

**New Methods**  
**in**

**The Theory of  $m$ - $Q$ -Quasi-Invariants**

(joint work with J. Bell and N. Wallach)

## The divided difference operator

Set  $\mathbf{X}_n = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ . For  $\mathbf{P} \in \mathbb{Q}[\mathbf{X}_n]$  we will write  $\mathbf{P}(\mathbf{x})$  for  $\mathbf{P}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$ .

Let  $s_{ij}$  denote the transposition which interchanges  $\mathbf{x}_i$  with  $\mathbf{x}_j$ .

### Proposition

*The divided difference operator*

$$\delta_{ij} = \frac{1}{\mathbf{x}_i - \mathbf{x}_j} (\mathbf{1} - s_{ij}) \quad (1)$$

*sends polynomials into polynomials and satisfies the “Leibnitz” identity*

$$\delta_{ij} (\mathbf{P}(\mathbf{x})\mathbf{Q}(\mathbf{x})) = (\delta_{ij} \mathbf{P}(\mathbf{x})) \mathbf{Q}(\mathbf{x}) + (s_{ij} \mathbf{P}(\mathbf{x})) \delta_{ij} \mathbf{Q}(\mathbf{x}) \quad (2)$$

### Proof

We trivially have

$$(\mathbf{1} - s_{ij})\mathbf{P}\mathbf{Q} = \mathbf{P}\mathbf{Q} - (s_{ij}\mathbf{P})(s_{ij}\mathbf{Q}) = (\mathbf{P} - s_{ij}\mathbf{P})\mathbf{Q} + (s_{ij}\mathbf{P})(\mathbf{Q} - s_{ij}\mathbf{Q})$$

and (2) follows upon dividing both sides by  $\mathbf{x}_i - \mathbf{x}_j$

## More identities

*Setting*

$$\mathbf{u} = \mathbf{x}_i + \mathbf{x}_j, \quad \mathbf{v} = \mathbf{x}_i - \mathbf{x}_j$$

*we can write*

$$\mathbf{x}_i = \frac{\mathbf{u} + \mathbf{v}}{2}, \quad \mathbf{x}_j = \frac{\mathbf{u} - \mathbf{v}}{2}$$

*Thus any polynomial  $P$  has an expansion of the form*

$$\mathbf{P} = \mathbf{a}_{000} + \sum_{r+s \geq 1} \mathbf{a}_{rs} \mathbf{u}^r \mathbf{v}^s \quad (\text{with } \mathbf{a}_{ij} \text{ polynomials in the remaining variables})$$

*Since  $\mathbf{s}_{ij} \mathbf{u} = \mathbf{u}$  and  $\mathbf{s}_{ij} \mathbf{v} = -\mathbf{v}$  it follows that*

$$(1 - \mathbf{s}_{ij})\mathbf{P} = \sum_{s \geq 1} \mathbf{a}_{rs} \mathbf{u}^r (1 - (-1)^s) \mathbf{v}^s = 2 \sum_{s \text{ odd}} \mathbf{a}_{rs} \mathbf{u}^r \mathbf{v}^s$$

*Thus the highest power of  $\mathbf{x}_i - \mathbf{x}_j$  that divides  $(1 - \mathbf{s}_{ij})\mathbf{P}$  is necessarily odd, in particular*

$$\frac{1}{\mathbf{x}_i - \mathbf{x}_j} (1 - \mathbf{s}_{ij})\mathbf{P}$$

*is a polynomial*

## The $m$ -Quasi-invariants

$\mathbf{P}$  is  $m$ -quasi-invariant if, for all  $1 \leq i < j \leq n$ ,  $(1 - s_{ij})\mathbf{P}$  is divisible by  $(\mathbf{x}_i - \mathbf{x}_j)^{2m+1}$

Some basic facts:

- (1) The space  $\mathbf{QI}_m[\mathbf{X}_n]$  of  $m$ -quasi-invariants in  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  is an algebra
- (2)  $\mathbb{Q}[\mathbf{X}_n] = \mathbf{QI}_0[\mathbf{X}_n] \supset \mathbf{QI}_1[\mathbf{X}_n] \supset \dots \supset \mathbf{QI}_m[\mathbf{X}_n] \supset \dots \supset \mathbf{QI}_\infty[\mathbf{X}_n] = \mathbf{SYM}_n$  (Symmetric polynomials)
- (3) Each  $\mathbf{QI}_m[\mathbf{X}_n]$  is an  $\mathbf{S}_n$  module

**Proof**

- (1):  $(1 - s_{ij})(\mathbf{PQ}) = (\mathbf{P} - s_{ij}\mathbf{P})\mathbf{Q} + (s_{ij}\mathbf{P})(\mathbf{Q} - s_{ij}\mathbf{Q})$
- (2):  $(\mathbf{x}_i - \mathbf{x}_j)^{2m+1}$  “divides”  $\mathbf{P} - s_{ij}\mathbf{P}$  for all  $m$  if and only if  $\mathbf{P} - s_{ij}\mathbf{P} = \mathbf{0}$
- (3): If  $\sigma \in \mathbf{S}_n$  then

$$\begin{aligned} (1 - s_{ij})\sigma^{-1}\mathbf{P} &= \sigma^{-1}\sigma(1 - s_{ij})\sigma^{-1}\mathbf{P} = \sigma^{-1}(1 - s_{\sigma_i\sigma_j})\mathbf{P} \\ &= \sigma^{-1}(\mathbf{x}_{\sigma_i} - \mathbf{x}_{\sigma_j})^{2m+1}\mathbf{Q} = (\mathbf{x}_i - \mathbf{x}_j)^{2m+1}\sigma^{-1}\mathbf{Q} \end{aligned}$$

### Further properties of $m$ -Quasi-invariants

$$(1) \quad \mathbf{QI}_m[\mathbf{X}_n] = \mathcal{H}_0[\mathbf{QI}_m] \oplus \mathcal{H}_1[\mathbf{QI}_m] \oplus \mathcal{H}_2[\mathbf{QI}_m] \oplus \cdots \oplus \mathcal{H}_r[\mathbf{QI}_m] \oplus \cdots$$

where  $\mathcal{H}_r[\mathbf{QI}_m]$  denotes the subspace of homogeneous  $m$ -Quasi-invariants of degree  $r$ .

$$(2) \quad \text{Setting } \Pi(\mathbf{x}) = \prod_{1 \leq i < j \leq n} (\mathbf{x}_i - \mathbf{x}_j) \text{ we have}$$

$$\Pi(\mathbf{x})^{2m+1} \in \mathbf{QI}_m[\mathbf{X}_n]$$

$$(3) \quad \text{More generally for every polynomial } \mathbf{P} \in \mathbb{Q}[\mathbf{X}_n] \text{ we have}$$

$$\Pi(\mathbf{x})^{2m} \mathbf{P}(\mathbf{x}) \in \mathbf{QI}_m[\mathbf{X}_n]$$

**Proof**

$$(1): \text{ If } \mathbf{P} = \sum_{r=0}^k \mathbf{P}_r \text{ and } \mathbf{Q} = \sum_{r=0}^h \mathbf{Q}_r \text{ with } \mathbf{P}_r, \mathbf{Q}_r \text{ homogeneous of degree } i \text{ then}$$

$$(1 - s_{ij})\mathbf{P} = (\mathbf{x}_i - \mathbf{x}_j)^{2m+1} \mathbf{Q} \implies \sum_{i=0}^k (1 - s_{ij})\mathbf{P}_r = \sum_{r=0}^h (\mathbf{x}_i - \mathbf{x}_j)^{2m+1} \mathbf{Q}_r$$

(This also shows that the first  $2m$  components of  $P$  must be symmetric)

$$(3): (1 - s_{ij})\Pi(\mathbf{x})^{2m} \mathbf{P}(\mathbf{x}) = \Pi(\mathbf{x})^{2m} (1 - s_{ij})\mathbf{P}(\mathbf{x}) = \Pi(\mathbf{x})^{2m} (\mathbf{x}_i - \mathbf{x}_j) \mathbf{Q}(\mathbf{x})$$

