Shift Differential Operators
in
The Theory of m-Quasi-Invariants

by

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I. Preamble

Let $X_n = \{x_1, x_2, \ldots, x_n\}$ and set $\mathbf{R} = \mathbb{Q}[X_n]$. For a polynomial $P(x) = P(x_1, x_2, \ldots, x_n) \in \mathbf{R}$ and $\sigma \in S_n$ we set

$$\sigma P(x_1, x_2, \ldots, x_n) \;=\; P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n})$$

we shall also denote by $P[\partial_x]$ the differential operator

$$P(\partial_x) \;=\; P(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})\,.$$

The transposition that interchanges x_i and x_j will be denoted s_{ij}. It is easily shown that for any $P \in \mathbf{R}$ and $1 \leq i < j \leq n$ we have the factorization

$$(1 - s_{ij})P(x) \;=\; (x_i - x_j)^{1+2r} P_{ij}(x) \qquad \text{I.1}$$

with $r \geq 0$, $P_{ij}(x)$ prime with $(x_i - x_j)$ and symmetric in x_i, x_j.

This given, a polynomial $P(x) \in \mathbb{Q}[X_n]$ is said to be "*m-quasi-invariant*" if and only if the difference

$$(1 - s_{ij})P(x)$$

is divisible by $(x_i - x_j)^{2m+1}$. The space of m-quasi-invariant polynomials in $x_1, x_2, \ldots, x_n$ will here and after be denoted "$\mathcal{QI}_m[X_n]$" or briefly "$\mathcal{QI}_m$". Clearly $\mathcal{QI}_m$ is a vector space over $\mathbb{Q}$, moreover the simple identity

$$(1 - s_{ij})\, PQ \;=\; ((1 - s_{ij})\ P)Q \;+\; (s_{ij}P)(1 - s_{ij})\ Q \qquad \text{I.2}$$

shows that $\mathcal{QI}_m$ is also a ring. Note that we have the inclusions

$$\mathbb{Q}[X_n] = \mathcal{QI}_0[X_n] \supset \mathcal{QI}_1[X_n] \supset \mathcal{QI}_2[X_n] \supset \cdots \supset \mathcal{QI}_m[X_n] \supset \cdots \supset \mathcal{QI}_\infty[X_n] = \mathcal{SYM}[X_n]$$

where we have denoted by "$\mathcal{SYM}[X_n]$" the space of symmetric polynomials in $x_1, x_2, \ldots, x_n$.

It was recently proved by Etingof and Ginzburg in [] that for each m and n, $\mathcal{QI}_m[X_n]$ is a free module over $\mathcal{SYM}[X_n]$ or rank $n!$. This may be viewed as a beautiful extension of the well known analogous result for the pair $\mathbb{Q}[X_n]$, $\mathcal{SYM}[X_n]$. Infact the identities derived in [] show a that remarkable multitude of properties connected with the study of this classical pair generalize almost verbatim to each of the pairs $\mathcal{QI}_m[X_n]$, $\mathcal{SYM}[X_n]$. The proofs in [] contain a variety of identities and properties of various differential operators on $\mathbb{Q}[X_n]$ and $\mathcal{QI}_m[X_n]$ that are of independent interest. In these notes we shall endeavour to provide a completely self contained presentation of these developments.

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1. Dunkl operators and their basic properties

Since the factorization in I.1 plays an ubiquitous role in our developments, for sake of completeness, it will be good to establish it at the onset.

Proposition 1.1

For any polynomial $P[X_n]$ and for all for all $1 \leq i < j \leq n$ we have a factorization of the form

$$(1 - s_{ij})P(x) \;\; = \;\; (x_i - x_j)^{2r+1} P_{ij}(x) \qquad 1.1$$

with $r \geq 0$ and $P_{ij}(x)$ prime with $x_i - x_j$. In particular it follows that the divided difference operator

$$\delta_{ij} = \frac{1}{x_i - x_j}(1 - s_{ij})$$

sends polynomials into polynomials symmetric in x_i, x_j.

Proof

Note that for any pair i, j and exponents a, b we have the identities

$$x_i^a x_j^b - x_j^a x_i^b \;\; = \;\; \begin{cases} x_i^a x_j^a (x_i - x_j)(\sum_{r=0}^{b-a-1} x_j^r x_i^{b-a-1-r}) & \text{if } a \leq b \;\;, \\[2ex] x_i^b x_j^b (x_i - x_j)(\sum_{r=0}^{a-b-1} x_i^r x_j^{a-b-1-r}) & \text{if } a > b. \end{cases}$$

This may be rewritten as

$$\frac{x_i^a x_j^b - x_j^a x_i^b}{x_i - x_j} \;\; = \;\; \begin{cases} x_i^a x_j^a \, h_{b-a-1}[x_i, x_j] & \text{if } a \leq b \;\;, \\[2ex] x_i^b x_j^b h_{a-b-1}[x_i, x_j] & \text{if } a > b, \end{cases} \qquad 1.2$$

where $h_m[x_i, x_j]$ denotes the so called "homogeneous symmetric function" of degree m in x_i, x_j. This shows that the ratio in 1.2 is always a polynomial that is symmetric in x_i, x_j. For $P \in \mathbb{Q}[X_n]$ we thus have a factorization of the form

$$(1 - s_{ij})P(x) = (x_i - x_j)^{1+r} Q(x)$$

with $Q(x)$ prime with $x_i - x_j$ and $(x_i - x_j)^r Q(x)$ symmetric in x_i, x_j. Now if r were an odd number then we would have

$$(x_i - x_j)^r (1 + s_{ij}) Q(x) \;\; = \;\; (1 - s_{ij})(x_i - x_j)^r Q(x) \;\; = \;\; 0$$

which implies that

$$(1 + s_{ij}) Q(x) \Big|_{x_j \to x_i} \;\; = \;\; 2\, Q(x) \Big|_{x_j \to x_i} \;\; = \;\; 0$$

and this forces $Q(x)$ to have $x_i - x_j$ as a factor. This contraddiction forces r to be even and completes our proof.

It follows from I.2 that the operators δ_{ij} satisfy the "*Leibniz*" formula

$$\delta_{ij}\, PQ \;\; = \;\; (\delta_{ij} P) Q \;\; + \;\; (s_{ij} P) \delta_{ij} Q \qquad 1.3$$

The Dunkl operators are defined by setting for $1 \leq i \leq n$

$$\nabla_i(m) \;=\; \partial_{x_i} - m \sum_{j=1}^{n}{}^{(i)} \tfrac{1}{x_i - x_j}\,(1 - s_{ij}) \qquad 1.4$$

where the symbol "$\sum^{(i)}$" is to indicate that the sum omits the term $j = i$.

For notational brevity, when the value of m is not an issue, we shall simply write ∇_r for $\nabla_r(m)$. It will also be convenient to set

$$\theta_i \;=\; \sum_{j=1}^{n}{}^{(i)} \tfrac{1}{x_i - x_j}\,(1 - s_{ij}) \;=\; \sum_{j=1}^{n}{}^{(i)} \partial_{ij} \qquad 1.5$$

and write

$$\nabla_i(m) \;=\; \partial_{x_i} - m\;\theta_i\,. \qquad 1.6$$

These operators have remarkable properties. To begin note that we have

Proposition 1.2

For any $\sigma \in S_n$

$$a)\quad \sigma\;\partial_{x_i}\,\sigma^{-1} \;=\; \partial_{x_{\sigma_i}}\,, \qquad\qquad b)\quad \sigma\;\partial_{ij}\,\sigma^{-1} \;=\; \partial_{\sigma_i\sigma_j}\,, \qquad 1.7$$

thus in particular

$$a)\quad \sigma\;\theta_i\,\sigma^{-1} \;=\; \theta_{\sigma_i}\,, \qquad\qquad b)\quad \sigma\;\nabla_i\,\sigma^{-1} \;=\; \nabla_{\sigma_i} \qquad 1.8$$

Proof

Note that

$$\sigma\;\partial_{x_i}\,\sigma^{-1}\,x_{\sigma_i}^a \;=\; \sigma\;\partial_{x_i}\;x_i^a \;=\; a\;\sigma\,x_i^{a-1} \;=\; a\;x_{\sigma_i}^{a-1} \;=\; \partial_{x_{\sigma_i}}x_{\sigma_i}^a\,.$$

This proves 1.7 a). Note next that

$$\begin{aligned}\sigma\;\partial_{ij}\,\sigma^{-1} \;&=\; \sigma\tfrac{1}{x_i - x_j}\;(1 - s_{ij})\sigma^{-1}\\ &=\; \tfrac{1}{x_{\sigma_i} - x_{\sigma_j}}\;\sigma(1 - s_{ij})\sigma^{-1}\\ &=\; \tfrac{1}{x_{\sigma_i} - x_{\sigma_j}}\;(1 - s_{\sigma_i\sigma_j}) \;=\; \partial_{\sigma_i,\sigma_j}\,,\end{aligned}$$

This proves 1.7 b). This given, 1.8 a) and b) then follow immediately from 1.5 and 1.6 and our proof is thus complete.

