# Hilbert Series of Invariants, Constant terms
and
# Kostka-Foulkes Polynomials
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**Abstract.** A problem that arose in the study of the mass of the neutrino led us to the evaluation of a constant term with a variety of ramifications into several areas from Invariant Theory, Representation Theory, the Theory of Symmetric Functions and Combinatorics. A significant by-product of our evaluation is the construction of a trigraded Cohen Macaulay basis for the Invariants under an action of $SL_n(\mathbb{C})$ on a space of $2n+n^2$ variables.

## Introduction

This paper covers a variety of topics encountered in the construction of a proof of the following constant term identity

**Theorem I.1**

$$\frac{1}{(1-q)^n}\prod_{i=1}^{n}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{1\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_i/x_j)(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} = \frac{1+q^{\binom{n+1}{2}}}{(1-q)\prod_{i=2}^{n}(1-q^i)^2(1-q^{n+1})\big(1-q^{\binom{n+1}{2}}\big)} \qquad \text{I.1}$$

This problem arose in the determination of the ring of invariants under an action of $SL_n[\mathbb{C}]$. on the polynomial ring $\mathbb{Q}[U,V,X]$ in the $2n+n^2$ variables

$$\{u_i, v_j, x_{i,j}\}_{i,j=1}^n \qquad \text{I.2}$$

Here, a matrix $g\in SL_n[\mathbb{C}]$ is made to act on the row vector $U=(u_1,u_2,\ldots,u_n)$ by right multiplication, on the column vector $V=(v_1,v_2,\cdots,v_n)$ by left multiplication and on the matrix $X=\|x_{i,j}\|_{i,j=1}^n$ by conjugation. More precisely, the action of $g$ on a polynomial $P(U,V,X)\in\mathbb{Q}[U,V,X]$ is defined by setting

$$T_gP(U,V,X) \;=\; P(Ug, g^{-1}V, g^{-1}Xg). \qquad \text{I.3}$$

It follows from well known results of Invariant Theory that the ring of invariants $\mathbb{Q}[U,V,X]^{SL_n[\mathbb{C}]}$ is Cohen Macaulay. This means that we must be able to find a basic set of invariants $\{\theta_1,\ldots,\theta_M\,;\,\eta_1,\ldots,\eta_N\}$ such that every invariant can be uniquely expanded as a linear combination of $\eta_1,\ldots,\eta_N$ with coefficients polynomials in $\theta_1,\ldots,\theta_M$. We shall here and after refer to the task of constructing such a basic set as the "*UVX Problem* " and the polynomials $P(U,V,X)\in\mathbb{C}[U,V,X]^{SL_n[\mathbb{C}]}$ will be called "*UVX invariants* ".

A useful tool in identifying a basic set in a Cohen Macaulay ring is the Hilbert series of the ring. That is the generating function of the dimension of the successive homogeneous components of the ring. In this case, denoting by $\mathcal{H}_m(U,V,X)$ the subspace of homogeneous elements of degree $m$ in $\mathbb{C}[U,V,X]^{SL_n[\mathbb{C}]}$. the Hilbert series is simply the rational function $F_{UVX}(q)$ with Taylor expansion

$$F_{UVX}(q) \;=\; \sum_{m\ge 0}\dim\,\mathcal{H}_m(U,V,X)\,q^m.$$

The constant term in I.1 arises precisely in the construction of this rational function. That is we will show that

### Theorem I.2

$$F_{UVX}(q) \;=\; \frac{1}{(1-q)^n}\prod_{i=1}^{n}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{1\le i<j\le n}\frac{1-x_i/x_j}{(1-qx_i/x_j)(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}$$

In particular by combining Theorems I.1 and I.2 we obtain

### Theorem I.3

*The Hilbert series of the ring of invariants $\mathbb{C}[U,VX]^{SL_n[\mathbb{C}]}$ is the rational function*

$$F_{UVX}(q)=\frac{1+q^{\binom{n+1}{2}}}{(1-q)\prod_{i=2}^{n}(1-q^i)^2(1-q^{n+1})\big(1-q^{\binom{n+1}{2}}\big)} \qquad \text{I.4}$$

This somewhat surprising result strongly suggests the nature of a possible basic set. Indeed, a Cohen Macaulay ring with homogeneous basic set $\{\theta_1,\ldots,\theta_M\,;\,\eta_1,\ldots,\eta_N\}$ will necessarily have as Hilbert series the rational function

$$\frac{\sum_{i=1}^{N} q^{deg(\eta_i)}}{\prod_{j=1}^{M}\big(1-q^{deg(\theta_j)}\big)}$$

Calling the $\theta_j$ "*quasi-generators*" and the $\eta_i$ "*separators*", I.4 suggests that our ring should have $2n$ quasi-generators of degrees $1,2,\ldots,n;2,n,\ldots,n+1$, a quasi-generator of degree $\binom{n+1}{2}$ and two separators, one a constant and one of degree $\binom{n+1}{2}$. The first set of $2n$ potential quasi-generators is not difficult to construct, Indeed, the invariance of a trace under conjugation yields that the following $n$ polynomials are all UVX invariant

$$\Pi_1=traceX\ ,\ \Pi_2=traceX^2\ ,\ \Pi_3=traceX^3\ ,\ \ldots\ ,\ \Pi_n=traceX^n \qquad \text{I.5}$$

The same is easily shown to be true for the polynomials

$$\theta_1=UV\ ,\ \ \theta_2=UXV\ ,\ \ \theta_2=UX^2V\ ,\ \ldots\ ,\ \theta_n=UX^{n-1}V, \qquad \text{I.6}$$

here all these expressions should be interpreted as matrix products.

The search for two further homogeneous invariants of degree $\binom{n+1}{2}$ as suggested by I.4, after some efforts, yielded the following surprising pair of polynomials

$$\Phi(U,X) \;=\; \det\begin{Vmatrix} U\\ UX\\ UX^2\\ \vdots\\ UX^{n-1}\end{Vmatrix} \qquad \text{and} \qquad \Psi(V,X) \;=\; \det\Big\| V,XV,X^2V,\ldots,X^{n-1}V\Big\|$$

In fact note that for any $g\in SL_n(\mathbb{C})$ we have

$$T_g\Phi(U,X) \;=\; \det\begin{Vmatrix} Ug^{-1}\\ UXg^{-1}\\ UX^2g^{-1}\\ \vdots\\ UX^{n-1}g^{-1}\end{Vmatrix} \;=\; \det\begin{Vmatrix} U\\ UX\\ UX^2\\ \vdots\\ UX^{n-1}\end{Vmatrix}\det g^{-1} \;=\; \Phi(U,X)$$

and

$$T_g\Psi(V,X) = \det\left\|gV, gXV, gX^2V, \ldots, gX^{n-1}V\right\| = \det g\ \det\left\|V, XV, X^2V, \ldots, X^{n-1}V\right\| = \Psi(V,X)$$

Before we show how to construct a basic set from these UVX invariants, it will be good to give an overview of the developments that yielded the evaluation of the constant term in I.1. The most natural approach to proving I.1 is to start with a representation theoretical interpretation of the kernel involved in the constant term. In fact as we shall see this kernel(†) is none other than a graded character of $SL_n[\mathbb{C}]$. Following this approach required the decomposition of this character into its irreducible constituents, i.e. computing the Schur expansion of this kernel. This is precisely where the so-called Kostka-Foulkes polynomials make their appearance. This done, the completion of the proof may be carried out by a fascinating combination of tools from Representation Theory, the Theory of Symmetric function and Combinatorics. Although all this is natural and possibly quite revealing, we were compelled to find a shorter path. The first path we pursued is to condense the essential ideas of this approach into a succession of symmetric function identities. We give this proof in full detail in section 2. Nevertheless, so that the flavour of the first proof is not entirely lost we give some of the highlights of the representation theoretical proof in section 3. There is however another equally natural path that can be pursued, that is to use the algorithmic machinery of constant term evaluations. In fact, if we simply process the kernel in the left hand side of I.1 by the MAPLE software of G. Xin (††), out pops the right hand side of I.1 in a matter of seconds for $n = 2, 3, 4$ and these instances are sufficient for a formulation of the general result. The problem then arises whether the identity in I.1, in full generality, can be obtained manually by means of the partial fraction algorithm of G. Xin. Following this path yielded unexpected surprises: To begin it showed the power of the partial fraction algorithm, yielding the constant term in a few lines and avoiding almost all the sophisticated machinery of the previous proofs. Next but not least it yielded a tri-graded version of the constant term and consequently also a tri-graded Hilbert series. This development is presented in section 4. Its by-products can be stated as follows.

## Theorem I.4

*For $u, v, q$ variables and $n \geq 2$ we have*

$$\frac{1}{(1-q)^n}\prod_{i=1}^{n}\frac{1}{(1-ux_i)(1-v/x_i)}\prod_{1\leq i<j\leq n}\frac{1-x_i/x_j}{(1-qx_i/x_j)(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} = \frac{1}{\prod_{i=1}^{n}(1-q^i)\ \prod_{i=1}^{n}(1-uvq^{i-1})}\left(\frac{v^nq^{\binom{n}{2}}}{1-v^nq^{\binom{n}{2}}}+\frac{1}{1-u^nq^{\binom{n}{2}}}\right). \qquad \text{I.7}$$

A post hoc examination of this identity immediately suggested a natural tri-grading of the UVX invariants. More precisely let us denote by $\mathcal{H}_{r,s,m}(UVX)$ the subspace of UVX invariants that are tri-homogeneous of degree $r$ in $u_1, u_2, \ldots, u_n$, of degree $s$ in $v_1, v_2, \ldots, v_n$ and degree $m$ in the $x'_{i,j}s$ and set

$$F_{UVX}(u,v,q) \;=\; \sum_{r\geq 0}\sum_{s\geq 0}\sum_{m\geq 0} u^rv^sq^m \dim \mathcal{H}_{r,s,m}(UVX)$$

(†) Except for the factor $\prod_{1\leq i<j\leq n}(1-x_j/x_i)$

(††) (downloadable from *www.combinatorics.net.cn/homepage/xin/*

Then as a Corollary of Theorem I.4 we derive

## Theorem I.5

$$F_{UVX}(u,v,q) \;=\; \frac{1}{\prod_{i=1}^n(1-q^i)\,\prod_{i=1}^n(1-uvq^{i-1})}\left(\frac{v^n q^{\binom{n}{2}}}{1-v^n q^{\binom{n}{2}}}+\frac{1}{1-u^n q^{\binom{n}{2}}}\right). \qquad \text{I.8}$$

These two results turn out to be precisely the refinements of Theorems I.2 and I.3 needed for a surprisingly simple approach to the construction of bases for our UVX invariants. For example, we derive from I.8

## Theorem I.6

*The UVX invariants have the tri-graded basis*

$$\mathcal{B}^{ab} \;=\; \Big\{\Phi^m \Pi_1^{r_1}\Pi_2^{r_2}\cdots\Pi_n^{r_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_n^{s_n}\,;\;\Psi^{m+1}\Pi_1^{r_1}\Pi_2^{r_2}\cdots\Pi_n^{r_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_n^{s_n}\;:m\geq 0, s_i\geq 0, r_i\geq 0\Big\} \qquad \text{I.9}$$

and this in turn yields

## Theorem I.7

*Setting*

$$\Gamma^+(U;V;X)=\Phi(U;X)+\Psi(V;X) \qquad \textit{and} \qquad \Gamma^-(U;V;X)=\Phi(U;X)-\Psi(V;X) \qquad \text{I.10}$$

*both collections*

$$\mathcal{B}^{+} \;=\; \Big\{(\Gamma^+)^a(\Gamma^-)^b\Pi_1^{r_1}\Pi_2^{r_2}\cdots\Pi_n^{r_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_n^{s_n}\;: a=0,1\,;\,b\geq 0\,;\,r_i,s_j\geq 0\Big\} \qquad \text{I.11}$$

*and*

$$\mathcal{B}^{-} \;=\; \Big\{(\Gamma^-)^a(\Gamma^+)^b\Pi_1^{r_1}\Pi_2^{r_2}\cdots\Pi_n^{r_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_n^{s_n}\;: a=0,1\,;b\geq 0\,;\,r_i,s_j\geq 0\Big\} \qquad \text{I.12}$$

*are vector space bases for the UVX invariants*

Remarkably, as we shall see, this path can be reversed and derive the identity in I.8 from the following result that may be proved directly from the singly graded Hilbert series in I.4

## Theorem I.8

*The UVX invariants have the tri-graded basis*

$$\mathcal{B}^{uv} \;=\; \Big\{\theta_n^m\,\Pi_1^{r_1}\Pi_2^{r_2}\cdots\Pi_n^{r_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_{n-1}^{s_{n-1}}\,\Phi^u\,\Psi^v\;\;: s_i\geq 0, r_i\geq 0;\; u\geq 0, v\geq 0;\;\; 0\leq m\leq n-1\Big\} \qquad \text{I.9}$$

These three results are shown in section 5. The paper starts in the next section with a proof of Theorem I.2.

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## 1. Molien's Theorem and constant terms

The relation between Hilbert series and constant terms brought to the fore in the examples studied in [4] is not an isolated accident. In fact, the path

*Hilbert series* $\longrightarrow$ *Moliens Theorem* $\longrightarrow$ *Integral*$\longrightarrow$*Constant Term*

can be followed verbatim in a variety of cases leading to constant term problems gravid with algebraic and combinatorial ramifications. Another example in point is given by the present UVX problem.

