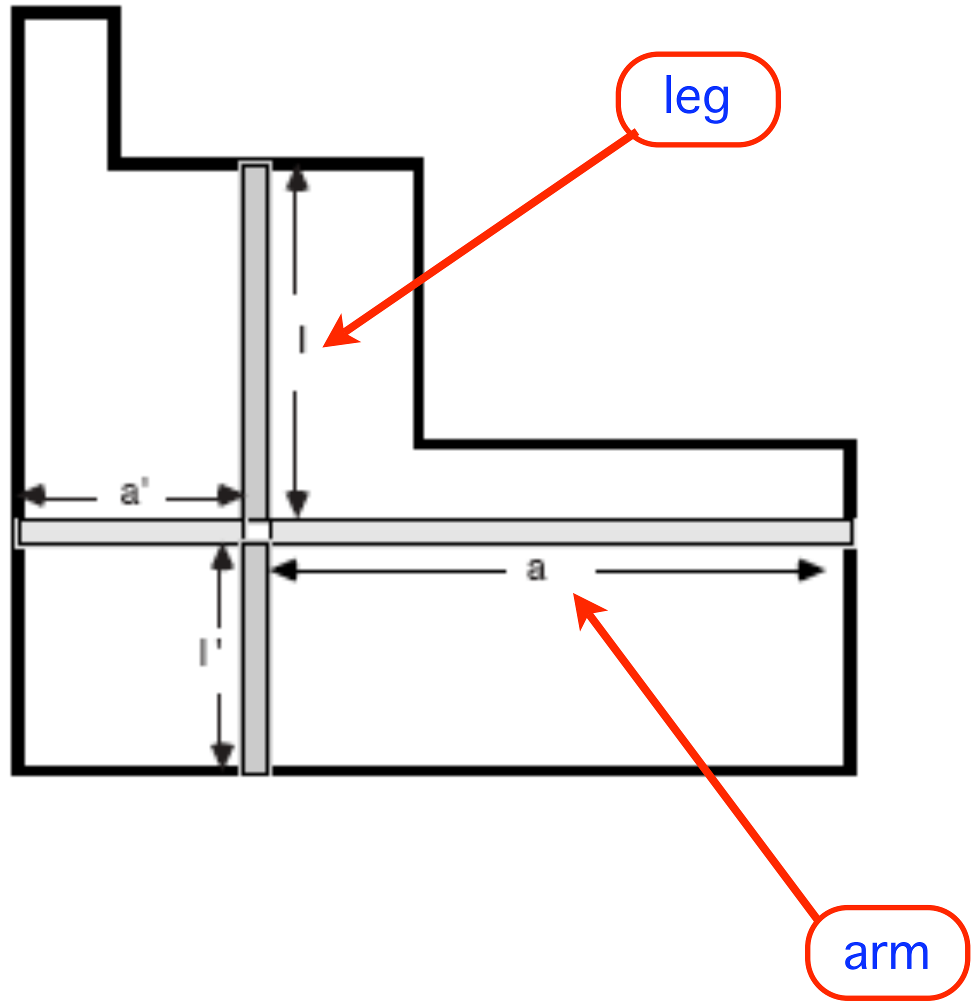

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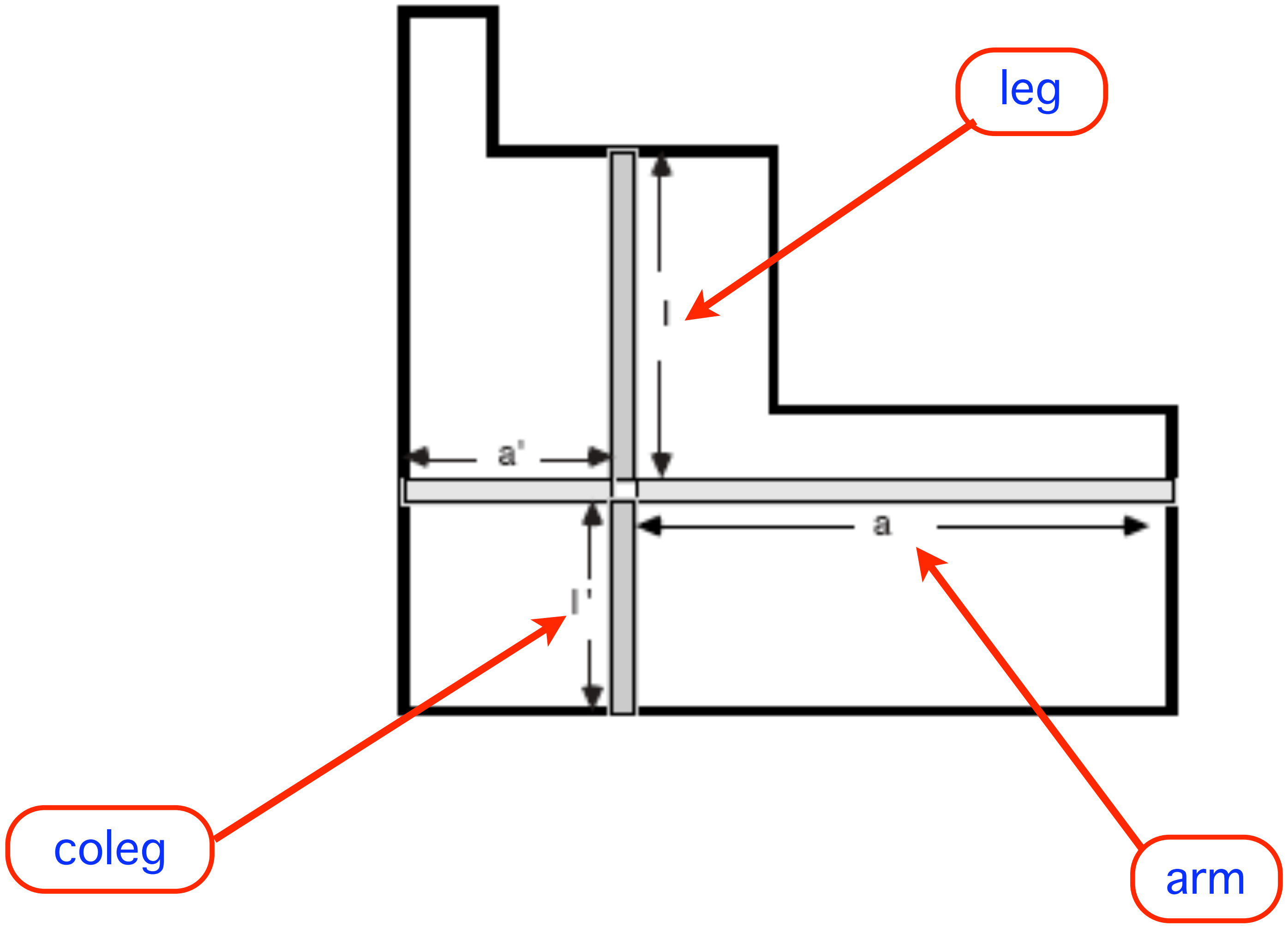

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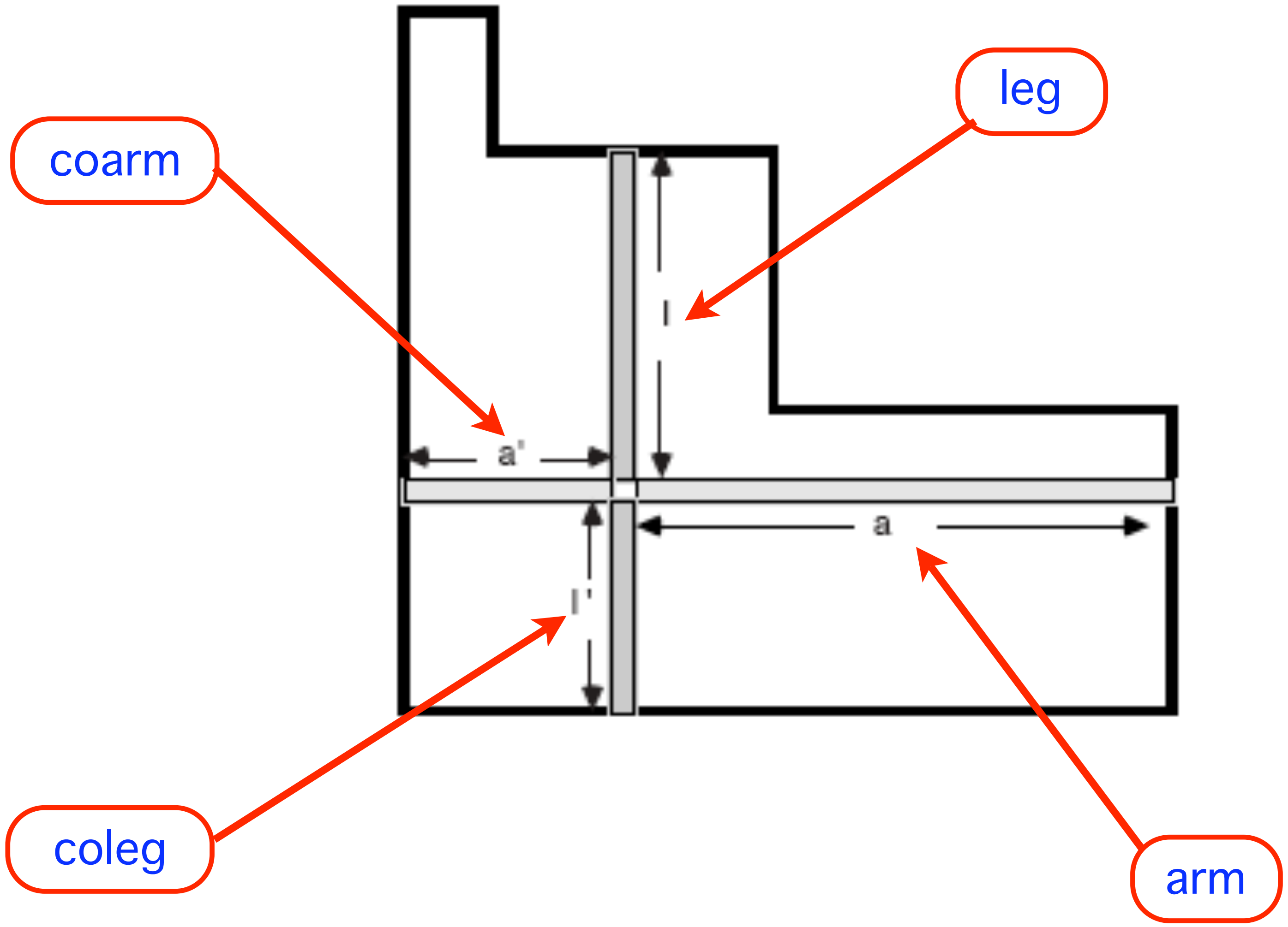

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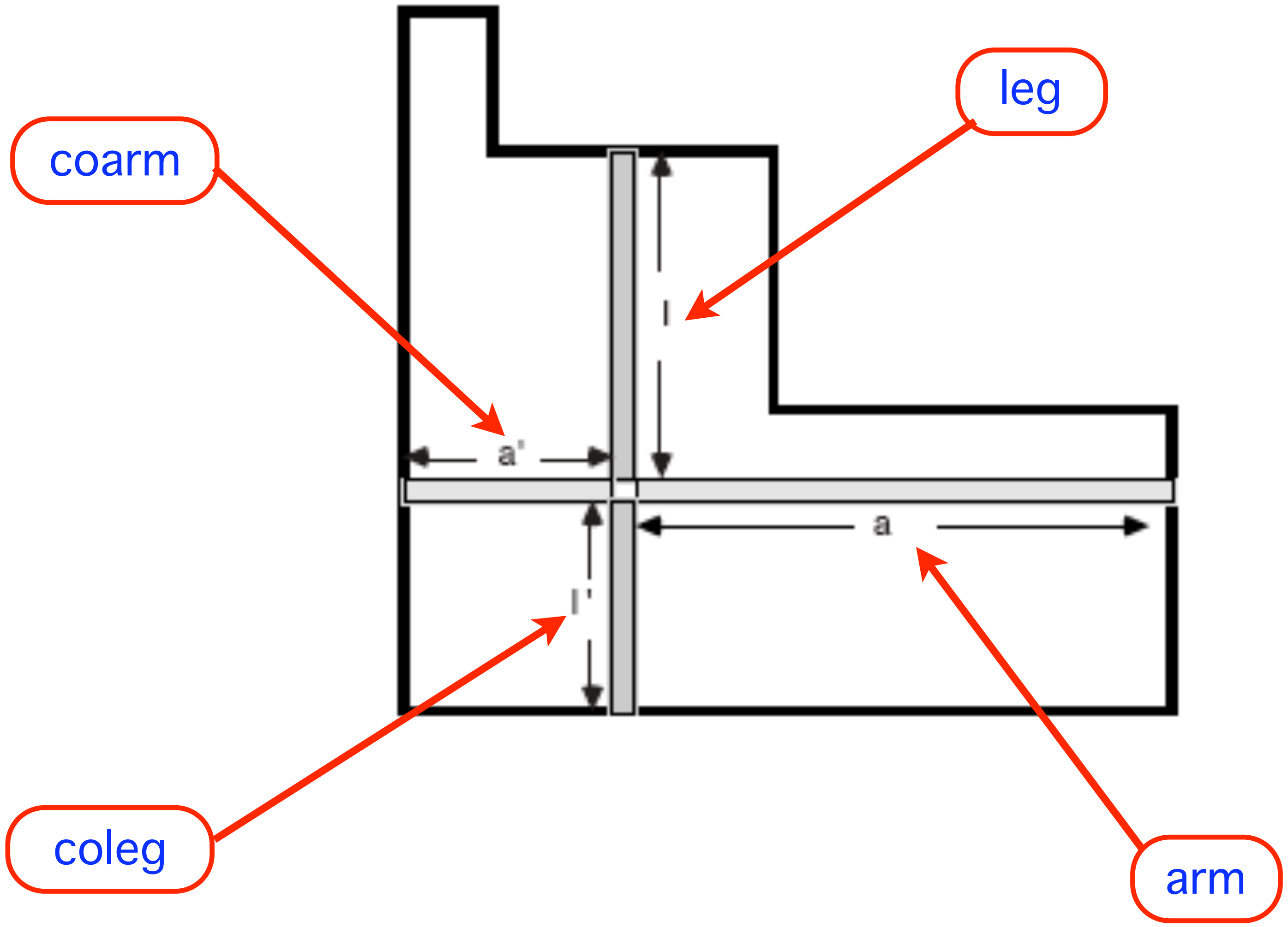

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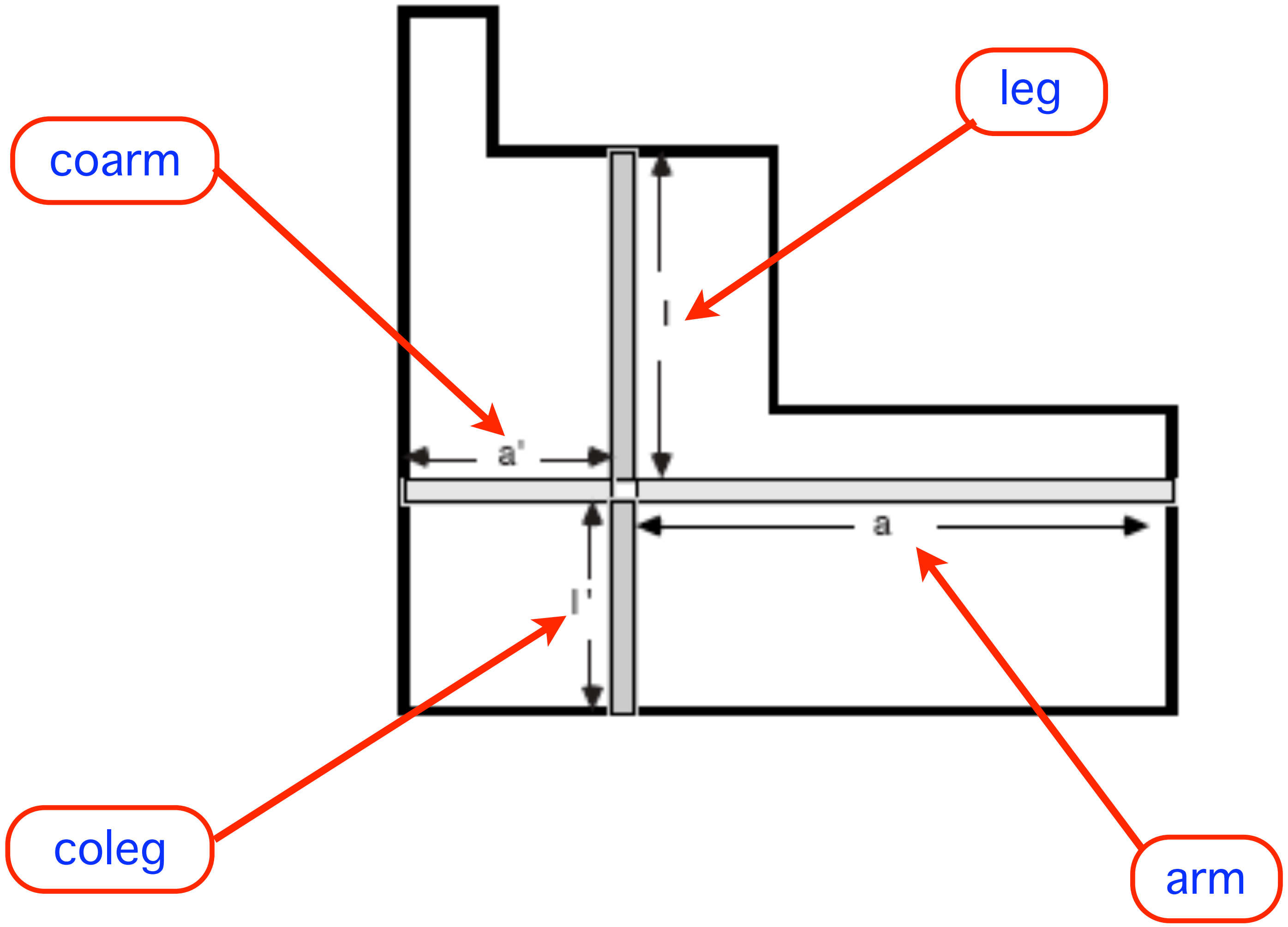

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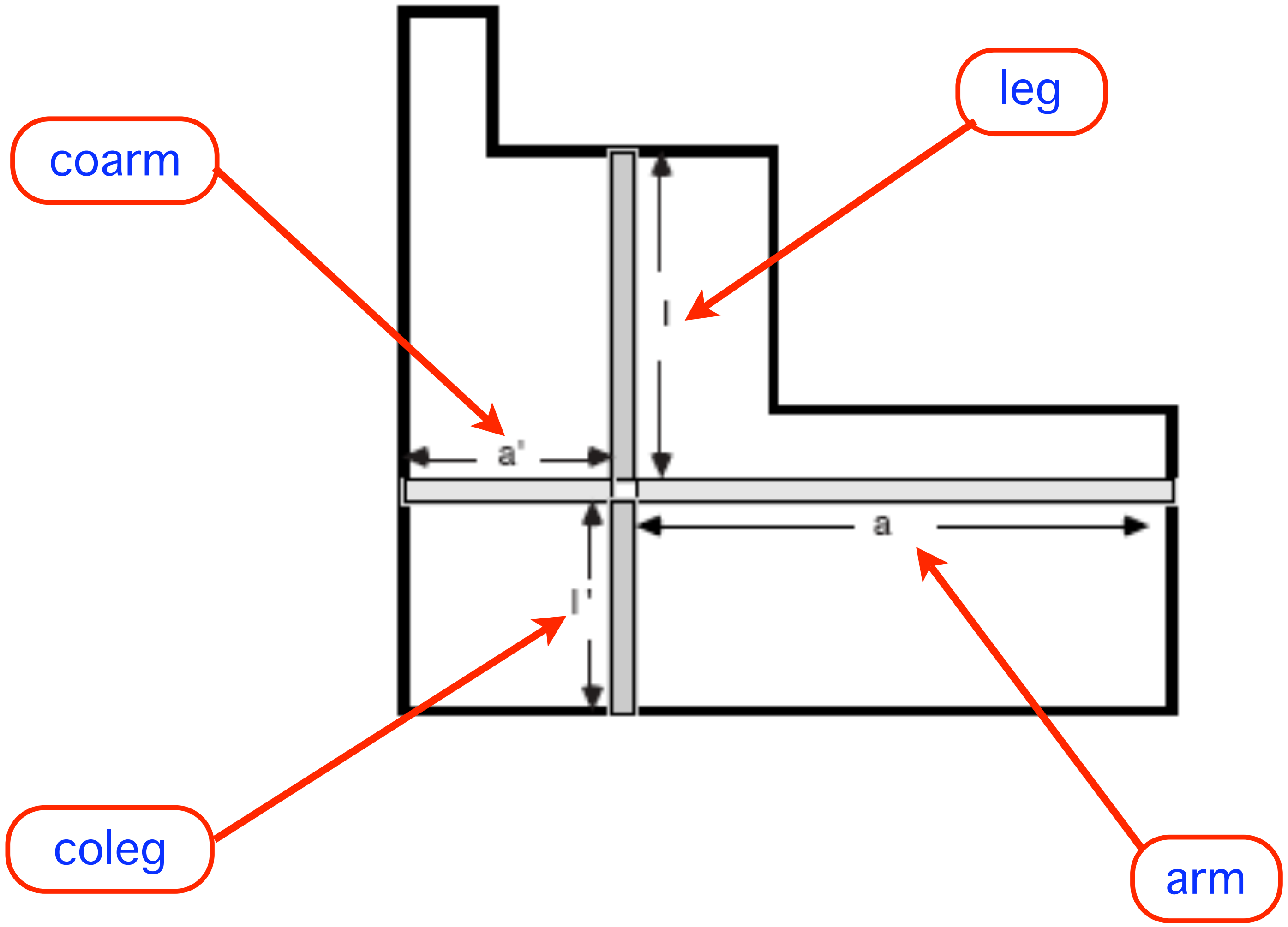

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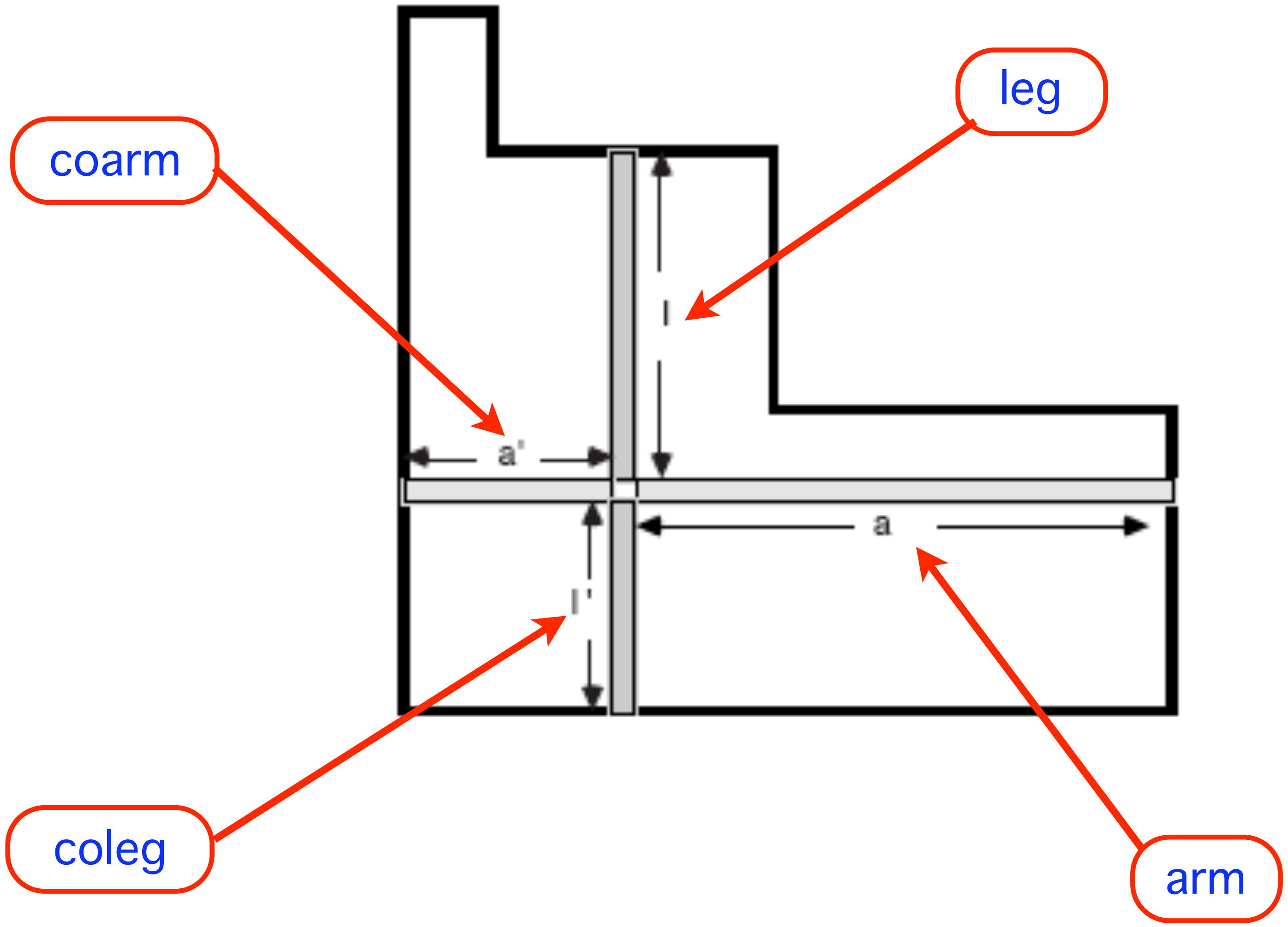

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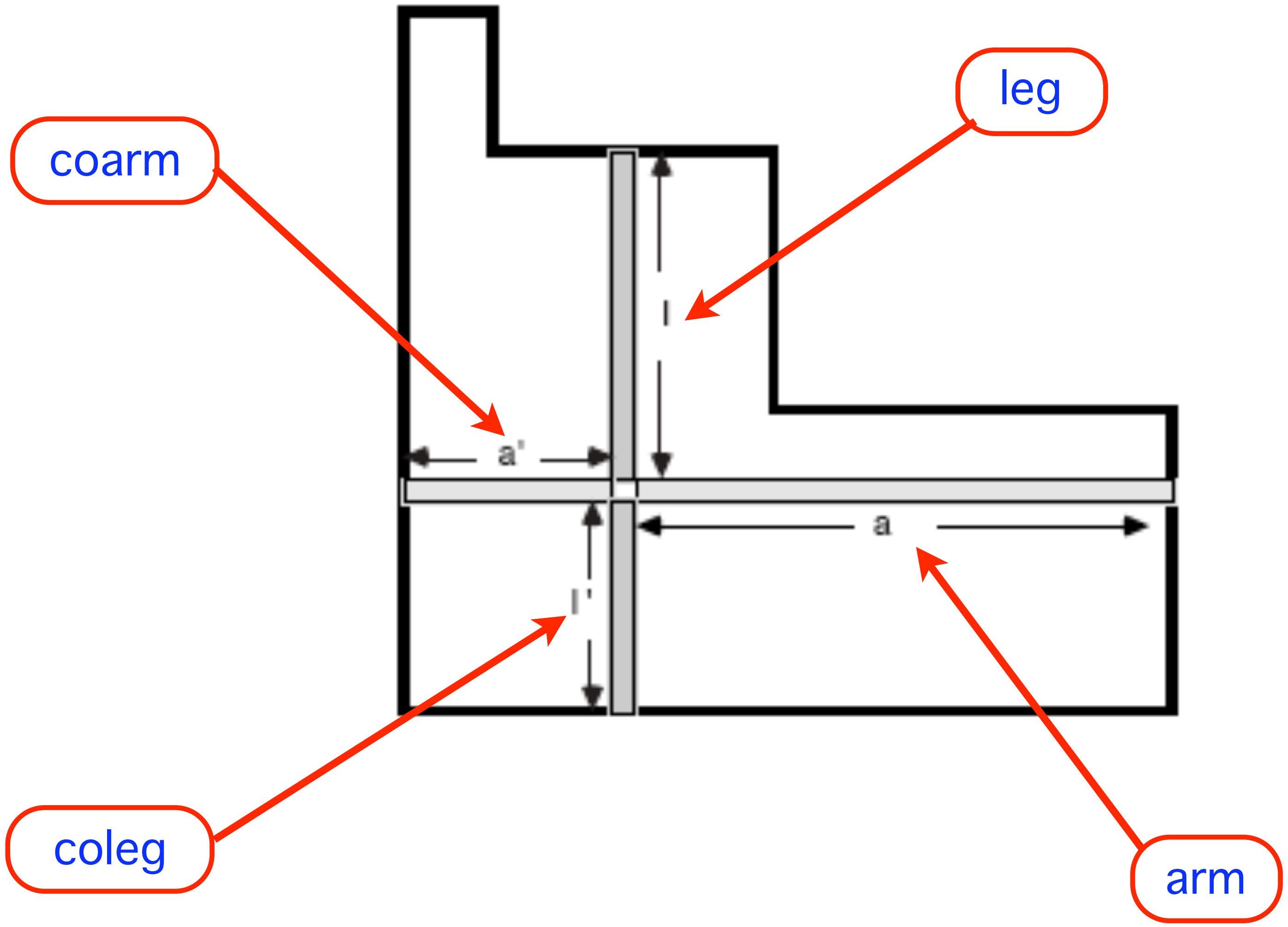

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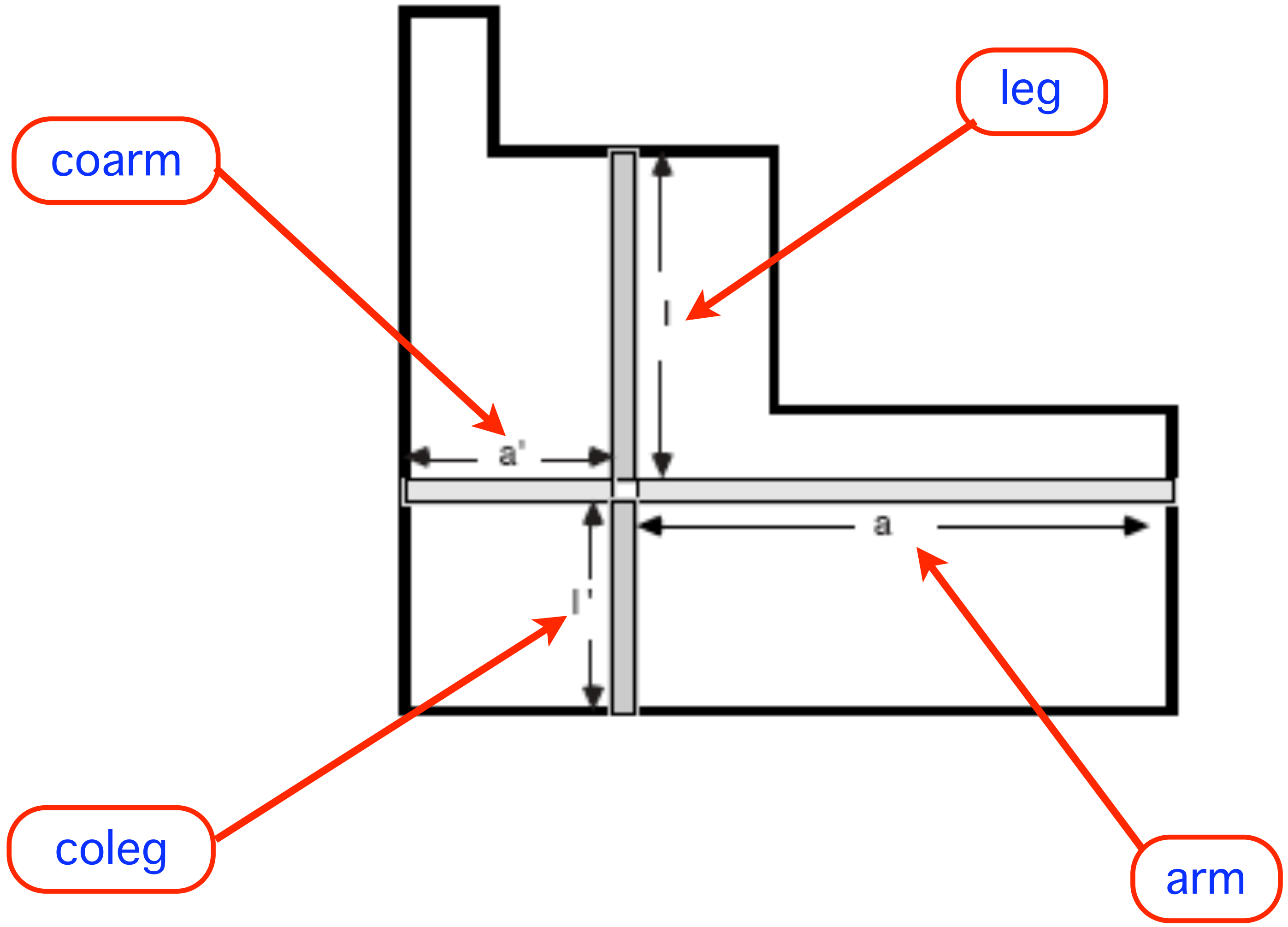

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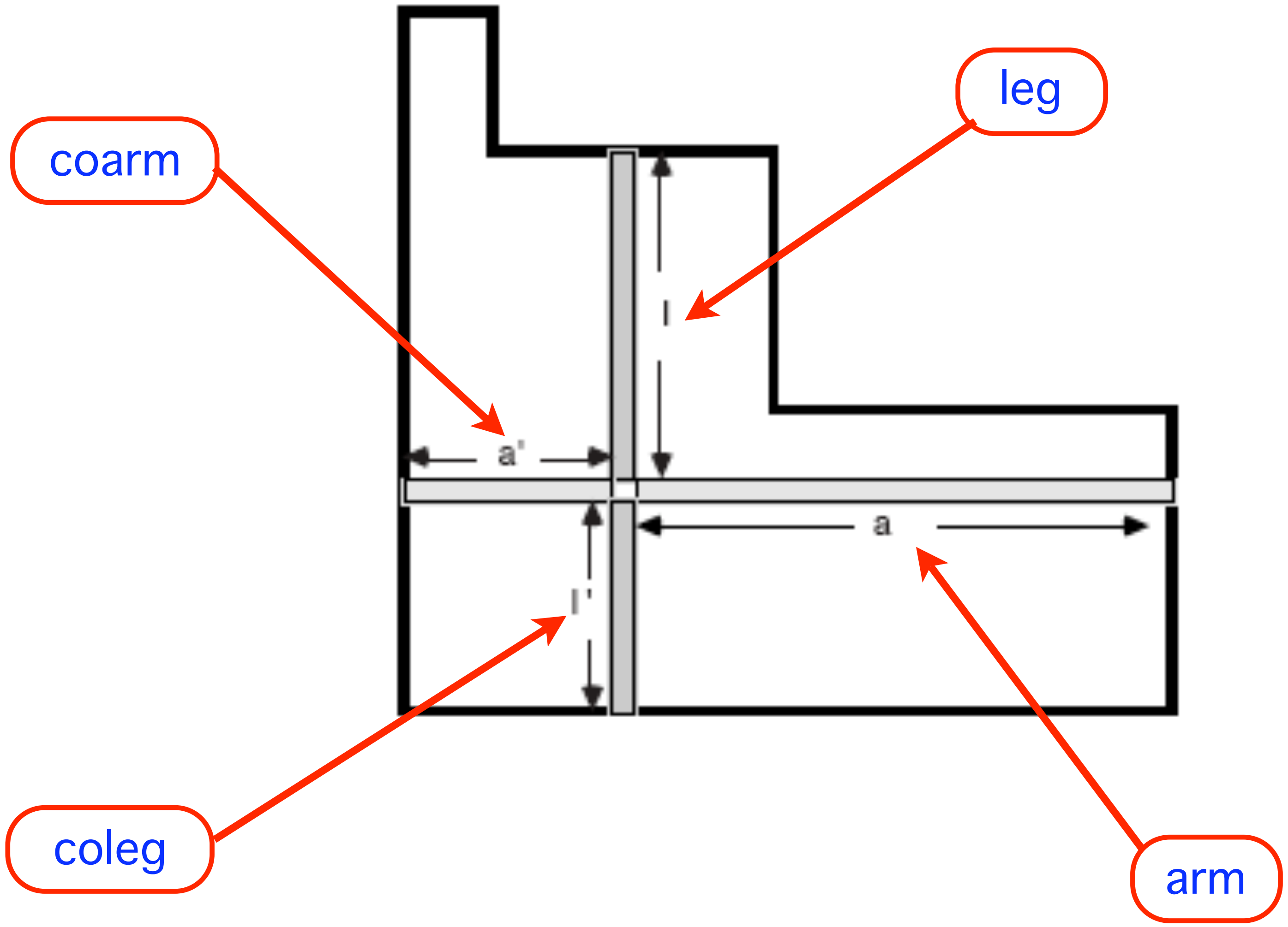

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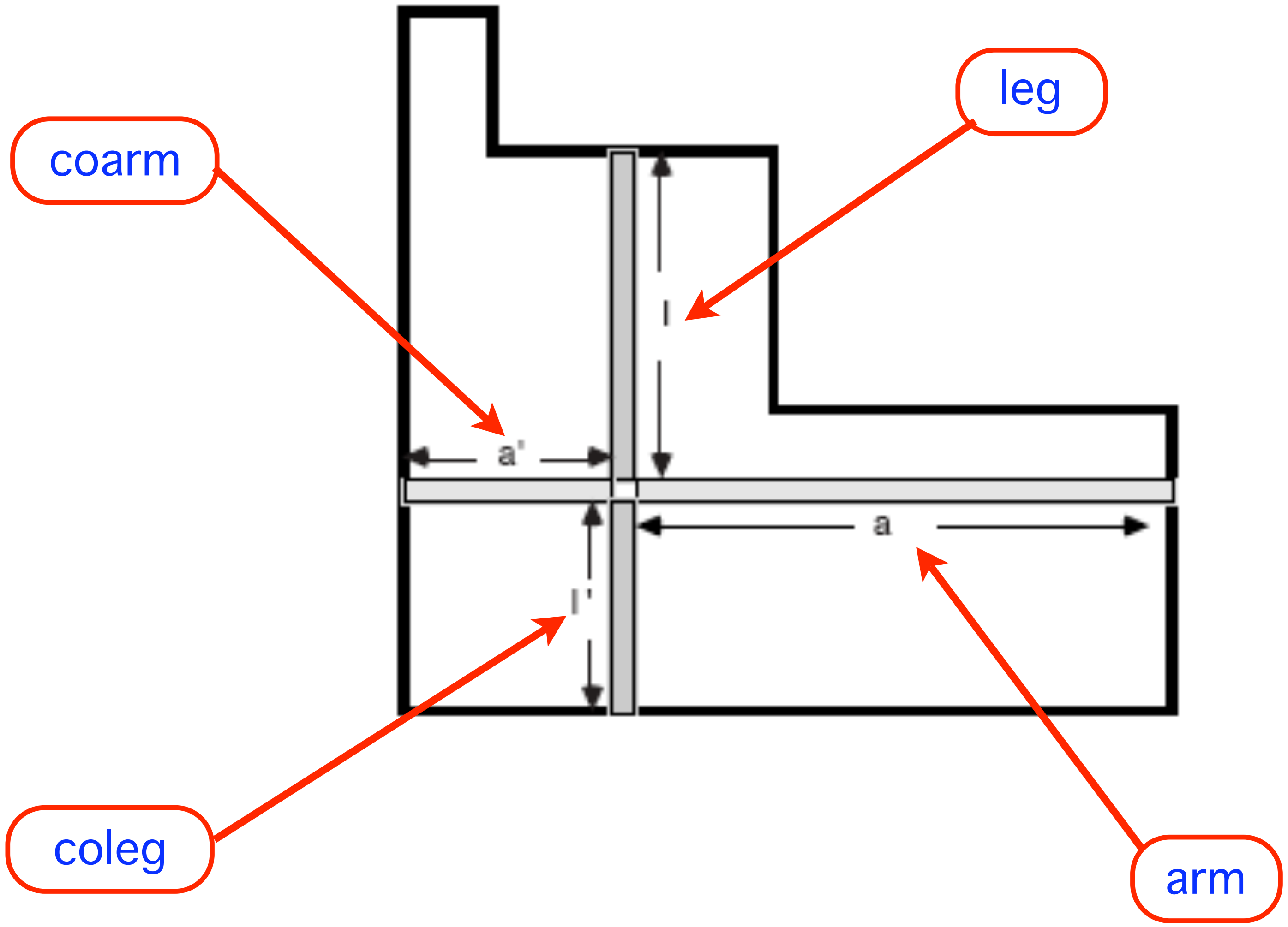

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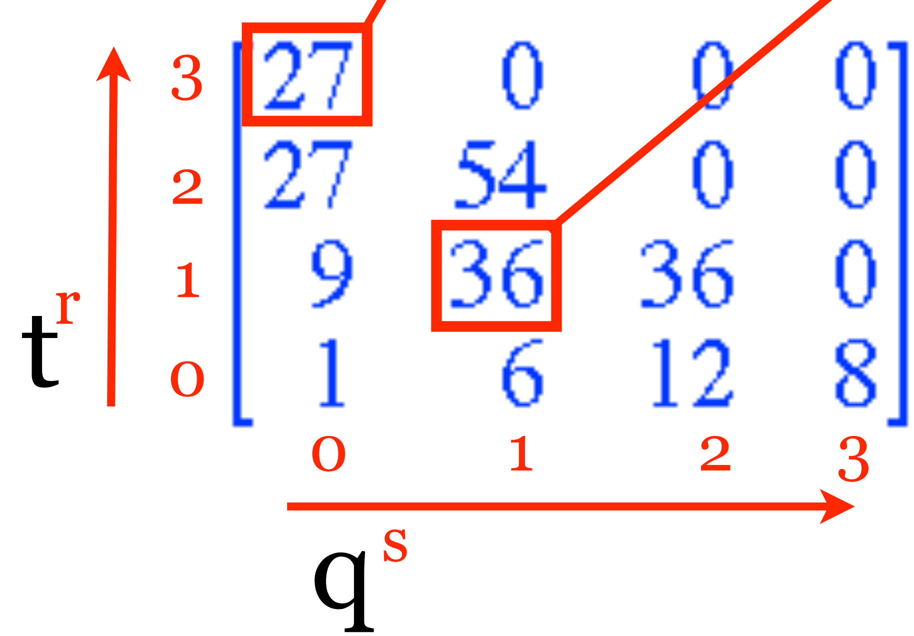

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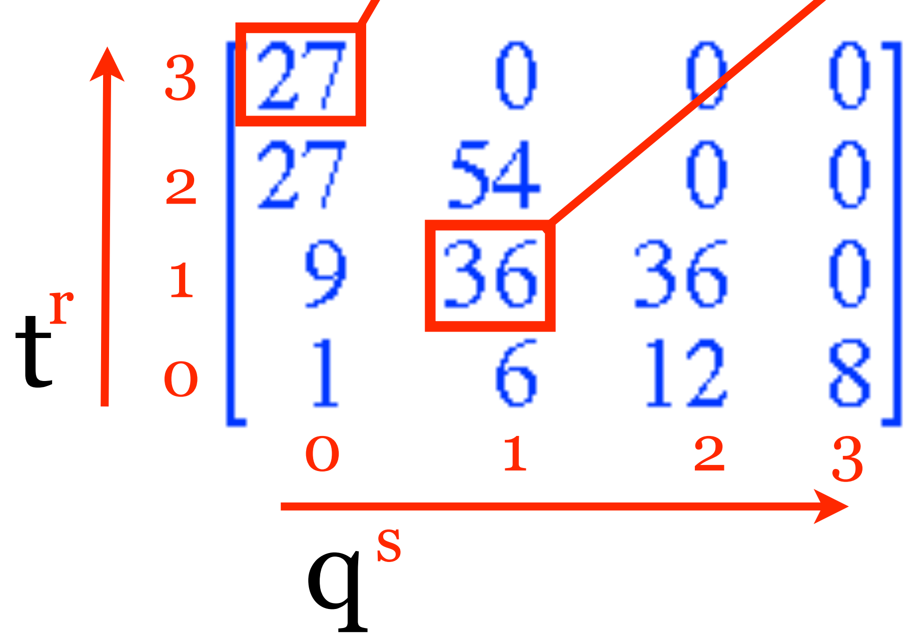

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$$+ s_{2,2,1}\;q\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} + s_{2,1,1,1}\;t\,q^2\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix} + s_{1,1,1,1,1}\;t^2\,q^4$$

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next

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The Jig Saw Puzzle

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Schur Function Expansions

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$\tilde{H}_{[1,1,1,1]}[X;q,t]$

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Schur Function Expansions

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The Jig Saw Puzzle

$\tilde{H}_{[1,1,1,1]}[X; q, t]$

$$\begin{bmatrix} s_{1,1,1,1} \\ s_{2,1,1} \\ s_{2,2} + s_{2,1,1} \\ s_{3,1} + s_{2,1,1} \\ s_{3,1} + s_{2,2} \\ s_{3,1} \\ s_{4} \end{bmatrix}$$

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$\tilde{H}_{[1,1,1,1]}[X;q,t]$

$$\begin{bmatrix} s_{1,1,1,1} \\ s_{2,1,1} \\ s_{2,2}+s_{2,1,1} \\ s_{3,1}+s_{2,1,1} \\ s_{3,1}+s_{2,2} \\ s_{3,1} \\ s_4 \end{bmatrix}$$

Frobenius characteristic of the linear span of derivatives of the Vandermonde determinant

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$\tilde{H}_{[2,1,1]}[X;q,t]$

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$\tilde{H}_{[1,1,1,1]}[X;q,t]$

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$\tilde{H}_{[4]}[X;q,t]$

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$$\begin{bmatrix} s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1}+s_{2,2} & s_{2,1,1} \\ s_{3,1} & s_{2,2}+s_{2,1,1} \\ s_4 & s_{3,1} \end{bmatrix}$$

$\tilde{H}_{[4]}[X;q,t]$

$$\begin{bmatrix} s_{2,2} & s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1} & s_{3,1}+s_{2,1,1} & s_{2,1,1} \\ s_4 & s_{3,1} & s_{2,2} \end{bmatrix}$$

next

The Jig Saw Puzzle

of

Schur Function Expansions

$\tilde{H}_{[1,1,1,1]}[X;q,t]$

Frobenius characteristic of the linear span of derivatives of the Vandermonde determinant

$$\begin{bmatrix} s_{1,1,1,1} \\ s_{2,1,1} \\ s_{2,2}+s_{2,1,1} \\ s_{3,1}+s_{2,1,1} \\ s_{3,1}+s_{2,2} \\ s_{3,1} \\ s_4 \end{bmatrix}$$

$\tilde{H}_{[2,1,1]}[X;q,t]$

$$\begin{bmatrix} s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1}+s_{2,2} & s_{2,1,1} \\ s_{3,1} & s_{2,2}+s_{2,1,1} \\ s_4 & s_{3,1} \end{bmatrix}$$

$\tilde{H}_{[4]}[X;q,t]$

$$\begin{bmatrix} s_{2,2} & s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1} & s_{3,1}+s_{2,1,1} & s_{2,1,1} \\ s_4 & s_{3,1} & s_{2,2} \end{bmatrix}$$

$\tilde{H}_{[3,1]}[X;q,t]$

next

The Jig Saw Puzzle

of

Schur Function Expansions

$\tilde{H}_{[1,1,1,1]}[X;q,t]$

$$\begin{bmatrix} s_{1,1,1,1} \\ s_{2,1,1} \\ s_{2,2}+s_{2,1,1} \\ s_{3,1}+s_{2,1,1} \\ s_{3,1}+s_{2,2} \\ s_{3,1} \\ s_4 \end{bmatrix}$$

Frobenius characteristic of the linear span of derivatives of the Vandermonde determinant

$\tilde{H}_{[2,1,1]}[X;q,t]$

$$\begin{bmatrix} s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1}+s_{2,2} & s_{2,1,1} \\ s_{3,1} & s_{2,2}+s_{2,1,1} \\ s_4 & s_{3,1} \end{bmatrix}$$

$\tilde{H}_{[4]}[X;q,t]$

$$\begin{bmatrix} s_{2,2} & s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1} & s_{3,1}+s_{2,1,1} & s_{2,1,1} \\ s_4 & s_{3,1} & s_{2,2} \end{bmatrix}$$

$\tilde{H}_{[3,1]}[X;q,t]$

$$\begin{bmatrix} s_{3,1} & s_{2,2}+s_{2,1,1} & s_{2,1,1} & s_{1,1,1,1} \\ s_4 & s_{3,1} & s_{3,1}+s_{2,2} & s_{2,1,1} \end{bmatrix}$$

next

The Jig Saw Puzzle

of

Schur Function Expansions

$\tilde{H}_{[1,1,1,1]}[X;q,t]$

Frobenius characteristic of the linear span of derivatives of the Vandermonde determinant

$$\begin{bmatrix} s_{1,1,1,1} \\ s_{2,1,1} \\ s_{2,2}+s_{2,1,1} \\ s_{3,1}+s_{2,1,1} \\ s_{3,1}+s_{2,2} \\ s_{3,1} \\ s_4 \end{bmatrix}$$

$\tilde{H}_{[2,1,1]}[X;q,t]$

$$\begin{bmatrix} s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1}+s_{2,2} & s_{2,1,1} \\ s_{3,1} & s_{2,2}+s_{2,1,1} \\ s_4 & s_{3,1} \end{bmatrix}$$

$\tilde{H}_{[4]}[X;q,t]$

$$\begin{bmatrix} s_{2,2} & s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1} & s_{3,1}+s_{2,1,1} & s_{2,1,1} \\ s_4 & s_{3,1} & s_{2,2} \end{bmatrix}$$

$\tilde{H}_{[3,1]}[X;q,t]$

$$\begin{bmatrix} s_{3,1} & s_{2,2}+s_{2,1,1} & s_{2,1,1} & s_{1,1,1,1} \\ s_4 & s_{3,1} & s_{3,1}+s_{2,2} & s_{2,1,1} \end{bmatrix}$$

$\tilde{H}_{[2,2]}[X;q,t]$

next

The Jig Saw Puzzle

of

Schur Function Expansions

$\tilde{H}_{[1,1,1,1]}[X;q,t]$

Frobenius characteristic of the linear span of derivatives of the Vandermonde determinant

$$\begin{bmatrix} s_{1,1,1,1} \\ s_{2,1,1} \\ s_{2,2}+s_{2,1,1} \\ s_{3,1}+s_{2,1,1} \\ s_{3,1}+s_{2,2} \\ s_{3,1} \\ s_4 \end{bmatrix}$$

$\tilde{H}_{[2,1,1]}[X;q,t]$

$$\begin{bmatrix} s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1}+s_{2,2} & s_{2,1,1} \\ s_{3,1} & s_{2,2}+s_{2,1,1} \\ s_4 & s_{3,1} \end{bmatrix}$$

$\tilde{H}_{[4]}[X;q,t]$

$$\begin{bmatrix} s_{2,2} & s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1} & s_{3,1}+s_{2,1,1} & s_{2,1,1} \\ s_4 & s_{3,1} & s_{2,2} \end{bmatrix}$$

$\tilde{H}_{[3,1]}[X;q,t]$

$$\begin{bmatrix} s_{3,1} & s_{2,2}+s_{2,1,1} & s_{2,1,1} & s_{1,1,1,1} \\ s_4 & s_{3,1} & s_{3,1}+s_{2,2} & s_{2,1,1} \end{bmatrix}$$

$\tilde{H}_{[2,2]}[X;q,t]$

$$[s_4, s_{3,1}, s_{3,1}+s_{2,2}, s_{3,1}+s_{2,1,1}, s_{2,2}+s_{2,1,1}, s_{2,1,1}, s_{1,1,1,1}]$$

next

The Jig Saw Puzzle

of

Schur Function Expansions

$\tilde{H}_{[1,1,1,1]}[X;q,t]$

$$\begin{bmatrix} s_{1,1,1,1} \\ s_{2,1,1} \\ s_{2,2}+s_{2,1,1} \\ s_{3,1}+s_{2,1,1} \\ s_{3,1}+s_{2,2} \\ s_{3,1} \\ s_4 \end{bmatrix}$$

Frobenius characteristic of the linear span of derivatives of the Vandermonde determinant

$\tilde{H}_{[2,1,1]}[X;q,t]$

$$\begin{bmatrix} s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1}+s_{2,2} & s_{2,1,1} \\ s_{3,1} & s_{2,2}+s_{2,1,1} \\ s_4 & s_{3,1} \end{bmatrix}$$

$\tilde{H}_{[4]}[X;q,t]$

$$\begin{bmatrix} s_{2,2} & s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1} & s_{3,1}+s_{2,1,1} & s_{2,1,1} \\ s_4 & s_{3,1} & s_{2,2} \end{bmatrix}$$

They are all deformations of the Sn Harmonics

$\tilde{H}_{[3,1]}[X;q,t]$

$$\begin{bmatrix} s_{3,1} & s_{2,2}+s_{2,1,1} & s_{2,1,1} & s_{1,1,1,1} \\ s_4 & s_{3,1} & s_{3,1}+s_{2,2} & s_{2,1,1} \end{bmatrix}$$

$\tilde{H}_{[2,2]}[X;q,t]$

$$[s_4, s_{3,1}, s_{3,1}+s_{2,2}, s_{3,1}+s_{2,1,1}, s_{2,2}+s_{2,1,1}, s_{2,1,1}, s_{1,1,1,1}]$$

next

The Jig Saw Puzzle

of

Schur Function Expansions

$\tilde{H}_{[1,1,1,1]}[X;q,t]$

$$\begin{bmatrix} s_{1,1,1,1} \\ s_{2,1,1} \\ s_{2,2}+s_{2,1,1} \\ s_{3,1}+s_{2,1,1} \\ s_{3,1}+s_{2,2} \\ s_{3,1} \\ s_4 \end{bmatrix}$$

Frobenius characteristic of the linear span of derivatives of the Vandermonde determinant

What makes the Schur functions move???

$\tilde{H}_{[2,1,1]}[X;q,t]$

$$\begin{bmatrix} s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1}+s_{2,2} & s_{2,1,1} \\ s_{3,1} & s_{2,2}+s_{2,1,1} \\ s_4 & s_{3,1} \end{bmatrix}$$

$\tilde{H}_{[4]}[X;q,t]$

$$\begin{bmatrix} s_{2,2} & s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1} & s_{3,1}+s_{2,1,1} & s_{2,1,1} \\ s_4 & s_{3,1} & s_{2,2} \end{bmatrix}$$

They are all deformations of the Sn Harmonics

$\tilde{H}_{[3,1]}[X;q,t]$

$$\begin{bmatrix} s_{3,1} & s_{2,2}+s_{2,1,1} & s_{2,1,1} & s_{1,1,1,1} \\ s_4 & s_{3,1} & s_{3,1}+s_{2,2} & s_{2,1,1} \end{bmatrix}$$

$\tilde{H}_{[2,2]}[X;q,t]$

$$[s_4, s_{3,1}, s_{3,1}+s_{2,2}, s_{3,1}+s_{2,1,1}, s_{2,2}+s_{2,1,1}, s_{2,1,1}, s_{1,1,1,1}]$$

next

The Jig Saw Puzzle

of

Schur Function Expansions

$\tilde{\mathrm{H}}_{[1,1,1,1]}[\mathbf{X};\mathbf{q},\mathbf{t}]$

$$\begin{bmatrix} s_{1,1,1,1} \\ s_{2,1,1} \\ s_{2,2}+s_{2,1,1} \\ s_{3,1}+s_{2,1,1} \\ s_{3,1}+s_{2,2} \\ s_{3,1} \\ s_4 \end{bmatrix}$$

Frobenius characteristic of the linear span of derivatives of the Vandermonde determinant

$\tilde{\mathrm{H}}_{[2,1,1]}[\mathbf{X};\mathbf{q},\mathbf{t}]$

$$\begin{bmatrix} s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1}+s_{2,2} & s_{2,1,1} \\ s_{3,1} & s_{2,2}+s_{2,1,1} \\ s_4 & s_{3,1} \end{bmatrix}$$

What makes the Schur functions move???

Macdonald Reciprocity

$\tilde{\mathrm{H}}_{[4]}[\mathbf{X};\mathbf{q},\mathbf{t}]$

$$\begin{bmatrix} s_{2,2} & s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1} & s_{3,1}+s_{2,1,1} & s_{2,1,1} \\ s_4 & s_{3,1} & s_{2,2} \end{bmatrix}$$

They are all deformations of the Sn Harmonics

$\tilde{\mathrm{H}}_{[3,1]}[\mathbf{X};\mathbf{q},\mathbf{t}]$

$$\begin{bmatrix} s_{3,1} & s_{2,2}+s_{2,1,1} & s_{2,1,1} & s_{1,1,1,1} \\ s_4 & s_{3,1} & s_{3,1}+s_{2,2} & s_{2,1,1} \end{bmatrix}$$

$\tilde{\mathrm{H}}_{[2,2]}[\mathbf{X};\mathbf{q},\mathbf{t}]$

$$[s_4, s_{3,1}, s_{3,1}+s_{2,2}, s_{3,1}+s_{2,1,1}, s_{2,2}+s_{2,1,1}, s_{2,1,1}, s_{1,1,1,1}]$$

next

The Jig Saw Puzzle

of

Schur Function Expansions

$\tilde{H}_{[1,1,1,1]}[X;q,t]$

$$\begin{bmatrix} s_{1,1,1,1} \\ s_{2,1,1} \\ s_{2,2}+s_{2,1,1} \\ s_{3,1}+s_{2,1,1} \\ s_{3,1}+s_{2,2} \\ s_{3,1} \\ s_4 \end{bmatrix}$$

Frobenius characteristic of the linear span of derivatives of the Vandermonde determinant

$\tilde{H}_{[2,1,1]}[X;q,t]$

$$\begin{bmatrix} s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1}+s_{2,2} & s_{2,1,1} \\ s_{3,1} & s_{2,2}+s_{2,1,1} \\ s_4 & s_{3,1} \end{bmatrix}$$

$\tilde{H}_{[4]}[X;q,t]$

$$\begin{bmatrix} s_{2,2} & s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1} & s_{3,1}+s_{2,1,1} & s_{2,1,1} \\ s_4 & s_{3,1} & s_{2,2} \end{bmatrix}$$

$\tilde{H}_{[3,1]}[X;q,t]$

$$\begin{bmatrix} s_{3,1} & s_{2,2}+s_{2,1,1} & s_{2,1,1} & s_{1,1,1,1} \\ s_4 & s_{3,1} & s_{3,1}+s_{2,2} & s_{2,1,1} \end{bmatrix}$$

$\tilde{H}_{[2,2]}[X;q,t]$

$$[s_4, s_{3,1}, s_{3,1}+s_{2,2}, s_{3,1}+s_{2,1,1}, s_{2,2}+s_{2,1,1}, s_{2,1,1}, s_{1,1,1,1}]$$

What makes the Schur functions move???

Macdonald Reciprocity

$$\frac{\tilde{H}_\mu[B_\lambda(q,t)]}{\Pi_\mu(q,t)} = \frac{\tilde{H}_\lambda[B_\mu(q,t)]}{\Pi_\lambda(q,t)}$$

They are all deformations of the Sn Harmonics

next

The Jig Saw Puzzle

of

Schur Function Expansions

$\tilde{\mathrm{H}}_{[1,1,1,1]}[\mathbf{X};\mathbf{q},\mathbf{t}]$

Frobenius characteristic of the linear span of derivatives of the Vandermonde determinant

$$\begin{bmatrix} s_{1,1,1,1} \\ s_{2,1,1} \\ s_{2,2}+s_{2,1,1} \\ s_{3,1}+s_{2,1,1} \\ s_{3,1}+s_{2,2} \\ s_{3,1} \\ s_4 \end{bmatrix}$$

$\tilde{\mathrm{H}}_{[2,1,1]}[\mathbf{X};\mathbf{q},\mathbf{t}]$

$$\begin{bmatrix} s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1}+s_{2,2} & s_{2,1,1} \\ s_{3,1} & s_{2,2}+s_{2,1,1} \\ s_4 & s_{3,1} \end{bmatrix}$$

$\tilde{\mathrm{H}}_{[4]}[\mathbf{X};\mathbf{q},\mathbf{t}]$

$$\begin{bmatrix} s_{2,2} & s_{2,1,1} & s_{1,1,1,1} \\ s_{3,1} & s_{3,1}+s_{2,1,1} & s_{2,1,1} \\ s_4 & s_{3,1} & s_{2,2} \end{bmatrix}$$

$\tilde{\mathrm{H}}_{[3,1]}[\mathbf{X};\mathbf{q},\mathbf{t}]$

$$\begin{bmatrix} s_{3,1} & s_{2,2}+s_{2,1,1} & s_{2,1,1} & s_{1,1,1,1} \\ s_4 & s_{3,1} & s_{3,1}+s_{2,2} & s_{2,1,1} \end{bmatrix}$$

$\tilde{\mathrm{H}}_{[2,2]}[\mathbf{X};\mathbf{q},\mathbf{t}]$

$$[s_4, s_{3,1}, s_{3,1}+s_{2,2}, s_{3,1}+s_{2,1,1}, s_{2,2}+s_{2,1,1}, s_{2,1,1}, s_{1,1,1,1}]$$

What makes
the Schur functions
move???

Macdonald Reciprocity

$$\frac{\tilde{\mathrm{H}}_\mu[\mathrm{B}_\lambda(\mathbf{q},\mathbf{t})]}{\Pi_\mu(\mathbf{q},\mathbf{t})} = \frac{\tilde{\mathrm{H}}_\lambda[\mathrm{B}_\mu(\mathbf{q},\mathbf{t})]}{\Pi_\lambda(\mathbf{q},\mathbf{t})}$$

What else ?

