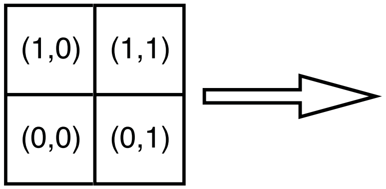
Seeking for n! Derivatives

$1,000$

Reward

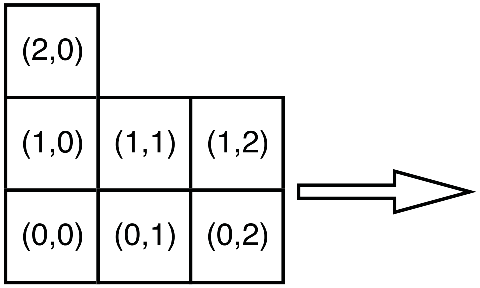
next

A remarkable determinant



(1,0)	(1,1)
(0,0)	(0,1)

$$\Delta_{2,2}(X,Y) = \det\begin{pmatrix} x_1^0y_1^0 & x_2^0y_2^0 & x_3^0y_3^0 & x_4^0y_4^0 \\ x_1^0y_1^1 & x_2^0y_2^1 & x_3^0y_3^1 & x_4^0y_4^1 \\ x_1^1y_1^0 & x_2^1y_2^0 & x_3^1y_3^0 & x_4^1y_4^0 \\ x_1^1y_1^1 & x_2^1y_2^1 & x_3^1y_3^1 & x_4^1y_4^1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix}$$



(2,0)		
(1,0)	(1,1)	(1,2)
(0,0)	(0,1)	(0,2)

$$\Delta_{331}(X,Y) = \det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_5 & y_7 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 & y_7^2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6^2 & x_7^2 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 & x_6y_6 & x_7y_7 \\ x_1y_1^2 & x_2y_2^2 & x_3y_3^2 & x_4y_4^2 & x_5y_5^2 & x_6y_6^2 & x_7y_7^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 & x_7^2 \end{pmatrix}$$

General definition

If $(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)$ are the cells of the Ferrers diagram of $\mu \vdash n$ then

$$\Delta_\mu(X,Y) = \det \left\| x_j^{p_i} y_j^{q_i} \right\|_{i,j=1}^{n}$$

next

As a starter .....

Theorem (easy)

For any $\mu \vdash n$ *the dimension of the linear span of the derivatives of* $\Delta_\mu(X,Y)$ *is at most* $\mathbf{n!}$

In symbols

$$\dim \mathbf{M}_\mu[\mathbf{X},\mathbf{Y}] \leq \mathbf{n!}$$

where

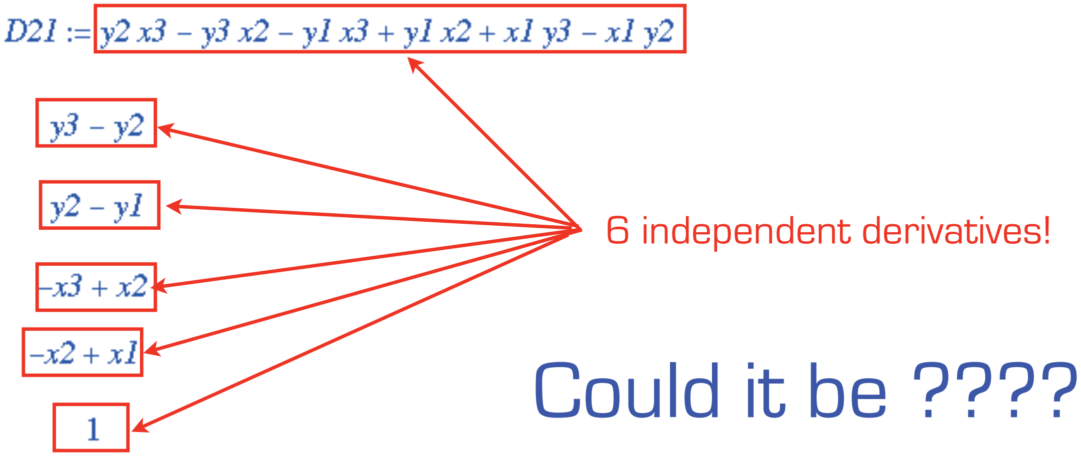
$$M_\mu[X,Y] \;=\; L\big[\delta_{x_1}^{r_1}\delta_{y_1}^{s_1}\delta_{x_2}^{r_2}\delta_{y_2}^{s_2}\cdots\delta_{x_n}^{r_n}\delta_{y_n}^{s_n}\Delta_\mu(X,Y) \;:\; r_i,s_i \geq 0\big]$$

For example (using MAPLE)

```
DDmu([2,1]);
```

$$\begin{bmatrix} 1 & 1 & 1 \\ y1 & y2 & y3 \\ x1 & x2 & x3 \end{bmatrix}$$

```
D21:=det(");
```



$$D21 := \boxed{y2\,x3 - y3\,x2 - y1\,x3 + y1\,x2 + x1\,y3 - x1\,y2}$$

```
diff(D21,x1);
```

$$\boxed{y3 - y2}$$

```
diff(D21,x3);
```

$$\boxed{y2 - y1}$$

```
diff(D21,y1);
```

$$\boxed{-x3 + x2}$$

```
diff(D21,y3);
```

$$\boxed{-x2 + x1}$$

```
diff(D21,x3,y2);
```

$$\boxed{1}$$

6 independent derivatives!

Could it be ?????

next

The n! Conjecture

(1990 - 2000)

For $\mu \vdash$ n

$$\dim M_\mu[X,Y] = n!$$

Proved by Mark Haiman using algebraic geometry

1,000$

Reward

for an "elementary" proof

"Elementary" means:

By calculus or/and combinatorics

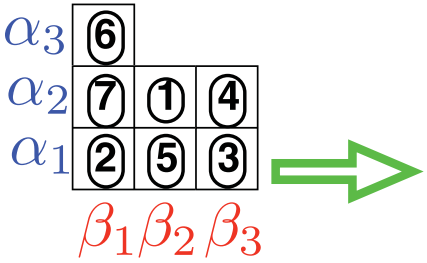
give an algorithm

that produces a "triangular" set of n! derivatives

next

The basic construction

from a tableau to a point in 2n dimensional space

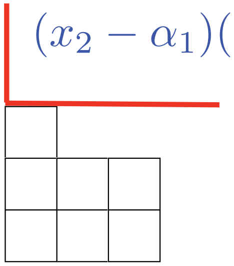


$$\left(\frac{\alpha_2\ \ \alpha_1\ \ \alpha_1\ \ \alpha_2\ \ \alpha_1\ \ \alpha_3\ \ \alpha_2\ \ \beta_2\ \ \beta_1\ \ \beta_3\ \ \beta_3\ \ \beta_2\ \ \beta_1\ \ \beta_1}{x_1\ \ x_2\ \ x_3\ \ x_4\ \ x_5\ \ x_6\ \ x_7\ \ y_1\ \ y_2\ \ y_3\ \ y_4\ \ y_5\ \ y_6\ \ y_7}\right)$$