But before we proceed with our specific case we need to review the underlying general set up. To this end note that the action of an $m \times m$ matrix $A = \|a_{ij}\|_{i,i=1}^{m}$ on a polynomial $P(x) = P(x_1, x_2, \ldots, x_n)$ is denoted $T_A P(x)$ and is defined by setting

$$T_A P(x_1, x_2, \ldots, x_n) \;=\; P\Big(\sum_{i=1}^{m} x_i a_{i1}\ ,\ \sum_{i=1}^{m} x_i a_{i2}\ ,\ \ldots\ ,\ \sum_{i=1}^{m} x_i a_{im}\Big) \qquad 1.1$$

In matrix notation, (viewing $x = (x_1, x_2, \ldots, x_n)$ as a row vector), we may simply rewrite this as

$$T_A P(x) \;=\; P\big(xA\big). \qquad 1.2$$

Recall that if $G$ is a group of $m \times m$ matrices we say that $P \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ is "*G-invariant*" if and only if

$$T_A P(x) \;=\; P(x) \qquad \forall \quad A \in G \qquad 1.3$$

The subspace of $\mathbb{C}[x] = \mathbb{C}[x_1, x_2, \ldots, x_n]$ of $G$-invariant polynomials is usually denoted $\mathbb{C}[x]^G$. Clearly, the action in 1.1 preserves homogeneity and degree, thus we have the direct sum decomposition

$$\mathbb{C}[x]^G \;=\; \mathcal{H}_o\big(\mathbb{C}[x]^G\big) \oplus \mathcal{H}_1\big(\mathbb{C}[x]^G\big) \oplus \mathcal{H}_2\big(\mathbb{C}[x]^G\big) \oplus \cdots \oplus \mathcal{H}_d\big(\mathbb{C}[x]^G\big) \oplus \cdots \qquad 1.4$$

where $\mathcal{H}_d\big(\mathbb{C}[x]^G\big)$ denotes the subspace of $G$-invariants that are homogeneous of degree $d$. The "*Hilbert series*" of $\mathbb{C}[x]^G$ is simply given by formal power series

$$F_G(q) \;=\; \sum_{d \geq 0} q^d \dim\Big(\mathcal{H}_d\big(\mathbb{C}[x]^G\big)\Big) \qquad 1.5$$

Since $\dim \mathcal{H}_d\big(\mathbb{C}[x]^G\big) \leq \dim\Big(\mathcal{H}_d\big(\mathbb{C}[x]\big)\Big) = \binom{d+m-1}{m-1}$ we see that this is a well defined formal power series.

In the case that $G$ is a finite group the Hilbert series $F_G(q)$ is immediately obtained from Molien's formula

$$F_G(q) \;=\; \frac{1}{|G|} \sum_{A \in G} \frac{1}{\det\big(I - qA\big)}. \qquad 1.6$$

For an infinite group $G$ which posesses a unit invariant measure $\omega$ this identity becomes

$$F_G(q) \;=\; \int_{A \in G} \frac{1}{\det\big(I - qA\big)}\, d\omega. \qquad 1.7$$

To convert such an integral into a constant term in [4] we used the following easily stablished identity.

## Proposition 1.1

*If $Q(a_1, a_2, \ldots, a_k)$ is a polynomial in $\mathbb{Q}[a_1, a_2, \ldots, a_k; 1/a_1.1/a_2, \ldots, 1/a_k]$ then*

$$\left(\frac{1}{2\pi}\right)^k \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} Q[(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_k}]d\theta_1 d\theta_2\cdots d\theta_k \;=\; Q(a_1, a_2, \ldots, a_k)\Big|_{a_1^0}\Big|_{a_2^0}\cdots\Big|_{a_k^0} \qquad 1.8$$

*where the symbol "$\Big|_{a^0}$" denotes the operator of taking the constant term in a Laurent polynomial in $a_1, a_2, \ldots, a_k$*

Armed with this machinery we can now proceed with

## A proof of Theorem I.2

Passing from $SL_n[\mathbb{C}]$ to $SU[n]$ and using Moliens Theorem, we derive that

$$F_{UVX}(q) \;=\; \int_{T_n} \frac{1}{\det|1 - qD(g)|}\, d\omega(g) \qquad 1.9$$

with $D(g)$ giving the action of $T_g$ on the on the alphabet $\{u_i, v_j, x_{i,j}\}_{i,j=1}^n$ and $d\omega(g)$ giving the corresponding normalized Haar measure. Moreover, since the integrand is invariant under conjugation, the integral needs to be carried out only over the thorus $T_n$ of diagonal matrices

$$g = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix} \qquad 1.10$$

with

$$a_1 = e^{i\theta_1},\, a_2 = e^{i\theta_2},\, \ldots,\, a_n = e^{i\theta_n},$$

and

$$a_1 a_2 \cdots a_n = 1, \qquad 1.11$$

Now for $g$ as in 1.10, from I.3 we derive that

$$T_g\{u_i, v_j, x_{i,j}\}_{i,j=1}^n \;=\; \{u_i a_i, a_j^{-1} v_j, a_i^{-1} x_{i,j} a_j\}_{i,j=1}^n.$$

That is $T_g$ acts on the alphabet $\{u_i, v_j, x_{i,j}\}_{i,j=1}^n$ the by the diagonal matrix $D(g)$ with eigenvalues

$$a_1, \ldots, a_n;\; a_1^{-1}, \ldots, a_n^{-1};\; \left\{a_i a_j^{-1} \,:\, 1 \le i, j \le n\right\}$$

this gives

$$\det|1 - qD(g)| \;=\; \prod_{r=1}^n (1 - qa_r)(1 - q/a_r) \prod_{r,s=1}^n (1 - qa_r/a_s) \qquad 1.12$$

and 1.9 reduces to

$$F_{UVX}(q) \;=\; \int_{T_n} \prod_{r=1}^n \frac{1}{(1 - qa_r)(1 - q/a_r)} \prod_{r,s=1}^n \frac{1}{(1 - qa_r/a_s)}\, d\omega(g) \qquad 1.13$$

where the Haar measure here is

$$d\omega_g \;=\; \big|\Delta(g)\big|^2 \frac{d\theta_1 d\theta_2 \cdots d\theta_{n-1}}{n!(2\pi)^{n-1}}. \qquad 1.14$$

with $\Delta(g)=\prod_{1\le r<s\le n}(a_r-a_s)$ the Vandermonde determinant in the variables $a_r=e^{i\theta_r}$. Note next that Vandermonde determinant expansion gives

$$\begin{aligned}\big|\Delta(g)\big|^2 &= \Delta(g)\Delta(g^{-1})\\ &= \Delta(g)\sum_{\sigma\in S_n} sgn(\sigma)\prod_{j=1}^{n-1} a_{\sigma_j}^{-(n-j)}\\ &= \sum_{\sigma\in S_n}\sigma\Big(\Delta(g)\prod_{j=1}^{n-1}a_j^{-(n-j)}\Big) \;=\; \sum_{\sigma\in S_n}\sigma\Big(\prod_{1\le r<s\le n}\big(1-a_s/a_r\big)\Big)\end{aligned}$$

Using this in 1.14, 1.13 becomes

$$F_{UVX}(q)=\sum_{\sigma\in S_n}\int_{T_n}\prod_{r=1}^{n}\frac{1}{(1-qa_r)(1-q/a_r)}\prod_{r,s=1}^{n}\frac{1}{(1-qa_r/a_s)}\,\sigma\Big(\prod_{1\le r<s\le n}\big(1-a_s/a_r\big)\Big)\frac{d\theta_1 d\theta_2\cdots d\theta_{n-1}}{n!(2\pi)^{n-1}}.$$

However, we see that the symetry of the expression to the left of $\sigma$ allows us to move $\sigma$ all the way to the left of the integrand and reduce this integral to

$$F_{UVX}(q)=\sum_{\sigma\in S_n}\int_{T_n}\sigma\Big(\prod_{r=1}^{n}\frac{1}{(1-qa_r)(1-q/a_r)}\prod_{r,s=1}^{n}\frac{1}{(1-qa_r/a_s)}\prod_{1\le r<s\le n}\big(1-a_s/a_r\big)\Big)\frac{d\theta_1 d\theta_2\cdots d\theta_{n-1}}{n!(2\pi)^{n-1}}.$$

But with the substitution $a_n=(a_1a_2\cdots a_{n-1})^{-1}$ the integrand is still symmetric in $a_1,a_2,\ldots,a_{n-1}$, and the action of $\sigma$ cannot affect the value of the integral. Thus

$$F_{UVX}(q)=\int_{T_n}\prod_{r=1}^{n}\frac{1}{(1-qa_r)(1-q/a_r)}\prod_{r,s=1}^{n}\frac{1}{(1-qa_r/a_s)}\prod_{1\le r<s\le n}\big(1-a_s/a_r\big)\frac{d\theta_1 d\theta_2\cdots d\theta_{n-1}}{(2\pi)^{n-1}}.$$

and this can be further simplified to

$$F_{UVX}(q)=\frac{1}{(1-q)^n}\int_{T_n}\prod_{r=1}^{n}\frac{1}{(1-qa_r)(1-q/a_r)}\prod_{1\le r<s\le n}\frac{\big(1-a_s/a_r\big)}{(1-qa_r/a_s)(1-qa_s/a_r)}\,\frac{d\theta_1 d\theta_2\cdots d\theta_{n-1}}{(2\pi)^{n-1}}.$$

The identity in I.1 is thus an immediate consequence of Proposition 1.1.

## 2. Computing the constant term by symmetric function methods

The object of this section is to evaluate the constant term

$$Q \;=\; \frac{1}{(1-q)^n}\prod_{i=1}^{n}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{1\le i<j\le n}\frac{1-x_i/x_j}{(1-qx_i/x_j)(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}. \qquad 2.1$$

using the Theory of Symmetric Functions.

To begin note that we can write

$$F(x;q) \;=\; \prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \;=\; \sum_{a\geq 0}\sum_{b\geq 0} q^{a+b} h_a(x) h_b(1/x)$$

where we have set $h_b(1/x) = h(1/x_1, 1/x_2, \ldots, 1/x_n)$. We can thus split this factor of 2.1 into three summands

$$F(x;q) \;=\; F_0(x;q) \,+\, F_1(x;q) \,+\, F_2(x;q) \tag{2.2}$$

where

$$F_0(x;q) = \sum_{a\geq 0} q^{2a} h_a(x) h_a(1/x)$$

and

$$F_1(x;q) = \sum_{0\leq a<b} q^{a+b} h_a(x) h_b(1/x)\;, \qquad F_2(x;q) = \sum_{0\leq b<a} q^{a+b} h_a(x) h_b(1/x)$$

Using 2.2 in 2.1 we get the decomposition

$$Q \;=\; Q_0 \,+\, Q_1 \,+\, Q_2.$$

with

$$Q_i \;=\; \frac{1}{(1-q)^n}\, F_i(x;q) \prod_{1\leq i<j\leq n} \frac{1-x_i/x_j}{(1-qx_i/x_j)(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} \qquad (\text{for } i=0,1,2)$$

Note that

$$F_1(x;q) = \sum_{0\leq b<a} q^{a+b} h_b(x) h_a(1/x) \;=\; F_2(1/x;q)$$

Thus

$$\begin{aligned} Q_1 \;&=\; \frac{1}{(1-q)^n}\, F_2(1/x;q) \prod_{1\leq i<j\leq n} \frac{1-x_i/x_j}{(1-qx_i/x_j)(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} \\ &=\; \frac{1}{(1-q)^n}\, F_2(x;q) \prod_{1\leq i<j\leq n} \frac{1-x_j/x_i}{(1-qx_j/x_i)(1-qx_i/x_j)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} \\ &=\; Q_2 \end{aligned}$$

The last equality due to the fact that any permutation of the variables cannot affect this constant term. In summary we have

$$Q \;=\; Q_0 \,+\, 2\,Q_2 \tag{2.3}$$

Now it is easy to show that

$$h_b(1/x) \;=\; \frac{1}{(x_1x_2\ldots x_n)^b}\, S_{b^{n-1}}(x) \cong S_{b^{n-1}}(x) \tag{2.4}$$

where here and after the symbol "$\cong$" represents congruence "*modulo* $x_1x_2\ldots x_n$". It follows from 2.4 that

$$h_a(x)h_b(1/x) \cong \sum_{d=0}^{a\wedge b} S_{(b+a-d,b^{n-2},d)} \cong \sum_{d=0}^{a\wedge b} S_{(b-d+a-d,(b-d)^{n-2},0)}.$$

Using this gives

$$\begin{aligned} F_0(x;q) &= \sum_{a\geq 0} q^{2a} \sum_{d=0}^{a} S_{2(a-d),(a-d)^{n-2})}(x) \\ &= \sum_{a\geq 0} q^{2a} \sum_{d=0}^{a} S_{2d,d^{n-2}}(x) = \frac{1}{1-q^2} \sum_{d\geq 0} q^{2d} S_{2d,d^{n-2}}(x). \end{aligned} \qquad 2.5$$

Likewise

$$\begin{aligned} F_2(x;q) &= \sum_{0\leq b<a} q^{a+b} \sum_{d=0}^{b} S_{b-d+a-d,(b-d)^{n-2}} \\ &= \sum_{b\geq 0}\sum_{a>b} q^{a+b} \sum_{d=0}^{b} S_{2d+a-b,d^{n-2}} \end{aligned}$$

and making the substitution $a=b+k$ we get

$$F_2(x;q) = \sum_{d\geq 0}\sum_{k\geq 1} q^k S_{2d+k,d^{n-2}} \sum_{b\geq d} q^{2b}$$