They are all
deformations
of
the Sn Harmonics

next

k-Schur expansion of Macdonald Polynomials

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

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H([2,2])
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k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $q^2 A_{1,1,1,1} + (q + t\,q) A_{2,1,1} + A_{2,2}$

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $\quad q^2 A_{1,1,1,1} + (q + t\,q) A_{2,1,1} + A_{2,2} \quad \begin{bmatrix} 0 & A_{2,1,1} & 0 \\ A_{2,2} & A_{2,1,1} & A_{1,1,1,1} \end{bmatrix}$

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $q^2 A_{1,1,1,1} + (q + t\,q) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} 0 & A_{2,1,1} & 0 \\ A_{2,2} & A_{2,1,1} & A_{1,1,1,1} \end{bmatrix}$

H([2,1,1]

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $q^2 A_{1,1,1,1} + (q + t\,q)\,A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} 0 & A_{2,1,1} & 0 \\ A_{2,2} & A_{2,1,1} & A_{1,1,1,1} \end{bmatrix}$

H([2,1,1] $t\,q\,A_{1,1,1,1} + (q + t^2)\,A_{2,1,1} + A_{2,2}$

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $q^2 A_{1,1,1,1} + (q + t\,q) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} 0 & A_{2,1,1} & 0 \\ A_{2,2} & A_{2,1,1} & A_{1,1,1,1} \end{bmatrix}$

H([2,1,1] $t\,q A_{1,1,1,1} + (q + t^2) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{2,1,1} & 0 \\ 0 & A_{1,1,1,1} \\ A_{2,2} & A_{2,1,1} \end{bmatrix}$

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $q^2 A_{1,1,1,1} + (q + t\,q) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} 0 & A_{2,1,1} & 0 \\ A_{2,2} & A_{2,1,1} & A_{1,1,1,1} \end{bmatrix}$

H([2,1,1] $t\,q A_{1,1,1,1} + (q + t^2) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{2,1,1} & 0 \\ 0 & A_{1,1,1,1} \\ A_{2,2} & A_{2,1,1} \end{bmatrix}$

H([1,1,1,1])

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $q^2 A_{1,1,1,1} + (q + t\,q) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} 0 & A_{2,1,1} & 0 \\ A_{2,2} & A_{2,1,1} & A_{1,1,1,1} \end{bmatrix}$

H([2,1,1] $t\,q\,A_{1,1,1,1} + (q + t^2) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{2,1,1} & 0 \\ 0 & A_{1,1,1,1} \\ A_{2,2} & A_{2,1,1} \end{bmatrix}$

H([1,1,1,1]) $t^4 A_{1,1,1,1} + (t^2 + t^3) A_{2,1,1} + A_{2,2}$

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $q^2 A_{1,1,1,1} + (q + t\,q)\, A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} 0 & A_{2,1,1} & 0 \\ A_{2,2} & A_{2,1,1} & A_{1,1,1,1} \end{bmatrix}$

H([2,1,1] $t\,q\, A_{1,1,1,1} + (q + t^2)\, A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{2,1,1} & 0 \\ 0 & A_{1,1,1,1} \\ A_{2,2} & A_{2,1,1} \end{bmatrix}$

H([1,1,1,1]) $t^4 A_{1,1,1,1} + (t^2 + t^3)\, A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{1,1,1,1} \\ A_{2,1,1} \\ A_{2,1,1} \\ 0 \\ A_{2,2} \end{bmatrix}$

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $q^2 A_{1,1,1,1} + (q + t\,q)\,A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} 0 & A_{2,1,1} & 0 \\ A_{2,2} & A_{2,1,1} & A_{1,1,1,1} \end{bmatrix}$

H([2,1,1] $t\,q\,A_{1,1,1,1} + (q + t^2)\,A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{2,1,1} & 0 \\ 0 & A_{1,1,1,1} \\ A_{2,2} & A_{2,1,1} \end{bmatrix}$

H([1,1,1,1]) $t^4 A_{1,1,1,1} + (t^2 + t^3)\,A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{1,1,1,1} \\ A_{2,1,1} \\ A_{2,1,1} \\ 0 \\ A_{2,2} \end{bmatrix}$

next

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $q^2 A_{1,1,1,1} + (q + t\,q) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} 0 & A_{2,1,1} & 0 \\ A_{2,2} & A_{2,1,1} & A_{1,1,1,1} \end{bmatrix}$

H([2,1,1] $t\,q A_{1,1,1,1} + (q + t^2) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{2,1,1} & 0 \\ 0 & A_{1,1,1,1} \\ A_{2,2} & A_{2,1,1} \end{bmatrix}$

H([1,1,1,1]) $t^4 A_{1,1,1,1} + (t^2 + t^3) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{1,1,1,1} \\ A_{2,1,1} \\ A_{2,1,1} \\ 0 \\ A_{2,2} \end{bmatrix}$

(for 2 bounded partitions of 6)

next

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $q^2 A_{1,1,1,1} + (q + t\,q)\, A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} 0 & A_{2,1,1} & 0 \\ A_{2,2} & A_{2,1,1} & A_{1,1,1,1} \end{bmatrix}$

H([2,1,1] $t\,q\, A_{1,1,1,1} + (q + t^2)\, A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{2,1,1} & 0 \\ 0 & A_{1,1,1,1} \\ A_{2,2} & A_{2,1,1} \end{bmatrix}$

H([1,1,1,1]) $t^4 A_{1,1,1,1} + (t^2 + t^3)\, A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{1,1,1,1} \\ A_{2,1,1} \\ A_{2,1,1} \\ 0 \\ A_{2,2} \end{bmatrix}$

(for 2 bounded partitions of 6)

$H_{2,2,2}$, --> , , $\begin{bmatrix} 0 & A_{2,2,1,1} & A_{2,1,1,1,1} & 0 \\ 0 & A_{2,2,1,1} & A_{2,1,1,1,1} & 0 \\ A_{2,2,2} & A_{2,2,1,1} & A_{2,1,1,1,1} & A_{1,1,1,1,1,1} \end{bmatrix}$

next

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $q^2 A_{1,1,1,1} + (q + t\,q)\, A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} 0 & A_{2,1,1} & 0 \\ A_{2,2} & A_{2,1,1} & A_{1,1,1,1} \end{bmatrix}$

H([2,1,1] $t\,q\, A_{1,1,1,1} + (q + t^2)\, A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{2,1,1} & 0 \\ 0 & A_{1,1,1,1} \\ A_{2,2} & A_{2,1,1} \end{bmatrix}$

H([1,1,1,1]) $t^4 A_{1,1,1,1} + (t^2 + t^3)\, A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{1,1,1,1} \\ A_{2,1,1} \\ A_{2,1,1} \\ 0 \\ A_{2,2} \end{bmatrix}$

(for 2 bounded partitions of 6)

$H_{2,2,2}$, --> , , $\begin{bmatrix} 0 & A_{2,2,1,1} & A_{2,1,1,1,1} & 0 \\ 0 & A_{2,2,1,1} & A_{2,1,1,1,1} & 0 \\ A_{2,2,2} & A_{2,2,1,1} & A_{2,1,1,1,1} & A_{1,1,1,1,1,1} \end{bmatrix}$

$H_{2,2,1,1}$, --> , , $\begin{bmatrix} A_{2,2,1,1} & A_{2,1,1,1,1} & 0 \\ 0 & A_{2,1,1,1,1} & 0 \\ 0 & A_{2,2,1,1} & A_{1,1,1,1,1,1} \\ A_{2,2,2} & A_{2,2,1,1} & A_{2,1,1,1,1} \end{bmatrix}$

next

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $q^2 A_{1,1,1,1} + (q + t\,q) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} 0 & A_{2,1,1} & 0 \\ A_{2,2} & A_{2,1,1} & A_{1,1,1,1} \end{bmatrix}$

H([2,1,1] $t\,q A_{1,1,1,1} + (q + t^2) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{2,1,1} & 0 \\ 0 & A_{1,1,1,1} \\ A_{2,2} & A_{2,1,1} \end{bmatrix}$

H([1,1,1,1]) $t^4 A_{1,1,1,1} + (t^2 + t^3) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{1,1,1,1} \\ A_{2,1,1} \\ A_{2,1,1} \\ 0 \\ A_{2,2} \end{bmatrix}$

(for 2 bounded partitions of 6)

$H_{2,2,2},$ --> , , $\begin{bmatrix} 0 & A_{2,2,1,1} & A_{2,1,1,1,1} & 0 \\ 0 & A_{2,2,1,1} & A_{2,1,1,1,1} & 0 \\ A_{2,2,2} & A_{2,2,1,1} & A_{2,1,1,1,1} & A_{1,1,1,1,1,1} \end{bmatrix}$

$H_{2,2,1,1},$ --> , , $\begin{bmatrix} A_{2,2,1,1} & A_{2,1,1,1,1} & 0 \\ 0 & A_{2,1,1,1,1} & 0 \\ 0 & A_{2,2,1,1} & A_{1,1,1,1,1,1} \\ A_{2,2,2} & A_{2,2,1,1} & A_{2,1,1,1,1} \end{bmatrix}$

$H_{2,1,1,1,1},$ --> , , $\begin{bmatrix} A_{2,1,1,1,1} & 0 \\ 0 & 0 \\ A_{2,2,1,1} & A_{1,1,1,1,1,1} \\ A_{2,2,1,1} & A_{2,1,1,1,1} \\ 0 & A_{2,1,1,1,1} \\ 0 & 0 \\ A_{2,2,2} & A_{2,2,1,1} \end{bmatrix}$

next

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $q^2 A_{1,1,1,1} + (q + t\,q) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} 0 & A_{2,1,1} & 0 \\ A_{2,2} & A_{2,1,1} & A_{1,1,1,1} \end{bmatrix}$

H([2,1,1] $t\,q\,A_{1,1,1,1} + (q + t^2) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{2,1,1} & 0 \\ 0 & A_{1,1,1,1} \\ A_{2,2} & A_{2,1,1} \end{bmatrix}$

H([1,1,1,1]) $t^4 A_{1,1,1,1} + (t^2 + t^3) A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{1,1,1,1} \\ A_{2,1,1} \\ A_{2,1,1} \\ 0 \\ A_{2,2} \end{bmatrix}$

(for 2 bounded partitions of 6)

$H_{2,2,2}$, -->, , $\begin{bmatrix} 0 & A_{2,2,1,1} & A_{2,1,1,1,1} & 0 \\ 0 & A_{2,2,1,1} & A_{2,1,1,1,1} & 0 \\ A_{2,2,2} & A_{2,2,1,1} & A_{2,1,1,1,1} & A_{1,1,1,1,1,1} \end{bmatrix}$

$H_{2,2,1,1}$, -->, , $\begin{bmatrix} A_{2,2,1,1} & A_{2,1,1,1,1} & 0 \\ 0 & A_{2,1,1,1,1} & 0 \\ 0 & A_{2,2,1,1} & A_{1,1,1,1,1,1} \\ A_{2,2,2} & A_{2,2,1,1} & A_{2,1,1,1,1} \end{bmatrix}$

$H_{2,1,1,1,1}$, -->, , $\begin{bmatrix} A_{2,1,1,1,1} & 0 \\ 0 & 0 \\ A_{2,2,1,1} & A_{1,1,1,1,1,1} \\ A_{2,2,1,1} & A_{2,1,1,1,1} \\ 0 & A_{2,1,1,1,1} \\ 0 & 0 \\ A_{2,2,2} & A_{2,2,1,1} \end{bmatrix}$

$H_{1,1,1,1,1,1}$, -->, , $\begin{bmatrix} A_{1,1,1,1,1,1} \\ A_{2,1,1,1,1} \\ A_{2,1,1,1,1} \\ A_{2,1,1,1,1} \\ A_{2,2,1,1} \\ A_{2,2,1,1} \\ A_{2,2,1,1} \\ 0 \\ 0 \\ A_{2,2,2} \end{bmatrix}$

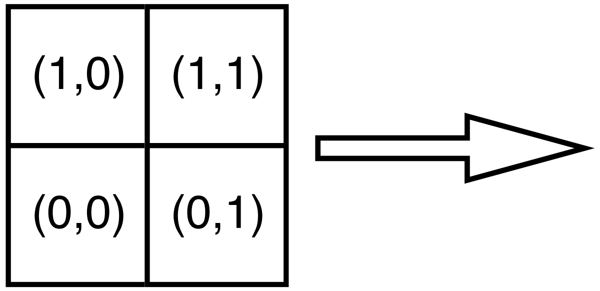
next

Some remarkable determinants

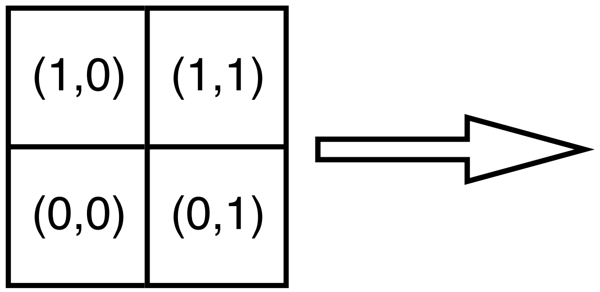
Some remarkable determinants

(1,0)	(1,1)
(0,0)	(0,1)

Some remarkable determinants

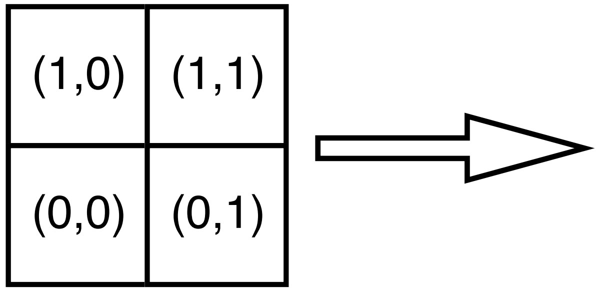


Some remarkable determinants



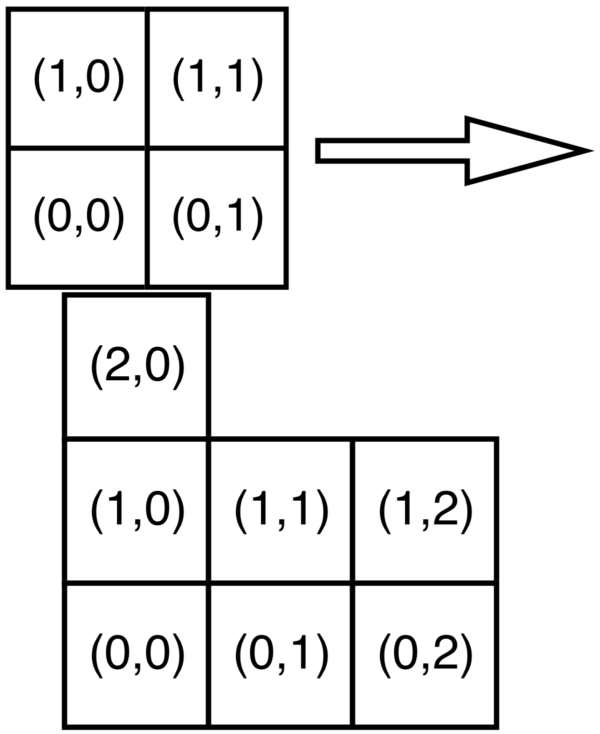
$$\Delta_{2,2}(X,Y) = \det \begin{pmatrix} x_1^0 y_1^0 & x_2^0 y_2^0 & x_3^0 y_3^0 & x_4^0 y_4^0 \\ x_1^0 y_1^1 & x_2^0 y_2^1 & x_3^0 y_3^1 & x_4^0 y_4^1 \\ x_1^1 y_1^0 & x_2^1 y_2^0 & x_3^1 y_3^0 & x_4^1 y_4^0 \\ x_1^1 y_1^1 & x_2^1 y_2^1 & x_3^1 y_3^1 & x_4^1 y_4^1 \end{pmatrix}$$

Some remarkable determinants



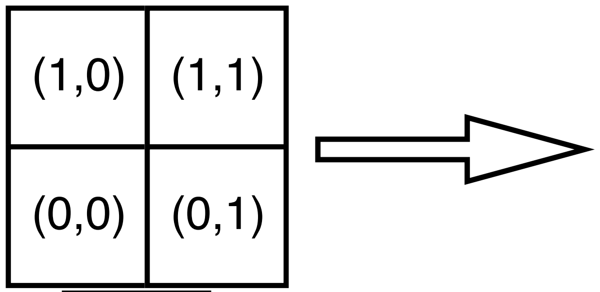
$$\Delta_{2,2}(X,Y) = \det\begin{pmatrix} x_1^0y_1^0 & x_2^0y_2^0 & x_3^0y_3^0 & x_4^0y_4^0 \\ x_1^0y_1^1 & x_2^0y_2^1 & x_3^0y_3^1 & x_4^0y_4^1 \\ x_1^1y_1^0 & x_2^1y_2^0 & x_3^1y_3^0 & x_4^1y_4^0 \\ x_1^1y_1^1 & x_2^1y_2^1 & x_3^1y_3^1 & x_4^1y_4^1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix}$$

Some remarkable determinants

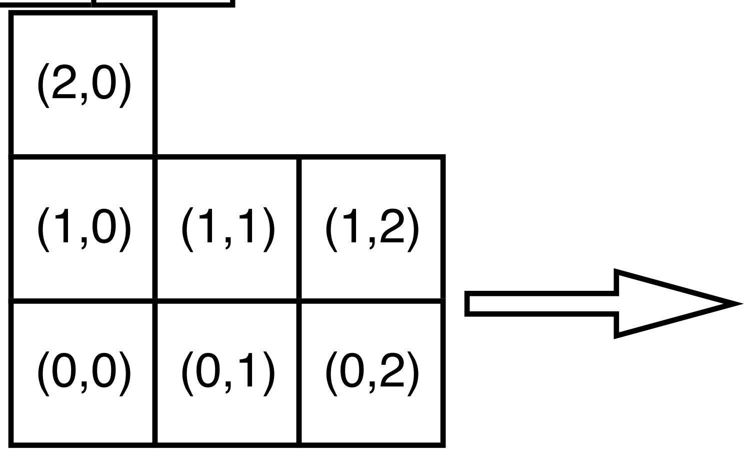


$$\Delta_{2,2}(X,Y) = \det\begin{pmatrix} x_1^0y_1^0 & x_2^0y_2^0 & x_3^0y_3^0 & x_4^0y_4^0 \\ x_1^0y_1^1 & x_2^0y_2^1 & x_3^0y_3^1 & x_4^0y_4^1 \\ x_1^1y_1^0 & x_2^1y_2^0 & x_3^1y_3^0 & x_4^1y_4^0 \\ x_1^1y_1^1 & x_2^1y_2^1 & x_3^1y_3^1 & x_4^1y_4^1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix}$$

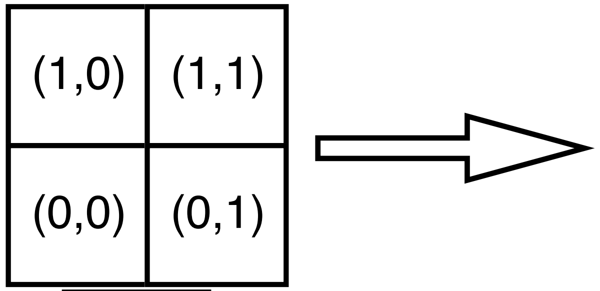
Some remarkable determinants



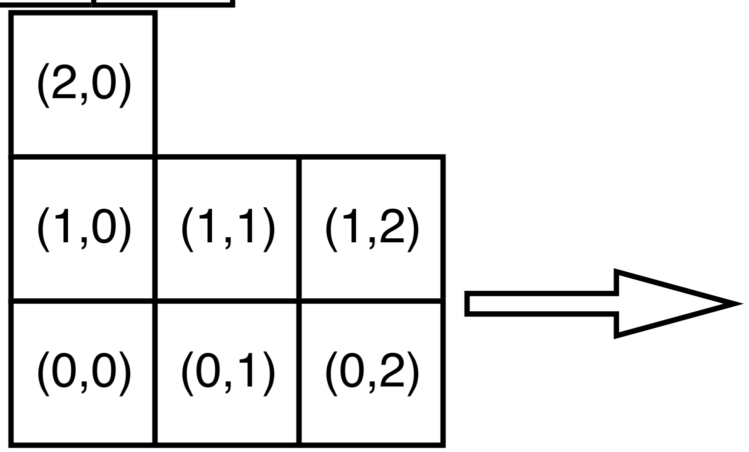
$$\mathbf{\Delta_{2,2}(X,Y)} = \det\begin{pmatrix} x_1^0y_1^0 & x_2^0y_2^0 & x_3^0y_3^0 & x_4^0y_4^0 \\ x_1^0y_1^1 & x_2^0y_2^1 & x_3^0y_3^1 & x_4^0y_4^1 \\ x_1^1y_1^0 & x_2^1y_2^0 & x_3^1y_3^0 & x_4^1y_4^0 \\ x_1^1y_1^1 & x_2^1y_2^1 & x_3^1y_3^1 & x_4^1y_4^1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix}$$



Some remarkable determinants

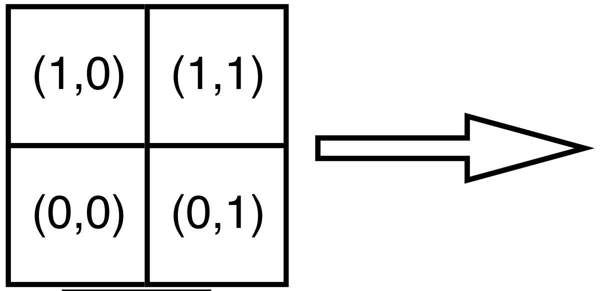


$$\Delta_{2,2}(X,Y) = \det\begin{pmatrix} x_1^0y_1^0 & x_2^0y_2^0 & x_3^0y_3^0 & x_4^0y_4^0 \\ x_1^0y_1^1 & x_2^0y_2^1 & x_3^0y_3^1 & x_4^0y_4^1 \\ x_1^1y_1^0 & x_2^1y_2^0 & x_3^1y_3^0 & x_4^1y_4^0 \\ x_1^1y_1^1 & x_2^1y_2^1 & x_3^1y_3^1 & x_4^1y_4^1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix}$$

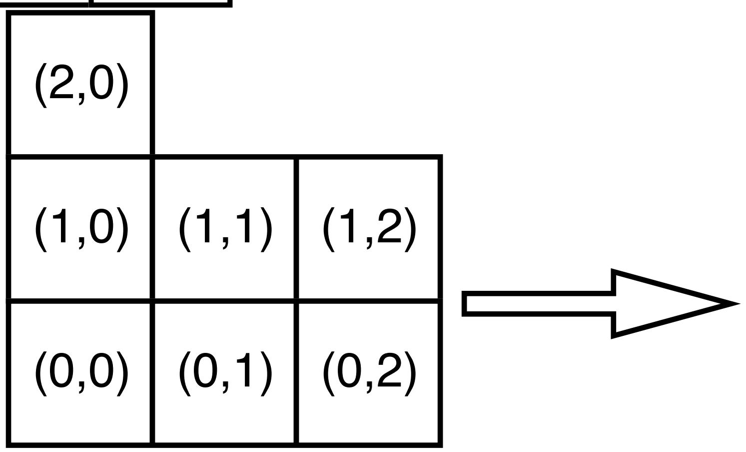


$$\Delta_{331}(X,Y) = \det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_5 & y_7 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 & y_7^2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6^2 & x_7^2 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 & x_6y_6 & x_7y_7 \\ x_1y_1^2 & x_2y_2^2 & x_3y_3^2 & x_4y_4^2 & x_5y_5^2 & x_6y_6^2 & x_7y_7^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 & x_7^2 \end{pmatrix}$$

Some remarkable determinants



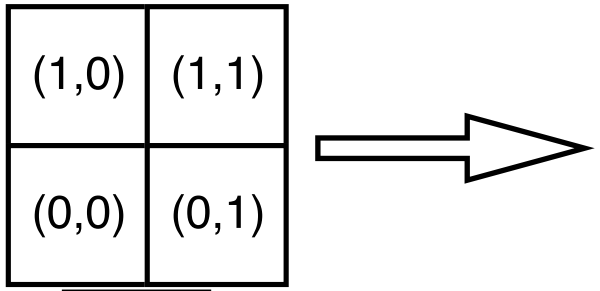
$$\Delta_{2,2}(X,Y) = \det\begin{pmatrix} x_1^0y_1^0 & x_2^0y_2^0 & x_3^0y_3^0 & x_4^0y_4^0 \\ x_1^0y_1^1 & x_2^0y_2^1 & x_3^0y_3^1 & x_4^0y_4^1 \\ x_1^1y_1^0 & x_2^1y_2^0 & x_3^1y_3^0 & x_4^1y_4^0 \\ x_1^1y_1^1 & x_2^1y_2^1 & x_3^1y_3^1 & x_4^1y_4^1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix}$$



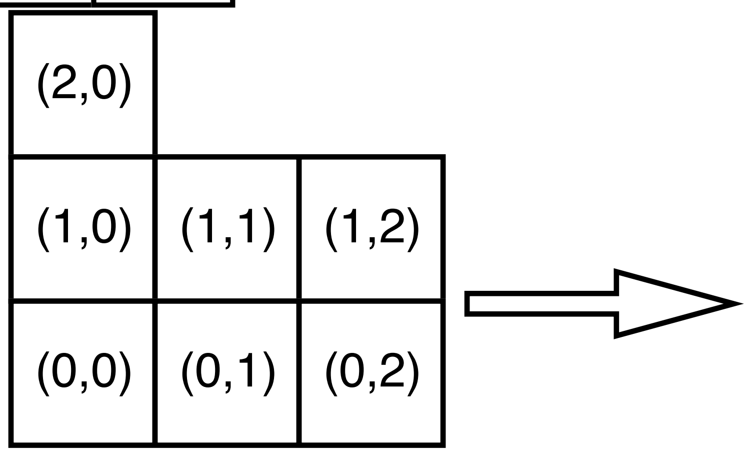
$$\Delta_{331}(X,Y) = \det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_5 & y_7 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 & y_7^2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6^2 & x_7^2 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 & x_6y_6 & x_7y_7 \\ x_1y_1^2 & x_2y_2^2 & x_3y_3^2 & x_4y_4^2 & x_5y_5^2 & x_6y_6^2 & x_7y_7^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 & x_7^2 \end{pmatrix}$$

General definition

Some remarkable determinants



$$\Delta_{2,2}(X,Y) = \det\begin{pmatrix} x_1^0y_1^0 & x_2^0y_2^0 & x_3^0y_3^0 & x_4^0y_4^0 \\ x_1^0y_1^1 & x_2^0y_2^1 & x_3^0y_3^1 & x_4^0y_4^1 \\ x_1^1y_1^0 & x_2^1y_2^0 & x_3^1y_3^0 & x_4^1y_4^0 \\ x_1^1y_1^1 & x_2^1y_2^1 & x_3^1y_3^1 & x_4^1y_4^1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix}$$

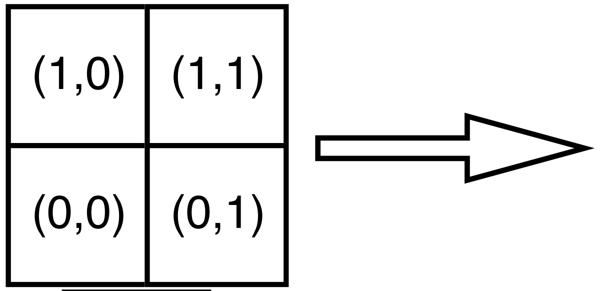


$$\Delta_{331}(X,Y) = \det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_5 & y_7 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 & y_7^2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6^2 & x_7^2 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 & x_6y_6 & x_7y_7 \\ x_1y_1^2 & x_2y_2^2 & x_3y_3^2 & x_4y_4^2 & x_5y_5^2 & x_6y_6^2 & x_7y_7^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 & x_7^2 \end{pmatrix}$$

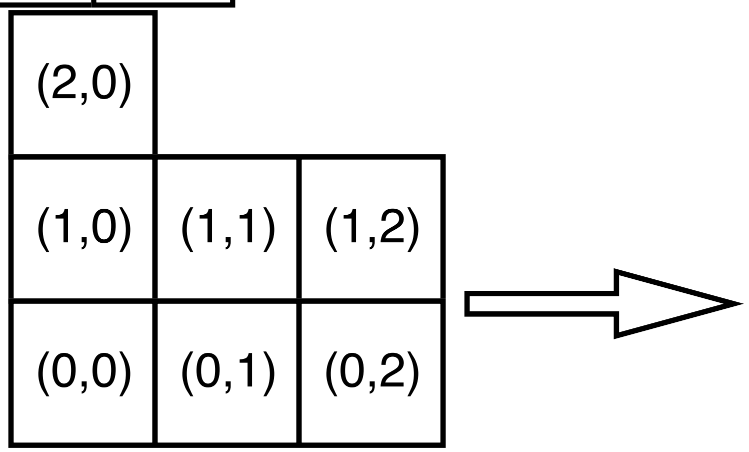
General definition

If $(p_1,q_1),(p_2,q_2),\ldots,(p_n,q_n)$ are the cells of the Ferrers diagram of $\mu \vdash n$ then

Some remarkable determinants



$$\Delta_{2,2}(X,Y) = \det \begin{pmatrix} x_1^0y_1^0 & x_2^0y_2^0 & x_3^0y_3^0 & x_4^0y_4^0 \\ x_1^0y_1^1 & x_2^0y_2^1 & x_3^0y_3^1 & x_4^0y_4^1 \\ x_1^1y_1^0 & x_2^1y_2^0 & x_3^1y_3^0 & x_4^1y_4^0 \\ x_1^1y_1^1 & x_2^1y_2^1 & x_3^1y_3^1 & x_4^1y_4^1 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix}$$



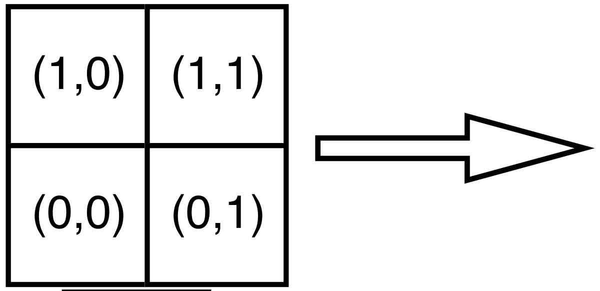
$$\Delta_{331}(X,Y) = \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_5 & y_7 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 & y_7^2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6^2 & x_7^2 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 & x_6y_6 & x_7y_7 \\ x_1y_1^2 & x_2y_2^2 & x_3y_3^2 & x_4y_4^2 & x_5y_5^2 & x_6y_6^2 & x_7y_7^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 & x_7^2 \end{pmatrix}$$

General definition

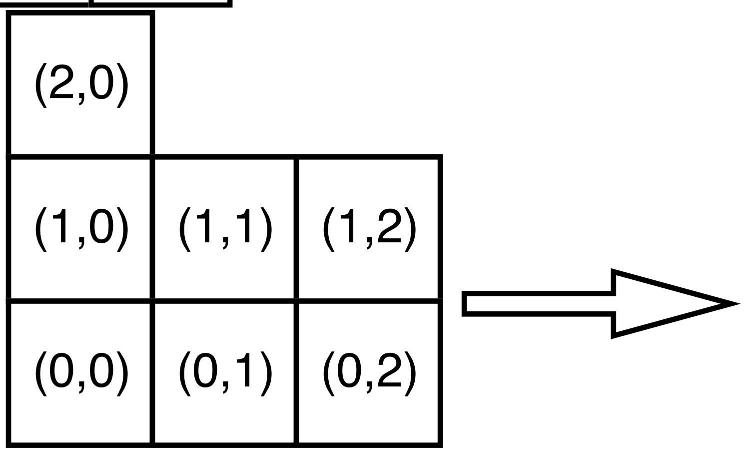
If $(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)$ are the cells of the Ferrers diagram of $\mu \vdash n$ then

$$\Delta_\mu(x,y) = \det \| x_j^{p_i} y_j^{q_i} \|_{i,j=1}^{n}$$

Some remarkable determinants



$$\Delta_{2,2}(X,Y) = \det\begin{pmatrix} x_1^0y_1^0 & x_2^0y_2^0 & x_3^0y_3^0 & x_4^0y_4^0 \\ x_1^0y_1^1 & x_2^0y_2^1 & x_3^0y_3^1 & x_4^0y_4^1 \\ x_1^1y_1^0 & x_2^1y_2^0 & x_3^1y_3^0 & x_4^1y_4^0 \\ x_1^1y_1^1 & x_2^1y_2^1 & x_3^1y_3^1 & x_4^1y_4^1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix}$$



$$\Delta_{331}(X,Y) = \det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_5 & y_7 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 & y_7^2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6^2 & x_7^2 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 & x_6y_6 & x_7y_7 \\ x_1y_1^2 & x_2y_2^2 & x_3y_3^2 & x_4y_4^2 & x_5y_5^2 & x_6y_6^2 & x_7y_7^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 & x_7^2 \end{pmatrix}$$

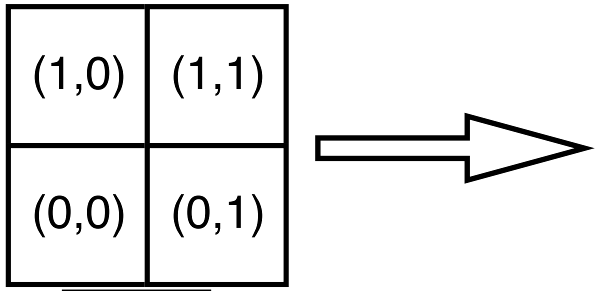
General definition

If $(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)$ are the cells of the Ferrers diagram of $\mu \vdash n$ then

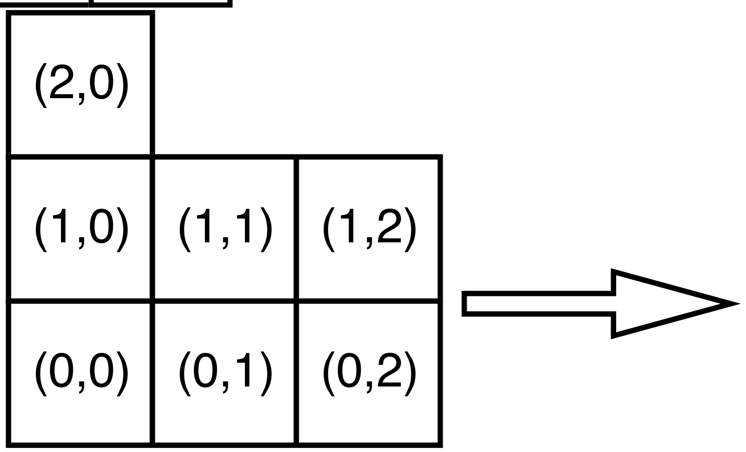
$$\Delta_\mu(x,y) = \det \| x_j^{p_i} y_j^{q_i} \|_{i,j=1}^{n}$$

The linear span of all the partial derivatives of $\Delta_\mu(x,y)$ is denoted $M_\mu[X,Y]$

Some remarkable determinants



$$\Delta_{2,2}(X,Y) = \det \begin{pmatrix} x_1^0y_1^0 & x_2^0y_2^0 & x_3^0y_3^0 & x_4^0y_4^0 \\ x_1^0y_1^1 & x_2^0y_2^1 & x_3^0y_3^1 & x_4^0y_4^1 \\ x_1^1y_1^0 & x_2^1y_2^0 & x_3^1y_3^0 & x_4^1y_4^0 \\ x_1^1y_1^1 & x_2^1y_2^1 & x_3^1y_3^1 & x_4^1y_4^1 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix}$$



$$\Delta_{331}(X,Y) = \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_5 & y_7 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 & y_7^2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6^2 & x_7^2 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 & x_6y_6 & x_7y_7 \\ x_1y_1^2 & x_2y_2^2 & x_3y_3^2 & x_4y_4^2 & x_5y_5^2 & x_6y_6^2 & x_7y_7^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 & x_7^2 \end{pmatrix}$$

General definition

If $(p_1,q_1),(p_2,q_2),\ldots,(p_n,q_n)$ are the cells of the Ferrers diagram of $\mu \vdash n$ then

$$\Delta_\mu(x,y) = \det \| x_j^{p_i} y_j^{q_i} \|_{i,j=1}^{n}$$

The linear span of all the partial derivatives of $\Delta_\mu(x,y)$ is denoted $M_\mu[X,Y]$

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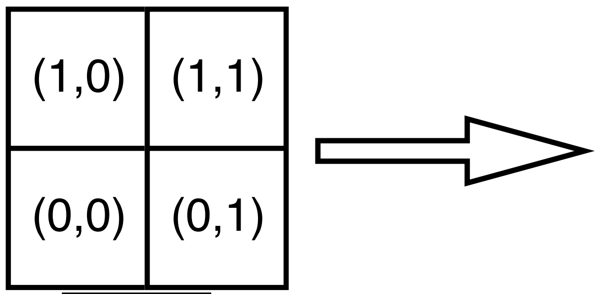

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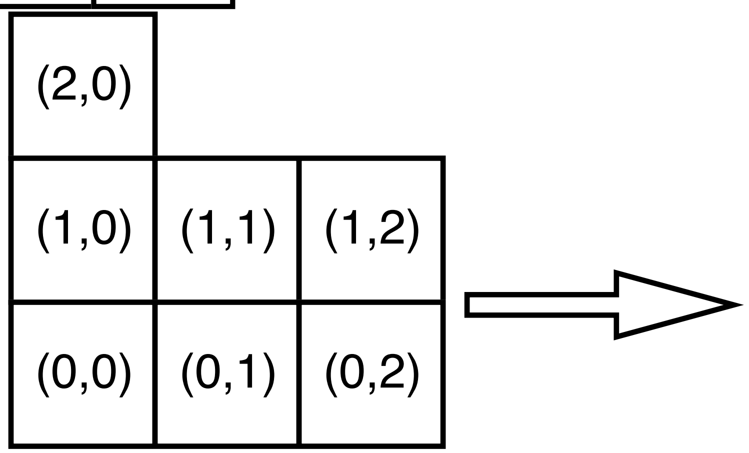

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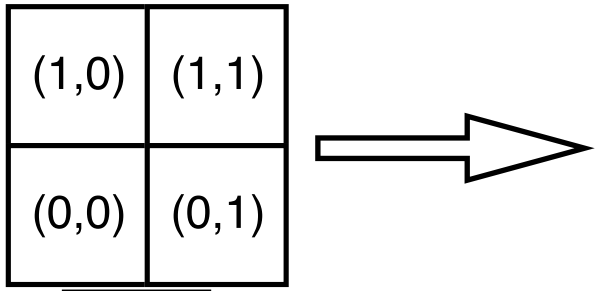

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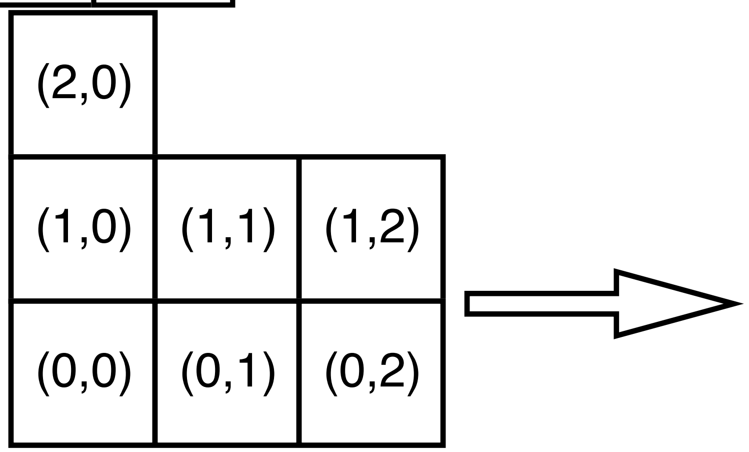

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next

Brief review

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Theorem

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Theorem (easy)

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For any $\mu \vdash n$ the dimension of the linear span of the derivatives of $\Delta_\mu(X,Y)$ is at most n!

In symbols $$\dim \mathbf{M}_\mu[\mathbf{X},\mathbf{Y}] \leq \mathbf{n}!$$

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next

Hilbert series

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the "*Hilbert series*" of V

For instance for $R = \mathbb{Q}[x_1, x_2, \ldots, x_n]$ we have $R = H_0(R) \oplus H_1(R) \oplus H_2(R) \oplus \cdots$ with

$$H_m(R) \; = \; L\left[x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \; : \; p_1 + p_2 + \cdots + p_n = m\right]$$

In this case

$$\dim H_m(R) \; = \; \binom{m+n-1}{n-1} \quad \text{and} \quad F_R(t) \; = \; \frac{1}{(1-t)^n}$$

Our spaces $M[X,Y]$ are "*bigraded*" that is we have the double decomposition

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With $H_{r,s}(M_\mu[X,Y])$ the linear span of derivatives of $\Delta_\mu(x,y)$

that are homogeneous of degree r in $x_1, x_2, \ldots, x_n$ and degree s in $y_1, y_2, \ldots, y_n$

Here and after we set

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Hilbert series

A vector space V is called "*graded*" if and only if

$$V = H_o(V) \oplus H_1(V) \oplus H_2(V) \oplus \cdots \oplus H_m(V) \oplus \cdots$$

The subspace "$H_m(V)$" is called the "m^{th} *homogeneous component*" of V.

its elements are called homogeneous of degree m

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next

The Macdonald Polynomials as Frobenius Characteristics

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⟵ **Restriction to $\mathcal{S}_3$**

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How does this decompose in terms of the basis $\{\tilde{H}_\mu[X;q,t]\}_{\mu\vdash 3}$

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next

The Macdonald-Stanley (dual) Pieri Rules

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$$\partial_{\mathbf{p_1}} \tilde{\mathrm{H}}_\mu[\mathbf{X};\mathbf{q},\mathbf{t}] \quad = \quad \sum_{\nu \to \mu} \mathbf{c}_{\mu,\nu}(\mathbf{q},\mathbf{t}) \tilde{\mathrm{H}}_\nu[\mathbf{X};\mathbf{q},\mathbf{t}]$$

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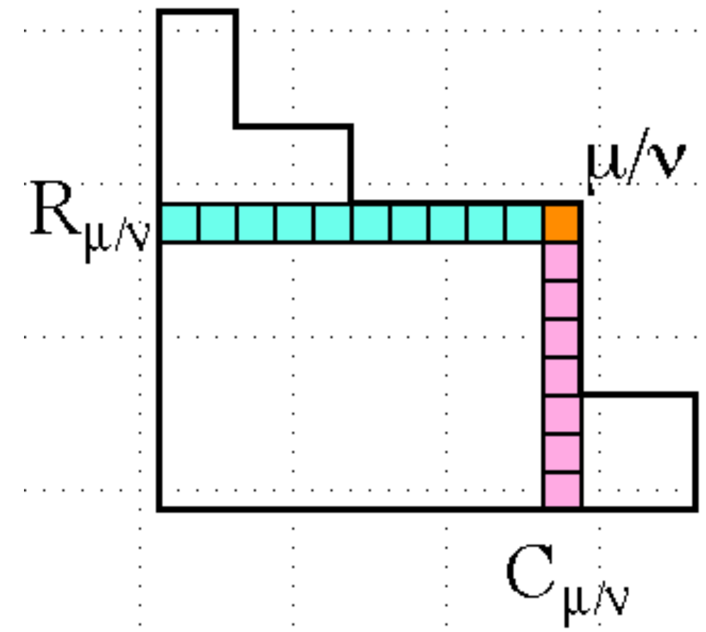

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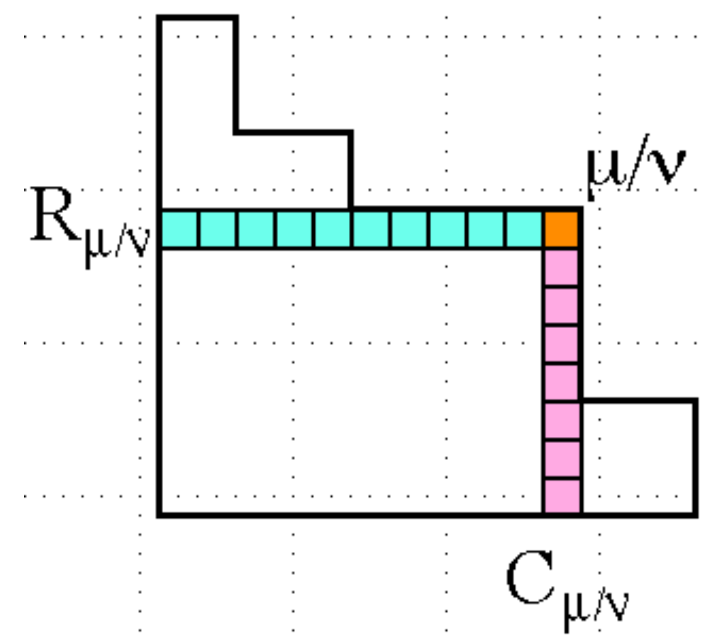

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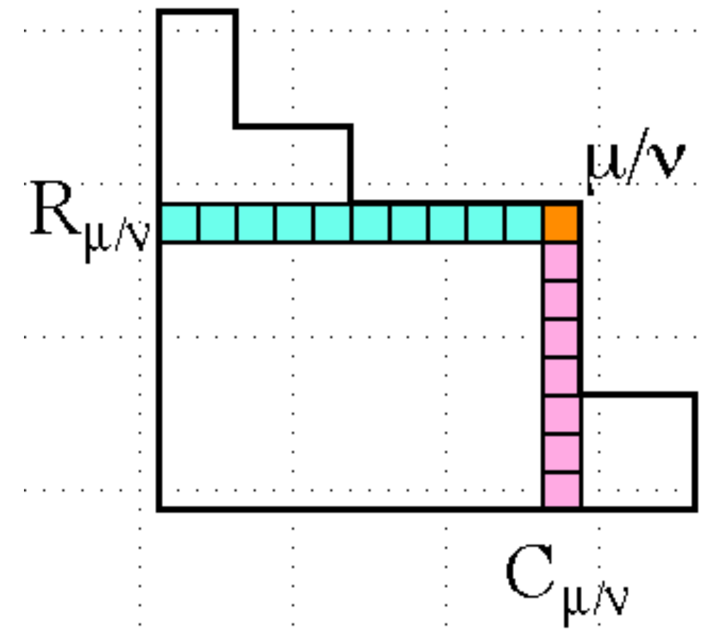

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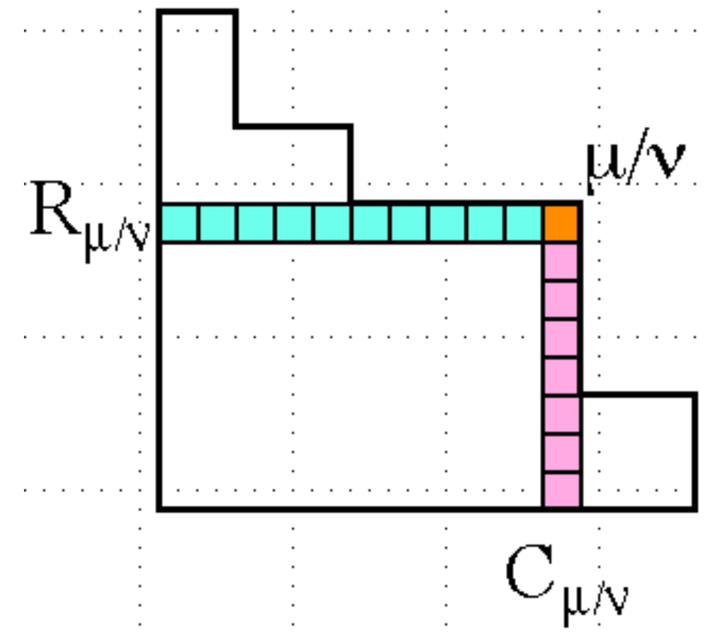

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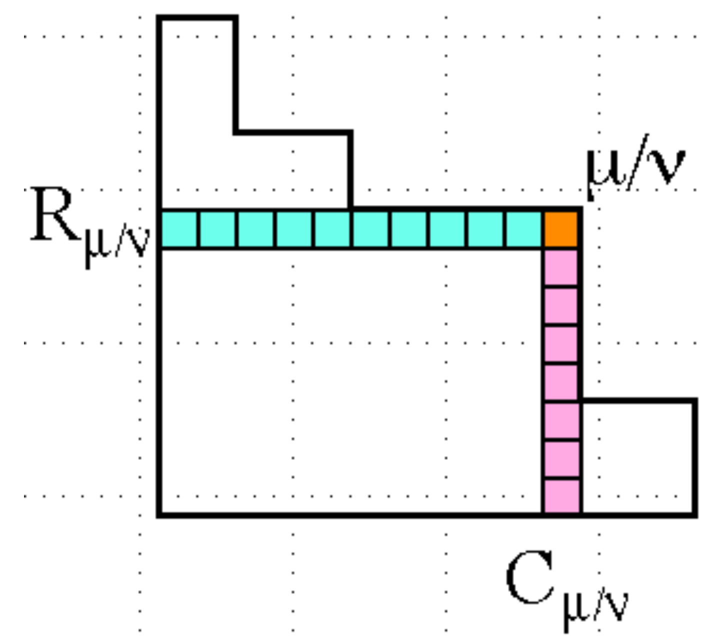

where $R_{\mu\nu}$ and $C_{\mu\nu}$ denote the collections of cells of ν that are respectively in the row and the column of the cell μ/ν.

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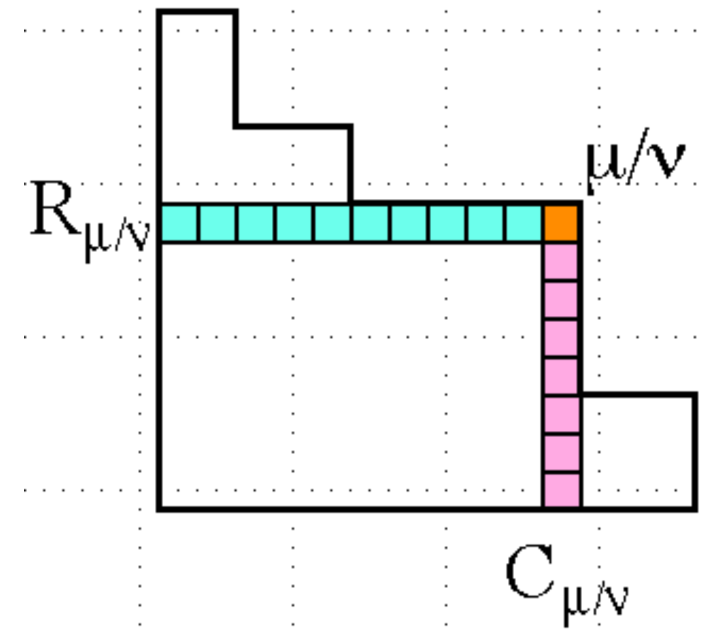


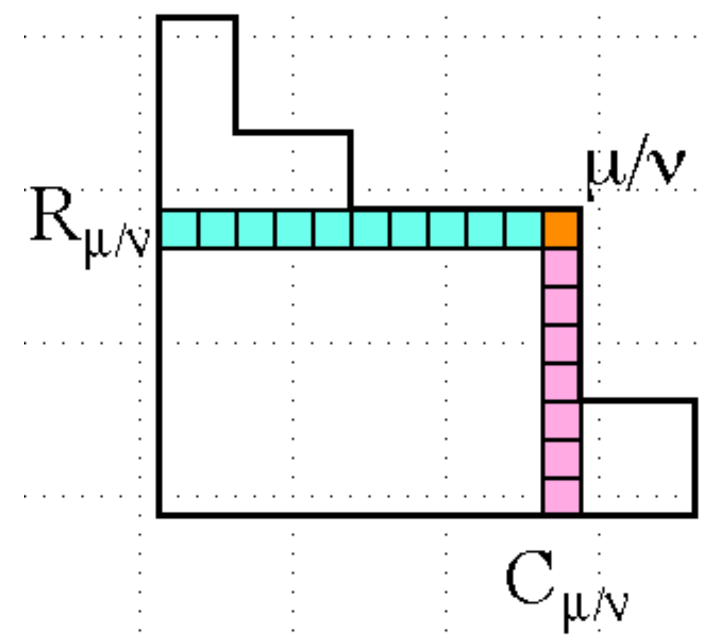
where $R_{\mu\nu}$ and $C_{\mu\nu}$ denote the collections of cells of ν that are respectively in the row and the column of the cell μ/ν.

The Hilbert series

The Macdonald-Stanley (dual) Pieri Rules

$$\partial_{p_1}\tilde{H}_\mu[X;q,t] = \sum_{\nu\to\mu} c_{\mu,\nu}(q,t)\tilde{H}_\nu[X;q,t]$$

$$c_{\mu\nu}(q,t) = \prod_{s\in R_{\mu\nu}} \frac{t^{l_\mu(s)} - q^{a_\mu(s)+1}}{t^{l_\nu(s)} - q^{a_\nu(s)+1}} \prod_{s\in C_{\mu\nu}} \frac{q^{a_\mu(s)} - t^{l_\mu(s)+1}}{q^{a_\nu(s)} - t^{l_\nu(s)+1}},$$



where $R_{\mu\nu}$ and $C_{\mu\nu}$ denote the collections of cells of ν that are respectively in the row and the column of the cell μ/ν.

The Hilbert series

$F_\mu(q,t)$

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$$\partial_{p_1}\tilde{H}_\mu[X;q,t] = \sum_{\nu\to\mu} c_{\mu,\nu}(q,t)\tilde{H}_\nu[X;q,t]$$

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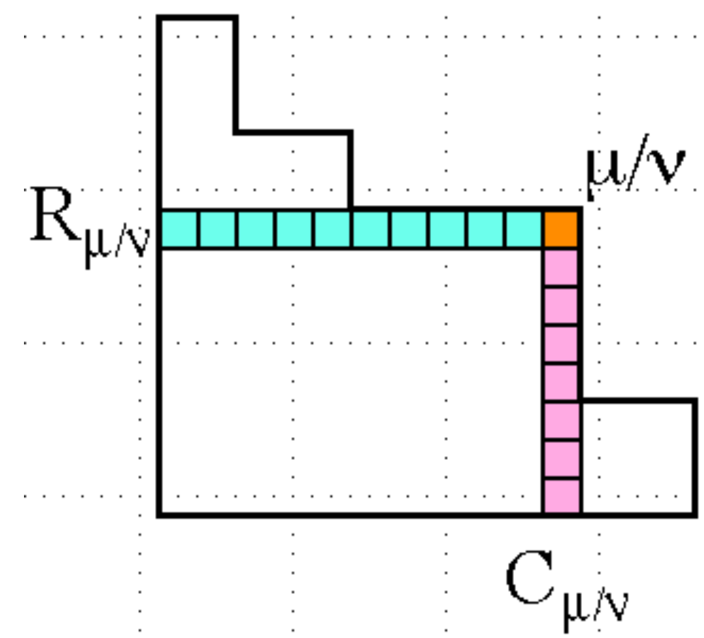

where $R_{\mu\nu}$ and $C_{\mu\nu}$ denote the collections of cells of ν that are respectively in the row and the column of the cell μ/ν.

The Hilbert series

$$F_\mu(q,t) = \partial_{p_1}^n \tilde{H}_\mu[X;q,t]$$

The Macdonald-Stanley (dual) Pieri Rules

$$\partial_{p_1}\tilde{H}_\mu[X;q,t] = \sum_{\nu\to\mu} c_{\mu,\nu}(q,t)\tilde{H}_\nu[X;q,t]$$

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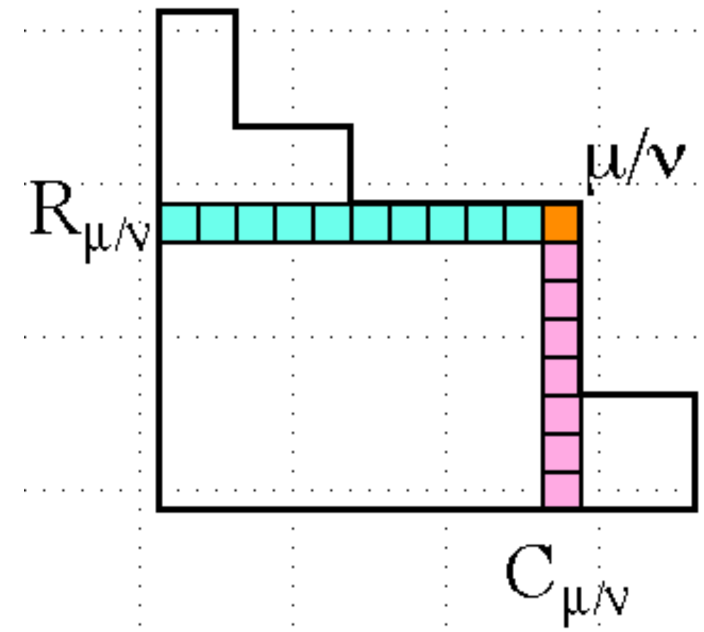

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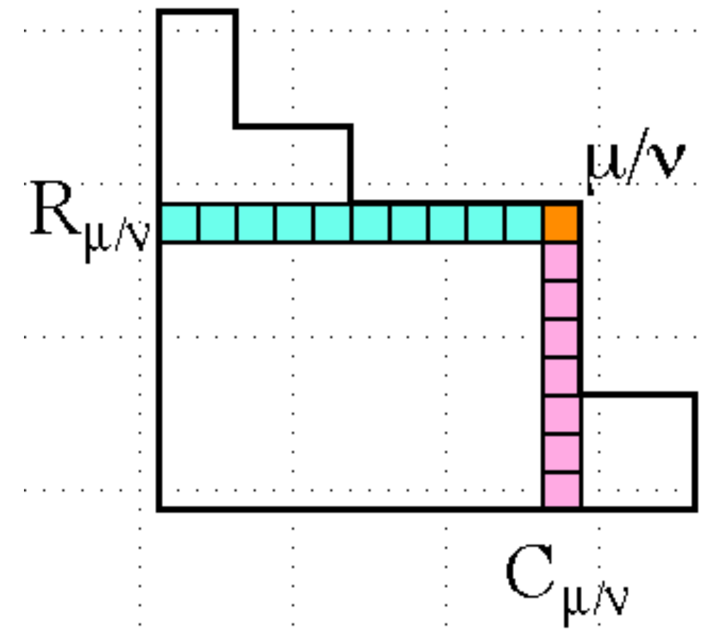

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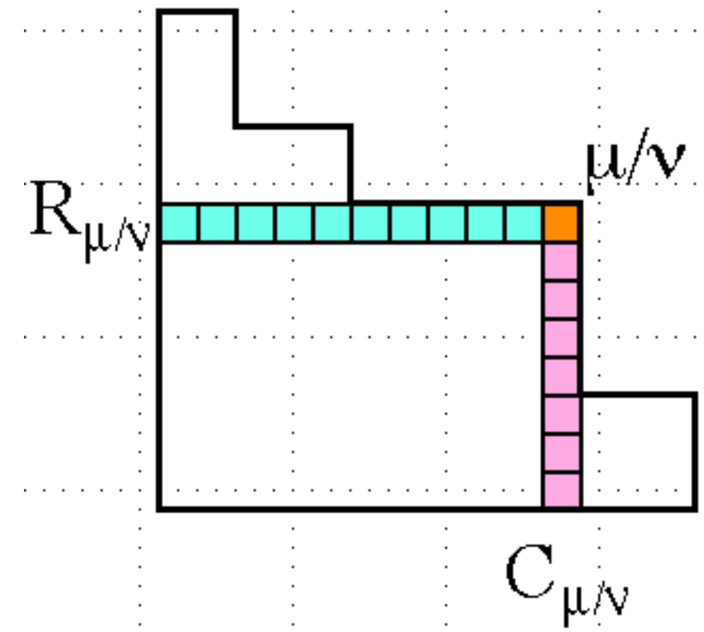
The Macdonald-Stanley (dual) Pieri Rules

$$\partial_{\mathbf{p_1}} \tilde{\mathrm{H}}_\mu[\mathbf{X};\mathbf{q},\mathbf{t}] = \sum_{\nu\to\mu} \mathbf{c}_{\mu,\nu}(\mathbf{q},\mathbf{t}) \tilde{\mathrm{H}}_\nu[\mathbf{X};\mathbf{q},\mathbf{t}]$$

$$\mathbf{c}_{\mu\nu}(\mathbf{q},\mathbf{t}) = \prod_{\mathbf{s}\in \mathbf{R}_{\mu\nu}} \frac{\mathbf{t}^{\mathbf{l}_\mu(\mathbf{s})} - \mathbf{q}^{\mathbf{a}_\mu(\mathbf{s})+1}}{\mathbf{t}^{\mathbf{l}_\nu(\mathbf{s})} - \mathbf{q}^{\mathbf{a}_\nu(\mathbf{s})+1}} \prod_{\mathbf{s}\in \mathbf{C}_{\mu\nu}} \frac{\mathbf{q}^{\mathbf{a}_\mu(\mathbf{s})} - \mathbf{t}^{\mathbf{l}_\mu(\mathbf{s})+1}}{\mathbf{q}^{\mathbf{a}_\nu(\mathbf{s})} - \mathbf{t}^{\mathbf{l}_\nu(\mathbf{s})+1}},$$



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$$\mathrm{F}_\mu(\mathbf{q},\mathbf{t}) = \partial^{\mathbf{n}}_{\mathbf{p_1}} \tilde{\mathrm{H}}_\mu[\mathbf{X};\mathbf{q},\mathbf{t}] = \sum_{\nu\to\mu} c_{\mu\nu}(q,t) \partial^{n-1}_{p_1} \widetilde{H}_\mu[x;q,t] = \sum_{\nu\to\mu} \mathbf{c}_{\mu,\nu}(\mathbf{q},\mathbf{t}) \mathrm{F}_\nu(\mathbf{q},\mathbf{t})$$

$$\boxed{F_\mu(q,t) = \sum_{\nu\to\mu} c_{\mu\nu}(q,t) F_\nu(q,t)}$$

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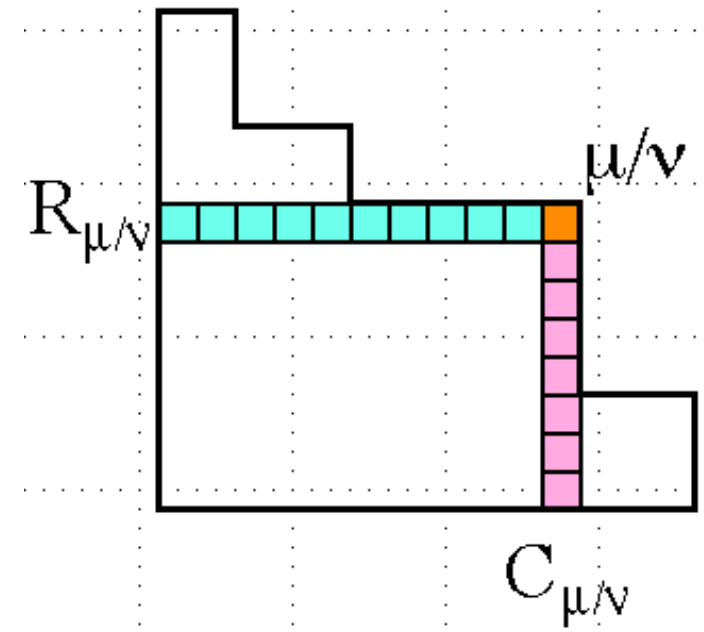
$$\partial_{\mathbf{p_1}} \tilde{\mathrm{H}}_\mu[\mathbf{X};\mathbf{q},\mathbf{t}] = \sum_{\nu\to\mu} \mathbf{c}_{\mu,\nu}(\mathbf{q},\mathbf{t}) \tilde{\mathrm{H}}_\nu[\mathbf{X};\mathbf{q},\mathbf{t}]$$

$$\mathbf{c}_{\mu\nu}(\mathbf{q},\mathbf{t}) = \prod_{s\in \mathbf{R}_{\mu\nu}} \frac{\mathbf{t}^{\mathbf{l}_\mu(s)} - \mathbf{q}^{\mathbf{a}_\mu(s)+1}}{\mathbf{t}^{\mathbf{l}_\nu(s)} - \mathbf{q}^{\mathbf{a}_\nu(s)+1}} \prod_{s\in \mathbf{C}_{\mu\nu}} \frac{\mathbf{q}^{\mathbf{a}_\mu(s)} - \mathbf{t}^{\mathbf{l}_\mu(s)+1}}{\mathbf{q}^{\mathbf{a}_\nu(s)} - \mathbf{t}^{\mathbf{l}_\nu(s)+1}},$$



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$$F_\mu(q,t) = \sum_{\nu\to\mu} c_{\mu\nu}(q,t) F_\nu(q,t)$$

Using Maple

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$$\partial_{p_1} \tilde{H}_\mu[X;q,t] = \sum_{\nu\to\mu} c_{\mu,\nu}(q,t)\tilde{H}_\nu[X;q,t]$$

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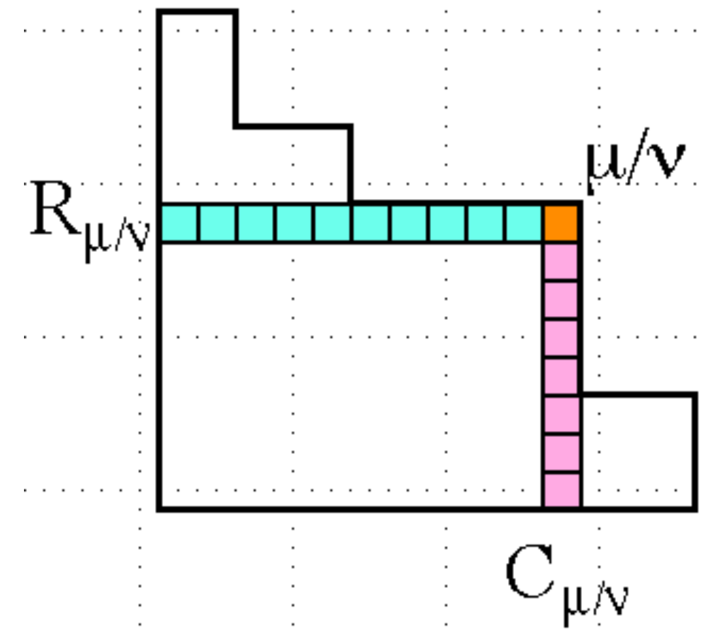

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$$F_\mu(q,t) = \partial^n_{p_1} \tilde{H}_\mu[X;q,t] = \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\partial^{n-1}_{p_1}\widetilde{H}_\mu[x;q,t] = \sum_{\nu\to\mu} c_{\mu,\nu}(q,t)F_\nu(q,t)$$

$$F_\mu(q,t) = \sum_{\nu\to\mu} c_{\mu\nu}(q,t)F_\nu(q,t)$$

Using Maple

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hilb([3,2]);
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The Macdonald-Stanley (dual) Pieri Rules

$$\partial_{p_1}\tilde{H}_\mu[X;q,t] = \sum_{\nu\to\mu} c_{\mu,\nu}(q,t)\tilde{H}_\nu[X;q,t]$$

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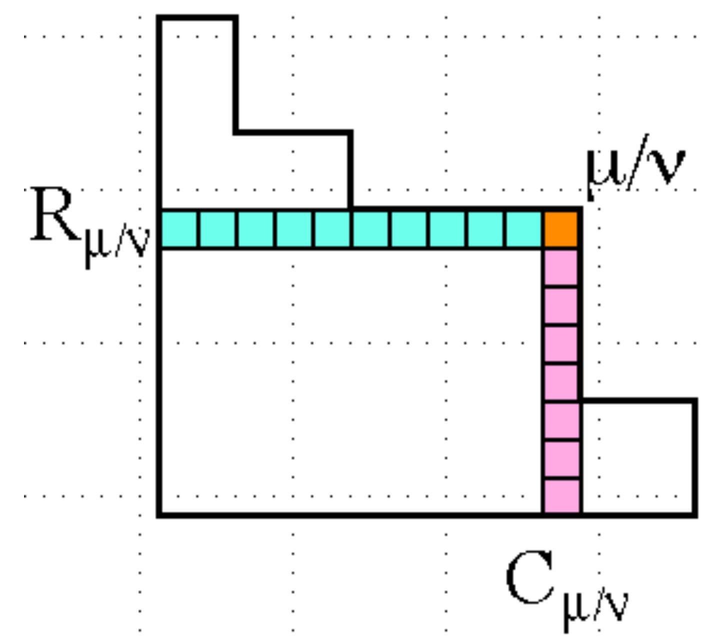

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Using Maple

```
hilb([3,2]);
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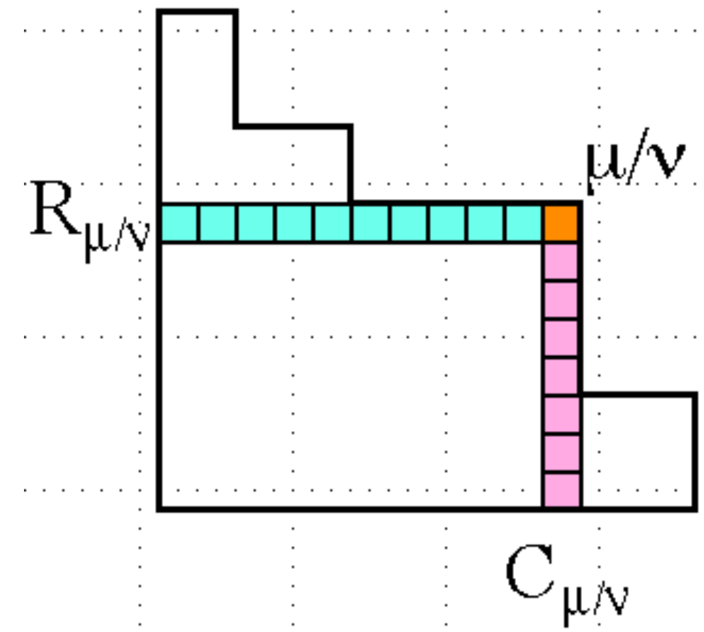
$$q^4t^2 + 4q^4t + 4q^3t^2 + 5q^4 + 15q^3t + 9q^2t^2 + 11q^3 + 22q^2t + 11qt^2 + 9q^2 + 15qt + 5t^2 + 4q + 4t + 1$$

The Macdonald-Stanley (dual) Pieri Rules

$$\partial_{p_1} \tilde{H}_\mu[X;q,t] = \sum_{\nu \to \mu} c_{\mu,\nu}(q,t) \tilde{H}_\nu[X;q,t]$$

$$c_{\mu\nu}(q,t) = \prod_{s \in R_{\mu\nu}} \frac{t^{l_\mu(s)} - q^{a_\mu(s)+1}}{t^{l_\nu(s)} - q^{a_\nu(s)+1}} \prod_{s \in C_{\mu\nu}} \frac{q^{a_\mu(s)} - t^{l_\mu(s)+1}}{q^{a_\nu(s)} - t^{l_\nu(s)+1}},$$



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$$F_\mu(q,t) = \sum_{\nu \to \mu} c_{\mu\nu}(q,t) F_\nu(q,t)$$

Using Maple

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hilb([3,2]);
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$$q^4 t^2 + 4 q^4 t + 4 q^3 t^2 + 5 q^4 + 15 q^3 t + 9 q^2 t^2 + 11 q^3 + 22 q^2 t + 11 q t^2 + 9 q^2 + 15 q t + 5 t^2 + 4 q + 4 t + 1$$

$$\begin{bmatrix} 5 & 11 & 9 & 4 & 1 \\ 4 & 15 & 22 & 15 & 4 \\ 1 & 4 & 9 & 11 & 5 \end{bmatrix}$$

The Macdonald-Stanley (dual) Pieri Rules

$$\partial_{p_1}\tilde{H}_\mu[X;q,t] = \sum_{\nu\to\mu} c_{\mu,\nu}(q,t)\tilde{H}_\nu[X;q,t]$$

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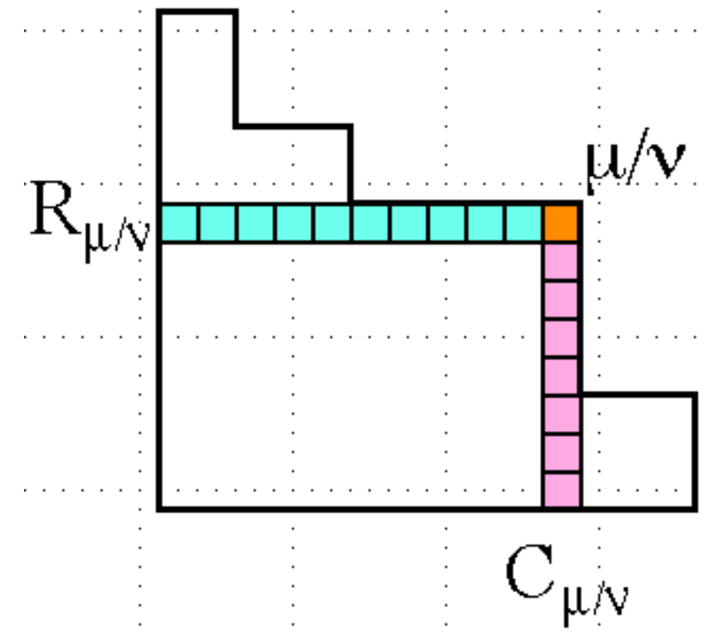

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Using Maple

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	0	1	2	3	4
2 →	5	11	9	4	1
1 →	4	15	22	15	4
0 →	1	4	9	11	5

The Macdonald-Stanley (dual) Pieri Rules

$$\partial_{p_1}\tilde{H}_\mu[X;q,t] = \sum_{\nu\to\mu} c_{\mu,\nu}(q,t)\tilde{H}_\nu[X;q,t]$$

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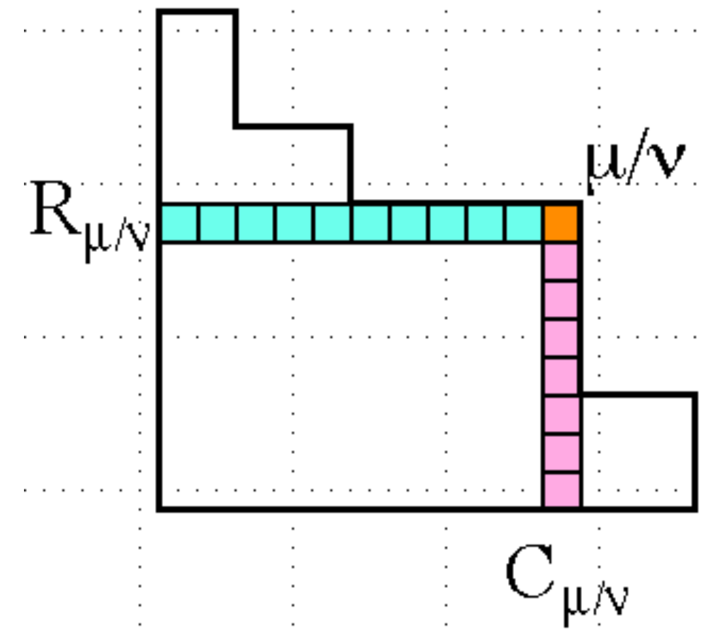

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$$F_\mu(q,t) = \partial_{p_1}^n \tilde{H}_\mu[X;q,t] = \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\partial_{p_1}^{n-1}\widetilde{H}_\mu[x;q,t] = \sum_{\nu\to\mu} c_{\mu,\nu}(q,t)F_\nu(q,t)$$

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Using Maple

```
hilb([3,2]);
```

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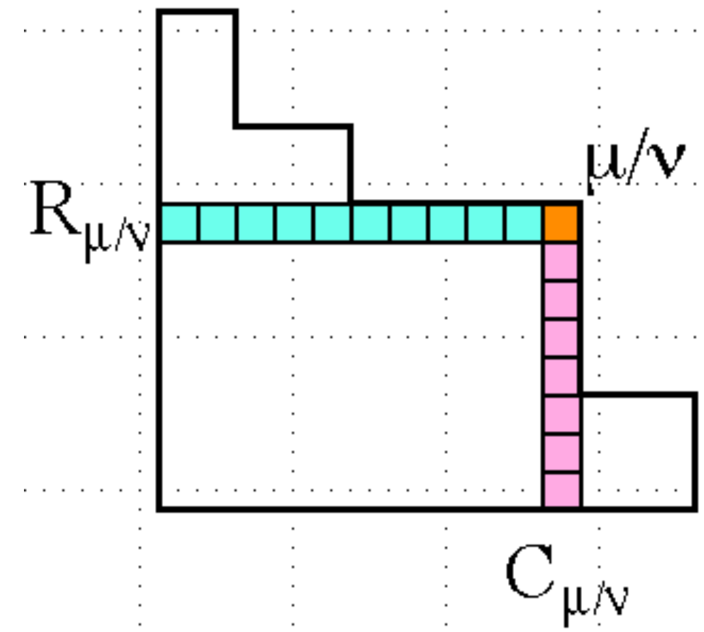
The dimension of $H_{2,1}(M_{3,2}[X,Y])$

	0	1	2	3	4
2	5	11	9	4	1
1	4	15	22	15	4
0	1	4	9	11	5

The Macdonald-Stanley (dual) Pieri Rules

$$\partial_{\mathbf{p_1}} \tilde{\mathbf{H}}_\mu[\mathbf{X};\mathbf{q},\mathbf{t}] = \sum_{\nu\to\mu} \mathbf{c}_{\mu,\nu}(\mathbf{q},\mathbf{t})\tilde{\mathbf{H}}_\nu[\mathbf{X};\mathbf{q},\mathbf{t}]$$

$$\mathbf{c}_{\mu\nu}(\mathbf{q},\mathbf{t}) = \prod_{s\in \mathbf{R}_{\mu\nu}} \frac{\mathbf{t}^{\mathbf{l}_\mu(\mathbf{s})} - \mathbf{q}^{\mathbf{a}_\mu(\mathbf{s})+1}}{\mathbf{t}^{\mathbf{l}_\nu(\mathbf{s})} - \mathbf{q}^{\mathbf{a}_\nu(\mathbf{s})+1}} \prod_{s\in \mathbf{C}_{\mu\nu}} \frac{\mathbf{q}^{\mathbf{a}_\mu(\mathbf{s})} - \mathbf{t}^{\mathbf{l}_\mu(\mathbf{s})+1}}{\mathbf{q}^{\mathbf{a}_\nu(\mathbf{s})} - \mathbf{t}^{\mathbf{l}_\nu(\mathbf{s})+1}},$$



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Using Maple

```
hilb([3,2]);
```

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$$\begin{array}{c} 2\rightarrow \\ 1\rightarrow \\ 0\rightarrow \end{array}\begin{bmatrix} 5 & \boxed{11} & 9 & 4 & 1 \\ 4 & 15 & 22 & \boxed{15} & 4 \\ 1 & 4 & 9 & 11 & 5 \end{bmatrix}$$

$$\uparrow\ \ \uparrow\ \ \uparrow\ \ \uparrow\ \ \uparrow$$
$$0\quad 1\quad 2\quad 3\quad 4$$

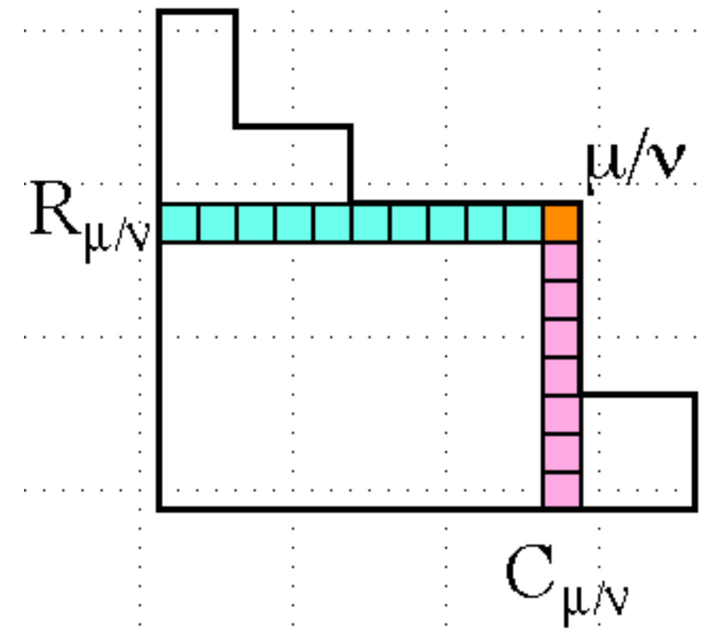
The dimension of $\mathbf{H}_{2,1}(\mathbf{M}_{3,2}[\mathbf{X},\mathbf{Y}])$

The dimension of $\mathbf{H}_{1,3}(\mathbf{M}_{3,2}[\mathbf{X},\mathbf{Y}])$

The Macdonald-Stanley (dual) Pieri Rules

$$\partial_{p_1}\tilde{H}_\mu[X;q,t] = \sum_{\nu\to\mu} c_{\mu,\nu}(q,t)\tilde{H}_\nu[X;q,t]$$

$$c_{\mu\nu}(q,t) = \prod_{s\in R_{\mu\nu}} \frac{t^{l_\mu(s)} - q^{a_\mu(s)+1}}{t^{l_\nu(s)} - q^{a_\nu(s)+1}} \prod_{s\in C_{\mu\nu}} \frac{q^{a_\mu(s)} - t^{l_\mu(s)+1}}{q^{a_\nu(s)} - t^{l_\nu(s)+1}},$$



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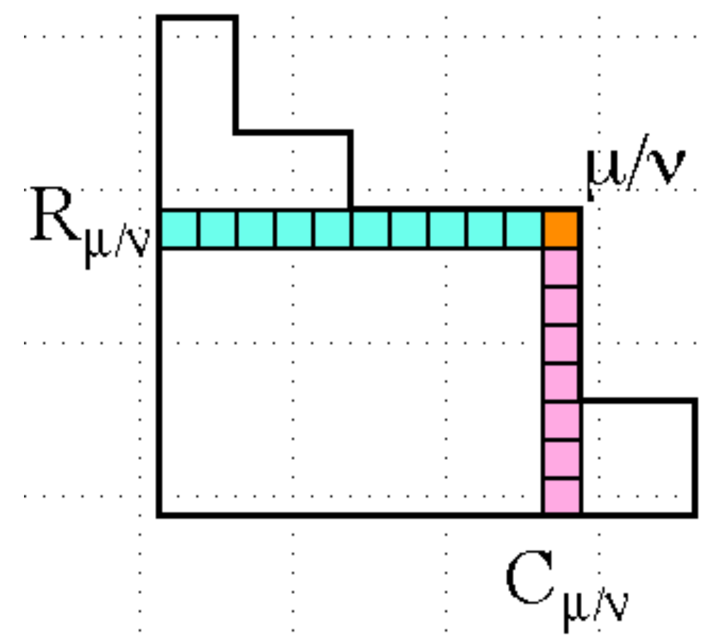
$\uparrow$ 0 $\quad\uparrow$ 1 $\quad\uparrow$ 2 $\quad\uparrow$ 3 $\quad\uparrow$ 4

The dimension of $H_{1,3}(M_{3,2}[X,Y])$

The Macdonald-Stanley (dual) Pieri Rules

$$\partial_{p_1}\tilde{H}_\mu[X;q,t] = \sum_{\nu\to\mu} c_{\mu,\nu}(q,t)\tilde{H}_\nu[X;q,t]$$

$$c_{\mu\nu}(q,t) = \prod_{s\in R_{\mu\nu}} \frac{t^{l_\mu(s)} - q^{a_\mu(s)+1}}{t^{l_\nu(s)} - q^{a_\nu(s)+1}} \prod_{s\in C_{\mu\nu}} \frac{q^{a_\mu(s)} - t^{l_\mu(s)+1}}{q^{a_\nu(s)} - t^{l_\nu(s)+1}},$$



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$$F_\mu(q,t) = \sum_{\nu\to\mu} c_{\mu\nu}(q,t)F_\nu(q,t)$$

Using Maple

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hilb([3,2]);
```

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The Macdonald-Stanley (dual) Pieri Rules

$$\partial_{p_1}\tilde{H}_\mu[X;q,t] = \sum_{\nu\to\mu} c_{\mu,\nu}(q,t)\tilde{H}_\nu[X;q,t]$$

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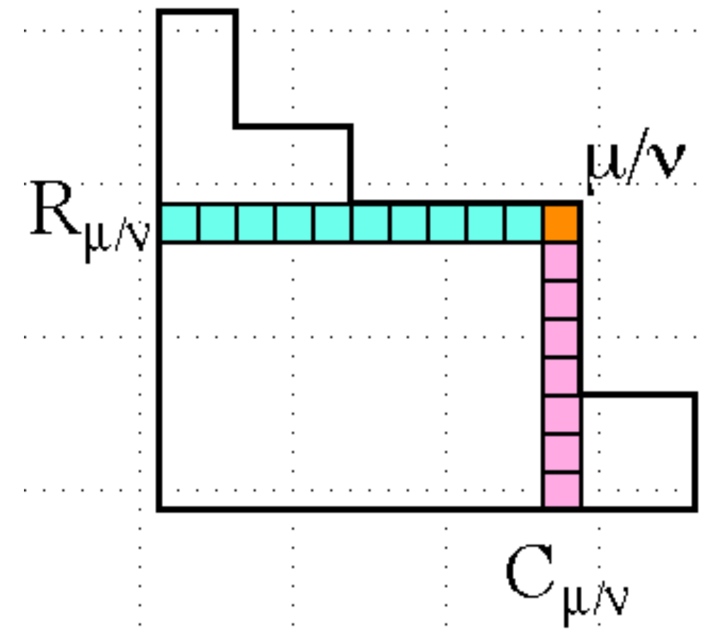

where $R_{\mu\nu}$ and $C_{\mu\nu}$ denote the collections of cells of ν that are respectively in the row and the column of the cell μ/ν.