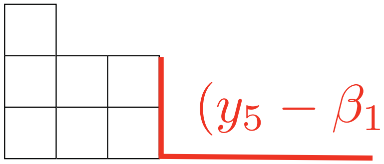
some polynomials vanishing at this orbit point

$$x_1-\alpha_2 \qquad x_4-\alpha_2 \qquad x_7-\alpha_2 \qquad y_2-\beta_1 \qquad y_5-\beta_2$$

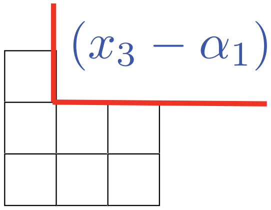
Polynomials vanishing at all these tableau points:



$(x_2-\alpha_1)(x_2-\alpha_2)(x_2-\alpha_3)$



$(y_5-\beta_1)(y_5-\beta_2)(y_5-\beta_3)$



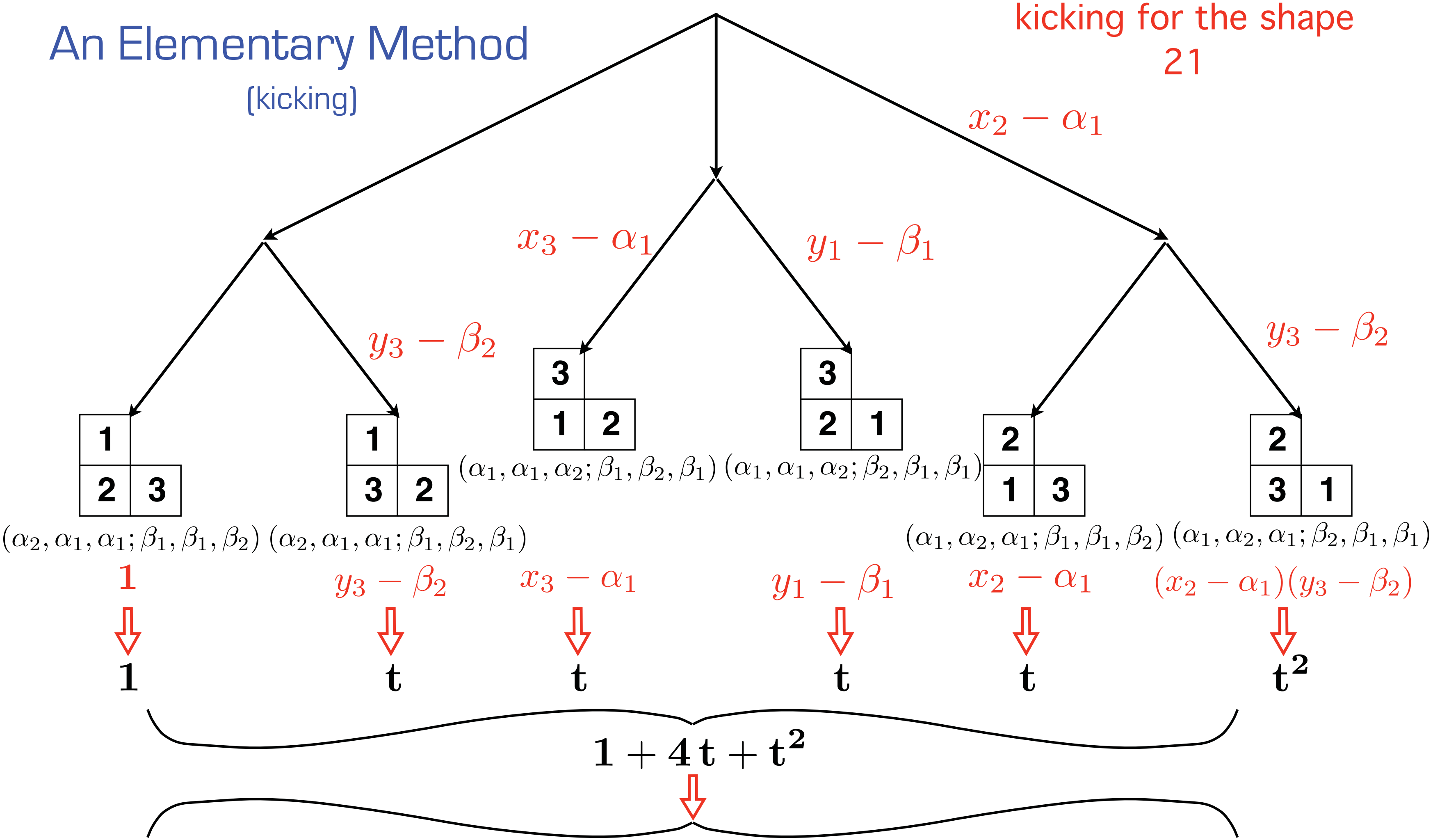
$(x_3-\alpha_1)(x_3-\alpha_2)(y_3-\beta_1)$

next

An Elementary Method

(kicking)