## Some deeper facts

**Theorem** (Etingof-Ginzburg, etc )

$\mathbf{QI}_m[\mathbf{X}_n]$  is free over  $\mathbf{SYM}[\mathbf{X}_n]$  and the quotient

$$\mathbf{QI}_m[\mathbf{X}_n]/(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)\mathbf{QI}_m$$

is a graded version of the Regular representation of  $S_n$

**Theorem** (Felder-Veselov using a KZ connection)

(KZ=Knizhnik-Zamolodchikov)

The Frobenius characteristic of  $\mathbf{QI}_m[X]_n$  as a graded  $S_n$ -module is

$$\mathbf{c}_\lambda = \sum_{(\mathbf{i}, \mathbf{j}) \in \lambda} \frac{\sum_{\lambda \vdash n} \mathbf{S}_\lambda(\mathbf{x}) \sum_{\mathbf{T} \in \mathbf{ST}(\lambda)} \mathbf{q}^{\mathbf{co}(\mathbf{T}) + m \binom{n}{2} - c_\lambda}}{(\mathbf{1} - \mathbf{q})(\mathbf{1} - \mathbf{q}^2) \cdots (\mathbf{1} - \mathbf{q}^n)}$$

(There are  $m$ -analogs of everything in sight!)

## Classical Harmonics of $S_n$

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$$\mathbf{H}_n = \left\{ \mathbf{q}(\partial_{\mathbf{x}}) \Pi(\mathbf{x}) : \mathbf{q} \in \mathbb{R} \right\} \quad \text{with} \quad \Pi(\mathbf{x}) = \prod_{1 \leq i < j \leq n} (\mathbf{x}_i - \mathbf{x}_j)$$

$m$ - Harmonics of  $S_n$

Note if  $\mathbf{q}(\mathbf{x})$  is homogeneous of degree  $d$

$$\mathbf{q}(\partial_{\mathbf{x}}) = \frac{1}{2^d d!} \sum_{r=0}^d \binom{d}{2} (-1)^r \Delta^{d-r} \mathbf{q}(\mathbf{x}) \Delta^r \quad \Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$$

For  $\mathbf{q} \in \mathbf{QI}_m[\mathbf{X}_n]$  set

$$\mathbf{L}_q(\mathbf{m}) = \frac{1}{2^d d!} \sum_{r=0}^d \binom{d}{2} (-1)^r \Delta(\mathbf{m})^{d-r} \mathbf{q}(\mathbf{x}) \Delta(\mathbf{m})^r$$

with

$$\Delta(\mathbf{m}) = \Delta - 2\mathbf{m} \sum_{1 \leq i < j \leq n} \frac{1}{\mathbf{x}_i - \mathbf{x}_j} (\partial_{x_i} - \partial_{x_j})$$

Then

$$\mathbf{H}_n(\mathbf{m}) = \left\{ \mathbf{L}_q(\mathbf{m}) \Pi(\mathbf{x})^{2m+1} : \mathbf{q} \in \mathbf{QI}_m[\mathbf{X}_n] \right\}$$

## A bit on Finitely generated graded algebras

(1)  $\mathbf{A} = \mathcal{H}_0(\mathbf{A}) \oplus \mathcal{H}_1(\mathbf{A}) \oplus \mathcal{H}_2(\mathbf{A}) \oplus \mathcal{H}_3(\mathbf{A}) \oplus \cdots$  with

$$\mathcal{H}_r(\mathbf{A}) \times \mathcal{H}_s(\mathbf{A}) \subseteq \mathcal{H}_{r+s}(\mathbf{A})$$

(2) For some  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m \in \mathbf{A}$  the collection  $\{\mathbf{a}_1^{\mathbf{P}_1} \mathbf{a}_2^{\mathbf{P}_2} \cdots \mathbf{a}_m^{\mathbf{P}_m}\}$  spans  $\mathbf{A}$ .