The Hilbert series

$$F_\mu(q,t) = \partial_{p_1}^{n}\tilde{H}_\mu[X;q,t] = \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\partial_{p_1}^{n-1}\widetilde{H}_\mu[x;q,t] = \sum_{\nu\to\mu} c_{\mu,\nu}(q,t)F_\nu(q,t)$$

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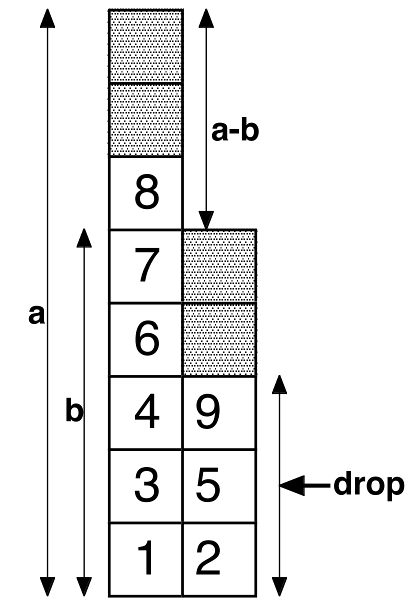

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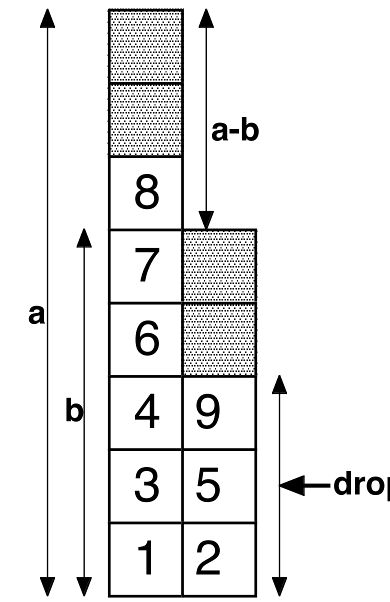

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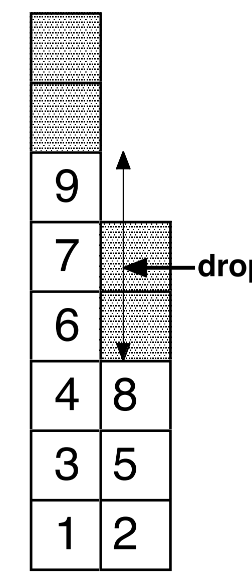

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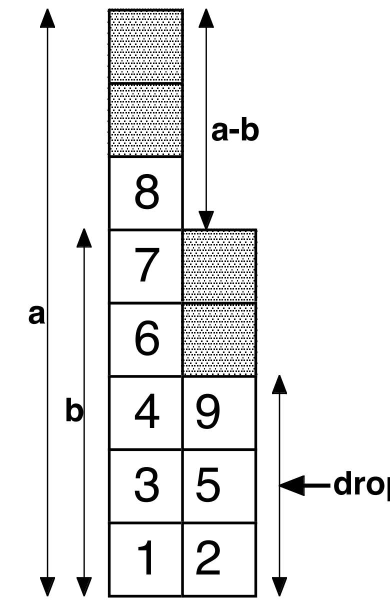

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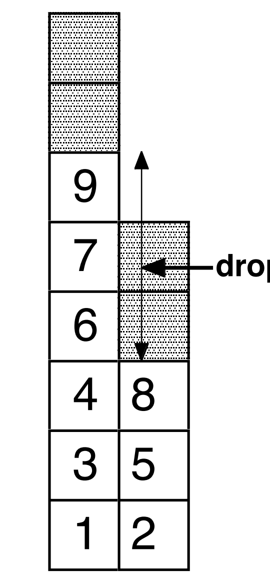

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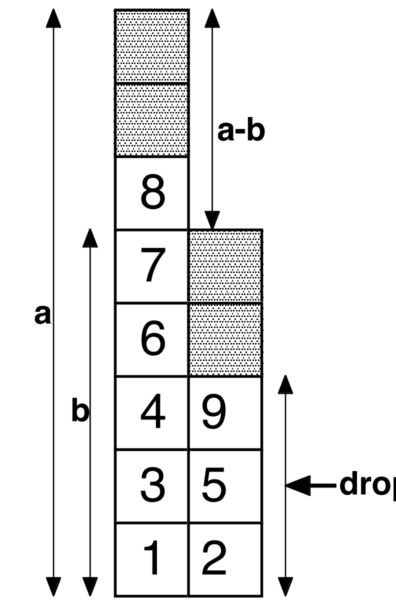

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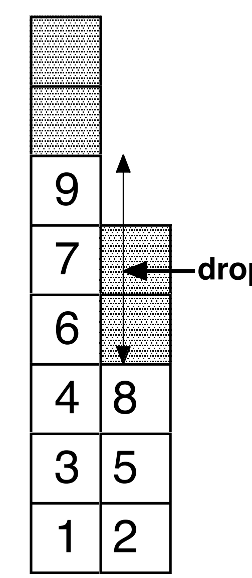

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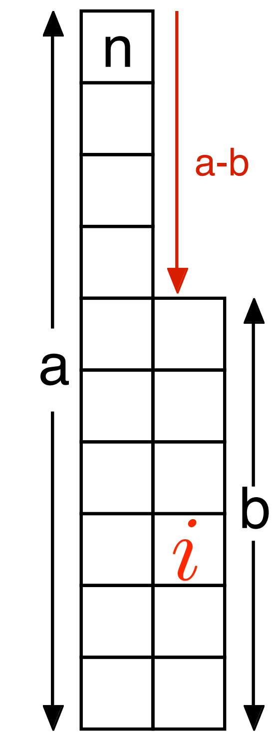

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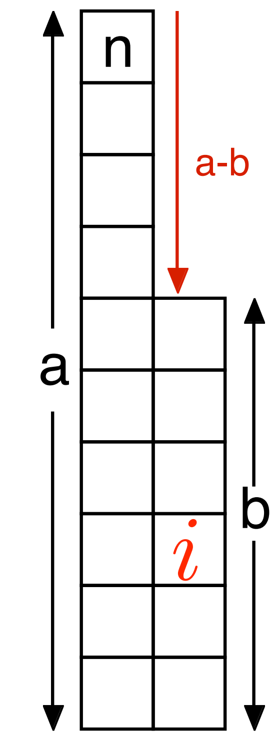

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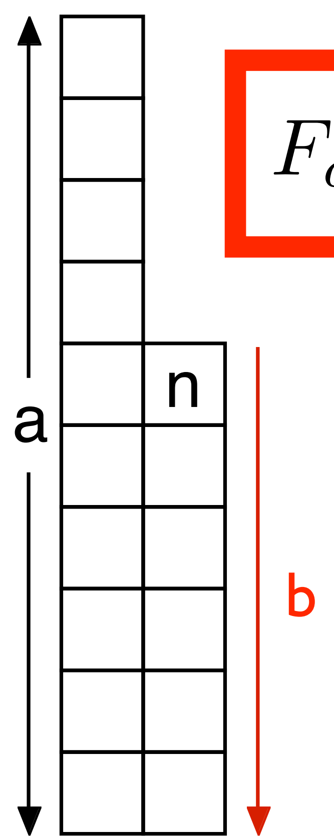

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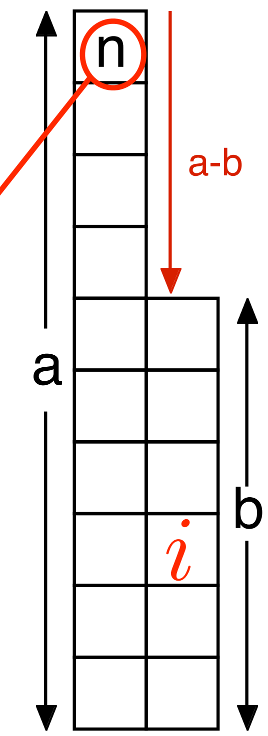

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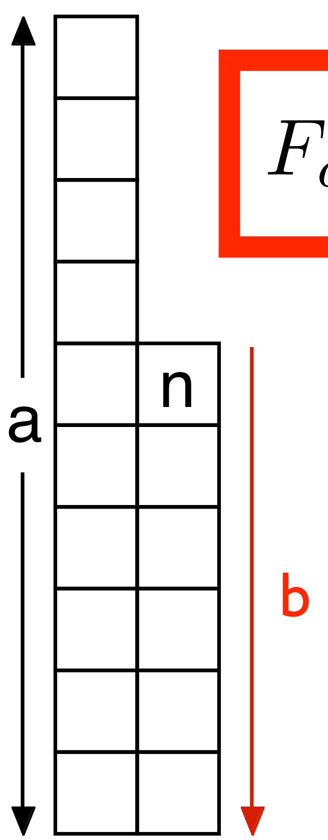

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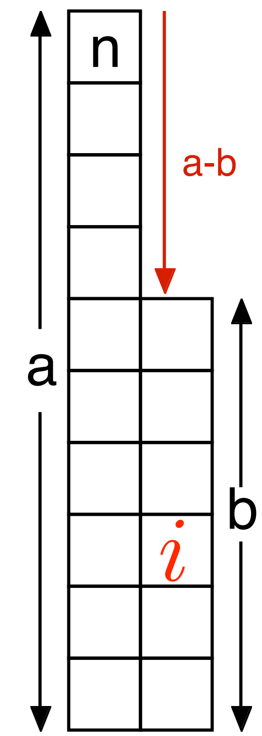

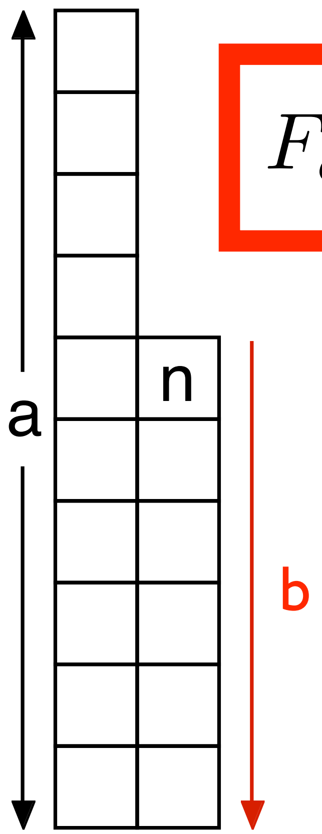

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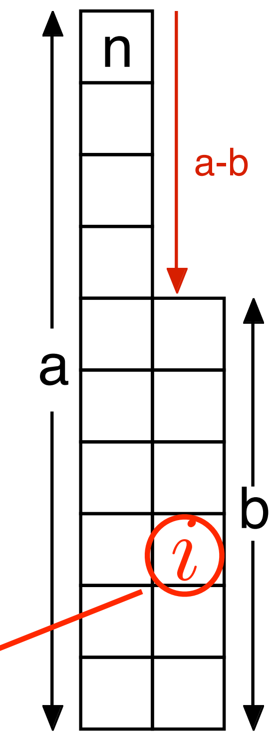

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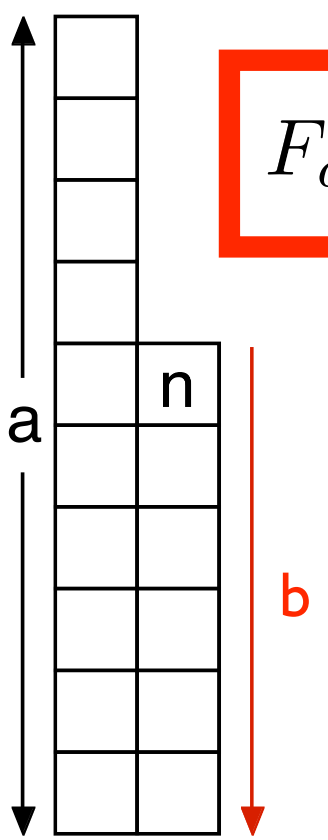

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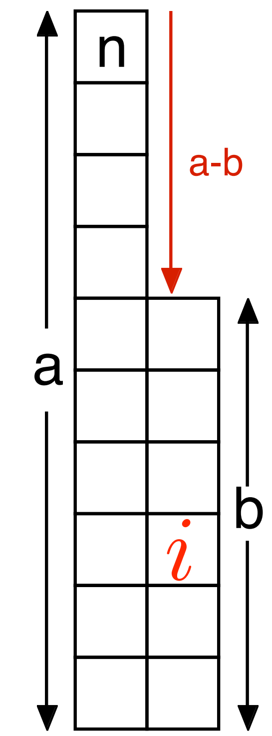

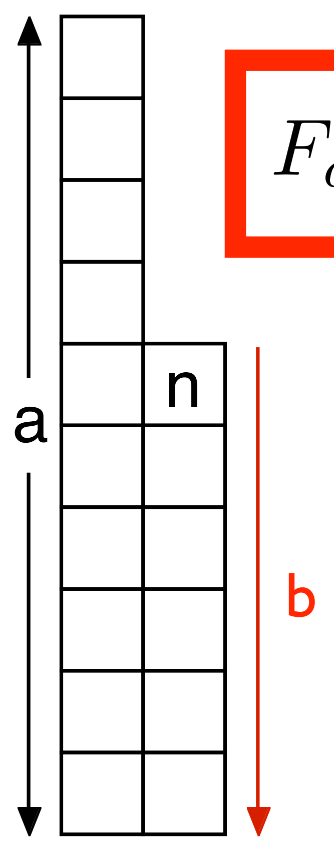

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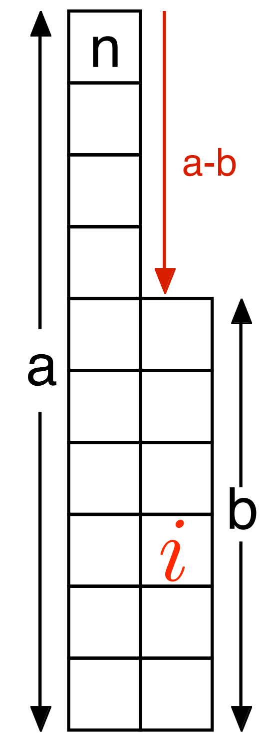

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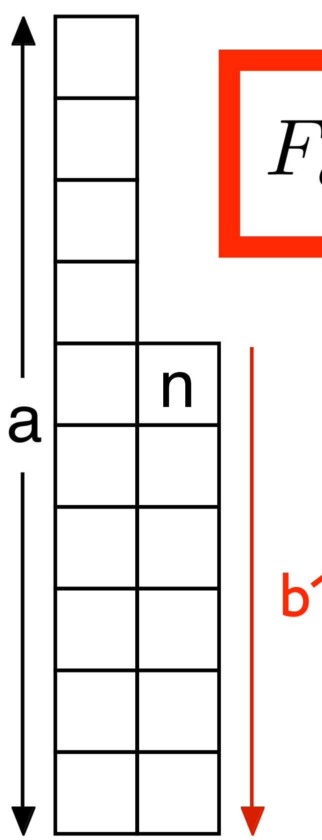

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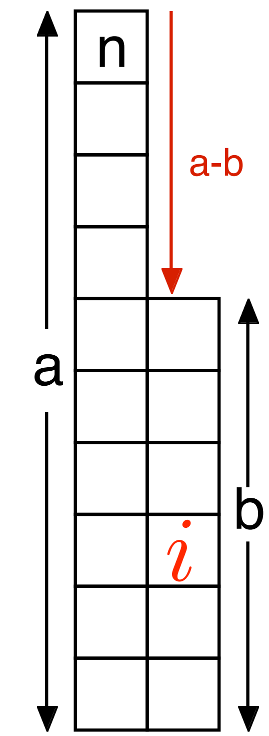

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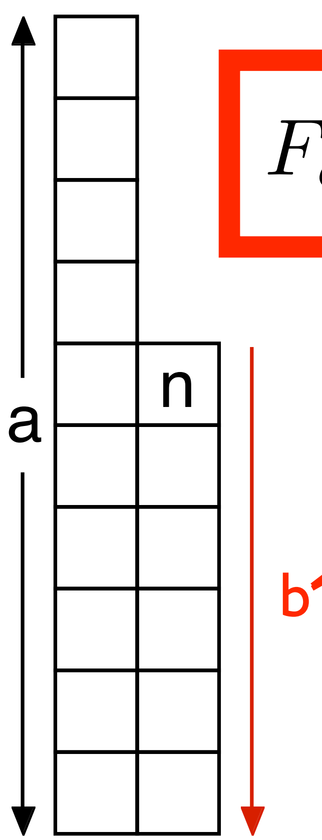

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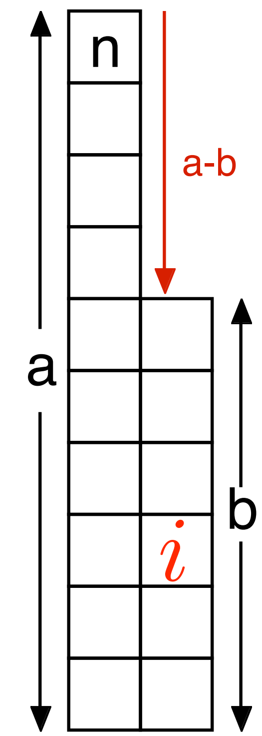

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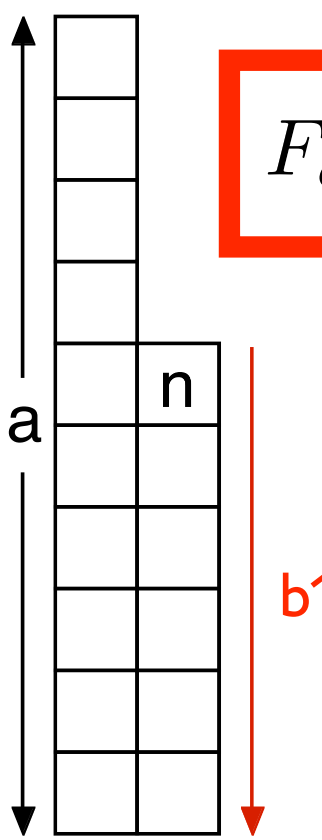

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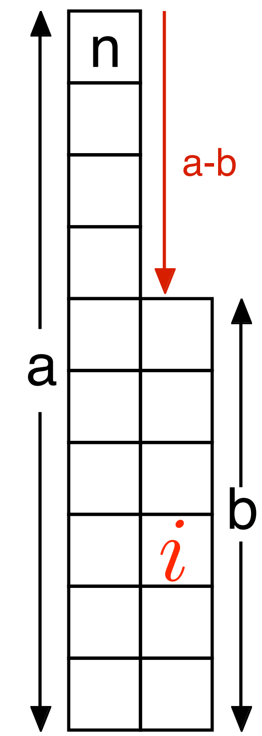

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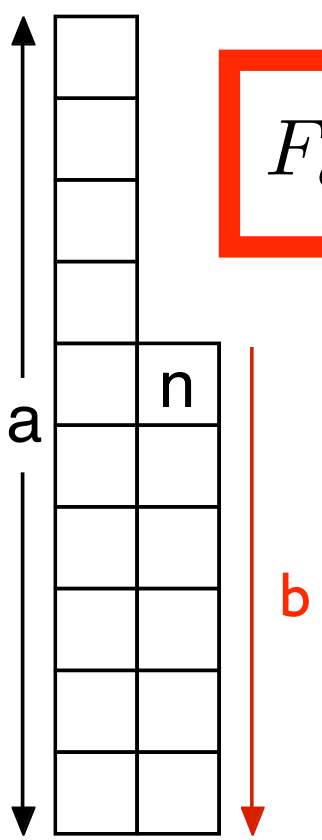

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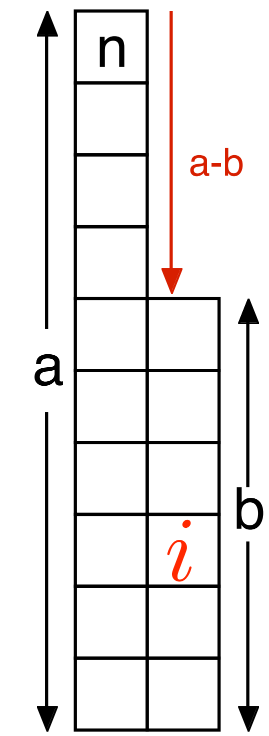

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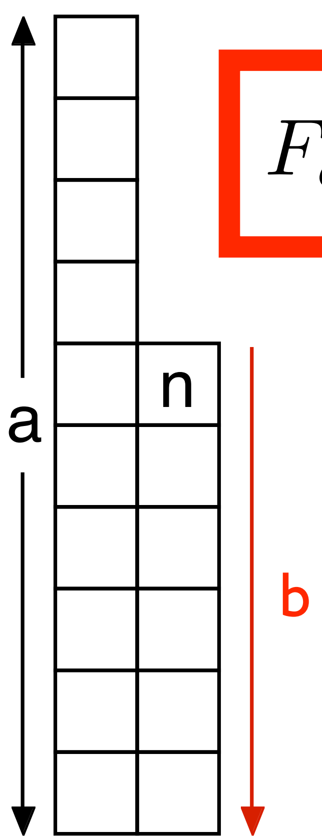

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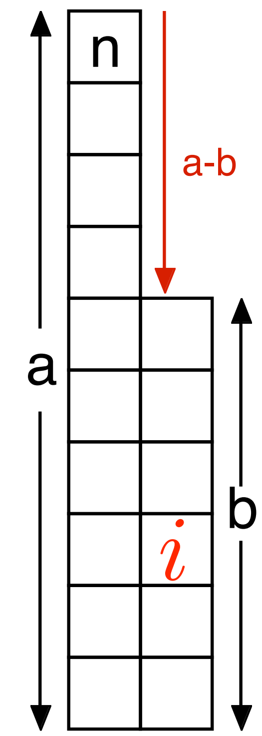

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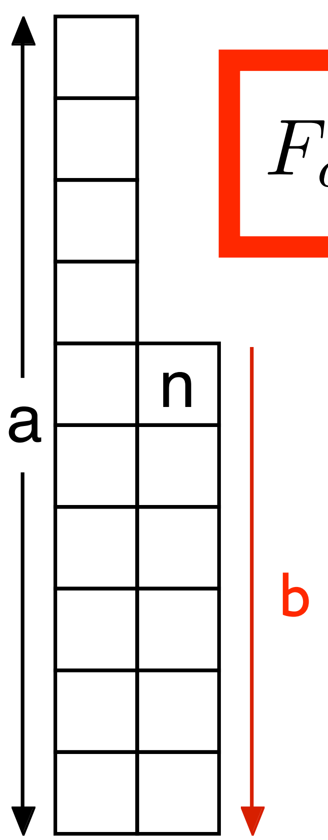

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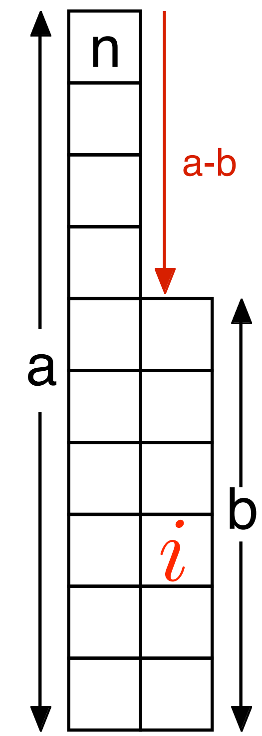

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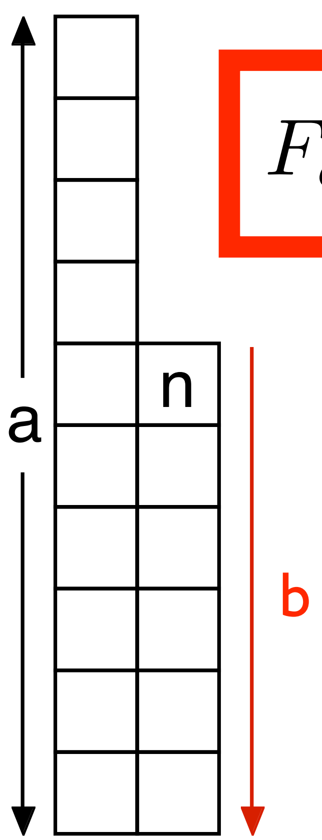

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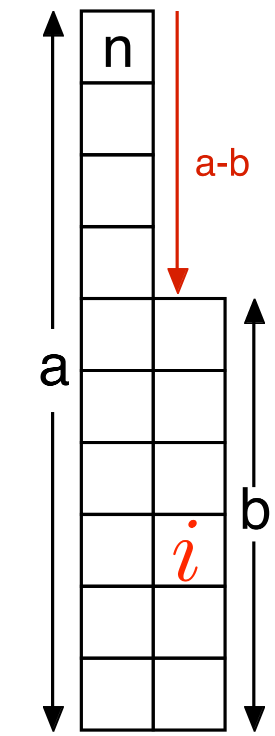

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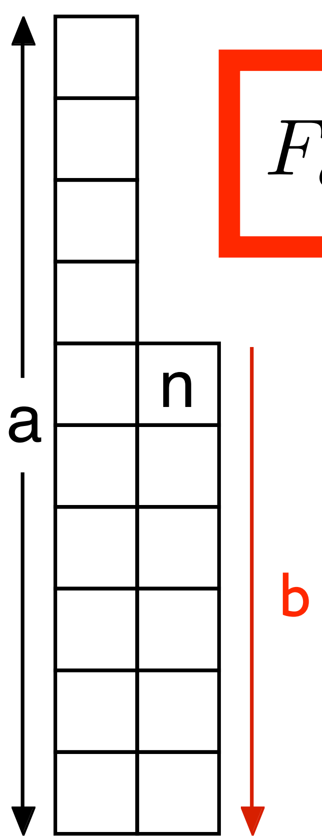

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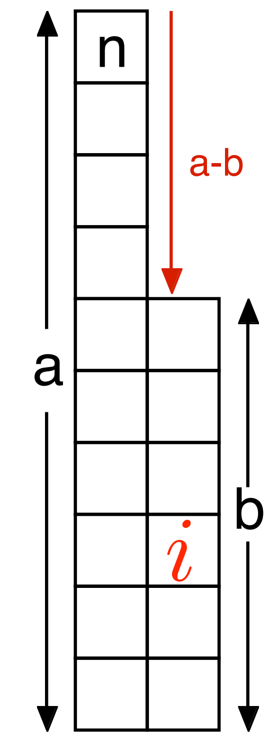

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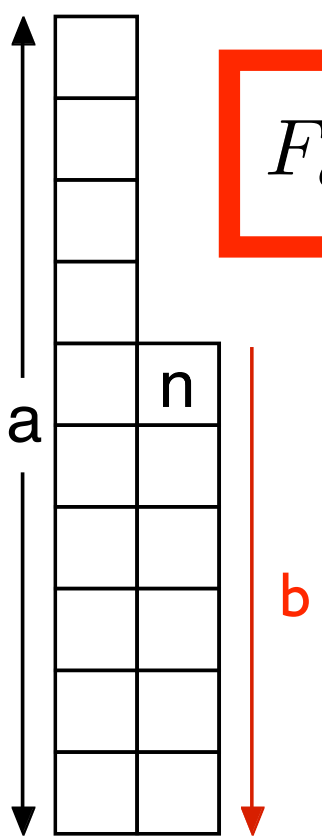

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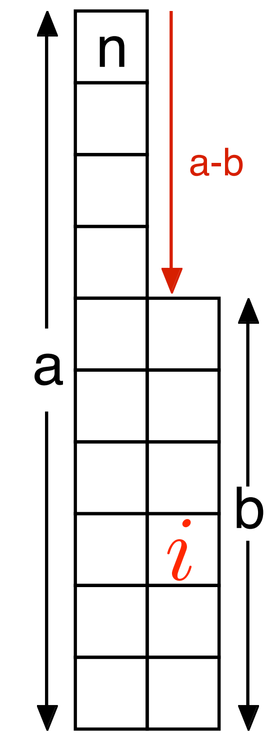

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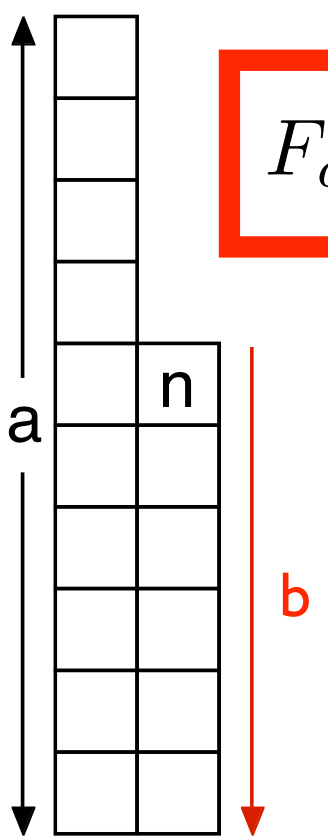

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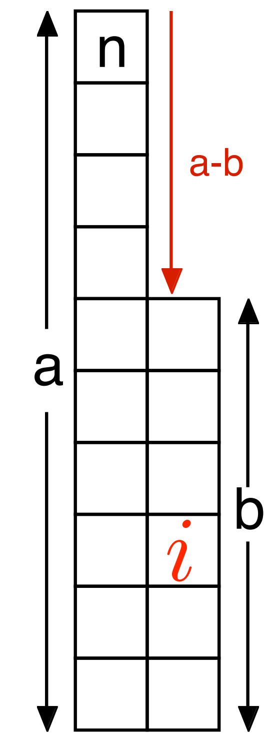

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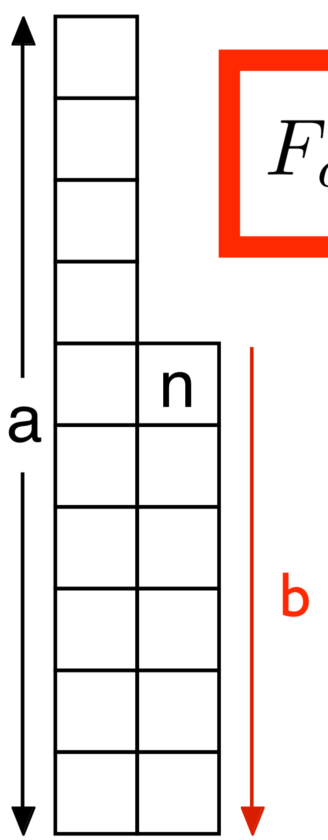

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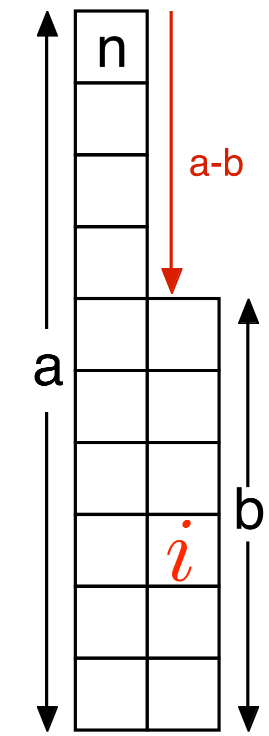

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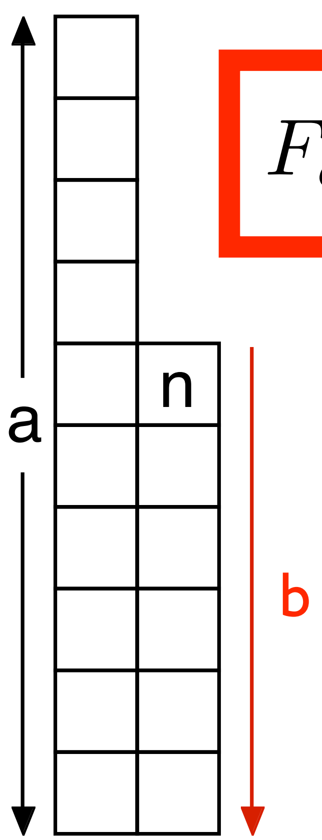

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next

The underlying representation theoretical identity

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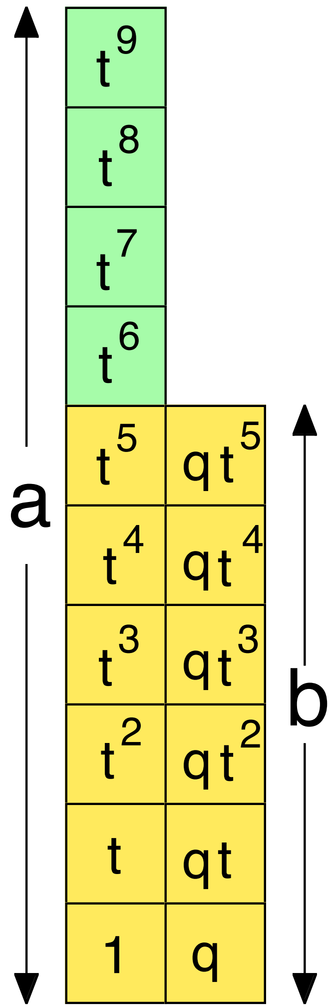

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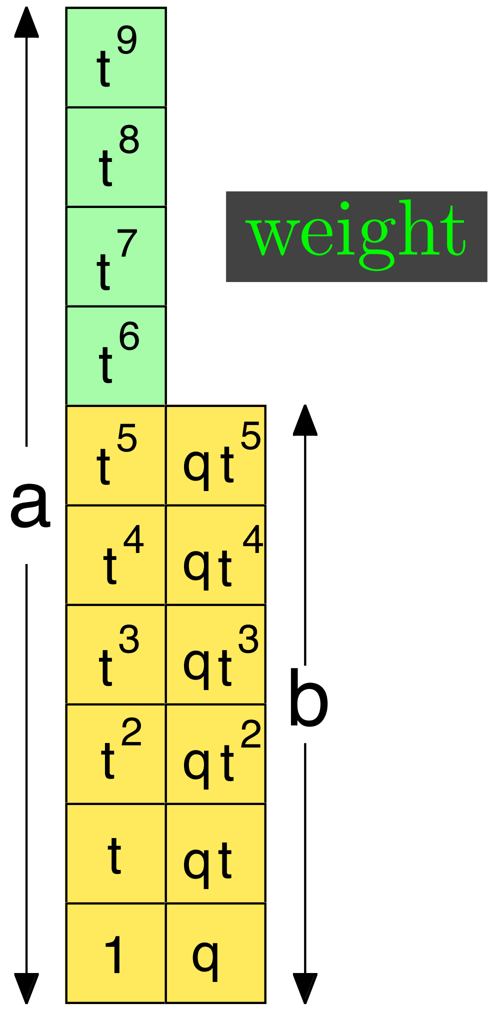

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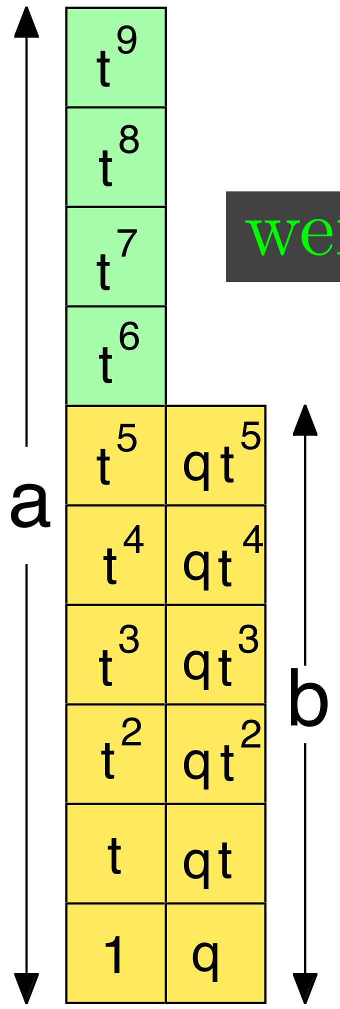

weight $t^6(1+t+t^2+t^3)$

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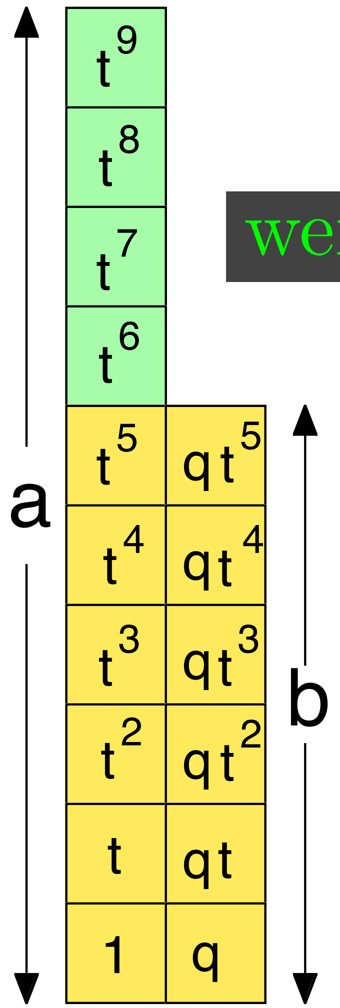

weight $\mathbf{t^6(1+t+t^2+t^3) = t^b[a-b]_t}$

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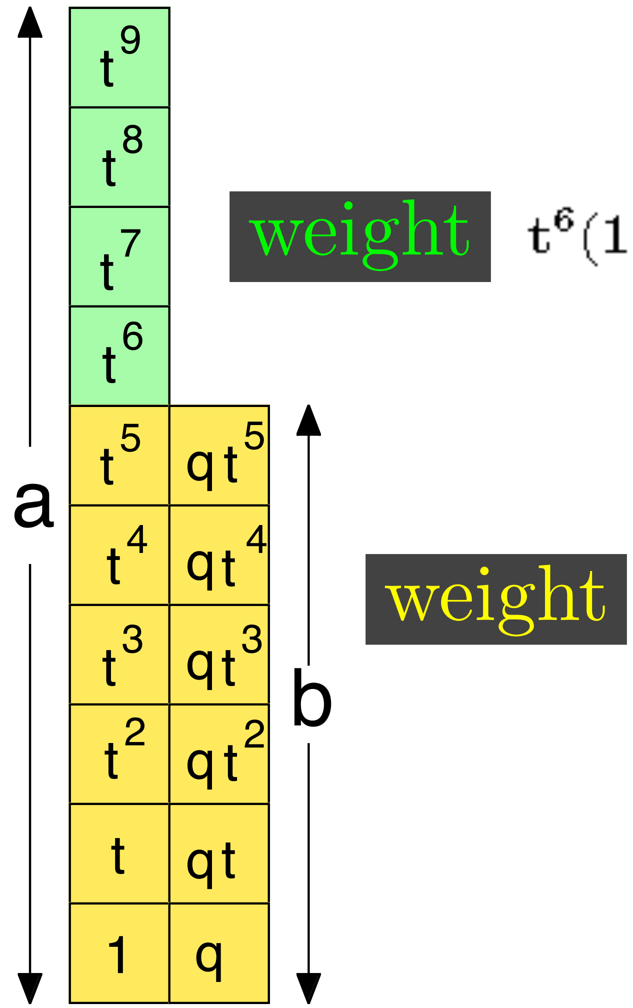

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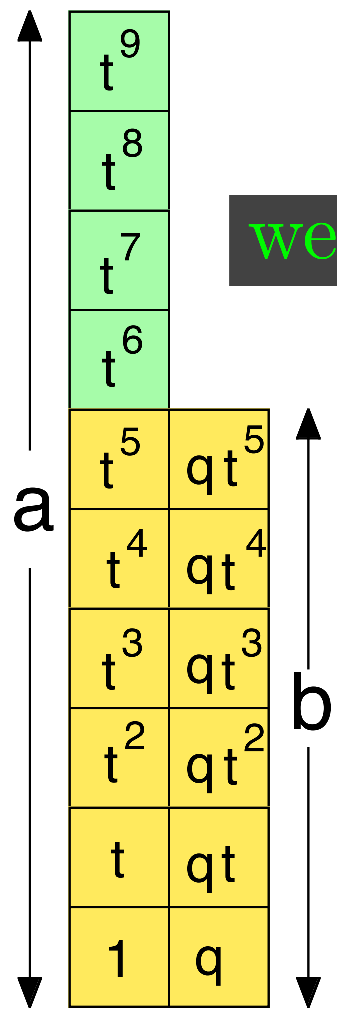

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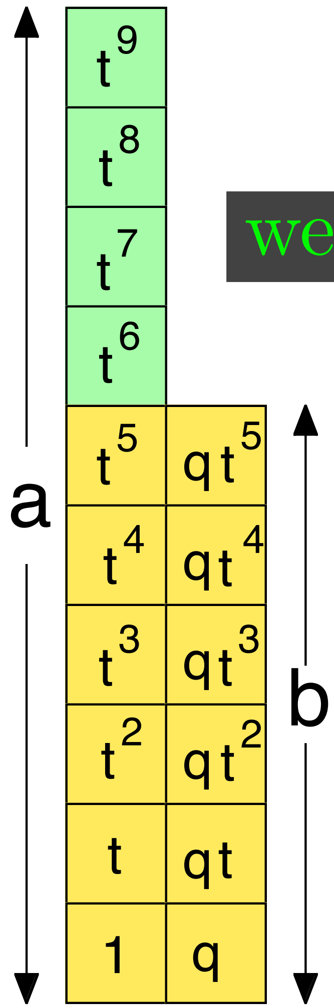

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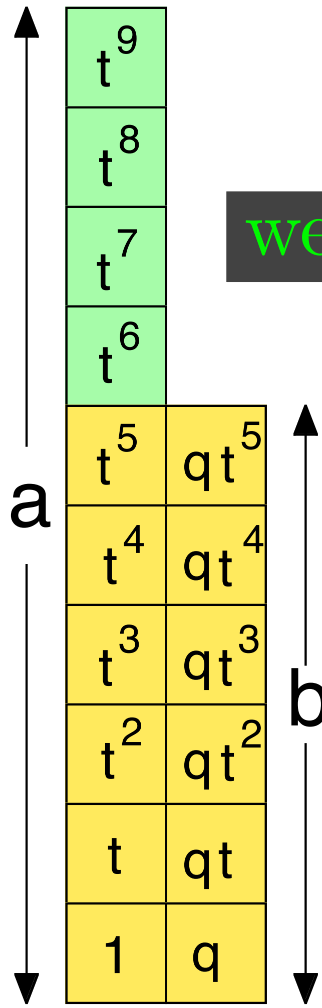

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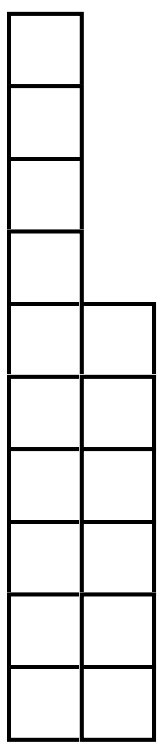
∂_{p_1}



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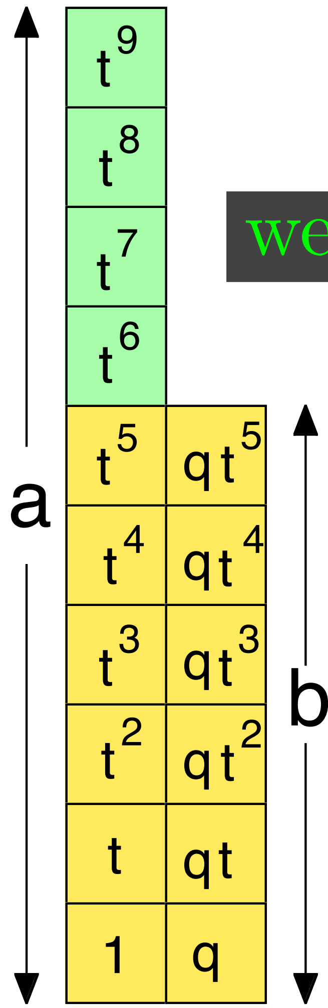

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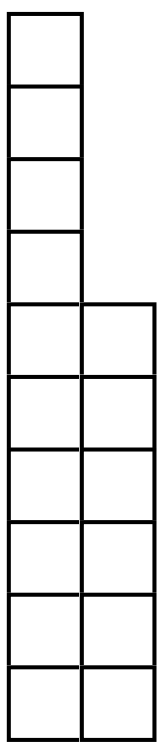
∂_{p_1}  $=$

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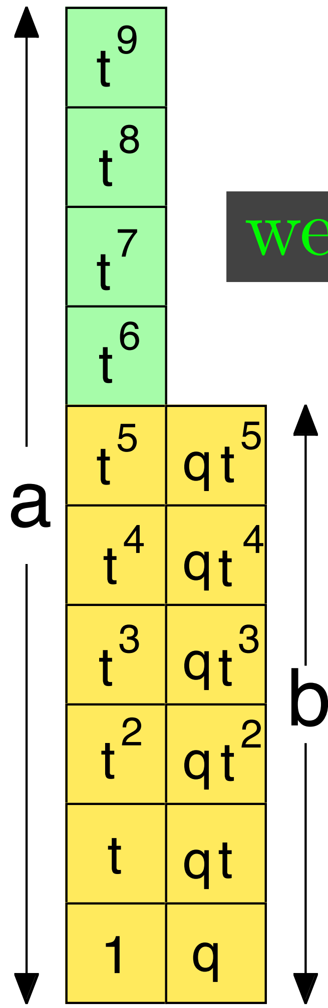

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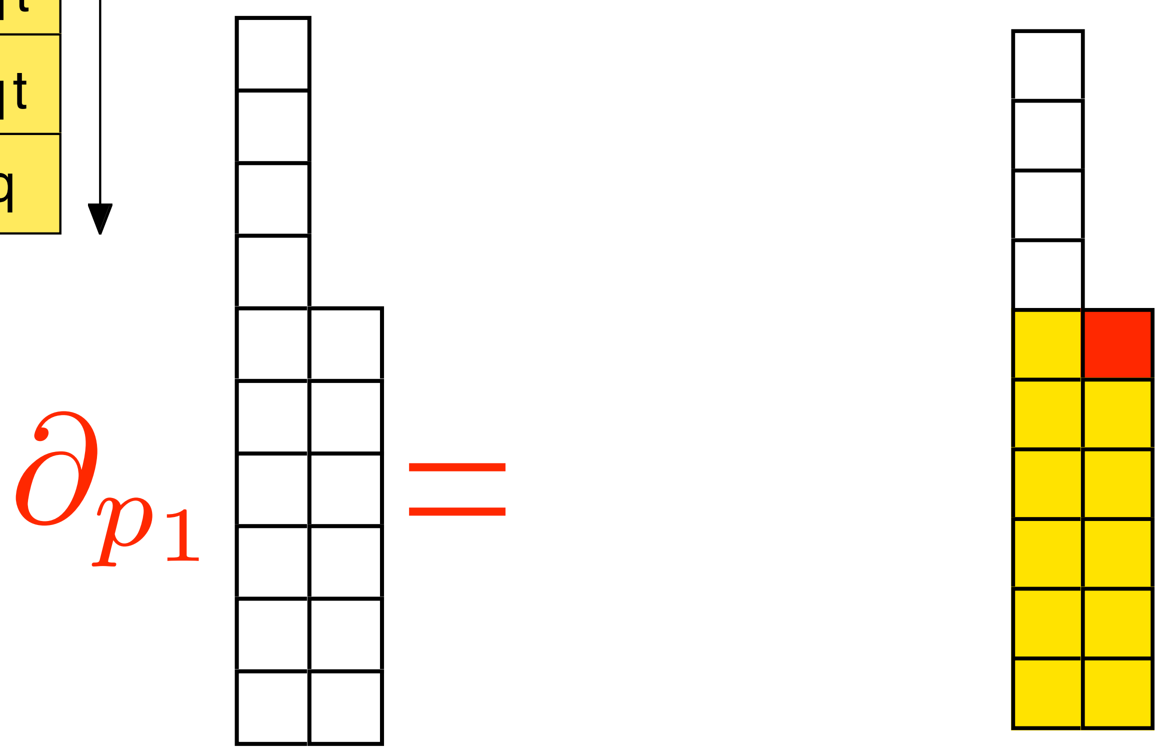

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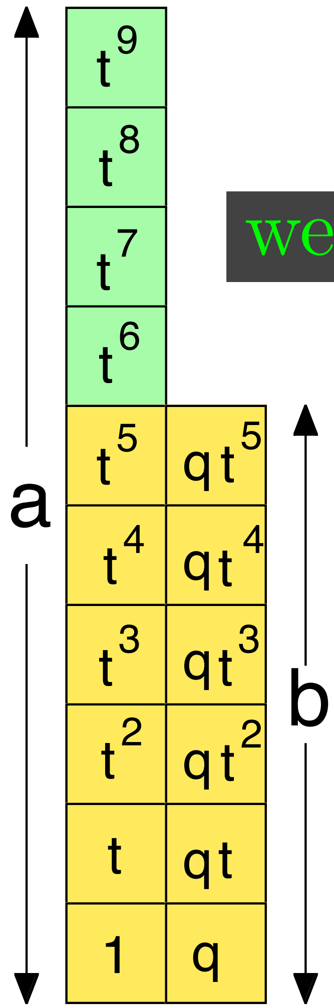

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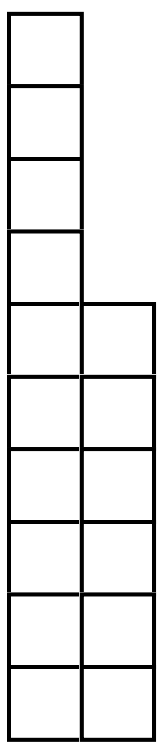
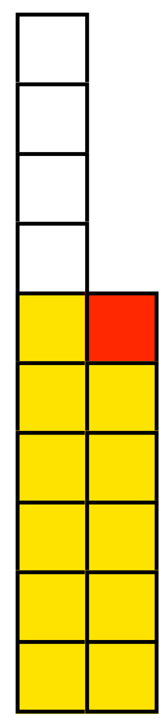
∂_{p_1}  $= (1+q)\,[b]_t$ 

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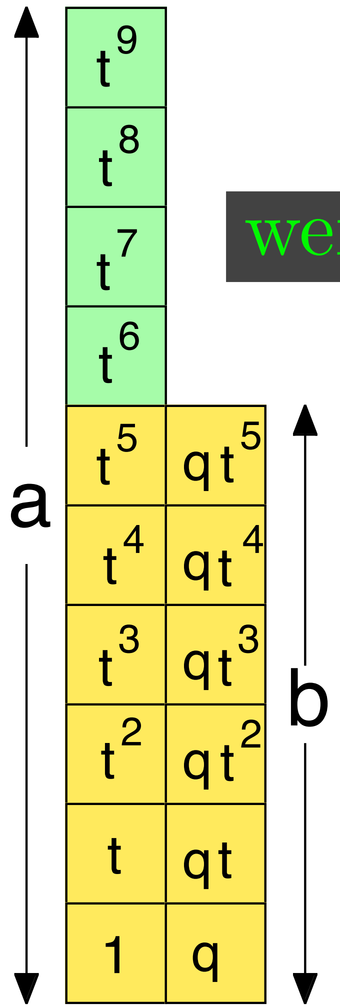

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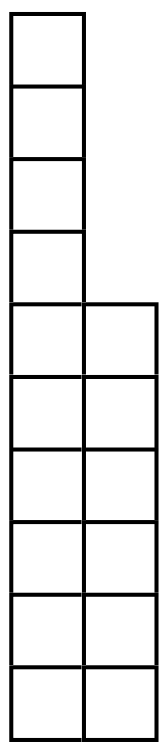
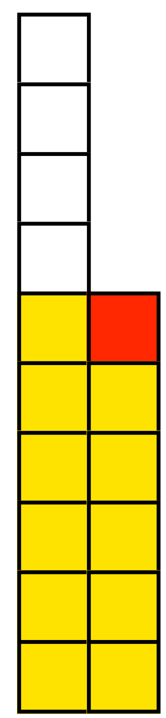
∂_{p_1}  $= (1+q)\,[b]_t$  $+$

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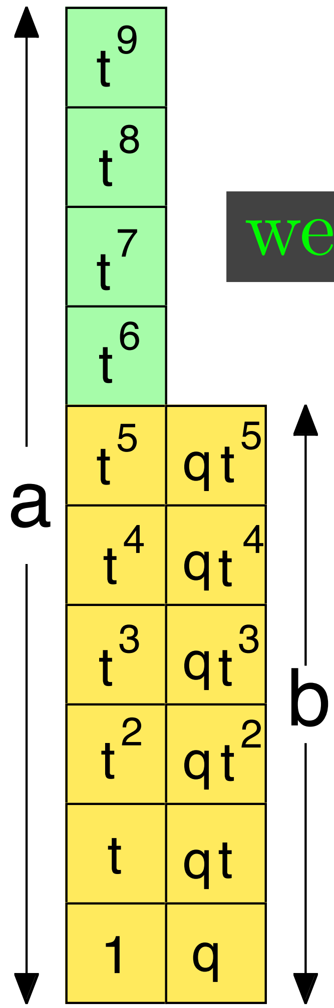

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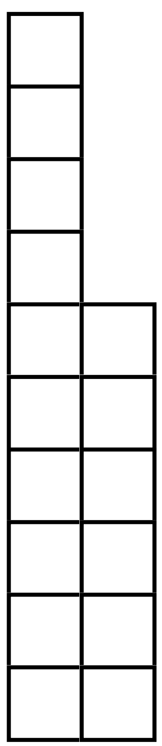
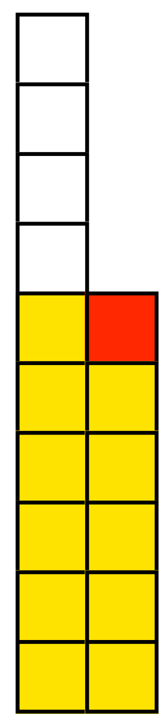
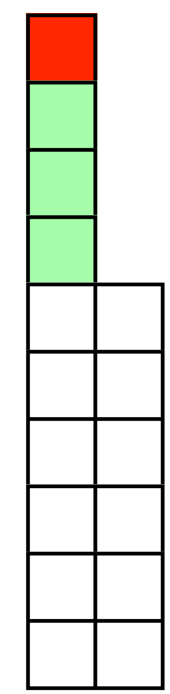
∂_{p_1}  $= (1+q)\,[b]_t$  $+$ 

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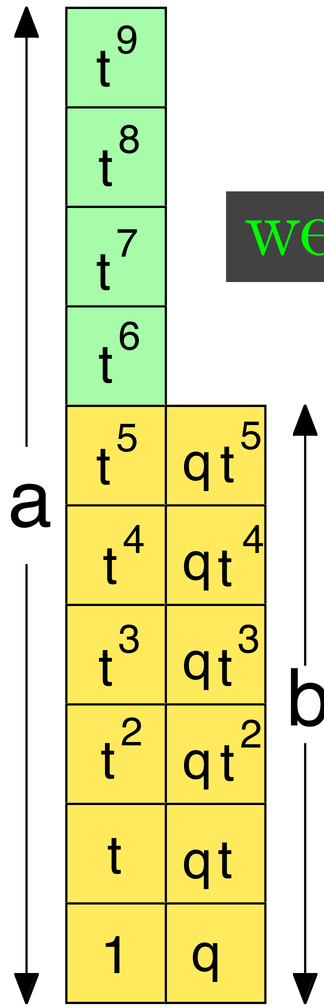

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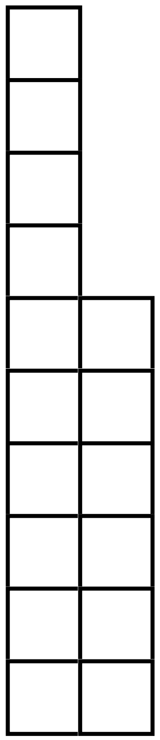
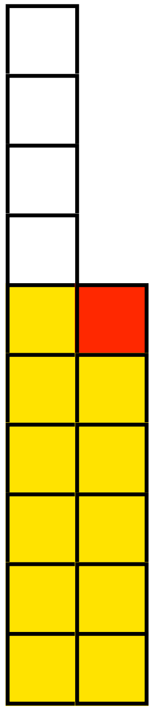
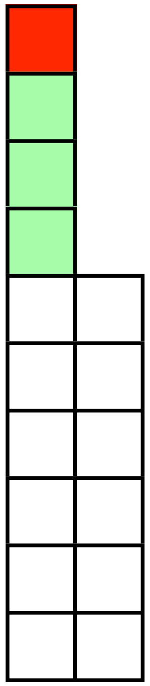
∂_{p_1}  $= (1+q)\,[b]_t$  $+\ t^b\,[a-b]_t$ 

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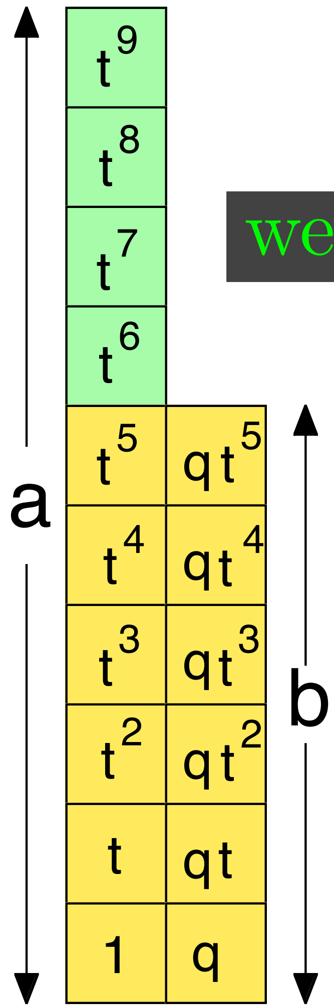

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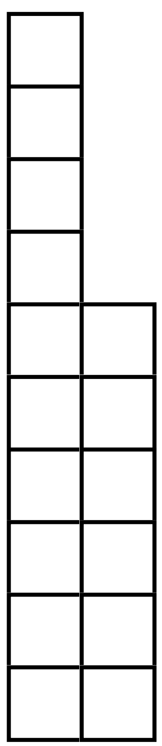
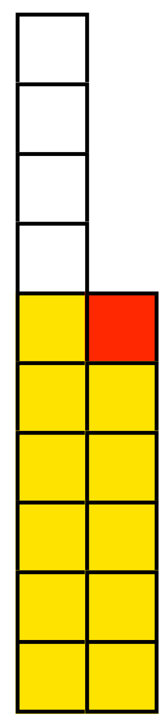
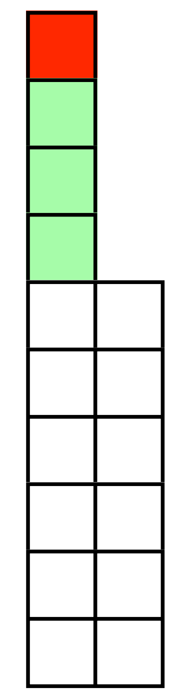
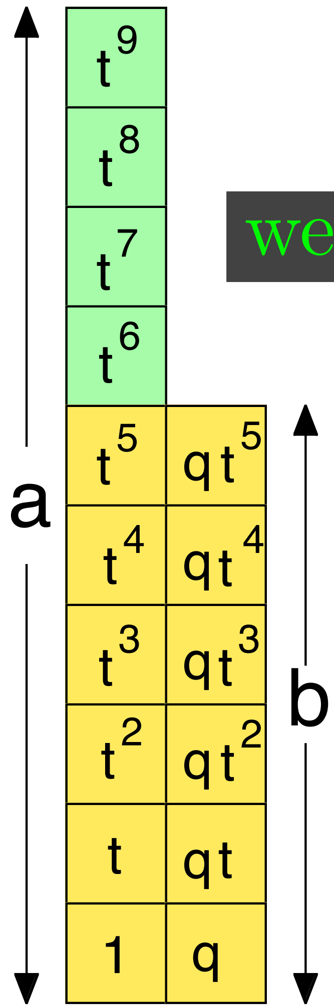
∂_{p_1}  $= (1+q)\,[b]_t$  $+\ t^b\,[a-b]_t$  $(q/t,t)$

The underlying representation theoretical identity



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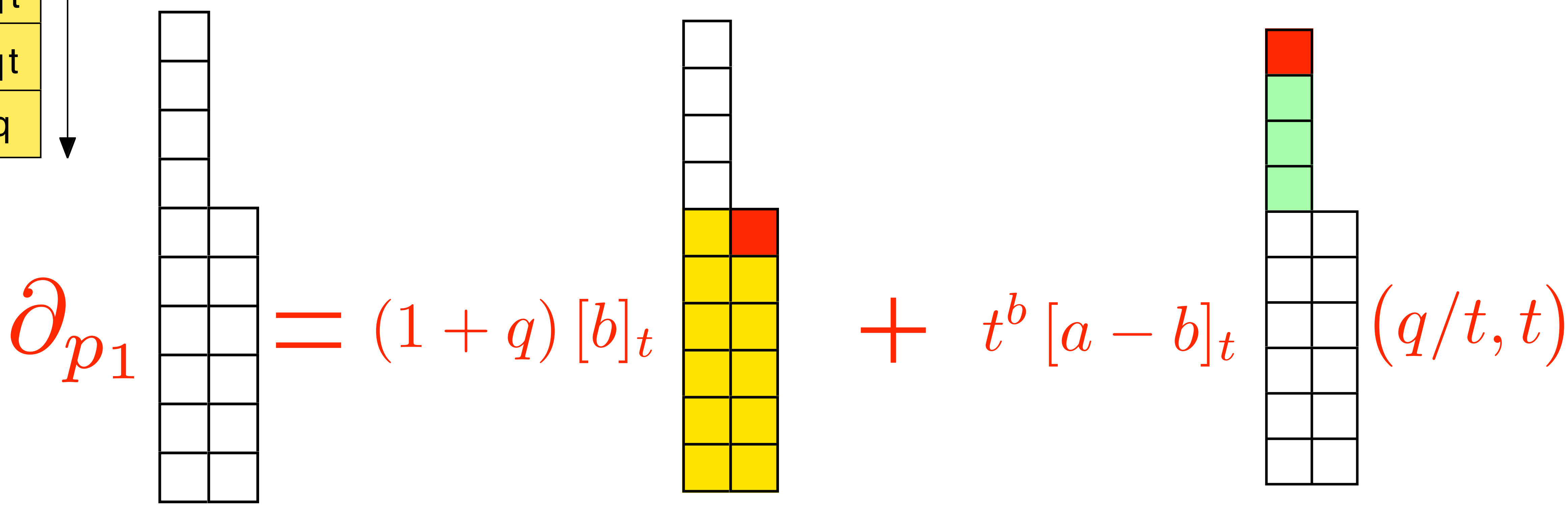
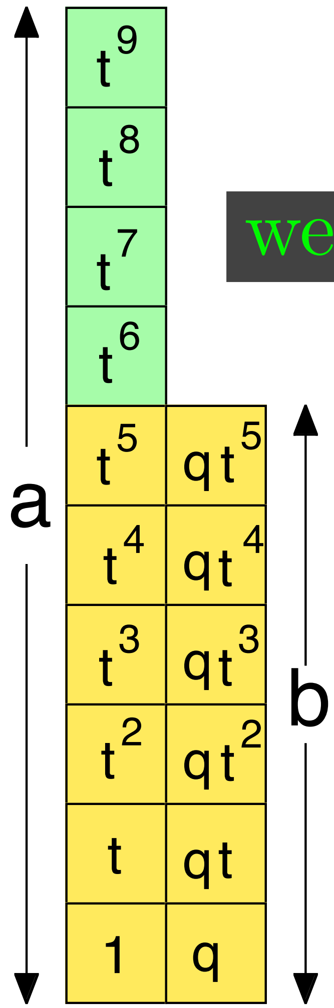


