Some Problems and Conjectures

in the

(manipulatorial)

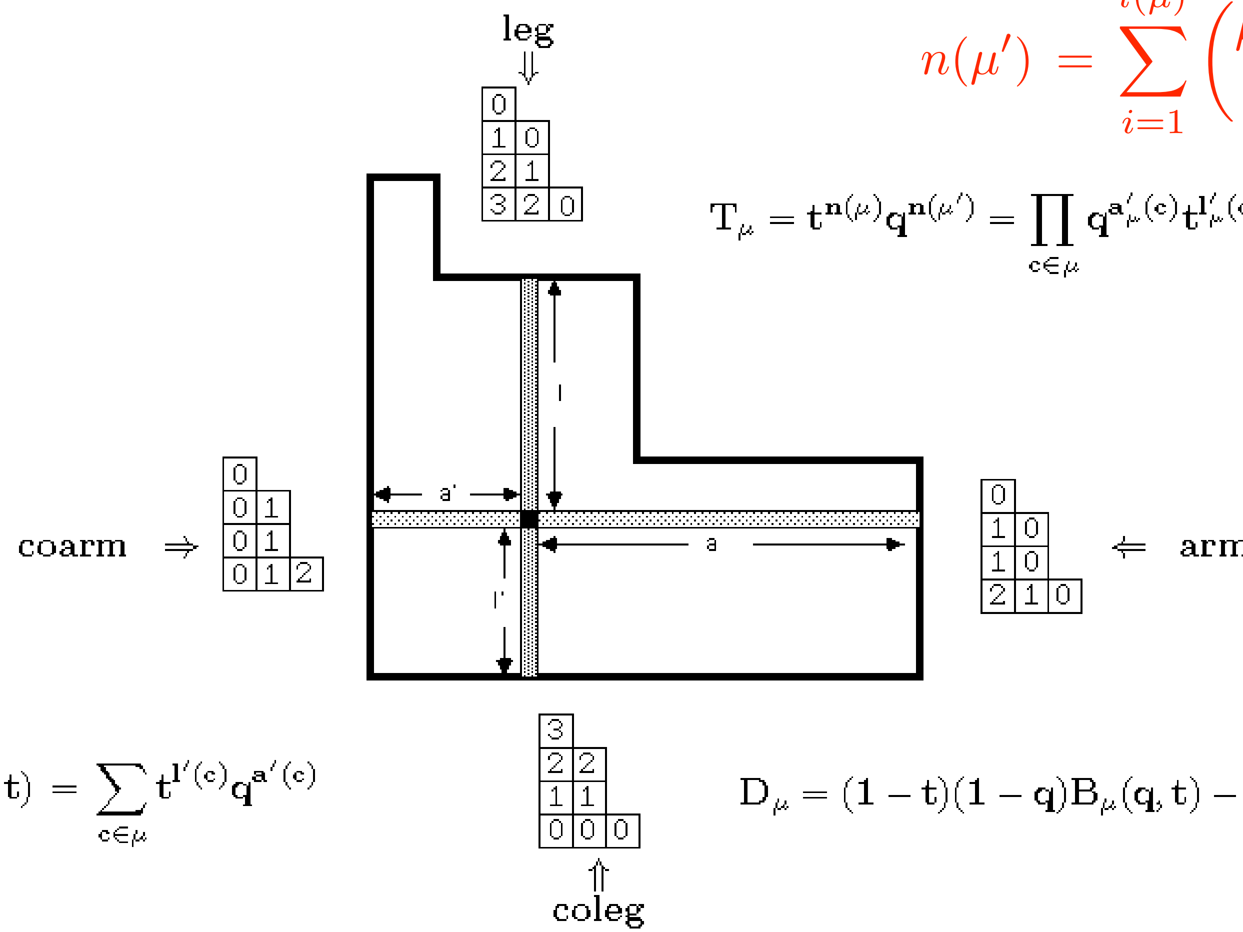
Theory of Macdonald Polynomials

Notation

$$\mathrm{n}(\mu) = \sum_{c\in\mu} \mathrm{l}'(c) = \sum_{c\in\mu} \mathrm{l}(c) \qquad \mathrm{n}(\mu') = \sum_{c\in\mu} \mathrm{a}'(c) = \sum_{c\in\mu} \mathrm{a}(c)$$

$$n(\mu') = \sum_{i=1}^{l(\mu)} \binom{\mu_i}{2}$$

$$\mathrm{T}_\mu = \mathrm{t}^{\mathrm{n}(\mu)}\mathrm{q}^{\mathrm{n}(\mu')} = \prod_{c\in\mu} \mathrm{q}^{\mathrm{a}'_\mu(c)}\mathrm{t}^{\mathrm{l}'_\mu(c)}$$



$$\mathrm{B}_\mu(\mathrm{q},\mathrm{t}) = \sum_{c\in\mu} \mathrm{t}^{\mathrm{l}'(c)}\mathrm{q}^{\mathrm{a}'(c)}$$

$$\mathrm{D}_\mu = (1-\mathrm{t})(1-\mathrm{q})\mathrm{B}_\mu(\mathrm{q},\mathrm{t}) - 1 \quad .$$

next

Preliminaries

The Macdonald polynomials we workwith here are those whose Schur function expansion is

$$\tilde{H}_\mu(x;q,t) = \sum_\lambda S_\lambda(x)\,\tilde{K}_{\lambda,\mu}(q,t)$$

where

$$\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)} K_{\lambda\mu}(q,1/t)$$

with $K_{\lambda\mu}(q,t)$ the Macdonald q,t-Kostka coefficient.

$$\tilde{H}_{[3,2]}(x;q,t) = s_5 + s_{4,1}\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix} + s_{3,2}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\end{bmatrix} + s_{3,1,1}\,q\begin{bmatrix}1 & 0 & 0\\ 1 & 2 & 1\\ 0 & 0 & 1\end{bmatrix} +$$

$$+\ s_{2,2,1}\ q\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} + s_{2,1,1,1}\ t\,q^2\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix} + s_{1,1,1,1,1}\ t^2 q^4$$

$$t^2q^3(a + bq + cq^2 + dt + etq + ft^2) \Rightarrow t^2q^3\begin{pmatrix}f & 0 & 0\\ d & e & 0\\ a & b & c\end{pmatrix}$$