kicking for the shape
21

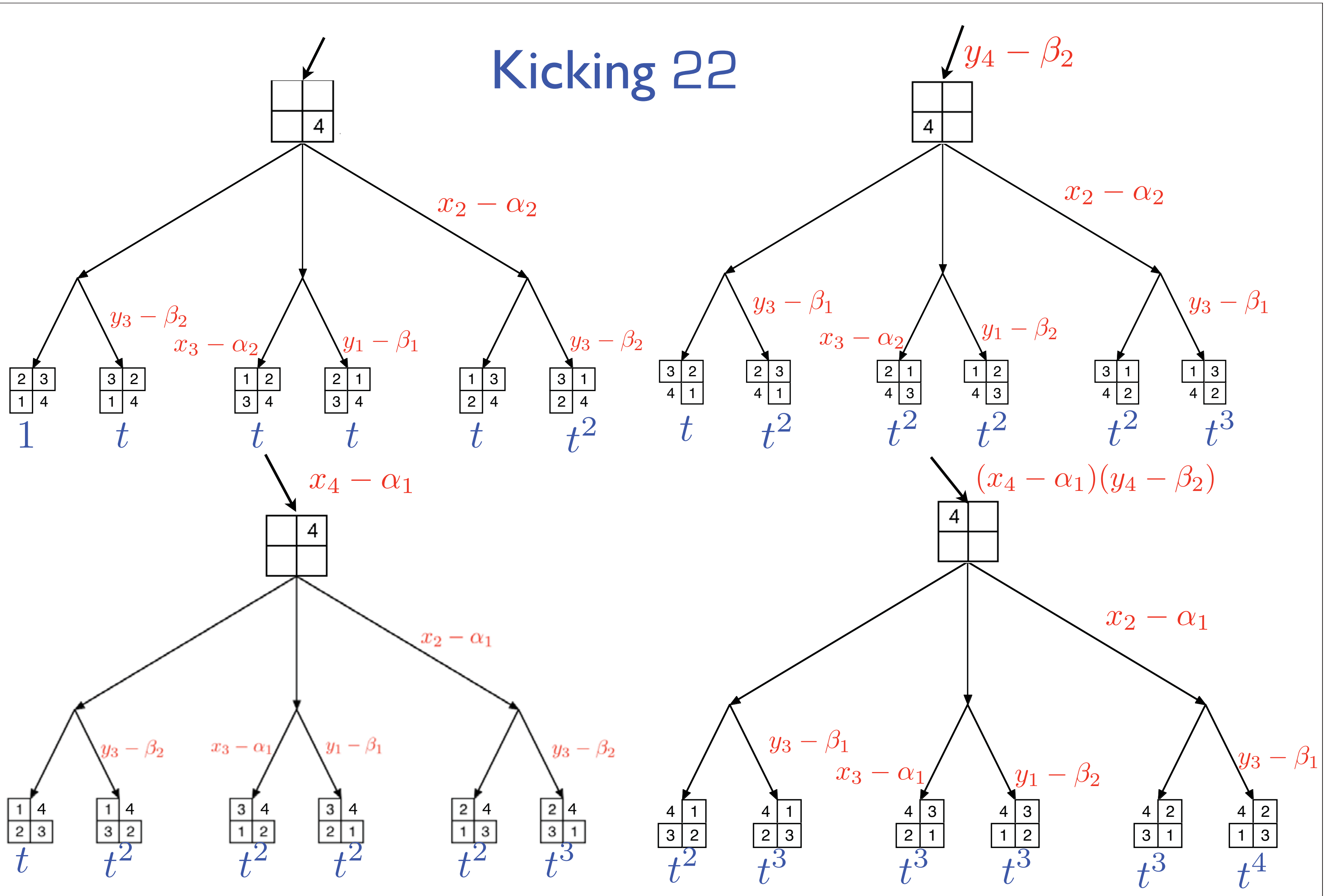


$\Delta_{21}[X,Y]$	$\partial_{y_3}\Delta_{21}[X,Y]$	$\partial_{x_3}\Delta_{21}[X,Y]$	$\partial_{y_1}\Delta_{21}[X,Y]$	$\partial_{x_2}\Delta_{21}[X,Y]$	$\partial_{x_2}\partial_{y_3}\Delta_{21}[X,Y]$
$\boxed{\begin{bmatrix} 1 & 1 & 1 \\ y1 & y2 & y3 \\ x1 & x2 & x3 \end{bmatrix}}$	$\boxed{-x2+x1}$	$\boxed{y2-y1}$	$\boxed{-x3+x2}$	$\boxed{-y_3+y_1}$	$\boxed{1}$

AN INDEPENDENT SET WITH 6=3! ELEMENTS

next

Kicking 22

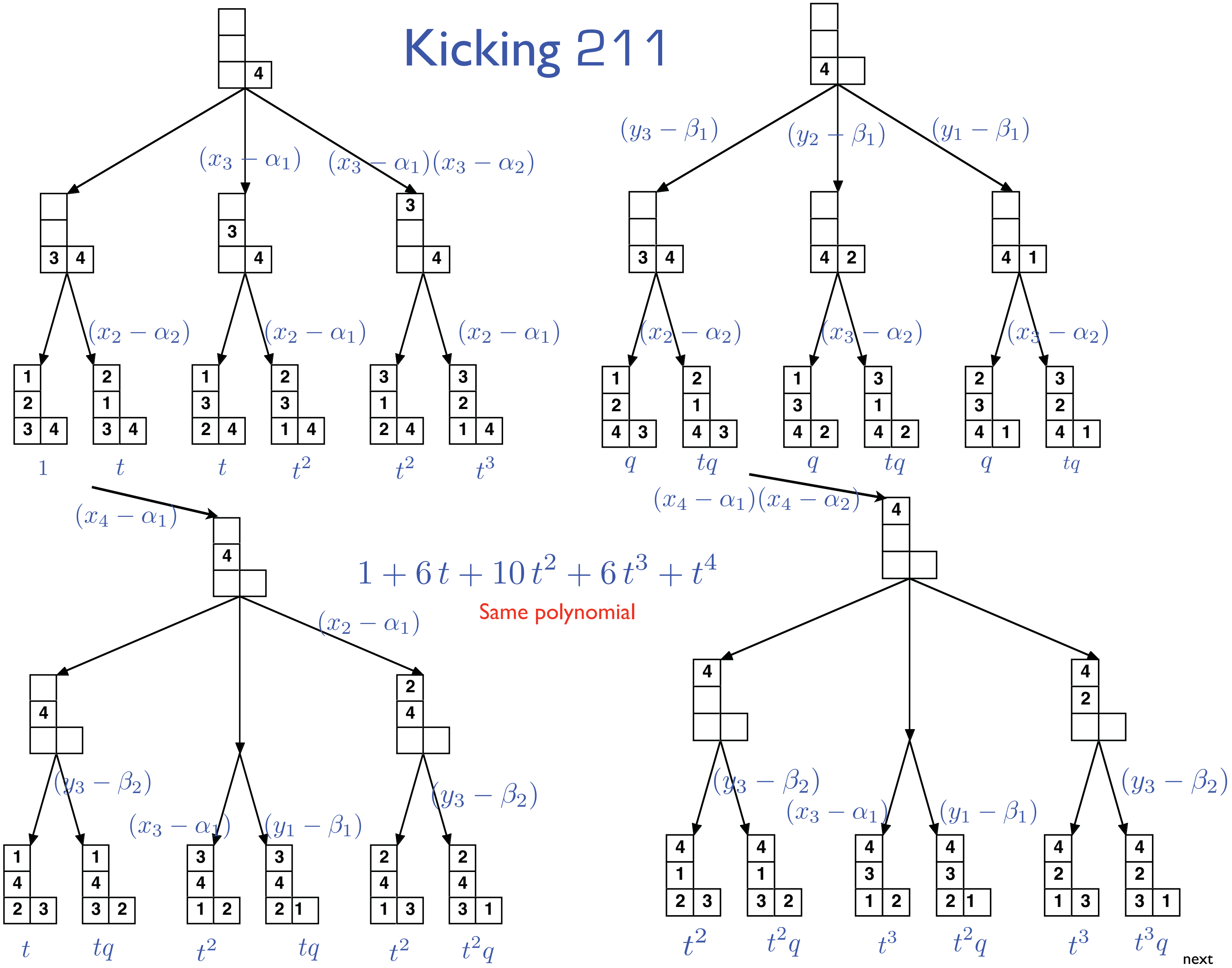


$$1+6\,t+10\,t^2+6\,t^3+t^4$$

The symmetry condition is satisfied

next

Kicking 211



$$1+6\,t+10\,t^2+6\,t^3+t^4$$

Same polynomial

next

The Algorithm

Step 1

Fill the Ferrers diagram of μ *with* $1, 2, \ldots, \mathrm{n}$ *in all possible ways to get the* $\mathrm{n}!$ *tableaux.*