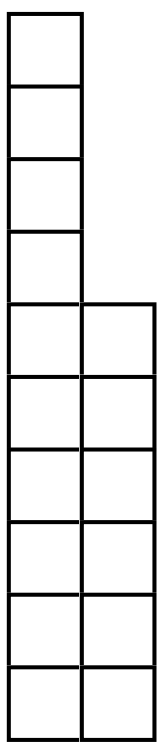
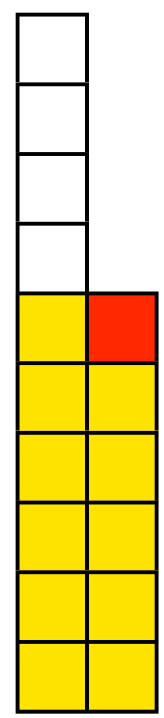
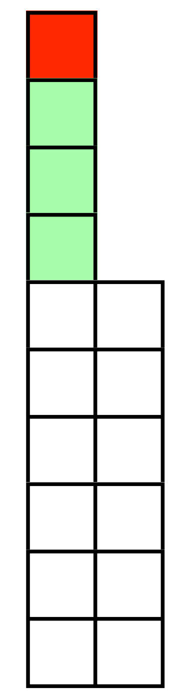
$$\partial_{p_1}\phi_{a,b}(x;q,t) \;=\; (1+q)\,[b]_t\phi_{a,b-1}(x;q,t) \;+\; t^b[a-b]_t\,\phi_{a-1,b}(x;q/t,t)$$

The underlying representation theoretical identity



weight $\mathbf{t^6(1+t+t^2+t^3) = t^b[a-b]_t}$

weight $\mathbf{(1+q)(1+t+t^2+\cdots+t^5) = (1+q)[b]_t}$

∂_{p_1}  $= (1+q)\,[b]_t$  $+\ t^b\,[a-b]_t$  $(q/t,t)$

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next

Tuesday, March 24, 2009

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next

next the miracles

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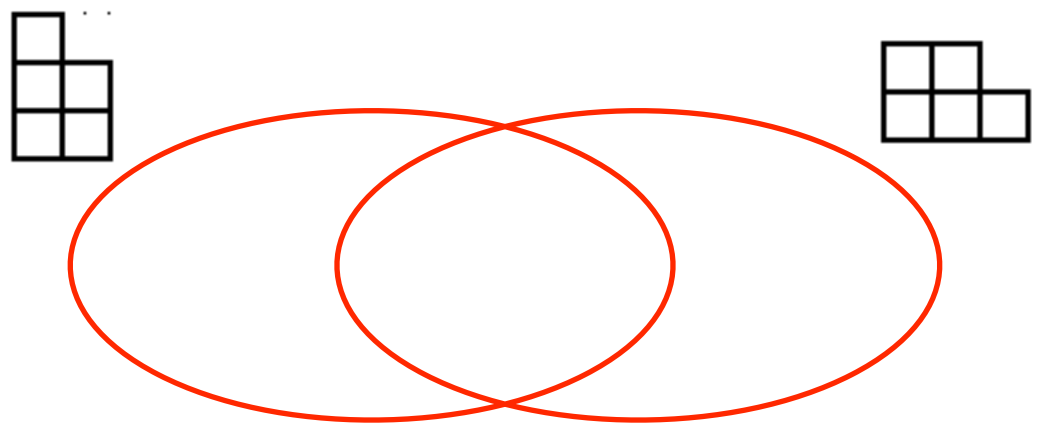

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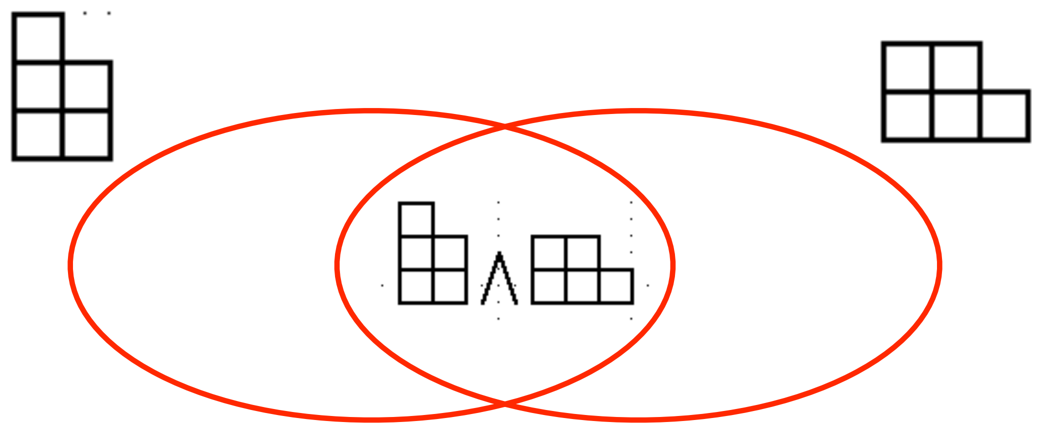

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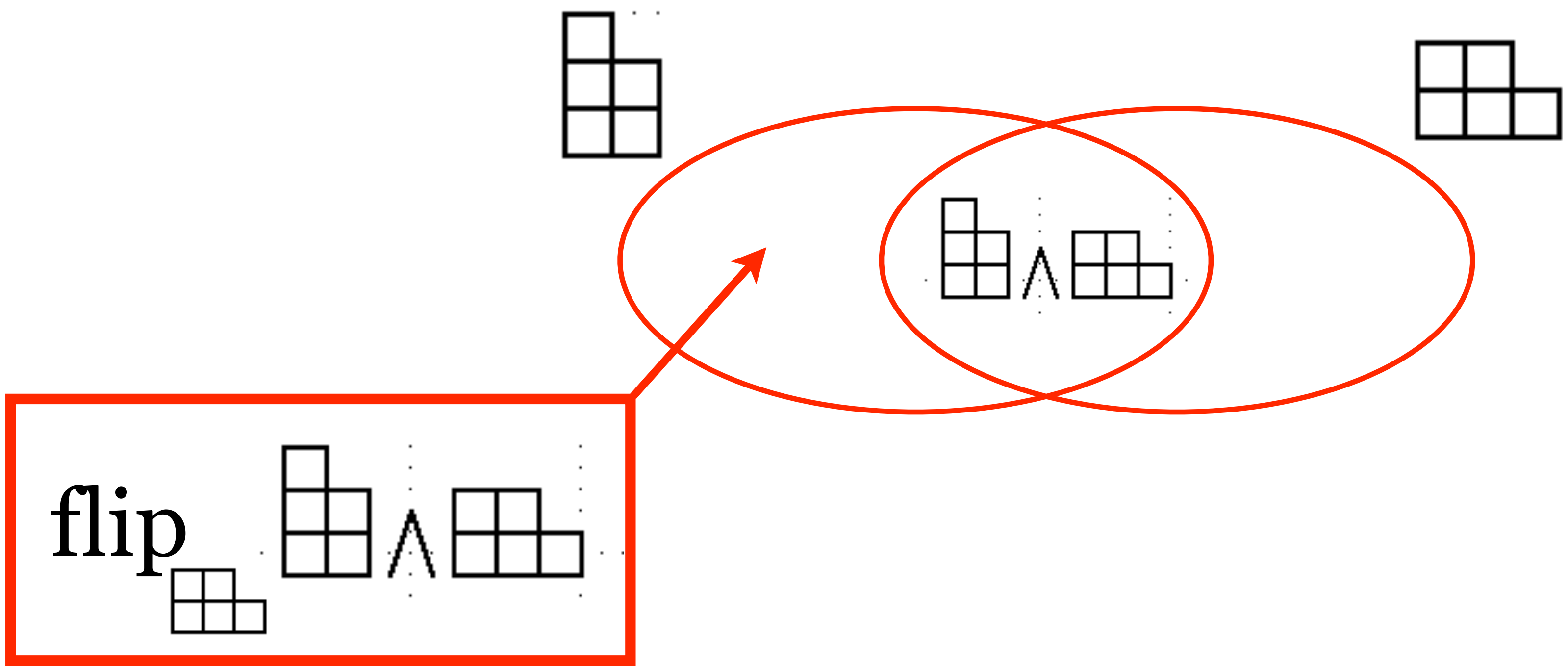

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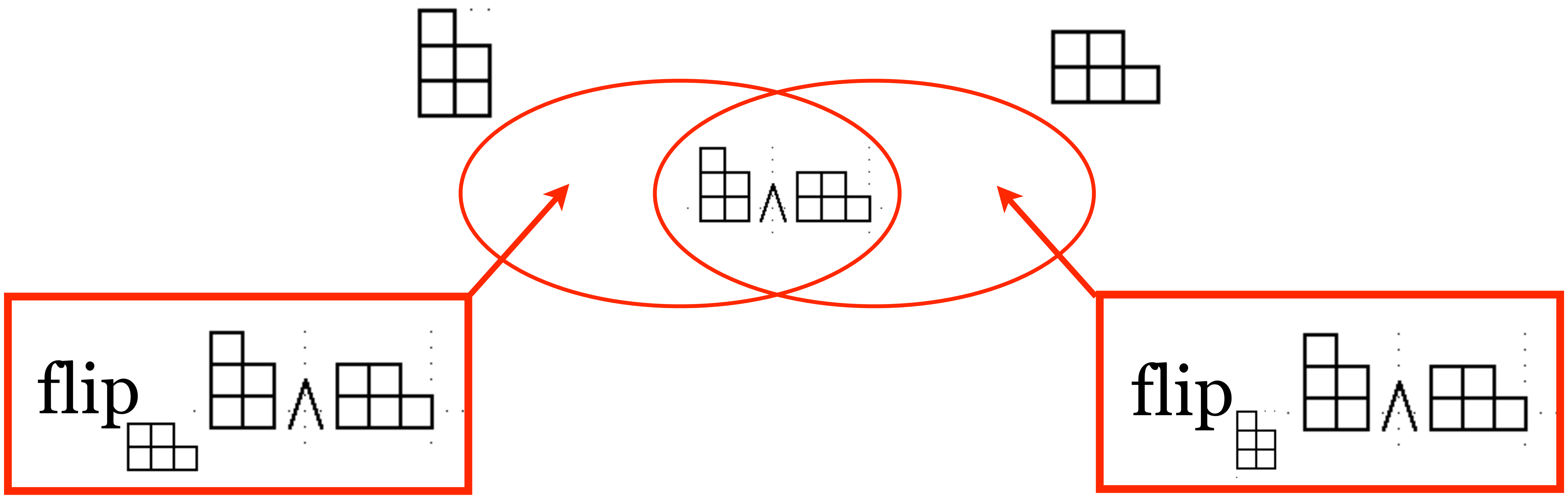

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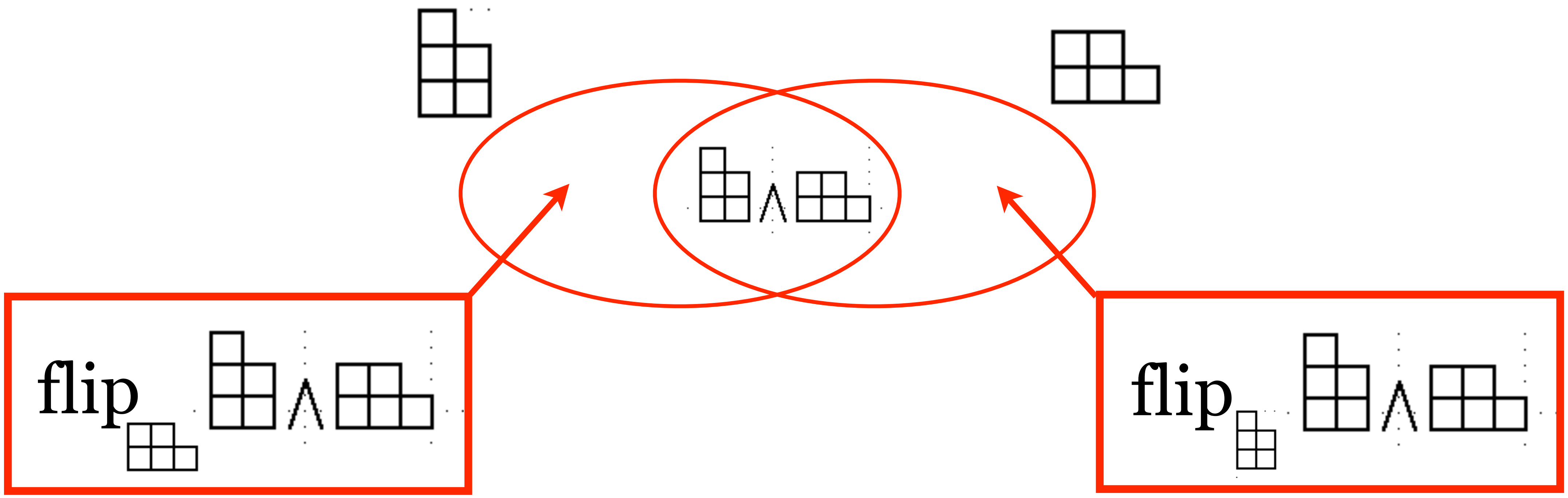

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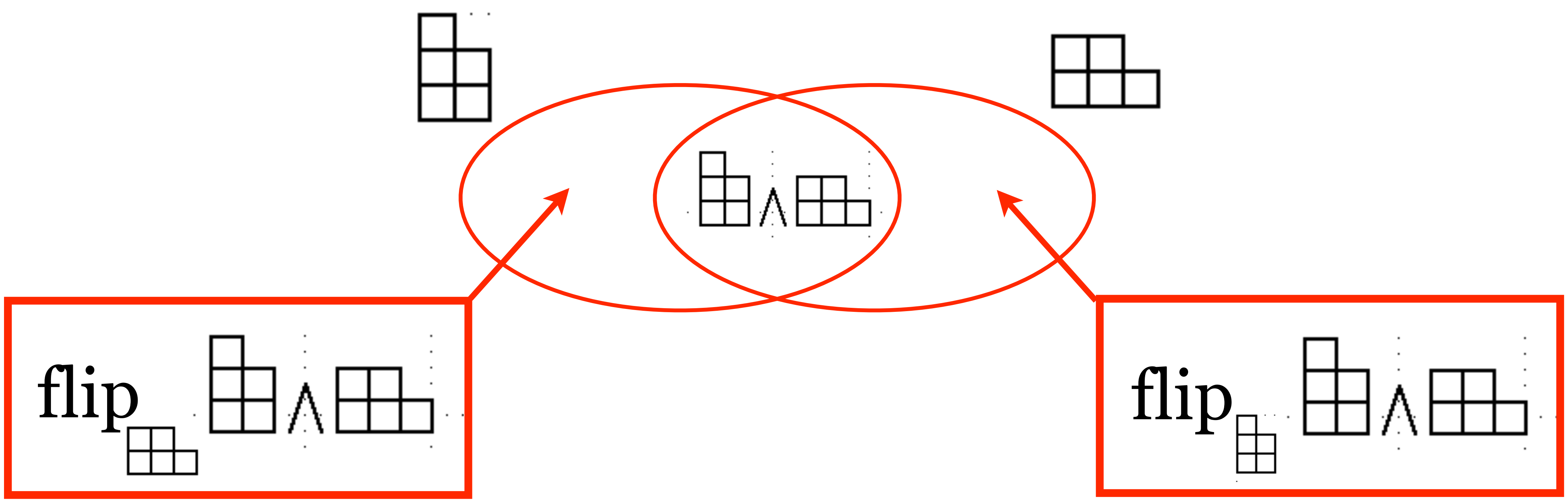

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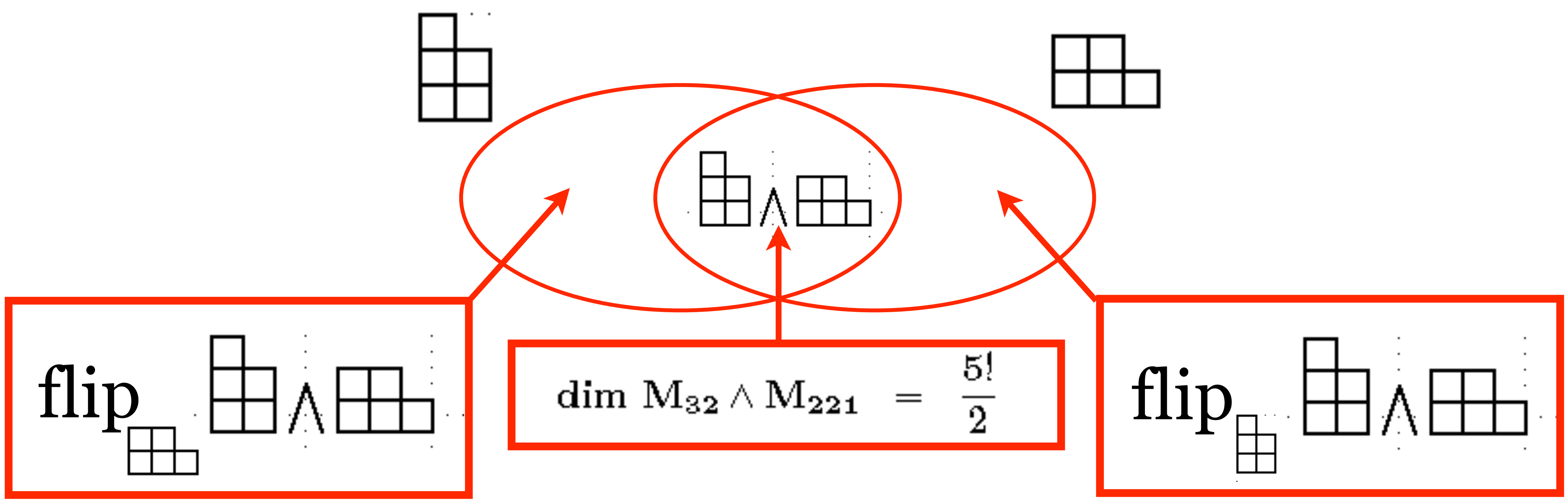

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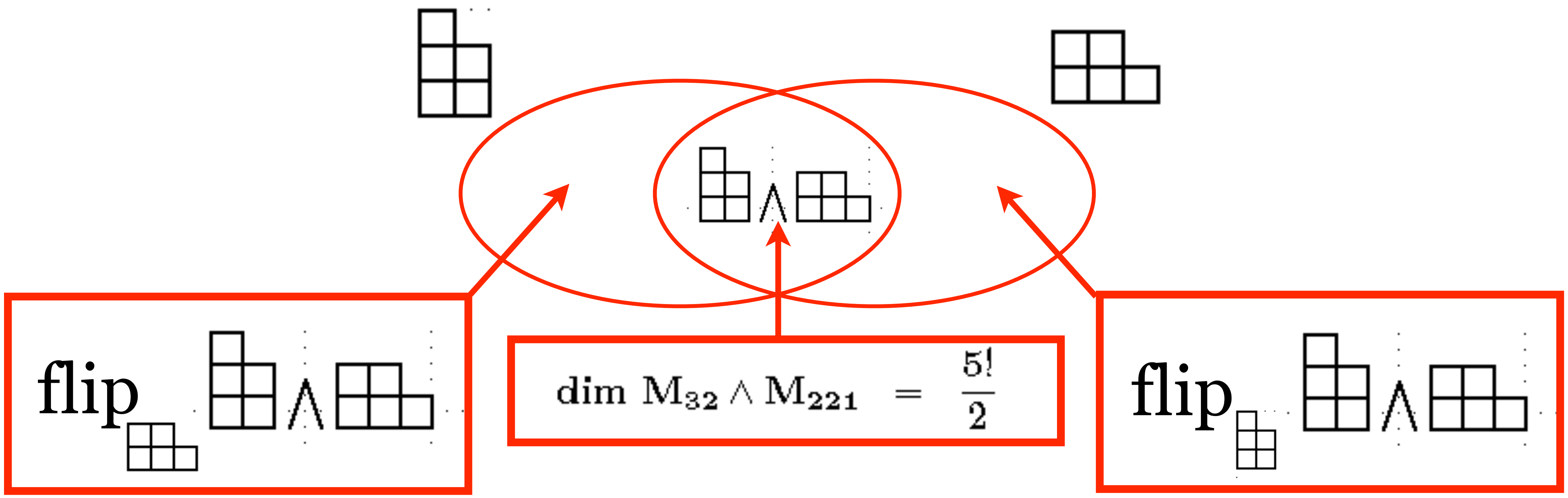

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$\frac{n!}{2}$ **Conjecture**

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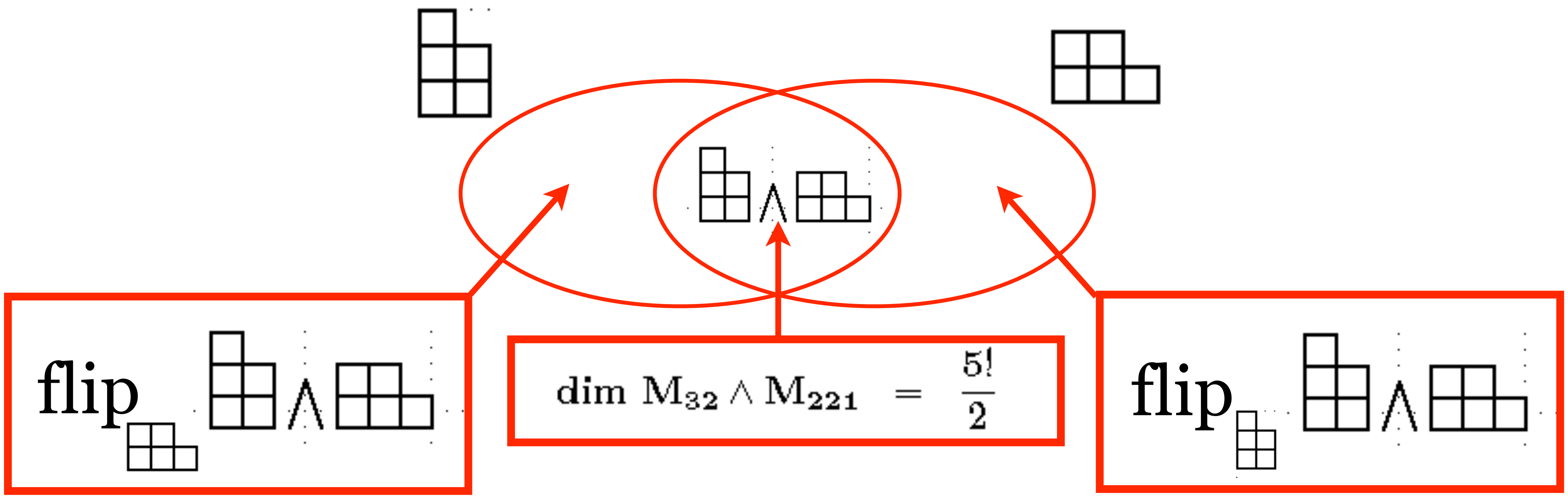

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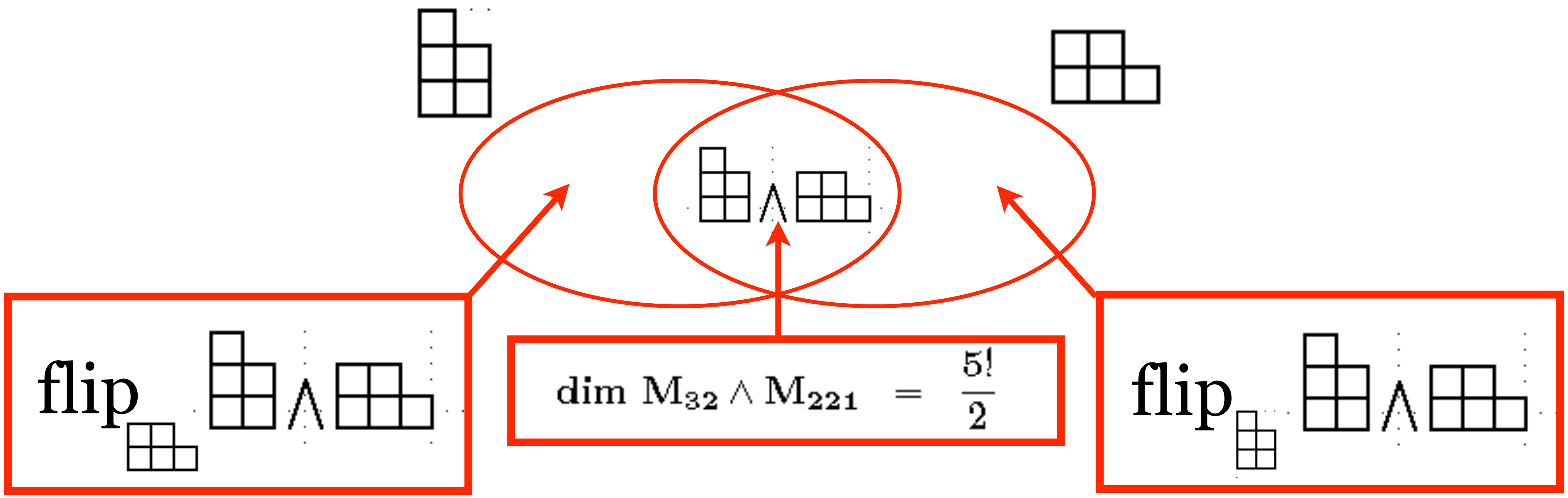

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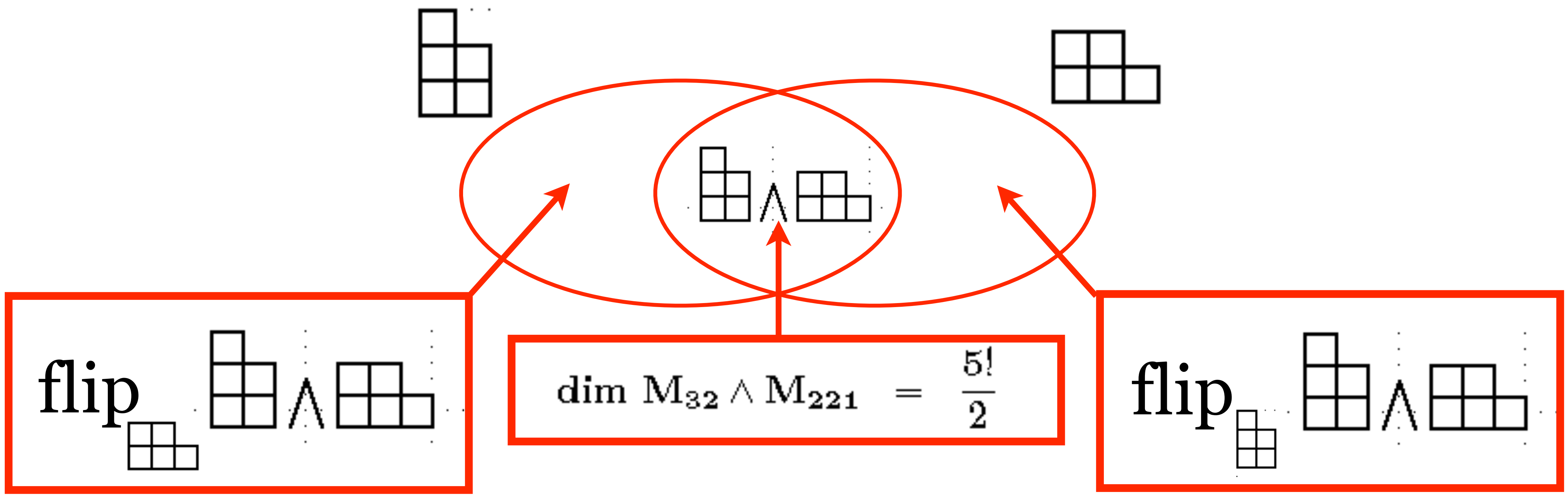

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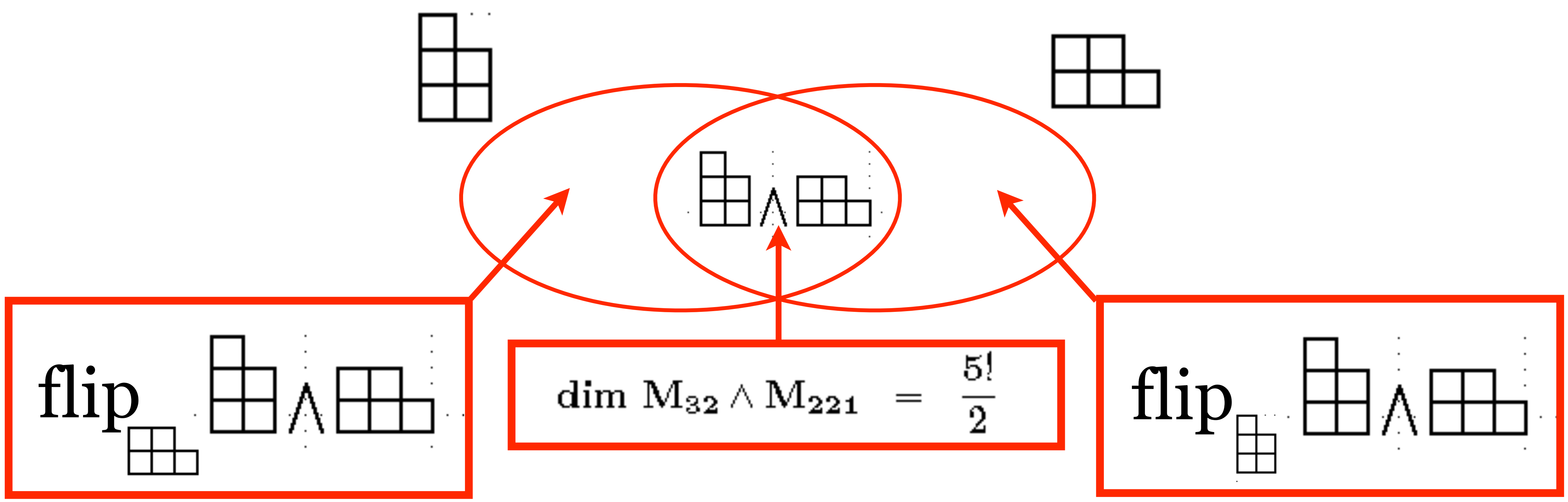
SF Conjecture

next the miracles



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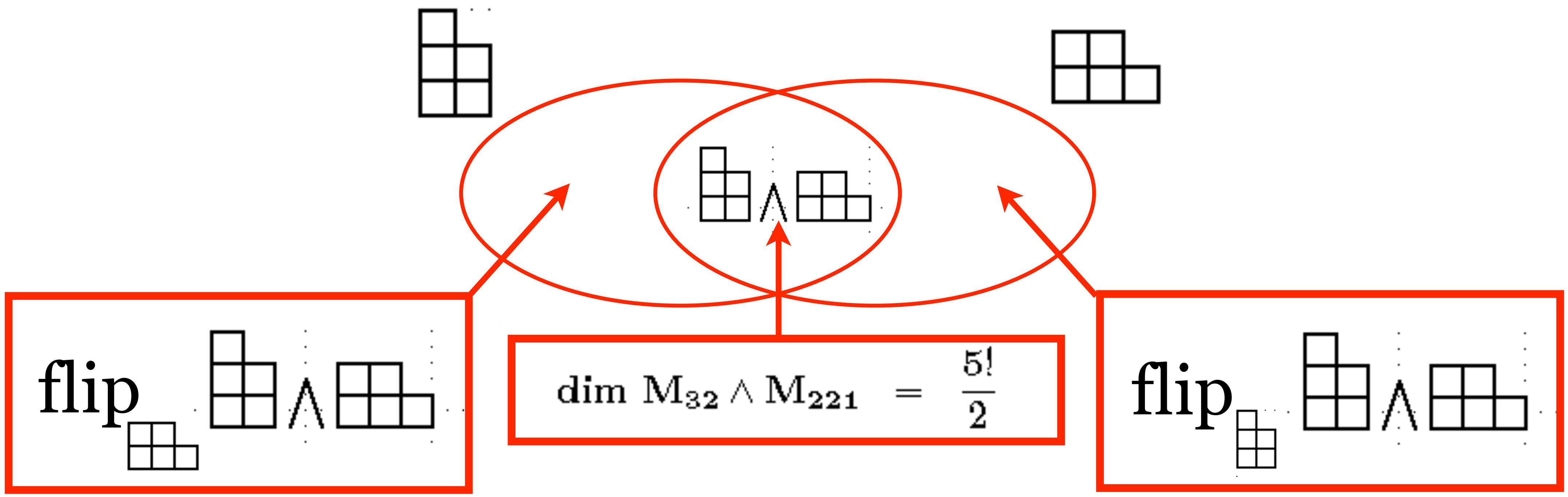

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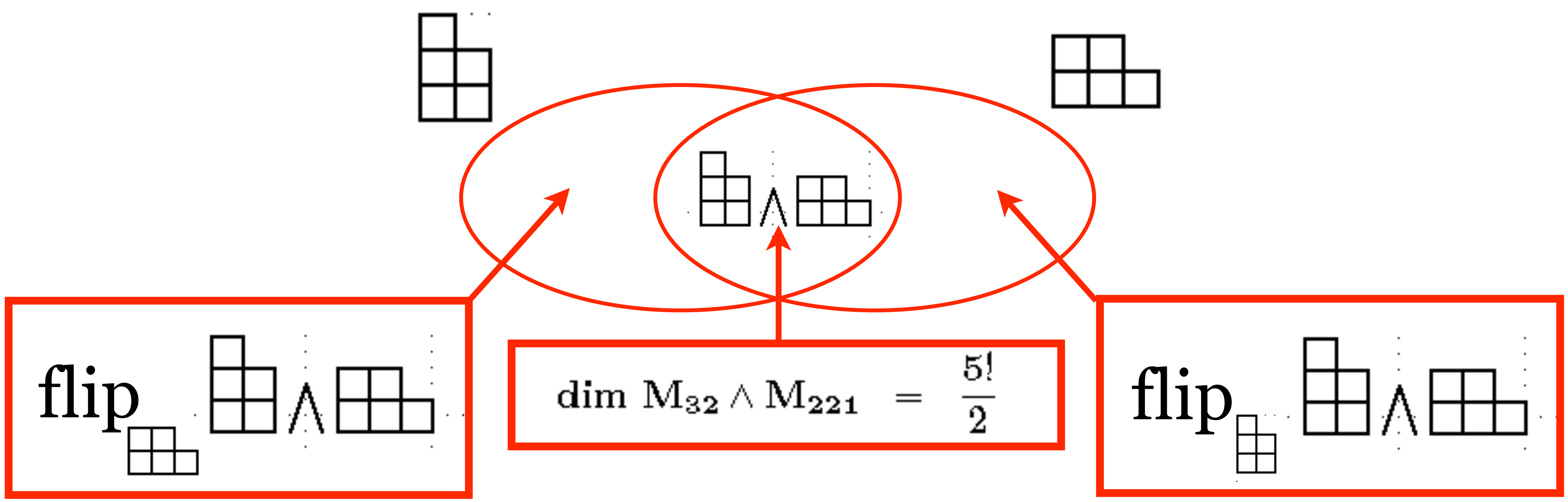

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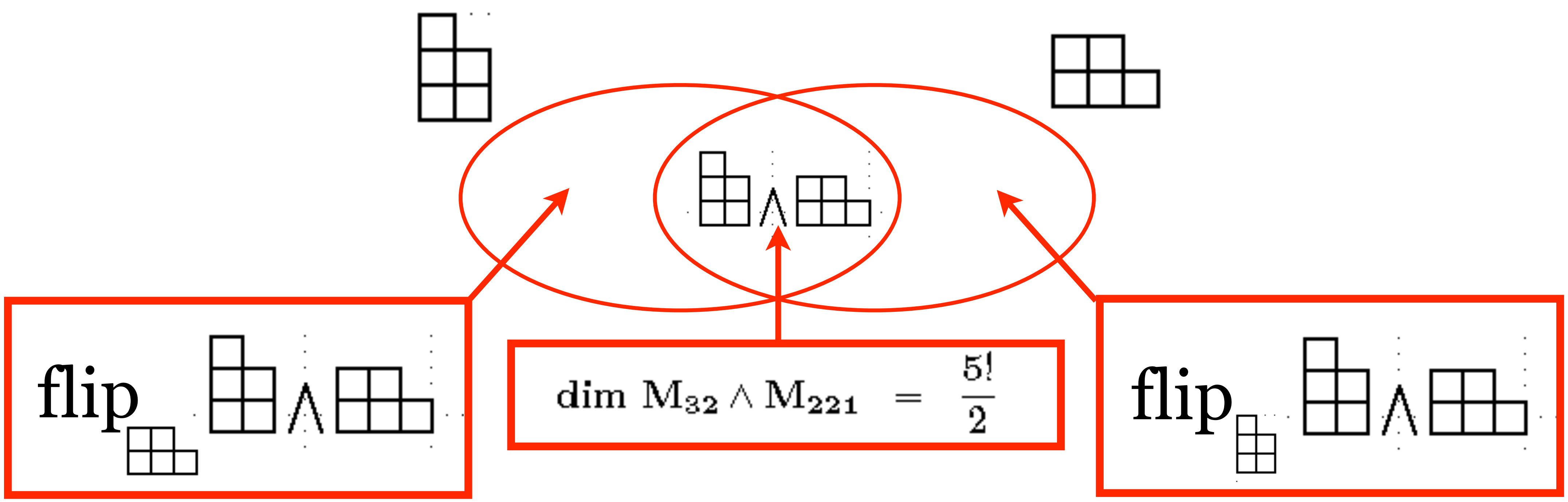

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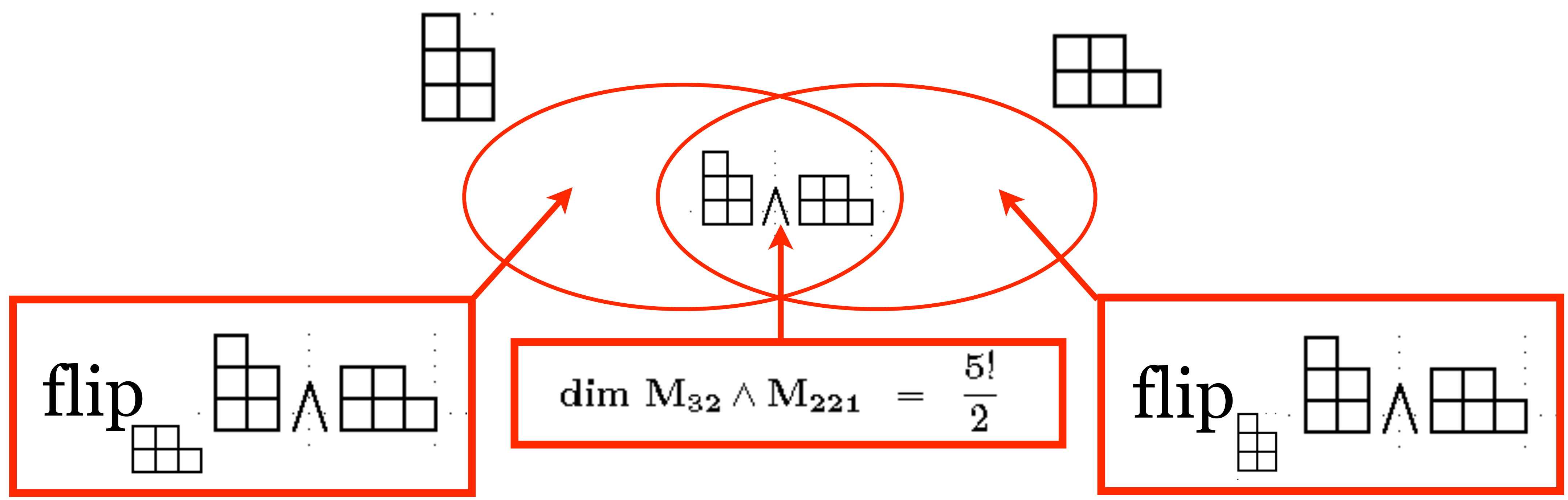

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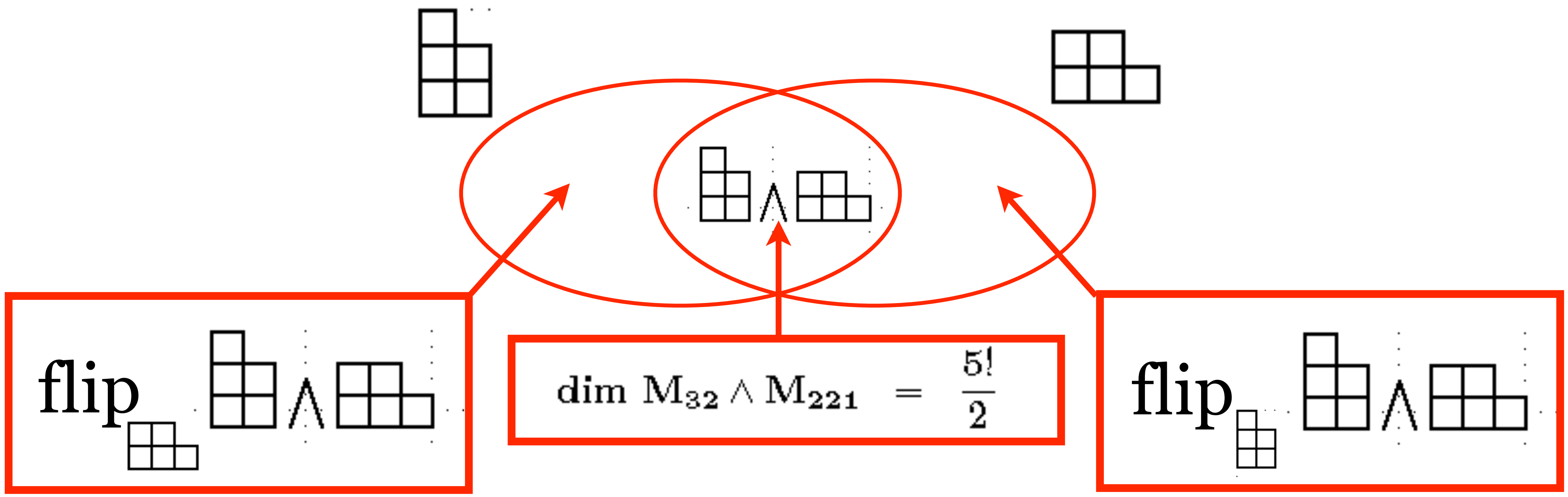

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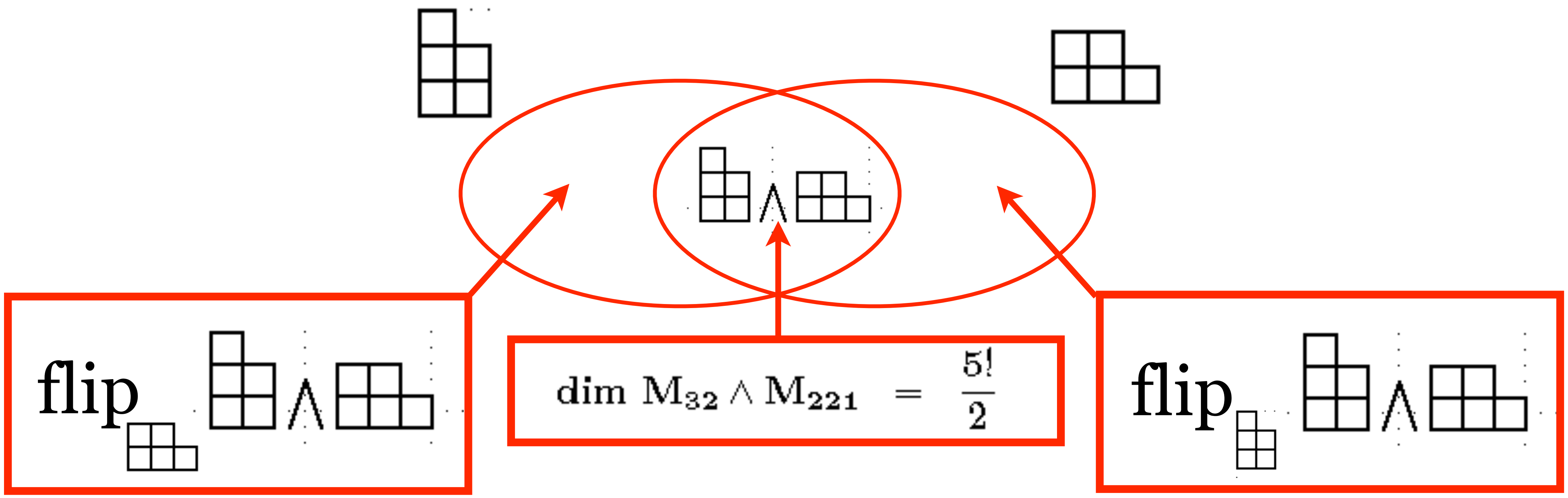

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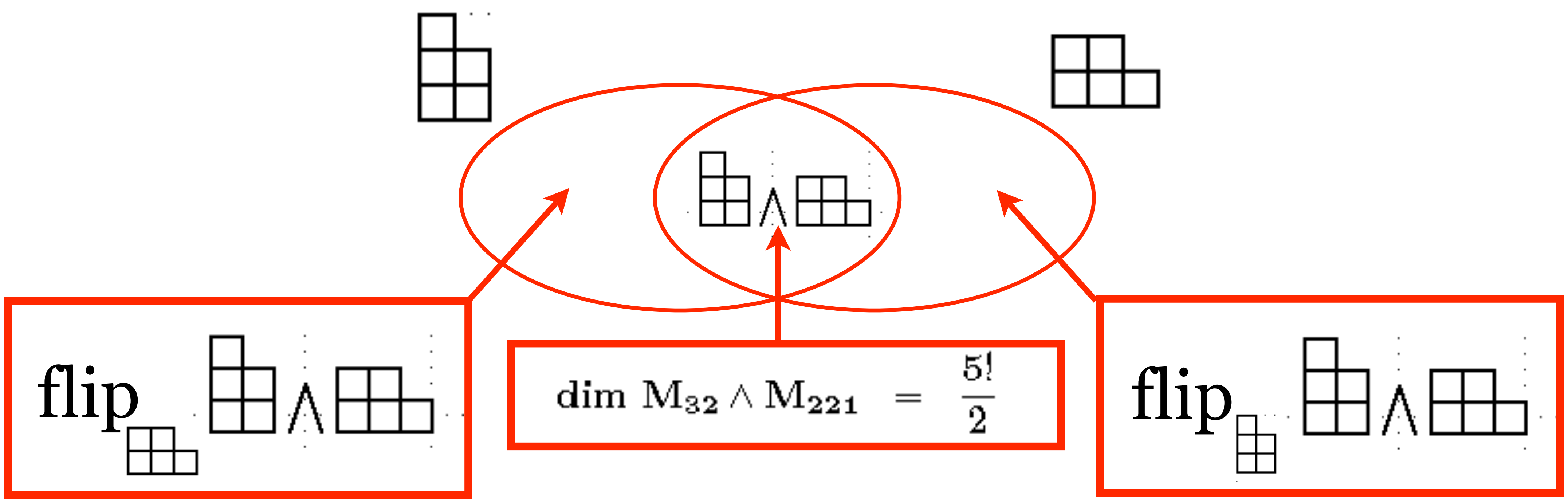

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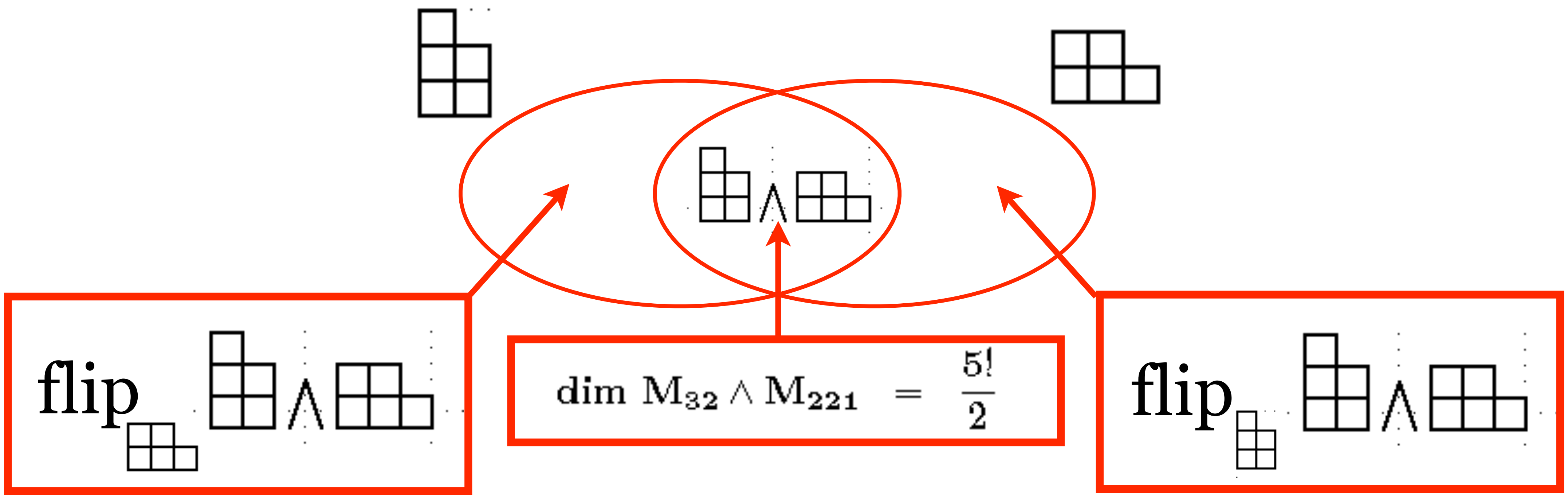

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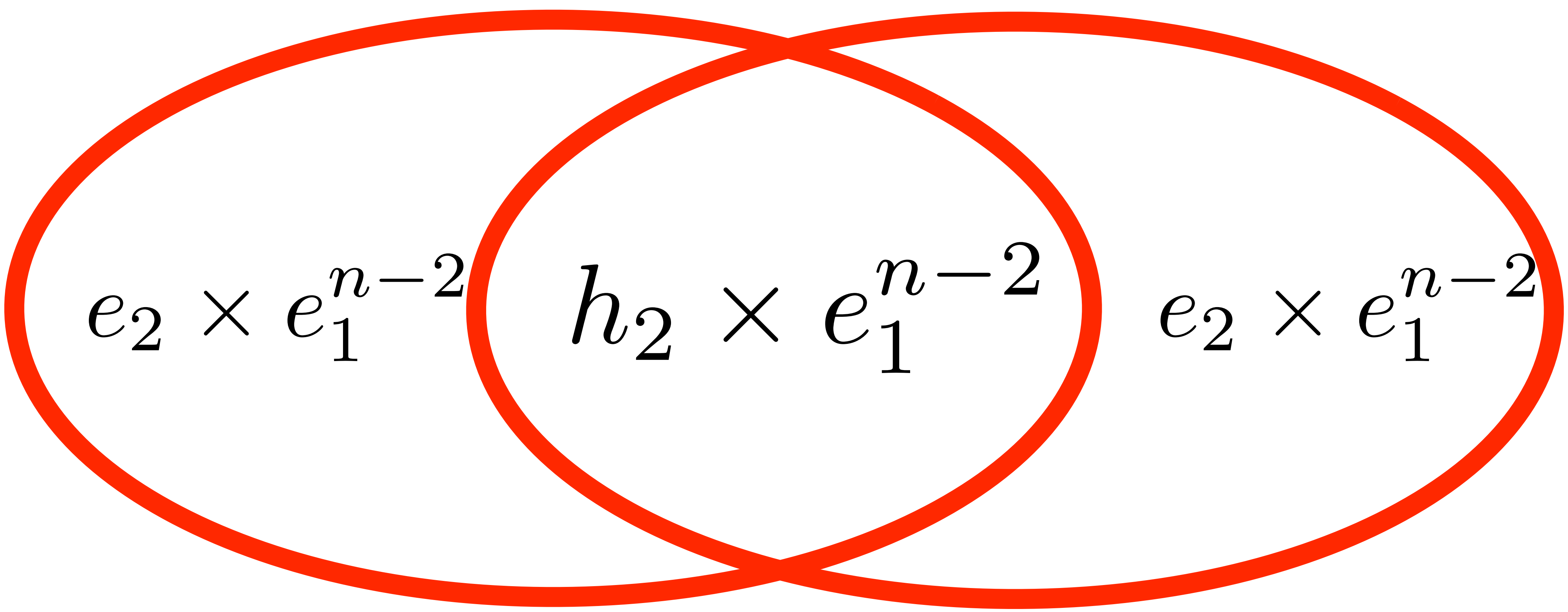

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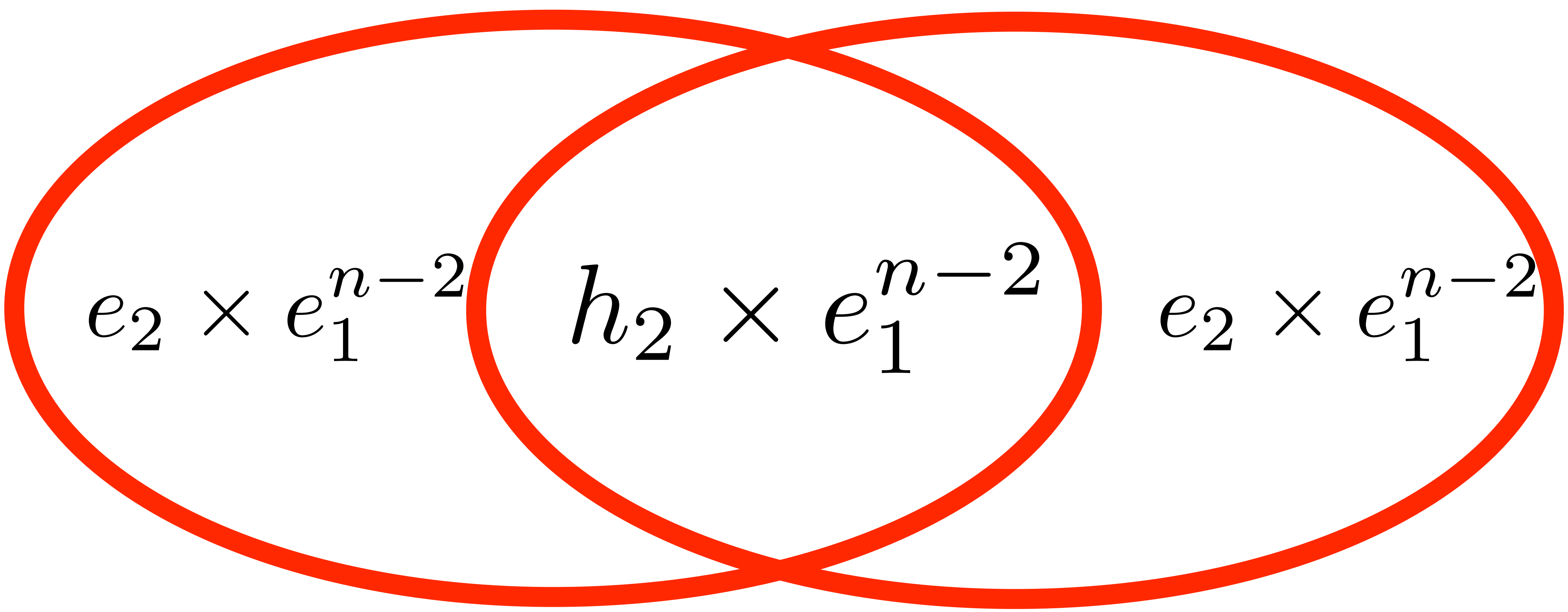

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$$\begin{cases} (0) \quad \mathbf{G_D(x;q,t)} = \tilde{\mathbf{H}}_\mu(\mathbf{x;q,t}) & \textbf{if} \quad \mathbf{D} \textbf{ is the diagram of } \mu \\ (1) \quad \mathbf{G_{D_1}(x;q,t)} = \mathbf{G_{D_2}(x;q,t)} & \textbf{if} \quad \mathbf{D_1} \approx \mathbf{D_2} \\ (2) \quad \mathbf{G_{D_1}(x;q,t)} = \mathbf{G_{D_2}(x;t,q)} & \textbf{if} \quad \mathbf{D_2} \approx \mathbf{D_1'} \\ (3) \quad \mathbf{G_D(x;q,t)} = \mathbf{G_{D_1}(x;q,t)G_{D_2}(x;q,t)} & \textbf{if} \quad \mathbf{D} \approx \mathbf{D_1} \times \mathbf{D_2} \\ (4) \quad \partial_{\mathbf{p_1}} \mathbf{G_D(x;q,t)} = \sum_{\mathbf{s} \in \mathbf{D}} \mathbf{w_{s,D}(q,t)} \; \mathbf{G_{D/s}(x;q,t)} \;, & \end{cases}$$

Representation theoretical reasons suggest that,

in the case that $\mathbf{D}$ *is a skew diagram,*

$$\textbf{a)} \quad \mathbf{w_1[s,D]} = \mathbf{t^{l_D(s)} q^{a'_D(s)}} \qquad \textbf{and} \qquad \textbf{b)} \quad \mathbf{w_2[s,D]} = \mathbf{t^{l'_D(s)} q^{a_D(s)}}$$

Note: these properties overdetermine the family $\{\mathbf{G_D(x;q,t)}\}_\mathbf{D}$,

The basic tool : "gistol" Macdonalds

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Representation theoretical reasons suggest that,

in the case that D *is a skew diagram,*

$$\textbf{a)}\quad w_1[s,D] = t^{l_D(s)}q^{a'_D(s)} \qquad \textbf{and} \qquad \textbf{b)}\quad w_2[s,D] = t^{l'_D(s)}q^{a_D(s)}$$

Note: these properties overdetermine the family $\{G_D(x;q,t)\}_D$*,*

existence is by no means guaranteed.

The basic tool : "gistol" Macdonalds

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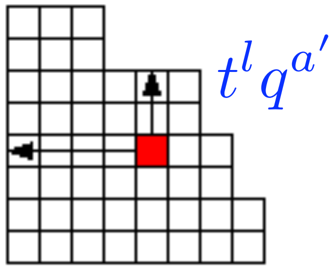
Note: these properties overdetermine the family $\{G_D(x;q,t)\}_D$,

existence is by no means guaranteed.

next

An example of a use of gistols

An example of a use of gistols



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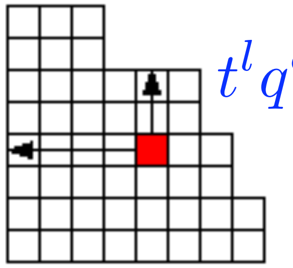
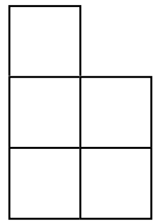


$t^l q^{a'}$

∂_{p_1}  $=$

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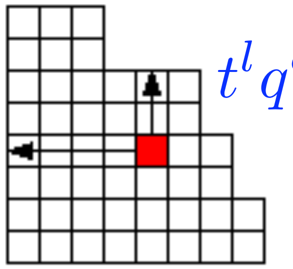

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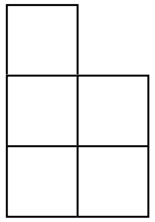
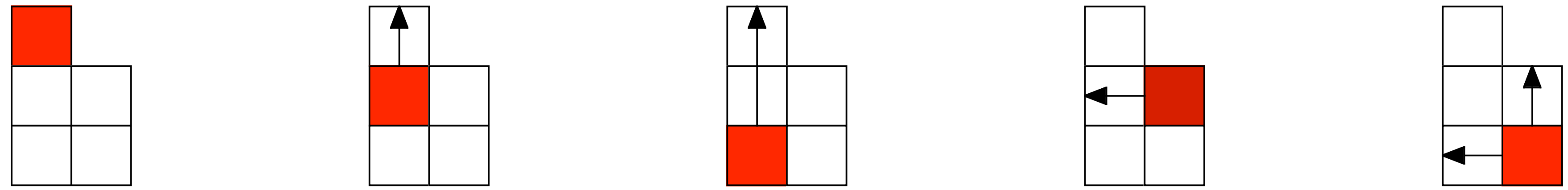
∂_{p_1}  $=$ 

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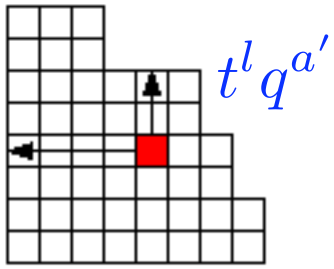

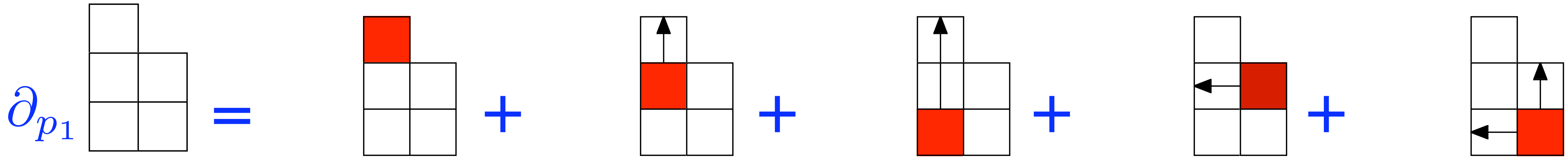

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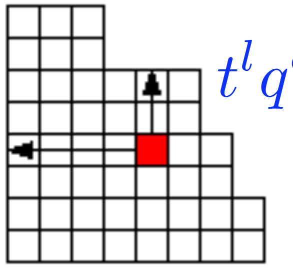

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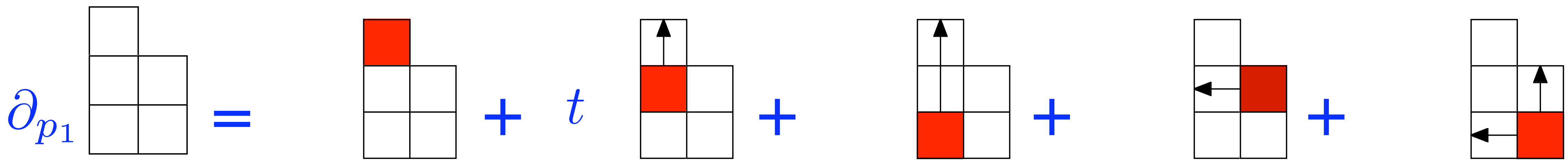

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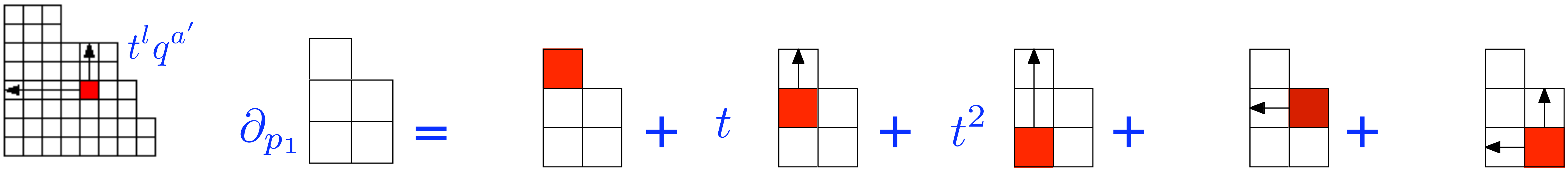

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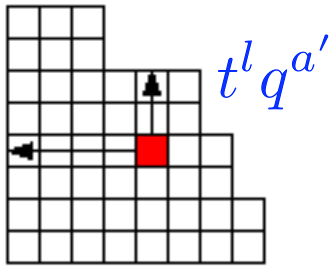

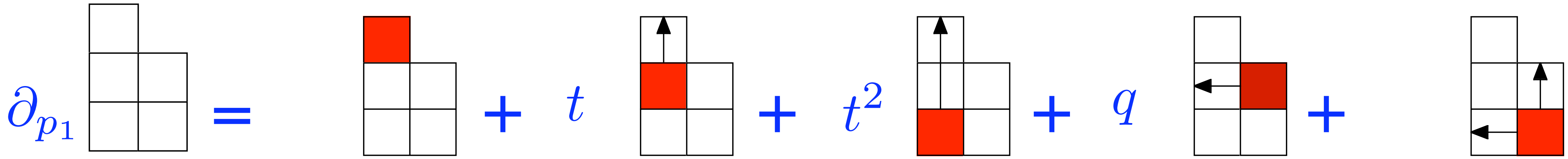

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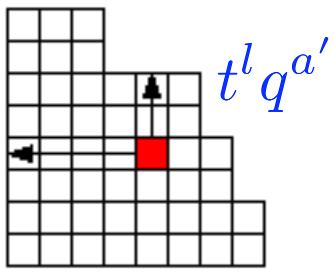

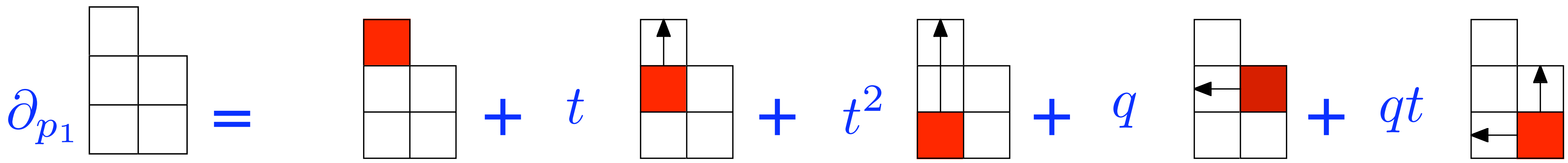

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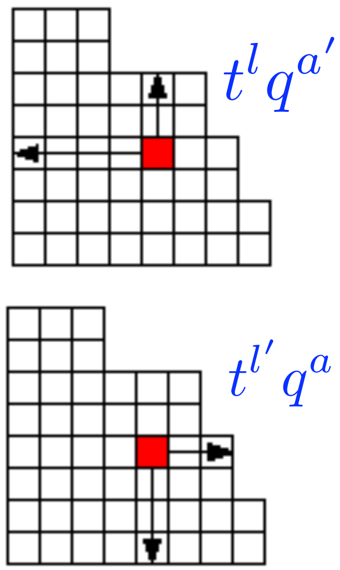

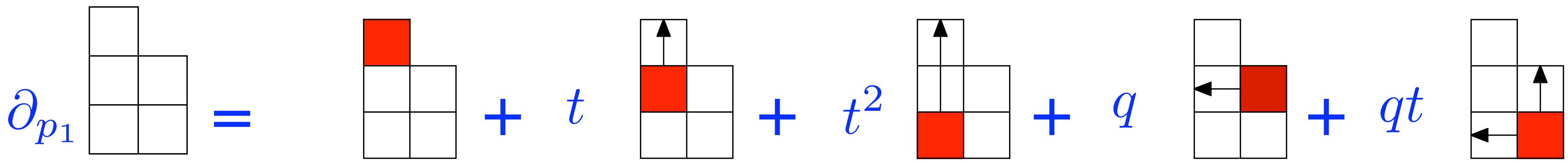

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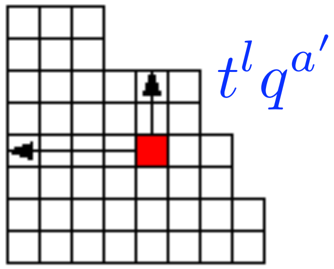

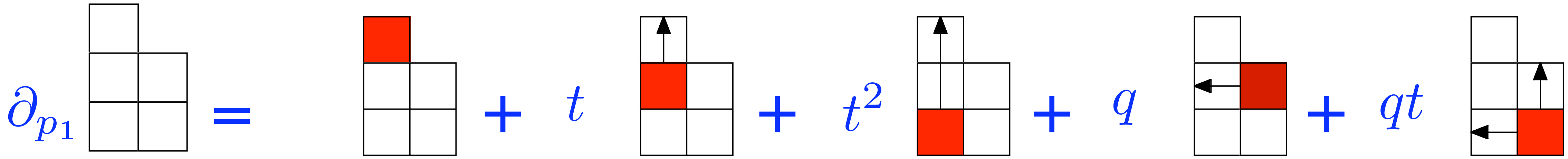

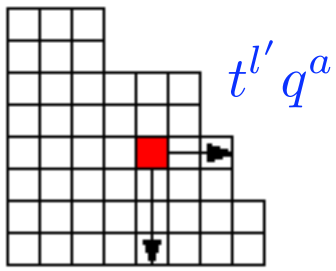

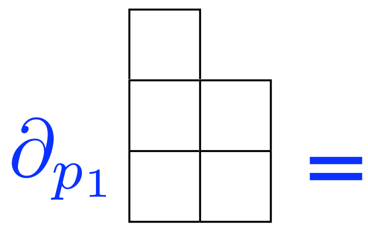

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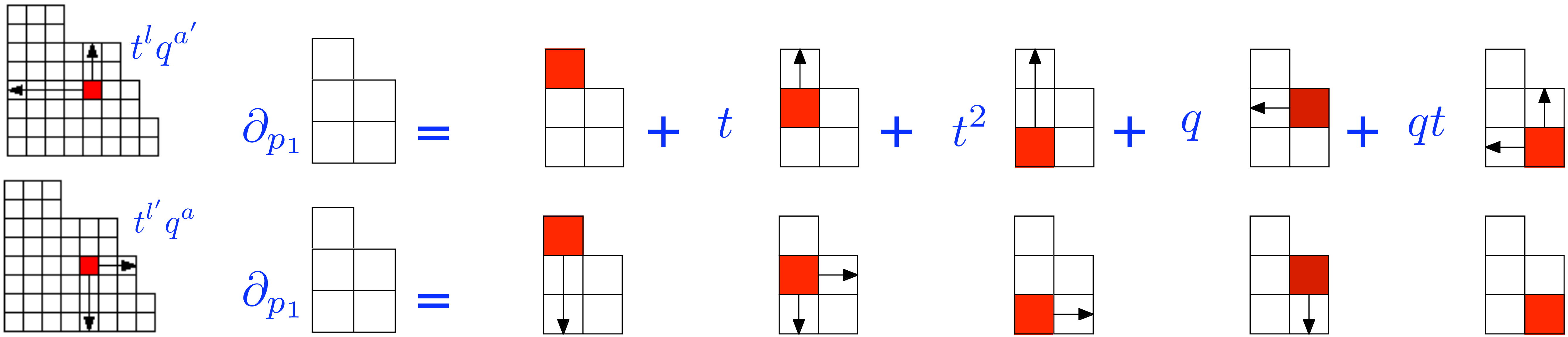