### The Hilbert Series

$$\mathbf{F}_{\mathbf{A}}(t) = \sum_{r \geq 0} t^r \dim \mathcal{H}_r(\mathbf{A})$$

### Theorem

$\mathbf{F}_{\mathbf{A}}(t)$  is a rational function and if  $\mathcal{B}$  is a homogeneous basis for  $\mathbf{A}$  then

$$\mathbf{F}_{\mathbf{A}}(t) = \sum_{\mathbf{b} \in \mathcal{B}} t^{\text{degree}(\mathbf{b})}$$

If  $\mathbf{a}_i$  is homogeneous of degree  $d_i$  then

$$(2) : \implies \mathbf{F}_{\mathbf{A}}(t) << \frac{1}{(1-t^{d_1})(1-t^{d_2}) \cdots (1-t^{d_m})}$$

and thus  $\mathbf{F}_{\mathbf{A}}(q)$  has a pole of order at most  $m$  at  $t = 1$



## The Krull dimension

### Theorem

The following three integers are equal (with  $\mathbf{b}_i \in \mathbf{A}$  homogeneous )

$\mathbf{n}_1$  : The order of the pole of  $F_{\mathbf{A}}(t)$  at  $t = 1$

$\mathbf{n}_2$  : The maximum  $\mathbf{m}$  such that  $\{\mathbf{b}_1^{\mathbf{p}_1} \mathbf{b}_2^{\mathbf{p}_2} \cdots \mathbf{b}_m^{\mathbf{p}_m}\}_{\mathbf{p}_i \geq 0}$  is an independent set

$\mathbf{n}_3$  : The minimum  $\mathbf{m}$  such that  $\mathbf{d} = \dim \mathbf{A}/(\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m)_{\mathbf{A}} < \infty$  (\*)

Set  $\mathbf{m}_{\mathbf{A}} = \mathbf{n}_1 = \mathbf{n}_2 = \mathbf{n}_3$

Note that if  $\{\mathbf{Q}_1, \mathbf{Q}_2, \dots, \mathbf{Q}_d\} \subseteq \mathbf{A}$  is any basis for the quotient in (\*) then (\*) implies that the collection

$$\left\{ \mathbf{Q}_i \mathbf{b}_1^{\mathbf{p}_1} \mathbf{b}_2^{\mathbf{p}_2} \cdots \mathbf{b}_m^{\mathbf{p}_m} \right\}_{1 \leq i \leq d; \mathbf{p}_i \geq 0} \quad (**)$$

spans  $\mathbf{A}$ . When  $\mathbf{m} = \mathbf{m}_{\mathbf{A}}$  we say that  $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m\}$  is an HSOP HSOP = "Homogeneous System Of Parameters"

Note that if (\*\*) is a basis for  $\mathbf{A}$  then  $\mathbf{A}$  is called "Cohen-Macaulay" and of course then  $\mathbf{A}$  is a free module over  $\mathbb{Q}[\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m]$  of rank  $\mathbf{d}$  moreover if  $\mathbf{d}_i = \text{degree}(\mathbf{b}_i)$  then

$$\mathbf{F}_{\mathbf{A}}(\mathbf{q}) = \frac{\sum_{i=1}^m t^{\text{degree}(\mathbf{Q}_i)}}{(1-t^{\mathbf{d}_1})(1-t^{\mathbf{d}_2}) \cdots (1-t^{\mathbf{d}_n})}$$

## The Classic Example

$$\mathbf{R}_n = \mathbb{Q}[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$$

with

$$\mathbf{F}_{\mathbf{R}_n}(\mathbf{t}) = \frac{1}{(1-t)^n} = \frac{[n]_t!}{(1-t)(1-t^2)\cdots(1-t^n)} \quad [r]_t = 1 + t + \cdots + t^{r-1} \quad (*)$$

### Theorem

The collection  $ARR_n = \{\mathbf{x}^{\mathbf{r}} = \mathbf{x}_2^{r_2} \mathbf{x}_3^{r_3} \cdots \mathbf{x}_n^{r_n}\}$  with  $\mathbf{0} \leq r_i \leq i-1$  spans the quotient

$$\mathbf{R}_n / (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{\mathbf{R}}$$

with

$$\mathbf{e}_{\mathbf{k}} = \sum_{1 \leq i_1 < i_2, \dots, < i_k \leq n} \mathbf{x}_{i_1} \mathbf{x}_{i_2} \cdots \mathbf{x}_{i_k}$$

Then it follows from (\*) that the collection

$$\{\mathbf{x}^{\mathbf{r}} \mathbf{e}_1^{\mathbf{p}_1} \mathbf{e}_2^{\mathbf{p}_2} \cdots \mathbf{e}_n^{\mathbf{p}_n}\}_{\mathbf{x}^{\mathbf{r}} \in ARR_n}$$

is a basis for  $\mathbf{R}$ .