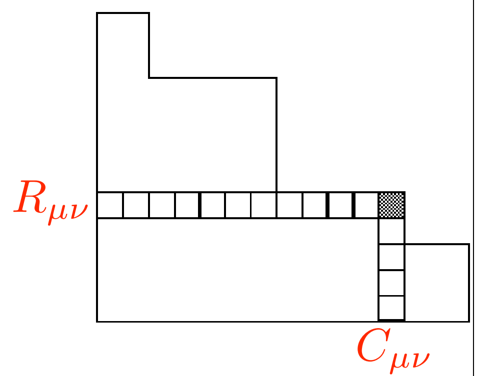
next

The Dual Pieri Rule

$$\partial_{p_1} \tilde{H}_\mu(x;q,t) \;=\; \sum c_{\mu\nu}(q,t)\, \tilde{H}_\nu(x;q,t)$$

with

$$c_{\mu\nu}(q,t) \;=\; \prod_{s\in R_{\mu/\nu}} \frac{t^{l_\mu(s)} - q^{a_\mu(s)+1}}{t^{l_\mu(s)} - q^{a_\mu(s)}} \prod_{s\in C_{\mu/\nu}} \frac{q^{a_\mu(s)} - t^{l_\mu(s)+1}}{q^{a_\mu(s)} - t^{l_\mu(s)}}$$



We have a combinatorial proof that $\displaystyle\sum_{\nu\to\mu} \mathbf{c}_{\mu\nu}(\mathbf{q},\mathbf{t}) \;=\; \mathbf{B}_\mu(\mathbf{q},\mathbf{t})$

We have a purely combinatorial description of the polynomial solutions of the recurrence

$$\mathbf{B_k}(\mathbf{q},\mathbf{t}) \;=\; \sum_{\nu\to\mu} \mathbf{c}_{\mu\nu}(\mathbf{q},\mathbf{t})\, \mathbf{B_{k-1}}(\mathbf{q},\mathbf{t}) \qquad \big(\mathbf{B_0}(\mathbf{q},\mathbf{t}) = \mathbf{B}(\mathbf{q},\mathbf{t})\big)$$

Problem

Give a combinatorial proof. More generally explain combinatorially why so many recurrences involving these dual Pieri coefficients have integral polynomial solutions.

next

NABLA

$$\nabla \tilde{H}_\mu = T_\mu \tilde{H}_\mu$$

	$\mathbf{S}4$	$\mathbf{S}31$	$\mathbf{S}22$	$\mathbf{S}211$
$\nabla\mathbf{S}4\rightarrow$	0	$-t^3\,q^3$	$-t^3\,q^3\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$-t^3\,q^3\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix}$
$\nabla\mathbf{S}31\rightarrow$	0	$t^2\,q^2\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$t^2\,q^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$t^2\,q^2\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&2&1&0\\0&0&1&1\end{pmatrix}$
$\nabla\mathbf{S}22\rightarrow$	0	$-t^2\,q^2$	0	$-t^2\,q^2\begin{pmatrix}1&0\\0&1\end{pmatrix}$
$\nabla\mathbf{S}211\rightarrow$	0	$-t\,q\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$-t\,q\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&0&1\end{pmatrix}$	$-t\,q\begin{pmatrix}1&0&0&0&0\\1&1&0&0&0\\0&2&1&0&0\\0&0&2&1&0\\0&0&0&1&1\end{pmatrix}$

next

Totally positive or totally negative

Conjecture I

For any pair of partitions λ, μ and for a every positive integer m we have

$$(-1)^{b(\lambda')} \langle \nabla S_\lambda \, , \, S_\mu \rangle \in N[q,t] \qquad \textbf{(*)}$$

with $\langle \, , \, \rangle$ the Hall inner product and

$$b(\lambda) = \binom{l(\lambda)}{2} + \sum_{\lambda_i < (i-1)} (i - 1 - \lambda_i)$$

The sign in (*) was identified by M. Bousquet-Melou who gave a combinatorial interpretation to the left hand side of (*), for $m = 1$ and $\mu = 1^n$.

Note for $\lambda = 1^n$ (*) is a corollary of a general result of Mark Haiman.

Conjecture II (A. Lascoux)

For any partitions λ, μ we have

$$(-1)^{|\mu| - l(\mu)} \langle \nabla \tilde{H}_\mu(x; 0, t) \, , \, S_\lambda(x) \rangle \in \mathbf{N}[q,t]$$

next

NABLA at t=1

$$(1) \qquad \tilde{\nabla}\tilde{h}_m\left[\frac{X}{1-q}\right] = q^{\binom{m}{2}} h_m\left[\frac{X}{1-q}\right]$$

$$(2) \qquad \tilde{\nabla}\big(F(x)\times G(x)\big) = \tilde{\nabla}F(x)\times\tilde{\nabla}G(x)$$

$$T_\mu\Big|_{t=1} = \prod_{i=1}^{l(\mu)} q^{\binom{\mu_i}{2}}$$

$$\tilde{H}_\mu(x;q,1) = \prod_{i=1}^{l(\mu)}(1-q)\cdots(1-q^{\mu_i})h_{\mu_i}\left[\frac{X}{1-q}\right]$$

$$\tilde{\nabla}\tilde{H}_\mu(x;q,1) = \prod_{i=1}^{l(\mu)}(1-q)\cdots(1-q^{\mu_i})q^{\binom{\mu_i}{2}}h_{\mu_i}\left[\frac{X}{1-q}\right]$$

next

What is known

Theorem(M. Haiman)

For all μ

$$\langle \nabla \mathbf{e_n} \ , \ \mathbf{S}_\mu \rangle \in \mathbf{N}[\mathbf{q},\mathbf{t}]$$

Theorem(C.Lennart)

For all λ, μ

$$(-\mathbf{1})^{\mathbf{b}(\lambda)} \langle \tilde{\nabla} \mathbf{S}_\lambda \ , \ \mathbf{S}_\mu \rangle \in \mathbf{N}[\mathbf{q},\mathbf{t}]$$

Note Jacobi-Trudi plus multiplicativity gives

$$\tilde{\nabla} \mathbf{S}_\lambda = \mathbf{det} \left\| \tilde{\nabla} \mathbf{e}_{\lambda'_\mathbf{i}+\mathbf{j}-\mathbf{i}} \right\|_{\mathbf{i},\mathbf{j}}$$

(Lennart and Bousquet-Melou use this to prove their result)

next

PROBLEM: Nabla "Lagrange inverts" what?