What makes the Dunkl operators remarkable is that they commute. More precisely we have the following surprising identities

Theorem 4.1

For any pair $1 \leq a < b \leq n$

$$a) \quad \partial_{x_a}\theta_b \, - \, \theta_b\partial_{x_a} \, = \, \partial_{x_b}\theta_a \, - \, \theta_a\partial_{x_b} \, , \qquad b) \quad \theta_a\theta_b \, = \, \theta_b\theta_a \tag{1.9}$$

and the latter are equivalent to the validity of

$$\nabla_a(m)\,\nabla_b(m) \;=\; \nabla_b(m)\,\nabla_a(m) \qquad \text{(for all } m,a,b) \, . \tag{1.10}$$

Proof

Note that using 1.6 we can write

$$\nabla_a(m)\nabla_a(m) \;=\; (\partial_{x_a} - m\,\theta_a)(\partial_{x_b} - m\,\theta_b) \;=\; \partial_{x_a}\partial_{x_b} - m\,\partial_{x_a}\theta_b - m\,\theta_a\partial_{x_b} + m^2\theta_a\theta_b$$

similarly we get

$$\nabla_b(m)\nabla_a(m) \;=\; (\partial_{x_b} - m\,\theta_b)(\partial_{x_a} - m\,\theta_a) \;=\; \partial_{x_b}\partial_{x_a} - m\,\partial_{x_b}\theta_a - m\,\theta_b\partial_{x_a} + m^2\theta_b\theta_a$$

Thus we see that in order for 1.10 to be valid for all m, a, b it is necessary and sufficient that 1.9 a) and b) be valid for all a, b. Now to prove 1.9 a) and b), in view of Proposition 1.1, we only need to prove the identities

$$a) \quad \partial_{x_1}\theta_2 \, - \, \theta_2\partial_{x_1} \, = \, \partial_{x_2}\theta_1 \, - \, \theta_1\partial_{x_2} \, , \qquad b) \quad \theta_1\theta_2 \, = \, \theta_2\theta_1 \, . \tag{1.11}$$

For the same reason, to prove 1.11 a) & b) we need only verify that we have

$$a) \quad \partial_{x_1}\theta_2 \, - \, \theta_2\partial_{x_1} \, = \, s_{12}\,\big(\partial_{x_1}\theta_2 \, - \, \theta_2\partial_{x_1}\big)s_{12} \, , \qquad b) \quad \theta_1\theta_2 \, = \, s_{12}\,\Big(\theta_1\theta_2\Big)s_{12}. \tag{1.12}$$

We will start with 1.12 a). So choose $P \in \mathbb{Q}[X_n]$ and note that

$$\begin{aligned} \partial_{x_1}\theta_2\,P \;&=\; \partial_{x_1}\sum_{i=1}^{n}{}^{(2)}\frac{1}{x_2 - x_i}(1 - s_{2i})P \\ &=\; \partial_{x_1}\frac{1}{x_2 - x_1}(1 - s_{21})P \;+\; \sum_{i=3}^{n}\frac{1}{x_2 - x_i}(1 - s_{2i})\,\partial_{x_1}P \\ &=\; \frac{1}{(x_2 - x_1)^2}(1 - s_{21})P \;+\; \frac{1}{x_2 - x_1}(\partial_{x_1}P - s_{21}\partial_{x_2}P) \;+\; \sum_{i=3}^{n}\frac{1}{x_2 - x_i}(1 - s_{2i})\,\partial_{x_1}P \, . \end{aligned} \tag{1.13}$$

Similarly we get

$$\begin{aligned} \theta_2\partial_{x_1}\,P \;&=\; \sum_{i=1}^{n}{}^{(2)}\frac{1}{x_2 - x_i}(1 - s_{2i})\partial_{x_1}P \\ &=\; \frac{1}{x_2 - x_1}(1 - s_{21})\partial_{x_1}P \;+\; \sum_{i=3}^{n}\frac{1}{x_2 - x_i}(1 - s_{2i})\,\partial_{x_1}P \end{aligned} \tag{1.14}$$

Combining 1.13 and 1.14 gives

$$\begin{aligned} \big(\partial_{x_1}\theta_2 - \theta_2\partial_{x_1}\big)\,P \;&=\; \frac{1}{(x_2 - x_1)^2}(1 - s_{21})P \;+\; \frac{1}{x_1 - x_2}\big(s_{21}\partial_{x_2}P - s_{21}\partial_{x_1}P\big) \\ &=\; \frac{1}{(x_2 - x_1)^2}(1 - s_{21})P \;+\; \frac{1}{x_1 - x_2}\big(\partial_{x_1} - \partial_{x_2}\big)s_{21}P \end{aligned}$$

Thus

$$\big(\partial_{x_1}\theta_2 - \theta_2\partial_{x_1}\big) \;=\; \frac{1}{(x_2-x_1)^2}(1-s_{21}) \;+\; \frac{1}{x_1-x_2}\big(\partial_{x_1}-\partial_{x_2}\big)s_{21}\,. \qquad 1.15$$

Which easily shows that the left hand side is invariant under conjugation by s_{21}, proving 1.16 a) and a fortiori establishing 1.13 a). The proof of 1.13 b) is more elaborate. To begin let us set

$$\begin{aligned} A_1 &= \frac{1}{x_1-x_2}(1-s_{12})\,, &\qquad B_1 &= \sum_{i=3}^{n}\frac{1}{x_1-x_i}(1-s_{1i}) \\ A_2 &= \frac{1}{x_2-x_1}(1-s_{12})\,, &\qquad B_2 &= \sum_{i=3}^{n}\frac{1}{x_2-x_i}(1-s_{2i}) \\ \theta_1 &= A_1+B_1\,, &\qquad \theta_2 &= A_2+B_2\,. \end{aligned} \qquad 1.16$$

So that

$$\theta_1\theta_2 \;=\; A_1A_2 + A_1B_2 + B_1A_2 + B_1B_2\,. \qquad 1.17$$

Now we have

$$\begin{aligned} A_1A_2 &= \frac{1}{x_1-x_2}(1-s_{12})\frac{1}{x_2-x_1}(1-s_{12}) \\ &= \frac{1}{x_1-x_2}\frac{1}{x_2-x_1}(1-s_{12}) - \frac{1}{x_1-x_2}\frac{1}{x_1-x_2}(1-s_{12})s_{12} \\ &= -\frac{1}{(x_1-x_2)^2}(1-s_{12}) - \frac{1}{(x_1-x_2)^2}(1-s_{12})s_{12} \\ &= -\frac{1}{(x_1-x_2)^2}(1-s_{12})(1+s_{12}) \;=\; 0\,, \end{aligned} \qquad 1.18$$

$$\begin{aligned} A_1B_2 &= \frac{1}{x_1-x_2}(1-s_{12})\sum_{i=3}^{n}\frac{1}{x_2-x_i}(1-s_{2i}) \\ &= \frac{1}{x_1-x_2}\sum_{i=3}^{n}\frac{1}{x_2-x_i}(1-s_{2i}) - \frac{1}{x_1-x_2}\sum_{i=3}^{n}\frac{1}{x_1-x_i}(1-s_{1i})s_{12}\,, \end{aligned} \qquad 1.19$$

and

$$\begin{aligned} B_1A_2 &= \sum_{i=3}^{n}\frac{1}{x_1-x_i}(1-s_{1i})\frac{1}{x_2-x_1}(1-s_{21}) \\ &= \sum_{i=3}^{n}\frac{1}{x_1-x_i}\frac{1}{x_2-x_1}(1-s_{21}) - \sum_{i=3}^{n}\frac{1}{x_1-x_i}\frac{1}{x_2-x_i}(1-s_{2i})s_{1i}\,. \end{aligned} \qquad 1.20$$

To complete the picture, we break up B_1B_2 into two parts, the first $(B_1B_2)^*$ (which is clearly preserved by conjugation by s_{12}) and a remainder B_{12}. More precisely, we have

$$(B_1B_2)^* \;=\; \sum_{i=3}^{n}\sum_{j=3}^{n}\frac{\chi(i\neq j)}{(x_1-x_i)(x_2-x_j)}(1-s_{1i})(1-s_{2j}) \qquad 1.21$$

and

$$\begin{aligned} B_{12} &= \sum_{i=3}^{n}\frac{1}{(x_1-x_i)}(1-s_{1i})\frac{1}{(x_2-x_i)}(1-s_{2i}) \\ &= \sum_{i=3}^{n}\frac{1}{(x_1-x_i)}\frac{1}{(x_2-x_i)}(1-s_{2i}) - \sum_{i=3}^{n}\frac{1}{(x_1-x_i)}\frac{1}{(x_2-x_1)}(1-s_{21})s_{1i} \end{aligned} \qquad 1.22$$

Note that we may write

$$\frac{1}{(x_1-x_i)}\frac{1}{(x_2-x_i)} \;=\; \frac{1}{(x_1-x_2)}\Big(\frac{1}{x_2-x_i}-\frac{1}{x_1-x_i}\Big)\,.$$

Using this identity in 1.20 and 1.22 we now have

$$A_1B_2 \;=\; \frac{1}{x_1-x_2}\sum_{i=3}^{n}\frac{1}{x_2-x_i}(1-s_{2i}) \;-\; \frac{1}{x_1-x_2}\sum_{i=3}^{n}\frac{1}{x_1-x_i}(1-s_{1i})s_{12}\,, \tag{1.23}$$

$$B_1A_2 = \frac{1}{x_1-x_2}\Big(-\sum_{i=3}^{n}\frac{1}{x_1-x_i}(1-s_{21}) \;-\; \sum_{i=3}^{n}\big(\frac{1}{x_2-x_i}-\frac{1}{x_1-x_i}\big)(1-s_{2i})s_{1i}\Big)\,, \tag{1.24}$$

$$B_{12} = \frac{1}{x_1-x_2}\Big(\sum_{i=3}^{n}\big(\frac{1}{x_2-x_i}-\frac{1}{x_1-x_i}\big)(1-s_{2i}) \;+\; \sum_{i=3}^{n}\frac{1}{x_1-x_i}(1-s_{21})s_{1i}\Big)\,. \tag{1.25}$$

Combining 1.17, 1.23,1.24 and 1.25 we get

$$\begin{aligned}\theta_1\theta_2-(B_1B_2)^* \;=&\\ &\frac{1}{x_1-x_2}\Big(\sum_{i=3}^{n}\frac{1}{x_1-x_i}\big(-(1-s_{1i})s_{12}-(1-s_{21})-(1-s_{2i})+(1-s_{2i})s_{1i}+(1-s_{21})s_{1i}\big)\Big)\\ &\qquad+\;\frac{1}{x_1-x_2}\Big(\sum_{i=3}^{n}\frac{1}{x_2-x_i}\big((1-s_{2i})-(1-s_{2i})s_{1i}+(1-s_{2i})\big)\Big)\,.\end{aligned}$$

Carrying out the simplifications we finally obtain

$$\begin{aligned}\theta_1\theta_2 \;=\; (B_1B_2)^* \;+\; &\frac{1}{x_1-x_2}\Big(\sum_{i=3}^{n}\frac{1}{x_1-x_i}\big(-2+2s_{1i}+s_{2i}-(1,i,2)\big)\Big)\\ &\qquad+\;\frac{1}{x_1-x_2}\Big(\sum_{i=3}^{n}\frac{1}{x_2-x_i}\big(2-2s_{2i}-s_{1i}+(1,2,i)\big)\Big)\end{aligned}$$

where $(1,i,2)$ and $(1,2,i)$ customarily represent 3-cycles. The expression on the right hand side of this identity makes it quite obvious that conjugation by s_{12} preserves $\theta_1\theta_2$. This proves 1.9 b) and completes our proof.