Step 2

Construct the corresponding orbit points

5040

In this case

Step 3

Order the orbit points and construct the kicking polynomials.

Step 4

Compute the kicking statistics.

3		
6	1	2
4	7	5

$$\begin{bmatrix} 1 \\ 12 \\ 75 \\ 306 \\ 807 \\ 1319 \\ 1319 \\ 807 \\ 306 \\ 75 \\ 12 \\ 1 \end{bmatrix}$$

Theorem

If the kicking statistic are symmetric then the top components of the kicking polynomials give the $\mathrm{n}!$ *independent derivatives of* $\Delta_\mu(X,Y) = \det \left\| x_j^{p_i} y_j^{q_i} \right\|_{i,j=1}^{n}$

MAGICS?

No

Only Algebraic Combinatorics

next

Triangularity

A typical term order

The degree-Lexicographic order

$$x^p = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} <_{dlex} x^q = x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$$

$$\Updownarrow$$

$$\text{degree}(x^p) < \text{degree}(x^q) \quad \text{or} \quad \text{degree}(x^p) = \text{degree}(x^q) \quad \text{and} \quad p <_{lex} q$$

Where we have $p <_{lex} q$ if and only if for some $1 \le i \le n-1$

$$p_1 = q_1\ ,\ p_2 = q_2\ ,\ \ldots,\ ,\ p_i = q_i\ , \quad \text{and} \quad p_{i+1} < q_{i+1}$$

The "*Leading monomial*" in a polynomial $P = \sum_p c_p x^p$ is the dlex highest monomial in P

Definition

In a vector space V *of polynomials a collection* $\mathcal{C} = \{P_1, P_2, \ldots, P_m\}$ *is said to be "triangular" if*

$$\text{ldm}(P_1) <_{dlex} \text{ldm}(P_2) <_{dlex} \cdots <_{dlex} \text{ldm}(P_m)$$

Note:

The polynomials in a triangular collection are necessarily independent!

next

A basic tool

Let V be a vector space of polynomials and suppose that $\boxed{\dim V \leq d}$

then $\boxed{\dim V = d}$ *if and only if we can find in V*

a collection of $\boxed{d}$ *polynomials with* $\boxed{\text{distinct}}$ *leading monomials.*

Theorem

The vandermonde determinant $\Delta_n(x_1, x_2, \ldots, x_n)$ *has* $n!$ *independent derivatives*

Proof

$$\Delta_n(x_1, x_2, \ldots, x_n) = \det\|x_i^{n-j}\|_{i,j=1}^{n} = x_1^{n-1}x_2^{n-2}x_3^{n-3}\cdots x_n^{n-n} + \geq_{dlex} \cdots$$

thus if

$$0 \leq a_1 \leq n-1\,,\quad 0 \leq a_2 \leq n-2\,,\quad 0 \leq a_3 \leq n-3\,,\quad \cdots\,,\quad 0 \leq a_{n-1} \leq 1$$

then

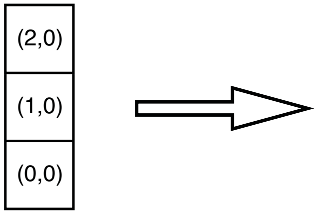
$$\partial_{x_1}^{a_1}\partial_{x_2}^{a_2}\partial_{x_3}^{a_3}\cdots\partial_{x_{n-1}}^{a_{n-1}}\Delta_n(x_1, x_2, \ldots, x_n) = x_1^{n-1-a_1}x_2^{n-2-a_2}x_3^{n-3-a_3}\cdots x_{n-1}^{1-a_{n-1}} + \geq_{dlex} \cdots$$

because

$$\boxed{x^p \leq_{dlex} x^q \quad \Longrightarrow \quad x^{p-a} \leq_{dlex} x^{q-a}}$$

Note:

This proves the $n!$ conjecture for $\lambda = 1^n$



$$\Delta_{1^3}[X] = \det\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}$$

next

Hilbert series

A vector space V is called "*graded*" if and only if

$$V \;=\; H_\circ(V) \oplus H_1(V) \oplus H_2(V) \oplus \cdots \oplus H_m(V) \oplus \cdots$$

The subspace "$H_m(V)$" is called the "m^{th} *homogeneous component*" of V.

its elements are called homogeneous of degree m

If $\dim H_m(V) < \infty$ for all m, we set

the "*Hilbert series*" of V

$$F_V(t) \;=\; \sum_{m>0} t^m \dim H_m(V)$$

For instance for $R = \mathbb{Q}[x_1, x_2, \ldots, x_n]$ we have $R = H_0(R) \oplus H_1(R) \oplus H_2(R) \oplus \cdots$ with

$$H_m(R) \;=\; L\big[x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \;:\; p_1 + p_2 + \cdots + p_n = m\big]$$

In this case

$$\dim H_m(R) \;=\; \binom{m+n-1}{n-1} \qquad \text{and} \qquad F_R(t) \;=\; \frac{1}{(1-t)^n}$$

Our spaces $M_\mu[X,Y]$ are "*bigraded* that is we have the double decomposition

$$\boxed{M_\mu[X,Y] \;=\; \bigoplus_{r=0}^{n(\mu)} \bigoplus_{s=0}^{n(\mu')} H_{r,s}(M_\mu[X,Y])}$$

With $H_{r,s}(M_\mu[X,Y])$ the linear span of derivatives of $\Delta_\mu(x,y)$

that are homogeneous of degree r in $x_1, x_2, \ldots, x_n$ and degree s in $y_1, y_2, \ldots, y_n$