At t=1

Theorem

The formal series

$$\mathbf{f(z)} \;=\; \mathbf{z}\sum_{\mathbf{n>0}} \mathbf{q^n}\,\tilde{\nabla}\mathbf{e_n}\,\mathbf{z^n} \qquad \left(\tilde{\nabla}\mathbf{e_n} \;=\; \sum_{\mu\vdash \mathbf{n}} \Big(\prod_{\mathbf{i}} \mathbf{q}^{\binom{\mu_i}{2}}\mathbf{h}_{\mu_i}\big[\tfrac{\mathbf{X}}{(1-\mathbf{q})}\big]\Big)\,\mathbf{f}_\mu[1-\mathbf{q}]\right)$$

is the q-Lagrange inverse of the series

$$\mathbf{F(z)} \;=\; \sum_{\mathbf{n\geq 1}} \mathbf{F_n}\,\mathbf{z^n} \;=\; \frac{\mathbf{z}}{\mathbf{E(z)}} \qquad \left(\;\mathbf{E(z)} \;=\; \sum_{\mathbf{n>0}} \mathbf{e_n(x)}\,\mathbf{z^n}\;\right).$$

More precisely we show that $\displaystyle\sum_{\mathbf{n\geq 1}} \mathbf{F_n}\,\mathbf{f(z)f(zq)\cdots f(zq^{n-1})} \;=\; \mathbf{z}\;.$

What if t =1/q ?

$$f(z) \;=\; z\sum_{n>0} q^{\binom{n}{2}}\nabla e_n\Big|_{t=1/q}\; z^n$$

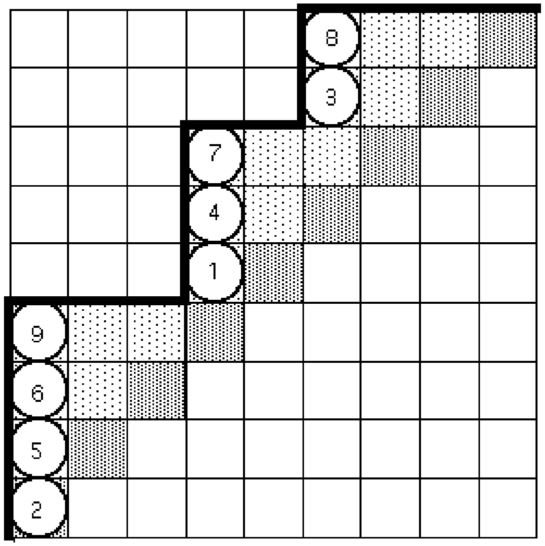
$$q^{\binom{n}{2}}\nabla e_n\Big|_{t=1/q} \;=\; \frac{e_n\big[X(1+q+\cdots+q^n)\big]}{1+q+\cdots+q^n} \;=\; \frac{1}{1+q+\cdots+q^n}\,R(z)R(zq)\cdots R(zq^n)\Big|_{z^n}$$

$$R(z) \;=\; \sum_{m\geq 0} e_m(x)\,z^n$$

$$f(z) \;=\; z\sum_{n\geq 0} \frac{1}{1+q+\cdots+q^n}\,R(z)R(zq)\cdots R(zq^n)\Big|_{z^n}\;,$$

next

NABLA AND PARKING FUNCTIONS



$$\Longrightarrow \quad q^{17}\, e_4(x)\, e_3(x)\, e_2(x)$$

$$\tilde{\nabla} e_n = \sum_{D \in \mathcal{D}_n} q^{a(D)} \prod_{i=1}^{n} e_{\alpha_i(D)}(x)$$

$\tilde{\nabla} e_n$ is a q-analogue of the Frobenius characteristic of a

sign twisted version of the action of S_n on parking functions

next

Two Important Operators to be further explored

$$\mathbf{D_k\, F[X] = F\big[X + \tfrac{(1-t)(1-q)}{z}\big] \sum_{m\geq 0} (-z)^m e_m[X] \Big|_{z^k}}$$

$$\mathbf{D_k^*\, F[X] = F\big[X - \tfrac{(1-1/t)(1-1/q)}{z}\big] \sum_{m\geq 0} z^m h_m[X] \Big|_{z^k}} .$$

(i)	$\mathbf{D_0\, \tilde{H}_\mu = -D_\mu(q,t)\, \tilde{H}_\mu}$,	(i)*	$\mathbf{D_0^*\, \tilde{H}_\mu = -D_\mu(1/q,1/t)\, \tilde{H}_\mu}$
(ii)	$\mathbf{D_k\, \underline{e}_1 - \underline{e}_1\, D_k = M\, D_{k+1}}$	(ii)*	$\mathbf{D_k^*\, \underline{e}_1 - \underline{e}_1\, D_k^* = -\widetilde{M}\, D_{k+1}^*}$
(iii)	$\mathbf{\nabla\, \underline{e}_1 \nabla^{-1} = -D_1}$	(iii)*	$\mathbf{\nabla\, D_1^* \nabla^{-1} = \underline{e}_1}$
(iv)	$\mathbf{\nabla^{-1}\, \partial_1 \nabla = \tfrac{1}{M} D_{-1}}$	(iv)*	$\mathbf{\nabla^{-1}\, D_{-1}^* \nabla = -\widetilde{M}\, \partial_1}$

$$\mathbf{M = (1-t)(1-q)\ ,\qquad \widetilde{M} = (1-1/t)(1-/q)}$$

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s, 1^a)$ **with** $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_s \geq 2$ **and** $a \geq 0$ **set**

$$W_\lambda[X;q,t] \;=\; \underline{e}_1^a D_1^* \underline{e}_1^{\lambda_1 -1} D_1^* \underline{e}_1^{\lambda_2 -1} \cdots D_1^* \underline{e}_1^{\lambda_s -1}$$

$$U_\lambda[X;q,t] \;=\; D_1^a \underline{e}_1 D_1^{\lambda_1 -1} \underline{e}_1 D_1^{\lambda_2 -1} \cdots \underline{e}_1 D_1^{\lambda_s -1}\, 1$$