The commutativity in 1.10 makes it unambiguous the evaluation of a polynomial $P(x_1,x_2,\ldots,x_n)$ at the operators $\nabla_1,\nabla_2,\ldots,\nabla_n$. The result is an operator we denote by $P(\nabla_1,\nabla_2,\ldots,\nabla_n)$ or simply $P(\nabla)$. We may also write $P\big(\nabla(m)\big)$ if the dependence on m is an issue. It will be convenient to refer to $P(\nabla)$ as the "*Dunklized P* ". The operators $P(\nabla)$ belong to a family of operators which combine differentiation and S_n action which we call "*Shift-Differential operators* ". Our study of m-quasi-invariantss requires the use a variety of these operators. In the next section we give definitions and derive some of the basic properties.

2. Basics on Shift-Differential operators

Here and after we let $\mathcal{R}_n$ the field of rational functions in $x_1, x_2, \ldots, x_n$. In symbols

$$\mathcal{R}_n \;=\; \{a = N(x)/D(x) : N(x), D(x) \in \mathbb{Q}[X_n]\}$$

It will also be convenient to denote by $\mathcal{SR}_n(x)$, the ring generated by the variables x_i together with the fractions $1/(x_i - x_j)$. In symbols

$$\mathcal{SR}_n(x) \;=\; \mathbb{Q}\Big[x_i,\, \tfrac{1}{x_i - x_j} \;:\; 1 \le i \le n\,, 1 \le i < j \le n\Big]$$

Similarly, we shall also set

$$\mathcal{SR}_n(y) \;=\; \mathbb{Q}\Big[y_i,\, \tfrac{1}{y_i - y_j} \;:\; 1 \le i \le n\,, 1 \le i < j \le n\Big]\,.$$

Clearly, every rational function in $f(x) \in \mathcal{SR}_n(x)$ has an expansion of the form

$$f(x) \;=\; \sum_{q=\{q_{ij}\}} a_q(x) \;=\; \sum_{q=\{q_{ij}\}} \frac{P_q(x)}{\prod_{1\le i<j\le n}(x_i - x_j)^{q_{ij}}}\,, \qquad 2.1$$

where each $P_q(x)$ is a polynomial. A term

$$a_q(x) \;=\; \frac{P_q(x)}{\prod_{1\le i<j\le n}(x_i - x_j)^{q_{ij}}}$$

with $P_q(x)$ a homogeneous polynomial is called "*homogeneous* " and its degree is simply taken to be

$$deg(a_p) \;=\; deg(P_q) - \sum_{ij} q_{ij} \qquad 2.2$$

Accordingly we shall say that $f(x)$ is homogeneous of degree d if each term $a_q(x)$ is homogeneous of degree d. It is easily seen that this gives $\mathcal{SR}_n(x)$ the structure of a graded algebra. That is, denoting by $\mathcal{H}_d(\mathcal{SR}_n(x))$ the subspace of elements of $\mathcal{SR}_n(x)$ of degree d we have the relations

$$\mathcal{H}_{d_1}(\mathcal{SR}_n(x))\mathcal{H}_{d_2}(\mathcal{SR}_n(x)) \;\subseteq\; \mathcal{H}_{d_1+d_2}(\mathcal{SR}_n(x))$$

and the direct sum decomposition

$$\mathcal{SR}_n(x) \;=\; \bigoplus_{d=-\infty}^{+\infty} \mathcal{H}_d(\mathcal{SR}_n(x))\,.$$

The differential operators we shall work with will be of the form

$$a(x, \partial_x) \;=\; \sum_p a_p(x)\partial_x^p \qquad (\text{ with } \partial_x^p = \partial_{x_1}^{p_1}\partial_{x_2}^{p_2}\cdots\partial_{x_n}^{p_n}) \qquad 2.3$$

with each $a_p(x) \in \mathcal{SR}_n(x)$. The family of these operators will be denoted "$\mathcal{D}_n$". The notion of degree defined by 2.2 makes also $\mathcal{D}_n$ into a graded algebra. We shall say that $a(x, \partial_x)$, as a differential operator, has degree d if each coefficient $a_p(x)$ is homogeneous and we have

$$deg(a_p) \;=\; |p| - d.$$

Note that for any $\sigma \in S_n$ and any $a(x, \partial_x)$ as in 2.3 we have

$$\sigma\, a(x, \partial_x)\, \sigma^{-1} \;=\; \sum_{|p| \leq N} a_p(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_2})\, \partial_{x_{\sigma_1}}^{p_1} \partial_{x_{\sigma_2}}^{p_2} \cdots \partial_{x_{\sigma_n}}^{p_n}\,.$$

Thus S_n acts on $\mathcal{D}_n$ by conjugation. Accordingly we shall call an operator $a(x, \partial_x) \in \mathcal{D}_n$ "*S_n-invariant*" or simply "*symmetric*" if

$$\sigma\, a(x, \partial_x)\, \sigma^{-1} \;=\; a(x, \partial_x) \qquad\qquad (\forall\;\, \sigma \in S_n\,)$$

When $a(x, \partial_x)$ is not known to be symmetric it is convenient to set

$$\sigma\, a(x, \partial_x)\, \sigma^{-1} \;=\; {}^{\sigma}a(x, \partial_x). \tag{2.4}$$

A "*shift-differential operator*" is an operator A acting $\mathcal{SR}_n(x)$ which can be expressed in the form

$$A \;=\; \sum_{\alpha \in S_n} a_\alpha(x, \partial_x)\alpha \qquad\qquad (\text{with each } a_\alpha(x, \partial_x) \in \mathcal{D}_n\,) \tag{2.5}$$

The family of shift-differential operators will be denoted "$\mathcal{SD}_n$". It is easily seen that $\mathcal{SD}_n$ is a ring. Indeed if A is as in 2.5 and

$$B \;=\; \sum_{\beta \in S_n} b_\beta(x, \partial_x)\beta \tag{2.6}$$

then we may write

$$\begin{aligned} AB \;&=\; \sum_{\alpha \in S_n} a_\alpha(x, \partial_x)\alpha \sum_{\beta \in S_n} b_\beta(x, \partial_x)\beta \\ &=\; \sum_{\alpha \in S_n} \sum_{\beta \in S_n} a_\alpha(x, \partial_x) \left(\alpha\, b_\beta(x, \partial_x)\alpha^{-1}\right) \alpha\beta \\ &=\; \sum_{\alpha \in S_n} \sum_{\beta \in S_n} a_\alpha(x, \partial_x)\; {}^{\alpha}b_\beta(x, \partial_x)\, \alpha\beta \\ &=\; \sum_{\gamma \in S_n} \Big(\sum_{\alpha\beta = \gamma} a_\alpha(x, \partial_x)\; {}^{\alpha}b_\beta(x, \partial_x) \Big)\, \gamma\,. \end{aligned} \tag{2.7}$$

A shift-differential operator A as in 2.5 is called "*symmetric*" if and only if

$$\sigma A \sigma^{-1} \;=\; A \qquad\qquad (\text{for all } \sigma \in S_n)$$

Note that this requires that

$$\sum_{\alpha \in S_n} {}^{\sigma}a_\alpha(x, \partial_x)\sigma\alpha\sigma^{-1} \;=\; \sum_{\alpha \in S_n} a_\alpha(x, \partial_x)\alpha \tag{2.8}$$

or equivalently

$$\sum_{\alpha\in S_n} {}^{\sigma}a_{\sigma^{-1}\alpha\sigma}(x,\partial_x)\alpha \;=\; \sum_{\alpha\in S_n} a_\alpha(x,\partial_x)\alpha\,.$$

There is a natural map $\Gamma:\mathcal{SD}_n\to\mathcal{D}_n$ we call the "*Forgetting Map*" that is simply obtained by setting

$$\Gamma A \;=\; \Gamma\sum_{\alpha\in S_n} a_\alpha(x,\partial_x)\alpha \;=\; \sum_{\alpha\in S_n} a_\alpha(x,\partial_x)\,. \qquad 2.9$$

It is important to note that

Proposition 2.1

ΓA is symmetric if and only if

$$\sum_{\alpha\in S_n} {}^{\sigma}a_\alpha(x,\partial_x) \;=\; \sum_{\alpha\in S_n} a_\alpha(x,\partial_x) \qquad (\textit{for all } \sigma\in S_n) \qquad 2.10$$

In particular if A is symmetric then ΓA is symmetric

Proof

From 2.9 we see that

$$\sigma\,\Gamma A\,\sigma^{-1} \;=\; A$$

if and only if

$$\sum_{\alpha\in S_n} \sigma\, a_\alpha(x,\partial_x)\sigma^{-1} \;=\; \sum_{\alpha\in S_n} a_\alpha(x,\partial_x) \qquad (\text{for all } \sigma\in S_n)$$

and this is 2.10. Finally, if A is symmetric then applying Γ to both sides of 2.8 gives 2.10 and completes our proof.