Here and after we set

$$\boxed{F_\mu(q,t) \;=\; \sum_{r=0}^{n(\mu)} \sum_{s=0}^{n(\mu')} t^r q^s \dim H_{r,s}(M_\mu[X,Y])}$$

next

Some Hilbert series

Theorem

If V *is graded and* B *is a homogeneous basis of* V *then*

$$F_V(t) \;=\; \sum_{b\in B} t^{\mathrm{degree}(b)}$$

Using this we get for the linear span of derivatives of the Vandermonde

$$F_{1^n}(t) \;=\; (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n-1}) \;=\; [n]_t!$$

Using Maple

```
hilb([3,2]);
```

$$q^4t^2+4q^4t+4q^3t^2+5q^4+\boxed{15q^3t}+9q^2t^2+11q^3+22q^2t+\boxed{11qt^2}+9q^2+15qt+5t^2+4q+4t+1$$

The dimension of $H_{2,1}(M_{3,2}[X,Y])$

$$\begin{array}{c|c}
2 \rightarrow & \begin{bmatrix} 5 & \boxed{11} & 9 & 4 & 1 \\ 4 & 15 & 22 & \boxed{15} & 4 \\ 1 & 4 & 9 & 11 & 5 \end{bmatrix} \\
1 \rightarrow & \\
0 \rightarrow & \\
 & \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 0 & 1 & 2 & 3 & 4 \end{array}
\end{array}$$

The dimension of $H_{1,3}(M_{3,2}[X,Y])$

The determinant $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ y1 & y2 & y3 & y4 & y5 \\ y1^2 & y2^2 & y3^2 & y4^2 & y5^2 \\ x1 & x2 & x3 & x4 & x5 \\ x1\,y1 & x2\,y2 & x3\,y3 & x4\,y4 & x5\,y5 \end{vmatrix}$ has 15 independent bihomogeneous derivatives of degree $\boxed{1}$ in $x_1, x_2 \ldots, x_5$ and degree $\boxed{3}$ in $y_1, y_2, \ldots, y_5$.

```
hilb([2,2,1]);
```

$$q^2t^4+4q^2t^3+4qt^4+9q^2t^2+15qt^3+5t^4+11q^2t+22qt^2+11t^3+5q^2+15qt+9t^2+4q+4t+1$$

$$\begin{bmatrix} 5 & 4 & 1 \\ 11 & 15 & 4 \\ 9 & 22 & 9 \\ 4 & 15 & 11 \\ 1 & 4 & 5 \end{bmatrix}$$

next

Flip

Definition

We say that a vector space $\mathbf{V}$ *of polynomials is a "cone " if it is the linear span of derivatives of a single homogeneous polynomial* $\mathbf{\Delta}(\mathrm{x})$.

In this case $\mathbf{V}$ *has an automorphism " Flip " defined by*

$$\mathrm{flip}\,\mathrm{P}(\mathrm{x}) \quad = \quad \mathrm{P}(\partial_{\mathrm{x}})\mathbf{\Delta}(\mathrm{x})$$

Note this is a non-singular linear map of $\mathbf{V}$ onto $\mathbf{V}$ since if

$$\mathrm{P}(\mathrm{x}) \quad = \quad \mathrm{Q}(\partial_{\mathrm{x}})\mathbf{\Delta}(\mathrm{x}) \qquad \text{and} \qquad \mathrm{P}(\partial_{\mathrm{x}})\mathbf{\Delta}(\mathrm{x}) = 0$$

then

$$\mathrm{P}(\partial_{\mathrm{x}})\mathrm{P}(\mathrm{x}) \;= \mathrm{P}(\partial_{\mathrm{x}})\mathrm{Q}(\partial_{\mathrm{x}})\mathbf{\Delta}(\mathrm{x}) \;= \mathrm{Q}(\partial_{\mathrm{x}})\mathrm{P}(\partial_{\mathrm{x}})\mathbf{\Delta}(\mathrm{x}) \;= 0$$

Thus

$$\mathrm{flip}\,\mathrm{P}(\mathrm{x}) = 0 \implies \mathrm{P}(\partial_{\mathrm{x}})\mathrm{P}(\mathrm{x}) = 0 \implies \mathrm{P}(\mathrm{x}) = 0$$

Note that this implies that if degree $\mathbf{\Delta}(\mathrm{x}) = \mathrm{n_o}$ then

$$\mathrm{F_V}(\mathrm{t}) \quad = \quad \mathrm{t^{no}}\,\mathrm{F_V}(1/\mathrm{t})$$

The same is true if $\mathbf{V}$ is a cone with summit a bihomogeneous polynomial $\mathbf{\Delta}(\mathrm{x},\mathrm{y})$ of bi-degree $\mathrm{n_x}, \mathrm{n_y}$ In this case

$$\boxed{\mathrm{F_V}(\mathrm{q},\mathrm{t}) \quad = \quad \mathrm{t^{n_x}}\mathrm{q^{n_y}}\mathrm{F_V}(1/\mathrm{q},1/\mathrm{t})}$$

This explains the symmetry

$$\mathrm{F_{32}}(\mathrm{q},\mathrm{t}) \quad = \quad \begin{bmatrix} \boxed{5} & 11 & \boxed{9} & \boxed{4} & 1 \\ 4 & 15 & 22 & 15 & 4 \\ 1 & \boxed{4} & \boxed{9} & 11 & \boxed{5} \end{bmatrix}$$

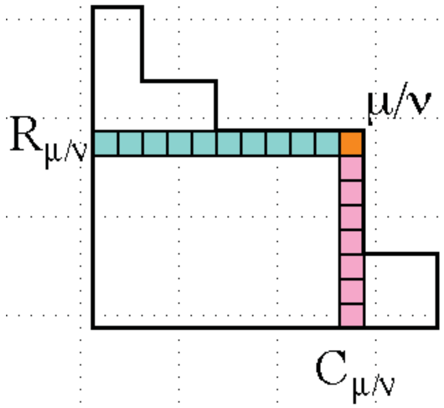
next

There is a formula for these Hilbert series!