Thus  $\mathbf{R}$  is a Cohen-Macaulay algebra, free over  $\mathbb{Q}[\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n] = \mathbf{SYM}_n$  of rank  $n!$

## A few words about **GORDDAN** (“Gröbner”) bases

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### **Theorem 1** (Gordan: *Liouville Journal* 1900)

*Any collection of monomials in  $x_1, x_2, \dots, x_n$  has a finite number of minimal elements under the division partial order.*

For  $\mathbf{P} \in \mathbf{R}$  denote by  $\mathbf{h}(\mathbf{R})$  the homogeneous component of highest degree in  $\mathbf{P}$  and let  $\text{lm}(\mathbf{P})$  be **dlex** largest monomial in  $\mathbf{P}$  (leading monomial)

### **Theorem 2** (Gordan: *Liouville Journal* 1900)

*If  $\mathbf{J}$  is any ideal in  $\mathbf{R}$  and  $\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k$  are the minimal elements of*

$$\mathcal{M}_{\mathbf{J}} = \{\text{lm}(\mathbf{P}) : \mathbf{P} \in \mathbf{J}\}$$

*and*

$$\mathbf{m}_1 = \mathbf{h}(\mathbf{f}_1), \mathbf{m}_2 = \mathbf{h}(\mathbf{f}_2), \dots, \mathbf{m}_k = \mathbf{h}(\mathbf{f}_k)$$

*then every  $\mathbf{P} \in \mathbf{J}$  may be expressed in the form*

$$\mathbf{P} = \mathbf{A}_1 \mathbf{f}_1 + \mathbf{A}_2 \mathbf{f}_2 + \dots + \mathbf{A}_k \mathbf{f}_k \quad (\text{with } \mathbf{A}_i \in \mathbf{R} \text{ and } \text{degree}(\mathbf{A}_i \mathbf{f}_i) \leq \text{degree}(\mathbf{P}))$$

*Moreover the complement of  $\mathcal{M}_{\mathbf{J}}$  yields a monomial basis for the quotient  $\mathbf{R}/\mathbf{J}$*

## An Application

### Theorem

For  $\mathbf{J} = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{\mathbf{R}}$  we have

$$\min \mathcal{M}_{\mathbf{J}} = \{ \mathbf{x}_1, \mathbf{x}_2^2, \dots, \mathbf{x}_n^n \}$$

### Proof

The “dlex” GORDAN basis of the ideal  $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{\mathbf{R}}$  is simply given by the complete homogeneous symmetric functions

$$\mathbf{h}_{\mathbf{k}}(\mathbf{x}_{\mathbf{k}}, \mathbf{x}_{\mathbf{k}+1}, \dots, \mathbf{x}_{\mathbf{n}}) = \mathbf{x}_{\mathbf{k}}^{\mathbf{k}} + \sum_{\mathbf{r}=0}^{\mathbf{k}-1} \mathbf{x}_{\mathbf{k}}^{\mathbf{r}} \mathbf{h}_{\mathbf{k}-\mathbf{r}}(\mathbf{x}_{\mathbf{k}+1}, \dots, \mathbf{x}_{\mathbf{n}}) \quad (\text{for } \mathbf{1} \leq \mathbf{k} \leq \mathbf{n})$$

It thus follows from the Gordan Theorem that the collection

$$\mathbf{ART}_{\mathbf{n}} = \{ \mathbf{x}_2^{\mathbf{r}_2} \mathbf{x}_3^{\mathbf{r}_3} \dots \mathbf{x}_n^{\mathbf{r}_n} \} \quad (\text{with } \mathbf{0} \leq \mathbf{r}_i \leq \mathbf{i} - \mathbf{1})$$

is a basis for the quotient

$$\mathbf{R}/(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)_{\mathbf{R}}$$