The map Γ is clearly linear but is not multiplicative. Yet it is so in a variety of special cases, an instance in point is given by the following basic fact

Theorem 2.1

If $A,B\in\mathcal{SD}_n$ and ΓB is symmetric then

$$\Gamma AB \;=\; (\Gamma A)(\Gamma B)\,. \qquad 2.11$$

In particular 2.11 will hold true if B itself is symmetric

Proof

Assuming that A and B are given by 2.5 and 2.6 from 2.7 we derive that

$$\Gamma AB \;=\; \sum_{\alpha\in S_n}\sum_{\beta\in S_n} a_\alpha(x,\partial_x)\,{}^{\alpha}b_\beta(x,\partial_x) \;=\; \sum_{\alpha\in S_n} a_\alpha(x,\partial_x)\sum_{\beta\in S_n} {}^{\alpha}b_\beta(x,\partial_x)$$

Thus the assertions are immediate consequences of Proposition 2.1 .

Remark 2.1

We must note that there is a certain asymmetry in this result. In fact, as we shall see 2.11 may fail if only ΓA is known to be symmetric. Much grief can ensue by a use of 2.11 when ΓA is not symmetric.

There is a powerful way of deriving differential operators identities specially when it concerns the action of operators ΓA with $A \in \mathcal{SD}_n$. To present it we need some auxiliary facts and observations concerning the action of differential operators on rational functions.

To begin note that in the case of a single variable x, our operators may be written in the form

$$P\big(x, \tfrac{d}{dx}\big) \;=\; \sum_{k=0}^{d} \frac{P_k(x)}{x^{q_k}} (\tfrac{d}{dx})^k$$

with $P_k(x)$ a polynomial. We thus see that algebra $\mathcal{D}_1$ is generated by $\frac{d}{dx}$ and multiplication by x and $\frac{1}{x}$. In the same vein we see that $\mathcal{D}_n$ is generated by $\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}$ and multiplication by

$$x_1, x_2, \ldots, x_n \quad \text{and} \left\{\tfrac{1}{x_i - x_j}\right\}_{1\leq i<j\leq n}.$$

For convenience, here and after the operator multiplication by a rational function $f(x)$ will be simply denoted "$f(\underline{x})$". We this notation we can state the following basic operator identity.

Theorem 2.2

In the case of a single variable x, if $f(x)$ is a rational functions and $P(x)$ is a polynomial, then

$$P(\tfrac{d}{dx})f(\underline{x}) \;-\; f(\underline{x})P(\tfrac{d}{dx}) \;=\; \sum_{r\geq 1} \frac{1}{r!} f^{(r)}(\underline{x}) P^{(r)}(\tfrac{d}{dx}) \tag{2.12}$$

where

$$f^{(r)}(x) = (\tfrac{d}{dx})^r f(x) \quad \textit{and} \quad P^{(r)}(y) = (\tfrac{d}{dy})^r P(y)\,.$$

.

Proof

We need only verify 2.12 for $P(x) = x^k$. This given, let $g(x)$ be any rational function and note that the Leibnitz formula gives

$$(\tfrac{d}{dx})^k f(\underline{x}) g(x) \;=\; \sum_{r=0}^{k} \binom{k}{r} \big((\tfrac{d}{dx})^r f(x)\big)\big((\tfrac{d}{dx})^{k-r} g(x)\big)$$

Thus

$$\begin{aligned}(\tfrac{d}{dx})^k f(\underline{x}) g(x) \;-\; f(\underline{x})(\tfrac{d}{dx})^k g(x) \;&=\; \sum_{r=1}^{k} \binom{k}{r} \big((\tfrac{d}{dx})^r f(x)\big)\big((\tfrac{d}{dx})^{k-r} g(x)\big) \\ &=\; \sum_{r=1}^{k} \frac{1}{r!} f^{(r)}(\underline{x}) k(k-1)\cdots(k-r+1)(\tfrac{d}{dx})^{k-r} g(x)\big)\end{aligned}$$

and this is simply another way of writing 2.12 with $P(x) = x^k$.

We are now in a position to prove the following surprising fact.

Theorem 2.3

If two differential operators in $\mathcal{D}_n$

$$A(x, \partial_x) \;=\; \sum_{|p|\leq d} a_p(x)\partial_x^p \quad \textit{and} \quad B(x, \partial_x) \;=\; \sum_{|p|\leq d} b_p(x)\partial_x^p$$

agree on symmetric polynomials they necessarily agree on all polynomials

Proof

We need only show that if

$$A(x,\partial_x)\, f(x) = \sum_{|p|\le d} a_p(x)\partial_x^p\, f(x) \;\;=\;\; 0 \tag{2.13}$$

for all symmetric $f(x)$ then the polynomial

$$A(x,y\,) \;=\; \sum_{|p|\le d} a_p(x) y^p$$

vanishes identically. There is nothing to prove if $A(x,y)$ is of degree 0 in y. So we may proceed by induction on the y degree of $A(x,y)$. So let us suppose that the Theorem is true for all $A(x,y\,)$ of y-degree less than d and suppose A satisfies 2.13 for all symmetric f with

$$\sum_{|p|=d} a_p(x)y^p \neq 0 \qquad \text{(for some } d\ge 1\text{)}\,. \tag{2.14}$$

Note that for any symmetric polynomial $P(x)$ then also the operator

$$A(x,\partial_x)P(\underline{x}) \,-\, P(\underline{x})A(x,\partial_x)$$

will kill all symmetric polynomials. Now Theorem 2.2 gives that for all $1\le i\le n$ we have

$$A(x,\partial_x)\underline{x}_i^k \,-\, \underline{x}_i^k A(x,\partial_x) \;\;=\;\; \sum_{r\ge 1} \frac{(k)_r}{r!}\, \underline{x}_i^{k-r} A_i^{(r)}(x,\partial_x) \tag{2.15}$$

where for convenience we have set

$$A_i^{(r)}(x,y) \;\;=\;\; \partial_{y_i}^r A(x,y)\,.$$

Thus if we set

$$B(x,\partial_x) \;\;=\;\; A(x,\partial_x)p_k(\underline{x}) \,-\, p_k(\underline{x})A(x,\partial_x)$$

where $p_k(x)=\sum_{i=1}^n x_i^k$, then it follows from 2.15 that the component of highest y-degree of $B(x,y)$ is

$$k\sum_{i=1}^n x_i^{k-1}\partial_{y_i}\Big(\sum_{|p|=d} a_p(x)y^p\Big)\,.$$

Since the inductive hypothesis forces $B(x,y)=0$ we necessarily must have

$$\sum_{i=1}^n x_i^{k-1}\partial_{y_i}\Big(\sum_{|p|=d} a_p(x)y^p\Big) \;\;=\;\; 0 \qquad\qquad \text{(for } k=1,2,\ldots,n\text{)}$$

and this forces

$$\partial_{y_i}\Big(\sum_{|p|=d} a_p(x)y^p\Big) \;\;=\;\; 0 \qquad\qquad \text{(for } i=1,2,\ldots,n\text{)}$$

which is in plain contraddiction with 2.14. So there can't be a non vanishing operator $A(x,\partial_x)$ which satisfies 2.13 for all symmetric f as asserted.

Theorem 2.3 yields us a crucial tool for identifying the images of Γ.

Theorem 2.4

For $A=\sum_{\alpha\in S_n} a_\alpha(x,\partial_x)\alpha \in \mathcal{A}_n$ and $B(x,\partial_x)\in\mathcal{D}_n$ we have

$$\Gamma A \;=\; B(x,\partial_x) \tag{2.16}$$

if and only if the equality

$$Af(x) \;=\; B(x,\partial_x)f(x) \tag{2.17}$$

holds true for all symmetric $f(x)$

Proof

Theorem 2.3 assures that we have 2.16 if and only if for all symmetric $f(x)$ we have

$$\Gamma Af(x) \;=\; B(x,\partial_x)f(x) \tag{2.18}$$

but when $f(x)$ is symmetric

$$\begin{aligned} Af(x) \;&=\; \sum_{\alpha\in S_n} a_\alpha(x,\partial_x)\,\alpha f \\ &=\; \sum_{\alpha\in S_n} a_\alpha(x,\partial_x)\,f \;=\; \Gamma A\,f(x)\,. \end{aligned}$$

Thus 2.17 is equivalent to 2.18 and 2.16 is forced.