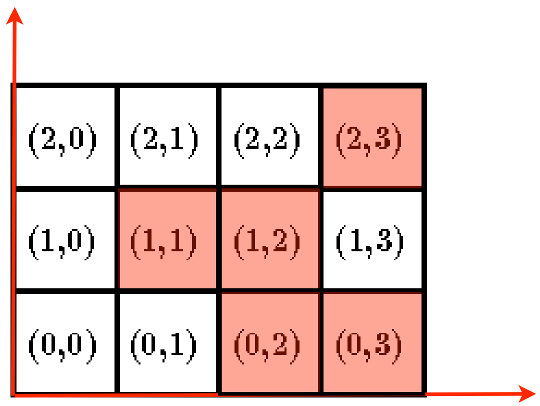
Theorem

The operator ∇ may be computed from the identity

$$\nabla\, W_\lambda[X;q,t] \;=\; (-1)^{a+\sum_{i=1}^{s}(\lambda_i -1)}\;\; U_\lambda[X;q,t]\ .$$

next

Lattice Polynomials



$$\mathbf{L} \;=\; \{\, (\mathbf{0},\mathbf{2})\,,\; (\mathbf{0},\mathbf{3})\,,\; (\mathbf{1},\mathbf{1})\,,\; (\mathbf{1},\mathbf{2})\,,\; (\mathbf{2},\mathbf{3}) \,\}$$

$$\boldsymbol{\Delta}_{\mathbf{L}} \;=\; \mathbf{det} \begin{pmatrix} y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 \\ y_1^3 & y_2^3 & y_3^3 & y_4^3 & y_5^3 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_4y_5 \\ x_1y_1^2 & x_2y_2^2 & x_3y_3^2 & x_4y_4^2 & x_4y_5^2 \\ x_1^2y_1^3 & x_2^2y_2^3 & x_3^2y_3^3 & x_4^2y_4^3 & x_5^2y_5^3 \end{pmatrix}$$

More generally if

$$\mathbf{L} \;=\; \{\, (\mathbf{p_1},\mathbf{q_1})\,,\; (\mathbf{p_2},\mathbf{q_2})\,,\; \ldots\,,\; (\mathbf{p_n},\mathbf{q_n}) \,\}$$

$$\Delta_L[X,Y] \;=\; \left\| x_i^{p_j} y_i^{q_j} \right\|_{i,j=1}^{n}$$

The Corresponding S_n Modules

$$\mathbf{M_L} \quad = \quad \mathcal{L}\left[\partial_{\mathbf{x}}^{\mathbf{p}} \partial_{\mathbf{y}}^{\mathbf{q}} \mathbf{\Delta_J}[\mathbf{X}, \mathbf{Y}]\right]$$

Definition

A Lattice diagram with n cells will be called "regular" if the corresponding module is a multiple of the regular representation of S_n.

Theorem (Mark Haiman)

Every Ferrers diagram is regular and its bigraded Frobenius characteristic is

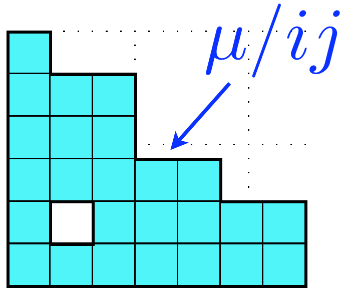
$$\tilde{\mathbf{H}}_\mu(\mathbf{x}; \mathbf{q}, \mathbf{t})$$

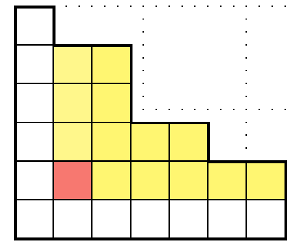
Corollary

The diagram $\mu/00$ obtained by removing the 00 cell from a Ferrers diagram of a partition $\mu \vdash n+1$ is regular and its bigraded Frobenius characteristic is

$$\partial_{\mathbf{p_1}} \tilde{\mathbf{H}}_\mu(\mathbf{x}; \mathbf{q}, \mathbf{t})$$

Pierced Diagrams

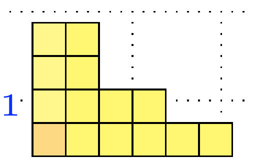
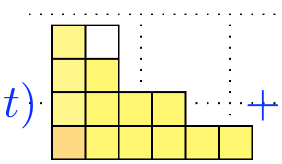
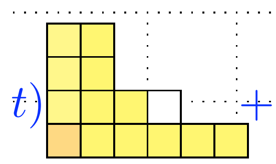
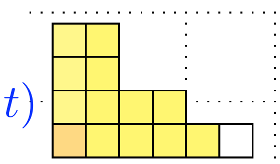




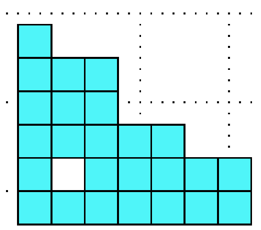
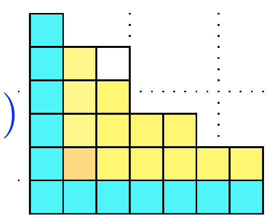
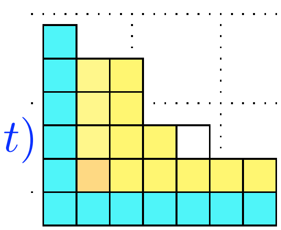
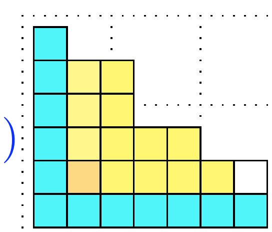
The shadow of the cell (i,j)

Conjecture III (Francois and Nantel)

For $\mu \vdash n+1$ the lattice diagram μ/ij is always regular and the module $\mathbf{M}_{\mu/ij}$ affords as many copies of the regular representation of S_n as there are cells in the shadow of (i,j)