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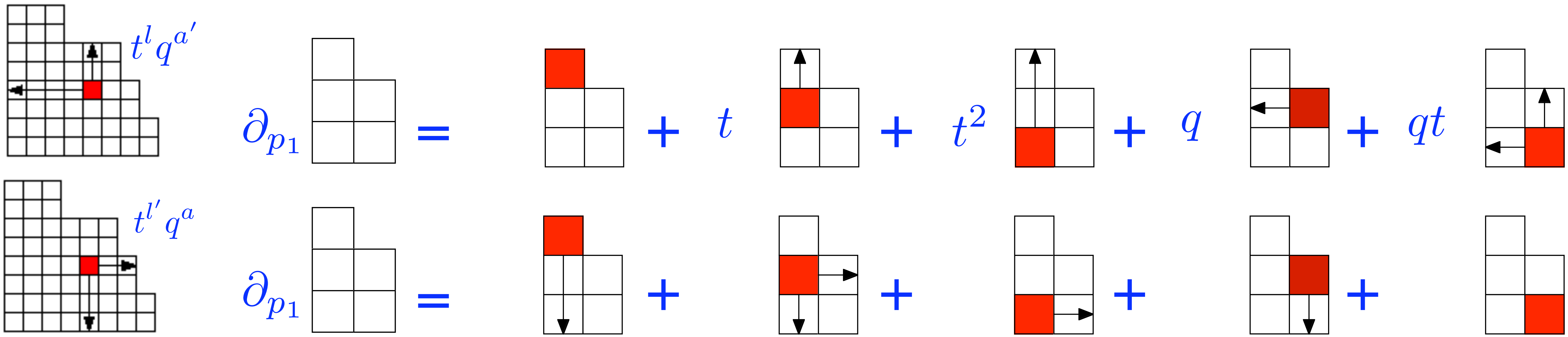

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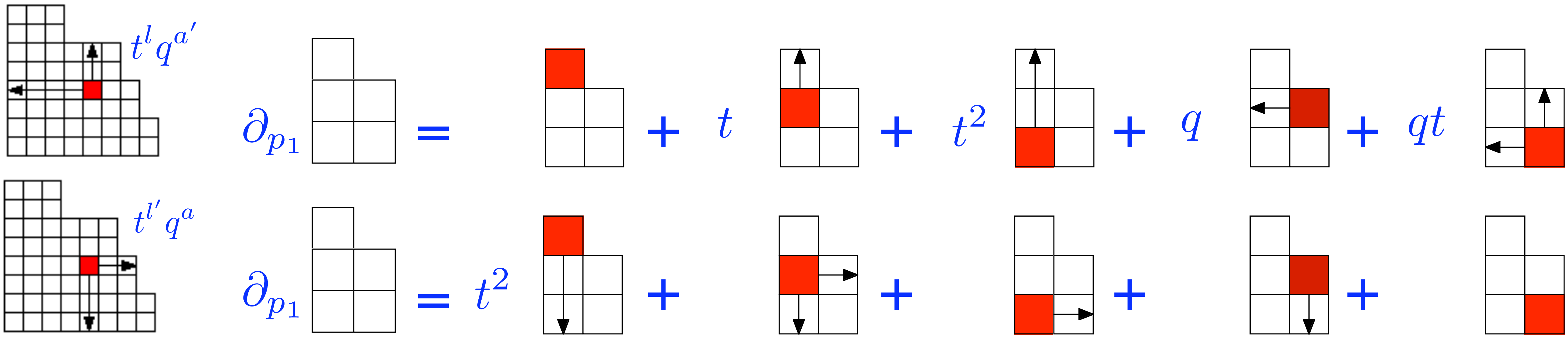

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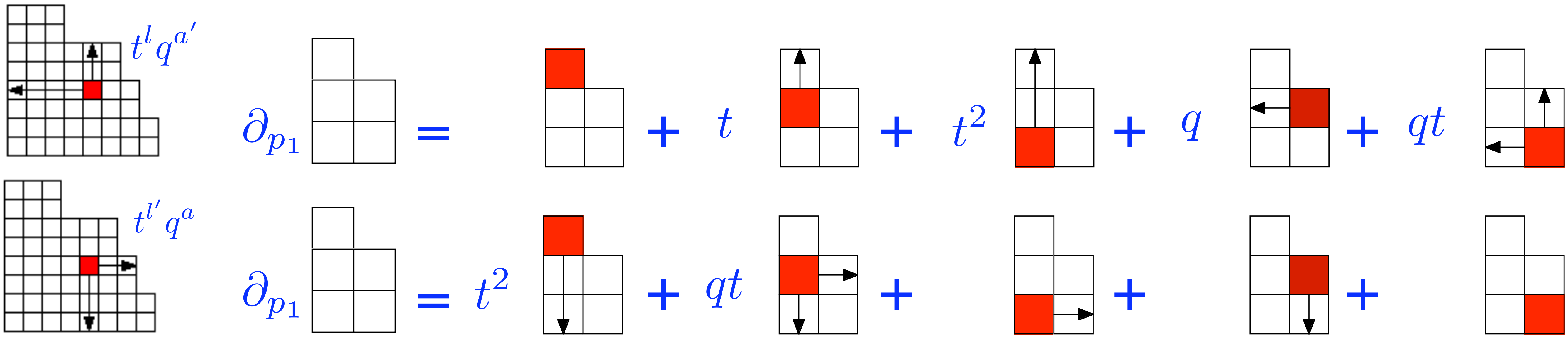

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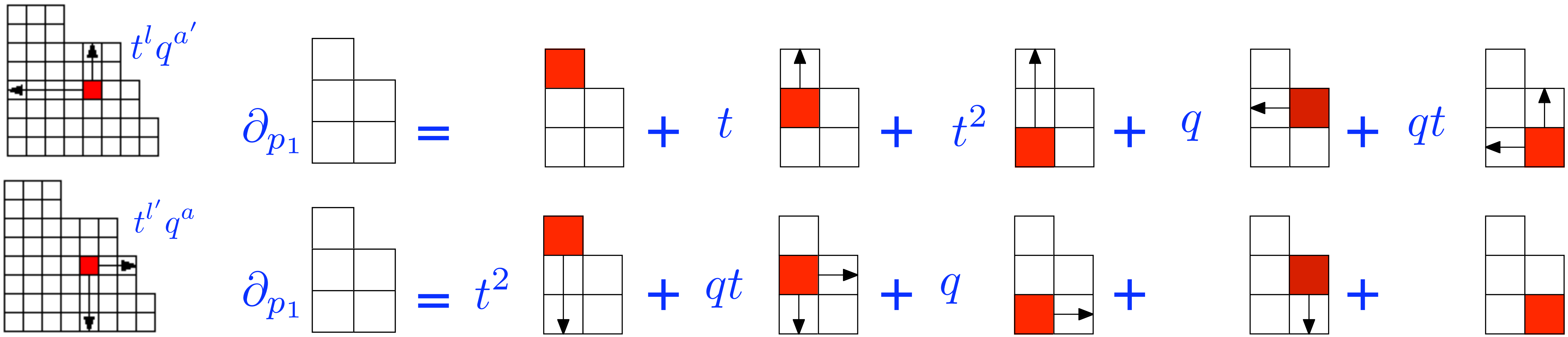

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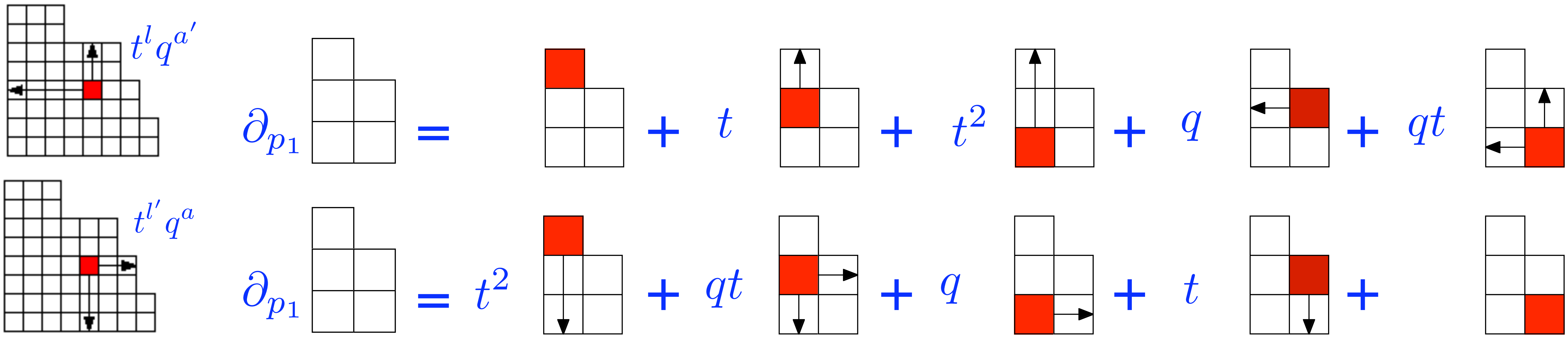

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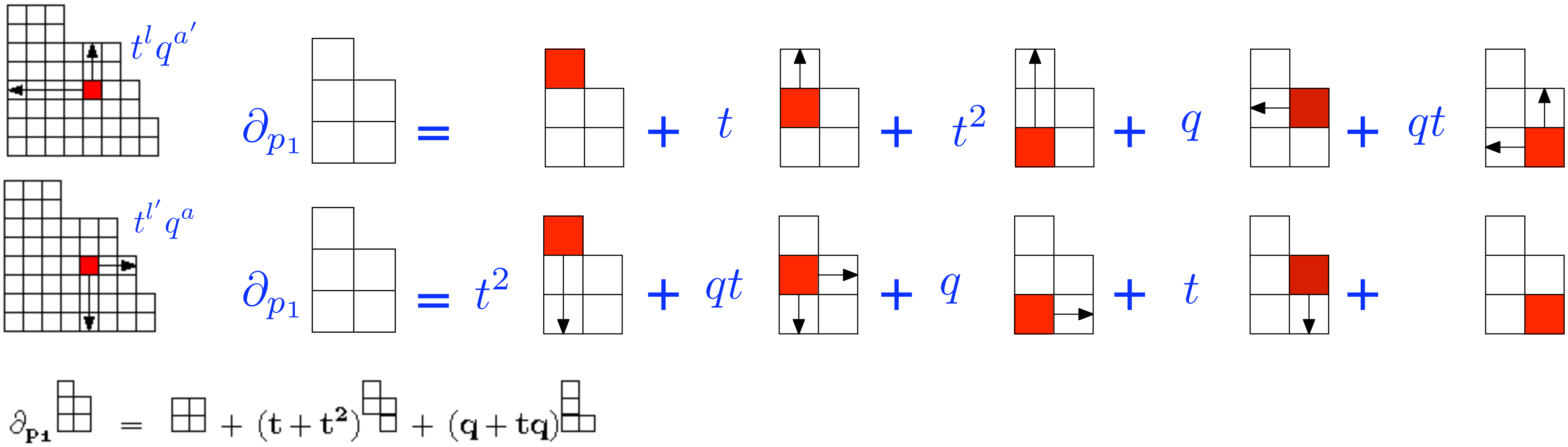

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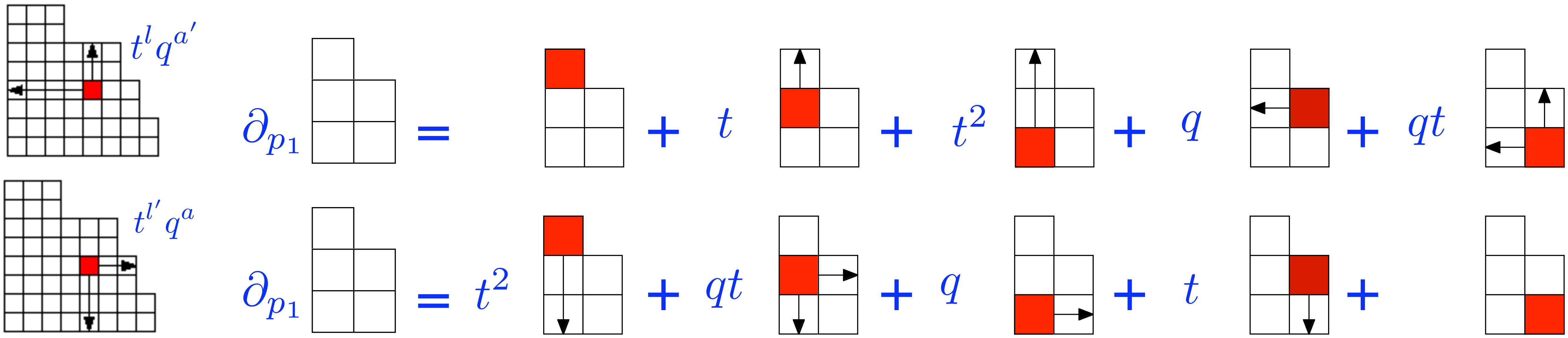

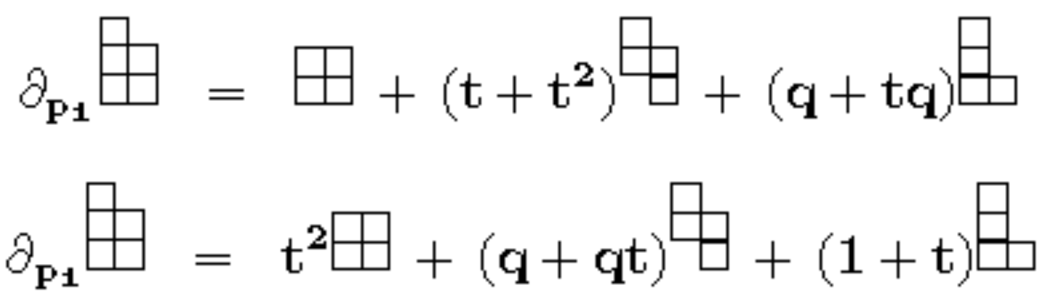

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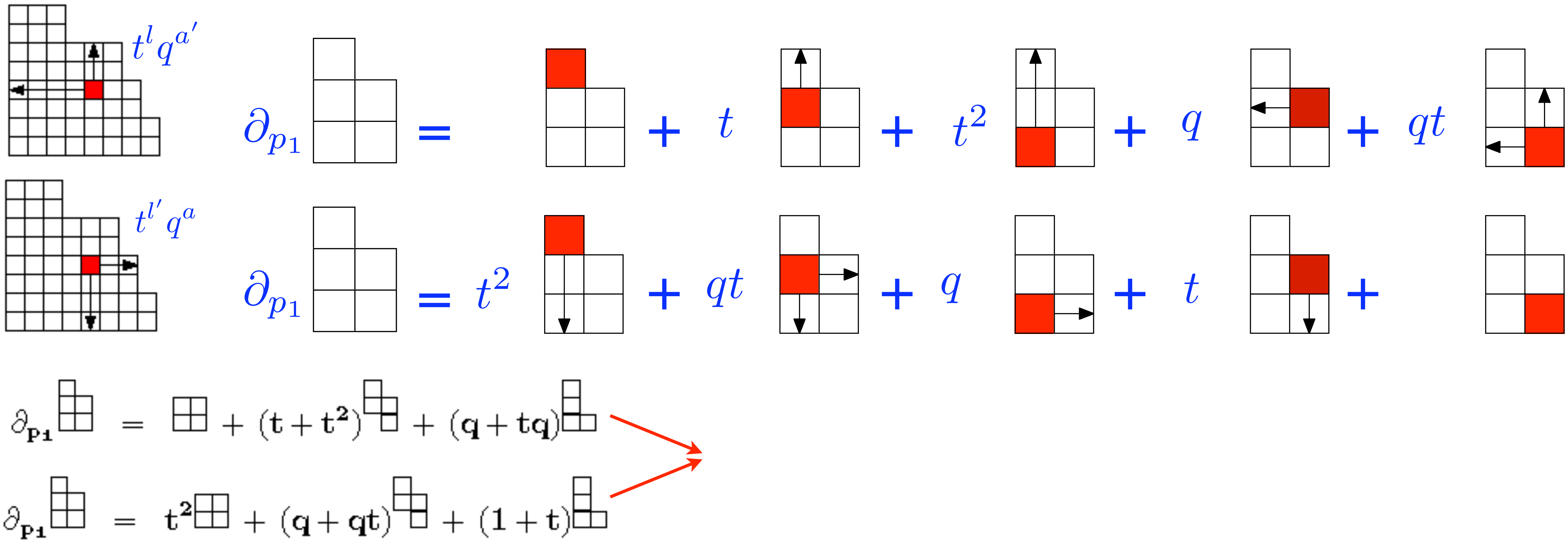

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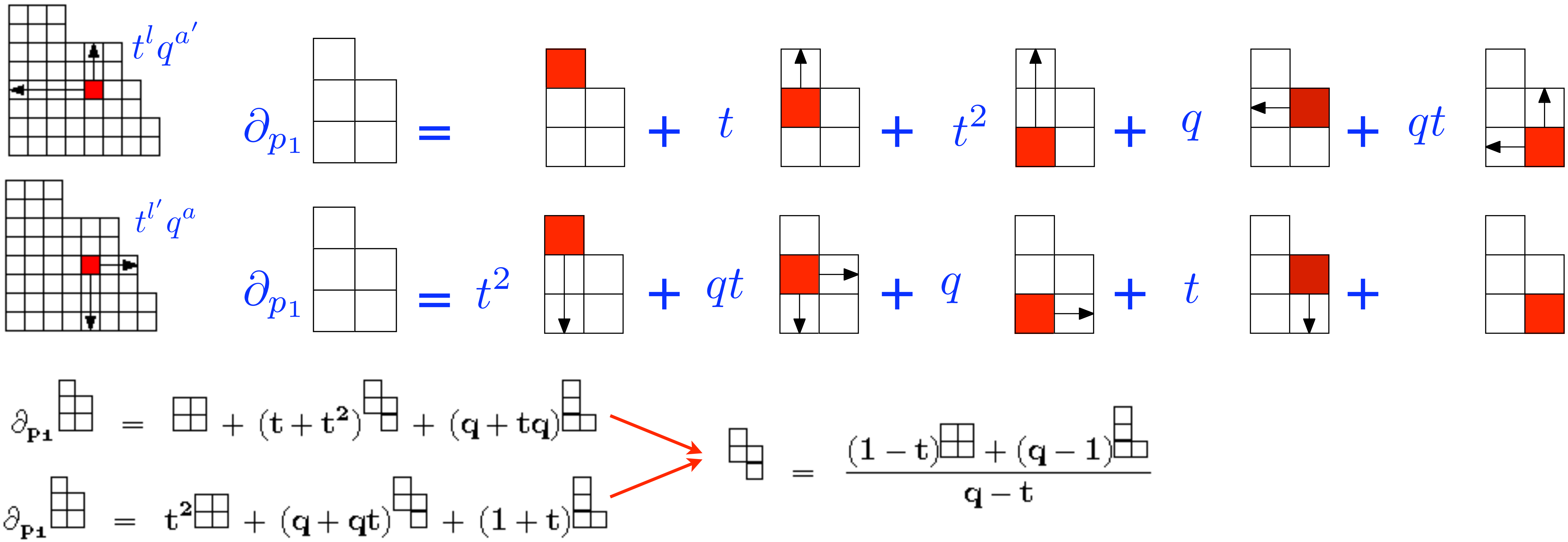

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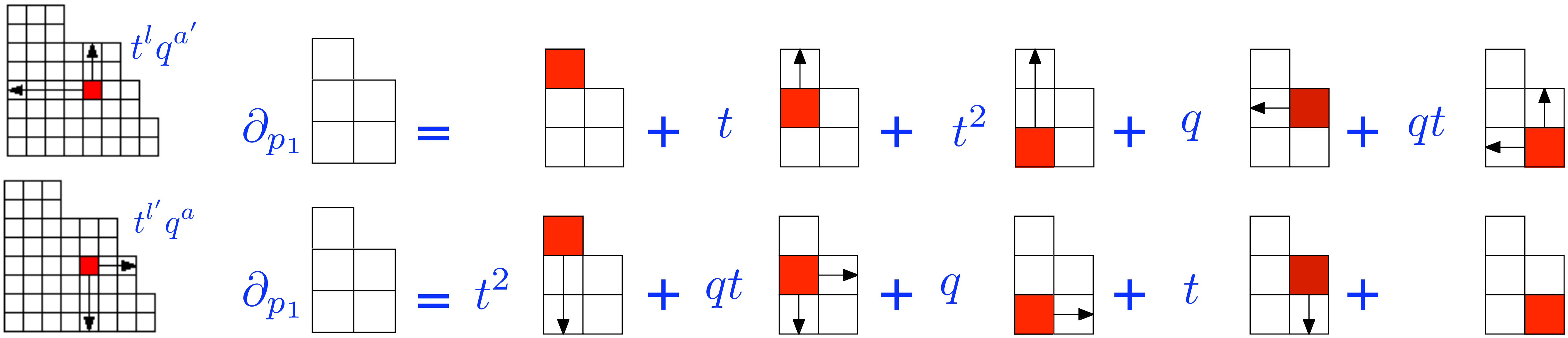

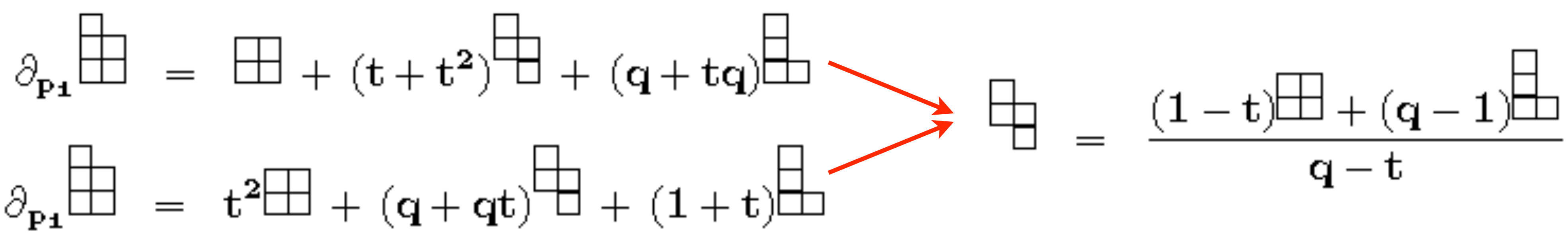

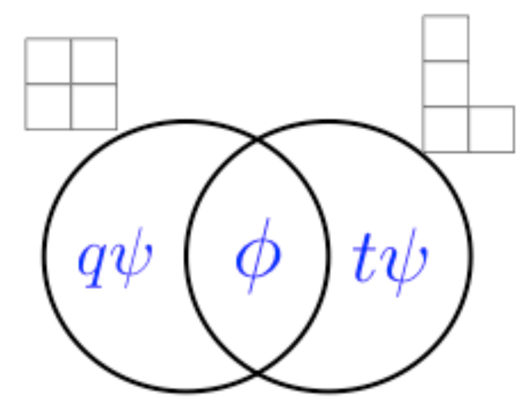

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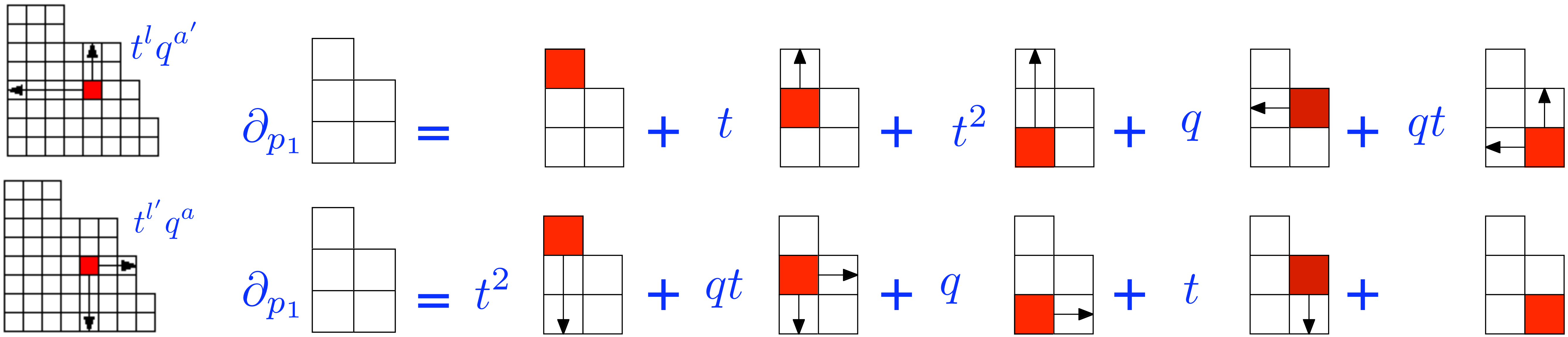

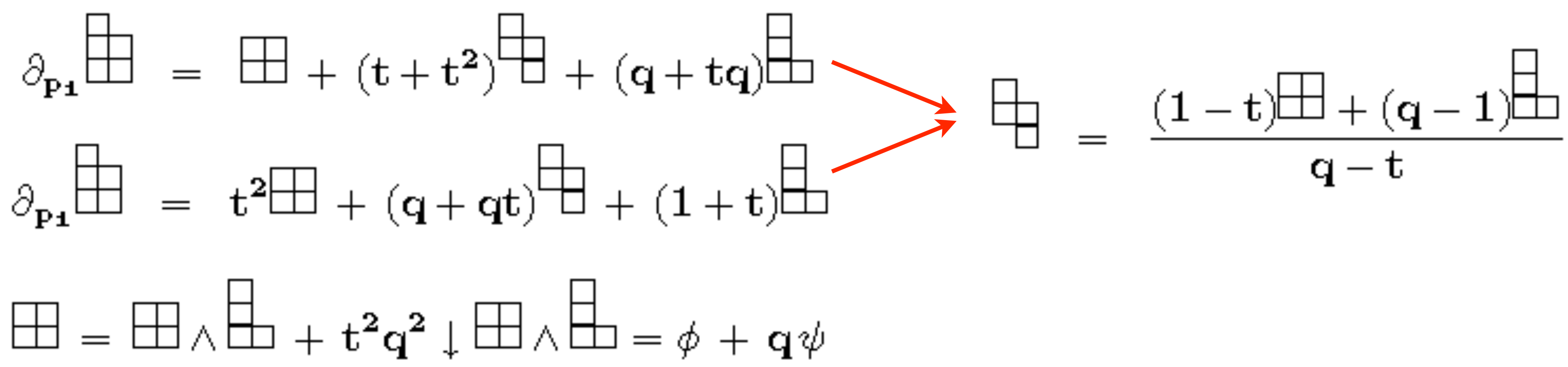

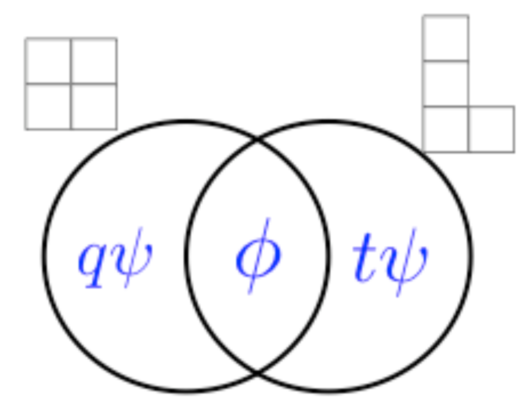

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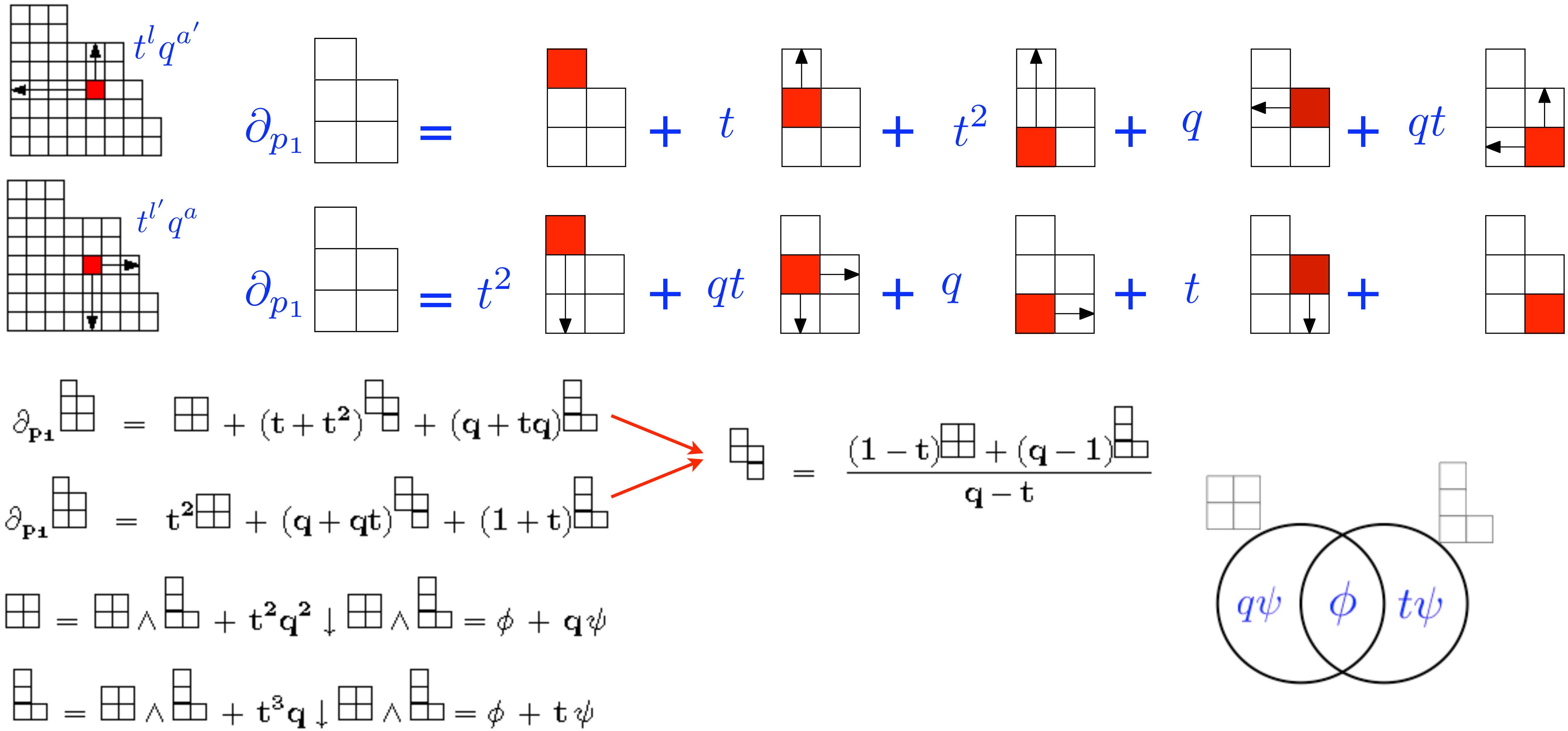

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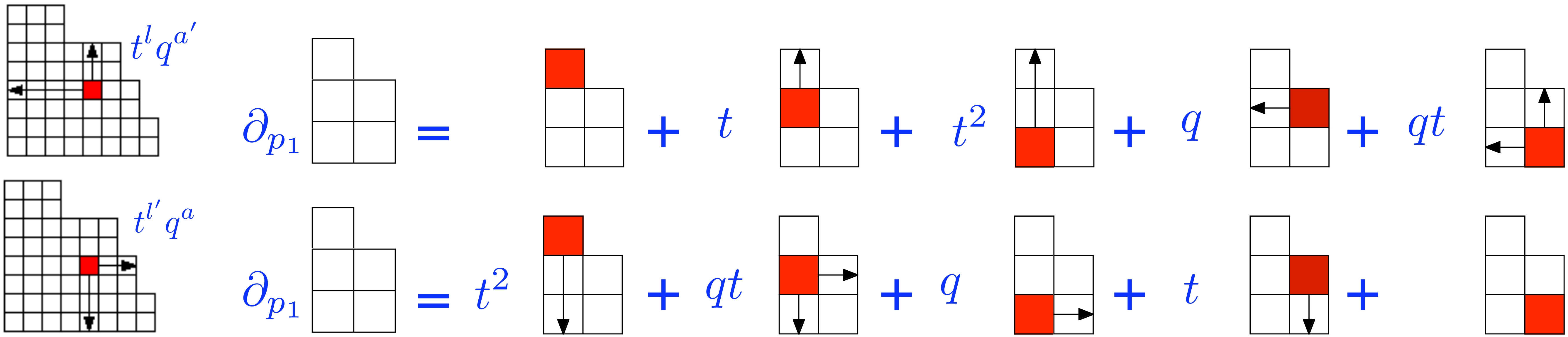

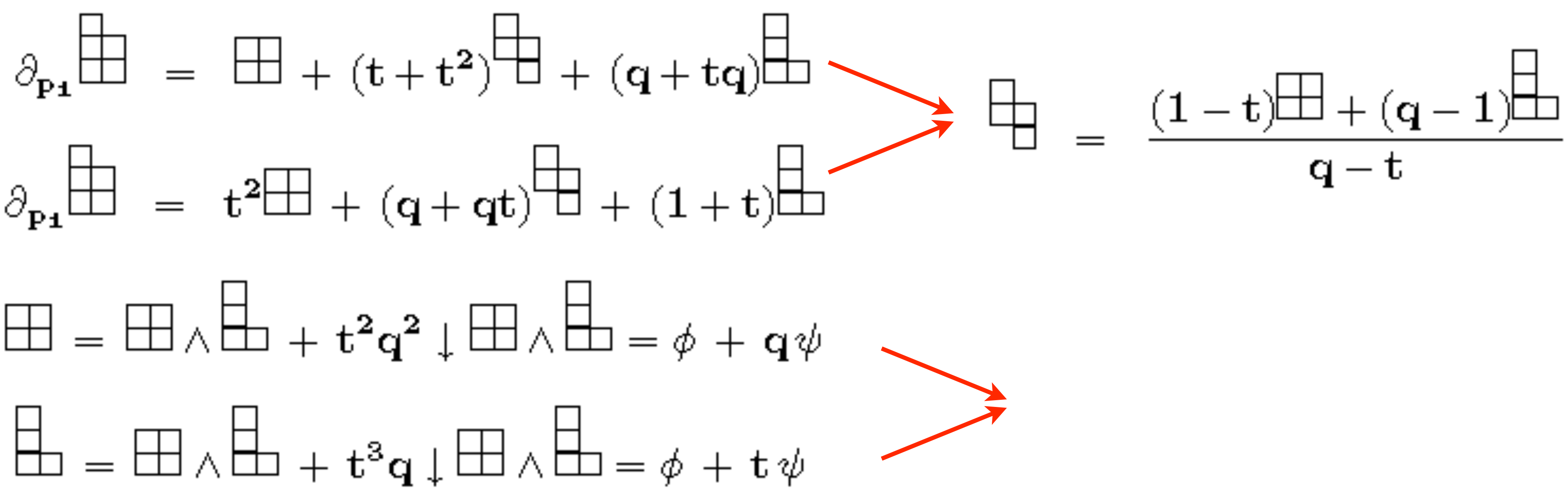

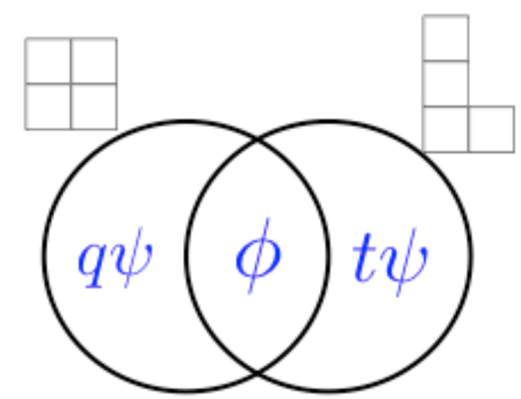

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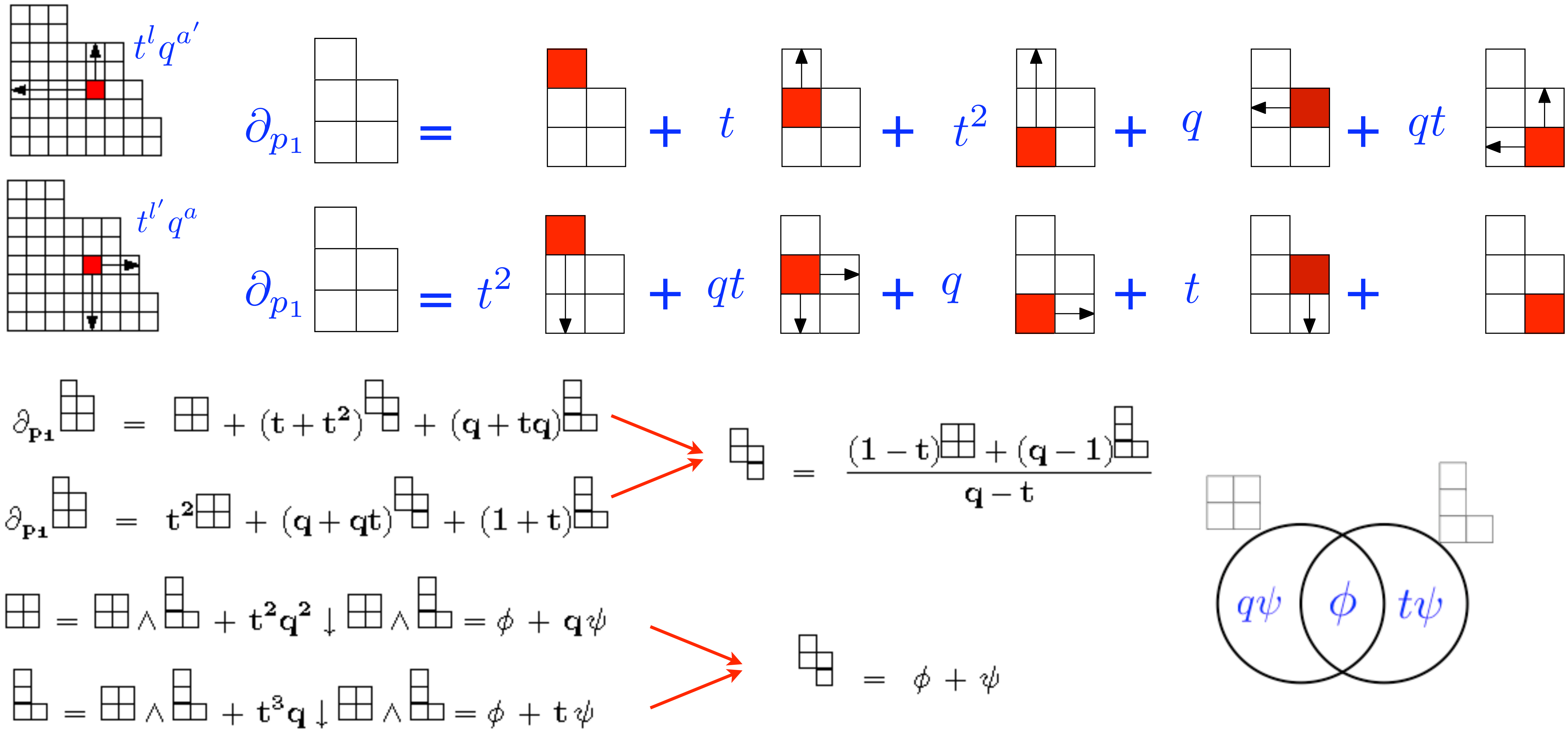

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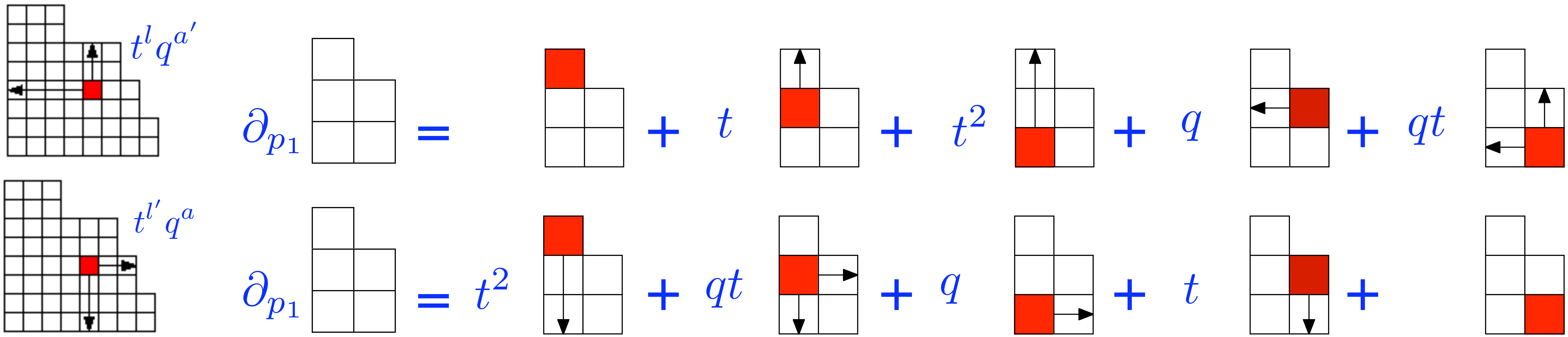

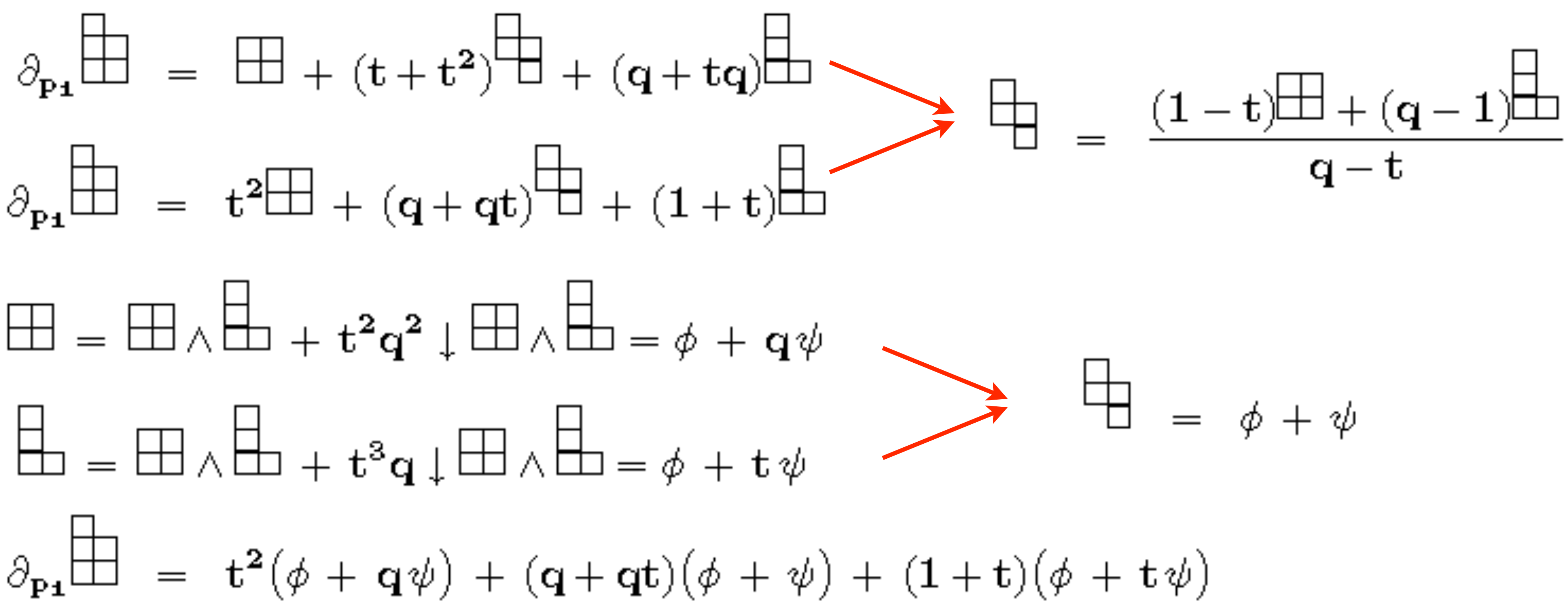

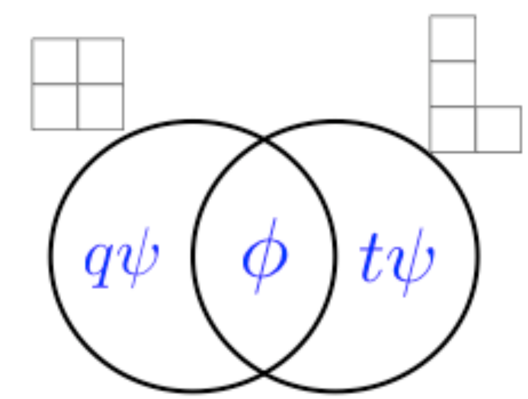

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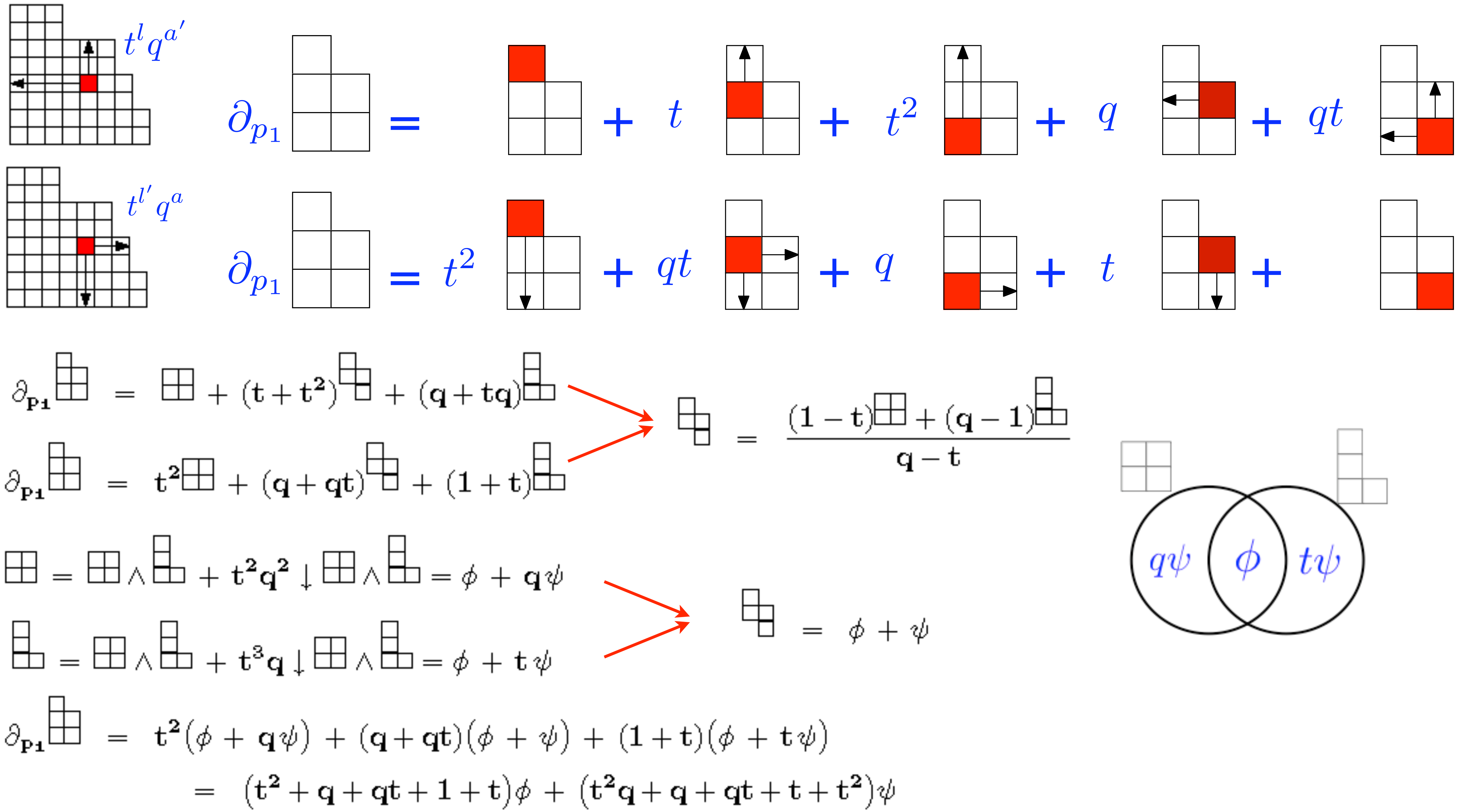

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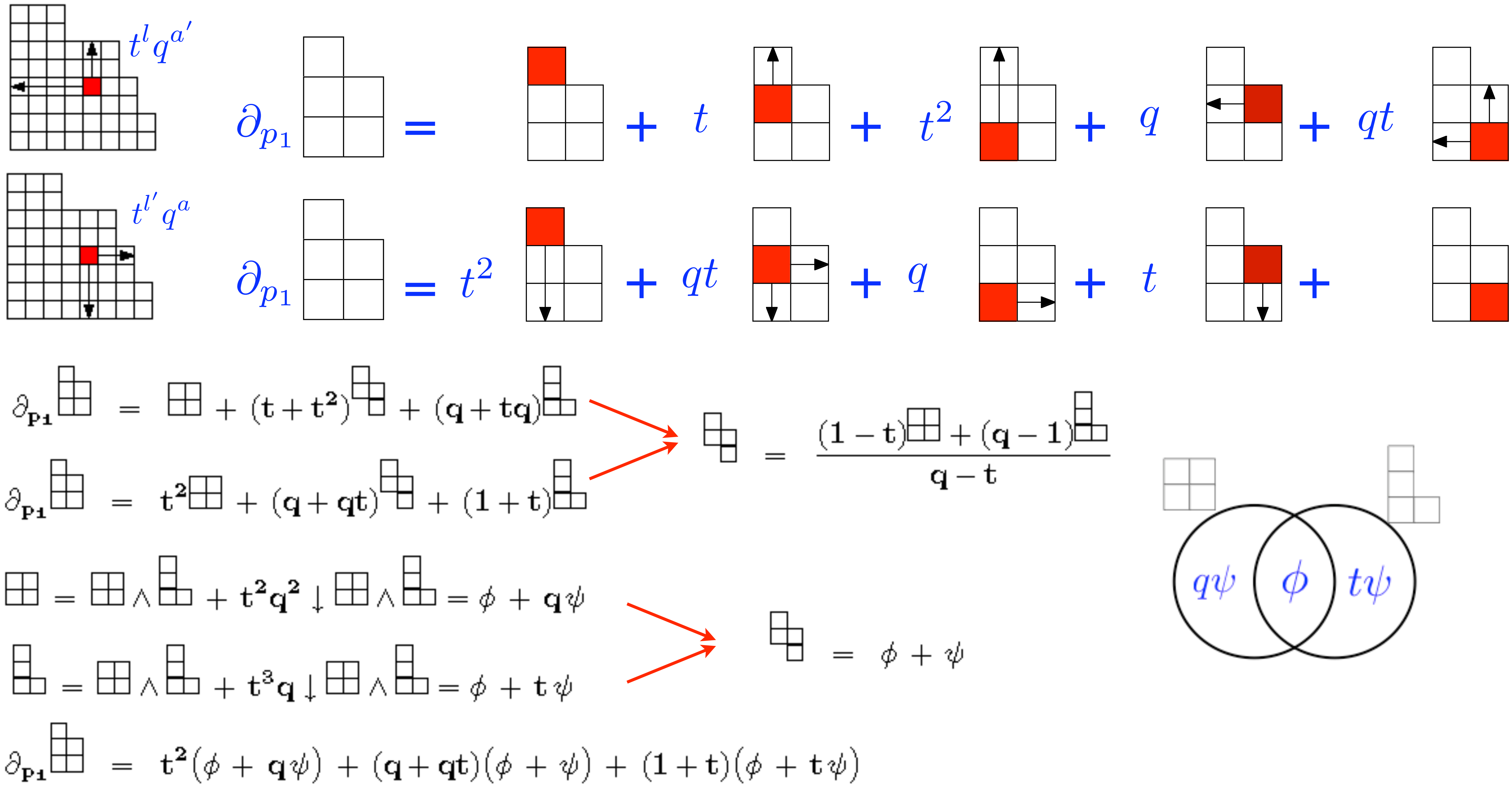

$$= \left(t^2 + q + qt + 1 + t\right)\phi + \left(t^2 q + q + qt + t + t^2\right)\psi$$

$$= \left((1+t)(1+q) + t^2\right)\phi + \left(t(1+t)(1+q) + q\right)\psi$$

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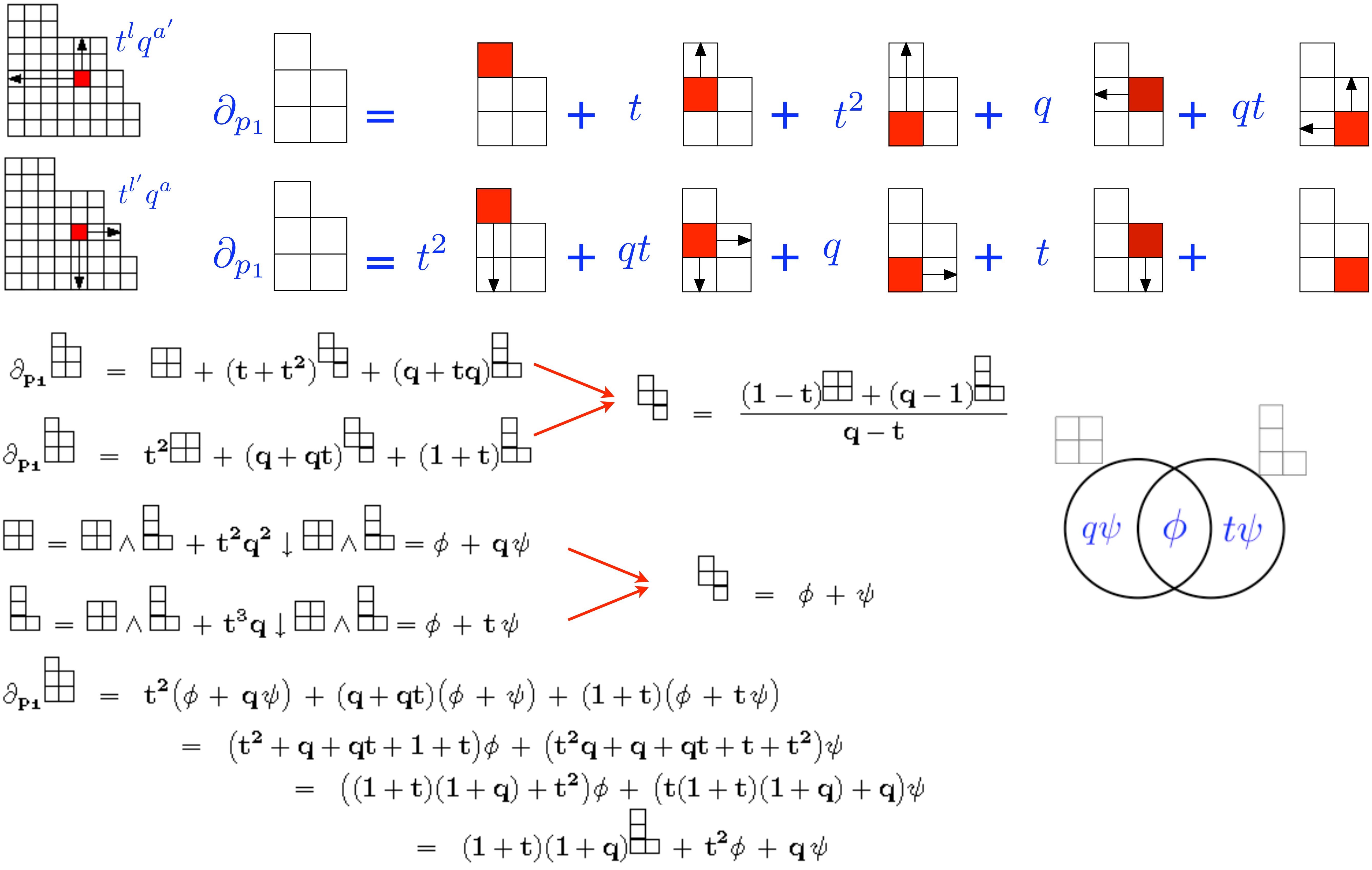

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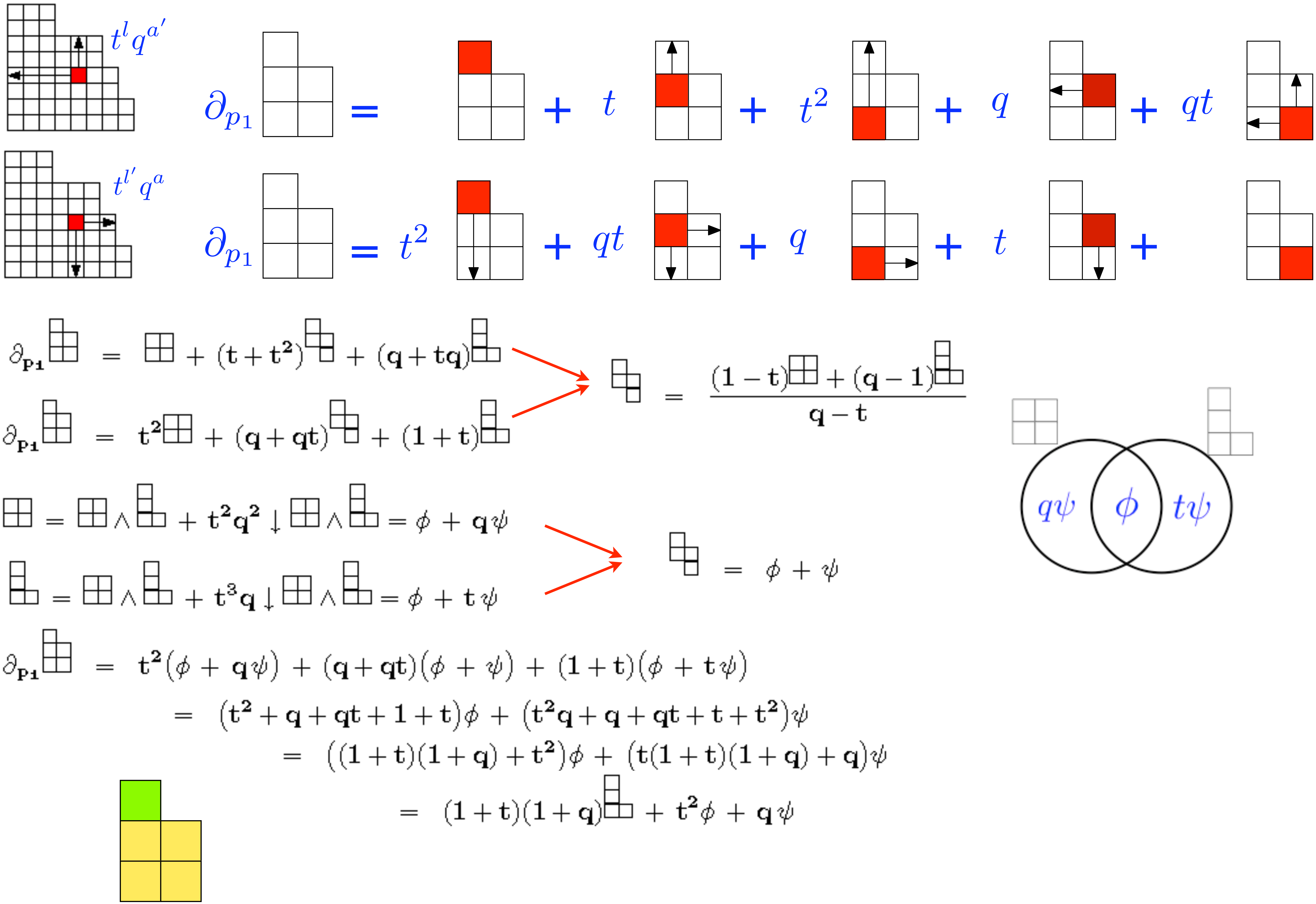

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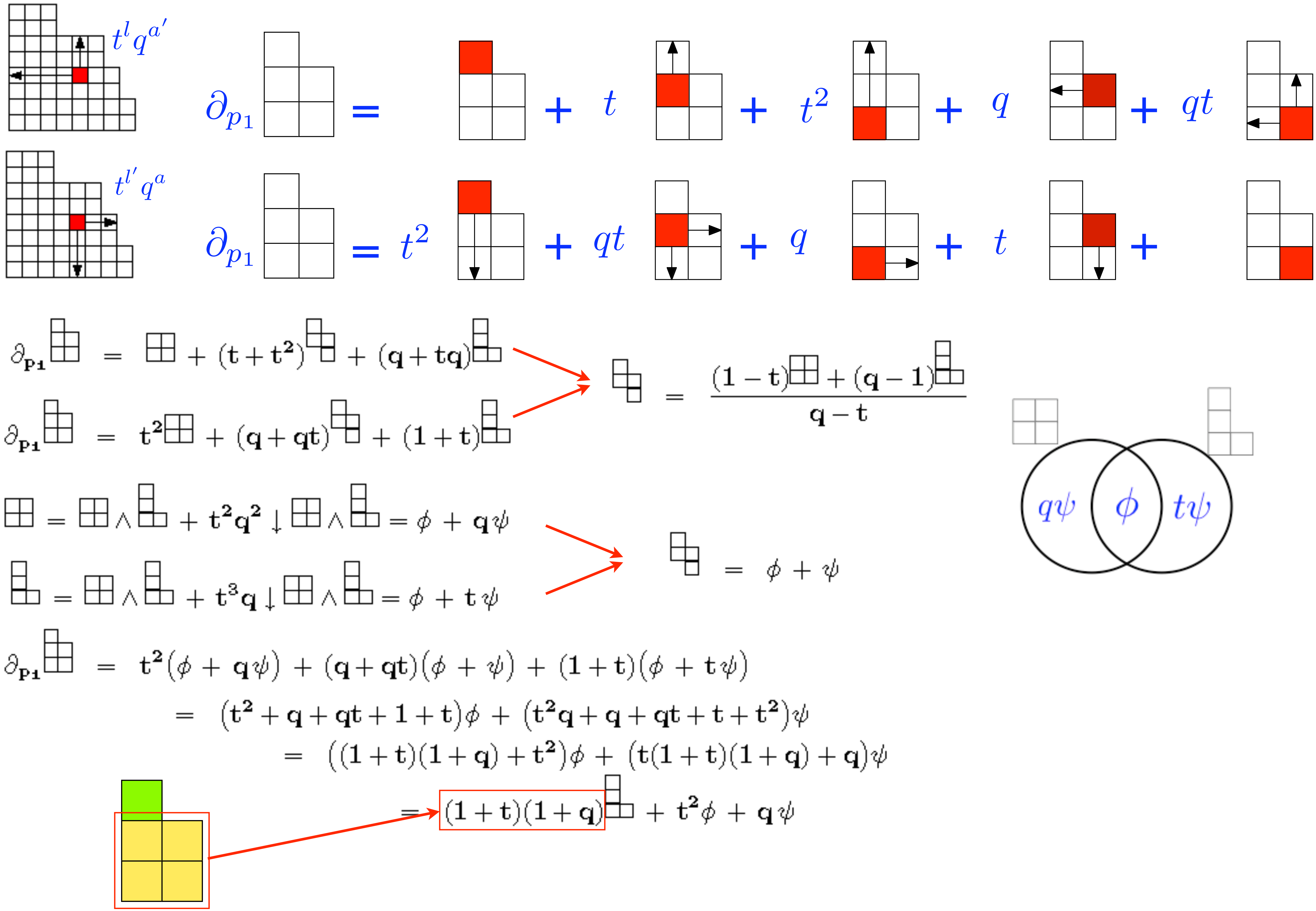

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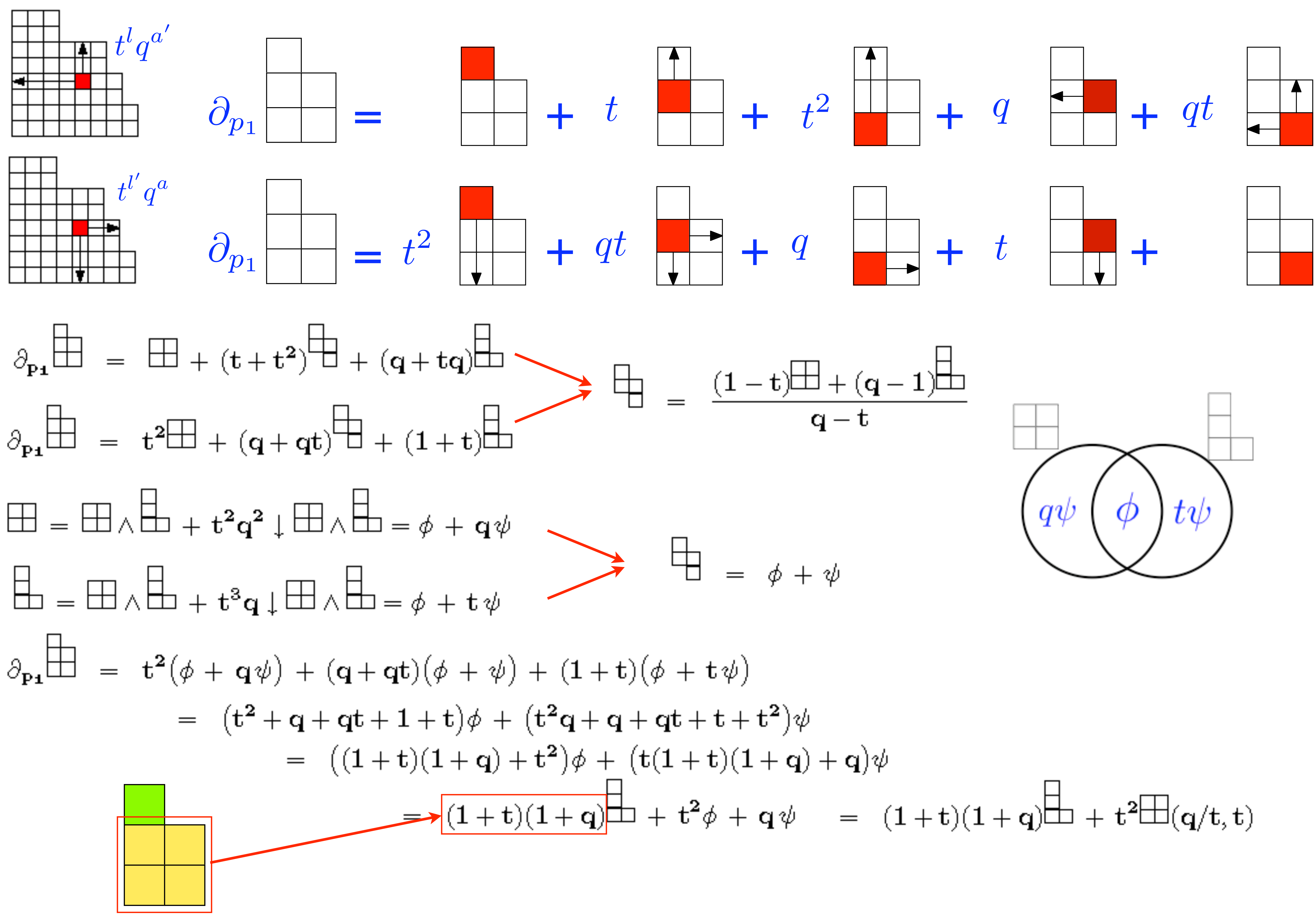

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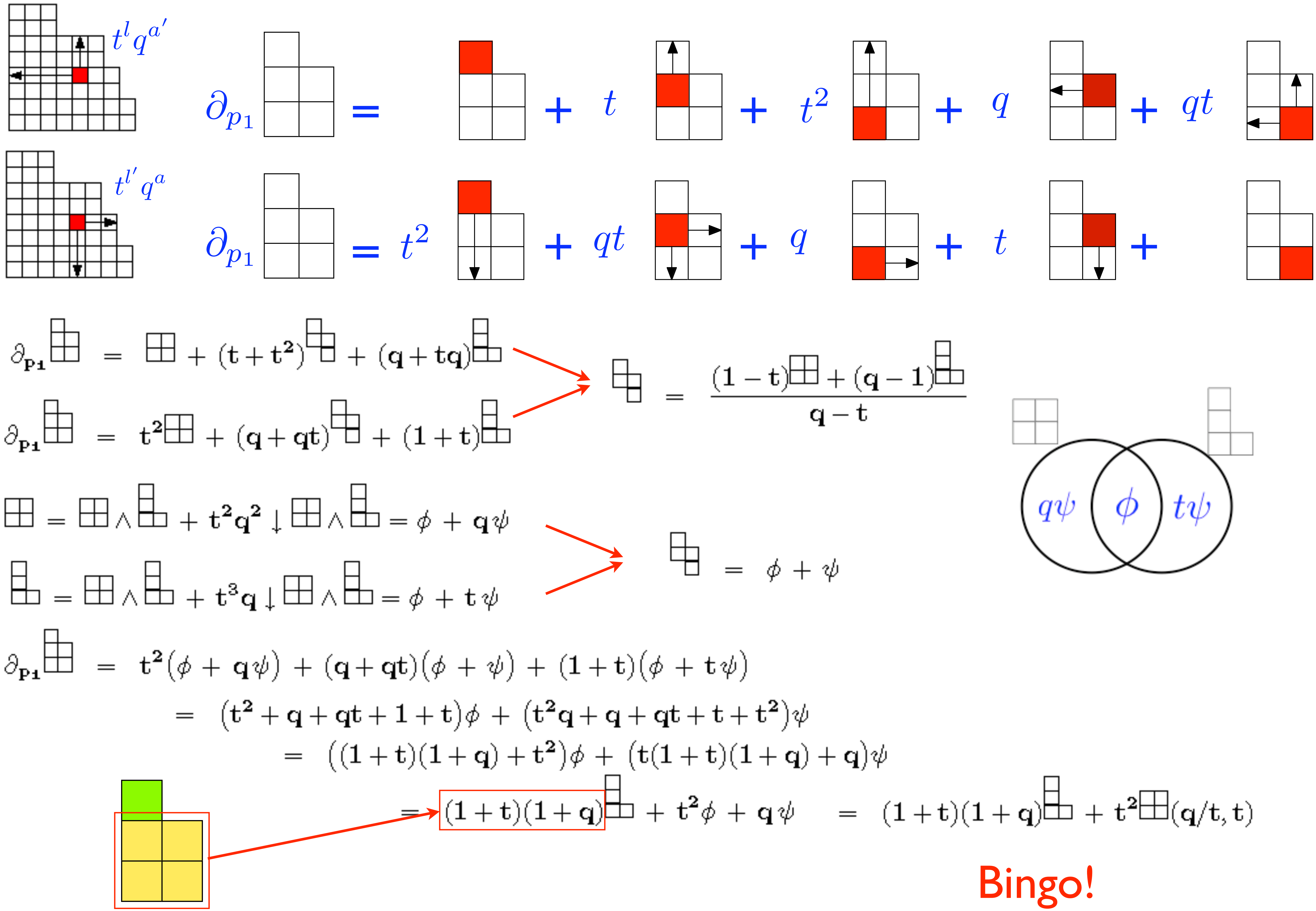
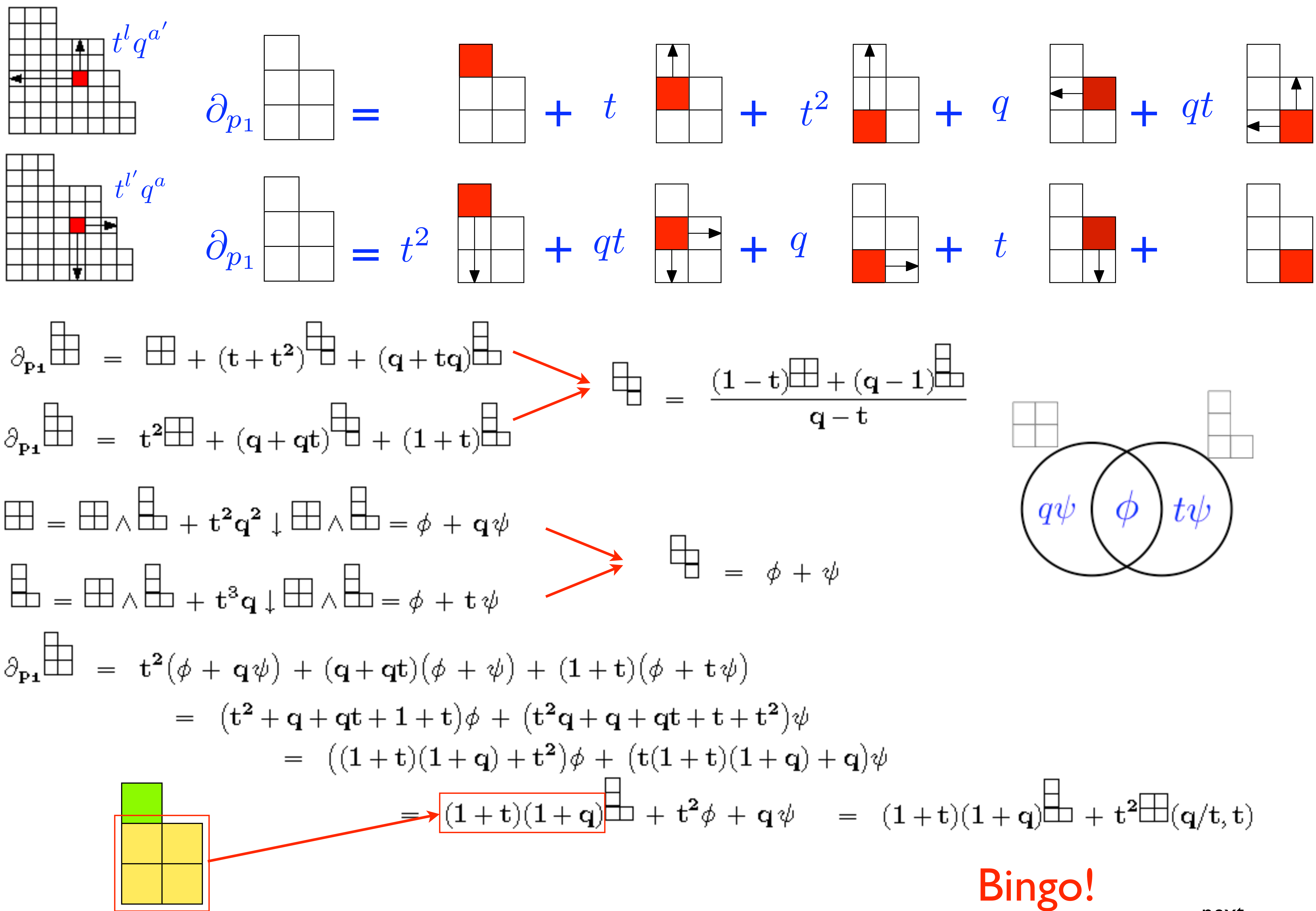


Bingo!