Theorem 2.4 immediately yields the following important identity

Theorem 2.5

For all $m\ge 0$ we have

$$\Gamma\Big(\nabla_1^2(m)+\nabla_2^2(m)+\cdots+\nabla_n^2(m)\Big) \;=\; \Delta_2 \,-\, 2m\sum_{1\le i<j\le n}\frac{1}{x_i-x_j}(\partial_{x_i}-\partial_{x_j}) \tag{2.19}$$

where Δ_2 denotes the ordinary Laplacian

$$\Delta_2 \;=\; \partial_{x_1}^2+\partial_{x_2}^2+\cdots+\partial_{x_n}^2\,.$$

Proof

When $f(x)$ is symmetric, the definitions in 2.3 and 2.4 give

$$\begin{aligned} \nabla_i^2(m)f(x) \;&=\; \nabla_i(m)\partial_{x_i}f(x) \\ &=\; \partial_{x_i}^2 f(x) - m\,\theta_i\partial_{x_i}f(x) \end{aligned}$$

but again the symmetry of $f(x)$ yields that for any $j\neq i$ we have

$$(1-s_{ij})\partial_{x_i}f(x) = \partial_{x_i}f(x) \,-\, s_{ij}\partial_{x_i}s_{ij}f(x) \;=\; \partial_{x_i}f(x) \,-\, \partial_{x_j}f(x)$$

and 2.4 gives

$$\nabla_i^2(m)f(x) \;=\; \partial_{x_i}^2 f(x) \,-\, m \sum_{j=1}^{n}{}^{(i)} \frac{1}{x_i - x_j}(\partial_{x_i} - \partial_{x_j})f(x)\,.$$

Summing these identities we get

$$\sum_{i=1}^{n} \nabla_i^2(m)f(x) \;=\; \sum_{i=1}^{n} \partial_{x_i}^2 f(x) \,-\, m \sum_{i=1}^{n}\sum_{j=1}^{n}{}^{(i)} \frac{1}{x_i - x_j}(\partial_{x_i} - \partial_{x_j})f(x)\,,$$

and Theorem 2.4 guarantees the identity

$$\Gamma \sum_{i=1}^{n} \nabla_i^2(m) \;=\; \Delta_2 \,-\, m \sum_{i=1}^{n}\sum_{j=1}^{n}{}^{(i)} \frac{1}{x_i - x_j}(\partial_{x_i} - \partial_{x_j})\,,$$

But this is simply another way of writing 2.19.

Here and after we shall set

$$L_m \;=\; \Delta_2 - 2\,m\,\mathcal{F}_n \tag{2.20}$$

with

$$\mathcal{F}_n \;=\; \sum_{1\le i<j\le n} \frac{1}{x_i - x_j}(\partial_{x_i} - \partial_{x_j})\,, \tag{2.21}$$

Theorem 2.6

If P is any symmetric polynomial then $P(\nabla)$ and $\Gamma P(\nabla)$ are necessarily also symmetric. Thus it follows that if both P and Q are symmetric then we also have the commutativity relation

$$\big(\Gamma P(\nabla)\big)\big(\Gamma Q(\nabla)\big) \;=\; \big(\Gamma Q(\nabla)\big)\big(\Gamma P(\nabla)\big) \tag{2.22}$$

in particular

$$\big(\Gamma P(\nabla)\big)\,L_m \;=\; L_m\,\big(\Gamma P(\nabla)\big) \tag{2.23}$$

holds true for all symmetric P.

Proof

The identity in 1.8 b) states that for all $\sigma \in S_n$ we have

$$\sigma\ \nabla_i\,\sigma^{-1} \;=\; \nabla_{\sigma_i}$$

This gives

$$\sigma P(\nabla_1, \nabla_2, \ldots, \nabla_n)\sigma^{-1} \;=\; P(\nabla_{\sigma_1}\nabla_{\sigma_2}, \cdots, \nabla_{\sigma_n}))$$

but if P is symmetric, we also have

$$P(\nabla_{\sigma_1}\nabla_{\sigma_2}, \cdots, \nabla_{\sigma_n})) \;=\; P(\nabla_1, \nabla_2, \ldots, \nabla_n)$$

combining these two relations proves that $P(\nabla)$ is symmetric.

Thus if both P an Q are symmetric then

$$\begin{aligned} \text{(by Theorem 2.1)} \quad \big(\Gamma P(\nabla)\big)\big(\Gamma Q(\nabla)\big) &= \Gamma\big(P(\nabla)Q(\nabla)\big) \\ \text{(by commutativity of the Dunkl operators)} &= \Gamma\big(Q(\nabla)P(\nabla)\big) \\ \text{(by Theorem 2.1)} &= \big(\Gamma Q(\nabla)\big)\big(\Gamma P(\nabla)\big) \end{aligned}$$

This proves 2.22 and the identity in 2.23 then follows from Theorem 2.5.

We are now ready to deal with one of the most important actors in the study of the m-quasi-invariants of S_n. This is the "*Opdam*" shift-differential operator

$$O_m \;=\; \Gamma\Pi\big(\nabla(m)\big)\Pi(\underline{x}) \qquad 2.24$$

with $\Pi(x)$ the Vandermonde determinant

$$\Pi(x) = \prod_{1\leq i<j\leq n} (x_i - x_j)\,.$$

We should note that this operator does not change the degree and that it is symmetric. In fact the operator $\Pi\big(\nabla(m)\big)\Pi(\underline{x})$ itself is symmetric. Indeed, the anti-symmetry of the Vandermonde determinant gives that for all $\sigma \in S_n$ we have

$$\sigma\,\Pi\big(\nabla(m)\big)\Pi(\underline{x})\sigma^{-1} = \sigma\,\Pi\big(\nabla(m)\big)\sigma^{-1}\sigma\;\Pi(\underline{x})\sigma^{-1} = (-1)^2\Pi\big(\nabla(m)\big)\Pi(\underline{x}) = \Pi\big(\nabla(m)\big)\Pi(\underline{x}) \qquad 2.25$$

Crudely speaking, the importance of O_m, is due to the fact that it converts L_{m-1} into L_m. It then follows that successive applications of $O_1, O_2, \ldots, O_m$ convert the Laplacian into L_m. In ultimate analysis, this enables us to obtain crucial information about the ring $\mathcal{QI}_m[X_n]$ of m-quasi-invariants from well known properties of the ordinary polynomial ring $\mathcal{QI}_0[X_n] = \mathbb{Q}[X_n]$. It will take a few sections to make all this precise and transparent. For the moment we begin by establishing the remarkable identity that makes all of this possible

Proposition 2.2

For all symmetric polynomials $f(x)$ we have

$$p_2\big(\nabla(m)\big)\Pi(\underline{x})f(x) \;=\; \Pi(\underline{x})\,p_2\big(\nabla(m-1)\big)f(x) \qquad 2.26$$

Proof

To show this note that from 1.6 we derive that

$$\begin{aligned} p_2\big(\nabla(m)\big)\Pi(\underline{x})f(x) &= \sum_{i=1}^{n}\big(\partial_{x_i}^2 - m\,\theta_i\partial_{x_i} - m\,\partial_{x_i}\theta_i + m^2\theta_i^2\big)\Pi(x)f(x) \\ &= A - m\,B - m\,C \;+\; m^2 D \end{aligned} \qquad 2.27$$

where for convenience we have set

$$A = \sum_{i=1}^{n}\partial_{x_i}^2\,\Pi(x)f(x)\,,\quad B = \sum_{i=1}^{n}\theta_i\partial_{x_i}\,\Pi(x)f(x)\,,\quad C = \sum_{i=1}^{n}\partial_{x_i}\theta_i\,\Pi(x)f(x)\,,\quad D = \sum_{i=1}^{n}\theta_i^2\,\Pi(x)f(x)\,. \qquad 2.28$$

We claim that we have

$$\begin{aligned} A &= 2\Pi(x)\mathcal{F}_n f(x) \;+\; \Pi(x)\Delta_2 f(x) \\ B &= 0 \\ C &= 2\Pi(x)\mathcal{F}_n f(x) \\ D &= 0 \end{aligned} \qquad 2.29$$

Acccepting for a moment these relations, from 2.27 we derive that

$$\begin{aligned} p_2\big(\nabla(m)\big)\Pi(\underline{x})f(x) &= 2\Pi(x)\mathcal{F}_n f(x) \;+\; \Pi(x)\Delta_2 f(x) - 2m\Pi(x)\mathcal{F}_n f(x) \\ &= \Pi(x)\big(\Delta_2 - 2(m-1)\mathcal{F}_n\big)f(x) \\ &= \Pi(x)L_{m-1}f(x) \;=\; \Pi(x)p_2(\nabla(m-1))f(x) \end{aligned}$$

and this is 2.26.

To complete our proof we need to establish the relations in 2.29. To begin, note that

$$A \;=\; \big(\Delta_2\,\Pi(x)\big)f(x) \;+\; 2\sum_{i=1}^{n}\big((\partial_{x_i}\Pi(x)\big)\partial_{x_i}f(x) \;+\; \Pi(x)\Delta_2\,f(x)\,. \qquad 2.30$$

Now

$$\begin{aligned} \sum_{i=1}^{n}\big(\partial_{x_i}\Pi(x)\big)\partial_{x_i}f(x) &= \Pi(x)\sum_{i=1}^{n}\tfrac{\partial_{x_i}\Pi(x)}{\Pi(x)}\,\partial_{x_i}f(x) \\ &= \Pi(x)\sum_{i=1}^{n}\big(\partial_{x_i}log\Pi(x)\big)\partial_{x_i}f(x) \\ &= \Pi(x)\sum_{1\le r<s\le n}\;\sum_{i=1}^{n}\tfrac{\partial_{x_i}(x_r-x_s)}{(x_r-x_s)}\,\partial_{x_i}f(x) \\ &= \Pi(x)\sum_{1\le r<s\le n}\tfrac{\partial_{x_r}f(x)-\partial_{x_s}f(x)}{(x_r-x_s)} \;=\; \Pi(x)\mathcal{F}_n\,f(x) \end{aligned}$$

On the other hand, $\Delta_2\,\Pi(x)$ must necessariy vanish since it is alternating and has degree $<\binom{n}{2}$, and 2.30 reduces to

$$A \;=\; 2\,\Pi(x)\mathcal{F}_n\,f(x) \;+\; \Pi(x)\Delta_2\,f(x)$$

This proves the first of 2.29.