In summary

$$F_2(x;q) = \frac{1}{1-q^2} \sum_{d\geq 0}\sum_{k\geq 1} q^{2d+k} S_{2d+k,d^{n-2}}(x). \qquad 2.6$$

Our next step is to obtain a more suitable version of the factor

$$G(x;q) = \frac{1}{(1-q)^n} \prod_{1\leq i<j\leq n} \frac{1}{(1-qx_i/x_j)(1-qx_j/x_i)} \qquad 2.7$$

Our point of departure is the following classical identity

## Proposition 2.1

*For any $n\geq 2$ we have*

$$\sum_{\sigma\in S_n} sign(\sigma)\sigma\Big(\prod_{1\leq i<j\leq n}(x_i-qx_j)\Big) = \Big(\prod_{i=1}^{n}\frac{1-q^i}{1-q}\Big) \prod_{1\leq i<j\leq n}(x_i-x_j) \qquad 2.8$$

## Proof

Note that for $n=2$ this identity reduces to

$$(x_1-qx_2)-(x_2-qx_1) = (1+q)(x_1-x_2)$$

which is patently true. We can thus proceed by induction on $n$. Let us assume that 2.8 is true for $n-1$. Letting $\sigma^{(s)}$ denote the left $S_{n-1}$-coset representative of $S_n$ that sends $n$ to $s$ and sends $1,2,\ldots,n-1$ onto the remaining integers in increasing order, we can rewrite the left hand side of 2.8 in the form

$$LHS \;=\; \sum_{s=1}^{n}(-1)^{n-s}\sigma^{(s)}\prod_{i=1}^{n-1}(x_i-qx_n)\sum_{\sigma\in S_{n-1}} sign(\sigma)\sigma\Big(\prod_{1\le i<j\le n-1}(x_i-qx_j)\Big)$$

and the inductive hypothesis gives that

$$LHS \;=\; \prod_{i=1}^{n-1}\frac{1-q^i}{1-q}\sum_{s=1}^{n}(-1)^{n-s}\sigma^{(s)}\prod_{i=1}^{n-1}(x_i-qx_n)\Big(\prod_{1\le i<j\le n-1}(x_i-x_j)\Big) \qquad 2.9$$

Using the identity

$$\prod_{i=1}^{n-1}(x_i-qx_n) \;=\; \sum_{r=0}^{n-1}(-qx_n)^{n-i-1}e_i(x_1,x_2,\ldots,x_{n-1})$$

2.9 becomes

$$LHS \;=\; \prod_{i=1}^{n-1}\frac{1-q^i}{1-q}\sum_{r=0}^{n-1}q^{n-1-i}\sum_{s=1}^{n}(-1)^{s-i-1}\sigma^{(s)}x_n^{n-i-1}e_i(x_1,x_2,\ldots,x_{n-1})\prod_{1\le i<j\le n-1}(x_i-x_j)$$

and 2.8 follows since we have

$$\sum_{s=1}^{n}(-1)^{s-i-1}\sigma^{(s)}x_n^{n-i-1}e_i(x_1,x_2,\ldots,x_{n-1})\prod_{1\le i<j\le n-1}(x_i-x_j) \;=\; \prod_{1\le i<j\le n}(x_i-x_j).$$

In fact the left hand side is none other than the expansion of the Vandermonde determinant with respect the row $x_1^{n-i-1},x_2^{n-i-1},\ldots,x_n^{n-i-1}$.

The identity in 2.8 has the following immediate corollary.

## Proposition 2.2

*For any $n\ge 2$ we have*

$$\begin{aligned}G(x;q)=\frac{1}{(1-q)^n}\prod_{1\le i<j\le n}\frac{1}{(1-qx_i/x_j)(1-qx_j/x_i)} \;&=\\ =\;\frac{1}{\prod_{i=1}^n(1-q^i)}\frac{1}{\Delta(x)}\sum_{\sigma\in S_n} sign(\sigma)\sigma\prod_{i-1}^{n}x_i^{n-i}&\Big(\prod_{1\le i<j\le n}\frac{1}{1-qx_i/x_j}\Big)\end{aligned} \qquad 2.10$$

*where $\Delta(x)$ denotes the Vandermonde determinant in $x_1,x_2,\ldots,x_n$.*

## Proof

Note that 2.8 can be rewritten in the form

$$\frac{1}{\prod_{i=1}^n(1-q^i)}\frac{1}{\Delta(x)}\sum_{\sigma\in S_n} sign(\sigma)\sigma\Big(\prod_{i-1}^{n}x_i^{n-i}\prod_{1\le i<j\le n}(1-qx_j/x_i)\Big) \;=\; \frac{1}{(1-q)^n}.$$

Next we divide both sides by the rational function $\prod_{i\neq j}(1-qx_i/x_j)$ and since this function is symmetric in $x_1, x_2, \ldots, x_n$, we can place it to the right of $\sigma$ in the summation side. This results in the identity

$$\frac{1}{\prod_{i=1}^n(1-q^i)}\frac{1}{\Delta(x)}\sum_{\sigma\in S_n} sign(\sigma)\sigma\Big(\prod_{i-1}^n x_i^{n-i}\prod_{1\le i<j\le n}\frac{1}{1-qx_i/x_j}\Big)=\frac{1}{(1-q)^n}\prod_{1\le i<j\le n}\frac{1}{(1-qx_i/x_j)(1-qx_j/x_i)}$$

and 2.10 then follows from 2.2.

## Remark 2.1

In using 2.8 to prove 2.17 we have followed an argument of Wallach-Willenbring [9] who prove the corresponding general result for all Weyl groups. The representational context which gives rise to these computations will be discussed in the next section.

The identity in 2.10 has the following remarkable consequence

## Proposition 2.3

*For $n\ge 2$ and for any symmetric rational funtion $A(x)$ we have*

$$\frac{1}{(1-q)^n}A(x)\prod_{1\le i<j\le n}\frac{1-x_i/x_j}{(1-qx_i/x_j)(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}= \frac{1}{\prod_{i=1}^n(1-q^i)}A(x)\prod_{1\le i<j\le n}\frac{1-x_i/x_j}{(1-qx_i/x_j)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}. \qquad 2.11$$

## Proof

Using 2.10 the left hand side of 2.11 becomes

$$\begin{aligned}LHS&=\frac{A(x)}{\prod_{i=1}^n(1-q^i)}\frac{\prod_{1\le i<j\le n}(1-x_i/x_j)}{\Delta(x)}\Big(\sum_{\sigma\in S_n}sign(\sigma)\sigma\prod_{i-1}^n x_i^{n-i}\prod_{1\le i<j\le n}\frac{1}{1-qx_i/x_j}\Big)\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}\\
&=\frac{A(x)}{\prod_{i=1}^n(1-q^i)}\frac{(-1)^{\binom{n}{2}}}{\prod_{j=1}^n x_j^{j-1}}\Big(\sum_{\sigma\in S_n}sign(\sigma)\sigma\prod_{i-1}^n x_i^{n-i}\prod_{1\le i<j\le n}\frac{1}{1-qx_i/x_j}\Big)\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}\\
&=\frac{A(x)}{\prod_{i=1}^n(1-q^i)}(-1)^{\binom{n}{2}}\prod_{j=1}^n x_j^{n-j}\Big(\sum_{\sigma\in S_n}sign(\sigma)\sigma\prod_{i-1}^n x_i^{1-i}\prod_{1\le i<j\le n}\frac{1}{1-qx_i/x_j}\Big)\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}\\
&=(-1)^{\binom{n}{2}}\sum_{\sigma\in S_n}sign(\sigma)\sigma\Big(\Big(\sigma^{-1}\prod_{j=1}^n x_j^{n-j}\Big)\frac{A(x)}{\prod_{i=1}^n(1-q^i)}\prod_{i-1}^n x_i^{1-i}\prod_{1\le i<j\le n}\frac{1}{1-qx_i/x_j}\Big)\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}.\end{aligned}$$

Since permuting the variables cannot affect this constant term we can remove the left most $\sigma$ and obtain

$$\begin{aligned}LHS&=(-1)^{\binom{n}{2}}\Big(\sum_{\sigma\in S_n}sign(\sigma)\sigma^{-1}\prod_{j=1}^n x_j^{n-j}\Big)\frac{A(x)}{\prod_{i=1}^n(1-q^i)}\prod_{i-1}^n x_i^{1-i}\prod_{1\le i<j\le n}\frac{1}{1-qx_i/x_j}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}\\
&=(-1)^{\binom{n}{2}}\Big(\prod_{1\le i<j\le n}(x_i-x_j)\Big)\frac{A(x)}{\prod_{i=1}^n(1-q^i)}\prod_{i-1}^n x_i^{1-i}\prod_{1\le i<j\le n}\frac{1}{1-qx_i/x_j}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}\\
&=\prod_{j=1}^n x_j^{j-1}\frac{A(x)}{\prod_{i=1}^n(1-q^i)}\prod_{i-1}^n x_i^{1-i}\prod_{1\le i<j\le n}\frac{1-x_i/x_j}{1-qx_i/x_j}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}\\
&=\frac{A(x)}{\prod_{i=1}^n(1-q^i)}\prod_{1\le i<j\le n}\frac{1-x_i/x_j}{1-qx_i/x_j}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}\end{aligned}$$

This proves 2.11.

Using 2.11 with $A(x)=F_0(x;q)$ and $A(x)=F_2(x;q)$ as given by 2.5 and 2.6 we get

$$Q_0 \;=\; \frac{1}{1-q^2}\frac{1}{\prod_{i=1}^n(1-q^i)}\sum_{d\geq 0} q^{2d}S_{2d,d^{n-2}}(x)\prod_{1\leq i<j\leq n}\frac{1-x_i/x_j}{(1-qx_i/x_j)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} \qquad 2.12$$

and

$$Q_2 \;=\; \frac{1}{1-q^2}\frac{1}{\prod_{i=1}^n(1-q^i)}\sum_{d\geq 0}\sum_{k\geq 1} q^{2d+k}S_{2d+k,d^{n-2}}(x)\prod_{1\leq i<j\leq n}\frac{1-x_i/x_j}{(1-qx_i/x_j)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} \qquad 2.13$$

This brings us to the evaluation of constant terms of the form

$$\Pi_\lambda(q) \;=\; S_\lambda(x)\prod_{1\leq i<j\leq n}\frac{1-x_i/x_j}{(1-qx_i/x_j)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} \qquad 2.14$$

with $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{n-1},0)$. Now note that the symmetry of $S_\lambda(x)$ and the invariance of our constant term under permutation of the variables allows us to rewrite 2.14 as

$$\begin{aligned} C_\lambda(q) \;&=\; S_\lambda(x)\prod_{1\leq i<j\leq n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} \\ &=\; \Big(\sum_{\sigma\in S_n} sign(\sigma)x^{\sigma(\lambda+\delta)-\delta}\Big)\prod_{1\leq i<j\leq n}\frac{1}{(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} \end{aligned} \qquad 2.15$$

with

$$\delta \;=\; (n-1,n-2,\ldots,1,0). \qquad 2.16$$

It will be convenient here and after to denote by $\mathcal{P}$ collection of vectors which may be written in the form

$$p=\sum_{1\leq i<j\leq n} a_{i,j}(e_i-e_j) \qquad 2.17$$

with $a_{i,j}\geq 0$ integers, and $e_1,e_2,\ldots,e_n$ the $n$-dimensional coordinate vectors. We may thus write

$$\prod_{1\leq i<j\leq n}\frac{1}{1-qx_i/x_j}=\sum_{p\in\mathcal{P}} q^{\|p\|}\,x^p. \qquad 2.18$$

where for $p$ as in 2.17 we set

$$\|p\| \;=\; \sum_{1\leq i<j\leq n} p_{ij}\,.$$

We should note that for any $p=(p_1,p_2,\ldots,p_n)\in\mathcal{P}$ we have

$$p_1+p_2+\cdots+p_n=0 \qquad 2.19$$

Using 2.18 in 2.15 gives

$$C_\lambda(q) \;=\; \sum_{p\in\mathcal{P}} q^{\|p\|} \sum_{\sigma\in S_n} sign(\sigma) x^{\sigma(\lambda+\delta)-\delta-p}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}. \tag{2.20}$$

This brings us to the following basic result

## Proposition 2.4

*The constant term $C_\lambda(q)$ vanishes unless the size of $\lambda$ is divisible by $n$, and if $\lambda_1+\lambda_2+\cdots+\lambda_n=nb$ then*

$$C_\lambda(q) \;=\; K_{\lambda,b^n}(q) \tag{2.21}$$

*the latter being the Kostka Foulkes polynomial with the given partition indexing.*

## Proof

Observe first that a constant term such as

$$x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}$$

fails to vanish if and only if $a_1=a_2=\cdots=a_n$. Indeed we have

$$x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} \;=\; x_1^{a_1-a_n}x_2^{a_2-a_n}\cdots x_{n-1}^{a_{n-1}-a_n}\Big|_{x_1^0x_2^0\cdots x_n^0} \;=\; \begin{cases} 1 & \text{if } a_i=a_n \text{ for } 1\le i\le n-1 \\ 0 & \text{otherwise} \end{cases}$$