An example of a use of gistols



The General case

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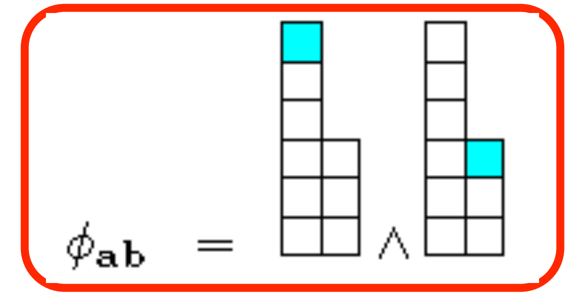
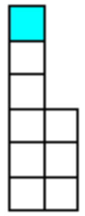
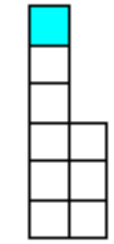
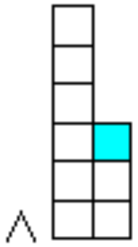
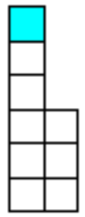


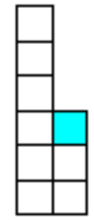
The General case

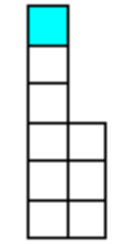
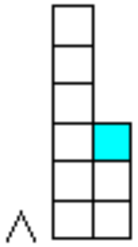
 $= \phi_{\mathbf{ab}} + \mathbf{q}\psi_{\mathbf{ab}}\ ,$

$\phi_{\mathbf{ab}} =$  $\wedge$ 

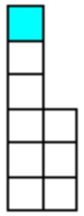
The General case

 $= \phi_{ab} + q\psi_{ab}$,

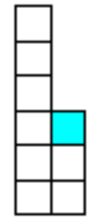
 $= \phi_{ab} + t^{a-b}\psi_{ab}$

$\phi_{ab} =$  $\wedge$ 

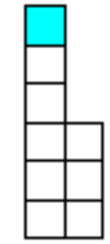
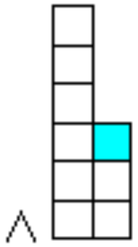
The General case

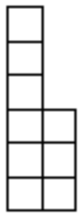
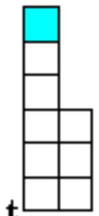
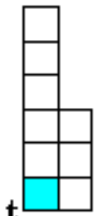


$= \phi_{ab} + q\psi_{ab}$,

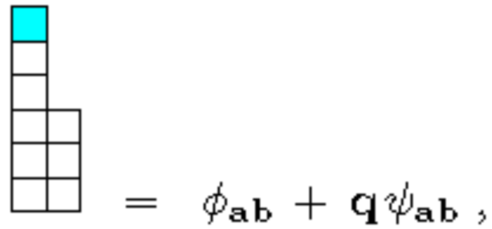


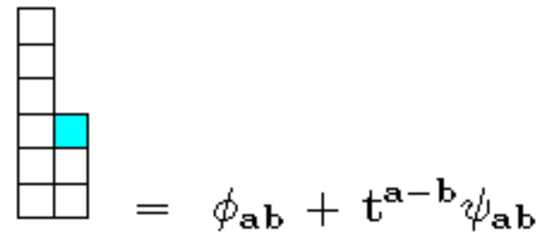
$= \phi_{ab} + t^{a-b}\psi_{ab}$

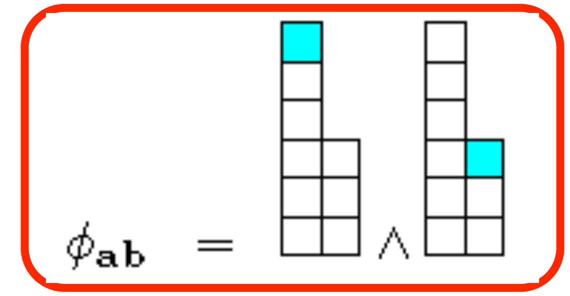
$\phi_{ab} =$  $\wedge$ 

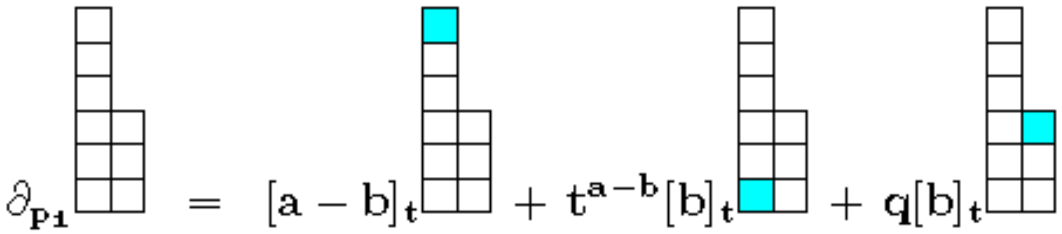
∂_{p_1}  $= [a-b]_t$  $+ t^{a-b}[b]_t$  $+ q[b]_t$ 

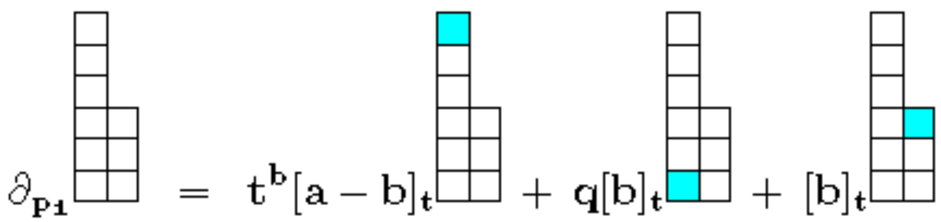
The General case



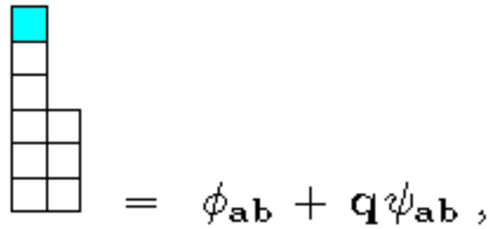


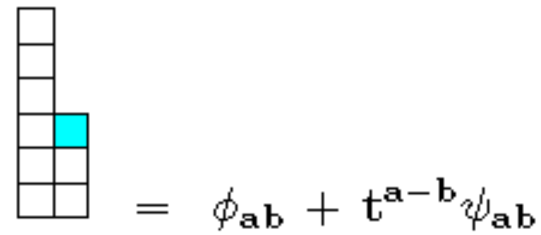


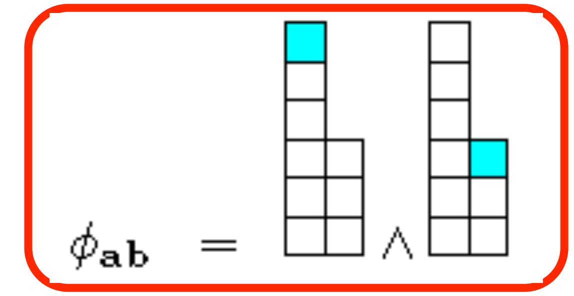


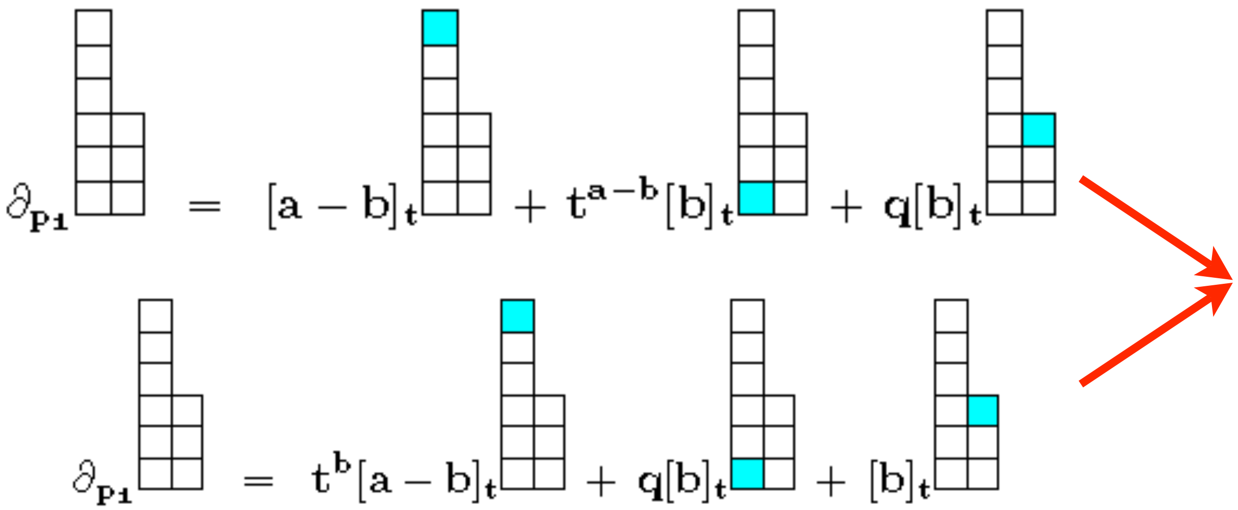


The General case

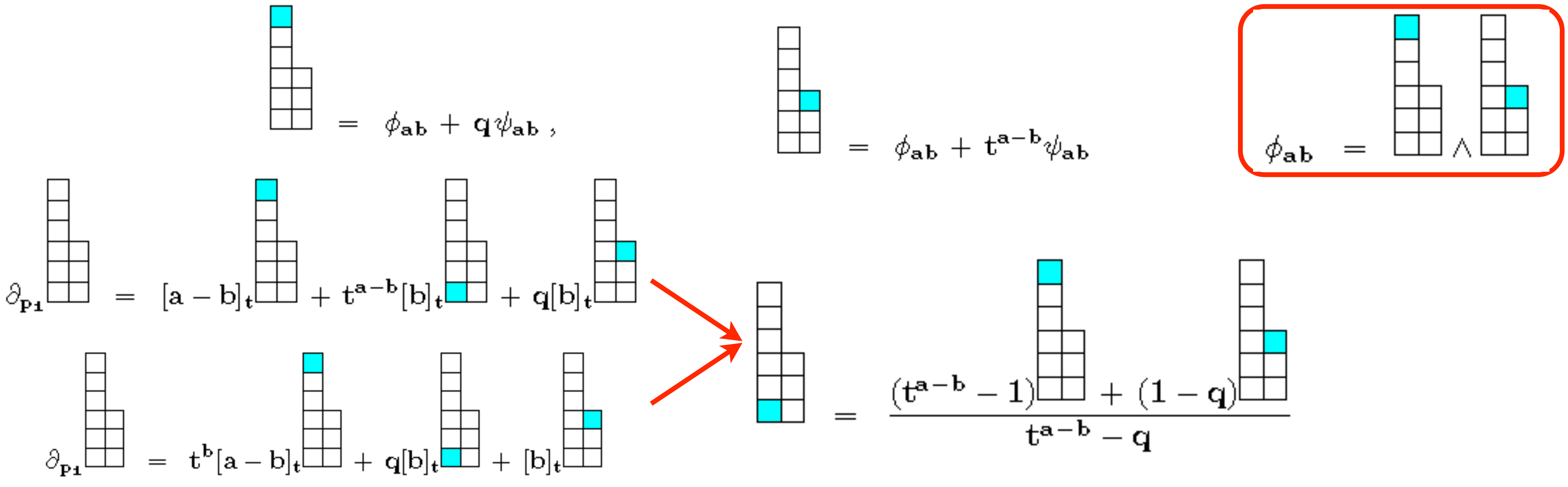




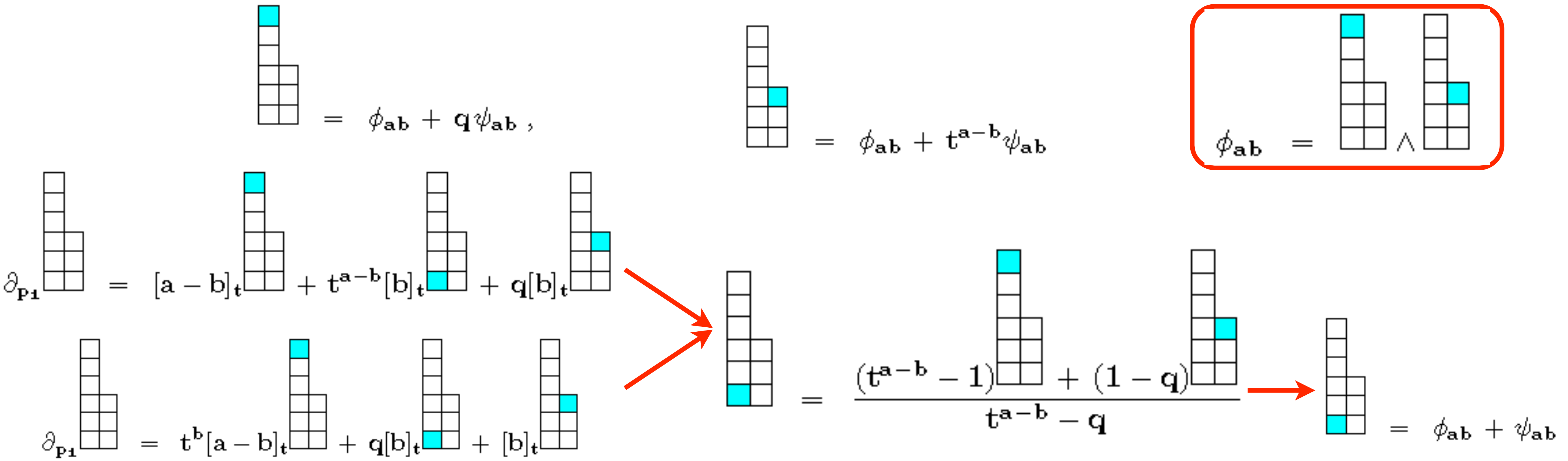




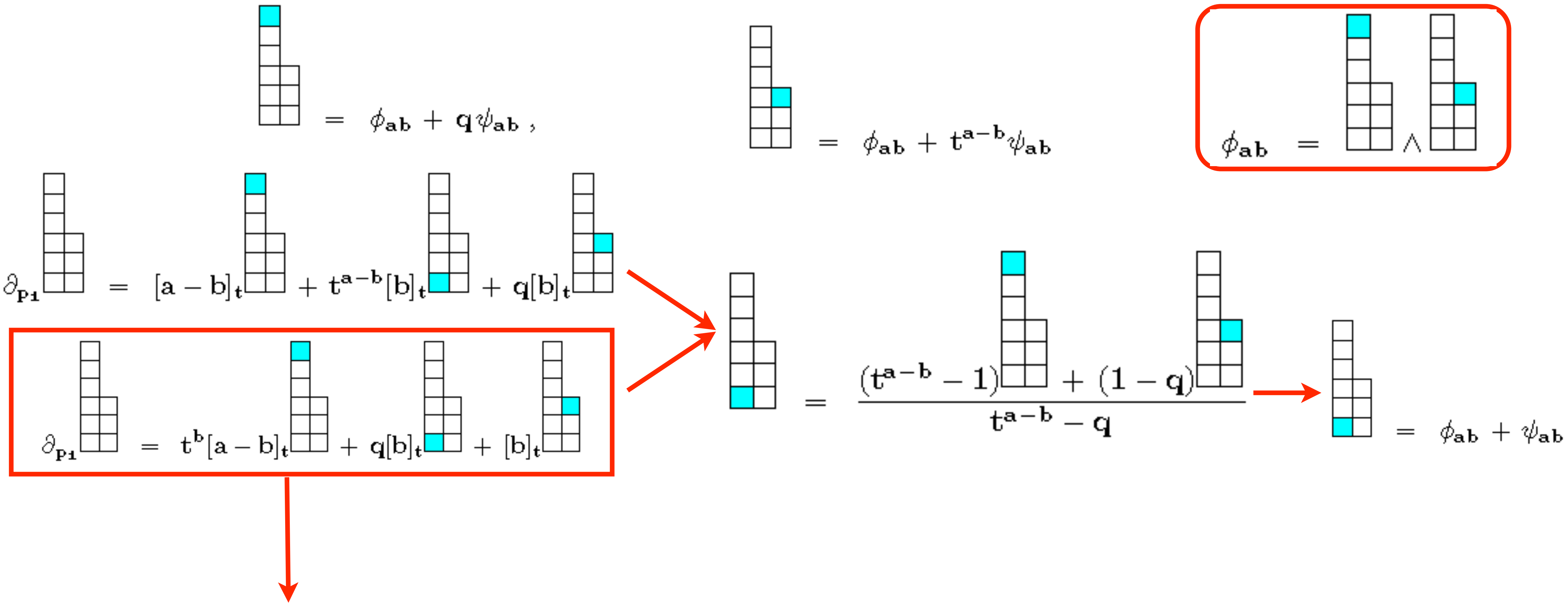
The General case



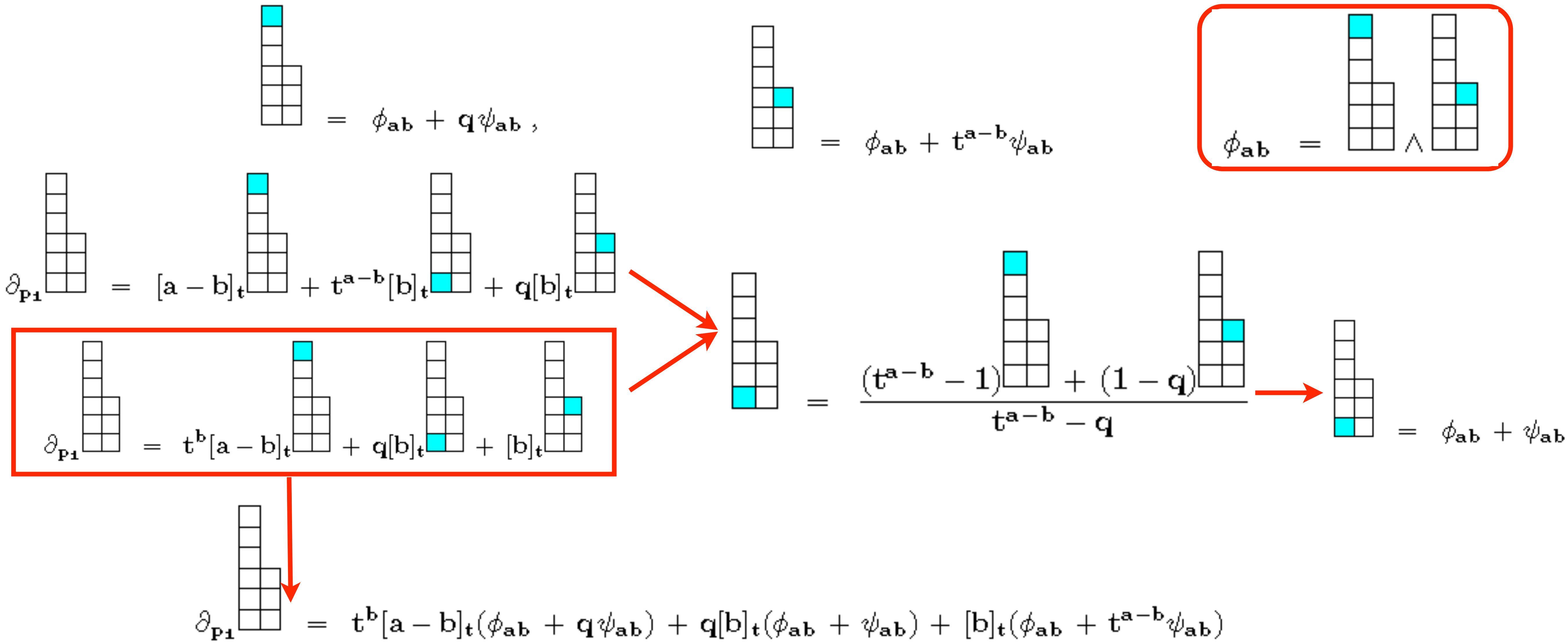
The General case



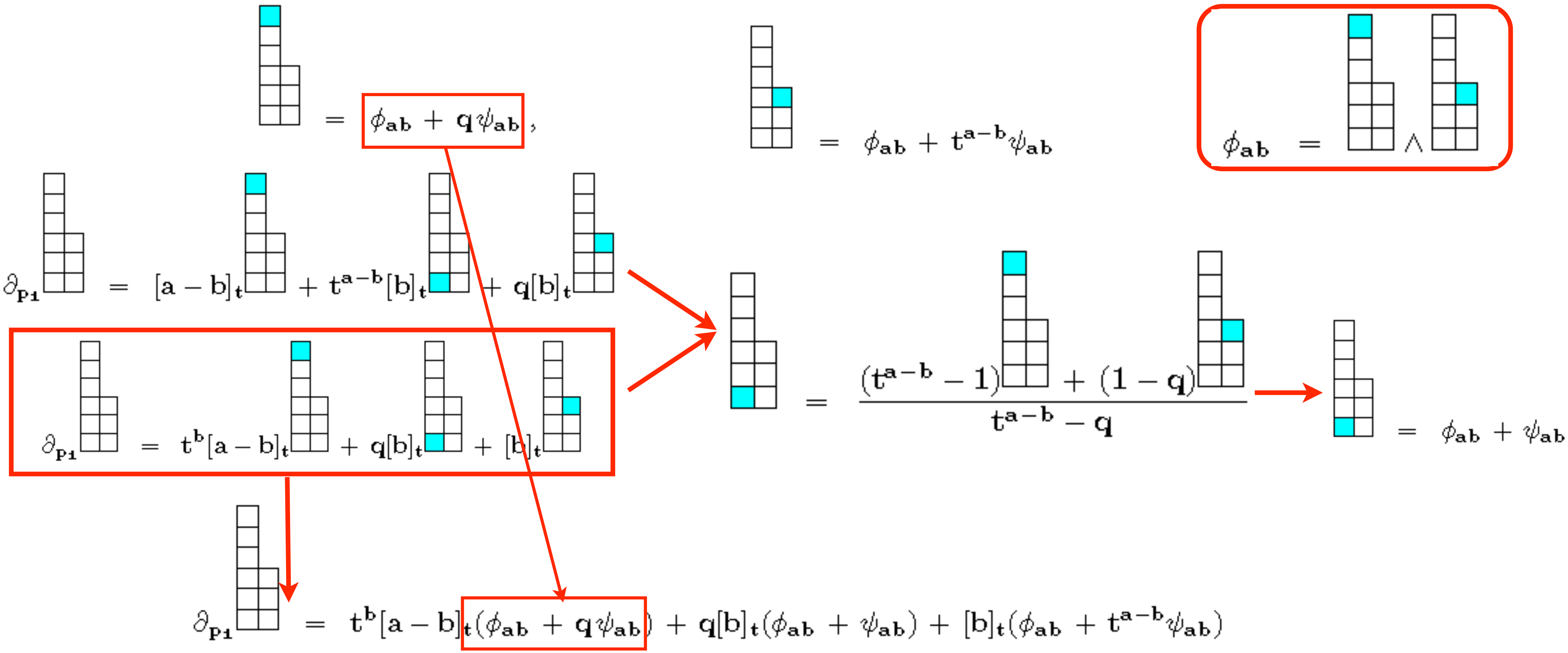
The General case



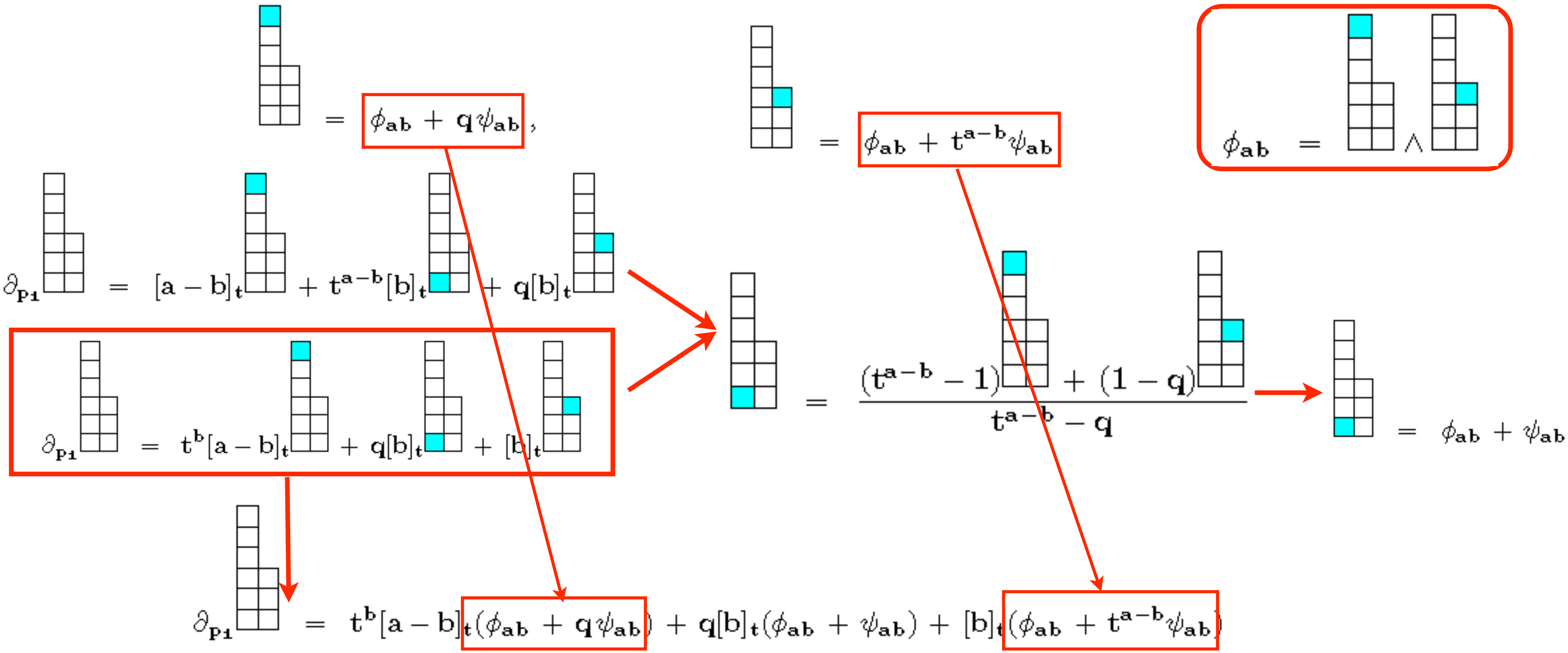
The General case



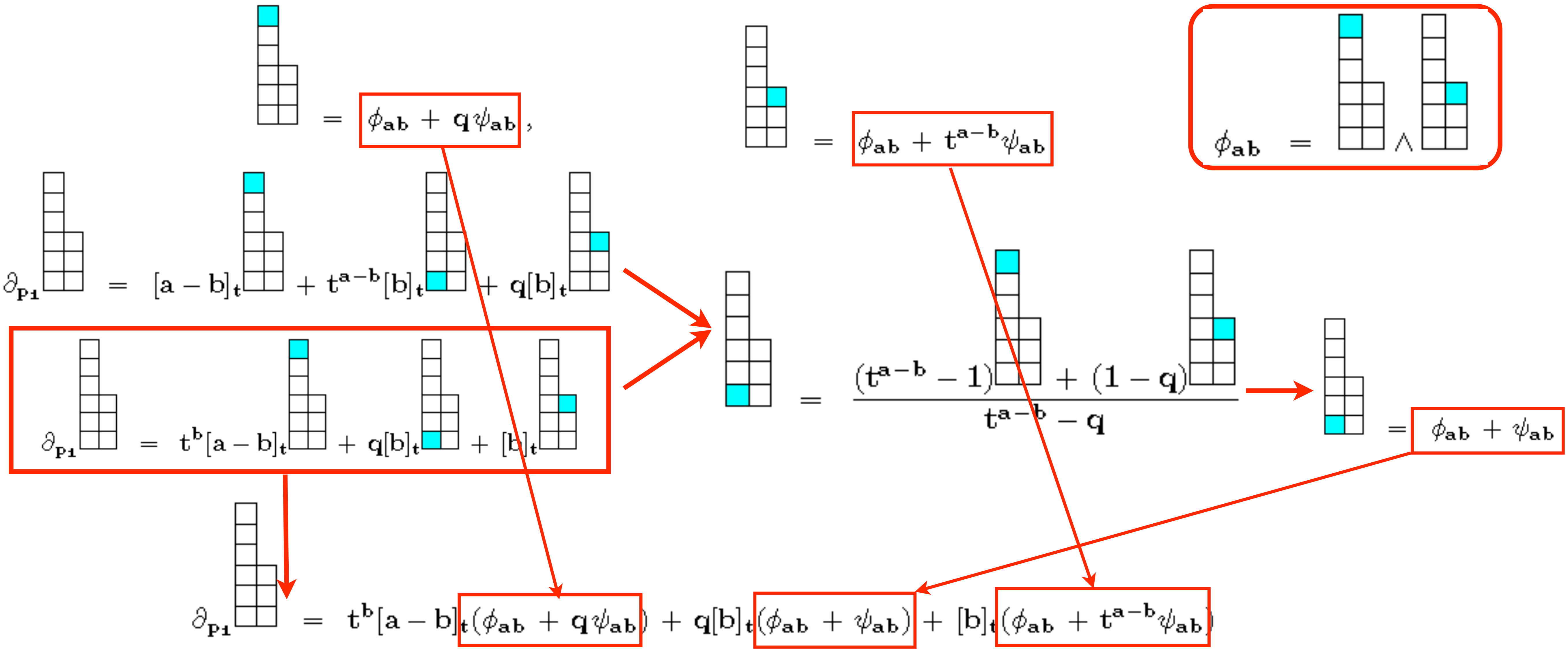
The General case



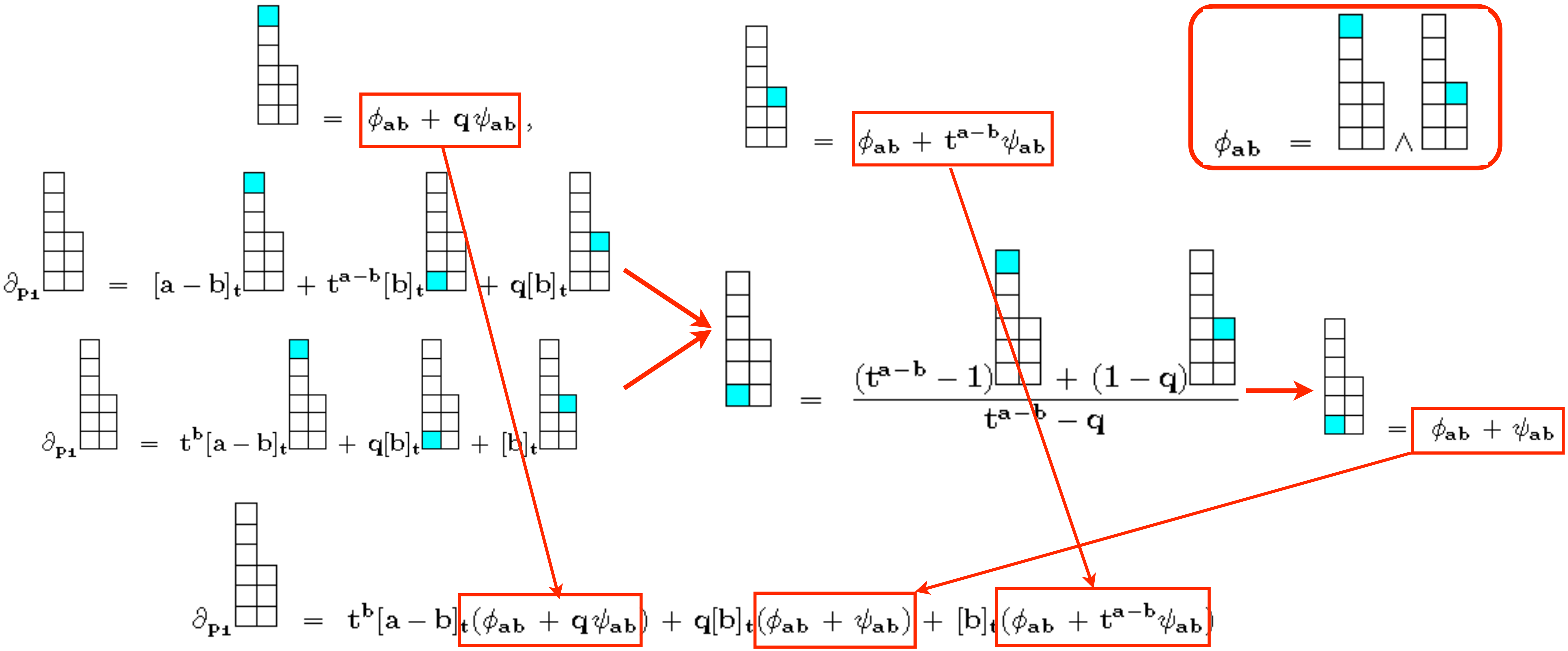
The General case



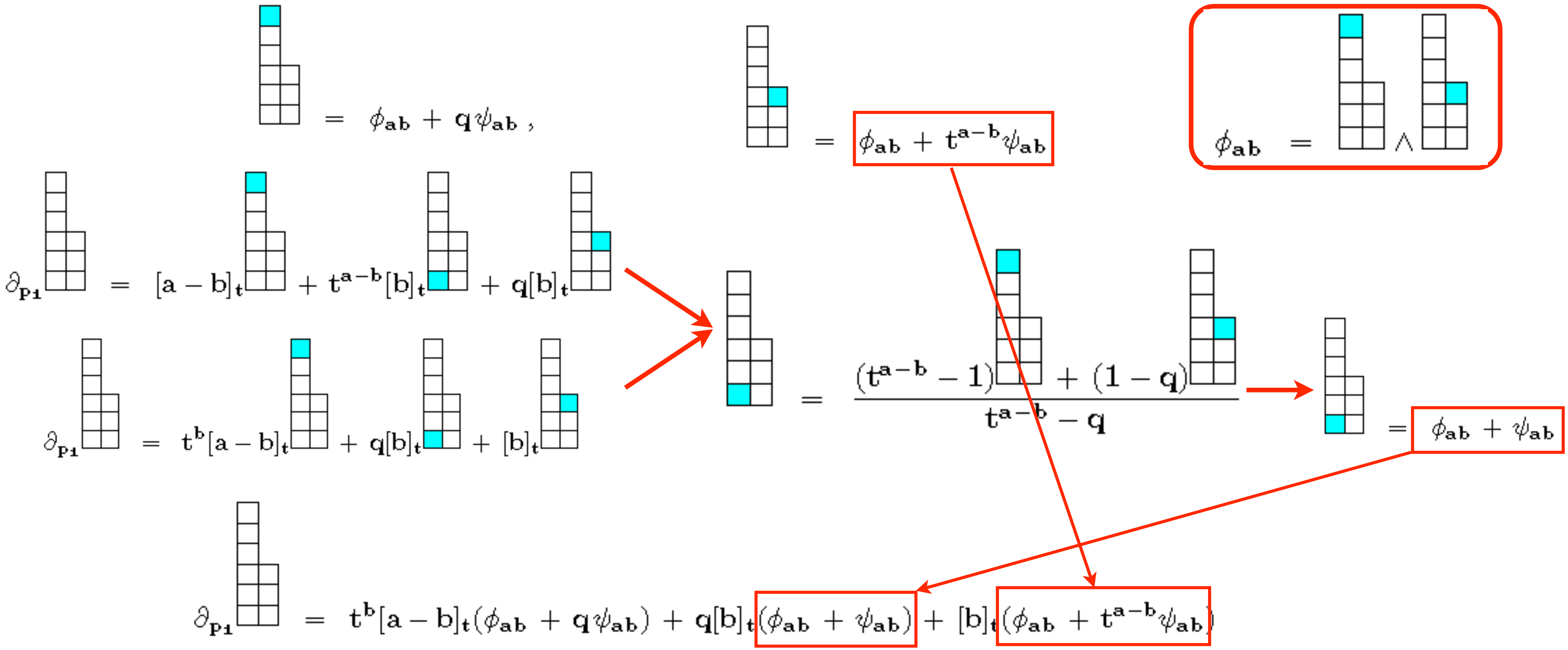
The General case



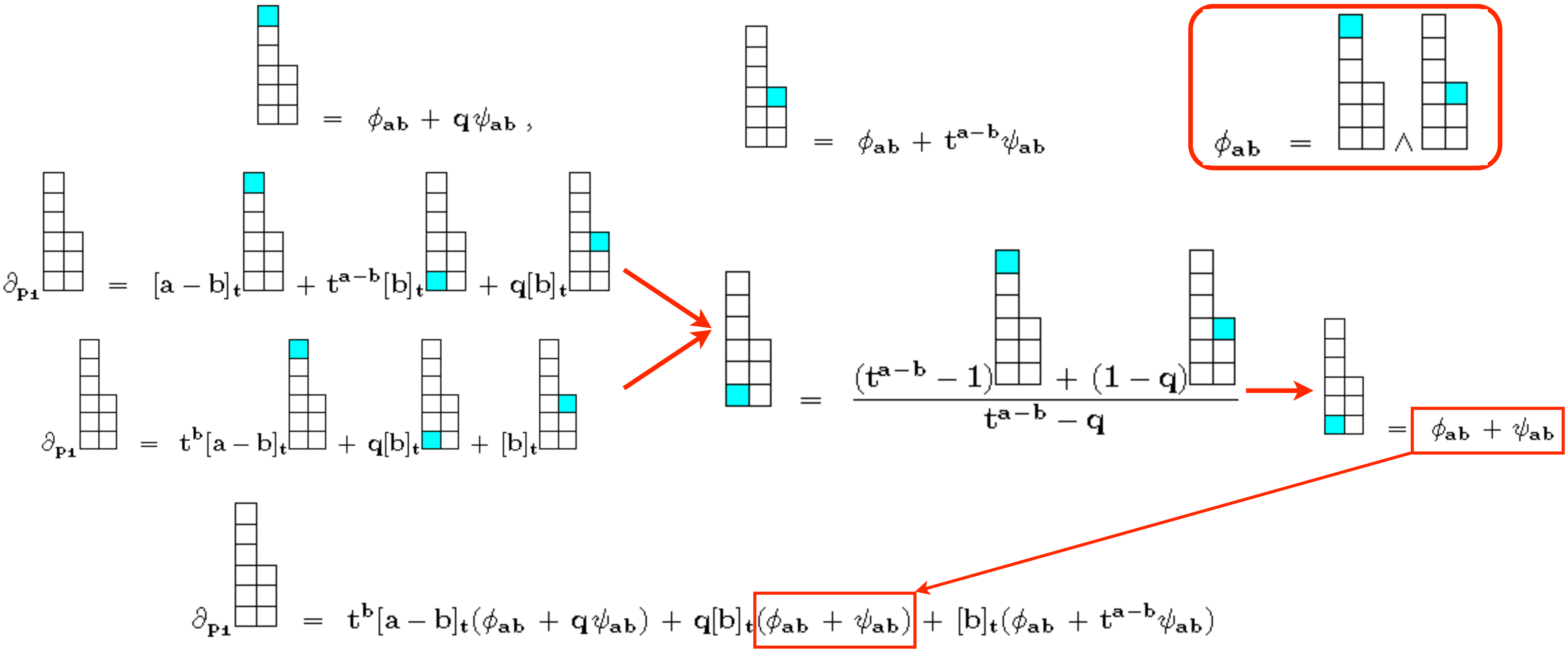
The General case



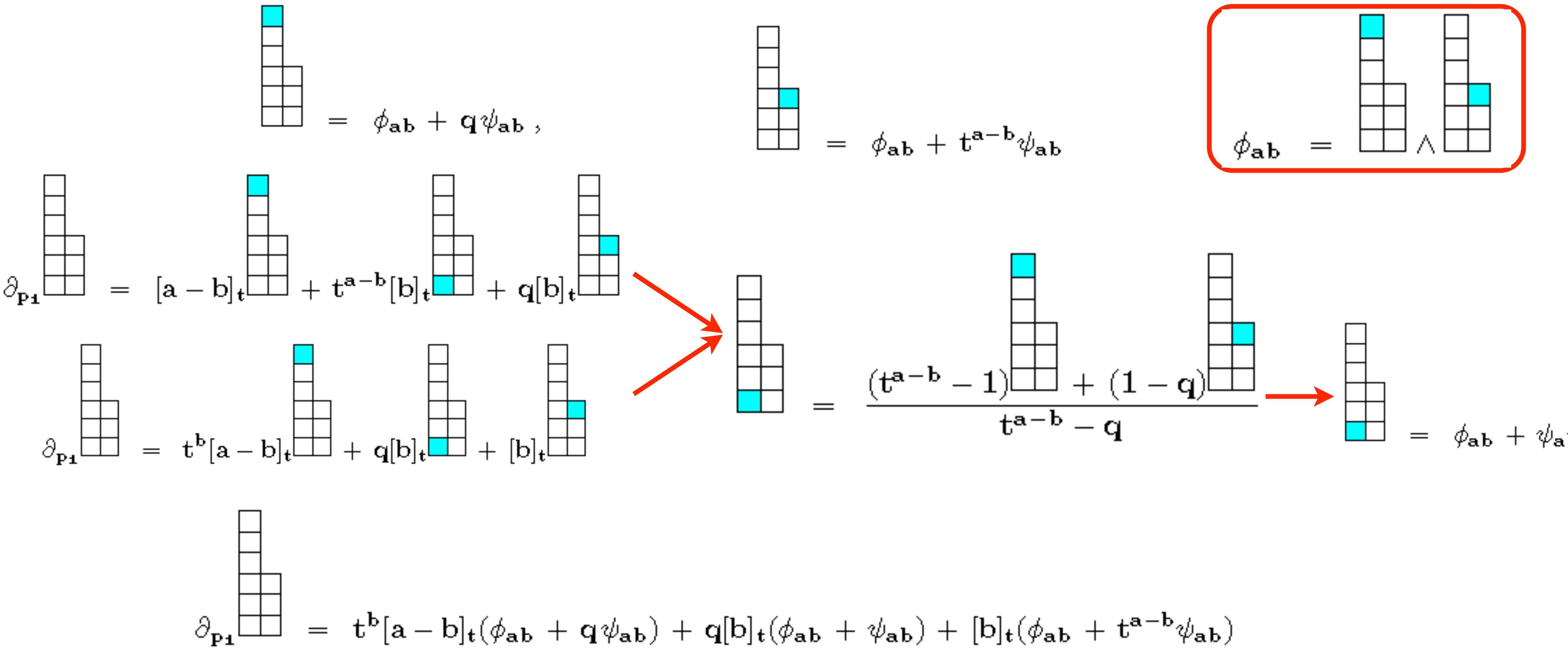
The General case



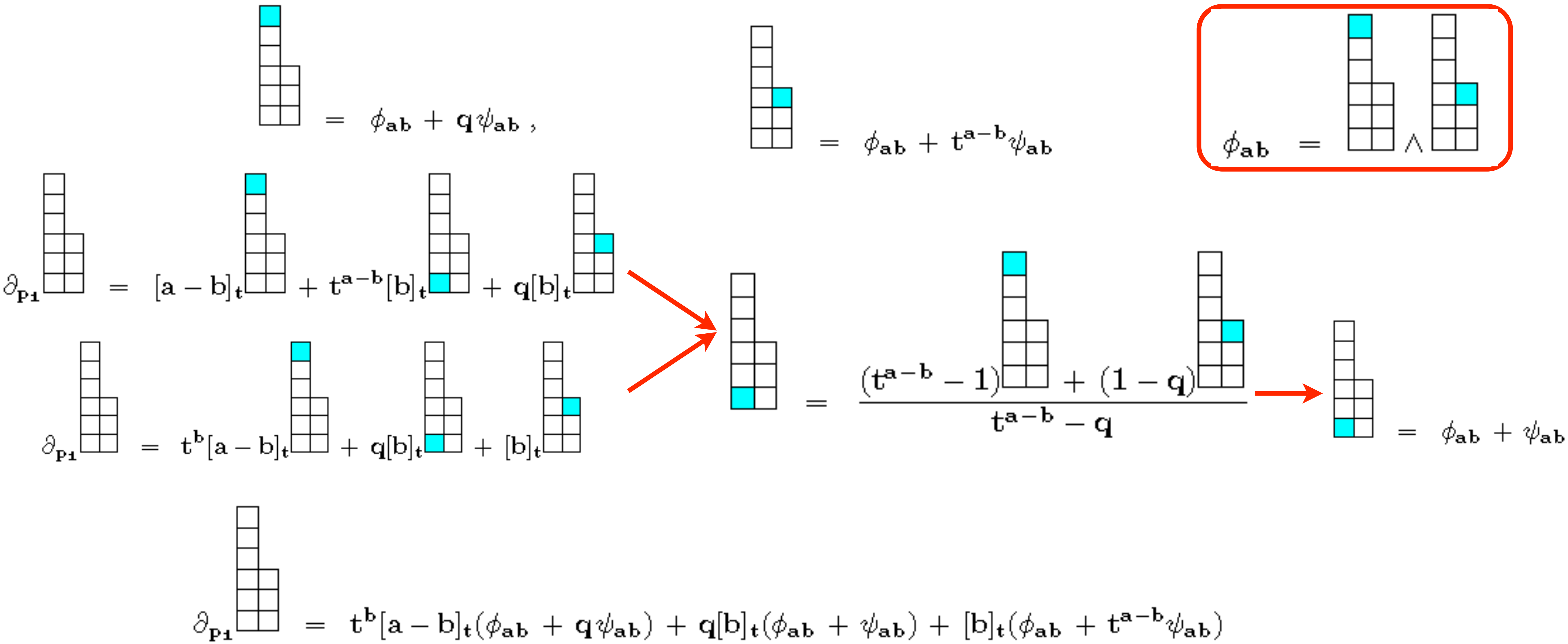
The General case



The General case

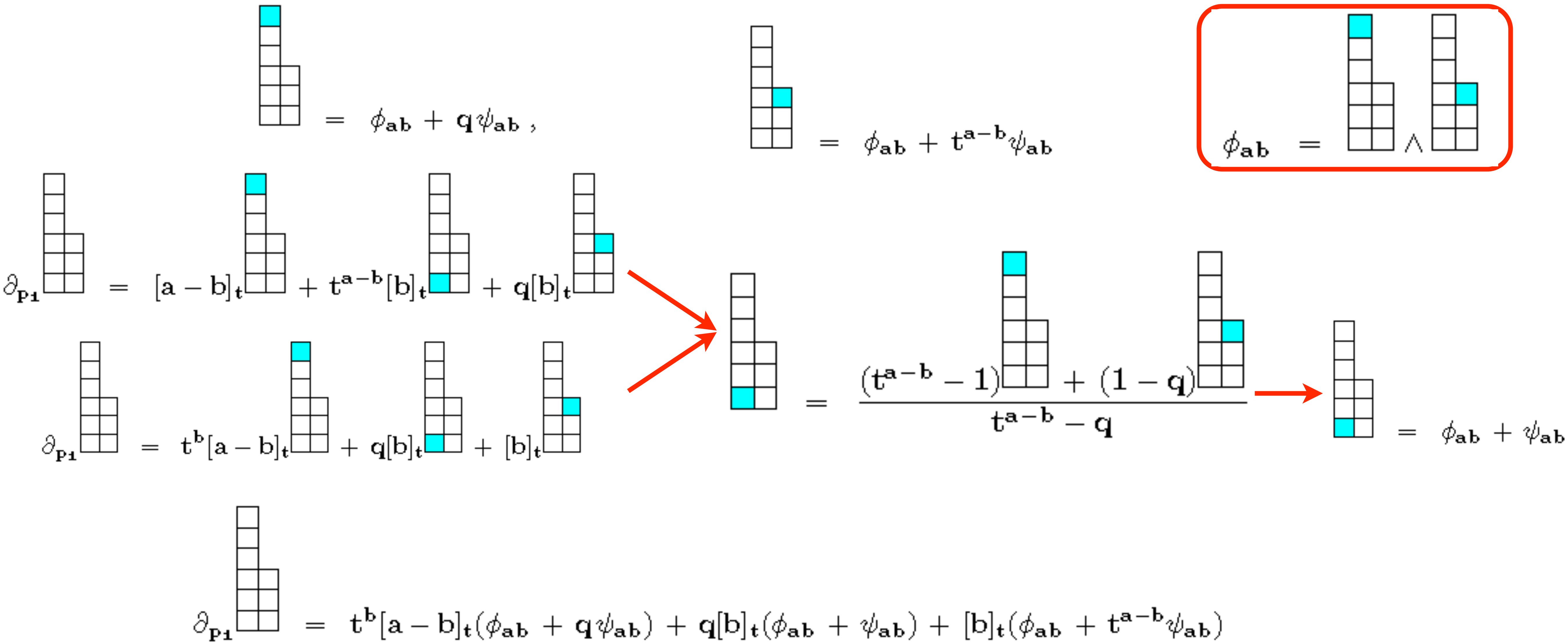


The General case



$$\psi_{\mathbf{ab}} : q(t^b[a-b]_t + [b]_t) + t^{a-b}[b]_t$$

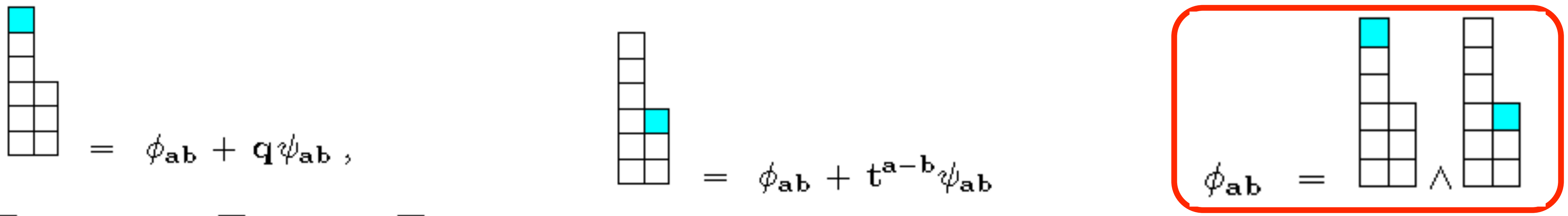
The General case

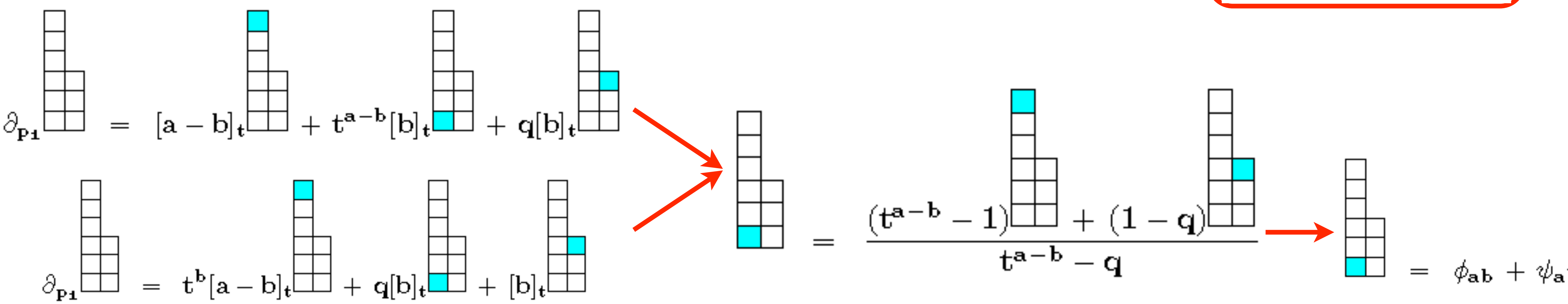


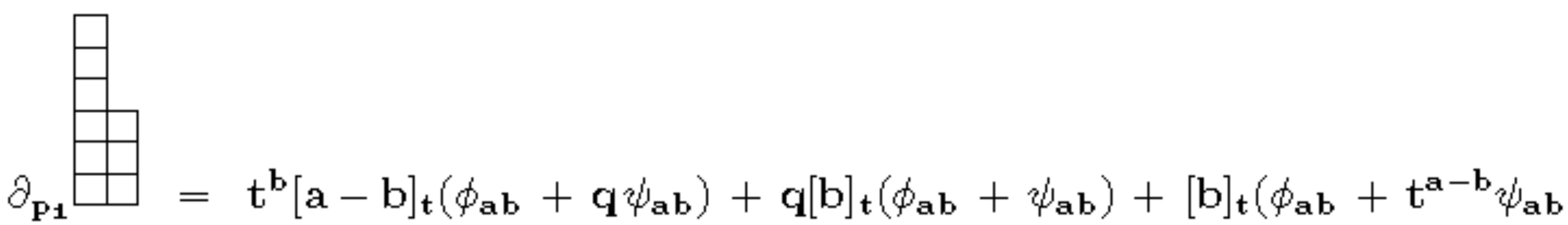
$$\psi_{ab} : q(t^b[a-b]_t + [b]_t) + t^{a-b}[b]_t$$

$$[b]_t + t^b[a-b]_t = 1 + t + \cdots + t^{a-1} = [a-b]_t + t^{a-b}[b]_t$$

The General case





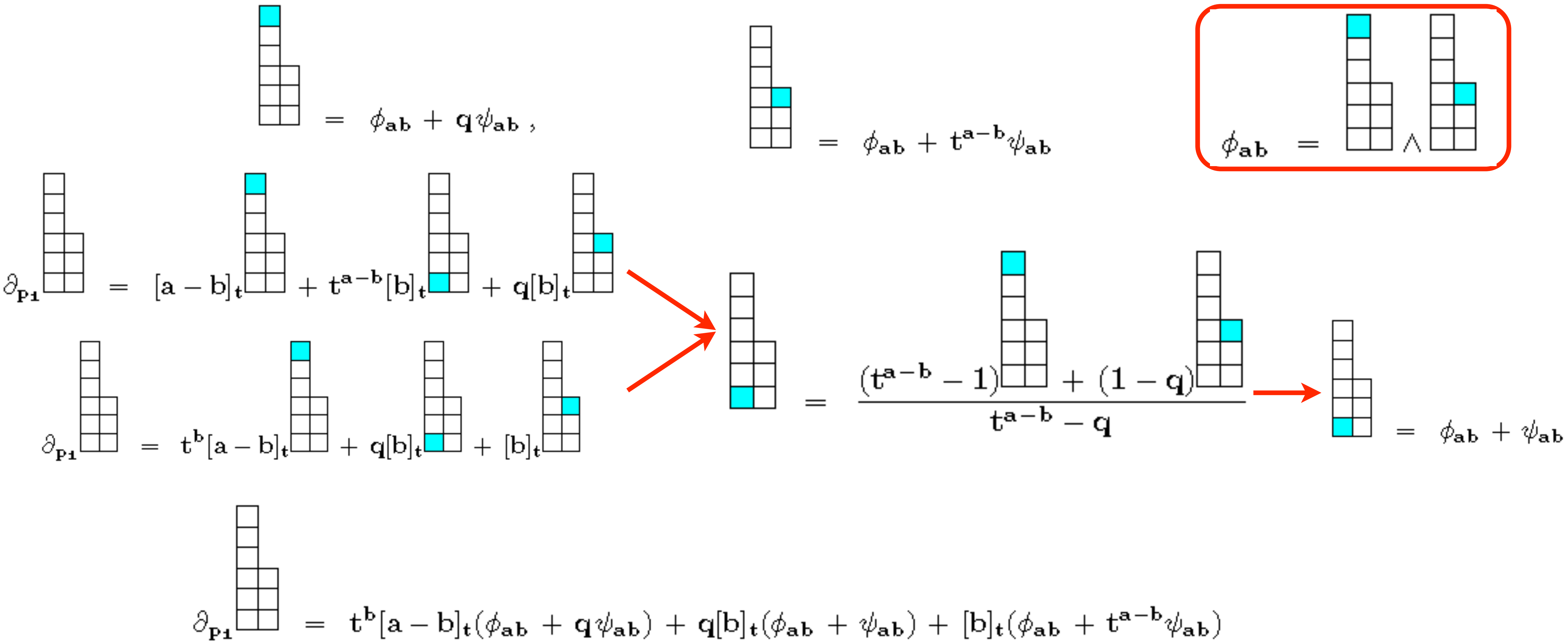


$$\psi_{ab} : q(t^b[a-b]_t + [b]_t) + t^{a-b}[b]_t$$

$$[b]_t + t^b[a-b]_t = 1 + t + \cdots + t^{a-1} = [a-b]_t + t^{a-b}[b]_t$$

$$\psi_{ab} : q([a-b]_t + t^{a-b}[b]_t) + t^{a-b}[b]_t$$

The General case



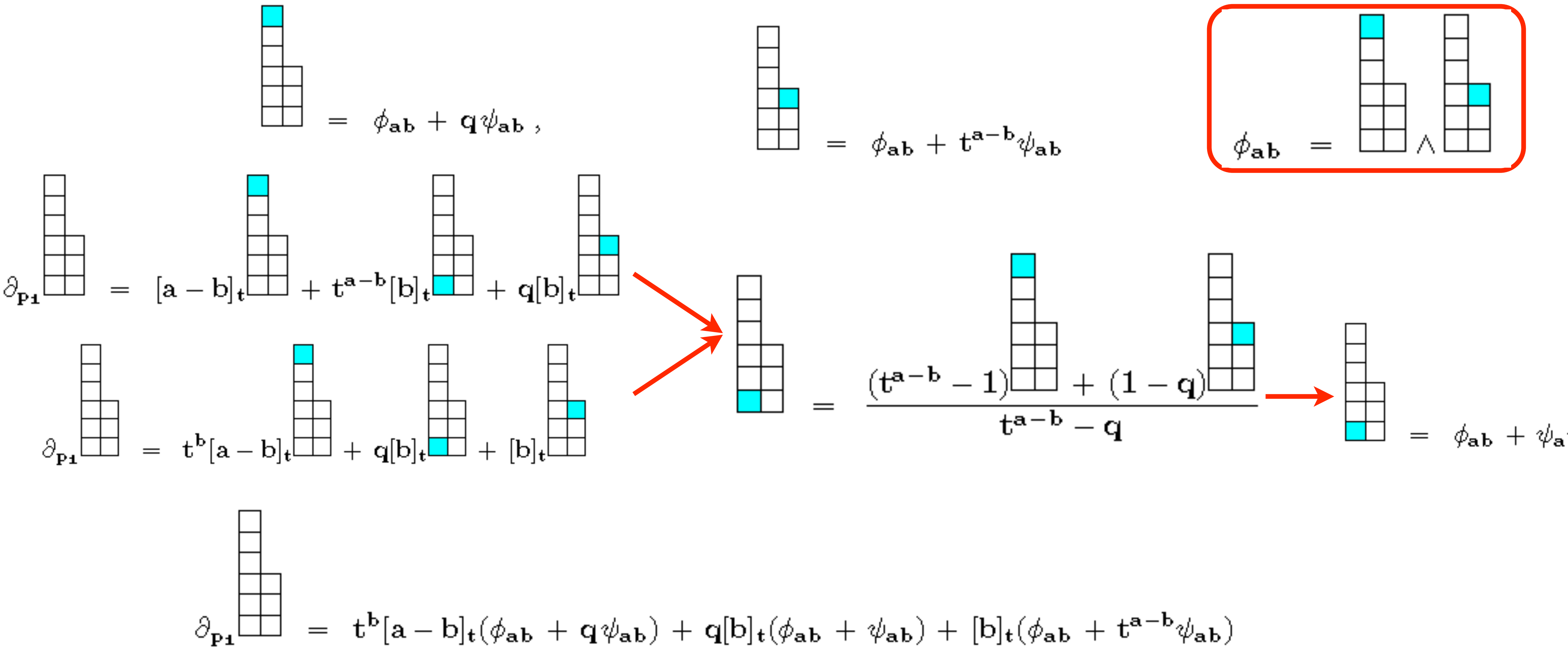
$$\psi_{\mathrm{ab}} : q(t^b[a-b]_t + [b]_t) + t^{a-b}[b]_t$$

$$[b]_t + t^b[a-b]_t = 1 + t + \cdots + t^{a-1} = [a-b]_t + t^{a-b}[b]_t$$

$$\psi_{\mathrm{ab}} : \; q([a-b]_t + t^{a-b}[b]_t) + t^{a-b}[b]_t$$

$$\psi_{\mathrm{ab}} : \; q[a-b]_t + (1+q)t^{a-b}[b]_t$$

The General case



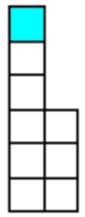
$$\psi_{ab} : q(t^b[a-b]_t + [b]_t) + t^{a-b}[b]_t$$

$$[b]_t + t^b[a-b]_t = 1 + t + \cdots + t^{a-1} = [a-b]_t + t^{a-b}[b]_t$$

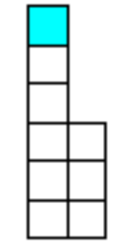
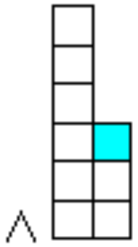
$$\psi_{ab} : q([a-b]_t + t^{a-b}[b]_t) + t^{a-b}[b]_t$$

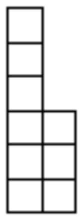
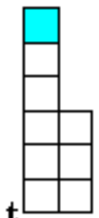
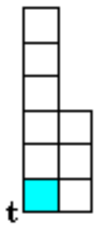
$$\phi_{ab} : t^b[a-b]_t + (1+q)[b]_t \qquad \psi_{ab} : q[a-b]_t + (1+q)t^{a-b}[b]_t$$

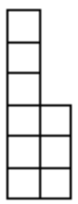
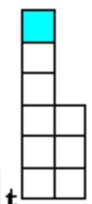
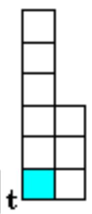
The General case

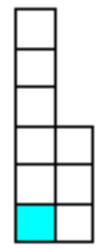
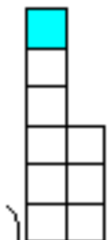
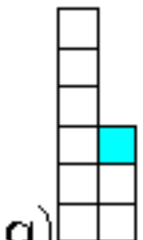
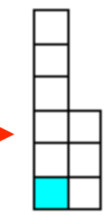
 $= \phi_{ab} + q\psi_{ab}$,

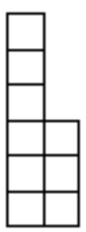
 $= \phi_{ab} + t^{a-b}\psi_{ab}$

$\phi_{ab} =$  $\wedge$ 

∂_{p_1}  $= [a-b]_t$  $+ t^{a-b}[b]_t$  $+ q[b]_t$ 

∂_{p_1}  $= t^b[a-b]_t$  $+ q[b]_t$  $+ [b]_t$ 

 $= \dfrac{(t^{a-b}-1)\,\square + (1-q)\,\square}{t^{a-b}-q}$ (diagrams: , ) $\rightarrow$  $= \phi_{ab} + \psi_{ab}$

∂_{p_1}  $= t^b[a-b]_t(\phi_{ab} + q\psi_{ab}) + q[b]_t(\phi_{ab} + \psi_{ab}) + [b]_t(\phi_{ab} + t^{a-b}\psi_{ab})$

$$\psi_{ab} : q(t^b[a-b]_t + [b]_t) + t^{a-b}[b]_t$$

$$[b]_t + t^b[a-b]_t = 1 + t + \cdots + t^{a-1} = [a-b]_t + t^{a-b}[b]_t$$

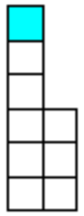
$$\psi_{ab} : q([a-b]_t + t^{a-b}[b]_t) + t^{a-b}[b]_t$$

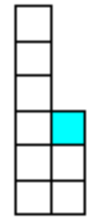
$$\phi_{ab} : t^b[a-b]_t + (1+q)[b]_t$$

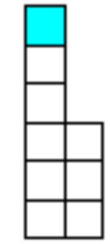
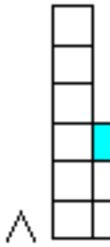
$$\psi_{ab} : q[a-b]_t + (1+q)t^{a-b}[b]_t$$