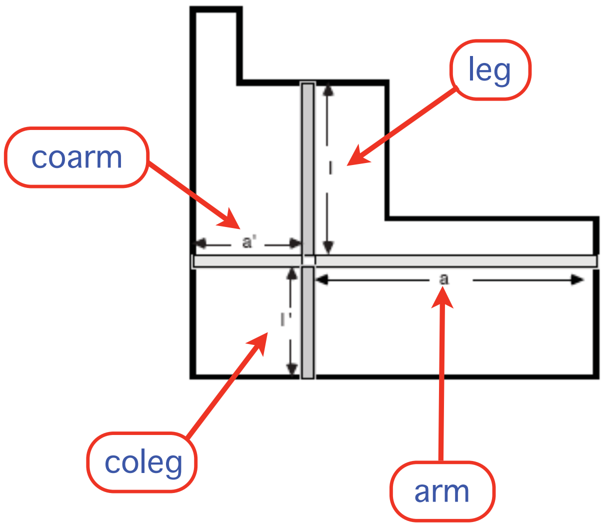
(brace yourself)

$$\boxed{F_\mu(q,t) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t)F_\nu(q,t)}$$

$$c_{\mu\nu}(q,t) \;=\; \prod_{s\in R_{\mu\nu}} \frac{t^{l_\mu(s)} - q^{a_\mu(s)+1}}{t^{l_\nu(s)} - q^{a_\nu(s)+1}} \prod_{s\in C_{\mu\nu}} \frac{q^{a_\mu(s)} - t^{l_\mu(s)+1}}{q^{a_\nu(s)} - t^{l_\nu(s)+1}} \;,$$



Some Basic Ingredients



next

Examples using MAPLE

`cmunu([3,2,1],[3,2]);`

$$\frac{(q^2-t^3)(q-t^2)}{(q^2-t^2)(q-t)}$$

`cmunu([3,2,1],[3,1,1]);`

$$\frac{(t-q^2)(q-t^2)}{(t-q)(q-t)}$$

`cmunu([3,2,1],[2,2,1]);`

$$\frac{(t^2-q^3)(t-q^2)}{(t^2-q^2)(t-q)}$$

$$F_{[n]}(q,t)=\prod_{i=1}^{n}(1+q+\cdots+q^{i-1})=[n]_q!$$

```
ricoF:=proc(mu)
local out,nus,nu;
if nops(mu)=1 then
out:=qfac(mu[1])/(1-q)^mu[1];
else
 nus:=preds(mu);
 out:=0;
for nu in nus do
  out:=out+cmunu(mu,nu)*ricoF(nu);
   od;
fi;
out;
end:
```

a simple recursive procedure

`> ricoF([3,2]);`

$$\frac{(t-q^3)(1-q^2)(1-t^2)\left(\frac{(t-q^2)(1-t^2)}{(t-q)(1-t)}+\frac{(q-t^2)(1-q^2)}{(q-t)(1-q)}\right)}{(t-q)(1-q)(1-t)}$$
$$+\frac{(q-t^2)(1-q^2)\left(\frac{(t-q^3)(1-q^2)\left(\frac{(t-q^2)(1-t^2)}{(t-q)(1-t)}+\frac{(q-t^2)(1-q^2)}{(q-t)(1-q)}\right)}{(t-q^2)(1-q)}+\frac{(q^2-t^2)(1-q^2)(1-q^3)}{(q^2-t)(1-q)^2}\right)}{(q-t)(1-q)}$$

`> factor(");`

$$(1+q)(t^2q^3+4tq^3+5q^3+6q^2+11tq^2+3t^2q^2+6t^2q+3q+11tq+5t^2+4t+1)$$

`> display(");`

$$\begin{bmatrix}5 & 11 & 9 & 4 & 1\\ 4 & 15 & 22 & 15 & 4\\ 1 & 4 & 9 & 11 & 5\end{bmatrix}$$

How could such a messy recursion yield such beautiful polynomials?

MAGICS?

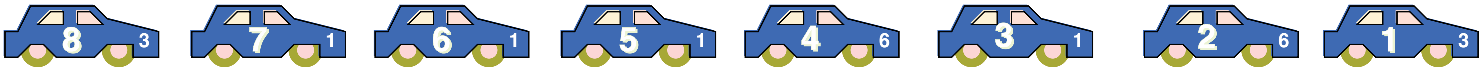
No

Only Algebraic Combinatorics

next

A one way street with 8 parking spaces

eight cars wish to park in it



The large number is their order of arrival

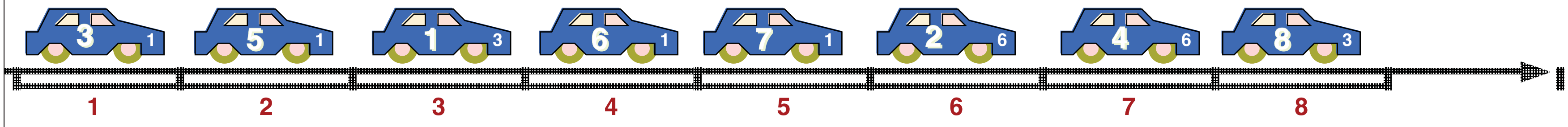
The small number is their preferred parking place

The corresponding Preference Function ⇒

x	f(x)
1	3
2	6
3	1
4	6
5	1
6	1
7	1
8	3

Each car proceeds to its preferred spot

if occupied it parks in the next available spot



next

A one way street with 8 parking spaces

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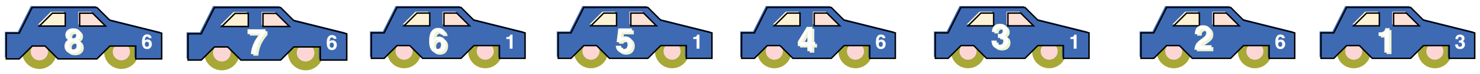