In particular, in the first case we will have

$$a_1+a_2+\ldots+a_n \;=\; (n-1)a_n+a_n \;=\; na_n.$$

In view of 2.19 we immediately derive from this that all the summands in 2.20 will identically vanish if $\lambda_1+\lambda_2+\cdots+\lambda_n$ is not divisible by $n$. On the other hand when $\lambda_1+\lambda_2+\cdots+\lambda_n=nb$ we can write

$$\begin{aligned} C_\lambda(q) \;&=\; \sum_{p\in\mathcal{P}} q^{\|p\|} \sum_{\sigma\in S_n} sign(\sigma) x^{\sigma(\lambda+\delta)-\delta-p}\Big|_{x_1^bx_2^b\cdots x_n^b} \\ &=\; S_\lambda(x) \prod_{1\le i<j\le n} \frac{1-x_j/x_i}{1-qx_j/x_i}\Big|_{x_1^bx_2^b\cdots x_n^b} \end{aligned}$$

and the latter is the well known formula for the Kostka-Foulkes polynomial $K_{\lambda,b^n}(q)$

To complete the evaluation of our constant term we need one more auxiliary result

## Proposition 2.5

*For $n\ge 2$, and $d,k\ge 0$ we have*

$$K_{(2d+kn,d^{n-2}),(k+d)^n}(q) \;=\; q^{k\binom{n}{2}+d} \begin{bmatrix} d+n-2 \\ d \end{bmatrix}_q \tag{2.22}$$

## Proof

We are to show that for $\lambda = (2d + nk, d^{n-2}, 0)$

$$S_\lambda[X_n] \prod_{1\le i<j\le n} \frac{1 - x_j/x_i}{1 - qx_j/x_i}\Big|_{x_1^{d+k}x_2^{d+k}\cdots x_n^{d+k}} = q^{k\binom{n}{2}+d} \begin{bmatrix} d+n-2 \\ d \end{bmatrix}_q$$

Note first that for $n = 2$, (setting $d + k = b$), this reduces to

$$S_{(2b,0)}[x_1 + x_2] \frac{1 - x_2/x_1}{1 - qx_2/x_1}\Big|_{x_1^b x_2^b} = \frac{(x_1^b x_2^{-b} - x_1^{-b-1}x_2^{b+1})}{1 - qx_2/x_1}\Big|_{x_1^b x_2^b} = q^b.$$

which is clearly true. So we can proceed by induction on $n \ge 2$ and assume 2.22 valid up to $n - 1$. This given, canceling the denominator of the Schur function we get

$$S_\lambda[X_n] \prod_{1\le i<j\le n} \frac{1 - x_j/x_i}{1 - qx_j/x_i}\Big|_{x_1^{d+k}x_2^{d+k}\cdots x_n^{d+k}} = \prod_{1\le i<j\le n} \frac{1}{1 - qx_j/x_i} \sum_{\sigma\in S_n} sgn(\sigma) \frac{x^{\sigma(\lambda+\delta)-\delta}}{\prod_{i=1}^n x_i^{d+k}}\Big|_{x_1^0 x_2^0\cdots x_n^0}$$

$$= \sum_{\sigma\in S_n} sgn(\sigma) \frac{x_1^{\lambda_{\sigma_1}-\sigma_1+1-d-k}}{\prod_{j=2}^n (1 - qx_j/x_1)}\Big|_{x_1^0} \frac{\prod_{i=2}^n x_i^{\lambda_{\sigma(i)}+\delta_{\sigma(i)}-\delta_i-(d+k)}}{\prod_{2\le i<j\le n} 1 - qx_j/x_i}\Big|_{x_2^0\cdots x_n^0}. \qquad 2.23$$

Now note that

$$\frac{x_1^{\lambda_{\sigma_1}-\sigma_1+1-d-k}}{\prod_{j=2}^n (1 - qx_j/x_1)} = \begin{cases} \frac{x_1^{-\sigma_1+1-k}}{\prod_{j=2}^n (1-qx_j/x_1)} & \text{if } 2 \le \sigma_1 \le n-1 \\ \frac{x_1^{-n+1-d-k}}{\prod_{j=2}^n (1-qx_j/x_1)} & \text{if } \sigma_1 = n \end{cases}$$

and we see that in either case this expression contains only negative powers of $x_1$. Thus only the terms with $\sigma_1 = 1$ contribute to the constant term in 2.23. Since for $\sigma_1 = 1$ we have

$$\frac{x_1^{\lambda_{\sigma_1}-\sigma_1+1-d-k}}{\prod_{j=2}^n (1 - qx_j/x_1)}\Big|_{x_1^0} = \frac{x_1^{2d+nk-d-k}}{\prod_{j=2}^n (1 - qx_j/x_1)}\Big|_{x_1^0} = q^{d+(n-1)k} h_{d+(n-1)k}(x_2 \ldots, x_n)$$

The constant term in 2.23 reduces to

$$q^{d+(n-1)k} h_{d+(n-1)k}(x_2, \ldots, x_n) \sum_{\sigma\in S_{(2,\ldots,n)}} sgn(\sigma) \frac{\prod_{i=2}^n x_i^{\lambda_{\sigma(i)}+\delta_{\sigma(i)}-\delta_i-(d+k)}}{\prod_{1<i<j\le n} 1 - qx_j/x_i}\Big|_{x_2^0\cdots x_n^0} =$$

$$= q^{d+(n-1)k} h_{d+(n-1)k}(x_2, \ldots, x_n) S_{d^{n-2}}(x_2, \ldots, x_n) \prod_{2\le i<j\le n} \frac{1 - x_j/x_i}{1 - qx_j/x_i}\Big|_{x_2^{d+k}\cdots x_n^{d+k}}$$

$$= q^{d+(n-1)k} \sum_{a=0}^d S_{(d+a+(n-1)k, d^{n-3}, d-a)}(x_2, \ldots, x_n) \prod_{2\le i<j\le n} \frac{1 - x_j/x_i}{1 - qx_j/x_i}\Big|_{x_2^{d+k}\cdots x_n^{d+k}}$$

$$= q^{d+(n-1)k} \sum_{a=0}^d S_{(2a+(n-1)k, a^{n-3}, 0)}(x_2, \ldots, x_n) \prod_{2\le i<j\le n} \frac{1 - x_j/x_i}{1 - qx_j/x_i}\Big|_{x_2^{a+k}\cdots x_n^{a+k}}$$

$$\text{(by induction)} \quad = q^{d+(n-1)k} \sum_{a=0}^d q^{k\binom{n-1}{2}+a} \begin{bmatrix} a+n-3 \\ a \end{bmatrix}_q = q^{k\binom{n}{2}+d} \sum_{a=0}^d q^a \begin{bmatrix} a+n-3 \\ a \end{bmatrix}_q,$$

and 2.22 follows from the $q$-binomial identity

$$\sum_{a=0}^{d} q^a \begin{bmatrix} a+n-3 \\ a \end{bmatrix}_q \;=\; \begin{bmatrix} d+n-2 \\ d \end{bmatrix}_q .$$

We now have all the ingrediens need to establish our final result here which, combined with Theorem 1.2, yields us our first proof of Theorem I.3. That is

## Theorem 2.1

$$Q \;=\; \frac{1}{(1-q)} \frac{1+q^{\binom{n+1}{2}}}{\prod_{i=2}^{n}(1-q^i)^2(1-q^{n+1})\big(1-q^{\binom{n+1}{2}}\big)} \qquad 2.24$$

## Proof

Proposition 2.3 gives

$$\begin{aligned}
Q_0 \;&=\; \frac{1}{\prod_{i=1}^{n}(1-q^i)}\, F_0(x;q) \prod_{1\le i<j\le n} \frac{1-x_i/x_j}{(1-qx_i/x_j)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} \\
(\text{using } 2.5\ ) \;&=\; \frac{1}{\prod_{i=1}^{n}(1-q^i)}\, \frac{1}{1-q^2} \sum_{d\ge 0} q^{2d} S_{2d,d^{n-2}}(x) \prod_{1\le i<j\le n} \frac{1-x_i/x_j}{(1-qx_i/x_j)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} \\
(\text{using } 2.21\ ) \;&=\; \frac{1}{\prod_{i=1}^{n}(1-q^i)}\, \frac{1}{1-q^2} \sum_{d\ge 0} q^{2d} K_{(2d,d^{n-2}),d^n}(q) \\
(\text{using } 2.22\ ) \;&=\; \frac{1}{\prod_{i=1}^{n}(1-q^i)}\, \frac{1}{1-q^2} \sum_{d\ge 0} q^{2d} q^d \begin{bmatrix} d+n-2 \\ d \end{bmatrix}_q \;=\; \frac{1}{1-q} \frac{1}{\prod_{i=2}^{n}(1-q^i)^2} \frac{1}{(1-q^{n+1})}. \qquad 2.25
\end{aligned}$$

where the last equality follows from the $q$-series identity

$$\sum_{d\ge 0} x^d \begin{bmatrix} d+m \\ d \end{bmatrix}_q \;=\; \frac{1}{(1-x)(1-xq)\cdots(1-xq^m)}. \qquad 2.26$$

Using again Proposition 2.3 we get

$$\begin{aligned}
Q_2 \;&=\; \frac{1}{\prod_{i=1}^{n}(1-q^i)}\, F_2(x;q) \prod_{1\le i<j\le n} \frac{1-x_i/x_j}{(1-qx_i/x_j)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} \\
(\text{using } 2.6\ ) \;&=\; \frac{1}{\prod_{i=1}^{n}(1-q^i)}\, \frac{1}{1-q^2} \sum_{d\ge 0}\sum_{k\ge 1} q^{2d+k} S_{2d+k,d^{n-2}}(x) \prod_{1\le i<j\le n} \frac{1-x_i/x_j}{(1-qx_i/x_j)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}
\end{aligned}$$

Now note that the size of the partition $(2d+k,d^{n-2})$ is a multiple of $n$ is if and only if $k$ itself is a multiple of $n$. Thus Proposition 4 reduces this constant term to

$$\begin{aligned}
Q_2 \;&=\; \frac{1}{\prod_{i=1}^{n}(1-q^i)}\, \frac{1}{1-q^2} \sum_{d\ge 0}\sum_{k\ge 1} q^{2d+nk} S_{2d+nk,d^{n-2}}(x) \prod_{1\le i<j\le n} \frac{1-x_i/x_j}{(1-qx_i/x_j)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} \\
(\text{using } 2.21\ ) \;&=\; \frac{1}{\prod_{i=1}^{n}(1-q^i)}\, \frac{1}{1-q^2} \sum_{d\ge 0}\sum_{k\ge 1} q^{2d+nk} K_{(2d+nk,d^{n-2}),(k+d)^n}(q) \\
(\text{using } 2.22\ ) \;&=\; \frac{1}{\prod_{i=1}^{n}(1-q^i)}\, \frac{1}{1-q^2} \sum_{d\ge 0}\sum_{k\ge 1} q^{2d+nk} q^{k\binom{n}{2}+d} \begin{bmatrix} d+n-2 \\ d \end{bmatrix}_q \\
(\text{using } 2.26\ ) \;&=\; \frac{1}{1-q} \frac{1}{\prod_{i=1}^{n}(1-q^i)^2} \frac{q^{\binom{n+1}{2}}}{\big(1-q^{\binom{n+1}{2}}\big)} \frac{1}{(1-q^{n+1})}
\end{aligned}$$

and from 2.3 we get

$$Q \;=\; Q_0 + 2Q_2 = \frac{1}{1-q}\frac{1}{\prod_{i=2}^{n}(1-q^i)^2}\frac{1}{(1-q^{n+1})}\left(1+\frac{2q^{\binom{n+1}{2}}}{1-q^{\binom{n+1}{2}}}\right),$$

proving 2.24 and completing our calculation of the constant term.

## 3. Computing the constant term by Representation Theory.

We give an overview of the original proof of Theorem I.1. We will see that the proof we gave in the previous section is the end product of a succession of efforts aimed at eliminating from the original proof all the steps that required more specialized knowledge. Our goal there was to produce an argument accessible to the general audience. Inevitably, some beautiful facts were lost in the process. To compensate, in this section, we will make available to the interested reader some of the identities that are needed for a representation theoretical proof of Theorem I.1.

We must point out that many of the tools need in this approach are well known to representation theorists. For sake of completeness, we will review them here recast in a language that is more familiar to the combinatorists.

Recall that the action of an $n\times n$ matrix $M = \|m_{i,j}\|_{i,j=1}^{n}$ on a polynomial $P(x) = P(x_1, x_2, \ldots, x_n)$ is defined by setting

$$T_M P(x) \;=\; P(xM)$$

The matrix expressing the action of $T_M$ on the homogeneous polynomials of degree $m$ in term of the monomial basis $\langle x^p \rangle_{|p|=m}$ is denoted here by $S^m(M)$ and its entries may be computed from the identities

$$T_M x^q \;=\; \sum_{|p|=m} x^p S^m_{p,q}(M)$$

That is

$$S^m_{p,q}(M) \;=\; (xM)^q\Big|_{x^p}.$$

It follows from the Macmahon Master Theorem that the generating function of the traces of the matrices $S^m(M)$ is given by the formula

$$\sum_{m\geq 0} q^m trace\, S^m(M) \;=\; \frac{1}{det(1-qM)}. \tag{3.1}$$

If $G$ is a group of $n\times n$ matrices then the right hand side of 3.1, as a function of $M\in G$, may be viewed as the "*graded character*" of $G$ as it acts on the polynomial ring $\mathbf{R} = \mathbb{C}[x_1, x_2, \ldots, x_n]$.