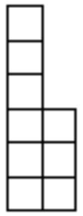
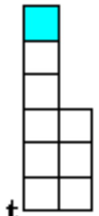
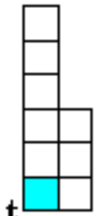
∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t(t^b\phi_{ab} + q\psi_{ab})$

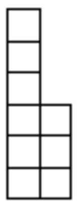
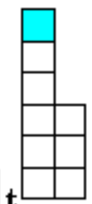
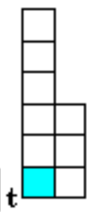
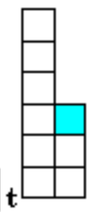
The General case

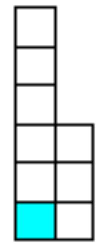
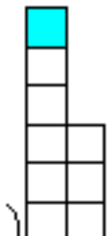
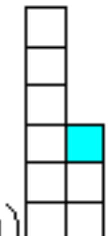
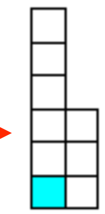
 $= \phi_{ab} + q\psi_{ab}$,

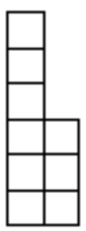
 $= \phi_{ab} + t^{a-b}\psi_{ab}$

$\phi_{ab} =$  $\wedge$ 

∂_{p_1}  $= [a-b]_t$  $+ t^{a-b}[b]_t$  $+ q[b]_t$ 

∂_{p_1}  $= t^b[a-b]_t$  $+ q[b]_t$  $+ [b]_t$ 

 $= \dfrac{(t^{a-b}-1)}{t^{a-b}-q}$  $+ \dfrac{(1-q)}{t^{a-b}-q}$  →  $= \phi_{ab} + \psi_{ab}$

∂_{p_1}  $= t^b[a-b]_t(\phi_{ab} + q\psi_{ab}) + q[b]_t(\phi_{ab} + \psi_{ab}) + [b]_t(\phi_{ab} + t^{a-b}\psi_{ab})$

$$\psi_{ab} : q(t^b[a-b]_t + [b]_t) + t^{a-b}[b]_t$$

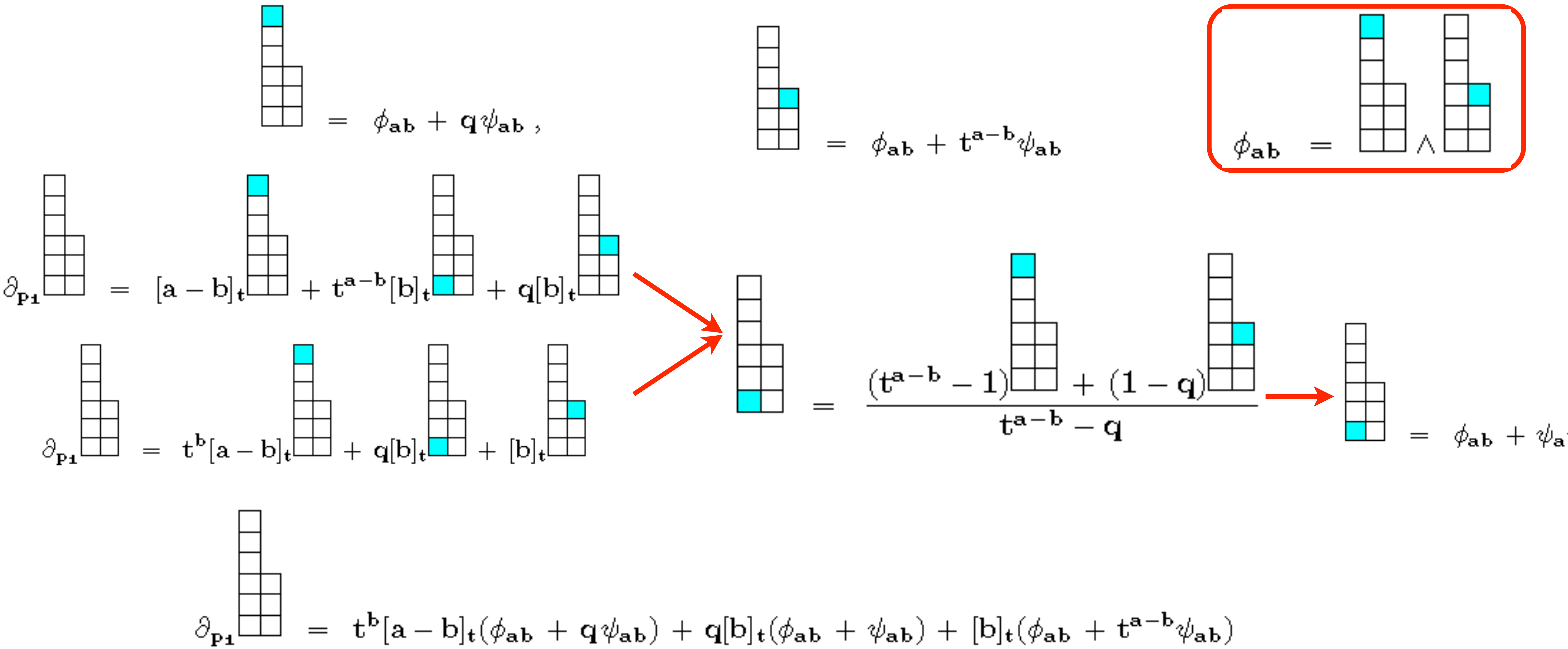
$$[b]_t + t^b[a-b]_t = 1 + t + \cdots + t^{a-1} = [a-b]_t + t^{a-b}[b]_t$$

$$\psi_{ab} : q([a-b]_t + t^{a-b}[b]_t) + t^{a-b}[b]_t$$

$$\phi_{ab} : t^b[a-b]_t + (1+q)[b]_t \qquad \psi_{ab} : q[a-b]_t + (1+q)t^{a-b}[b]_t$$

∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t(t^b\phi_{ab} + q\psi_{ab})$

The General case

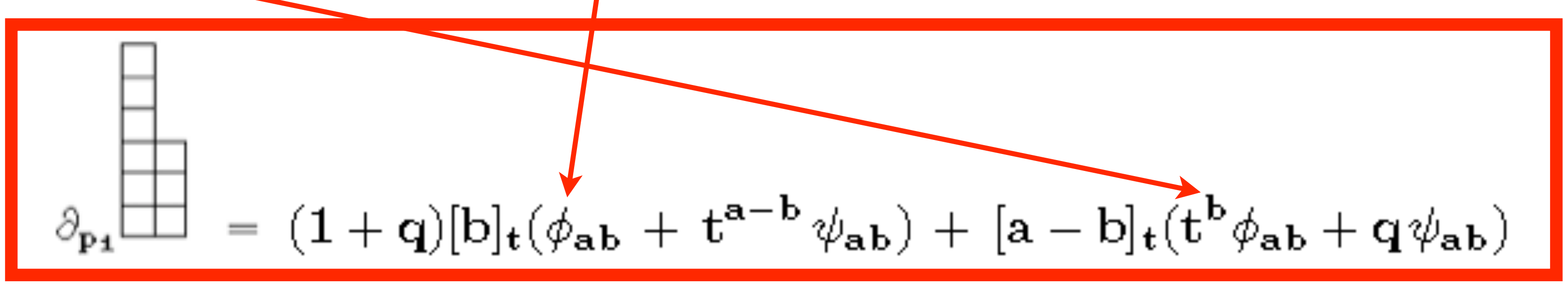


$$\psi_{\mathrm{ab}} : q(t^b[a-b]_t + [b]_t) + t^{a-b}[b]_t$$

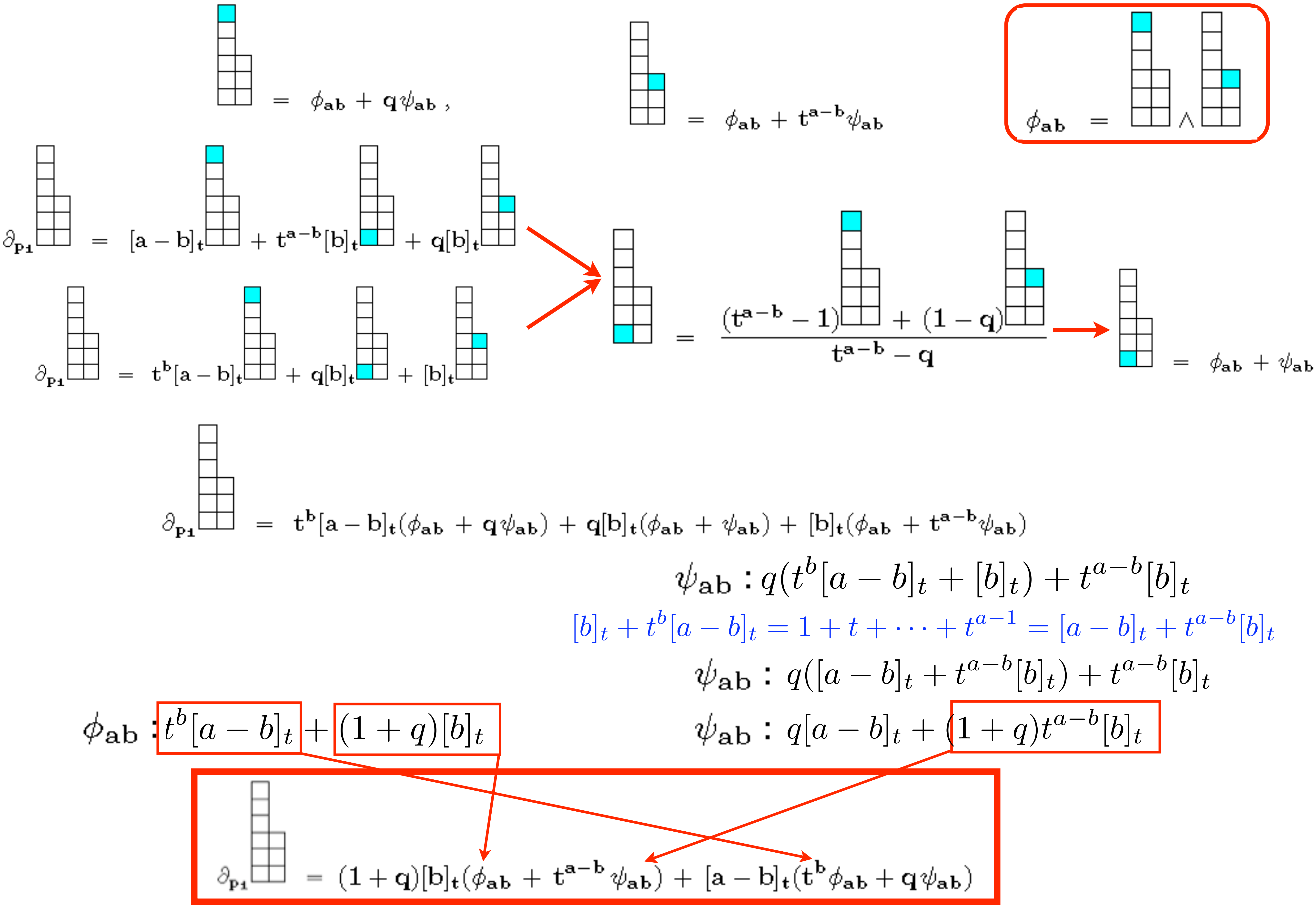
$$[b]_t + t^b[a-b]_t = 1 + t + \cdots + t^{a-1} = [a-b]_t + t^{a-b}[b]_t$$

$$\psi_{\mathrm{ab}} : q([a-b]_t + t^{a-b}[b]_t) + t^{a-b}[b]_t$$

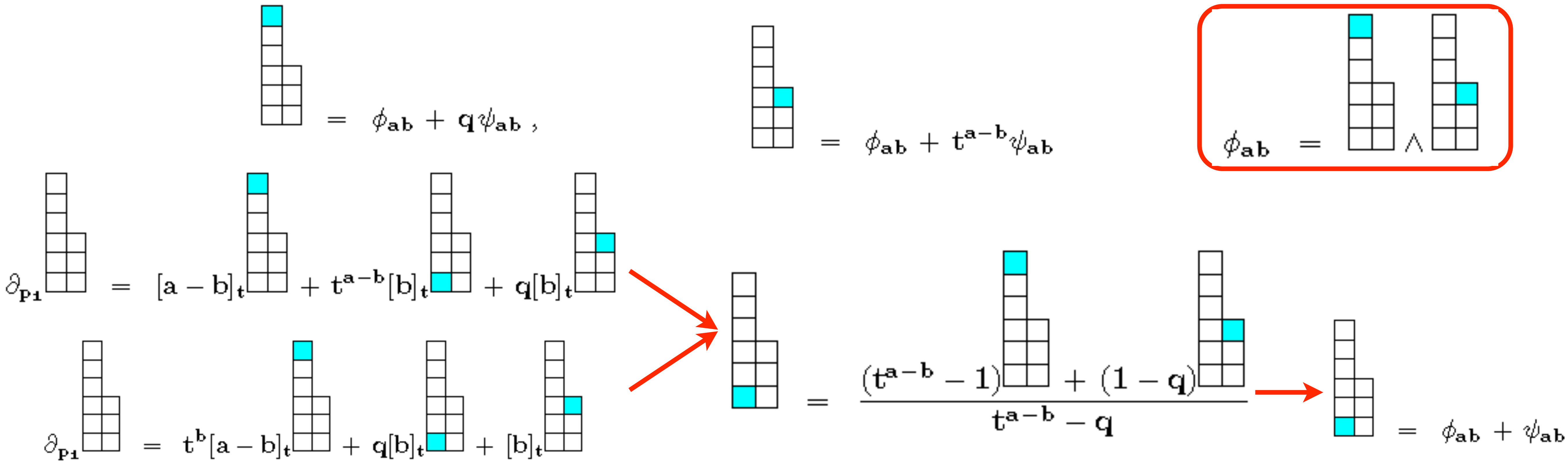
$$\phi_{\mathrm{ab}} : t^b[a-b]_t + (1+q)[b]_t \qquad \psi_{\mathrm{ab}} : q[a-b]_t + (1+q)t^{a-b}[b]_t$$

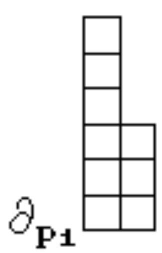


The General case



The General case





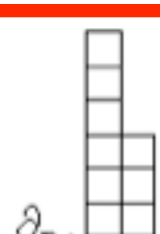
$$\partial_{p_1} = t^b[a-b]_t(\phi_{ab} + q\psi_{ab}) + q[b]_t(\phi_{ab} + \psi_{ab}) + [b]_t(\phi_{ab} + t^{a-b}\psi_{ab})$$

$$\psi_{ab} : q(t^b[a-b]_t + [b]_t) + t^{a-b}[b]_t$$

$$[b]_t + t^b[a-b]_t = 1 + t + \cdots + t^{a-1} = [a-b]_t + t^{a-b}[b]_t$$

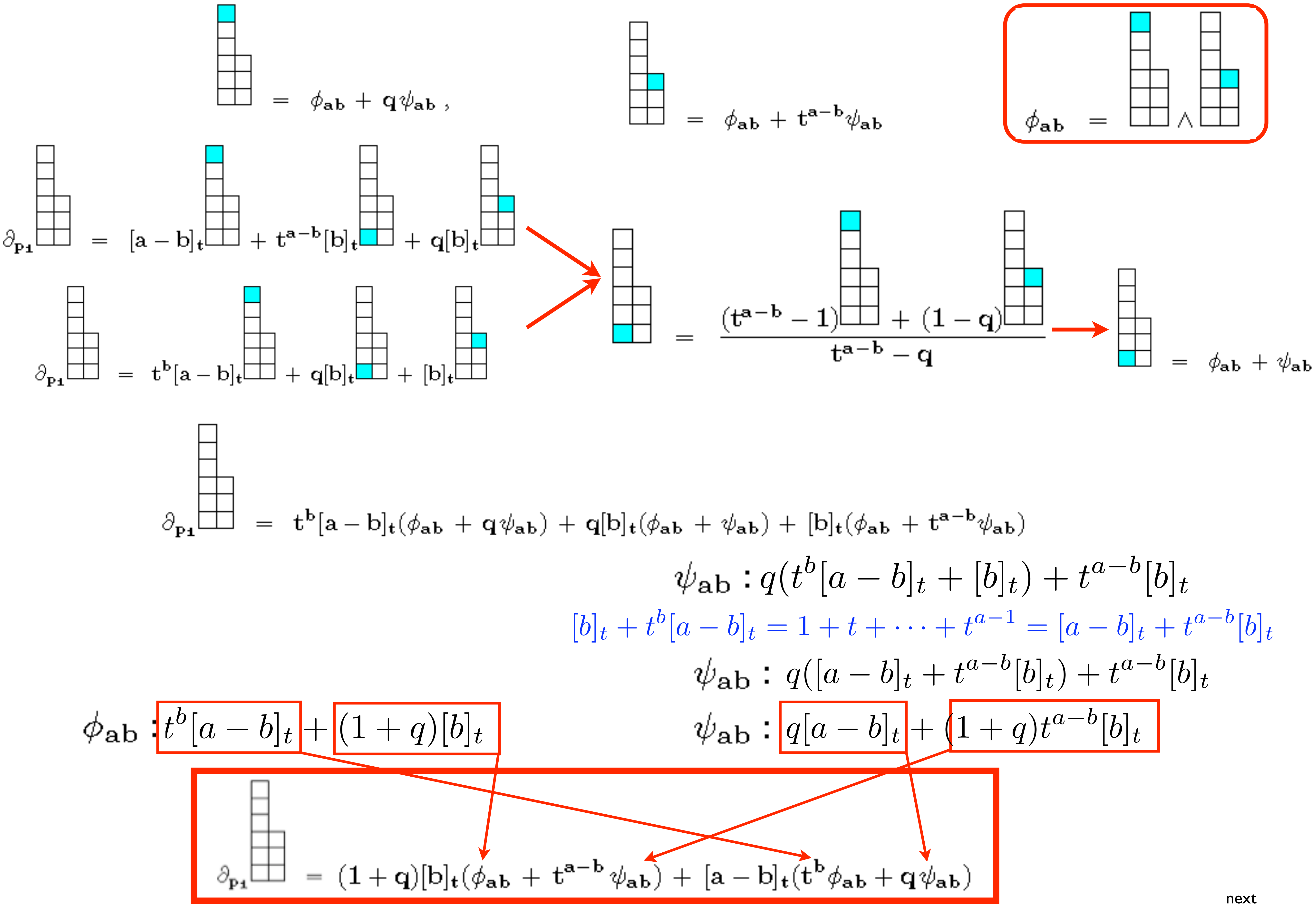
$$\psi_{ab} : q([a-b]_t + t^{a-b}[b]_t) + t^{a-b}[b]_t$$

$$\phi_{ab} : t^b[a-b]_t + (1+q)[b]_t \qquad \psi_{ab} : q[a-b]_t + (1+q)t^{a-b}[b]_t$$



$$\partial_{p_1} = (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t(t^b\phi_{ab} + q\psi_{ab})$$

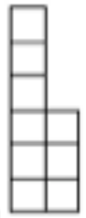
The General case



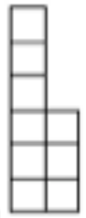
next

the next step

the next step

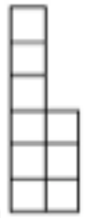
$\partial_{\mathbf{p_1}}$  $= (1+\mathbf{q})[\mathbf{b}]_{\mathbf{t}}(\phi_{\mathbf{ab}} + \mathbf{t}^{\mathbf{a-b}}\psi_{\mathbf{ab}}) + [\mathbf{a}-\mathbf{b}]_{\mathbf{t}}(\mathbf{t}^{\mathbf{b}}\phi_{\mathbf{ab}} + \mathbf{q}\psi_{\mathbf{ab}})$

the next step

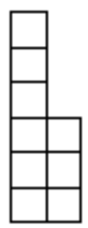
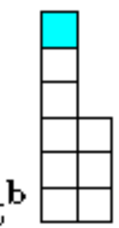
∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t(t^b\phi_{ab} + q\psi_{ab})$

While my representation theoretical extension of the Haglund conjecture is

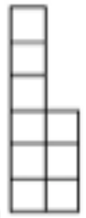
the next step

∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t(t^b\phi_{ab} + q\psi_{ab})$

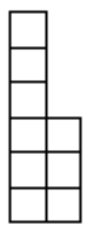
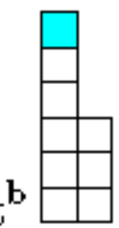
While my representation theoretical extension of the Haglund conjecture is

∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t\, t^b$  $(q/t, t)$

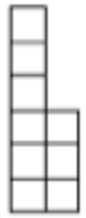
the next step

∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t(t^b\phi_{ab} + q\psi_{ab})$

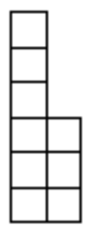
While my representation theoretical extension of the Haglund conjecture is

∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t\, t^b$  $(q/t, t)$

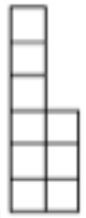
the next step

∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t(t^b\phi_{ab} + q\psi_{ab})$

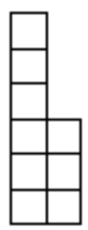
While my representation theoretical extension of the Haglund conjecture is

∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t\, t^b$  $(q/t, t)$

the next step

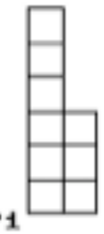
∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t(t^b\phi_{ab} + q\psi_{ab})$

While my representation theoretical extension of the Haglund conjecture is

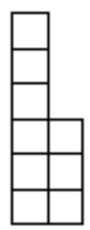
∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t\, t^b$  $(q/t, t)$

so we are reduced to proving the identity

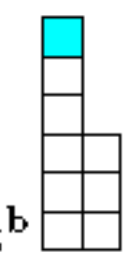
the next step

∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t(t^b\phi_{ab} + q\psi_{ab})$

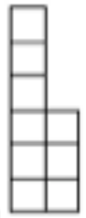
While my representation theoretical extension of the Haglund conjecture is

∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t\, t^b$  $(q/t, t)$

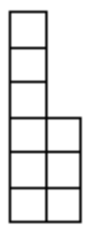
so we are reduced to proving the identity

t^b  $(q/t,t) \;=\; t^b\phi_{ab} + q\psi_{ab}$

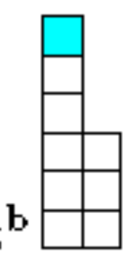
the next step

∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t(t^b\phi_{ab} + q\psi_{ab})$

While my representation theoretical extension of the Haglund conjecture is

∂_{p_1}  $= (1+q)[b]_t(\phi_{ab} + t^{a-b}\psi_{ab}) + [a-b]_t\, t^b$  $(q/t, t)$

so we are reduced to proving the identity

t^b  $(q/t, t) \;=\; t^b\phi_{ab} + q\psi_{ab}$

next

k-schur visualization of the a=3 b=2 case

k-schur visualization of the a=3 b=2 case

$$\partial_{\mathbf{p_1}} \yng(2,2,1)(\mathbf{q},\mathbf{t}) \;=\; (\mathbf{1}+\mathbf{t})(\mathbf{1}+\mathbf{q})\yng(2,1,1)(\mathbf{q},\mathbf{t}) + \mathbf{t}^{\mathbf{2}}\,\yng(2,2)(\mathbf{q}/\mathbf{t},\mathbf{t})$$

k-schur visualization of the a=3 b=2 case

$$\partial_{\mathbf{p_1}} \ydiagram{1,2,2}(\mathbf{q},\mathbf{t}) \;=\; (\mathbf{1}+\mathbf{t})(\mathbf{1}+\mathbf{q})\ydiagram{1,1,2}(\mathbf{q},\mathbf{t}) + \mathbf{t^2}\,\ydiagram{2,2}(\mathbf{q}/\mathbf{t},\mathbf{t})$$

$$\mathbf{t^2}\ydiagram{2,2}(\mathbf{q}/\mathbf{t},\mathbf{t}) \;=\; \mathbf{t^2}\phi_{\mathbf{32}} + \mathbf{q}\,\psi_{\mathbf{32}}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1} \ydiagram{1,2,2}(q,t) = (1+t)(1+q)\ydiagram{1,1,2}(q,t) + t^2\,\ydiagram{2,2}(q/t,t)$$

$$t^2\ydiagram{2,2}(q/t,t) = t^2\phi_{32} + q\,\psi_{32}$$

$$\ydiagram{1,1,2} = \phi_{32} + t\,\psi_{32}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1} \ydiagram{1,2,2}(q,t) \;=\; (1+t)(1+q)\ydiagram{1,1,2}(q,t) + t^2 \ydiagram{2,2}(q/t,t)$$

$$t^2 \ydiagram{2,2}(q/t,t) \;=\; t^2\phi_{32} + q\,\psi_{32}$$

$$\ydiagram{1,1,2} \;=\; \phi_{32} + t\,\psi_{32}$$

$$\ydiagram{2,2} \;=\; \phi_{32} + q\,\psi_{32}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1} \ydiagram{1,1,2}(q,t) \;=\; (1+t)(1+q)\ydiagram{1,1,2}(q,t) + t^2\, \ydiagram{2,2}(q/t,t)$$

$$t^2 \ydiagram{2,2}(q/t,t) \;=\; t^2 \phi_{32} + q\,\psi_{32}$$

$$\ydiagram{1,1,2} \;=\; \phi_{32} + t\,\psi_{32} \qquad\qquad \ydiagram{2,2} \;=\; \phi_{32} + q\,\psi_{32}$$

Now in terms of 2-Schur we get

k-schur visualization of the a=3 b=2 case

$$\partial_{\mathbf{p_1}} \ydiagram{1,1,2}(\mathbf{q},\mathbf{t}) = (1+\mathbf{t})(1+\mathbf{q})\ydiagram{1,2}(\mathbf{q},\mathbf{t}) + \mathbf{t}^2 \ydiagram{2,2}(\mathbf{q}/\mathbf{t},\mathbf{t})$$

$$\mathbf{t}^2 \ydiagram{2,2}(\mathbf{q}/\mathbf{t},\mathbf{t}) = \mathbf{t}^2 \phi_{32} + \mathbf{q}\,\psi_{32}$$

$$\ydiagram{1,2} = \phi_{32} + \mathbf{t}\,\psi_{32}$$

$$\ydiagram{2,2} = \phi_{32} + \mathbf{q}\,\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \ydiagram{1,2} \wedge \ydiagram{2,2} = \frac{\mathbf{q}\ydiagram{1,2} - \mathbf{t}\ydiagram{2,2}}{\mathbf{q}-\mathbf{t}}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1}\square_{211}(q,t) \;=\; (1+t)(1+q)\square_{211}(q,t) + t^2\,\square_{22}(q/t,t)$$

$$t^2\square_{22}(q/t,t) \;=\; t^2\phi_{32} + q\,\psi_{32}$$

$$\square_{211} \;=\; \phi_{32} + t\,\psi_{32} \qquad\qquad \square_{22} \;=\; \phi_{32} + q\,\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \square_{211} \wedge \square_{22} = \frac{q\,\square_{211} - t\,\square_{22}}{q-t} \;=\; \mathbf{A_{22} + qA_{211}}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{\mathbf{p_1}} \ydiagram{2,2,1}(\mathbf{q},\mathbf{t}) \;=\; (\mathbf{1}+\mathbf{t})(\mathbf{1}+\mathbf{q}) \ydiagram{2,1,1}(\mathbf{q},\mathbf{t}) + \mathbf{t^2} \ydiagram{2,2}(\mathbf{q}/\mathbf{t},\mathbf{t})$$

$$\mathbf{t^2} \ydiagram{2,2}(\mathbf{q}/\mathbf{t},\mathbf{t}) \;=\; \mathbf{t^2}\phi_{\mathbf{32}} + \mathbf{q}\,\psi_{\mathbf{32}}$$

$$\ydiagram{2,1,1} \;=\; \phi_{\mathbf{32}} + \mathbf{t}\,\psi_{\mathbf{32}}$$

$$\ydiagram{2,2} \;=\; \phi_{\mathbf{32}} + \mathbf{q}\,\psi_{\mathbf{32}}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \ydiagram{2,1,1} \wedge \ydiagram{2,2} = \frac{\mathbf{q}\ydiagram{2,1,1} - \mathbf{t}\ydiagram{2,2}}{\mathbf{q}-\mathbf{t}} = \mathbf{A_{22}} + \mathbf{q}\mathbf{A_{211}}$$

$$\phi_{\mathbf{32}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathbf{A_{22}} & \mathbf{A_{211}} & 0 \end{bmatrix}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1}[2,2,1](q,t) = (1+t)(1+q)[2,1,1](q,t) + t^2\,[2,2](q/t,t)$$

$$t^2[2,2](q/t,t) = t^2\phi_{32} + q\psi_{32}$$

$$[2,1,1] = \phi_{32} + t\psi_{32} \qquad [2,2] = \phi_{32} + q\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = [2,1,1] \wedge [2,2] = \frac{q\,[2,1,1] - t\,[2,2]}{q-t} = A_{22} + qA_{211}$$

$$\phi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{22} & A_{211} & 0 \end{bmatrix} \qquad t^2\phi_{32} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{\mathbf{p_1}} \ydiagram{2,2,1}(\mathbf{q},\mathbf{t}) \;=\; (1+\mathbf{t})(1+\mathbf{q})\ydiagram{2,1,1}(\mathbf{q},\mathbf{t}) + \mathbf{t}^2\, \ydiagram{2,2}(\mathbf{q}/\mathbf{t},\mathbf{t})$$

$$\mathbf{t}^2 \ydiagram{2,2}(\mathbf{q}/\mathbf{t},\mathbf{t}) \;=\; \mathbf{t}^2\phi_{32} + \mathbf{q}\,\psi_{32}$$

$$\ydiagram{2,1,1} \;=\; \phi_{32} + \mathbf{t}\,\psi_{32} \qquad\qquad \ydiagram{2,2} \;=\; \phi_{32} + \mathbf{q}\,\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \ydiagram{2,1,1} \wedge \ydiagram{2,2} = \frac{\mathbf{q}\ydiagram{2,1,1} - \mathbf{t}\ydiagram{2,2}}{\mathbf{q}-\mathbf{t}} = \mathbf{A_{22}} + \mathbf{q}\mathbf{A_{211}} \qquad\qquad \psi_{32} = \frac{\ydiagram{2,2} - \ydiagram{2,1,1}}{\mathbf{q}-\mathbf{t}}$$

$$\phi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathbf{A_{22}} & \mathbf{A_{211}} & 0 \end{bmatrix} \qquad \mathbf{t}^2\phi_{32} = \begin{bmatrix} \mathbf{A_{22}} & \mathbf{A_{211}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1}\square_{211}(q,t) \;=\; (1+t)(1+q)\square_{211}(q,t) + t^2\,\square_{22}(q/t,t)$$

$$t^2\square_{22}(q/t,t) \;=\; t^2\phi_{32} + q\,\psi_{32}$$

$$\square_{211} = \phi_{32} + t\,\psi_{32} \qquad\qquad \square_{22} = \phi_{32} + q\,\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \square_{211} \wedge \square_{22} = \frac{q\,\square_{211} - t\,\square_{22}}{q-t} = A_{22} + qA_{211} \qquad \psi_{32} = \frac{\square_{22} - \square_{211}}{q-t} = tA_{211} + qA_{1111}$$

$$\phi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{22} & A_{211} & 0 \end{bmatrix} \qquad t^2\phi_{32} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1}\ydiagram{2,2,1}(q,t) = (1+t)(1+q)\ydiagram{2,1,1}(q,t) + t^2\,\ydiagram{2,2}(q/t,t)$$

$$t^2\ydiagram{2,2}(q/t,t) = t^2\phi_{32} + q\,\psi_{32}$$

$$\ydiagram{2,1,1} = \phi_{32} + t\,\psi_{32} \qquad \ydiagram{2,2} = \phi_{32} + q\,\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \ydiagram{2,1,1} \wedge \ydiagram{2,2} = \frac{q\ydiagram{2,1,1} - t\ydiagram{2,2}}{q-t} = A_{22} + qA_{211} \qquad \psi_{32} = \frac{\ydiagram{2,2} - \ydiagram{2,1,1}}{q-t} = tA_{211} + qA_{1111}$$

$$\phi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{22} & A_{211} & 0 \end{bmatrix} \quad t^2\phi_{32} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ A_{211} & 0 & 0 \\ 0 & A_{1111} & 0 \end{bmatrix}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{\mathbf{p_1}} [2,2,1](\mathbf{q},\mathbf{t}) \;=\; (1+\mathbf{t})(1+\mathbf{q})[2,1,1](\mathbf{q},\mathbf{t}) + \mathbf{t}^2\,[2,2](\mathbf{q}/\mathbf{t},\mathbf{t})$$

$$\mathbf{t}^2 [2,2](\mathbf{q}/\mathbf{t},\mathbf{t}) \;=\; \mathbf{t}^2\phi_{32} + \mathbf{q}\,\psi_{32}$$

$$[2,1,1] \;=\; \phi_{32} + \mathbf{t}\,\psi_{32} \qquad\qquad [2,2] \;=\; \phi_{32} + \mathbf{q}\,\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = [2,1,1] \wedge [2,2] = \frac{\mathbf{q}\,[2,1,1] - \mathbf{t}\,[2,2]}{\mathbf{q}-\mathbf{t}} \;=\; \mathbf{A_{22}} + \mathbf{q}\mathbf{A_{211}}$$

$$\psi_{32} = \frac{[2,2] - [2,1,1]}{\mathbf{q}-\mathbf{t}} \;=\; \mathbf{t}\mathbf{A_{211}} + \mathbf{q}\mathbf{A_{1111}}$$

$$\phi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathbf{A_{22}} & \mathbf{A_{211}} & 0 \end{bmatrix} \quad \mathbf{t}^2\phi_{32} = \begin{bmatrix} \mathbf{A_{22}} & \mathbf{A_{211}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ \mathbf{A_{211}} & 0 & 0 \\ 0 & \mathbf{A_{1111}} & 0 \end{bmatrix} \quad \mathbf{q}\psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{A_{211}} & 0 \\ 0 & 0 & \mathbf{A_{1111}} \end{bmatrix}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1}\ydiagram{1,2,2}(q,t) = (1+t)(1+q)\ydiagram{1,1,2}(q,t) + t^2\,\ydiagram{2,2}(q/t,t)$$

$$t^2\ydiagram{2,2}(q/t,t) = t^2\phi_{32} + q\,\psi_{32}$$

$$\ydiagram{1,1,2} = \phi_{32} + t\,\psi_{32} \qquad \ydiagram{2,2} = \phi_{32} + q\,\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \ydiagram{1,1,2} \wedge \ydiagram{2,2} = \frac{q\ydiagram{1,1,2} - t\ydiagram{2,2}}{q-t} = A_{22} + qA_{211} \qquad \psi_{32} = \frac{\ydiagram{2,2} - \ydiagram{1,1,2}}{q-t} = tA_{211} + qA_{1111}$$

$$\phi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{22} & A_{211} & 0 \end{bmatrix} \quad t^2\phi_{32} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ A_{211} & 0 & 0 \\ 0 & A_{1111} & 0 \end{bmatrix} \quad q\psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

thus

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1} \begin{smallmatrix}\square\\ \square\\ \square\square\end{smallmatrix}(q,t) \;=\; (1+t)(1+q)\begin{smallmatrix}\square\\ \square\\ \square\square\end{smallmatrix}(q,t) + t^2\, \boxplus(q/t,t)$$

$$t^2 \boxplus(q/t,t) \;=\; t^2\phi_{32} + q\,\psi_{32}$$

$$\begin{smallmatrix}\square\\ \square\\ \square\square\end{smallmatrix} \;=\; \phi_{32} + t\,\psi_{32} \qquad\qquad \boxplus \;=\; \phi_{32} + q\,\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \begin{smallmatrix}\square\\ \square\\ \square\square\end{smallmatrix} \wedge \boxplus = \frac{q\begin{smallmatrix}\square\\ \square\\ \square\square\end{smallmatrix} - t\boxplus}{q-t} \;=\; A_{22} + qA_{211} \qquad\qquad \psi_{32} = \frac{\boxplus - \begin{smallmatrix}\square\\ \square\\ \square\square\end{smallmatrix}}{q-t} \;=\; tA_{211} + qA_{1111}$$

$$\phi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{22} & A_{211} & 0 \end{bmatrix} \quad t^2\phi_{32} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ A_{211} & 0 & 0 \\ 0 & A_{1111} & 0 \end{bmatrix} \quad q\psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

thus

$$\boxplus = A_{22} + qA_{211} + qtA_{211} + q^2A_{1111} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ A_{22} & A_{211} & A_{1111} \end{bmatrix}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1}\ydiagram{2,2,1}(q,t) \;=\; (1+t)(1+q)\ydiagram{2,1,1}(q,t) + t^2\,\ydiagram{2,2}(q/t,t)$$

$$t^2\ydiagram{2,2}(q/t,t) \;=\; t^2\phi_{32} + q\,\psi_{32}$$

$$\ydiagram{2,1,1} \;=\; \phi_{32} + t\,\psi_{32} \qquad\qquad \ydiagram{2,2} \;=\; \phi_{32} + q\,\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \ydiagram{2,1,1} \wedge \ydiagram{2,2} = \frac{q\,\ydiagram{2,1,1} - t\,\ydiagram{2,2}}{q-t} = A_{22} + qA_{211} \qquad \psi_{32} = \frac{\ydiagram{2,2} - \ydiagram{2,1,1}}{q-t} = tA_{211} + qA_{1111}$$

$$\phi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{22} & A_{211} & 0 \end{bmatrix} \quad t^2\phi_{32} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ A_{211} & 0 & 0 \\ 0 & A_{1111} & 0 \end{bmatrix} \quad q\psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

thus

$$\ydiagram{2,2} = A_{22} + qA_{211} + qtA_{211} + q^2A_{1111} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ A_{22} & A_{211} & A_{1111} \end{bmatrix}$$

and thus

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1}\ydiagram{2,2,1}(q,t) \;=\; (1+t)(1+q)\ydiagram{2,1,1}(q,t) + t^2\,\ydiagram{2,2}(q/t,t)$$

$$t^2\ydiagram{2,2}(q/t,t) \;=\; t^2\phi_{32} + q\,\psi_{32}$$

$$\ydiagram{2,1,1} \;=\; \phi_{32} + t\,\psi_{32} \qquad\qquad \ydiagram{2,2} \;=\; \phi_{32} + q\,\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \ydiagram{2,1,1} \wedge \ydiagram{2,2} = \frac{q\,\ydiagram{2,1,1} - t\,\ydiagram{2,2}}{q-t} = A_{22} + qA_{211} \qquad \psi_{32} = \frac{\ydiagram{2,2} - \ydiagram{2,1,1}}{q-t} = tA_{211} + qA_{1111}$$

$$\phi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{22} & A_{211} & 0 \end{bmatrix} \quad t^2\phi_{32} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ A_{211} & 0 & 0 \\ 0 & A_{1111} & 0 \end{bmatrix} \quad q\psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

thus

$$\ydiagram{2,2} = A_{22} + qA_{211} + qtA_{211} + q^2A_{1111} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ A_{22} & A_{211} & A_{1111} \end{bmatrix}$$

and thus

$$t^2\,\ydiagram{2,2}\,(q/t,t) = t^2A_{22} + tqA_{211} + t^2A_{211} + q^2A_{1111} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1}\ydiagram{1,2,2}(q,t) \;=\; (1+t)(1+q)\ydiagram{1,1,2}(q,t) + t^2\,\ydiagram{2,2}(q/t,t)$$

$$t^2\ydiagram{2,2}(q/t,t) \;=\; t^2\phi_{32} + q\,\psi_{32}$$

$$\ydiagram{1,1,2} \;=\; \phi_{32} + t\,\psi_{32} \qquad\qquad \ydiagram{2,2} \;=\; \phi_{32} + q\,\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \ydiagram{1,1,2}\wedge\ydiagram{2,2} = \frac{q\ydiagram{1,1,2} - t\ydiagram{2,2}}{q-t} \;=\; A_{22} + qA_{211} \qquad\qquad \psi_{32} = \frac{\ydiagram{2,2} - \ydiagram{1,1,2}}{q-t} \;=\; tA_{211} + qA_{1111}$$

$$\phi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{22} & A_{211} & 0 \end{bmatrix} \quad t^2\phi_{32} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ A_{211} & 0 & 0 \\ 0 & A_{1111} & 0 \end{bmatrix} \quad q\psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

thus

$$\ydiagram{2,2} = A_{22} + qA_{211} + qtA_{211} + q^2A_{1111} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ A_{22} & A_{211} & A_{1111} \end{bmatrix}$$

and thus

$$t^2\,\ydiagram{2,2}\,(q/t\,,t) = t^2A_{22} + tqA_{211} + t^2A_{211} + q^2A_{1111} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

$$t^2\phi_{32} \;+\; q\,\psi_{32}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1}\ydiagram{1,2,2}(q,t) \;=\; (1+t)(1+q)\ydiagram{1,1,2}(q,t) \;+\; t^2\,\ydiagram{2,2}(q/t,t)$$

$$t^2\ydiagram{2,2}(q/t,t) \;=\; t^2\phi_{32}+q\,\psi_{32}$$

$$\ydiagram{1,1,2} \;=\; \phi_{32}+t\,\psi_{32} \qquad\qquad \ydiagram{2,2} \;=\; \phi_{32}+q\,\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \ydiagram{1,1,2}\wedge\ydiagram{2,2} = \frac{q\,\ydiagram{1,1,2} - t\,\ydiagram{2,2}}{q-t} \;=\; A_{22}+qA_{211} \qquad\qquad \psi_{32} = \frac{\ydiagram{2,2} - \ydiagram{1,1,2}}{q-t} \;=\; tA_{211}+qA_{1111}$$

$$\phi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{22} & A_{211} & 0 \end{bmatrix} \quad t^2\phi_{32} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ A_{211} & 0 & 0 \\ 0 & A_{1111} & 0 \end{bmatrix} \quad q\psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

thus

$$\ydiagram{2,2} = A_{22}+qA_{211}+qtA_{211}+q^2A_{1111} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ A_{22} & A_{211} & A_{1111} \end{bmatrix}$$

and thus

$$t^2\,\ydiagram{2,2}\,(q/t\,,t) = t^2A_{22}+tqA_{211}+t^2A_{211}+q^2A_{1111} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

$$t^2\phi_{32} \;+\; q\,\psi_{32} \;=\; t^2(A_{22}+qA_{211}) \;+\; q(tA_{211} \;+\; qA_{1111})$$

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1}\ydiagram{2,2,1}(q,t) \;=\; (1+t)(1+q)\ydiagram{2,1,1}(q,t) \;+\; t^2\,\ydiagram{2,2}(q/t,t)$$

$$t^2\ydiagram{2,2}(q/t,t) \;=\; t^2\phi_{32} + q\,\psi_{32}$$

$$\ydiagram{2,1,1} \;=\; \phi_{32} + t\,\psi_{32} \qquad\qquad \ydiagram{2,2} \;=\; \phi_{32} + q\,\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \ydiagram{2,1,1} \wedge \ydiagram{2,2} = \frac{q\ydiagram{2,1,1} - t\ydiagram{2,2}}{q-t} \;=\; A_{22} + qA_{211} \qquad\qquad \psi_{32} = \frac{\ydiagram{2,2} - \ydiagram{2,1,1}}{q-t} \;=\; tA_{211} + qA_{1111}$$

$$\phi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{22} & A_{211} & 0 \end{bmatrix} \quad t^2\phi_{32} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ A_{211} & 0 & 0 \\ 0 & A_{1111} & 0 \end{bmatrix} \quad q\psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

thus

$$\ydiagram{2,2} = A_{22} + qA_{211} + qtA_{211} + q^2A_{1111} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ A_{22} & A_{211} & A_{1111} \end{bmatrix}$$

and thus

$$t^2\,\ydiagram{2,2}\,(q/t\,,t) = t^2A_{22} + tqA_{211} + t^2A_{211} + q^2A_{1111} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

$$t^2\phi_{32} + q\psi_{32} \;=\; t^2(A_{22} + qA_{211}) + q(tA_{211} + qA_{1111}) \;=\; \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1}\yng(1,2,2)(q,t) = (1+t)(1+q)\yng(1,1,2)(q,t) + t^2\,\yng(2,2)(q/t,t)$$

$$t^2\yng(2,2)(q/t,t) = t^2\phi_{32} + q\psi_{32}$$

$$\yng(1,1,2) = \phi_{32} + t\psi_{32} \qquad \yng(2,2) = \phi_{32} + q\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \yng(1,1,2) \wedge \yng(2,2) = \frac{q\,\yng(1,1,2) - t\,\yng(2,2)}{q-t} = A_{22} + qA_{211} \qquad \psi_{32} = \frac{\yng(2,2) - \yng(1,1,2)}{q-t} = tA_{211} + qA_{1111}$$

$$\phi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{22} & A_{211} & 0 \end{bmatrix} \quad t^2\phi_{32} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ A_{211} & 0 & 0 \\ 0 & A_{1111} & 0 \end{bmatrix} \quad q\psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

thus

$$\yng(2,2) = A_{22} + qA_{211} + qtA_{211} + q^2A_{1111} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ A_{22} & A_{211} & A_{1111} \end{bmatrix}$$

and thus

$$t^2\,\yng(2,2)\,(q/t,t) = t^2A_{22} + tqA_{211} + t^2A_{211} + q^2A_{1111} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

$$t^2\phi_{32} + q\psi_{32} = t^2(A_{22} + qA_{211}) + q(tA_{211} + qA_{1111}) = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

QED !

k-schur visualization of the a=3 b=2 case

$$\partial_{p_1}\yng(2,1,1)(q,t) \;=\; (1+t)(1+q)\yng(2,1,1)(q,t) + t^2\,\yng(2,2)(q/t,t)$$

$$t^2\yng(2,2)(q/t,t) \;=\; t^2\phi_{32} + q\,\psi_{32}$$

$$\yng(2,1,1) \;=\; \phi_{32} + t\,\psi_{32} \qquad\qquad \yng(2,2) \;=\; \phi_{32} + q\,\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} = \yng(2,1,1)\wedge\yng(2,2) = \frac{q\,\yng(2,1,1) - t\,\yng(2,2)}{q-t} \;=\; A_{22} + qA_{211} \qquad\qquad \psi_{32} = \frac{\yng(2,2) - \yng(2,1,1)}{q-t} \;=\; tA_{211} + qA_{1111}$$

$$\phi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{22} & A_{211} & 0 \end{bmatrix} \quad t^2\phi_{32} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ A_{211} & 0 & 0 \\ 0 & A_{1111} & 0 \end{bmatrix} \quad q\psi_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

thus

$$\yng(2,2) = A_{22} + qA_{211} + qtA_{211} + q^2A_{1111} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{211} & 0 \\ A_{22} & A_{211} & A_{1111} \end{bmatrix}$$

and thus

$$t^2\,\yng(2,2)\,(q/t,t) = t^2A_{22} + tqA_{211} + t^2A_{211} + q^2A_{1111} = \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

$$t^2\phi_{32} + q\psi_{32} \;=\; t^2(A_{22} + qA_{211}) + q(tA_{211} + qA_{1111}) \;=\; \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix}$$

QED !

next

A simpler question?

A simpler question?

Count the number of permutations of S_n

A simpler question?

Count the number of permutations of S_n
whose first $n-k$ entries are increasing

A simpler question?

Count the number of permutations of S_n
whose first $n - k$ entries are increasing
and have no increasing subsequence of length greater than $n - k$

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Count the number of permutations of S_n
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In symbols, we want $\#\Pi_{n,k}$ where

A simpler question?

Count the number of permutations of S_n
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In symbols, we want $\#\Pi_{n,k}$ where

$$\Pi_{n,k} = \{a = (a_1, a_2, \ldots, a_n) \in S_n : a_1 < a_2 < \cdots < a_{n-k} \ \&\ LI(a) = n-k\}$$

A simpler question?

Count the number of permutations of S_n
whose first $n-k$ entries are increasing
and have no increasing subsequence of length greater than $n-k$

In symbols, we want $\#\Pi_{n,k}$ where

$$\Pi_{n,k} = \{a = (a_1, a_2, \dots, a_n) \in S_n : a_1 < a_2 < \cdots < a_{n-k} \ \& \ LI(a) = n-k\}$$

$LI(a)$ denotes the length of the longest increasing subsequence

A simpler question?

Count the number of permutations of S_n
whose first $n-k$ entries are increasing
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In symbols, we want $\#\Pi_{n,k}$ where

$$\Pi_{n,k} = \{a = (a_1, a_2, \ldots, a_n) \in S_n : a_1 < a_2 < \cdots < a_{n-k} \ \& \ LI(a) = n-k\}$$

$LI(a)$ denotes the length of the longest increasing subsequence

Let $C_{n,k}$ be a collection of "*colored permutations*" of S_n such that

A simpler question?

Count the number of permutations of S_n
whose first $n-k$ entries are increasing
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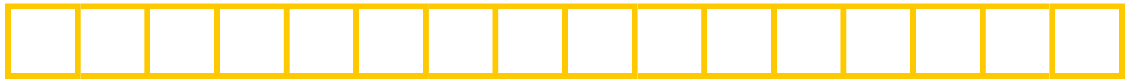

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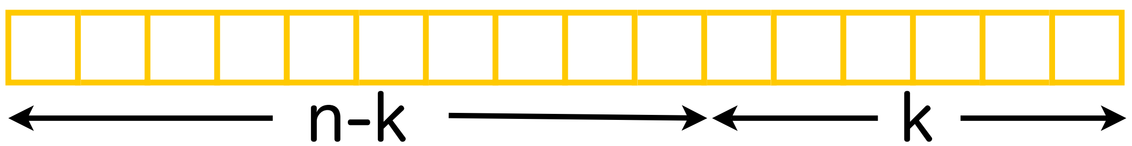

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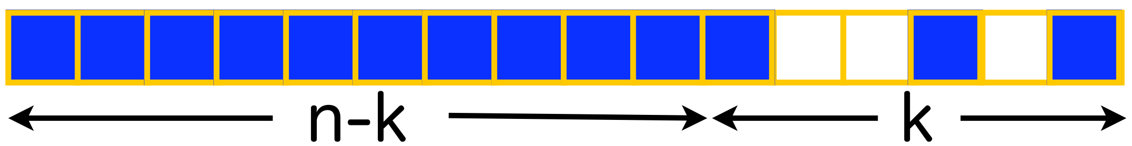

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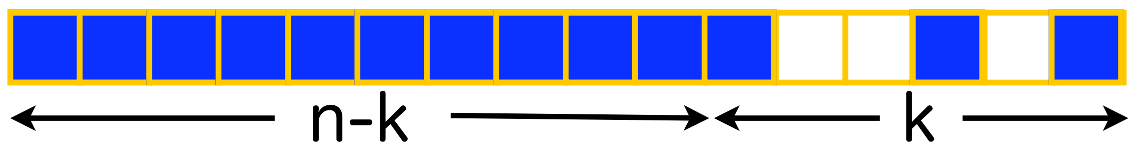

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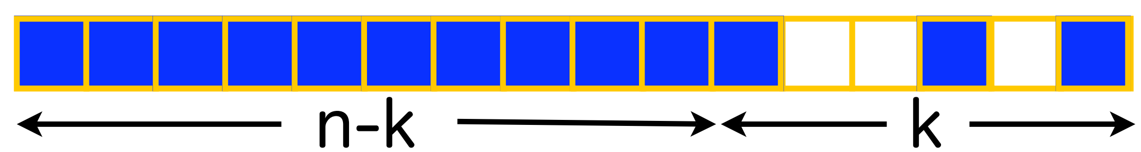

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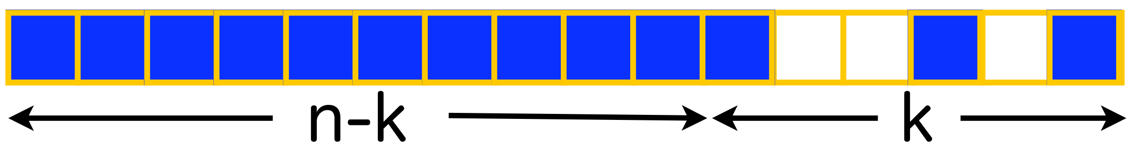

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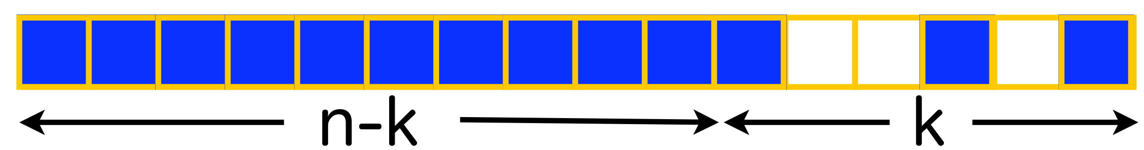

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100$

Reward

for an "elementary" proof

Do you think that was enough?

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here is more

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the Frobenius Characteristic of Sn Harmonics

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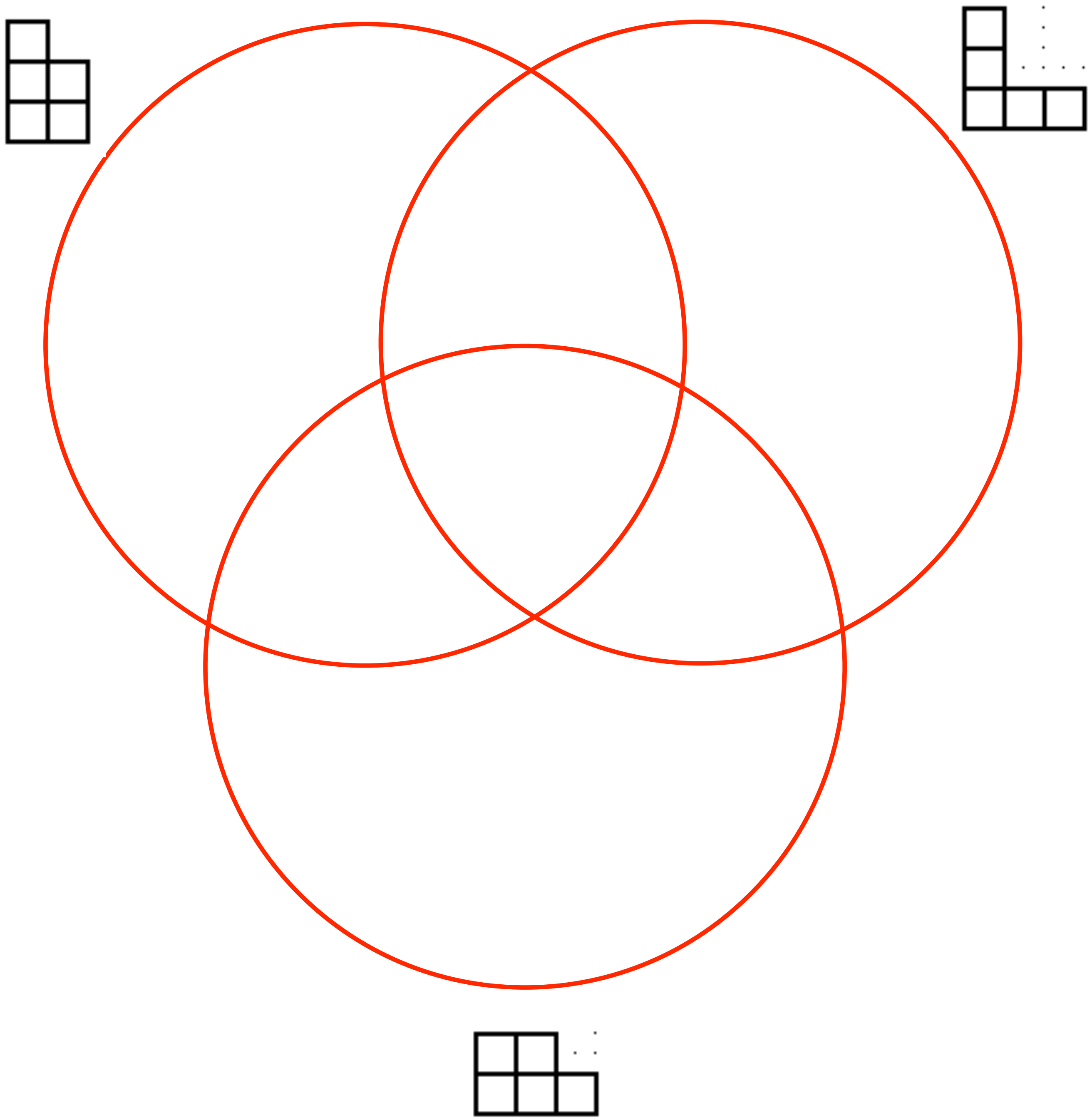
next

THE END

More miracles

More miracles



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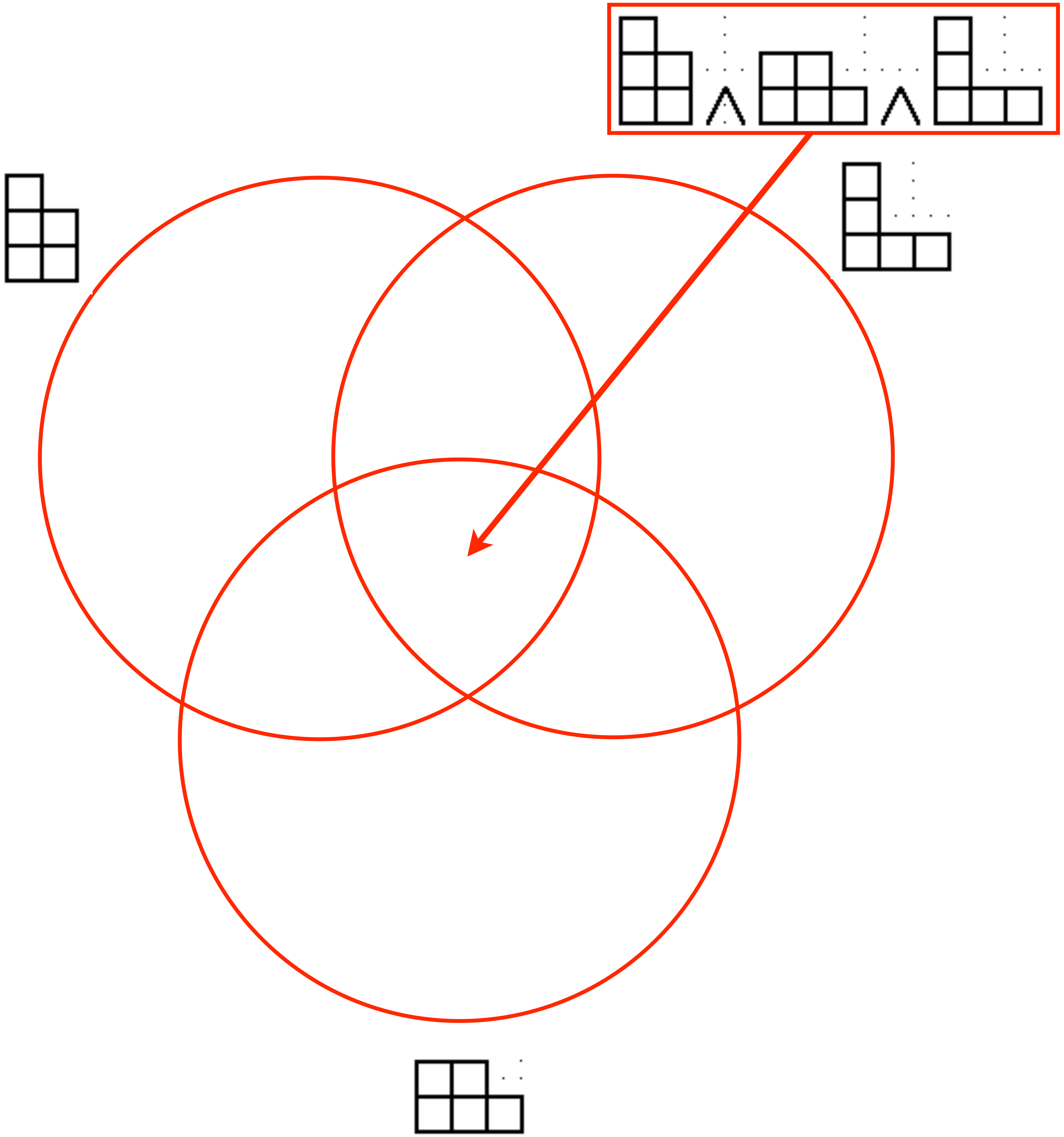
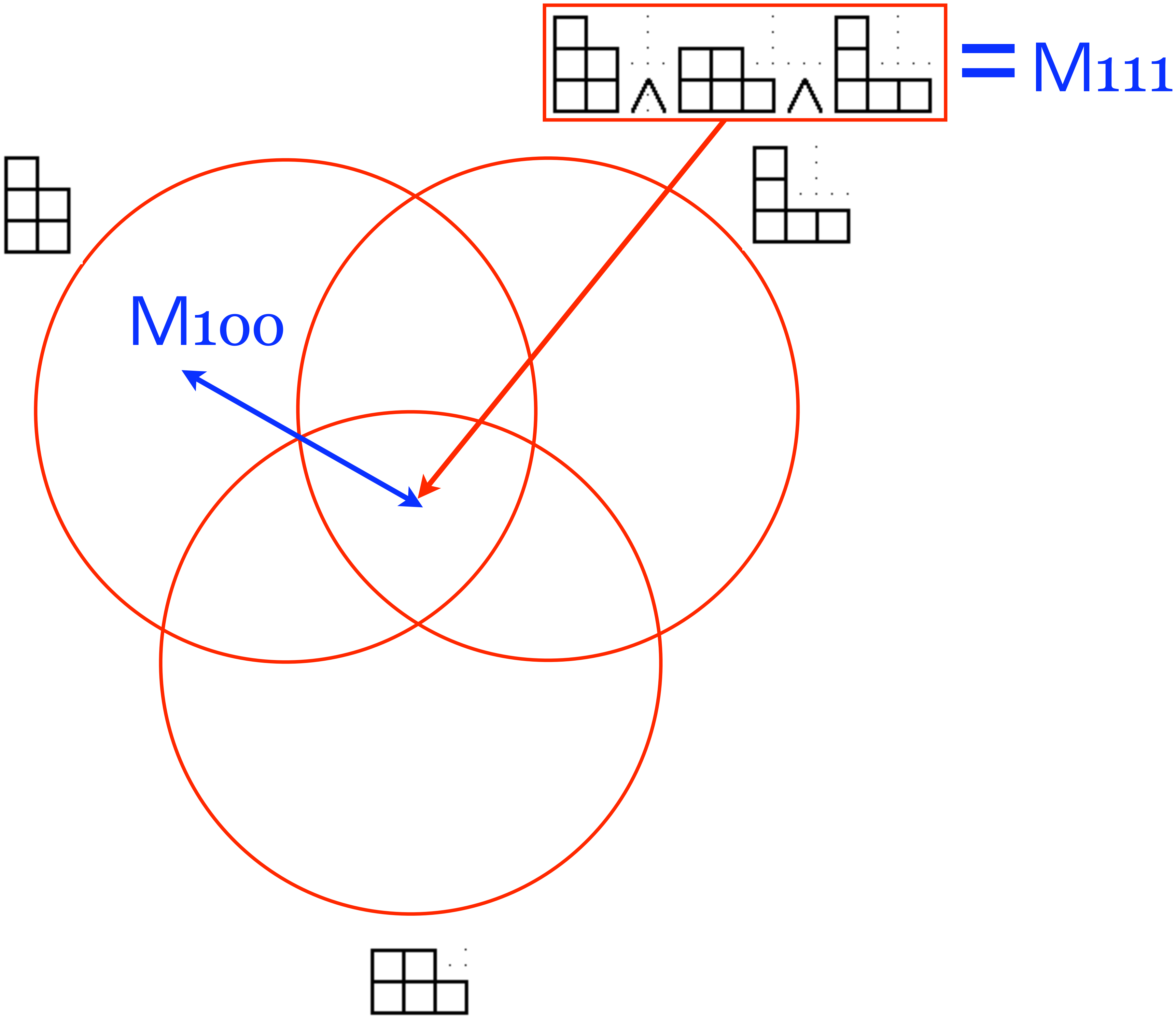


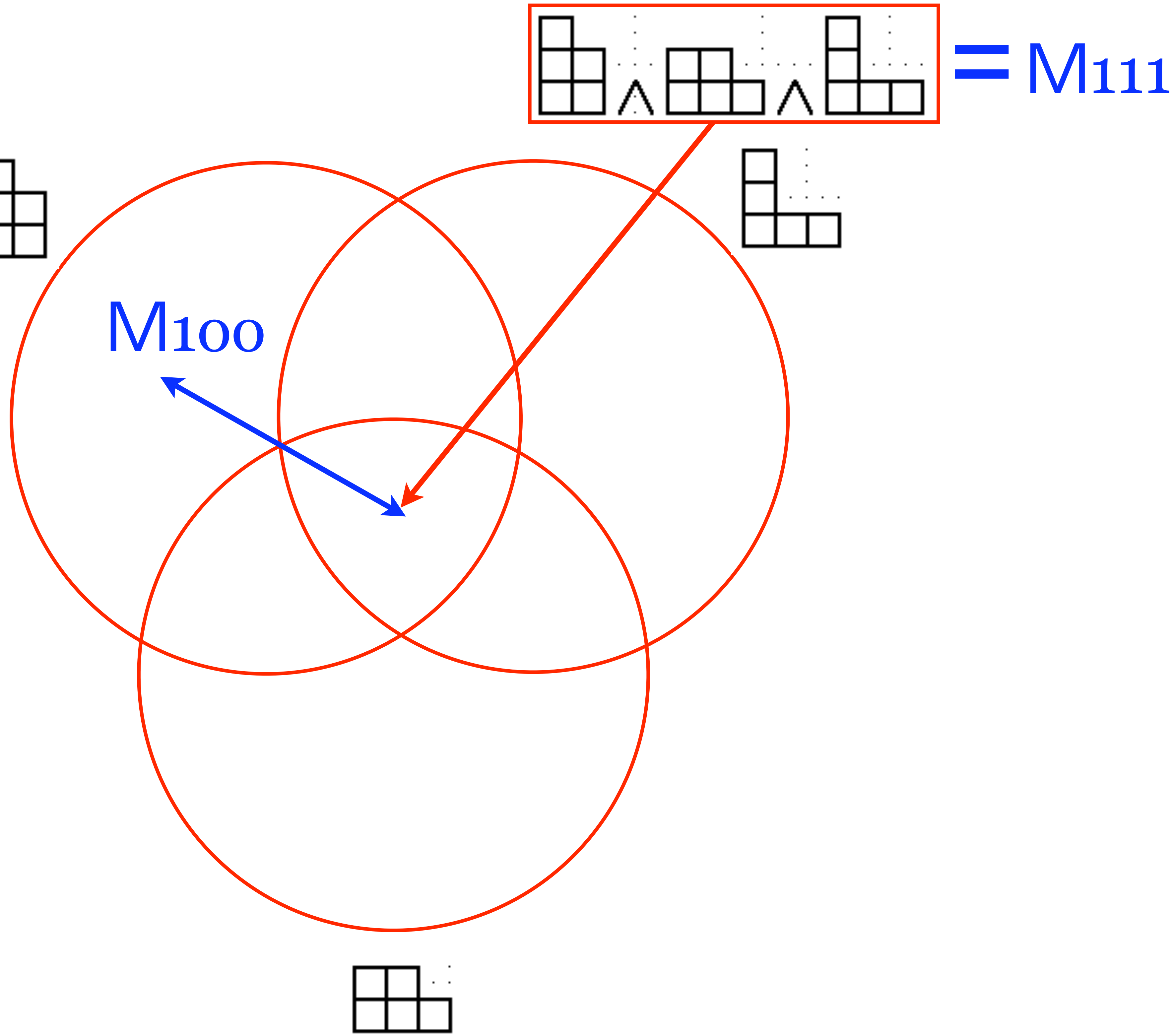
$= M_{111}$

More miracles



More miracles

$\mathrm{M}_{100} = \mathrm{flip}_{221}\mathrm{M}_{111}$



More miracles

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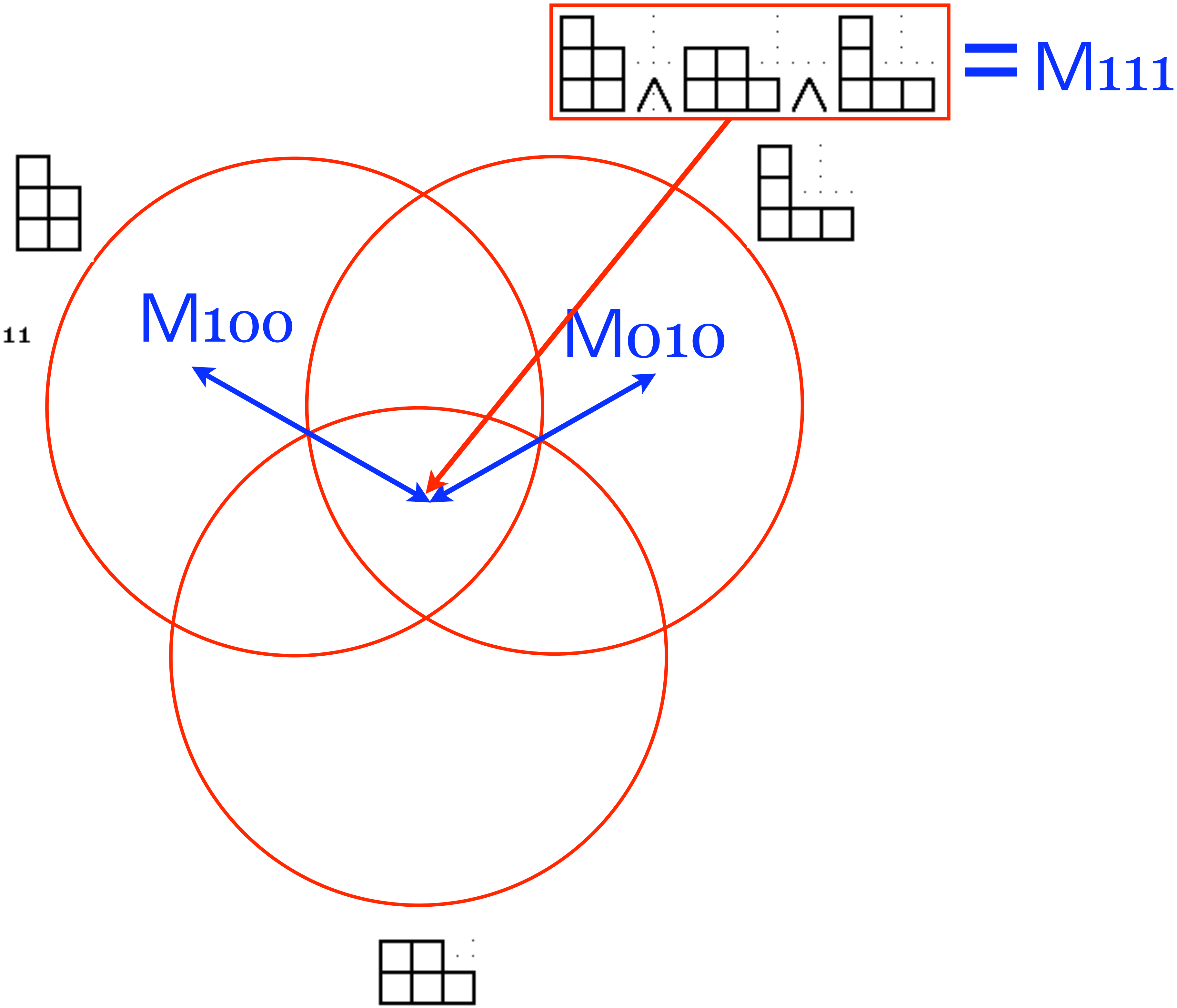