## More on m-Quasi-Invariants

It follows from the classic example that every polynomial  $\mathbf{P} \in \mathbb{Q}[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$  has an expansion of the form

$$\mathbf{P}(\mathbf{x}) = \sum_{\mathbf{b} \in AR\mathcal{T}_n} \mathbf{b}(\mathbf{x}) \mathbf{A}_{\mathbf{b}}(\mathbf{x})$$

where the coefficients  $\mathbf{A}_{\mathbf{b}}$  are symmetric polynomials uniquely determined by  $P$ .

**Example:** For  $n = 3$  any  $\mathbf{P} \in \mathbb{Q}[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$  may be uniquely expressed in the form

$$\mathbf{P} = \mathbf{A}_{000} + \mathbf{A}_{010} \mathbf{x}_2 + \mathbf{A}_{001} \mathbf{x}_3 + \mathbf{A}_{002} \mathbf{x}_3^2 + \mathbf{A}_{011} \mathbf{x}_2 \mathbf{x}_3 + \mathbf{A}_{012} \mathbf{x}_2 \mathbf{x}_3^2$$

Thus we may represent each  $\mathbf{P} \in \mathbb{Q}[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$  by a sextuple of symmetric polynomials

$$\mathbf{P} \iff (\mathbf{A}_{000}, \mathbf{A}_{010}, \mathbf{A}_{001}, \mathbf{A}_{002}, \mathbf{A}_{011}, \mathbf{A}_{012})$$

### Theorem

The algebra  $\mathbf{QI}_m[\mathbf{X}_n]$  of m-Quasi-Invariants is finitely generated

### Proof

It easily follows from the Hilbert basis theorem (or constructively from the Gordan Theorem) that a module of  $\mathbf{N}^{\text{tuples}}$  over a polynomial ring is finitely generated. (use induction on  $\mathbf{N}$  and apply the result to the ring  $\mathbb{Q}[\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n]$ .)

## A not so well known CM criterion

### Theorem

Let  $\mathbf{A}$  be a finitely generated graded algebra and let  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k$  be an HSOP of  $\mathbf{A}$  and let  $\mathbf{d}_i = \text{degree}(\mathbf{b}_i)$ . Suppose that

$$\text{a) } \lim_{q \rightarrow -1} (1 - q^{\mathbf{d}_1})(1 - q^{\mathbf{d}_2}) \cdots (1 - q^{\mathbf{d}_k}) F_{\mathbf{A}}(q) = \mathbf{N}$$

and

$$\text{b) } \dim \mathbf{A}/(\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k)_{\mathbf{A}} = \mathbf{N}$$

then  $\mathbf{A}$  is free over  $\mathbb{Q}[\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k]$  of rank  $\mathbf{N}$  and thus a Cohen-Macaulay algebra. In particular if  $\{\mathbf{Q}_1, \mathbf{Q}_2, \dots, \mathbf{Q}_N\} \subseteq \mathbf{A}$  is a homogeneous basis for  $\mathbf{A}/(\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k)_{\mathbf{A}}$  then

$$F_{\mathbf{A}}(q) = \frac{\sum_{i=1}^{\mathbf{N}} q^{\text{degree}(\mathbf{Q}_i)}}{(1 - q^{\mathbf{d}_1})(1 - q^{\mathbf{d}_2}) \cdots (1 - q^{\mathbf{d}_k})}$$

## Back to $m$ -Quasi-Invariants

### Corollary

The algebra  $\mathbf{QI}_m$  of  $m$ -Quasi-Invariants in  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  is a free module over  $\mathbb{Q}[\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n]$  of rank  $n!$

### Proof

It can be shown that

$$\dim \mathbf{QI}_m / (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n) \mathbf{QI}_m = n!$$