This simple observation yields

## Proposition 3.1

*The rational function*

$$\chi_n(x;q) = \chi_n(x_1, x_2, \ldots, x_n, q) \;=\; \frac{1}{(1-q)^n}\prod_{i\neq j}\frac{1}{1-qx_i/x_j} \tag{3.2}$$

*is the graded character of $SL_n(\mathbb{C})$ under the action on polynomials $P(X) \in \mathbb{C}[x_{i,j} : 1 \leq i, j \leq n]$ defined by*

$$T_g P(X) \;=\; P(g^{-1}Xg) \qquad ( \quad X = \|x_{ij}\|_{i,j=1}^{n}). \tag{3.3}$$

## Proof

Note that if $g$ is diagonal with eigenvalues $x_1, x_2, \ldots, x_n$ then

$$g^{-1}Xg \;=\; \|x_i^{-1} x_{ij} x_j\|_{ij}$$

thus in this case the character is given by 3.1 with $M$ the diagonal matrix with eigenvalues

$$x_i^{-1} x_j \qquad \text{for } 1 \leq i, j \leq n.$$

But then

$$det(1 - qM) \;=\; (1-q)^n \prod_{1 \leq i \neq j \leq n} \Big(1 - q x_j / x_i\Big).$$

Since $\det(1-qM)$ is invariant under conjugation, this proves 3.2 for a diagonalizable $g$ . The validity of 3.2 for all $g \in SL_n(\mathbb{C})$ then follows by a standard continuity argument.

In the same manner it follows from 3.1

## Proposition 3.2

*The rational function*

$$F(x;q) \;=\; \prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \;=\; \sum_{a \geq 0} \sum_{b \geq 0} q^{a+b} h_a(x) h_b(1/x) \tag{3.4}$$

*is the graded character of $SL_n(\mathbb{C})$ under the action on polynomials $P(U,V) \in \mathbb{C}[u_i, v_j : 1 \leq i, j \leq n]$ defined by*

$$T_g P(U,V) \;=\; P(Ug, g^{-1}V) \qquad (U = (u_1, u_2, \ldots, u_n) \quad V = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}). \tag{3.5}$$

## Proof

It suffices to note that when $g$ is diagonal with eigenvalues $x_1, x_2, \ldots, x_n$ then

$$T_g \big\langle u_1, u_2, \ldots, u_n; v_1, v_2, \ldots, v_n \big\rangle \;=\; \big\langle u_1 x_1, u_2 x_2, \ldots, u_n x_n; x_1^{-1} v_1, x_2^{-1} v_2, \ldots, x_n^{-1} v_n \big\rangle$$

Thus here $M$ is a diagonal matrix with eigenvalues $x_1, x_2, \ldots, x_n, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}$. and in this case 3.1 reduces to the right hand side of 3.4

By combinining these two results we obtain

## Theorem 3.1

*The rational function*

$$U_n(x;q) = \frac{1}{(1-q)^n} \prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{i\neq j} \frac{1}{1-qx_i/x_j} \qquad 3.6$$

*is the graded character of $SL_n(\mathbb{C})$ under the action on polynomials $P(U,V,X) \in \mathbb{C}[u_i, v_j, x_{ij} : 1 \le i,j \le n]$ defined by*

$$T_g P(U,V,X) \;=\; P(Ug, g^{-1}V, g^{-1}Xg) \qquad 3.7$$

*In particular 3.6 yields a representation theoretical proof of the identity*

$$F_{UVX}(q) = \frac{1}{(1-q)^n} \prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{1\le i<j\le n} \frac{1-x_i/x_j}{(1-qx_i/x_j)(1-qx_j/x_i)} \Big|_{x_1x_2\cdots x_n=1} \Big|_{x_1^0x_2^0\cdots x_n^0}. \qquad 3.8$$

## Proof

We need only show that 3.6 implies 3.8. To this end note that since $x_1, x_2, \ldots, x_n$ are the eigenvalues of a matrix in $SL_n(\mathbb{C})$ we necesarily have

$$x_1x_2\ldots x_n \;=\; 1 \qquad 3.9$$

Thus all computations of a character of $SL_n(\mathbb{C})$ should be carried out, modulo this relation. This implies that the irreducible characters of $SL_n(\mathbb{C})$ are Schur functions indexed by partitions of length $n-1$ at most. In fact, if $\lambda$ has $k$ columns of length $n$ and $\mu$ is the partition obtained by removing these columns then

$$s_\lambda(x) \;=\; (x_1x_2\ldots x_n)^k s_\mu(x) \cong s_\mu(x),$$

here again the symbol "$\cong$" represents "*congruence*" modulo 3.9. Since we have

$$s_\mu(x) \prod_{1\le i<j\le n} (1-x_i/x_j)\Big|_{x_1^0x_2^0\cdots x_n^0} \;=\; \begin{cases} 1 & \text{if } \mu=\phi \\ 0 & \text{otherwise} \end{cases}$$

we can see that the right hand side of 3.8 gives none other than the graded generating function of the multiplicities of the trivial representation of $SL_n(\mathbb{C})$ under the action in 3.7 on the polynomial ring $\mathbb{C}[u_i, v_j, x_{ij} : 1 \le i,j \le n]$. But this is only another way of saying that the right hand side of 3.8 is the Hilbert series of UVX invariants and our proof is thus complete.

These observations immediately yield a path for the computation of the constant term in 3.8 by symmetric function methods. Indeed, this computation can be carried out in three steps

(1) *Obtain the Schur function expansion of*

$$F(x;q) \;=\; \prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)}$$

(2) *Obtain the Schur function expansion of*

$$\chi_n(x;q)=\frac{1}{(1-q)^n}\prod_{i\neq j}^{n}\frac{1}{(1-qx_i/x_i)}$$

(3) *Multiply these two expansion by the Littlewood Richarson rule and then*
*a) set to 1 all the Schur functions indexed by rectangular partitions of height* $n$
*b) set to zero all the other Schur functions.*

We have seen how to carry out step (1) in the computations that yielded $F_o(x;q)$ and $F_2(x;q)$ (see 2.6). To carry out step (2) we can use a short cut yielded by a further representation theoretical argument.

To see this note that since traces are not affected by conjugation, it follows that $n$ polynomials

$$\Pi_1=traceX\ ,\ \Pi_2=traceX^2\ ,\ \ldots\ ,\ \Pi_n=traceX^n$$

are invariant under our action. Now from a general result of B. Kostant [8] it follows that the ring of polynomials in the $x_{i,j}$ is free over the ring of polynomials in $\Pi_1,\Pi_2,\ldots,\Pi_n$. From this fact we can immediately obtain the character of the action of $SL_n(\mathbb{C})$ on the quotient ring

$$\mathbb{C}[\,x_{ij}:1\leq i,j\leq n]/(\Pi_1,\Pi_2,\ldots,\Pi_n) \tag{3.10}$$

or equivalently on the space $\mathbf{H_n}$ of " $SL_n$-*Harmonic*" polynomials. That is the space polynomials in the $x_{ij}$ that are killed by the differential operators obtained from the $\Pi_i$ upon replacement of each $x_{ij}$ by $\partial_{x_{ij}}$. Denoting the graded character of this action by $\chi^{\mathbf{H_n}}(x;q)$,it follows from the theorem of Kostant that

$$\chi_n(x;q)\ =\ \frac{\chi^{\mathbf{H_n}}(x;q)}{(1-q^2)\cdots(1-q^n)}\,, \tag{3.11}$$

This given. Proposition 2.2 can be restated as

## Proposition 3.3

*For any* $n\geq 2$

$$\chi^{\mathbf{H_n}}(x;q)\ =\ \frac{1}{\Delta(x)}\sum_{\sigma\in S_n} sign(\sigma)\sigma\prod_{i-1}^{n}x_i^{n-i}\Big(\prod_{1\leq i<j\leq n}\frac{1}{1-qx_i/x_j}\Big) \tag{3.12}$$

*where* $\Delta(x)$ *denotes the Vandermonde determinant in* $x_1,x_2,\ldots,x_n$.

## Proof

In view of 3.2 the identity in 2.10 simply states that

$$\chi_n(x;q)\ =\ \frac{1}{\prod_{i=1}^n(1-q^i)}\frac{1}{\Delta(x)}\sum_{\sigma\in S_n} sign(\sigma)\sigma\prod_{i-1}^{n}x_i^{n-i}\Big(\prod_{1\leq i<j\leq n}\frac{1}{1-qx_i/x_j}\Big)$$

and 3.12 follows by combining this identity with 3.5.

This result has the following remarkable consequence of Kostant Theorem,

**Theorem 3.2** (first stated in this form by Ranee Gupta in [6], [7])

$$\chi^{\mathbf{H_n}}(x;q) \;=\; \sum_{b\geq 0}\sum_{\substack{\lambda\vdash nb\\ \lambda(\lambda)<n}} S_\lambda(x)K_{\lambda,b^n}(q) \tag{3.13}$$

*where $K_{\lambda,b^n}(q)$ is the so-called Kotska Foulkes polynomial.*

## Proof

The point of departure here is 3.6 which, using the notation introduced in 2.16, 2.17 and 2.18, becomes

$$\begin{aligned}\chi^{\mathbf{H_n}}(x;q) &= \frac{1}{\Delta(x)}\sum_{\sigma\in S_n} sign(\sigma)\sigma\prod_{i-1}^{n} x_i^{n-i}\Big(\prod_{1\leq i<j\leq n}\frac{1}{1-qx_i/x_j}\Big)\\ &= \sum_{p\in\mathcal{P}^+} q^{\|p\|}\frac{1}{\Delta(x)}\sum_{\sigma\in S_n} sign(\sigma)\sigma x^{\delta+p}\end{aligned} \tag{3.14}$$

Now note that for some $\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n)$ we have

$$\frac{1}{\Delta(x)}\sum_{\sigma\in S_n} sign(\sigma)\sigma x^{\delta+p} \;=\; \begin{cases} sign(\sigma)s_\lambda(x) & \text{if } \sigma(\lambda+\delta)=p+\delta \text{ for some } \sigma\in S_n\\ 0 & \text{otherwise}\end{cases}$$

Thus 3.14 may be rewritten as

$$\begin{aligned}\chi^{\mathbf{H_n}}(x;q) &= \sum_{p\in\mathcal{P}^+} q^{\|p\|}\sum_{\lambda} s_\lambda(x)\sum_{\sigma\in S_n} sign(\sigma)\chi(\sigma(\lambda+\delta)=\delta+p)\\ &= \sum_{p\in\mathcal{P}^+} q^{\|p\|}\sum_{\lambda} s_\lambda(x)\sum_{\sigma\in S_n} sign(\sigma)z^{\sigma(\lambda+\delta)-\delta-p}\Big|_{z_1^0z_2^0\cdots z_n^0}\\ &= \sum_{p\in\mathcal{P}^+} q^{\|p\|}\sum_{\lambda} s_\lambda(x)s_\lambda(z)\; z^{-\delta-p}\;\Big|_{z_1^0z_2^0\cdots z_n^0}\\ \text{(Using 2.18)} &= \sum_{\lambda} s_\lambda(x)s_\lambda(z)\; z^{-\delta}\Delta(z)\prod_{1\leq i<j\leq n}\frac{1}{1-qz_j/z_i}\Big|_{z_1^0z_2^0\cdots z_n^0}\end{aligned} \tag{3.15}$$

Now since $p_1+p_2+\cdots+p_n=0$ (see 2.10), it follows from the equality $\sigma(\lambda+\delta)-\delta=p$ that $\lambda_1+\lambda_2+\cdots+\lambda_n=0$. Thus we must have $-\lambda_n=b>0$ and, a fortiori, the vector $\lambda^*=(\lambda_1+b,\lambda_2+b,\ldots,\lambda_{n-1}+b,0)$ must be a partition of $b^n$. To convert $s_\lambda(x)$ into an customary Schur function, we then note that from the bideterminantal formula we get that

$$s_{\lambda^*}(x) \;=\; s_\lambda(x)(x_1x_2\cdots x_n)^b \cong s_\lambda(x)$$

Using this identity 3.15 becomes

$$\begin{aligned}\chi^{\mathbf{H_n}}(x;q) &\cong \sum_{b\geq 0}\sum_{\lambda^*\vdash b^n} s_{\lambda^*}(x)s_{\lambda^*}(z)\;\frac{\prod_{1\leq i<\leq n}(z_i-z_j)}{z_1^{b+n-1}z_2^{b+n-2}\cdots z_n^{b+n-n}}\prod_{1\leq i<j\leq n}\frac{1}{1-qz_j/z_i}\Big|_{z_1^0z_2^0\cdots z_n^0}\\ &\cong \sum_{b\geq 0}\sum_{\lambda^*} s_{\lambda^*}(x)s_{\lambda^*}(z)\prod_{1\leq i<j\leq n}\frac{1-z_j/z_i}{1-qz_j/z_i}\Big|_{z_1^bz_2^b\cdots z_n^b}\end{aligned}$$

This proves 3.13 since one of the classical formulas for the Kostka-Foulkes polynomial may be written in the form

$$K_{\lambda,\mu}(q) \;=\; S_\lambda(z_1,z_2,\ldots,z_n)\prod_{i\neq j}\frac{1-z_j/z_i}{1-qz_j/z_i}\bigg|_{z_1^{\mu_1}z_2^{\mu_2}\cdots z_n^{\mu_n}}$$

Having carried out step (1) and step (2) to carry out step (3), we need the following symmetric function fact.