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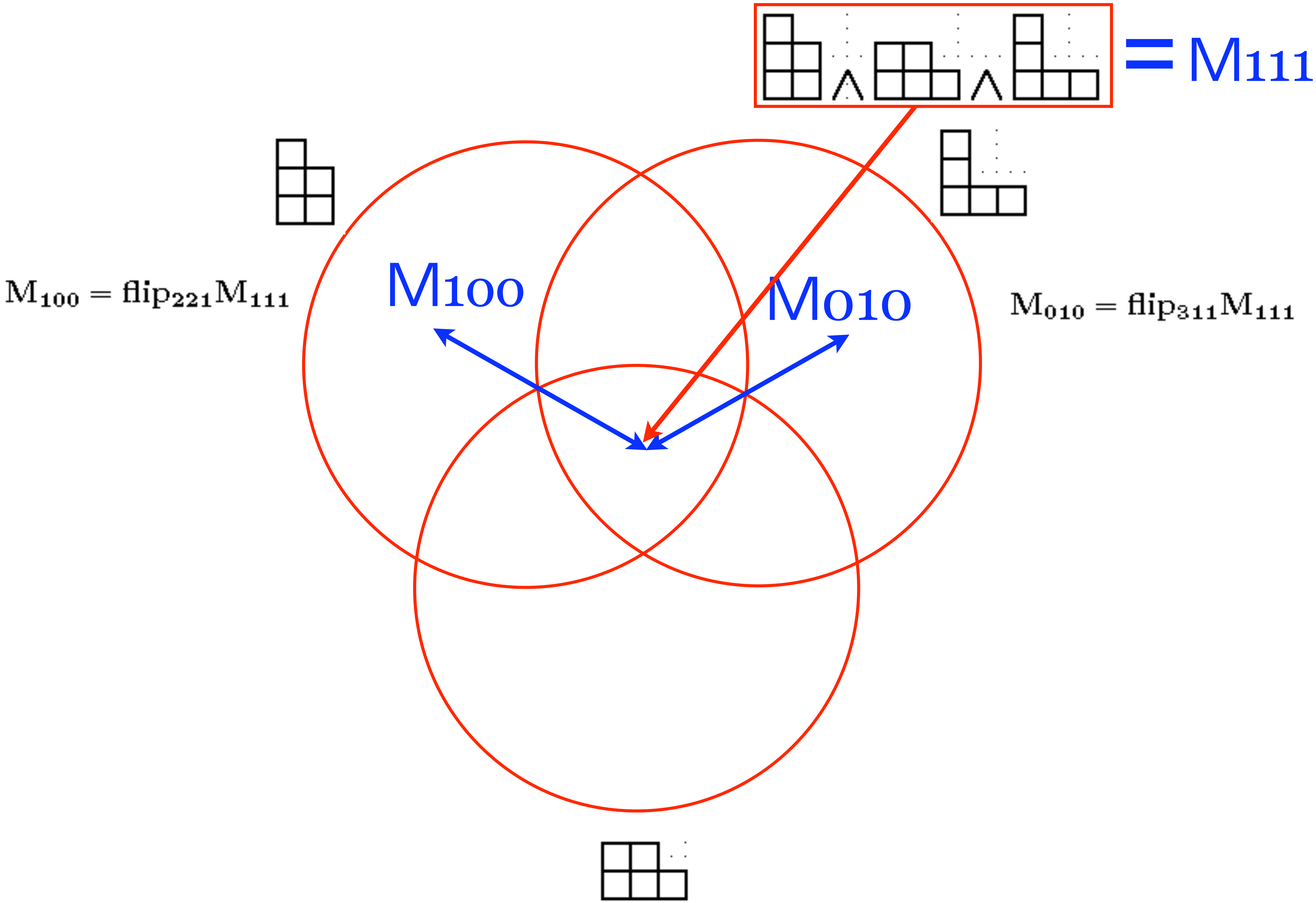

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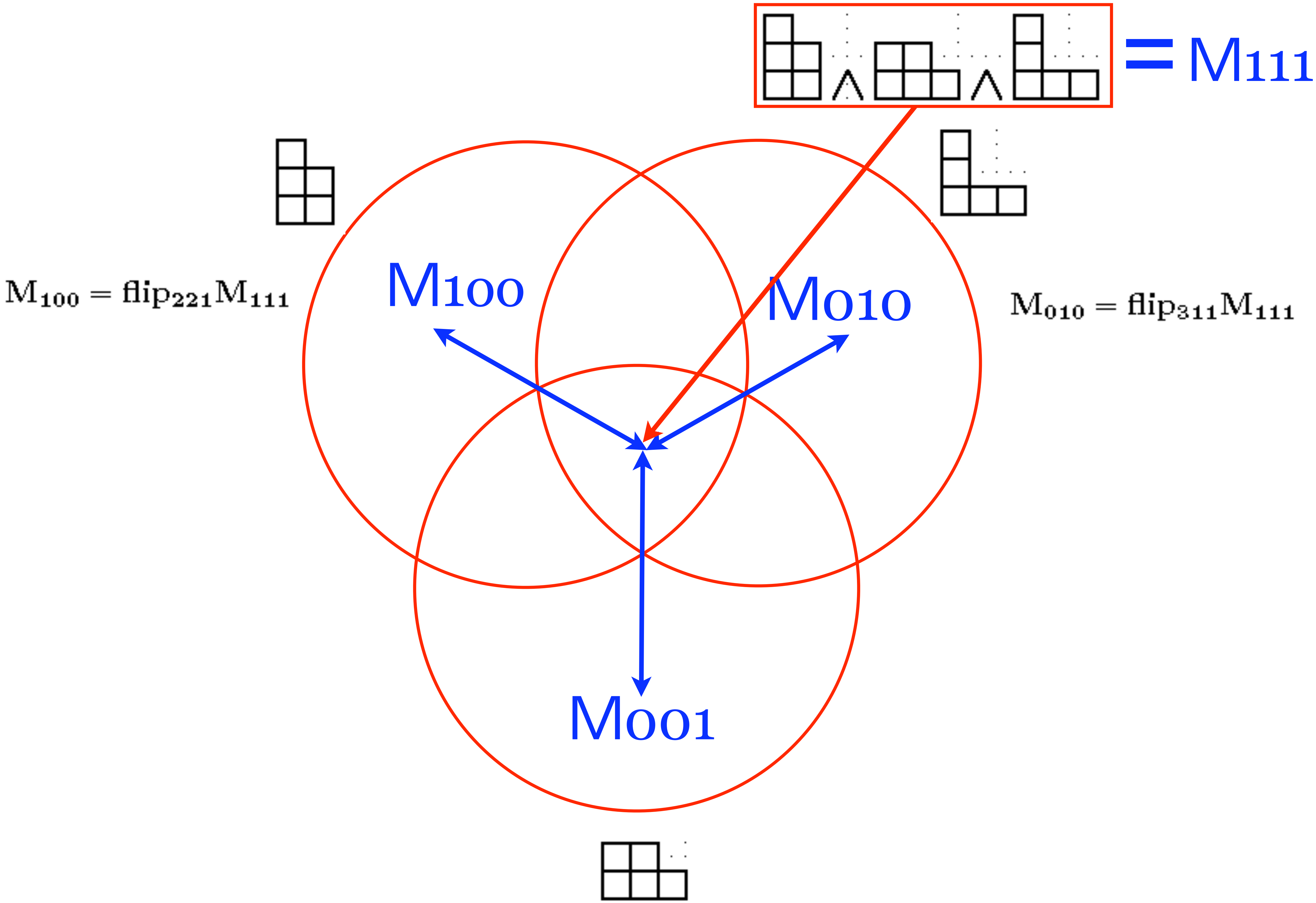

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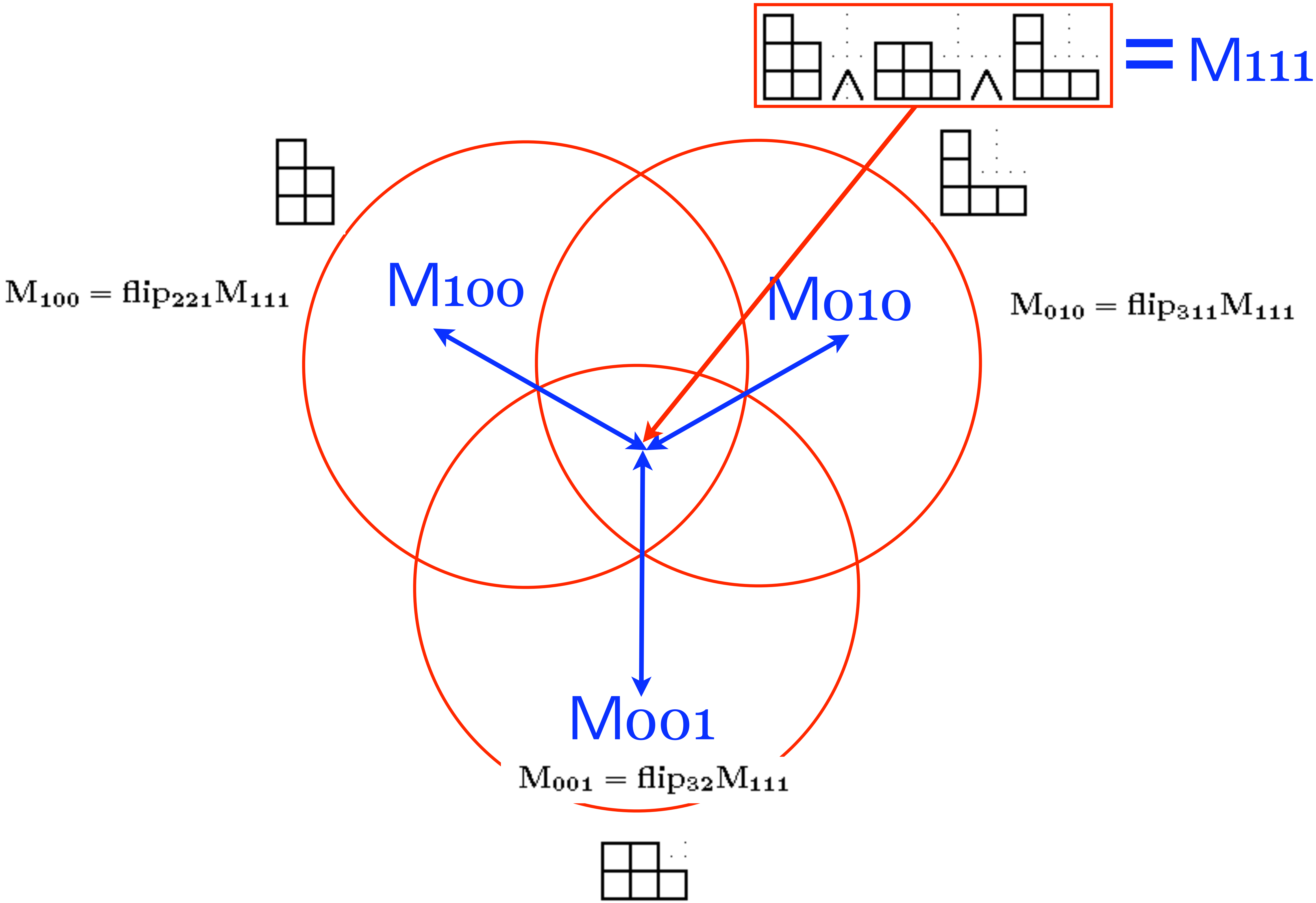

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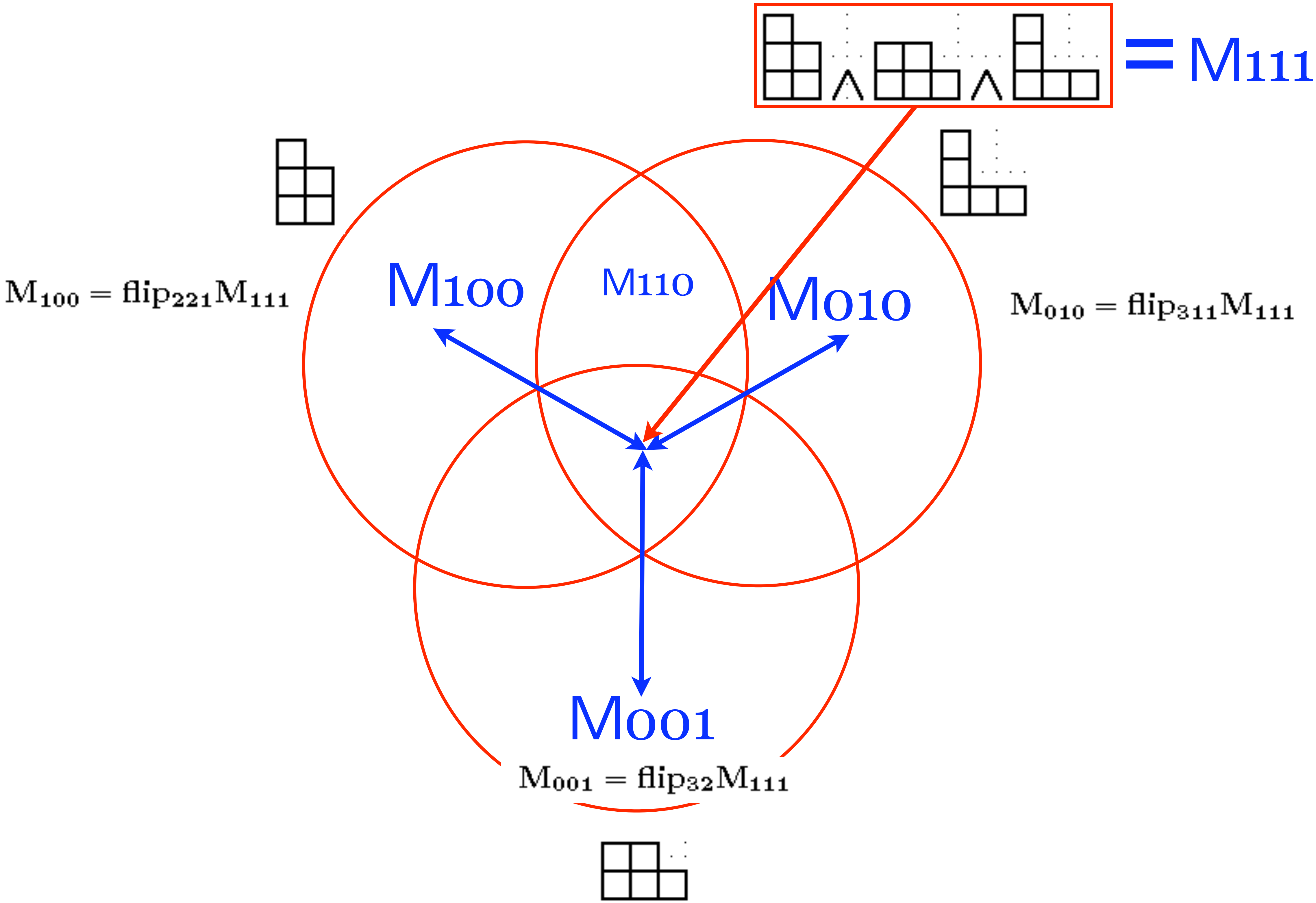

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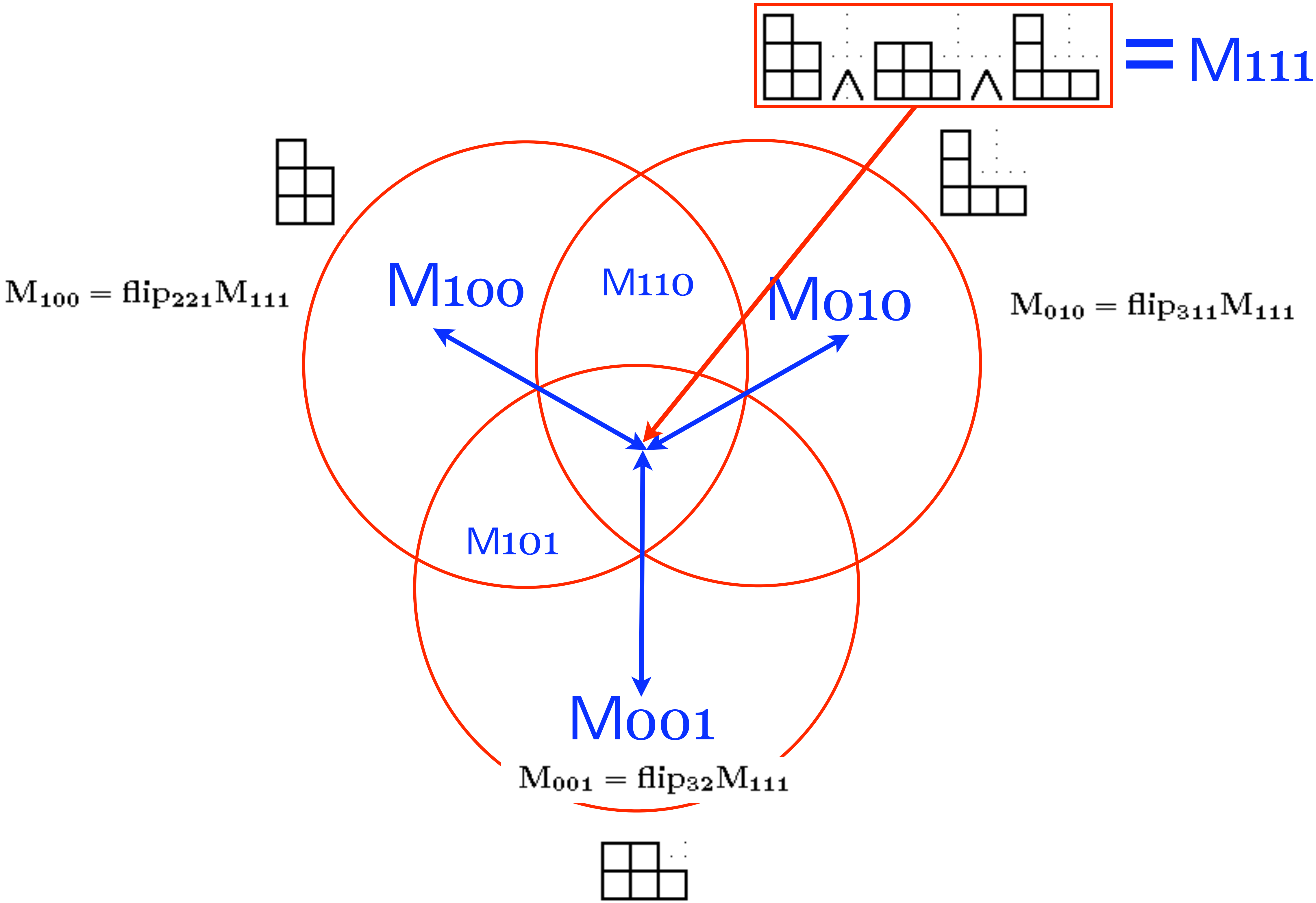

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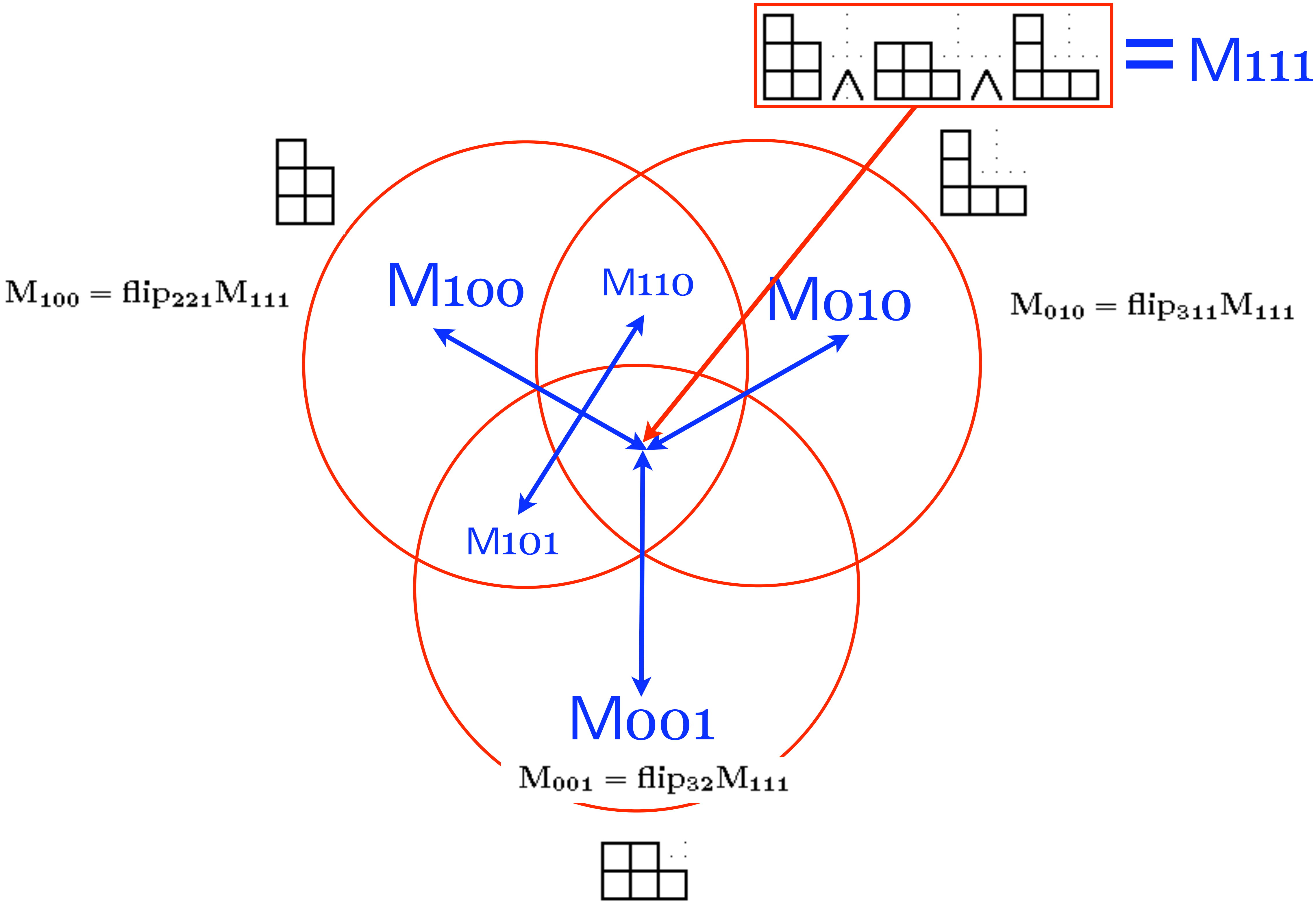
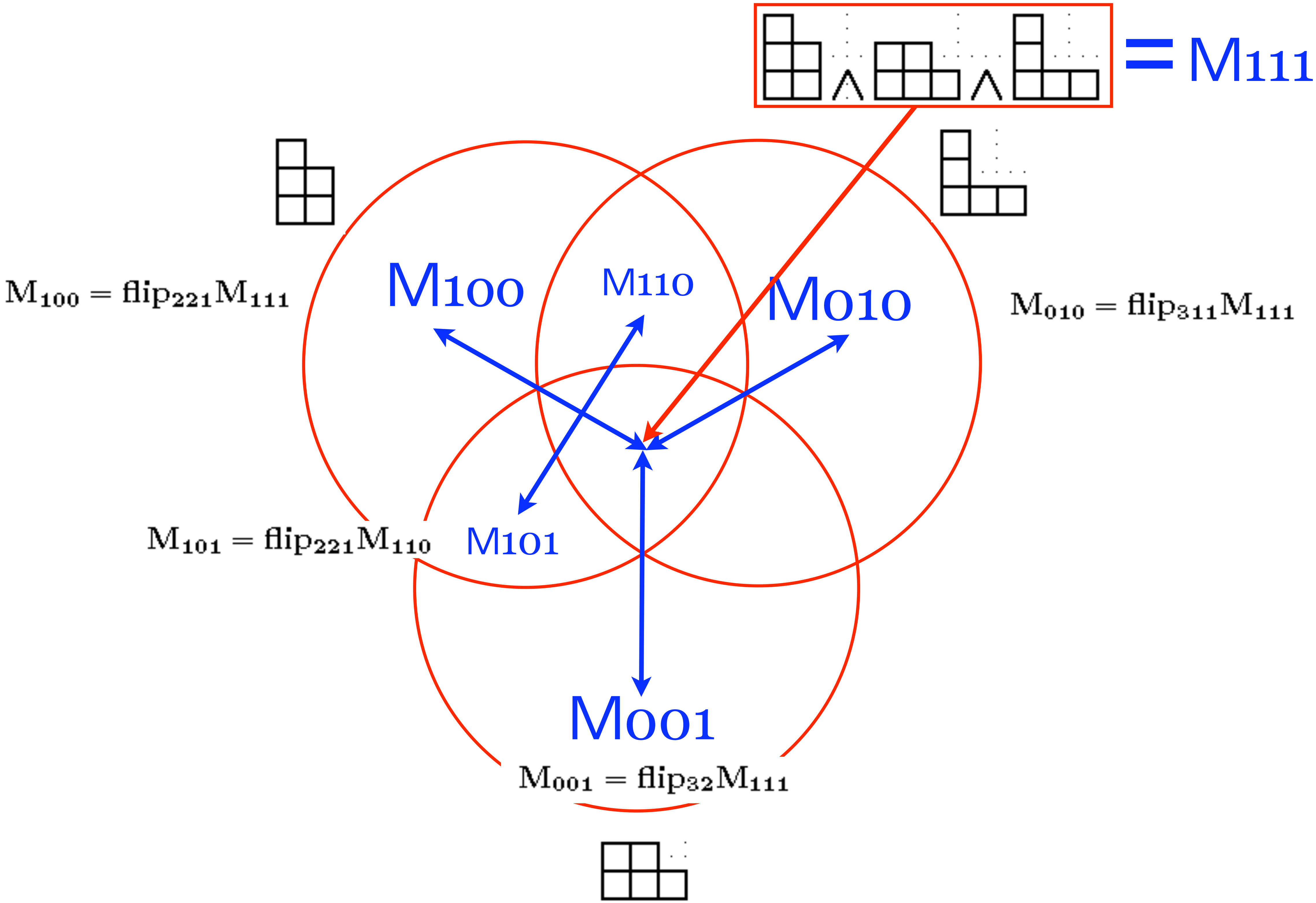
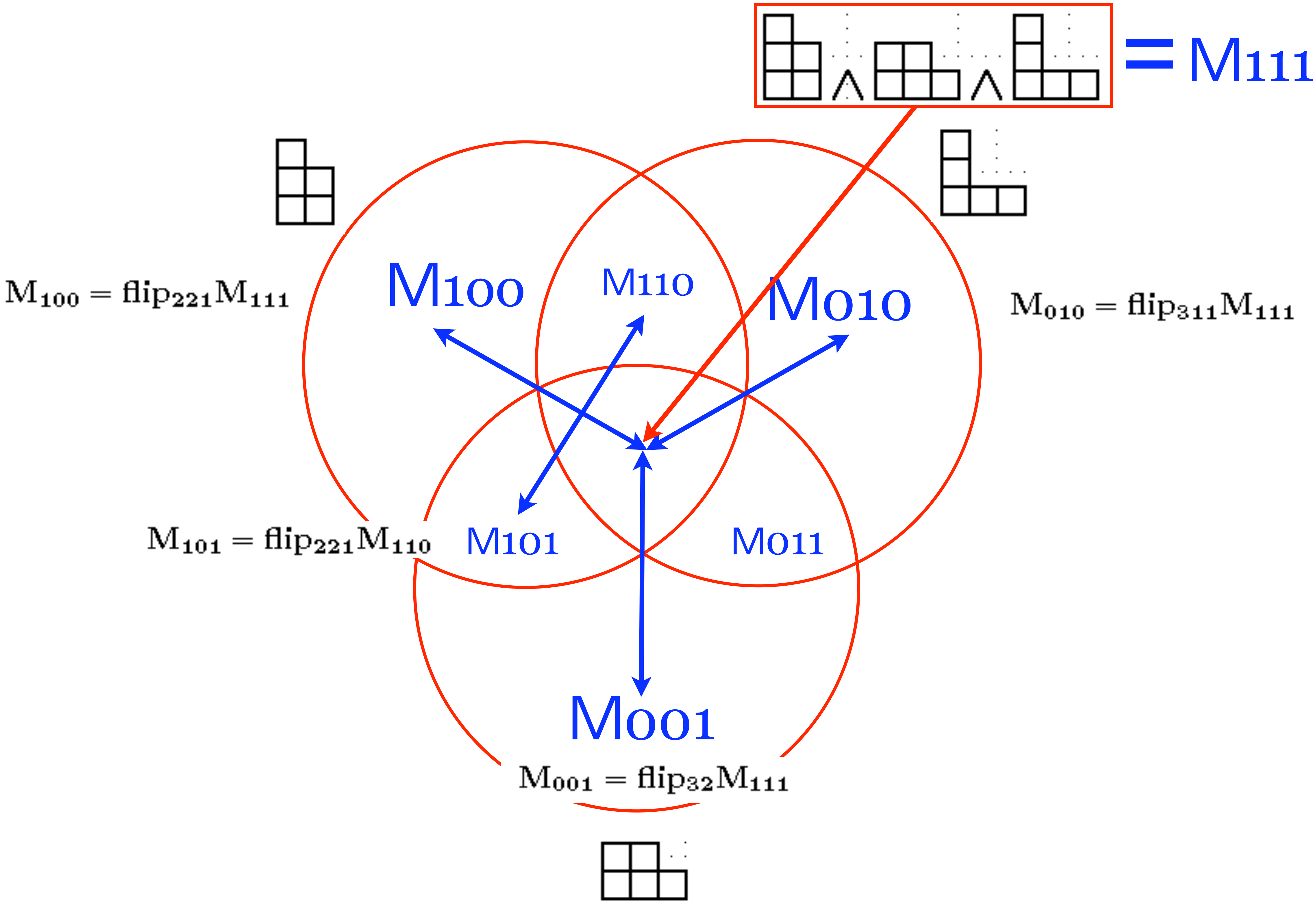


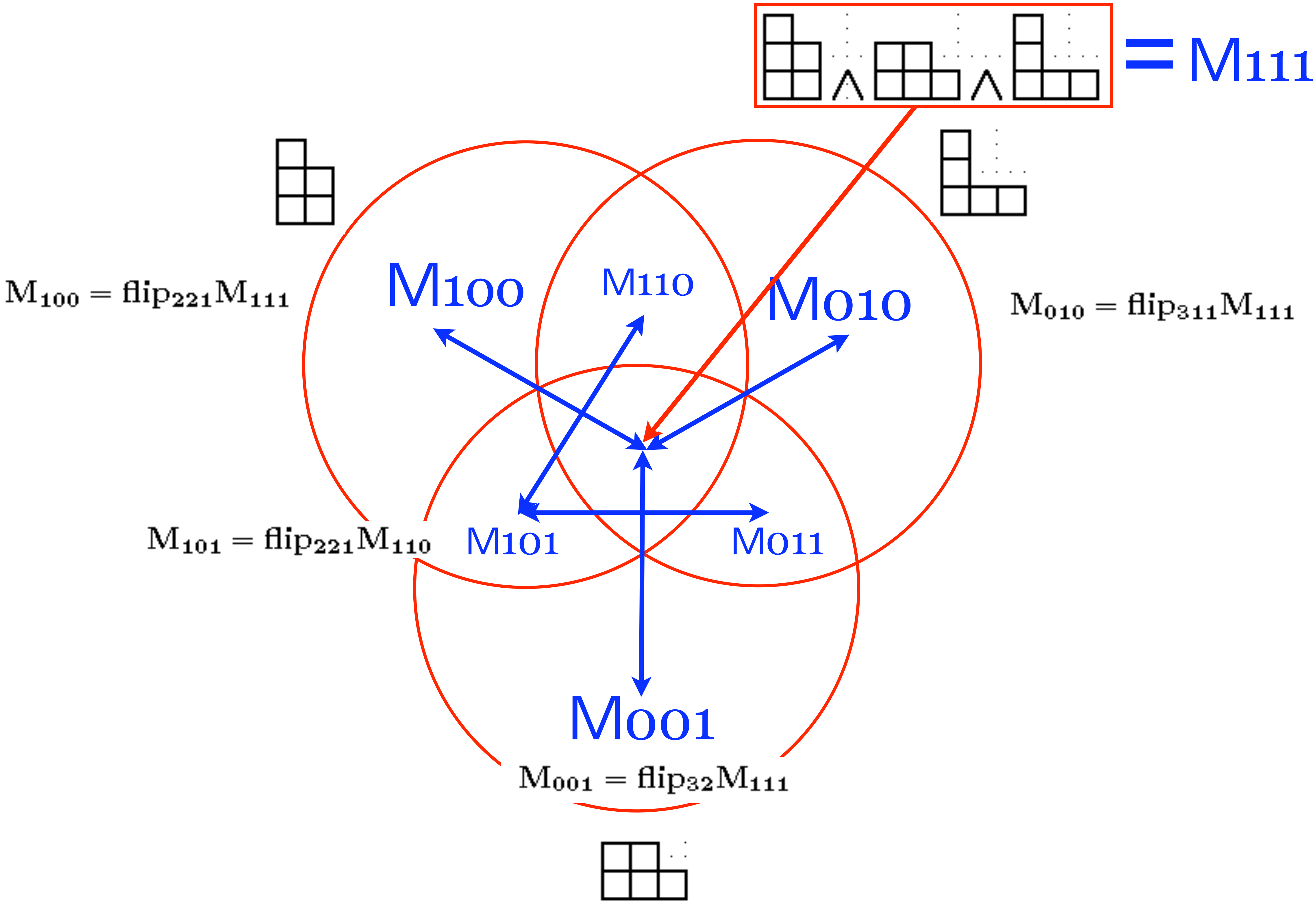
More miracles



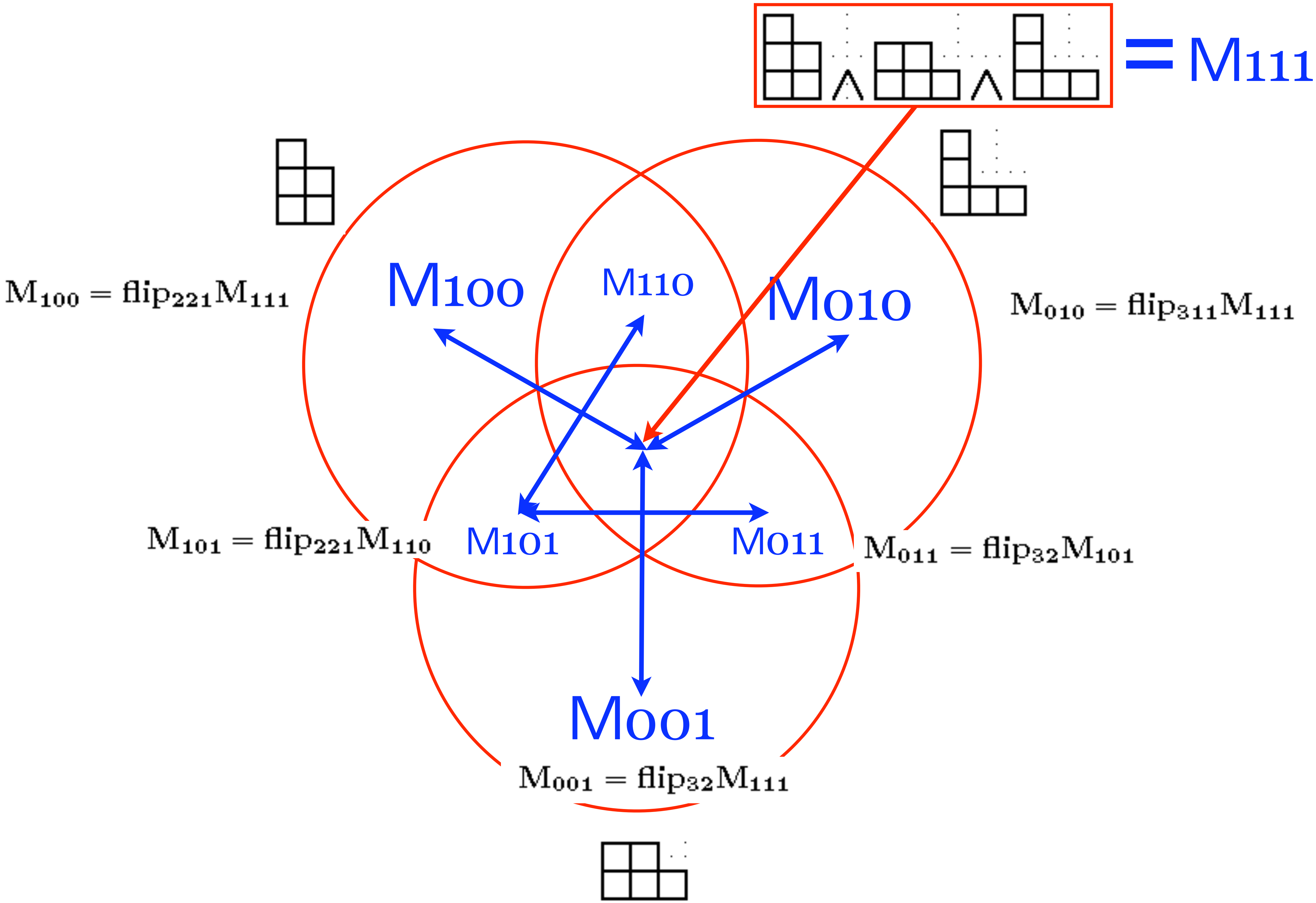
More miracles



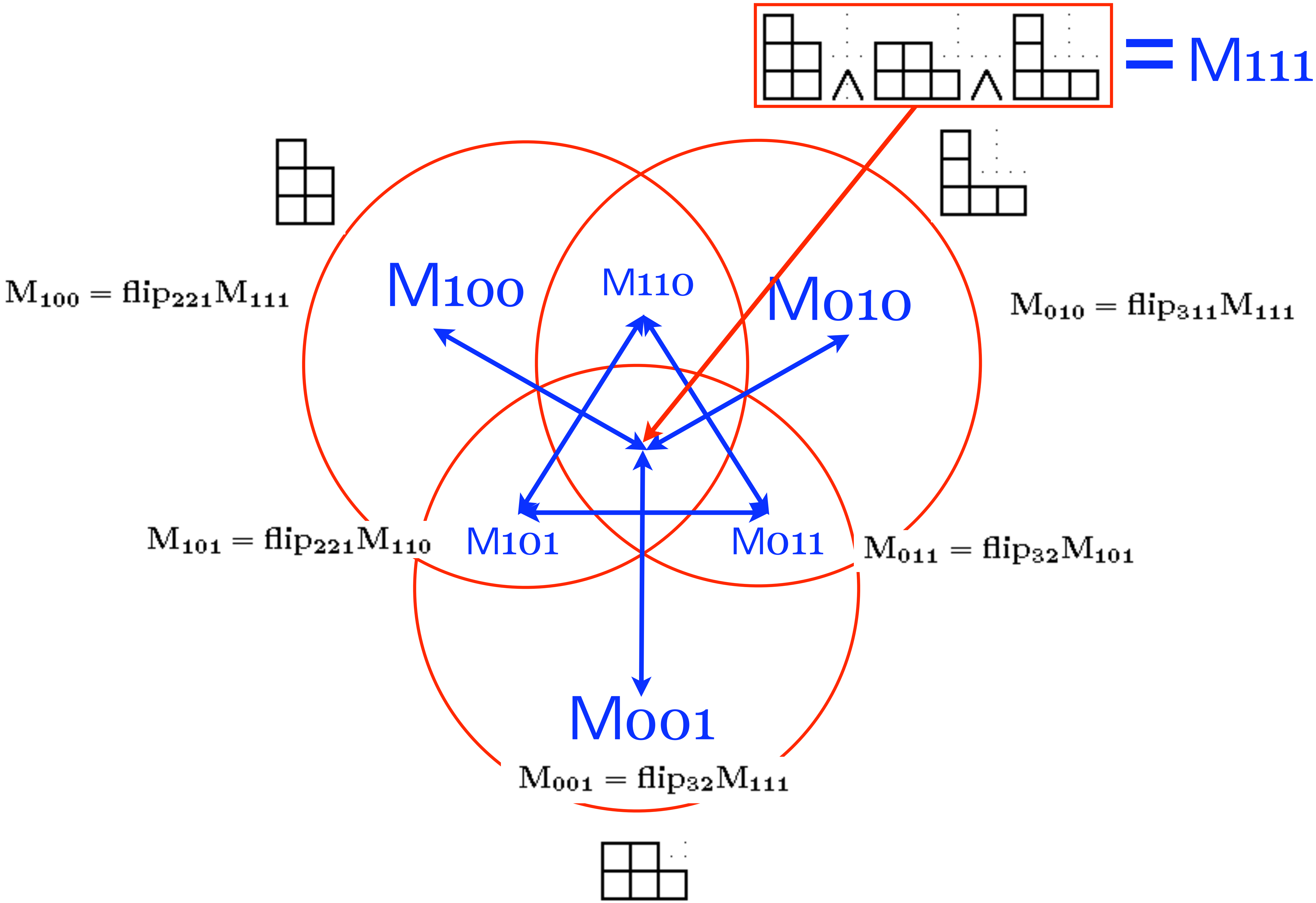
More miracles



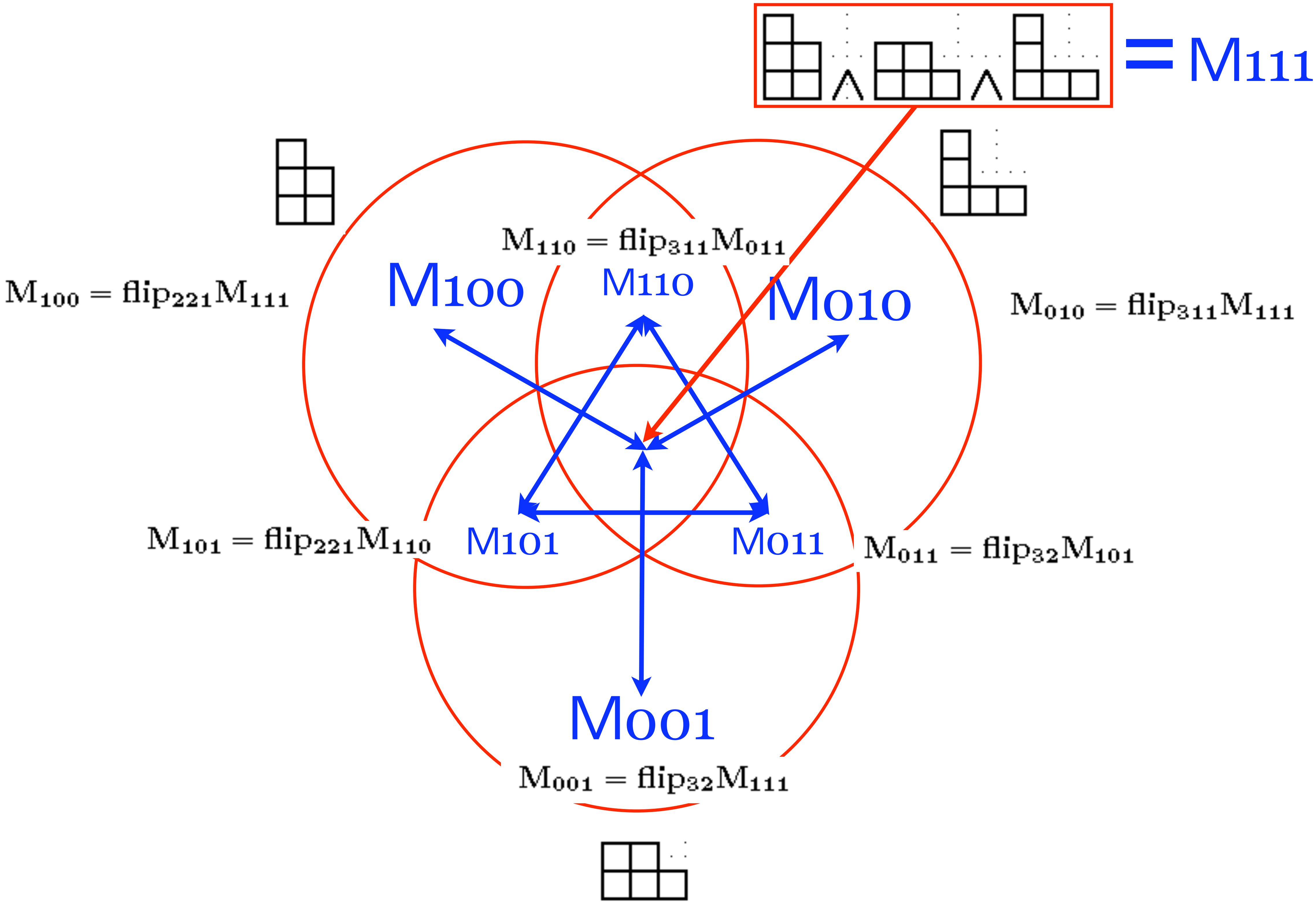
More miracles



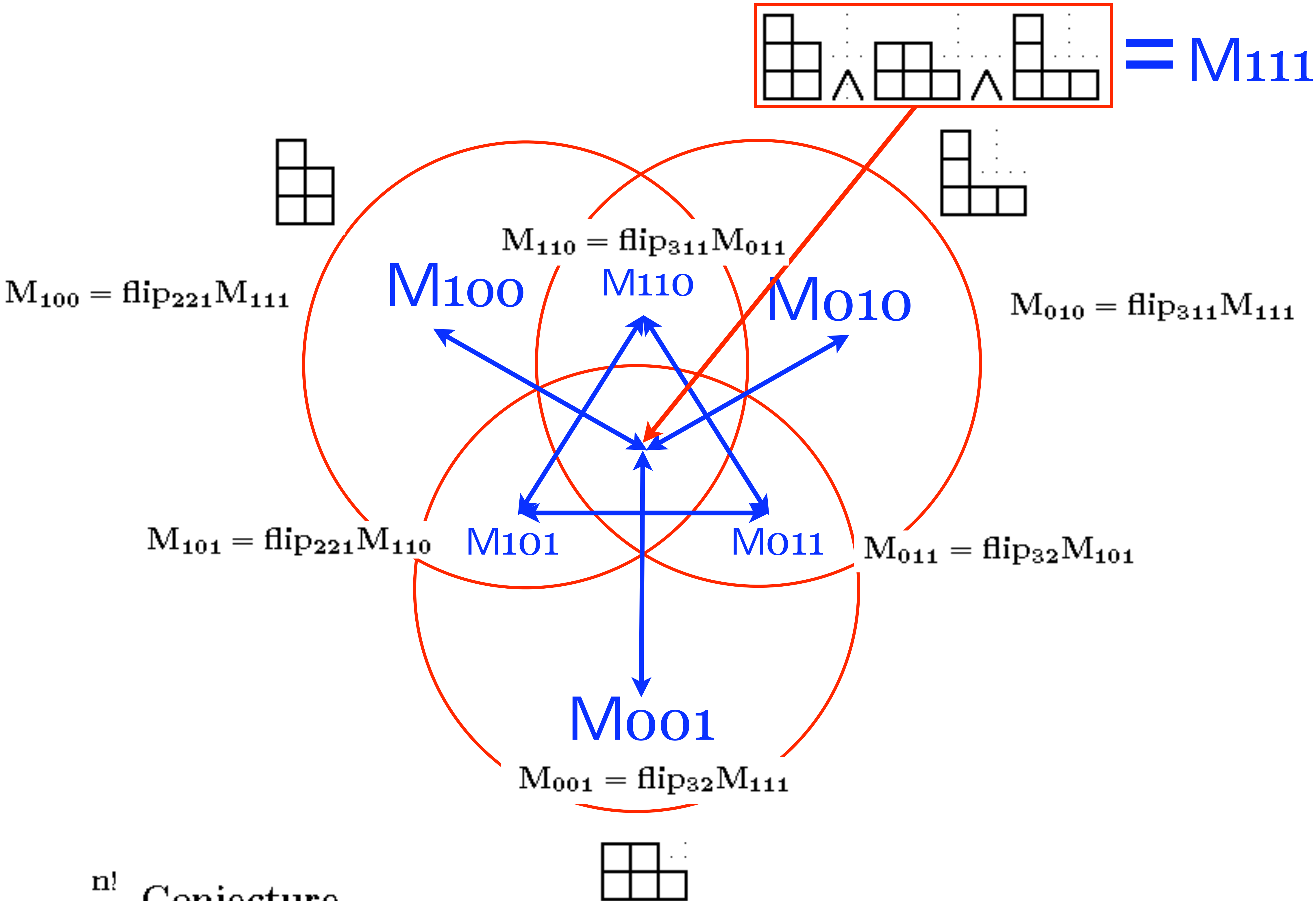
More miracles



More miracles

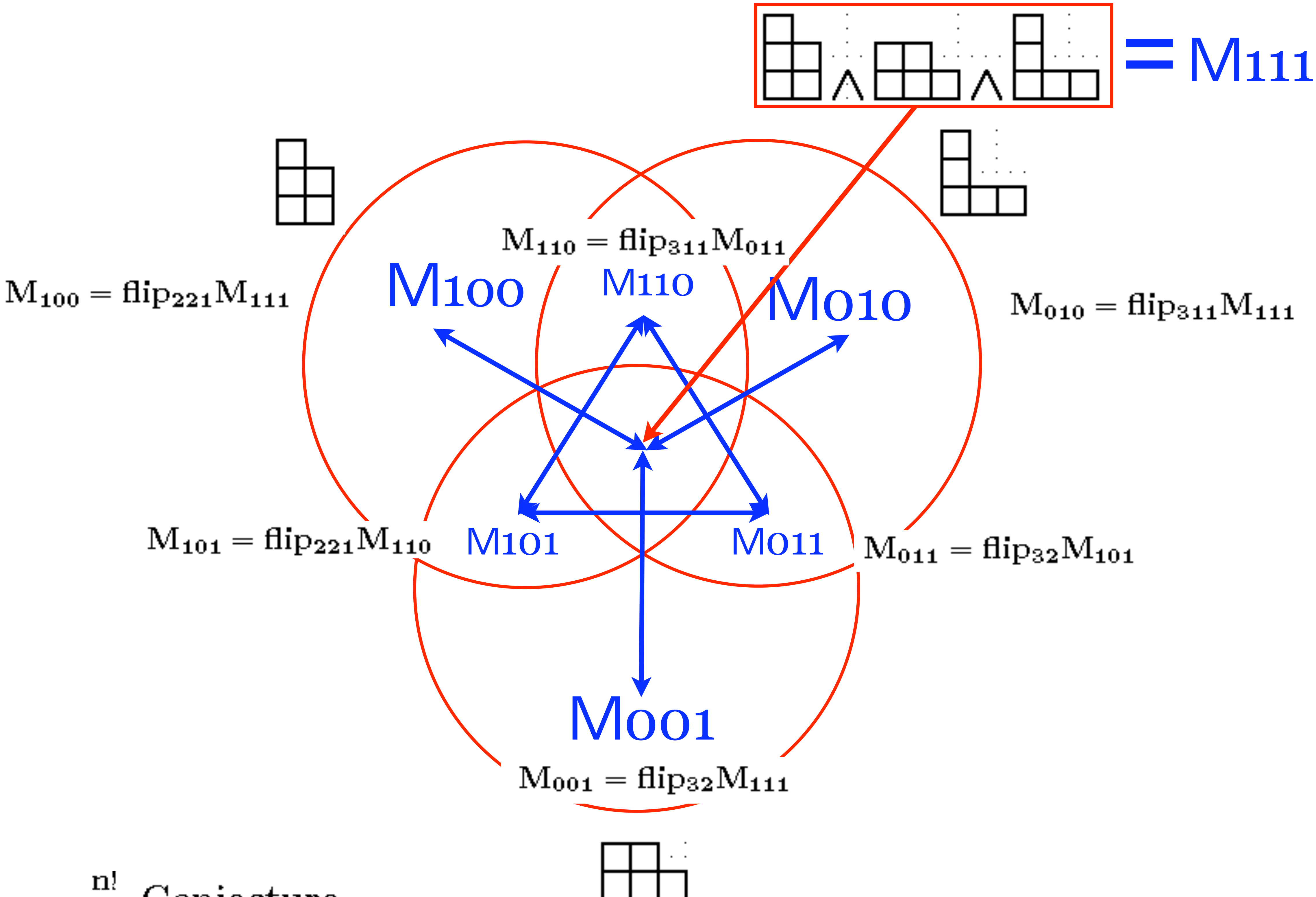


More miracles



$\frac{n!}{k!}$ Conjecture

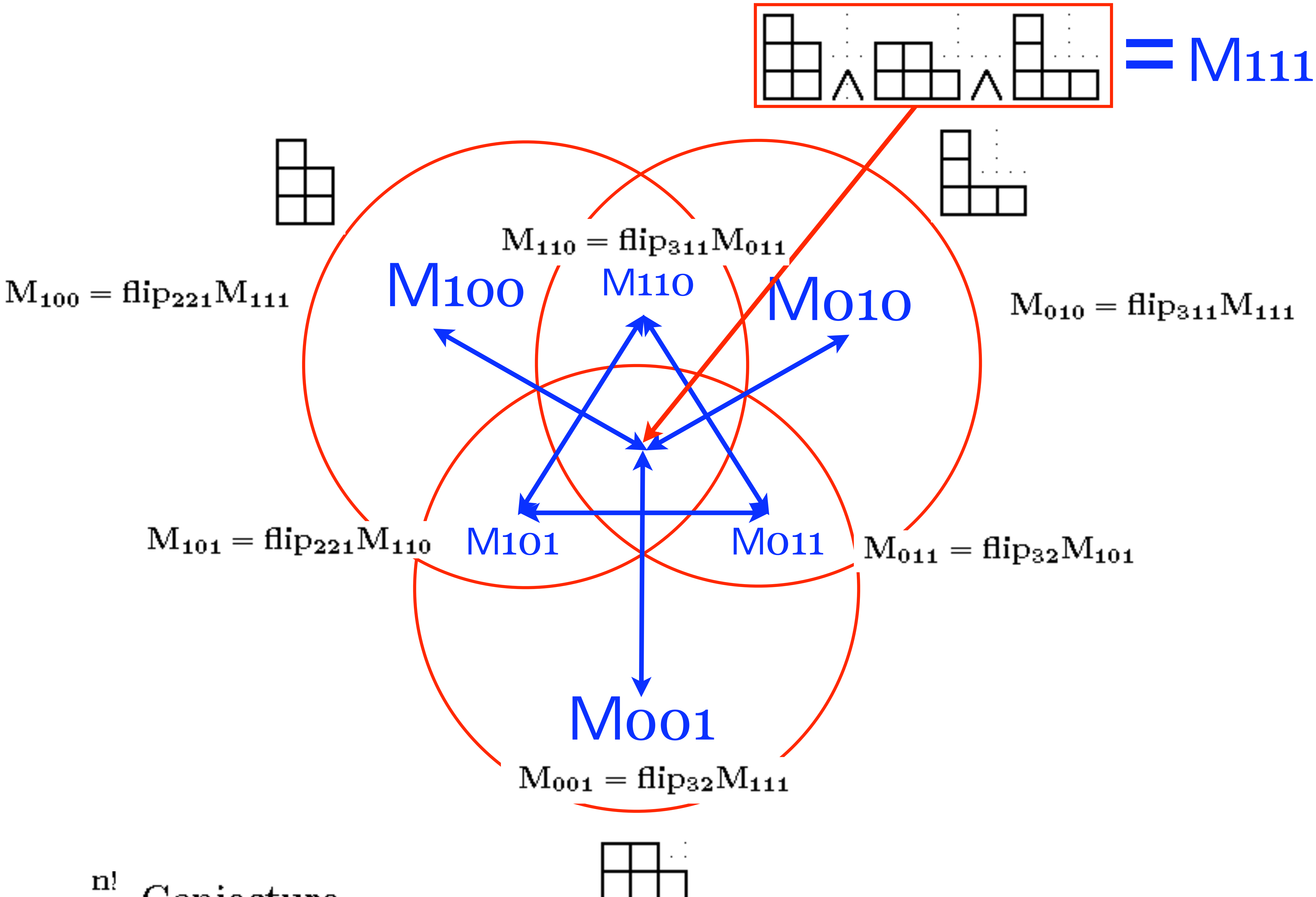
More miracles



$\frac{n!}{k!}$ **Conjecture**

If $\alpha_1, \alpha_2, \ldots, \alpha_k \vdash n$ *differ only by one cell from each other then*

More miracles



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If $\alpha_1, \alpha_2, \ldots, \alpha_k \vdash n$ *differ only by one cell from each other then*

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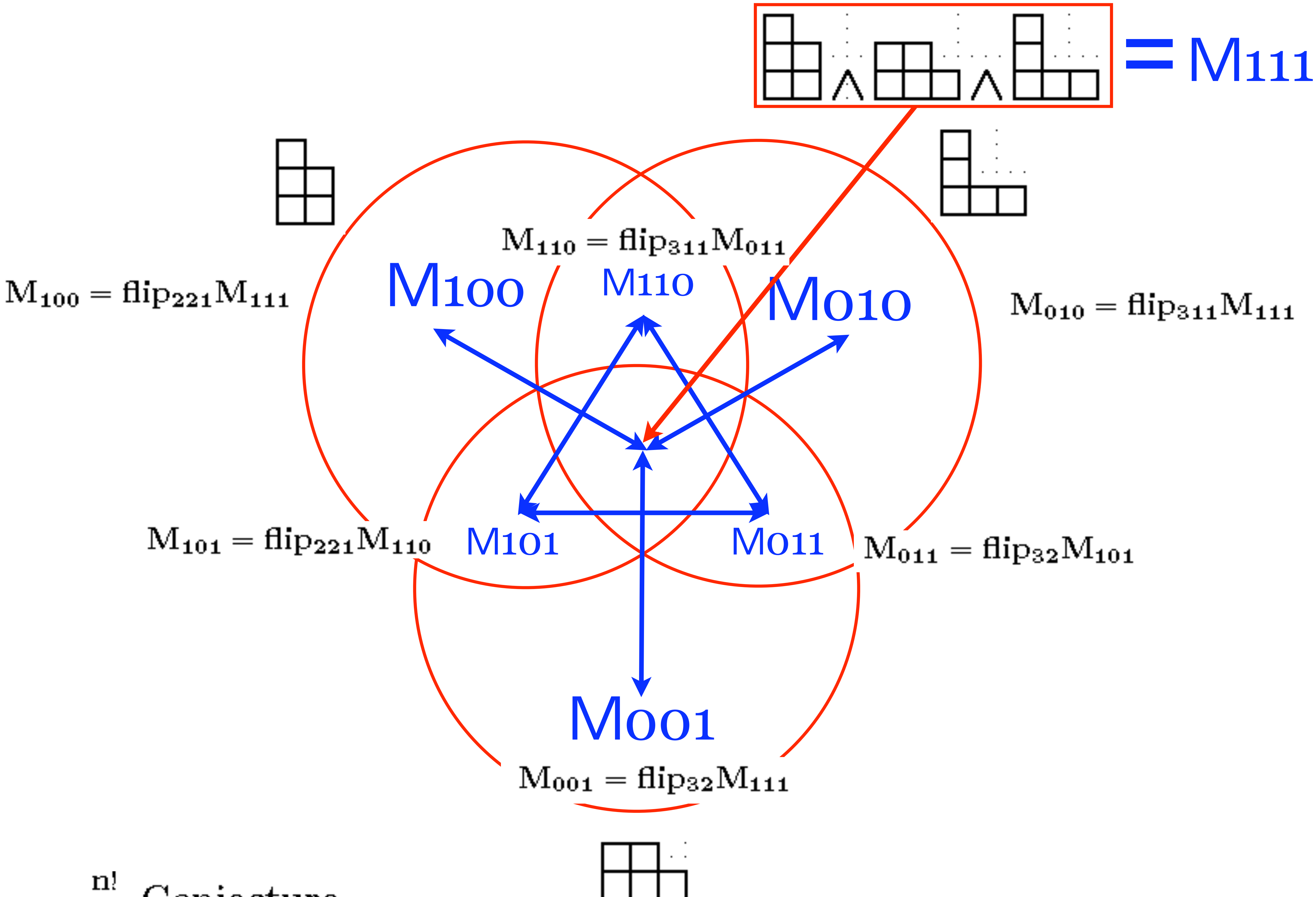
More miracles



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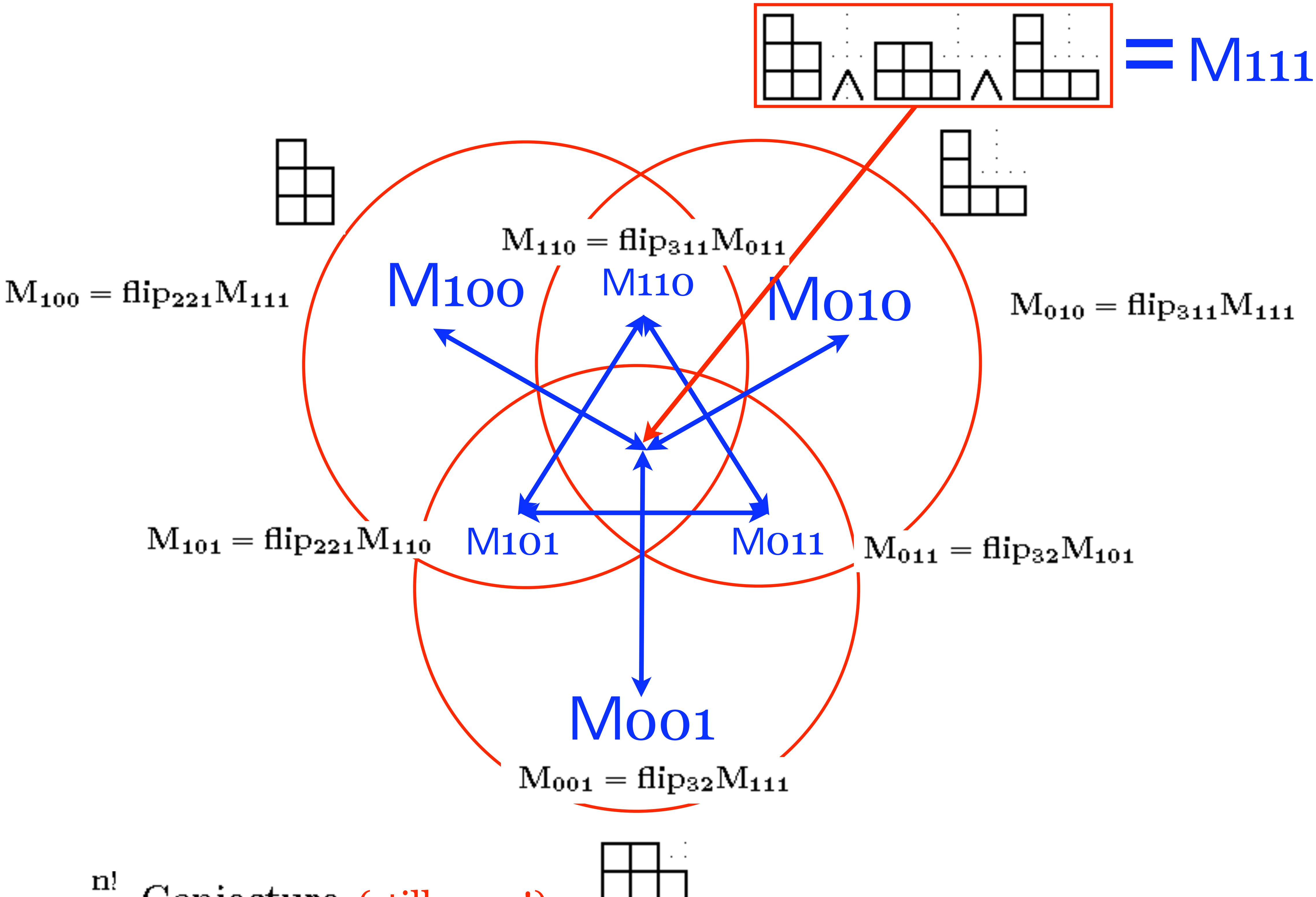
More miracles



$\frac{n!}{k!}$ **Conjecture** (still open!)

If $\alpha_1, \alpha_2, \ldots, \alpha_k \vdash n$ *differ only by one cell from each other then*

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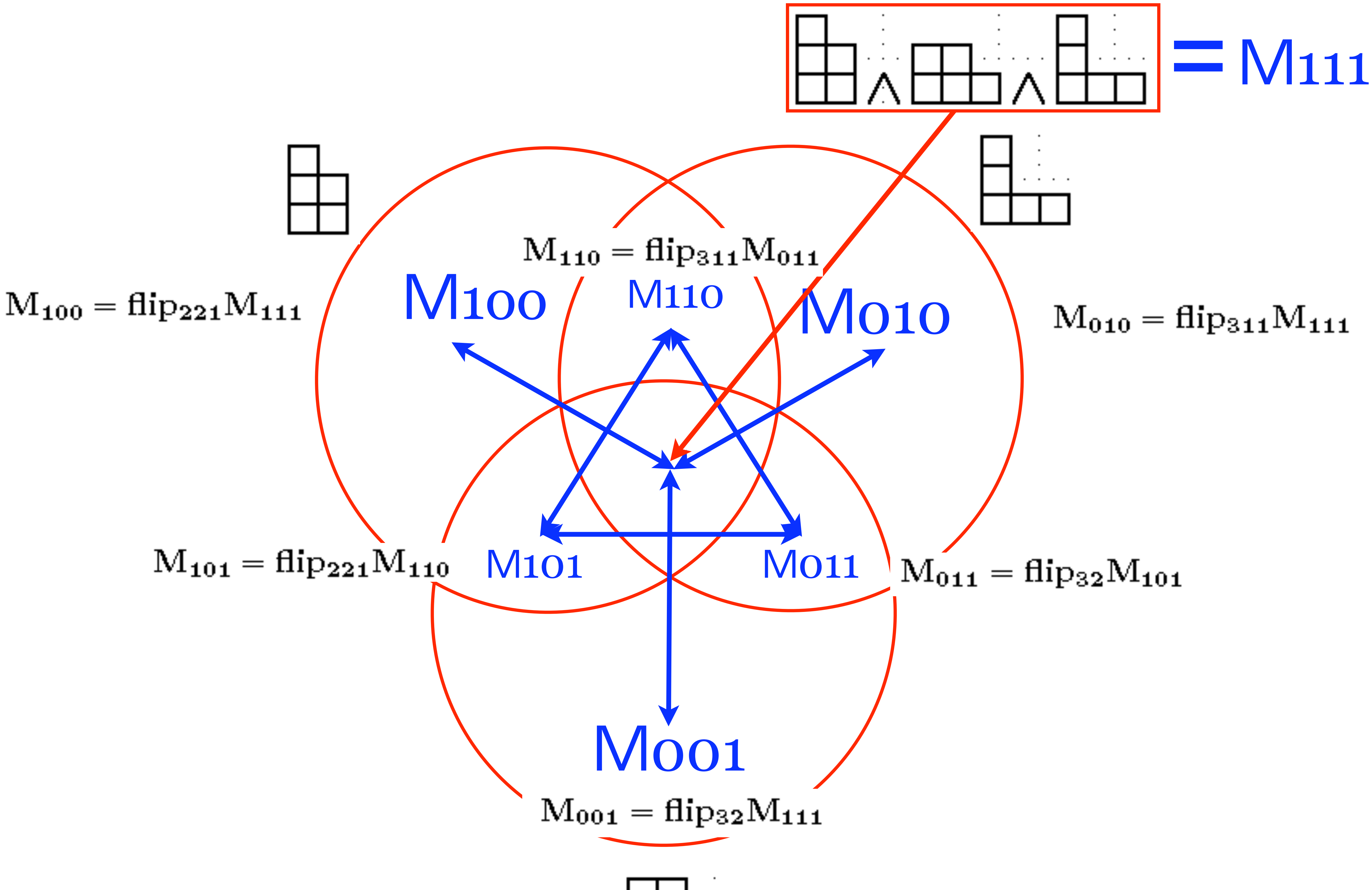

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next