∂_{p_1}  $= c_1(q,t)$  $+c_2(q,t)$  $+c_3(q,t)$ 

A Macdonald dual Pieri rule

 $= c_1(q,t)$  $+c_2(q,t)$  $+c_3(q,t)$ 

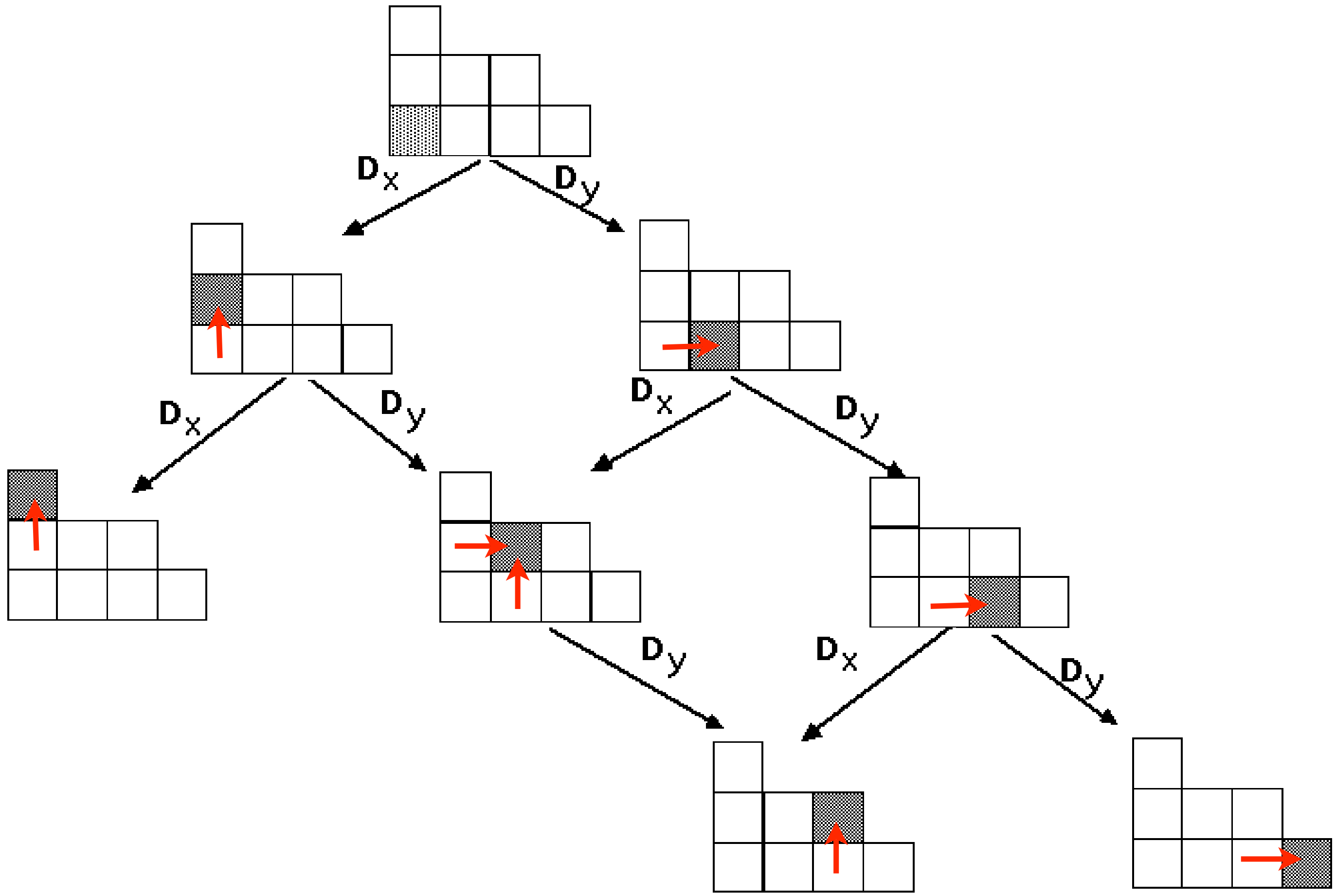
The hole propagates to one of the corners in its shadow!!

next

The action of the operators

$$D_x = \sum_{i=1}^{n} \partial_{x_i} \Uparrow \qquad D_y = \sum_{i=1}^{n} \partial_{y_i} \Rightarrow$$

THE CASE OF (4,3,1)



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THE CHARACTERISTIC OF PIERCED FERRERS MODULES

Conjecture IV

Denoting by $\mathbf{C}_{\mu/\mathbf{ij}}(\mathbf{x};\mathbf{q},\mathbf{t})$ *the characteristic of* $\mathbf{M}_{\mu/\mathbf{ij}}$*, we have*

$$\mathbf{C}_{\mu/\mathbf{ij}}(\mathbf{x};\mathbf{q},\mathbf{t}) \;=\; \sum_{\rho\to\tau} \mathbf{c}_{\tau\rho}(\mathbf{q},\mathbf{t})\,\tilde{\mathbf{H}}_{\mu-\tau+\rho}\,(\mathbf{x};\mathbf{q},\mathbf{t}) \qquad (*)$$

where τ *denotes the partition corresponding to the shadow of* $(\mathbf{i},\mathbf{j})$ *and the symbol "*$\mu-\tau+\rho$*" is to represent replacing* τ *by* ρ *in the shadow of* (i,j).

Theorem

Equation $(*)$ *above is equivalent to the four term recursion*

$$\mathbf{C}_{\mu/\mathbf{ij}} \;=\; \frac{\mathbf{t}^{\mathbf{l}}-\mathbf{q}^{\mathbf{a}+1}}{\mathbf{t}^{\mathbf{l}}-\mathbf{q}^{\mathbf{a}}}\,\mathbf{C}_{\mu/\mathbf{i},\mathbf{j}+1} + \frac{\mathbf{t}^{\mathbf{l}+1}-\mathbf{q}^{\mathbf{a}}}{\mathbf{t}^{\mathbf{l}}-\mathbf{q}^{\mathbf{a}}}\,\mathbf{C}_{\mu/\mathbf{i}+1,\mathbf{j}} - \frac{\mathbf{t}^{\mathbf{l}+1}-\mathbf{q}^{\mathbf{a}+1}}{\mathbf{t}^{\mathbf{l}}-\mathbf{q}^{\mathbf{a}}}\,\mathbf{C}_{\mu/\mathbf{i}+1,\mathbf{j}+1}\;,$$

with l *and* a *the leg an arm of* (i,j) *in* μ

next

KERNELS AND ATOMS

Let $\mathbf{K}^{x}_{ij}$ *denote the kernel of the operator* $\mathbf{D_x}$ *as a map of* $\mathbf{M}_{ij}$ *onto* $\mathbf{M}_{i+1,j}$.

Similarly, let $\mathbf{K}^{y}_{ij}$ *be the kernel of* $\mathbf{D_y}$ *as a map of* $\mathbf{M}_{ij}$ *onto* $\mathbf{M}_{i,j+1}$.