## Proposition 3.4

*Given two partitions $\lambda=(\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_{n-1}\geq 0)$ and $\mu=(\mu_1\geq\mu_2\geq\ldots\geq\mu_{n-1}\geq 0)$, the Schur fuction expansion of the product*

$$s_\lambda(x)s_\mu(x) \qquad 3.16$$

*contains a Schur function indexed by a rectangular partition of height $n$ if and only if*

$$\mu=(b-\lambda_n,\ldots,b-\lambda_2,b-\lambda_1,), \qquad 3.17$$

*where $b=max(\lambda_1,\mu_1)$ and $|\lambda|+|\mu|=nb$. In particular this Schur function occurs with multiplicity $1$.*

## Proof

The expansion of the product in 3.16 contains a Schur function $s_{b^n}(x)$ if and only if

$$\big\langle s_\lambda(x)s_\mu(x)\;,\;s_{b^n}(x)\big\rangle\neq 0 \qquad (\text{with } nb=|\lambda|+|\mu|)$$

now this is equivalent to

$$\big\langle s_\mu(x)\;,\;s_{b^n/\lambda}(x)\big\rangle\neq 0. \qquad 3.18$$

But a Schur function indexed a skew diagram obtained by removing a Ferrers diagram from a rectangle is identical to the Schur function indexed by the partition $\lambda^c$ obtained by a $180^o$ rotation of the skew diagram, (see figure where we depicted the case when $b=\lambda_1$). This geometric fact yields 3.17 as well as the multiplicity assertion.

Combining 3.11 with 3.13 and using the expansions in 2.4 and 2.6, it is not difficult to complete step 3 by means of Proposition 3.4. We shall not carry this out here since the remaining steps involve manipulations with symmetric functions that are quite similar to those we have seen in the previous section.

## 4. Computing the constant term by the partial fraction algorithm

A comprehensive introduction to the general form of the partial fraction algorithm can be found in [10]. A tutorial in the use of a restricted version this algorithm (sufficient for the present purposes) is given in [5]. In this paper we will strictly adhere to the notation and terminology introduced in [5] except that we will use the signs "$\prec$" and "$\succ$" for all monomial inequalities derived from our alphabet total order.

Our point of departure here is Proposition 2.3. More precisely, using the identity in 2.11 with

$$A(x) \;=\; \prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)}$$

gives

$$\frac{1}{(1-q)^n} \prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{1\le i<j\le n} \frac{1-x_i/x_j}{(1-qx_i/x_j)(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}$$
$$= \frac{1}{\prod_{i=1}^{n}(1-q^i)} \prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{1\le i<j\le n} \frac{1-x_i/x_j}{(1-qx_i/x_j)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}.$$

Thus to prove Theorem I.1, we need show that

$$\prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{1\le i<j\le n} \frac{1-x_i/x_j}{(1-qx_i/x_j)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} = \frac{1+q^{\binom{n+1}{2}}}{(1-q^{\binom{n+1}{2}})\prod_{i=2}^{n+1}(1-q^i)}$$

Since this constant term cannot change under any permutation of the variables, it will be equivalent to show

$$\prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{1\le i<j\le n} \frac{1-x_j/x_i}{(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} = \frac{1+q^{\binom{n+1}{2}}}{(1-q^{\binom{n+1}{2}})\prod_{i=2}^{n+1}(1-q^i)}. \tag{4.1}$$

The simplicity of the following purely manipulatorial derivation of 4.1 demonstrates the power of the partial fraction algorithm in the computation of constant terms.

Let $u, v$ and $w$ be three additional variables, and set

$$\mathcal{Q}_n(u,v,w) = \prod_{i=1}^{n} \frac{1}{(1-ux_i)(1-v/x_i)} \prod_{1\le i<j\le n} \frac{1-x_j/x_i}{(1-qx_j/x_i)}\Big|_{x_n=w/x_1\cdots x_{n-1}}.$$

$$\mathcal{Q}^*_n(u,v,w) = \prod_{i=1}^{n} \frac{1}{(1-ux_i)(1-v/x_i)} \prod_{1\le i<j\le n} \frac{1-x_j/x_i}{(1-qx_j/x_i)}\Big|_{x_1=w^{-1}/x_2\cdots x_n}. \tag{4.2}$$

To do this, we choose the total order of the variables to be $q \prec u \prec v \prec w \prec x_1 \prec x_2 \prec \ldots \prec x_n$, and define

$$R_n(u,v,w) = \frac{1}{1-w/x_1\cdots x_n} \prod_{i=1}^{n} \frac{1}{(1-ux_i)(1-v/x_i)} \prod_{1\le i<j\le n} \frac{1-x_j/x_i}{(1-qx_j/x_i)},$$

$$R^*_n(u,v,w) = \frac{1}{1-1/wx_1\cdots x_n} \prod_{i=1}^{n} \frac{1}{(1-ux_i)(1-v/x_i)} \prod_{1\le i<j\le n} \frac{1-x_j/x_i}{(1-qx_j/x_i)}.$$

Clearly we have $R_n(u,v,w) = R_n^*(u,v,1/w)$. In what follows, the argument $w$ may be replaced by a monomial $m$. We will always make the following choice: if $m/x_1\cdots x_n \prec 1$ or $m \prec 1$ for short, then we must choose $R_n(u,v,m)$, otherwise $m/x_1\cdots x_n \succ 1$ or $m \succ 1$ for short, we must choose $R_n^*(u,v,1/m)$.

We will evaluate the constant terms of $R_n(u,v,w)$ and $R_n^*(u,v,w)$ in two ways to obtain the constant terms of $Q_n(u,v,w)$ and $Q_n^*(u,v,w)$. This given, it is easy to check that $R_n(u,v,w)$ is proper in $x_i$ for all $i$ and vanishes when setting $x_i = 0$, so the tools of the tutorial in [5] may be applied for every $x_i$.

## Lemma 4.1

*We have*

$$R_n(u,v,w)\big|_{x_1^0\cdots x_n^0} = \frac{1}{1-uv}\, R_{n-1}(qu,b,uw)\big|_{x_1^0\cdots x_{n-1}^0}.$$

## Proof

We use Proposition 4.2 of [5] for the variable $x_1$. Among all factors containing $x_1$, the factors of the form $1-qx_j/x_1$ have a dual contribution; the factors $1-w/x_1\cdots x_n$ and $1-v/x_1$ have dual contribution; only the factor $1-ux_1$ has a contribution. Thus using the first equality in 4.10 of [5], this contribution is obtained by removing this factor and then replacing $x_1$ by $u^{-1}$. Carrying this out gives

$$\frac{1}{\left(1-\frac{uw}{x_2\cdots x_n}\right)(1-uv)} \prod_{i=2}^{n} \frac{1}{(1-x_iu)(1-v/x_i)} \prod_{2\le i<j\le n} \frac{1-x_j/x_i}{1-qx_j/x_i} \prod_{2\le j\le n} \frac{1-ux_j}{1-qux_j},$$

which simplifies to

$$\frac{1}{\left(1-\frac{uw}{x_2\cdots x_n}\right)(1-uv)} \prod_{i=2}^{n} \frac{1}{(1-x_iqu)(1-v/x_i)} \prod_{2\le i<j\le n} \frac{1-x_j/x_i}{1-qx_j/x_i}.$$

Since $uw \prec 1$ this is exactly $\frac{1}{1-uv}R_{n-1}(qu,v,uw)$ if we rename $x_i$ by $x_{i-1}$. Therefore the Lemma follows.

Iterating the above Lemma, together with the easy fact $R_0(u,v,w) = \frac{1}{1-w}$, we obtain

## Proposition 4.1

$$R_n(u,v,w)\big|_{x_1^0\cdots x_n^0} = \frac{1}{(1-uv)(1-uvq)\cdots(1-uvq^{n-1})(1-u^nwq^{\binom{n}{2}})}.$$

## Lemma 4.2

*We have*

$$R_n^*(u,v,w)\big|_{x_1^0\cdots x_n^0} = \frac{1}{1-uv}\, R_{n-1}^*(u,qv,qw)\big|_{x_1^0\cdots x_{n-1}^0}.$$

## Proof

We now use Proposition 4.2 of [5] for the variable $x_n$. Note that since $wx_1\cdots x_n \prec 1$ the proper form of $\frac{1}{\left(1-\frac{1}{wx_1\cdots x_n}\right)}$ is $\frac{-wx_1\cdots x_n}{(1-wx_1\cdots x_{n-1})}$. Thus among all the factors containing $x_n$, in the denominator of $R_n^*(u,v,w)$, only the factor $1-v/x_n$ has a dual contribution. Using the second equality in formula 4.10 of [5] we derive that the constant term of $R_n^*(u,v,w)$ in $x_n$ is

$$\frac{1}{\left(1-\frac{1}{vwx_1\cdots x_{n-1}}\right)(1-uv)} \prod_{i=1}^{n-1} \frac{1}{(1-x_iu)(1-v/x_i)} \prod_{1\le i<j\le n-1} \frac{1-x_j/x_i}{1-qx_j/x_i} \prod_{1\le i\le n-1} \frac{1-v/x_i}{1-qv/x_i},$$

This simplifies to

$$\frac{1}{(1-1/vwx_1\cdots x_{n-1})\,(1-uv)}\prod_{i=1}^{n-1}\frac{1}{(1-x_iu)(1-qv/x_i)}\prod_{1\le i<j\le n-1}\frac{1-x_j/x_i}{1-qx_j/x_i}.$$

Since $vw \prec 1$ this is exactly $\frac{1}{1-uv}R^*_{n-1}(u,qv,vw)$. Therefore the lemma follows.

Iterating the above Lemma, together with the easy fact $R_0(u,v,w)=\frac{1}{1-1/w}=-\frac{w}{1-w}$ (since $w\prec 1$), we obtain

## Proposition 4.2

$$R^*_n(u,v,w)\Big|_{x_1^0\cdots x_n^0}=-\frac{v^n w q^{\binom{n}{2}}}{(1-uv)(1-uvq)\cdots(1-uvq^{n-1})(1-v^n w q^{\binom{n}{2}})}.$$

Now we evaluate the constant terms of $R_n(u,v,w)$ and $R^*_n(u,v,w)$ in another way to obtain recurrences involving the constant terms of $Q_n(u,v,w)$ and $Q^*_n(u,v,w)$, and then compute these constant terms by solving the recurrences.

## Lemma 4.3

*We have*

$$R_n(u,v,w)\Big|_{x_1^0\cdots x_n^0}=Q_n(u,v,w)\Big|_{x_1^0\cdots x_{n-1}^0}+\frac{1}{1-uv}R^*_{n-1}(u,qv,v/w)\Big|_{x_1^0\cdots x_{n-1}^0}.$$

## Proof

We will use Proposition 4.2 of [5] with respect to $x_n$. Among all factors in the denominator containing $x_n$, only the factors $1-w/x_1\cdots x_n$ and $1-v/x_n$ have dual contributions. The dual contribution of the first factor is

$$\prod_{i=1}^{n}\frac{1}{(1-ux_i)(1-v/x_i)}\prod_{1\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}\Big|_{x_n=w/x_1\cdots x_{n-1}},$$

which is exactly $\mathcal{Q}_n(u,v,w)$. Using the second equality in formula 4.10 of [5], we deriove that the dual contribution of the second factor is

$$\frac{1}{\left(1-\frac{w/v}{x_1\cdots x_{n-1}}\right)(1-uv)}\prod_{i=1}^{n-1}\frac{1}{(1-ux_i)(1-v/x_i)}\prod_{1\le i<j\le n-1}\frac{1-x_j/x_i}{(1-qx_j/x_i)}\prod_{i=1}^{n-1}\frac{1-v/x_i}{1-qv/x_i},$$

which simplifies to

$$\frac{1}{\left(1-\frac{w/v}{x_1\cdots x_{n-1}}\right)(1-uv)}\prod_{i=1}^{n-1}\frac{1}{(1-ux_i)(1-qv/x_i)}\prod_{1\le i<j\le n-1}\frac{1-x_j/x_i}{(1-qx_j/x_i)}.$$

Since $w/v\succ 1$, this is clearly $\frac{1}{1-uv}R^*_{n-1}(u,qv,v/w)$. The Lemma then follows.

Applying Lemma 4.3, Propositions 4.1 and Proposition 4.2, we obtain

## Theorem 4.1

$$Q_n(u,v,w)\Big|_{x_1^0\cdots x_{n-1}^0} = \frac{1}{(uv)_n}\times\left(\frac{1}{1-u^nwq^{\binom{n}{2}}}+\frac{v^nw^{-1}q^{\binom{n}{2}}}{1-v^nw^{-1}q^{\binom{n}{2}}}\right),$$

where as customary we set

$$(x)_n \;=\; (1-x)(1-xq)\cdots(1-xq^{n-1})$$

## Lemma 4.4

$$R_n^*(u,v,w)|_{x_1^0\cdots x_n^0} = -\,Q_n^*(u,v,w)|_{x_2^0\cdots x_n^0} + \frac{1}{1-uv}R_{n-1}(qu,v,u/w)\Big|_{x_1^0\cdots x_{n-1}^0}$$

## Proof

We apply Proposition 4.2 of [5] with respect to $x_1$. Among all factors in the denominator containing $x_1$, only the factors $1-1/wx_1\cdots x_n$ and $1-ux_1$ have contributions. The first contribution is

$$-\prod_{i=1}^n \frac{1}{(1-ux_i)(1-v/x_i)}\prod_{1\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}\Big|_{x_1=1/wx_2\cdots x_n},$$

which is exactly $-\mathcal{Q}_n^*(u,v,w)$. The second contribution is

$$\frac{1}{\left(1-\frac{1}{w/ux_2\cdots x_n}\right)(1-uv)}\prod_{i=2}^n\frac{1}{(1-ux_i)(1-v/x_i)}\prod_{2\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}\prod_{j=2}^n\frac{1-x_ju}{1-x_jqa},$$

which simplifies to

$$\frac{1}{\left(1-\frac{1}{w/ux_2\cdots x_n}\right)(1-uv)}\prod_{i=2}^n\frac{1}{(1-qux_i)(1-v/x_i)}\prod_{2\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}.$$

Since $u/w\prec 1$, this is clearly $\frac{1}{1-uv}R_{n-1}(qu,v,u/w)$ if we rename $x_{i+1}$ by $x_i$. The lemma then follows.