On the other hand the inclusions

$$\prod_n (\mathbf{x})^{2m+1} \mathbb{Q}[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n] \subseteq \mathbf{QI}_m \subseteq \mathbb{Q}[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$$

imply the component wise Hilbert series inequalities

$$\frac{q^{(2m+1)\binom{n}{2}}}{(1-q)^n} \ll \mathbf{F}_{\mathbf{QI}_m}(q) \ll \frac{1}{(1-q)^n}$$

Thus

$$\lim_{q \rightarrow -1} (1-q)(1-q^2) \cdots (1-q^n) \mathbf{F}_{\Lambda}(q) = n!$$

## On $m$ -Quasi-Invariants of $S_3$

### Theorem

The polynomial

$$P_1 = A(x_2 + x_3) + Bx_2x_3$$

with  $A, B$  symmetric, is  $m$ -quasi-invariant if and if only we have

$$\text{a) } A = -\delta_{12}x_1(x_1 - x_3)^{2m}M(\mathbf{x}) \quad \text{B} = \delta_{12}(x_1 - x_3)^{2m}M(\mathbf{x})$$

where  $M$  is any polynomial that satisfies the two conditions

(\*)

$$\text{a) } s_{13}M(\mathbf{x}) = M(\mathbf{x}), \quad \text{b) } \delta_{23}\delta_{12}(x_1 - x_3)^{2m}M(\mathbf{x}) = 0,$$

### Problem

Construct these mysterious polynomials  $M(\mathbf{x})$



## The reduction to a Gordan basis problem

- a)  $\delta_{13}\delta_{12}(\mathbf{x}_2 - \mathbf{x}_3)^{2m} M(\mathbf{x}) = \mathbf{0}$ ,  
 b)  $\mathbf{s}_{23} M(\mathbf{x}) = M(\mathbf{x})$ . (\*)

Now b) implies that

$$M(\mathbf{x}) = \mathbf{a} + \mathbf{b}(\mathbf{x}_2 + \mathbf{x}_3) + \mathbf{c}\mathbf{x}_2\mathbf{x}_3 \quad (\text{with } \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{SYM}_3)$$

and we also have

$$(\mathbf{x}_2 - \mathbf{x}_3)^{2m} = \mathbf{A}_m + \mathbf{B}_m(\mathbf{x}_2 + \mathbf{x}_3) + \mathbf{C}_m\mathbf{x}_2\mathbf{x}_3 \quad (\text{with } \mathbf{A}_m, \mathbf{B}_m, \mathbf{C}_m \in \mathbf{SYM}_3)$$

Now it turns out that a) simply requires that

$$\mathbf{c}\bar{\mathbf{A}}_m + \mathbf{b}\bar{\mathbf{B}}_m + \mathbf{a}\bar{\mathbf{C}}_m = \mathbf{0}$$

with

$$\bar{\mathbf{A}}_m = \mathbf{A}_m + \mathbf{B}_m\mathbf{e}_1 + \mathbf{C}_m\mathbf{e}_2, \quad \bar{\mathbf{B}}_m = \mathbf{B}_m + \mathbf{C}_m\mathbf{e}_1, \quad \bar{\mathbf{C}}_m = \mathbf{C}_m$$

**Theorem 3.3**

The space of triplets of symmetric functions

$$\mathcal{M}_m(e) = \{(a, b, c) : c\bar{A}_m + b\bar{B}_m + a\bar{C}_m = 0\}$$

is a free  $\mathbb{Q}[e_1, e_2, e_3]$ -module of rank 2. With Hilbert series

$$F_{\mathcal{M}_m(e)}(t) = \frac{t^m + t^{m+1}}{(1-t)(1-t^2)(1-t^3)} \quad (*)$$

“Proof”

(\*) Follows from the two identities

$$\begin{aligned} t^{2m-2}F_{\mathcal{M}_m(e)}(t) &= F_{(A_m, B_m, C_M)}(t) + \frac{t^{2m-2} + t^{2m-1} + t^{2m}}{(1-t)(1-t^2)(1-t^3)} \\ F_{(A_m, B_m, C_M)}(t) &= \frac{(1 + t^m + t^{m-1})(1 - t^m)(1 - t^{m-1})}{(1-t)(1-t^2)(1-t^3)} \end{aligned}$$