Next we derive

$$\begin{aligned} B &= \sum_{i=1}^{n}\theta_i\partial_{x_i}\,\Pi(x)f(x) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}{}^{(i)}\frac{1}{x_i-x_j}(1-s_{ij})\partial_{x_i}\,\Pi(x)f(x) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}{}^{(i)}\frac{1}{x_i-x_j}(\partial_{x_i}\,\Pi(x)f(x) + s_{ij}\partial_{x_i}s_{ij}\,\Pi(x)f(x) \\ &= \sum_{i\neq j}\frac{1}{x_i-x_j}(\partial_{x_i}\Pi(x)f(x) + \partial_{x_j}\Pi(x)f(x)) \;=\; 0 \end{aligned} \qquad 2.31$$

since for any polynomial $Q(x)$ we have

$$\sum_{i\neq j}\frac{1}{x_i-x_j}(\partial_{x_i}Q+\partial_{x_j}Q) \;=\; \sum_{j\neq i}\frac{1}{x_j-x_i}(\partial_{x_j}Q+\partial_{x_j}Q) \;=\; -\sum_{i\neq j}\frac{1}{x_i-x_j}(\partial_{x_i}Q+\partial_{x_j}Q).$$

This proves the second of 2.29.

Next note that the symmetry of $f(x)$ yields that for any polynomial $Q(x)$

$$\frac{1}{x_i-x_j}(1-s_{ij})Q(x)f(x) \;=\; f(x)\frac{1}{x_i-x_j}(1-s_{ij})Q(x))$$

in particular we must also have

$$\theta_i Q(x)f(x) \;=\; f(x)\theta_i Q(x)\,. \tag{2.32}$$

This given

$$\begin{aligned} C \;&=\; \sum_{i=1}^n \partial_{x_i}\theta_i\,\Pi(x)f(x) \;=\; \sum_{i=1}^n \partial_{x_i}f(x)\theta_i\,\Pi(x)\\ &=\; \sum_{i=1}^n(\partial_{x_i}f(x))\theta_i\,\Pi(x) \;+\; f(x)\sum_{i=1}^n\partial_{x_i}\theta_i\,\Pi(x)\end{aligned} \tag{2.33}$$

Now the last term vanishes here since

$$\sum_{i=1}^n\partial_{x_i}\theta_i\,\Pi(x) \;=\; \sum_{i=1}^n\partial_{x_i}\sum_{j=1}^{n\;(i)}\frac{1}{x_i-x_j}(1-s_{ij})\Pi(x) \;=\; 2\sum_{i=1}^n\partial_{x_i}\sum_{j=1}^{n\;(i)}\frac{1}{x_i-x_j}\Pi(x)$$

and the latter is an alternating polynomial of degree $<\binom{n}{2}$. Now for the remaing term in 2.33 we have

$$\begin{aligned}\sum_{i=1}^n(\partial_{x_i}f(x))\theta_i\,\Pi(x) &= \sum_{i=1}^n(\partial_{x_i}f(x))\sum_{j=1}^{n\;(i)}\frac{1}{x_i-x_j}(1-s_{ij})\,\Pi(x)\\ &= 2\Pi(x)\sum_{i=1}^n(\partial_{x_i}f(x))\sum_{j=1}^{n\;(i)}\frac{1}{x_i-x_j} \;=\; 2\Pi(x)\sum_{i=1}^n\sum_{j=1}^{n\;(i)}\frac{\partial_{x_i}f(x)}{x_i-x_j}\\ &= 2\Pi(x)\sum_{i<j}\frac{\partial_{x_i}f(x)-\partial_{x_j}f(x)}{x_i-x_j} \;=\; 2\Pi(x)\,\mathcal{F}_n f(x)\end{aligned}$$

Using these two facts in 2.33 proves the third relation in 2.29.

Finally for D, using 2.32, we get

$$D \;=\; \sum_{i=1}^n\theta_i^2\,\Pi(x)f(x) \;=\; f(x)\sum_{i=1}^n\theta_i^2\,\Pi(x)\,. \tag{2.35}$$

But

$$\begin{aligned}\sum_{i=1}^n\theta_i^2\,\Pi(x) \;&=\; \sum_{i=1}^n\sum_{a=1}^{n\;(i)}\frac{1}{x_i-x_a}(1-s_{ia})\sum_{b=1}^{n\;(i)}\frac{1}{x_i-x_b}(1-s_{ib})\Pi(x)\\ &=\; \sum_{i=1}^n\sum_{a=1}^{n\;(i)}\frac{1}{x_i-x_a}(1-s_{ia})\sum_{b=1}^{n\;(i)}\frac{2\Pi(x)}{x_i-x_b}\\ &=\; \sum_{i=1}^n\sum_{a=1}^{n\;(i)}\frac{1}{x_i-x_a}(1-s_{ia})\frac{2\Pi(x)}{x_i-x_a} \;+\; \sum_{a\neq b\neq i}\frac{1}{x_i-x_a}(1-s_{ia})\frac{2\Pi(x)}{x_i-x_b}\end{aligned}$$

Now the first term vanishes since the ratio $2\Pi(x)/(x_i - x_a)$ is invariant under s_{ia}. This gives

$$\begin{aligned}\sum_{i=1}^{n} \theta_i^2 \, \Pi(x) &= \sum_{a \neq b \neq i} \frac{1}{x_i - x_a}(1 - s_{ia})\frac{2\Pi(x)}{x_i - x_b} \\ &= \sum_{a \neq b \neq i} \frac{2\Pi(x)}{(x_i - x_a)(x_i - x_b)} \;+\; \sum_{a \neq b \neq i} \frac{2\Pi(x)}{(x_i - x_a)(x_a - x_b)}\end{aligned}$$

and this term must also vanish since it is an alternating polynomial of degree $< \binom{n}{2}$. This gives

$$D \;=\; 0\,.$$

Thus all the relations in 2.29 have been established and the proof of 2.26 is now complete.

We shall later see that 2.26 implies a variety of symmetric function identities. But here we must be contented with the following immediate corollary of Proposition 2.2.

Theorem 2.7

$$L_m \, O_m \;=\; O_m \, L_{m-1} \qquad 2.36$$

Proof

Using 2.19 and 2.24 and setting $p_2(x) = \sum_{i=1}^{n} x_i^2$, we start by rewriting 2.36 in the form

$$\big(\Gamma p_2\big(\nabla(m)\big)\big)\Gamma\Pi\big(\nabla(m)\big)\Pi(\underline{x}) \;=\; \big(\Gamma\Pi\big(\nabla(m)\big)\Pi(\underline{x})\big)\Gamma p_2\big(\nabla(m-1)\big)\,.$$

Thus from the symmetry of $\Gamma\Pi\big(\nabla\big)\Pi(\underline{x})$, we derive that

$$\begin{aligned}\big(\Gamma p_2\big(\nabla(m)\big)\big)\,\Gamma\,\Pi\big(\nabla(m)\big)\Pi(\underline{x}) &= \Gamma\, p_2\big(\nabla(m)\big)\Pi\big(\nabla(m)\big)\Pi(\underline{x}) \\ &= \Gamma\,\Pi\big(\nabla(m)\big)p_2\big(\nabla(m)\big)\Pi(\underline{x})\end{aligned}$$

In the same manner, the symmetry of $p_2\big(\nabla(m-1)\big)$ gives

$$\big(\Gamma\Pi\big(\nabla(m)\big)\Pi(\underline{x})\big)\Gamma p_2\big(\nabla(m-1)\big) \;=\; \Gamma\,\Pi\big(\nabla(m)\big)\Pi(\underline{x})\,p_2\big(\nabla(m-1)\big)$$

Thus to prove 2.36 we need only verify that for all symmetric polynomials $f(x)$ we have

$$\Pi\big(\nabla(m)\big)p_2\big(\nabla(m)\big)\Pi(\underline{x})f(x) \;=\; \Pi\big(\nabla(m)\big)\Pi(\underline{x})\,p_2\big(\nabla(m-1)\big)f(x).$$

but this is simply obtained by applying the operator $\Pi(\nabla_(m))$ to both sides of 2.26.

Remark 2.2

Companion to O_m is the operator $\widetilde{O}_m$ defined by setting

$$\widetilde{O}_m \;=\; \Gamma\,\Pi(\underline{x})^{-1}\Pi(\nabla(m))\,. \qquad 2.37$$

We should note that $\widetilde{O}_m$ is well defined since for any symmetric polynomial $f(x)$ the polynomial $\Pi(\nabla(m))f(x)$ is alternating and therefore divisible by $\Pi(x)$. Now the identity in 2.26 immmediately yields that we also have

$$L_{m-1}\widetilde{O}_m \;=\; \widetilde{O}_m\,L_m \qquad 2.38$$

In fact, to show 2.38 we only need to verify that for all symmetric $f(x)$ we have

$$p_2(\nabla(m-1))\Pi(x)^{-1}\Pi(\nabla(m))f(x) \;=\; \Pi(x)^{-1}\Pi(\nabla(m))p_2(\nabla(m))f(x),. \qquad 2.39$$

Setting for a moment

$$\Pi(\nabla(m))f(x) \;=\; \Pi(x)A(x) \qquad\qquad (\text{with } A\in \leftarrow_n\ )\,, \qquad 2.40$$

we derive that

$$\begin{aligned} p_2(\nabla(m-1))\Pi(x)^{-1}\Pi(\nabla(m))f(x) &= p_2(\nabla(m-1))\Pi(\underline{x})^{-1}\Pi(x)A(x)\\ &= \Pi(x)^{-1}\Pi(x)\,p_2(\nabla(m-1))A(x)\\ (\text{by } 2.26\ ) &= \Pi(x)^{-1}p_2(\nabla(m))\Pi(x)\,A(x)\\ (\text{by } 2.40\ ) &= \Pi(x)^{-1}p_2(\nabla(m))\Pi(\nabla(m))f(x) \end{aligned}$$

This proves 2.39 and 2.38.