Applying Lemma 4.4, Propositions 4.1 and Proposition 4.2, we obtain

## Theorem 4.2

$$Q_n^*(u,v,w)\Big|_{x_2^0\cdots x_n^0} = \frac{1}{(uv)_n}\times\left(\frac{wv^nq^{\binom{n}{2}}}{1-wv^nq^{\binom{n}{2}}}+\frac{1}{1-u^nw^{-1}q^{\binom{n}{2}}}\right). \qquad 4.3$$

Note that the left hand side of 4.1 is none other than the constant term of $Q_n^*(q,q,1)$. But Theorem 4.2 gives

$$Q_n^*(q,q,1)\Big|_{x_2^0\cdots x_n^0} \;=\; \frac{1+q^{\binom{n+1}{2}}}{(q^2)_n(1-q^{\binom{n+1}{2}})}$$

completing the proof of 4.1.

Note further that the second case of 4.3, combined with the definition in 4.2 gives (setting $w = 1$)

$$\prod_{i=1}^{n} \frac{1}{(1-ux_i)(1-v/x_i)} \prod_{1\le i<j\le n} \frac{1-x_j/x_i}{(1-qx_j/x_i)}\Big|_{x_n=w^{-1}/x_1\cdots x_{n-1}}\Big|_{x_2^0\cdots x_n^0} = \frac{1}{(uv)_n}\left(\frac{v^n q^{\binom{n}{2}}}{1-v^n q^{\binom{n}{2}}} + \frac{1}{1-u^n q^{\binom{n}{2}}}\right). \quad 4.4$$

This proves Theorem I.4.

## 5. Our four bases for the UVX invariants.

Returning to UVX invariants we first need to derive Theorem I.5 from Theorem I.4. To this end note that if the variables $u_i, v_j$ and $x_{i,j}$ are respectively weighted by $u, v$ and $q$, then the corresponding tri-graded Hilbert series $F_{UVX}(u,v,q)$ should be given by the corresponding tri-graded version of Moliens theorem. This simply means that in the Molien integral we must replace the denominator factor

$$\det|1-qD(g)| \;=\; \prod_{r=1}^{n}(1-qa_r)(1-q/a_r)\prod_{r,s=1}^{n}(1-qa_r/a_s) \qquad 5.1$$

by a tri-graded factor that reflects the separate action of $g$ on the three sets of variables $u_i, v_j$ and $x_{i,j}$. Denoting by $D_1(g)$ $D_2(g)$ and $D_3(g)$ the three diagonal matrices with eigenvalues

$$a_1, a_2, \ldots a_n; \quad a_1^{-1}, a_2^{-1}, \ldots a_n^{-1}; \quad \text{and} \quad a_i a_j^{-1} \quad \text{for } 1 \le i,j \le n,$$

in the integral we must replace 5.1 by the product

$$\det|1-uD_1(g)|\det|1-vD_2(g)|\det|1-uD_r(q)| \;=\; \prod_{r=1}^{n}(1-ua_r)(1-v/a_r)\prod_{r,s=1}^{n}(1-qa_r/a_s) \qquad 5.2$$

This changes 1.13 to

$$F_{UVX}(u,v,q) \;=\; \int_{T_n}\prod_{r=1}^{n}\frac{1}{(1-ua_r)(1-v/a_r)}\prod_{r,s=1}^{n}\frac{1}{(1-qa_r/a_s)}\,d\omega(g). \qquad 5.3$$

This given, a close look at the proof of Theorem I.2 given in section 1, quickly reveals that the, replacements $q \to u$ and $q \to v$ in the first two factors does not affect the validity of any of the steps. Thus, with these replacements the proof in section 1 yields

$$F_{UVX}(u,v,q) \;=\; \frac{1}{(1-q)^n}\prod_{i=1}^{n}\frac{1}{(1-ux_i)(1-v/x_i)}\prod_{1\le i<j\le n}\frac{1-x_i/x_j}{(1-qx_i/x_j)(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}$$

Combining this with Theorem I.5 yields Theorem I.5:

$$F_{UVX}(u,v,q) \;=\; \frac{1}{\prod_{i=1}^{n}(1-q^i)\prod_{i=1}^{n}(1-uvq^{i-1})}\left(\frac{v^n q^{\binom{n}{2}}}{1-v^n q^{\binom{n}{2}}} + \frac{1}{1-u^n q^{\binom{n}{2}}}\right). \qquad 5.4$$

Now note that the tri-degrees of

$$\Pi_1 = traceX\ ,\ \Pi_2 = traceX^2\ ,\ \Pi_3 = traceX^3\ ,\ \ldots\ ,\ \Pi_n = traceX^n$$

are

$$(0,0,1),(0,0,2),(0,0,3),\ldots,(0,0,n),$$

the tri-degrees of

$$\theta_1 = UV\ ,\ \ \theta_2 = UXV\ ,\ \ \theta_2 = UX^2V\ ,\ \ldots\ ,\ \theta_n = UX^{n-1}V$$

are

$$(1,1,0),(1,1,1),\ldots,(1,1,n-1)$$

and those of the two determinants:

$$\Phi(U,X)\ =\ \det\begin{Vmatrix} U \\ UX \\ UX^2 \\ \vdots \\ UX^{n-1}\end{Vmatrix} \qquad \text{and} \qquad \Psi(V,X)\ =\ \det\left\| V, XV, X^2V,\ldots,X^{n-1}V\right\|$$

are

$$(n,0,\tbinom{n}{2}) \quad \text{and} \quad (0,n,\tbinom{n}{2})$$

Thus if we assign weights

$$w(\Pi_i) = q^i\ ,\quad w(\theta_j) = uvq^{j-1}\ ,\quad w(\Phi) = u^n q^{\binom{n}{2}} \quad \text{and} \quad w(\Psi) = v^n q^{\binom{n}{2}}$$

we see that 5.4 may be rewritten as the formal series

$$\begin{aligned} F_{UVX}(u,v,q) = &\sum_{r_1\geq 0}\cdots\sum_{r_n\geq 0}\sum_{s_1\geq 0}\cdots\sum_{s_n\geq 0}\sum_{m\geq 0} w(\Pi_1)^{r_1}\cdots w(\Pi_n)^{r_n}\times w(\theta_1)^{s_1}\cdots w(\theta_n)^{s_n} w(\Phi)^m \\ &+\sum_{r_1\geq 0}\cdots\sum_{r_n\geq 0}\sum_{s_1\geq 0}\cdots\sum_{s_n\geq 0}\sum_{m\geq 0} w(\Pi_1)^{r_1}\cdots w(\Pi_n)^{r_n}\times w(\theta_1)^{s_1}\cdots w(\theta_n)^{s_n} w(\Psi)^{m+1}\end{aligned} \qquad 5.5$$

This brings us in a position to prove

## Theorem I.6

*The UVX invariants have the tri-graded basis*

$$\mathcal{B}_1^{ab}\ =\ \Big\{\Phi^a\Pi_1^{r_1}\Pi_2^{r_2}\cdots\theta_n^{r_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_n^{s_n}\ ;\ \Psi^{b+1}\Pi_1^{r_1}\Pi_2^{r_2}\cdots\theta_n^{r_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_n^{s_n}\ : a,b\geq 0, r_i\geq 0, s_i\geq 0\Big\} \qquad 5.6$$

## Proof

The identity in 5.5 essentially says that the number of elements of the collection $\mathcal{B}_1^{ab}$ that are tri-homogeneous of tri-degree $r,s,m$ is exactly equal to the dimension of the subspace $\mathcal{H}_{r,s,m}(UVX)$. Thus to prove that $\mathcal{B}_1^{ab}$ is a basis it is sufficient to show independence.

To this end, suppose we had a vanishing linear combination $P$ of the monomials in 5.6. Since each of the tri-homogeneous components of $P$ would have to vanish separately, there is no loss in assuming that $P$ is tri-homogeneous. Now we have two important facts:

(1) The monomial $\Phi^a\Pi_1^{r_1}\Pi_2^{r_2}\cdots\theta_n^{r_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_n^{s_n}$ has tri-degree

$$(an,0,a\tbinom{n}{2})\ +\ \big(0,0,(\textstyle\sum_i ir_i)\big)+\big(\sum_i s_i,\sum_i s_i,(\sum_i is_i)\big) \qquad 5.7$$

(2) The monomial $\Psi^{b+1}\Pi_1^{r_1}\Pi_2^{r_2}\cdots\theta_n^{r_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_n^{s_n}$ has tri-degree

$$(0,(b+1)n,(b+1)\tbinom{n}{2}) + \big(0,0,(\textstyle\sum_i ir_i)\big) + \big(\textstyle\sum_i s_i, \sum_i s_i, (\sum_i is_i)\big) \qquad 5.8$$

This immediately shows that any tri-homogenous linear combination $P$ cannot contain both $\Phi$ and $\Psi$. Indeed we see from 5.7 and 5.8 that the terms of $P$ that contain $\Phi$ are of tri-degree $(r,s,m)$ with $r\geq s$ and those that contain $\Psi$ are of tri-degree $(r,s,m)$ with $r<s$. Now note that if $P$ contains only $\Phi$ and is of tri-degree $(r,s,m)$ then from 5.7 we derive we must have $an=r-s$ for each monomial in $P$. In other words, in each term of $P$, $\Phi$ must occur to the power $(r-s)/n$. Thus any vanishing tri-homogeneous $P$ that contains only $\Phi$ must factor as a product of $\Phi$ to some power times a vanishing linear combination that does not contain neither $\Phi$ nor $\Psi$. Of course we can reach the analogous conclusion interchanging $\Psi$ and $\Phi$ in the previous argument. In summary we thus obtain that by factoring out a power of $\Phi$ or $\Psi$ as the case may be any non trivial tri-homogeneous vanishing combination of the monomials in 4.2 will yield a vanishing polynomial in $\Pi_1,\Pi_2,\ldots,\Pi_n;\theta_1,\theta_2,\ldots,\theta_n$.

We are thus left to show that these polynomials are algebraically independent. But this is an immediate consequence of the fact that the Jacobian of $\Pi_1,\Pi_2,\ldots,\Pi_n;\theta_1,\theta_2,\ldots,\theta_n$ with respect to the variables $x_{11},x_{22},\ldots,x_{22};u_1,u_2,\ldots,u_n$ does not even vanish when we set to zero all the variable $x_{ij}$ with $i\neq j$. In fact we can easily see that carrying this out results in the Jacobian polynomial

$$v_1v_2,\ldots v_n\prod_{1\leq i<j\leq n}(x_{ii}-x_{jj}).$$

This completes our proof.