x	f(x)
1	③
2	6
3	①
4	6
5	①
6	①
7	6
8	6

The large number is their order of arrival

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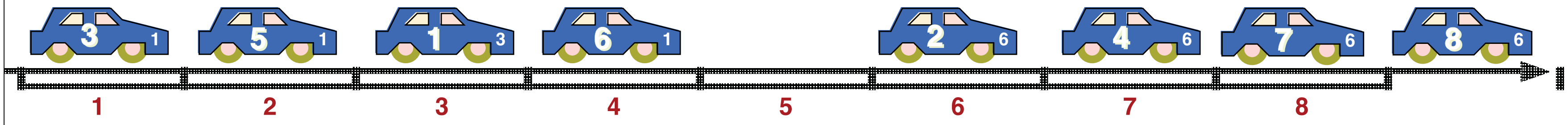
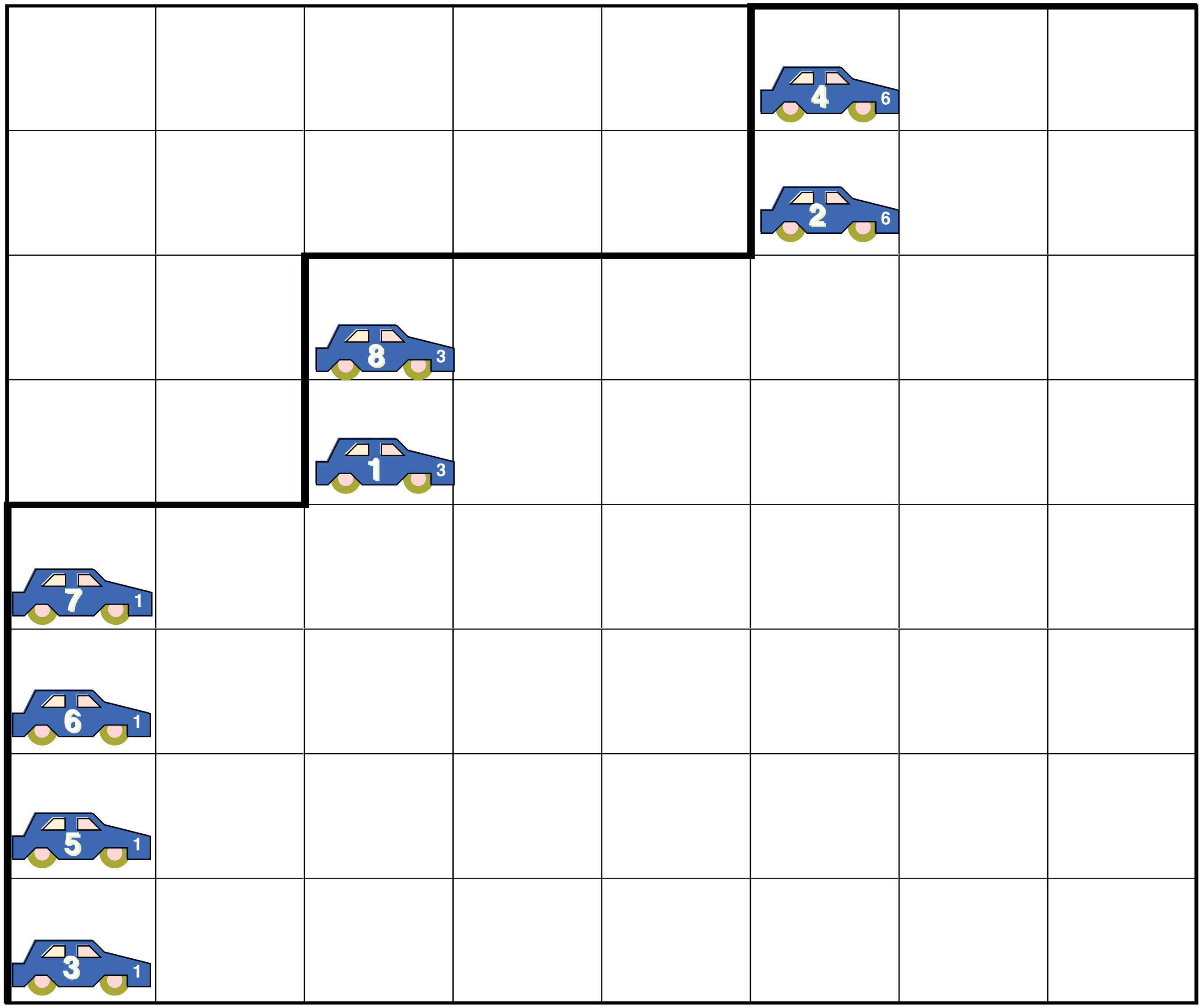


the number of cars that want to park in the first 5 places is less than 5 . . .

This preference function did not park the cars!!!

next

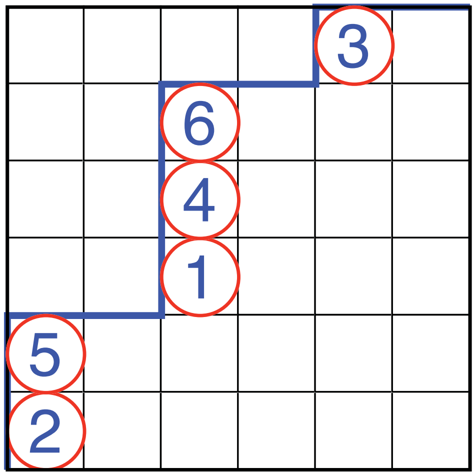
Parking functions and Dyck Paths



A Parking function is a preference function that parks the cars

The sufficiency

by induction



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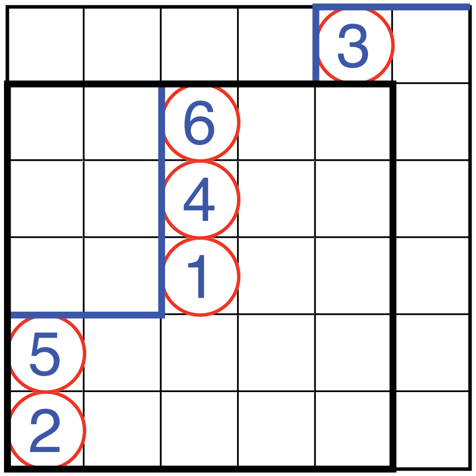

②	⑤	①	④	⑥	

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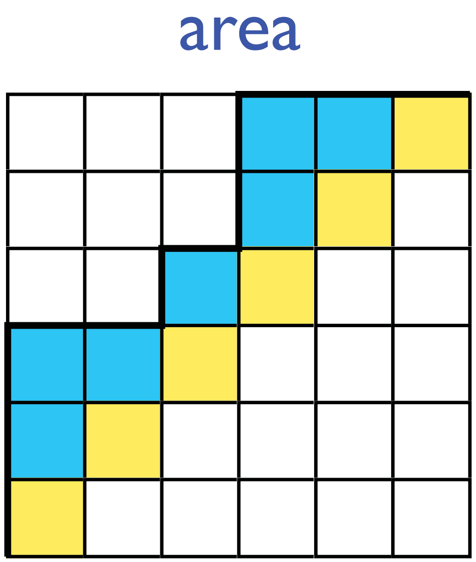
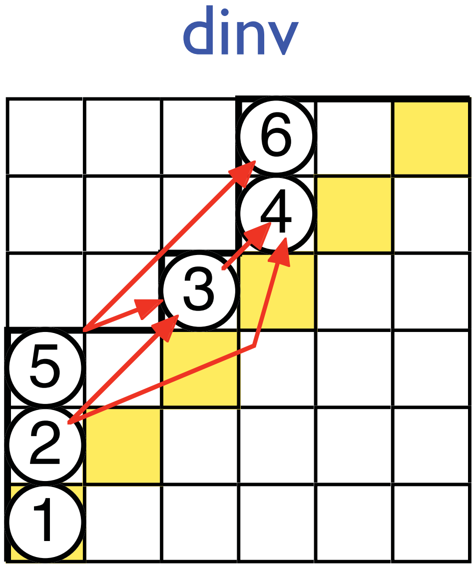
Two Statistics on Parking Functions