The operators L_m and O_m have remarkable properties and we will need to dedicate an entire section to them to begin to understand their action on the rational functions in $\mathcal{SR}_n(x)$.

3. The operators $\mathbf{L_m}$, $\mathbf{O_m}$ and $\mathbf{\Omega_m = O_mO_{m-1}\cdots O_1}$

Our ultimate goal in this section is to derive some of the basic properties of the fundamental operator

$$\Omega_m = O_mO_{m-1}\cdots O_1$$

and its relation to m-quasi-invariants. To carry this out we need to prove a few auxiliary results.

To begin there is a useful multivariate version of Leibnitz rule which will play an important role in the study of our operators. It may be stated as follows.

Proposition 3.1

For any $f,g\in\mathcal{SR}_n(x)$ and any multi-exponent $p=(p_1,p_2,\ldots,p_n)$ we have

$$\partial_x^p f(x)g(x) \;=\; \sum_{\alpha+\beta=p}\frac{p!}{\alpha!\ \beta!}\partial_x^\alpha f(x)\partial_x^\beta g(x) \qquad 3.1$$

Proof

The ordinary Leibnitz rule for the variable x_i gives

$$\partial_{x_i}^{p_i} f(x)g(x) \quad = \sum_{\alpha_i+\beta_i=p_i} \frac{p_i!}{\alpha_i!\ \beta_i!} \partial_x^{\alpha_i} f(x) \partial_x^{\beta_i} g(x) \tag{3.2}$$

from which it immediately follows that

$$\Big(\prod_{i=1}^n \partial_{x_i}^{p_i}\Big) f(x)g(x) \quad = \sum_{\alpha_1+\beta_1=p_1} \sum_{\alpha_2+\beta_2=p_2} \cdots \sum_{\alpha_n+\beta_1=p_n} \Big(\prod_{i=1}^n \frac{p_i!}{\alpha_i!\ \beta_i!}\Big)\partial_x^{\alpha_i}\Big)\Big(\prod_{i=1}^n \partial_x^{\alpha_i} f(x)\Big)\Big(\prod_{i=1}^n \partial_x^{\beta_i} g(x)\Big)$$

and 3.1 is is simply a compact way of writing this identity.

It is good to see right on the onset the uses we shall make of this identity. Basically, when we compose two differential operators, we have the problem of rewriting the result in the standard form of differentiations followed by multiplications. One of the basic uses we make of 3.1 is to determine what effect this rewriting has on the leading term of the resulting operator. An instance in point is given by the following identity.

Proposition 3.2

The product $c(x,\partial_x)$ of two differential operators

$$a(x,\partial_x) = f(\underline{x})P(\partial_x) + \sum_{|p|<d_1} f_p(\underline{x})\, \partial_x^p\,, \quad b(x,\partial_x) = g(\underline{x})Q(\partial_x) + \sum_{|q|<d_2} g_q(\underline{x})\, \partial_x^q$$

with $P(y)$,$Q(y)$ homogeneous polynomials of degree d_1 and d_2 respectively may be written in the form

$$c(x,\partial_x) \quad = \quad f(\underline{x})g(\underline{x})P(\partial_x)Q(\partial_x) \quad + \sum_{|r|<d_1+d_2} h_r(\underline{x})\, \partial_x^r \tag{3.3}$$

Proof

Of course we have for any $F(x) \in \mathcal{SR}_n(x)$

$$\begin{aligned} c(x,\partial_x)F(x) \quad = \quad & f(\underline{x})P(\partial_x)g(\underline{x})Q(\partial_x)F(x) + \sum_{|q|<d_2} f(\underline{x})P(\partial_x)\, g_q(\underline{x})\, \partial_x^q F(x) \ + \\ & + \sum_{|p|<d_1} f_p(\underline{x})\, \partial_x^p g(\underline{x})Q(\partial_x)F(x) \ + \sum_{|p|<d_1} \sum_{|q|<d_2} f_p(\underline{x})\, \partial_x^p\, g_q(\underline{x})\, \partial_x^q F(x) \end{aligned}$$

Now we immediately see that after carrying out all differentiations the three last terms in this expression can only contribute to the last term in 3.3 since the amount by which $F(x)$ gets differentiated remains less than d_1+d_2. So we need only study the term

$$f(\underline{x})P(\partial_x)g(\underline{x})Q(\partial_x)F(x)$$

Now if

$$P(y) \quad = \sum_{|p|=d_1} a_p \partial_x^p\,, \quad Q(y) \quad = \sum_{|q|=d_2} b_q\, \partial_x^q$$

then we may write

$$f(\underline{x})P(\partial_x)g(\underline{x})Q(\partial_x)F(x) \;=\; \sum_{|p|=d_1}\sum_{|q|=d_2} a_p b_q\, f(x)\,\partial_x^p\big(g(x)\;\partial_x^q F(x)\big)$$

and 3.1 gives that $f(x)\partial_x^p\big(g(x)\;\partial_x^q F(x)\big)$ is a $\mathbb{Z}$-linear combintion of terms of the form

$$f(x)\big(\partial_x^\alpha\, g(x)\big)\,\partial_x^{\beta+q}F(x)$$

with $\alpha+\beta=p$. This implies that the highest amount of differentiation on $F(x)$ is provided by the term

$$f(x)g(x)\partial_x^{p+q}F(x)$$

This gives that

$$f(\underline{x})P(\partial_x)g(\underline{x})Q(\partial_x)F(x) \;=\; f(x)g(x)\sum_{|p|=d_1}\sum_{|q|=d_2} a_p b_q\,\partial_x^{p+q}F(x) \\ +\;\; (\textit{terms in which } F(x) \textit{ is differentiated less than } d_1+d_2 \textit{ times})$$

This proves that the leading differentiating term in the product of these two operators is precisely as asserted in 3.3.

The following result reveals the nature of the operators $\Gamma P(\nabla(m))$ and provides a useful tool for their final identification.

Proposition 3.3

If $P(x)$ is a homogeneous polynomial of degree d then

$$\Gamma P(\nabla(m)) \;=\; P(\partial_x) \;+\; \sum_{|q|<d} f_q(x)\,\partial_x^q \qquad 3.4$$

where each $f_q(x)$ is of the form

$$f_q(x) \;=\; \sum_{|p|=d-|q|}\frac{c_{pq}}{\prod_{rs}(x_r-x_s)^{p_{rs}}}$$

with scalar coefficients c_{pq}. Thus, if Q is any homogeneous element of $\mathcal{SR}_n(x)$ of degree d_Q then $\Gamma P(\nabla(m))Q\in\mathcal{SR}_n(x)$ and is homogeneous of degree d_Q-d. In particular if $\Gamma P(\nabla(m))Q\in\mathbb{Q}[X_n]$ then it vanishes when $d_Q<d$.

Proof

Clearly it is sufficient to deal with case $P(x)=x^p$ with $|p|=d$. To begin we should note that since each Dunkl operator decreases the degree by 1 the last assertion is immediate for monomials in $\nabla_1,\nabla_2,\ldots,\nabla_n$. As for 3.4 it is trivially true for $d=1$ since for each i we have

$$\Gamma\nabla(m)_i \;=\; \partial_{x_i}\,.$$

Proceeding by induction on d, let us assume 3.4 to be true for monomials of degree $\le d-1$. Now let $|p|=d-1$ and set

$$\Gamma\nabla^p \;=\; \Gamma\nabla_1^{p_1}\nabla_2^{p_2}\cdots\nabla_n^{p_n} \;=\; \partial_x^p \;+\; \sum_{|q|<d} f_q(x)\,\partial_x^q\,. \qquad 3.5$$

Now note that for any symmetric polynomial Q we have

$$\nabla(m)_i\nabla^p Q = \nabla(m)_i\Gamma\nabla^p Q \;=\; \partial_{x_i}\partial_x^p Q + \sum_{|q|<d}\partial_{x_i} f_q(x)\,\partial_x^q Q \;-\; m\,\theta_i\partial_x^p Q \;-\; m\sum_{|q|<d}\theta_i\, f_q(x)\,\partial_x^q Q \qquad 3.6$$

Now we immediately see that the first term is of the desired form since the order of the differential operator given by the additional terms cannot exceed d. To complete our argument we shall deal individually with each of the remaining terms. To begin note that the induction hypothesis asserts that f_q is a linear combination of terms of the form

$$\prod_{1\le r<s\le n}\frac{1}{(x_r-x_s)^{p_{rs}}} \qquad\qquad (\text{with } \textstyle\sum_{1\le r<s\le n} = d-1-|q|\ )$$

since

$$\partial_{x_i}\prod_{1\le r<s\le n}\frac{1}{(x_r-x_s)^{p_{rs}}} \;=\; -\sum_{1\le a<b\le n}\frac{p_{ab}\partial_{x_i}(x_a-x_b)}{(x_a-x_b)^{p_{ab}+1}}\Big(\prod_{\substack{1\le r<s\le n\\ (r,s)\neq(a,b)}}\frac{1}{(x_r-x_s)^{p_{rs}}}\Big)$$

and

$$\partial_{x_i} f_q(x)\,\partial_x^q Q \;=\; \Big((\partial_{x_i} f_q(x))\,\partial_x^q \;+\; f_q(x)\,\partial_{x_i}\partial_x^q\Big)\,Q$$

we see that the first sum has the correct form. Next from the definition in 2.4 we get

$$\begin{aligned}\theta_i\partial_x^p Q \;&=\; \sum_{j=1}^{n}{}^{(i)}\,\frac{1}{x_i-x_j}(1-s_{ij})\partial_x^p Q\\ &=\; \Big(\sum_{j=1}^{n}{}^{(i)}\,\frac{1}{x_i-x_j}(\partial_x^p - s_{ij}\partial_x^p s_{ij})\Big)Q\end{aligned}$$

we see that the third term has the correct form.