Note next that an immediate by-product of this proof is that the collection

$$\Big\{\Phi^r\Psi^s\Pi_1^{r_1}\Pi_2^{r_2}\cdots\theta_n^{r_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_n^{s_n}\ :r,s,r_i,s_j\geq 0\Big\} \qquad 5.9$$

spans the UVX invariants. But since $\Phi=(\Gamma^+ +\Gamma^-)/2$ and $\Psi=(\Gamma^+ -\Gamma^-)/2$ the same will be true for the collection

$$\mathcal{B}^{\pm} \;=\; \Big\{(\Gamma^+)^r(\Gamma^-)^s\Pi_1^{r_1}\Pi_2^{r_2}\cdots\theta_n^{r_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_n^{s_n}\ :r,s,p_i,s_j\geq 0\Big\}.$$

Now it is important to note that $\Phi(U,X)$ and $\Psi(V,X)$ are not completely independent of the other invariants. More precisely we have

## Proposition 5.1

*The product $\Phi(U,X)\Psi(V,X)$ may be expresssed in terms of the parameters $\Pi_1,\Pi_2,\ldots,\Pi_n\,;\,\theta_1,\theta_2,\ldots,\theta_n$.*

## Proof

The Cayley-Hamilton theorem gives

$$\theta_k=UX^{k-1}V=U\Big(\sum_{i=1}^n(-1)^{i-1}e_i(X)X^{k-i-1}\Big)V=\sum_{i=1}^n(-1)^{i-1}e_i(X)\theta_{k-i}$$

since the polynomials $e_1(X),e_2(X),\cdots,e_n(X)$ (the elementary symmetric function of the eigenvalues of $X$) may be expressed as polynomials in $\Pi_1,\Pi_2,\ldots,\Pi_n$ it follows that the polynomials $\theta_k$, (for $k>n$), can all be expressed as polynomials in $\Pi_1, \Pi_2, \ldots, \Pi_n\,;\, \theta_1, \theta_2, \ldots, \theta_n$. This given, the assertion follows immediately from the determinantal identity

$$\Phi(U,X)\Psi(V,X) \;=\; det\Big\| U X^{i+j} V \Big\|_{0\le i,j\le n-1} \;=\; det\Big\| \theta_{i+j} \Big\|_{0\le i,j\le n-1}. \qquad 5.10$$

Note next that the two polynomials

$$\Gamma^+(U,V,X) = \Phi(U,X) + \Psi(V,X) \quad \text{and} \quad \Gamma^-(U,V,X) = \Phi(U,X) - \Psi(V,X)$$

satisfy the quadratic equation

$$(\Gamma^+)^2 \;=\; (\Gamma^-)^2 \,+\, 4\,\Phi\,\Psi. \qquad 5.11$$

This brings us in a position to prove Theorem I.7. That is to show that the two collections

$$\mathcal{B}^+ \;=\; \Big\{ (\Gamma^+)^a (\Gamma^-)^b \Pi_1^{r_1}\Pi_2^{r_2}\cdots\theta_n^{r_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_n^{s_n} \;: a\ge 1\,,\; b, r_i, s_j \ge 0 \Big\}$$

and

$$\mathcal{B}^- \;=\; \Big\{ (\Gamma^+)^a (\Gamma^-)^b \Pi_1^{r_1}\Pi_2^{r_2}\cdots\theta_n^{r_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_n^{s_n} \;: b\ge 1\,,\; a, r_i, s_j \ge 0 \Big\}$$

are vector space bases for the UVX invariants. To this end note that since $deg(\Pi_i) = i$, $deg(\theta_j) = j+1$ and $deg(\Gamma^+) = deg(\Gamma^-) = \binom{n+1}{2}$ it follows that

$$\sum_{b\in\mathcal{B}^+} q^{deg(b)} = \sum_{b\in\mathcal{B}^-} q^{deg(b)} = \frac{1+q^{\binom{n+1}{2}}}{(1-q)\prod_{i=2}^n (1-q^i)^2 (1-q^{n+1})\big(1-q^{\binom{n+1}{2}}\big)}$$

Thus our proof that

$$F_{UVX}(q;n) \;=\; \frac{1+q^{\binom{n+1}{2}}}{(1-q)\prod_{i=2}^n (1-q^i)^2 (1-q^{n+1})\big(1-q^{\binom{n+1}{2}}\big)}$$

is equivalent to each of the equalities

$$a)\;\sum_{b\in\mathcal{B}^+} q^{deg(b)} = F_{UVX}(q;n) \quad \text{and} \quad b)\;\sum_{b\in\mathcal{B}^-} q^{deg(b)} = F_{UVX}(q;n).$$

This means that both collections $\mathcal{B}^+$ and $\mathcal{B}^-$ have the correct number of elements in each degree. Thus to prove that they are bases we need only show that they span. Now we have seen that the collection

$$\mathcal{B}^\pm \;=\; \Big\{ (\Gamma^+)^r (\Gamma^-)^s \Pi_1^{p_1}\Pi_2^{p_2}\cdots\theta_n^{p_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_n^{s_n} \;: r, s, p_i, s_j \ge 0 \Big\}$$

spans the UVX invariants. This given, note that dropping from $\mathcal{B}^\pm$ all terms that contain $\Gamma^-$ to a power greater than 1 we get $\mathcal{B}^+$ while dropping all terms that contain $\Gamma^+$ to a power greater than 1 gives $\mathcal{B}^-$. Now Proposition 5.1 together with 5.11 assures that, in either case, the loss of these terms does not affect the spanning property and Theorem I.7 necessarily follows.

## Remark 5.1

Note that we can write

$$\frac{v^n q^{\binom{n}{2}}}{1-v^n q^{\binom{n}{2}}} + \frac{1}{1-u^n q^{\binom{n}{2}}} = \frac{v^n q^{\binom{n}{2}} - v^n q^{\binom{n}{2}} u^n q^{\binom{n}{2}} + 1 - v^n q^{\binom{n}{2}}}{(1-u^n q^{\binom{n}{2}})(1-v^n q^{\binom{n}{2}})} = \frac{1 - (uvq^{(n-1)})^n}{(1-u^n q^{\binom{n}{2}})(1-v^n q^{\binom{n}{2}})}$$

Thus the trigraded Hilbert series in I.8 may be rewritten in the form

$$\begin{aligned} F_{UVX}(u,v,q) &= \frac{1}{\prod_{i=1}^n (1-q^i)\,\prod_{i=1}^n (1-uvq^{i-1})} \frac{\big(1-\big(uvq^{n-1}\big)^n\big)}{(1-u^n q^{\binom{n}{2}})(1-v^n q^{\binom{n}{2}})} \\ &= \frac{1+uvq^{n-1}+\big(uvq^{n-1}\big)^2+\cdots+\big(uvq^{n-1}\big)^{n-1}}{\prod_{i=1}^n (1-q^i)\,\prod_{i=1}^{n-1} (1-uvq^{i-1})(1-u^n q^{\binom{n}{2}})(1-v^n q^{\binom{n}{2}})} . \end{aligned} \qquad 5.12$$

This alternate form of the Hilbert series suggests taking as quasi-generators of the ring of UVX invariants the polynomials

$$\Pi_1, \Pi_2, \ldots, \Pi_n \,;\, \theta_1, \theta_2, \ldots, \theta_{n-1} \,;\, \Phi, \Psi \qquad 5.13$$

and as separators

$$1, \theta_n, \theta_n^2, \ldots, \theta_n^{n-1}. \qquad 5.14$$

This is essentially the contents of Theorem I.8. To establish it we need only use the singly graded Hilbert series in I.4 which now can be rewritten in the form

$$F_{UVX}(q) = \frac{1+q^{n+1}+\big(q^{n+1}\big)^2+\cdots+\big(q^{n+1}\big)^{n-1}}{\prod_{i=1}^n (1-q^i)\,\prod_{i=1}^{n-1} (1-q^{i+1})(1-q^{\binom{n+1}{2}})(1-q^{\binom{n+1}{2}})}. \qquad 5.15$$

Now let us recall that we obtained, as a by product of the proof of of Theorem I.6, that the collection in 5.9, namely

$$\Big\{ \Phi^r \Psi^s \Pi_1^{r_1} \Pi_2^{r_2} \cdots \theta_n^{r_n} \theta_1^{s_1} \theta_2^{s_2} \cdots \theta_n^{s_n} \;: r, s, r_i, s_j \geq 0 \Big\} \qquad 5.16$$

spans the space of $UVX$ invariants. Thus in view of 5.15 to prove Theorem I.8 we need only show that all the powers $\theta_n^m$ (for $m \geq n$) can be removed from 5.16 without affecting the spanning property. Now this is an immediate consequence of the following

## Proposition 5.2

*There are polynomials* $a_0, a_1, \ldots a_{n-1}$ *in* $\Pi_1, \ldots, \Pi_n; \theta_1, \ldots, \theta_{n-1}; \Phi, \Psi$ *such that*

$$\theta_n = a_0 + a_1\theta_n + a_2\theta_n^2 + \ldots + a_{n-1}\theta_n^{n-1}$$

## Proof

We have seen in 5.10 that

$$\Phi\Psi = det\Big\|\theta_{i+j-1}\Big\|_{1\leq i,j\leq n}. \qquad 5.17$$

We have also seen in the proof of Proposition 5.1 that from the Caley-hamilton Theorem it follows that for $i+j-1 > n$ we have

$$\theta_{i+j-1} = s_{i+j-n}\theta_n + t_{i+j-n}$$

with the $s_m$ and $t_m$ polynomials in $\Pi_1,\ldots,\Pi_n;\theta_1,\ldots,\theta_{n-1}$. Using this in 5.17 gives

$$\Phi\psi \;=\; \det\begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_{n-2} & \theta_{n-1} & \theta_n \\ \theta_2 & \theta_3 & \cdots & \theta_{n-1} & \theta_{n-1} & \theta_n \\ \theta_3 & \theta_4 & \cdots & \theta_n & \theta_{n-1} & \theta_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \theta_n & s_1\theta_n+t_1 & \cdots & s_{n-3}\theta_n+t_{n-3} & s_{n-2}\theta_n+t_{n-2} & s_{n-1}\theta_n+t_{n-1} \end{bmatrix}$$

and we see, by expansion with respect to the first row, that all terms of this determinant except the term coming from second diagonal are of degree at most $n-1$ in $\theta_n$. This proves the result and completes the proof of Theorem I.8.

## Remark 5.2

Surprisingly, it is possible to establish Theorem I.8 without making use of the trigraded Hilbert series, and thus also obtain the identity in 5.4 itself as a by-product. To obtain such a proof we need establish the spanning property of the collection in 5.15 without using 5.4. Now note that this spanning property would itself be a consequence of Theorem I.7. Now Theorem I.7 can be established without using 5.4 by giving a 5.4 independent proof that the collection

$$\mathcal{B}^+ \;=\; \Big\{(\Gamma^+)^a(\Gamma^-)^b\Pi_1^{r_1}\Pi_2^{r_2}\cdots\Pi_n^{r_n}\theta_1^{s_1}\theta_2^{s_2}\cdots\theta_n^{s_n}\;:a=0,1\,;\,b\geq 0\,;\,r_i,s_j\geq 0\Big\} \qquad 5.18$$

is independent. More precisely, we need only show that the identity

$$F(\Pi_1,\ldots,\Pi_n;\theta_1,\ldots,\theta_n,\Phi-\Psi)\;+\;(\Phi+\Psi)\,G(\Pi_1,\ldots,\Pi_n;\theta_1,\ldots,\theta_n,\Phi-\Psi)\;=\;0 \qquad 5.19$$

with $F$ and $G$ polynomials in their arguments forces $F$ and $G$ to identically vanish.

Since the 5.4 independent proof of this result is not as simple nor as elementary as our previous proofs we will only give a brief sketch of the argument.

The idea is to show that 5.19 implies the vanishing of $F$ and $G$ even when we set $x_{i,j}=0$ for all $i\neq j$ and set $x_{i,i}=x_i$ for $1\leq i\leq n$. Note that these choices give

$$\Pi_k=\sum_{i=1}^n x_i^k\,,\qquad \theta_k\;=\;\sum_{i=1}^n u_iv_ix_i^k\,,\qquad \Phi=u_1u_2\cdots u_n\Delta(x)\,,\qquad \Psi=v_1v_2\cdots v_n\Delta(x) \qquad 5.20$$

with $\Delta(x)$ the Vandermonde determinant in $x_1,x_2,\ldots,x_n$. So that 5.18 can now be rewritten in the form

$$\begin{aligned} &f(x_1,\ldots,x_n\,;\,u_1v_1,\ldots,u_nv_n\,;\,u_1\cdots u_n-v_1\cdots v_n)\,+\\ &\qquad+\;(u_1\cdots u_n+v_1\cdots v_n)\,g(x,\ldots,x_n\,;\,u_1v_1,\ldots,u_nv_n\,;\,u_1\cdots u_n-v_1\cdots v_n)\;=\;0 \end{aligned} \qquad 5.21$$

Moreover, the relations

$$u_iv_i \;=\; \frac{1}{\Delta(x)}\sum_{i=1}^n \theta_i\,h_{i,j}(x)$$

which can be obtained by inverting the Vandemonde matrix, can be used to show that the vanishing of $f$ and $g$ forces the vanishing of $F$ and $G$.

This reduces us to showing that 5.21 forces the vanishing of $f$ and $g$.

To this end note that we also have the relation

$$(u_1 \cdots u_n + v_1 \cdots v_n))^2 \;=\; (u_1 \cdots u_n - v_1 \cdots v_n))^2 + 4u_1v_1u_2v_2 \dots u_nv_n \qquad 5.22$$

Furthermore, it is easy to show, (by computing the Jacobian), that the $2n+1$ polynomials

$$x_1, \dots, x_n \, ; \, u_1v_1, \dots, u_nv_n \, ; \, u_1 \cdots u_n + v_1 \cdots v_n$$

form a regular sequence in the ring $\mathbb{Q}[x_1,\dots,x_n;u_1,\dots,u_n;v_1,\dots,v_n]$. Thus, if $z_1, z_2, \dots, z_n$ and $\gamma, \delta$ are indeterminates then 5.21 and 5.22 are equivalent to

$$f(x_1,\dots,x_n\,;\,z_1,\dots,z_n\,;\,\gamma) \,+\, \delta\, g(x_1\dots,x_n\,;\,z_1,\dots,z_n\,;\,\gamma) \;=\; 0 \qquad \text{and} \qquad \delta^2 \;=\; \gamma^2 + 4z_1z_2\cdots z_n.$$

This would say that the rational function

$$\nu(x_1\dots,x_n\,;\,z_1,\dots,z_n\,;\,\gamma) \;=\; -\frac{f(x_1,\dots,x_n\,;\,z_1,\dots,z_n\,;\,\gamma)}{g(x_1,\dots,x_n\,;\,z_1,\dots,z_n\,;\,\gamma)}$$

satisfies

$$\nu(x_1\dots,x_n\,;\,z_1,\dots,z_n\,;\,\gamma)^2 \;=\; \gamma^2 + 4z_1z_2\cdots z_n.$$

for all $(x_1,\dots,x_n;z_1,\dots,z_n;\gamma)$ for which it is defined. This is impossible, Thus $f$ and $g$ must vanish identically and the desired independence of the collection in 5.18 necessarily follows.

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