Note we have

$$\mathbf{K}^{x}_{i,j+1} \subseteq \mathbf{K}^{x}_{ij} \qquad \textit{as well as} \qquad \mathbf{K}^{y}_{i+1,j} \subseteq \mathbf{K}^{y}_{ij}$$

Set

$$\mathbf{A}^{x}_{ij} = \mathbf{K}^{x}_{ij}/\mathbf{K}^{x}_{i,j+1} \qquad \textit{and} \qquad \mathbf{A}^{y}_{ij} = \mathbf{K}^{y}_{ij}/\mathbf{K}^{y}_{i+1,j}$$

and let A^{x}_{ij} *and* A^{y}_{ij} *denote their respective Frobenius characteristics.*

Proposition

$$\mathbf{K^x_{ij} = C_{\mu/ij} - t\, C_{\mu/i+1j} \qquad and \qquad K^y_{ij} = C_{\mu/ij} - q\, C_{\mu/ij+1}}$$

and

$$\mathbf{A^x_{ij} = K^x_{ij} - K^x_{ij+1} \qquad and \qquad A^y_{ij} = K^y_{ij} - K^y_{i+1j}}$$

In particular the four term recurrence may be rewritten in the simple form

$$\mathbf{t^l\, A^x_{ij} = q^a\, A^y_{ij}}$$

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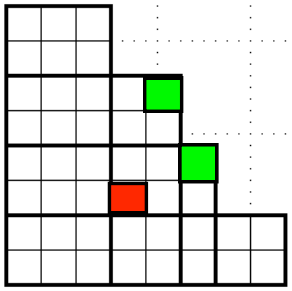
Computing Atoms
for the Ferrers Diagram of a $\mu \vdash n+1$



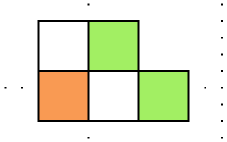
(1) Decompose μ into rectangles since (up to a scalar) atoms do not vary within a rectangle

(2) Atoms have explicit expressions in terms of the Macdonald polynomials indexed by partitions obtained from μ by removing one of the corners in the shadow of the atom.

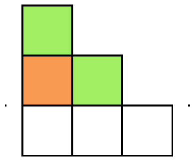
For the Atom $\mathbf{A_{00}}$ for $\mu = [3,2]$ the formula is

$$\mathbf{A_{00} = c_{32,31}\tilde{H}_{31} + c_{32,22}\tilde{H}_{22} - (c_{21,11}\tilde{H}_{22} + c_{21,2}\tilde{H}_{31} - t\tilde{H}_{31})}$$

$$\mathbf{= \frac{1-q}{t-q}\tilde{H}_{22} + \frac{t-1}{t-q}\tilde{H}_{31}}$$



For the Atom $\mathbf{A_{10}}$ for $\mu = [3,2,1]$ the formula is



$$\mathbf{A_{10} = c_{21,11}\tilde{H}_{311} + c_{21,2}\tilde{H}_{32} - t\tilde{H}_{32} \quad - \quad \tilde{H}_{311}}$$

$$\mathbf{= \frac{1-q}{t-q}\tilde{H}_{311} + \frac{t-1}{t-q}\tilde{H}_{32}}$$

Gistol Polynomials

(1) A "gistol" is a lattice diagram that can be transformed to a skew diagram by row and column interchanges

(2) We postulate the existence of a family of polynomials indexed by gistols with the following basic properties:

$$\begin{cases} (0) \quad G_D(x;q,t) = \tilde{H}_\mu(x;q,t) & \text{if } D \text{ is the diagram of } \mu \\ (1) \quad G_{D_1}(x;q,t) = G_{D_2}(x;q,t) & \text{if } D_1 \approx D_2 \\ (2) \quad G_{D_1}(x;q,t) = G_{D_2}(x;t,q) & \text{if } D_2 \approx D_1' \\ (3) \quad G_D(x;q,t) = G_{D_1}(x;q,t)G_{D_2}(x;q,t) & \text{if } D \approx D_1 \times D_2 \\ (4) \quad \partial_{p_1} G_D(x;q,t) = \sum_{s\in D} w_{s,D}(q,t)\ G_{D/s}(x;q,t)\ , & \end{cases}$$

Representation theoretical reasons suggest that, in the case that D *is a skew diagram,*

$$\text{a)} \quad w_1[s,D] = t^{l_D(s)} q^{a'_D(s)} \qquad \text{and} \qquad \text{b)} \quad w_2[s,D] = t^{l'_D(s)} q^{a_D(s)}$$

Note: these properties overdetermine the family $\{G_D(x;q,t)\}_D$*,*

existence is by no means guaranteed.

next

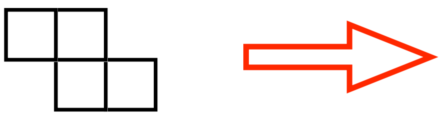
A Problem and a Conjecture

Problem

Construct a family of polynomials indexed by gistols that satisfies the gistol "axioms"

Conjecture

Gistol polynomials are Schur positive integral in $\mathbf{q}, \mathbf{t}$.



$$\begin{bmatrix} s_{3,1} + s_{2,2} & 2\, s_{2,1,1} & s_{1,1,1,1} \\ s_4 & 2\, s_{3,1} & s_{2,2} + s_{2,1,1} \end{bmatrix}$$

$$s_4 + 3\, s_{3,1} + 2\, s_{2,2} + 3\, s_{2,1,1} + s_{1,1,1,1}$$

next

More on Gistols

It is a cute exercise to show that for any skew Diagram $\mathbf{D}$ we have

$$\sum_{c\in D} t^{l(c)} q^{a'(c)} = \sum_{c\in D} t^{l'(c)} q^{a(c)}$$

It follows from this that Gistol Polynomials should be characteristics of left regular representations. In other words, for all gistols with $\mathbf{n}$ cells we have

$$\mathbf{G_D(x;q,t)}\big|_{\mathbf{t=q=1}} = \mathbf{e_1^n}$$

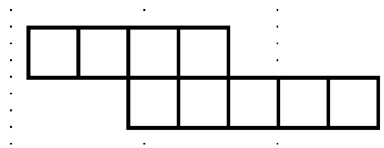
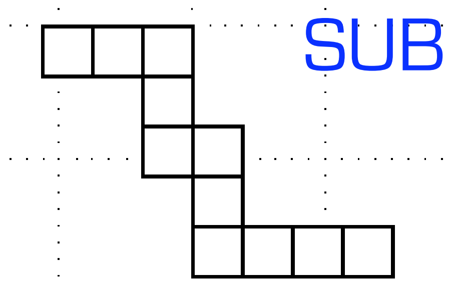
Problem 1:

Define the bigraded modules whose Frobenius characterstics are given by Gistol polynomials