By summing over all Parking Functions in the n × n square

t to the "area" and q to the "dinv" we get a beautiful polynomial

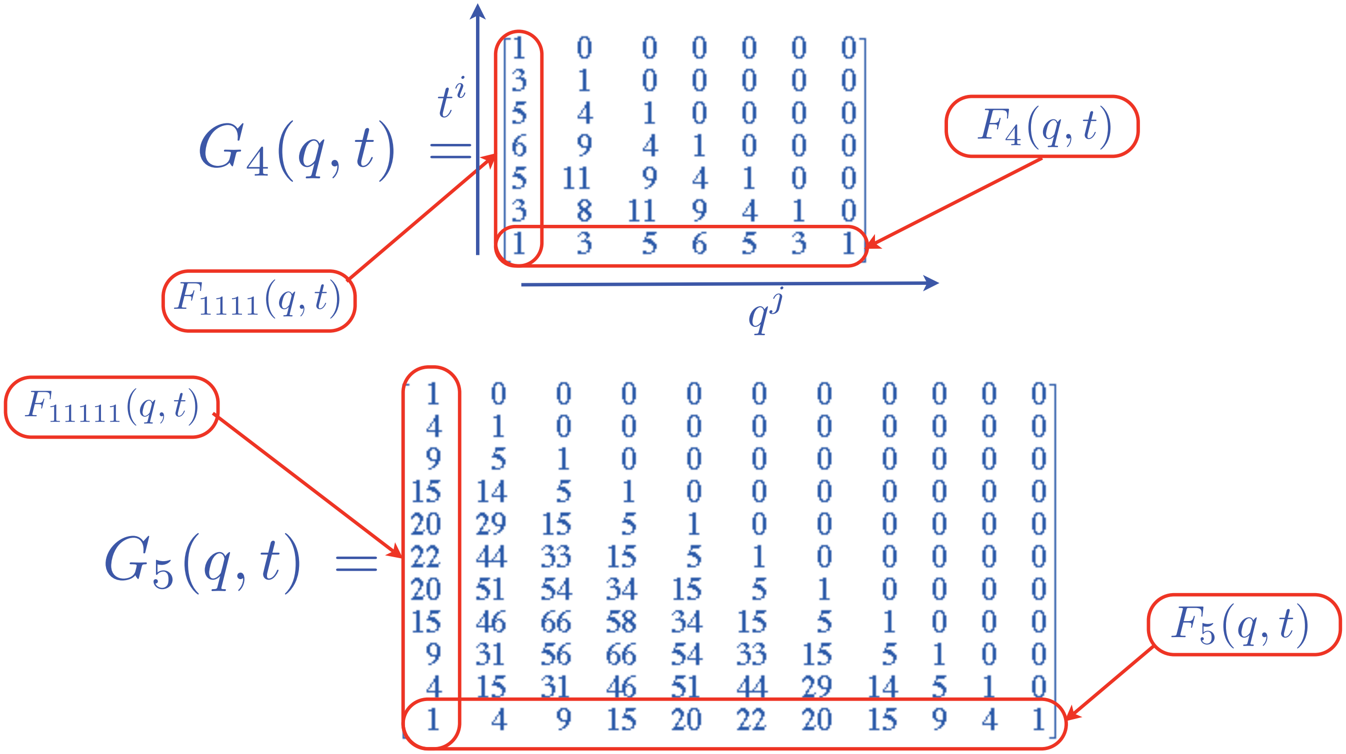
$$\boxed{G_n(q,t) = \sum_{PF \in \mathcal{PF}_n} t^{area(PF)} q^{dinv(PF)}}$$

To see how beautiful we need MAPLE

next

The MAPLE data

$$\boxed{G_n(q,t) = \sum_{PF \in \mathcal{PF}_n} t^{area(PF)} q^{dinv(PF)}}$$



$$G_4(q,t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 4 & 1 & 0 & 0 & 0 & 0 \\ 6 & 9 & 4 & 1 & 0 & 0 & 0 \\ 5 & 11 & 9 & 4 & 1 & 0 & 0 \\ 3 & 8 & 11 & 9 & 4 & 1 & 0 \\ 1 & 3 & 5 & 6 & 5 & 3 & 1 \end{bmatrix}$$

$$G_5(q,t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 15 & 14 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 20 & 29 & 15 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 22 & 44 & 33 & 15 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 20 & 51 & 54 & 34 & 15 & 5 & 1 & 0 & 0 & 0 & 0 \\ 15 & 46 & 66 & 58 & 34 & 15 & 5 & 1 & 0 & 0 & 0 \\ 9 & 31 & 56 & 66 & 54 & 33 & 15 & 5 & 1 & 0 & 0 \\ 4 & 15 & 31 & 46 & 51 & 44 & 29 & 14 & 5 & 1 & 0 \\ 1 & 4 & 9 & 15 & 20 & 22 & 20 & 15 & 9 & 4 & 1 \end{bmatrix}$$

Problem

For each $\mu \vdash n$, identify the subset $PF_\mu \subseteq \mathcal{PF}_n$, giving

$$\boxed{F_\mu(q,t) = \sum_{PF \in \mathcal{PF}_\mu} t^{area(PF)} q^{dinv(PF)}}$$

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A simpler question?

Count the number of permutations of S_n
whose first $n-k$ entries are increasing
and have no increasing subsequence of length greater than $n-k$

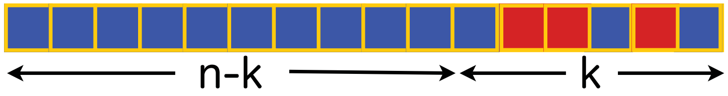
In symbols, we want $\#\Pi_{n,k}$ where

$$\Pi_{n,k} \;=\; \{a = (a_1, a_2, \ldots, a_n) \in S_n : a_1 < a_2 < \cdots < a_{n-k} \;\&\; LI(a) = n-k\}$$

$LI(a)$ denotes the length of the longest increasing subsequence

Let $C_{n,k}$ be a collection of "*colored permutations* " of S_n such that

1) each element is red or blue
2) the blue entries are increasing
3) only a subset of the last k entries can be red
4) the sign of a colored permutation is $(-1)^{\#\text{blue}}$



Theorem

$$\#\Pi_{n,k} = \sum_{a \in C_{n,k}} sign(a) = \sum_{r=0}^{k} \binom{k}{r} (-1)^{k-r} n(n-1)\cdots(n-r+1)$$

100$

Reward

for an "elementary" proof

next

Do you think that was enough?

no

here is more

$$\sum_{a\in\Pi_{n,k}} q^{maj(a^{-1})} = \sum_{r=0}^{k}(-1)^r\binom{k}{r}[n]_q[n-1]_q\cdots[n-r+1]_q$$

THE END

next

THE END