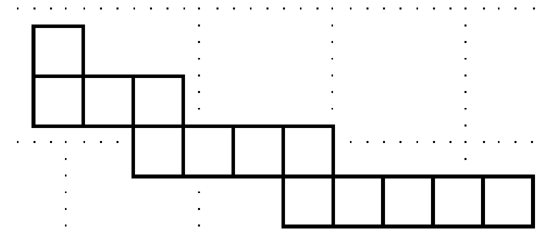
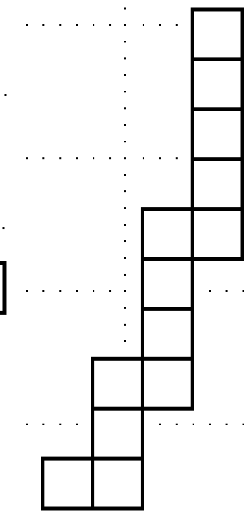
Problem 2:

Find by a combinatorial construction a family of polynomials which satisfies the Gistol "axioms"

next

SUB FAMILIES

A family of gistols is "complete" if it is closed under removal of cells

Closed families are "pistols", "ribbons", "down-one-ribbons", "right-one-ribbons", etc

Some facts:

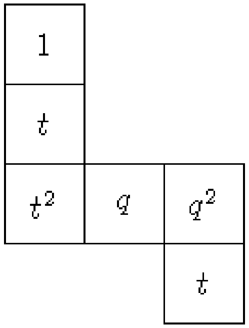
(1) In previous work we gave a construction of a family of pistol polynomials that satisfies the gistol axioms.

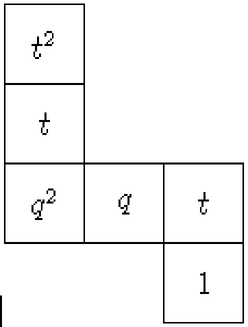
(2) Jason Bandlow showed that pistol as well as down-one ribbons can be constructed by an extension of the Jim Kaglund statistics.

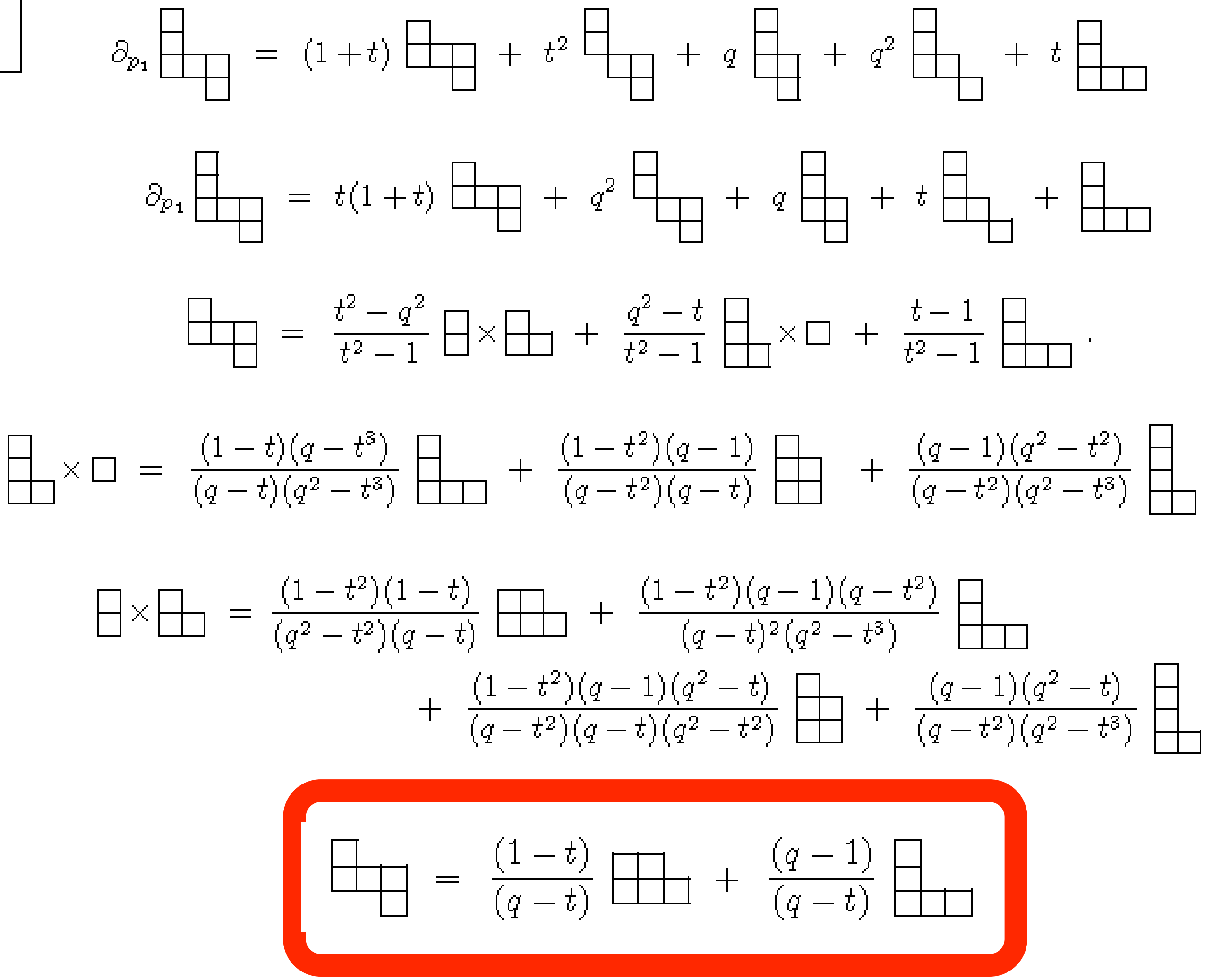
(3) Curiously, the right-one ribbons do not carry the Haglund statistics.

next

Computing Gistol Polynomials







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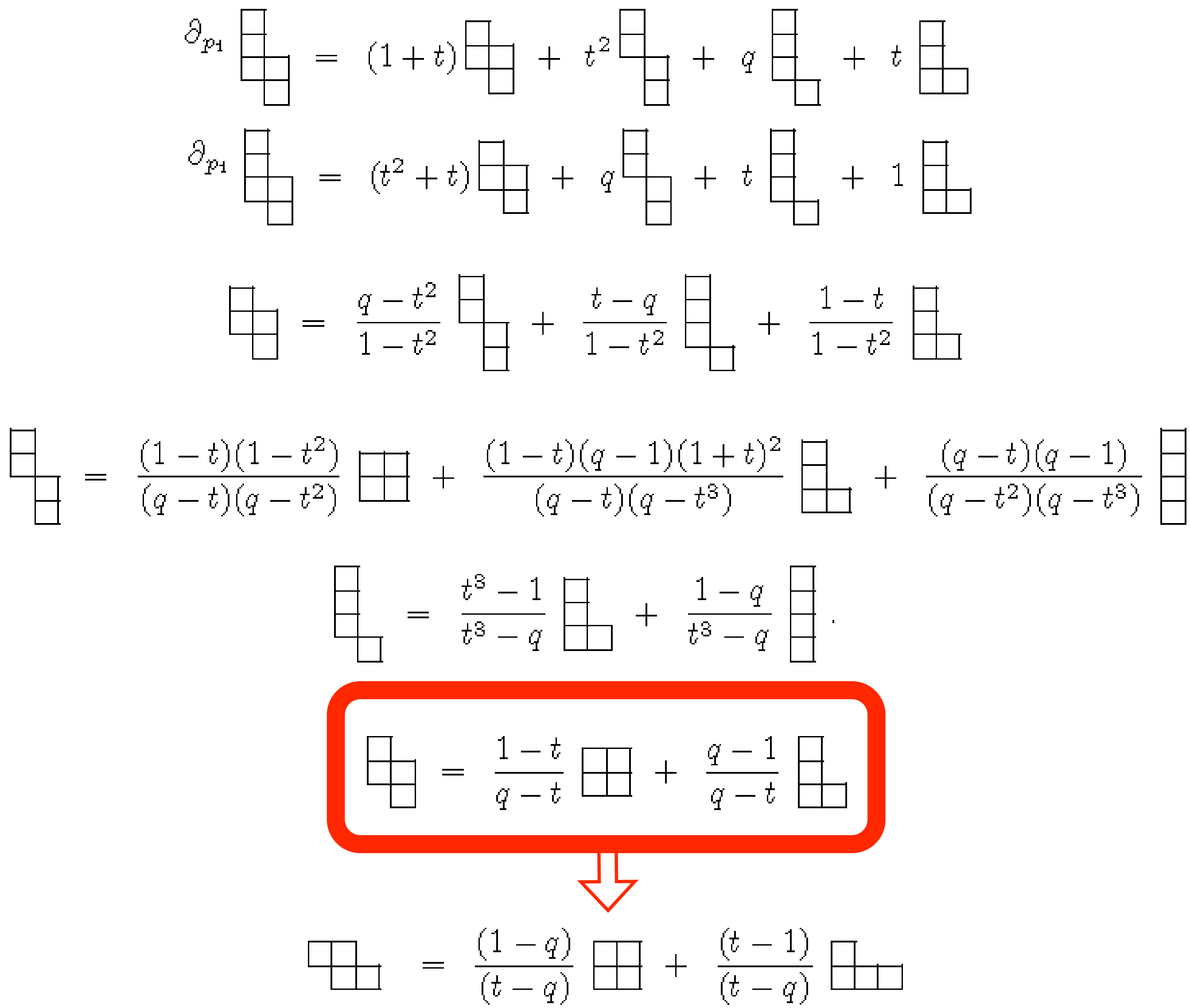
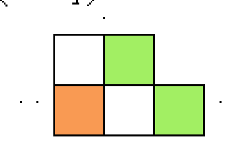
Yet another surprise



The Atom!!!!



next

Everybody heard about the n! conjecture

oops!!! Theorem!!!!!

But did you hear about the n!/k conjecture?

Let $\mu \models \mathbf{n+1}$ *be a* k *corner partition and*

$$\alpha_1, \alpha_2, \ldots, \alpha_k$$

be the partitions obtained by removing one of the corners of μ

Conjecture:

$$\dim \left(\mathbf{M}_{\alpha_1} \cap \mathbf{M}_{\alpha_2} \cap \cdots \cap \mathbf{M}_{\alpha_k} \right) = \frac{n!}{k}$$

next

Science Fiction?

$$Flip_\alpha \mathbf{M} = \{b(\partial_x, \partial_y)\Delta_\alpha(x,y) : b(x,y) \in \mathbf{M}\}$$

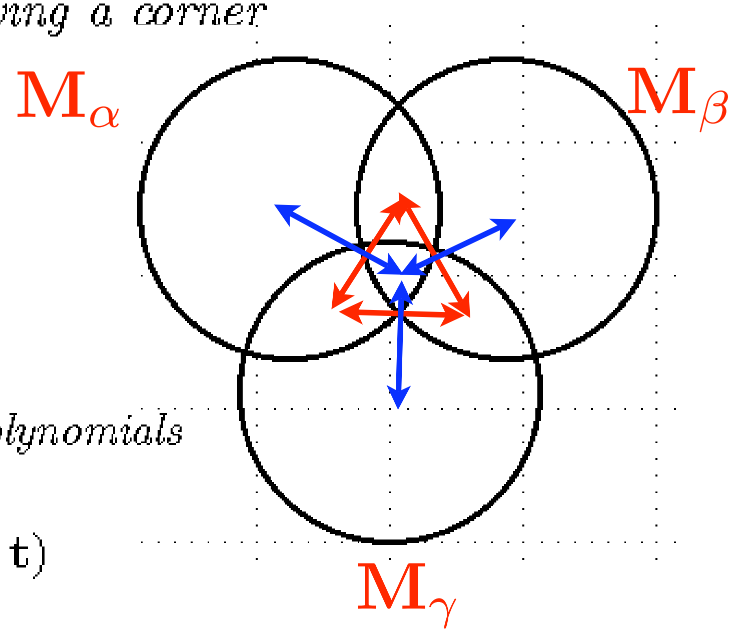
Theorem

If M *is a submodule of* M_α *with Frobenius characteristic* $\Phi_{\mathrm{M}}(\mathbf{x};\mathbf{q},\mathbf{t})$

then $\mathrm{Flip}_\alpha \mathrm{M}$ *has Frobenius characteristic*

$$\mathbf{T}_\alpha \omega \mathbf{\Phi}_{\mathbf{M}}(\mathbf{x}; \tfrac{\mathbf{1}}{\mathbf{q}}, \tfrac{\mathbf{1}}{\mathbf{t}})$$

Let α, β, γ *be the three partitions obtained by removing a corner from a three corner partition.*



As a result the Frobenius characteristics of each of these intersection submodules can be explicitely expressed in terms of the Macdonald polynomials

$$\tilde{\mathrm{H}}_\alpha(\mathbf{x};\mathbf{q},\mathbf{t}), \tilde{\mathrm{H}}_\beta(\mathbf{x};\mathbf{q},\mathbf{t}), \tilde{\mathrm{H}}_\gamma(\mathbf{x};\mathbf{q},\mathbf{t})$$

next