Finally, using 2.4 again the last term expands to

$$\begin{aligned}\theta_i\, f_q(x)\,\partial_x^q Q \;&=\; \sum_{j=1}^{n}{}^{(i)}\,\frac{1}{x_i-x_j}(1-s_{ij})f_q(x)\,\partial_x^q Q\\ &=\; \Big(\sum_{j=1}^{n}{}^{(i)}\,\frac{1}{x_i-x_j}(f_q(x)\,\partial_x^q - (s_{ij}f_q(x))\,s_{ij}\partial_x^q s_{ij}\Big)Q\end{aligned}$$

which is easily seen to be of the correct form as well. A use of Theorem 2.4 proves 3.4 for the monomials $x_i x^p$ and completes the induction.

We have a similar result for the operator O_m. It may be stated as follows

Proposition 3.4

$$O_m \;=\; \Pi(x)\Pi(\partial_x) \;+\; \sum_{|q|<\binom{n}{2}} a_q(x;m)\,\partial_x^q \qquad 3.7$$

where each $a_q(x;m)$ is a homogeneous element of $\mathcal{SR}_n(x)$ and

$$deg(a_q) = |q| \qquad\qquad (\text{for all } q) \qquad 3.8$$

Proof

For a symmetric polynomial $f(x)$ we have, by definition

$$O_m f(x) \;\; = \;\; \Pi(\nabla(m))\Pi(x)f(x)$$

and by expansion of the first Vandermonde determinant we get

$$O_m f(x) \;\; = \;\; \sum_{\sigma\in S_n} sign(\sigma)\nabla_1^{n-\sigma_1}(m)\nabla_2^{n-\sigma_2}(m)\cdots\nabla_n^{n-\sigma_n}(m)\Pi(x)f(x)$$

Thus 3.7 is established if we prove that for every multi-exponent $p=(p_1,p_2,\ldots,p_n)$ we have

$$\nabla_1^{p_1}(m)\nabla_2^{p_2}(m)\cdots\nabla_n^{p_n}(m)\Pi(x)f(x) = \Pi(x)\partial_{x_1}^{p_1}\partial_{x_2}^{p_2}\cdots\partial_{x_n}^{p_n}f(x) \;\; + \sum_{|q|<\binom{n}{2}} b_q^{(p)}(x;m)\,\partial_x^q f(x) \qquad 3.9$$

with the $b_q^{(p)}(x;m)$ polynomials in m with coefficients in $\mathcal{SR}_n(x)$ and

$$deg(b_q^{(p)}(x;m)) = |q|-|p|+\binom{n}{2} \qquad \text{(for all } q) \qquad 3.10$$

Note that for any $i=1,2,\ldots,n$ we have

$$\begin{aligned}
\nabla_i(m)\Pi(x)f(x) \;\; &= \;\; \Big(\partial_{x_i} - m\sum_{j=1}^{n}{}^{(i)}\tfrac{1}{x_i-x_j}(1-s_{ij})\Big)\Pi(x)f(x) \\
&= \;\; \partial_{x_i}\big(\Pi(x)f(x)\big) - m\sum_{j=1}^{n}{}^{(i)}\tfrac{1}{x_i-x_j}(1-s_{ij})\Pi(x)f(x) \\
&= \;\; (\Pi(x)\partial_{x_i}f(x) + \big(\partial_{x_i}\Pi(x)\big)f(x)\big) - m\sum_{j=1}^{n}{}^{(i)}\tfrac{1}{x_i-x_j}2\,\Pi(x)f(x) \\
&= \;\; \Pi(x)\partial_{x_i}f(x) + \Big(\partial_{x_i}\Pi(x) - 2m\sum_{j=1}^{n}{}^{(i)}\tfrac{\Pi(x)}{x_i-x_j}\Big)f(x)
\end{aligned}$$

and we clearly see that both 3.9 and 3.10 are trivially satisfied in this case. So proceeding by induction on the size of $|p|$ let us assume 3.9 and 3.10 true for $|p|=d$. Thus for any $i=1,2,\ldots,n$ we get (by the commutativity of the Dunkl operators)

$$\nabla_1^{p_1}(m)\cdots\nabla_i^{p_i+1}(m)\cdots\nabla_n^{p_n}(m)\Pi(x)f(x) = \nabla_i(m)\Pi(x)\partial_x^p f(x) \;\; + \sum_{|q|<\binom{n}{2}} \nabla_i(m)b_q^{(p)}(x;m)\,\partial_x^q f(x)$$

Now the first term is

$$\begin{aligned}
\nabla_i(m)\Pi(x)\partial_x^p f(x) \;\; &= \;\; \Big(\partial_{x_i} - m\sum_{j=1}^{n}{}^{(i)}\tfrac{1}{x_i-x_j}(1-s_{ij})\Big)\Pi(x)\partial_x^p f(x) \\
&= \;\; \partial_{x_i}\big(\Pi(x)\partial_x^p f(x)\big) - m\sum_{j=1}^{n}{}^{(i)}\tfrac{\Pi(x)}{x_i-x_j}\partial_x^p f(x) \; - \; m\sum_{j=1}^{n}{}^{(i)}\tfrac{1}{x_i-x_j}\Pi(x)s_{ij}\partial_x^p f(x) \\
&= \;\; \Pi(x)\partial_{x_i}\partial_x^p f(x) + \big(\partial_{x_i}\Pi(x)\big)\partial_x^p f(x) - m\sum_{j=1}^{n}{}^{(i)}\tfrac{\Pi(x)}{x_i-x_j}\big(\partial_x^p + s_{ij}\partial_x^p s_{ij}\big)f(x)
\end{aligned}$$

and we see that is of the desired form. Now for the summand in the second term at the right hand side of 3.9 we have

$$\begin{aligned}\nabla_i(m) b_q^{(p)}(x;m)\,\partial_x^q f(x) \;&=\; \Big(\partial_{x_i} - m\sum_{j=1}^{n}{}^{(i)}\tfrac{1}{x_i-x_j}(1-s_{ij})\Big) b_q^{(p)}(x;m)\,\partial_x^q f(x)\\ &=\; b_q^{(p)}(x;m)\,\partial_{x_i}\partial_x^q f(x) + \Big(\partial_{x_i} b_q^{(p)}(x;m)\Big)\,\partial_x^q f(x)\\ &\qquad\qquad - \Big(m\sum_{j=1}^{n}{}^{(i)}\tfrac{b_q^{(p)}(x;m)}{x_i-x_j}\Big)\,\partial_x^q f(x)\\ &\qquad\qquad\qquad + \Big(m\sum_{j=1}^{n}{}^{(i)}\tfrac{s_{ij}b_q^{(p)}(x;m)}{x_i-x_j}\; s_{ij}\partial_x^q s_{ij}\Big) f(x)\end{aligned}$$

Note that these sumands are of the proper form since differentiation by x_i of a homogeneous element of degree r in $\mathcal{SR}_n(x)$ yields a homogeneous element of degree $d-1$. Note further that since $|q| < d$ the amount of differentiation on $f(x)$ remains below $d+1$ in all of these terms. This completes the induction and the proof.

Proposition 3.4 yields us a first glimpse at the nature of the operator Ω_m. We shall later see that this result can be improved considerably.

Proposition 3.5

Recalling that, by definition, $\Omega_m = O_m O_{m-1}\cdots O_1$ we have

$$L_m\,\Omega_m \;=\; \Omega_m\,\Delta_2 \qquad 3.11$$

and

$$\Omega_m \;=\; \Pi(x)^m\;\Pi(\partial_x)^m \;+\; \sum_{|q|<m\binom{n}{2}} a_q(x;m)\,\partial_x^q \qquad 3.12$$

where each $a_q(x;m)$ is a homogeneous element of $\mathcal{SR}_n(x)$ and

$$deg(a_q) = |q| \qquad\qquad (\textit{for all } q) \qquad 3.13$$

Proof

The identity in 3.11 follows by a recursive application of Theorem 2.7 and then noting that for $m = 0$ the operator L_m reduces to the laplacian Δ_2.

Similarly the expansion in 3.12 together with 3.13 are immediately obtained by recursive applications of Propositions 3.2 and 3.4.

Our next task is a close study of the action of L_m on $\mathcal{QI}_m[X_n]$ but before we can do that we need some notation and some auxiliary identities. To begin, recall that we have set

$$\mathcal{SR}_n(x) \;=\; \mathbb{Q}\big[x_i, \tfrac{1}{(x_i-x_j)} : 1\le i\le n\,;\,1\le i<j\le n\big]$$

Now it will be convenient to denote by $\mathcal{SR}_n(x)^{r,s}$ the subalgebra

$$\mathcal{R}^{r,s} \;=\; \mathbb{Q}\big[x_i, \tfrac{1}{(x_i-x_j)} : 1\le i\le n\,;\,1\le i<j\le n\,;(i,j)\ne(r,s)\big]$$

In the rest of this section we shall work with L_m in the form given by Theorem 2.5. Namely, we will use the expansion

$$L_m \;=\; \Delta_2 \,-\, 2m\,\mathcal{F}_n \qquad 3.14$$

with

$$\mathcal{F}_n \;=\; \sum_{1\le i<j\le n} \frac{1}{x_i - x_j}(\partial_{x_i} - \partial_{x_j}) \qquad 3.15$$

Proposition 3.6

For any two rational functions $f, g \in \mathcal{SR}_n(x)$ we have

$$\Delta_2\, f\, g \;=\; (\Delta_2 f)\, g \;+\; 2\sum_{i=1}^{n}(\partial_{x_i} f)(\partial_{x_i} g) \;+\; f\,\Delta_2 g\,, \qquad 3.16$$

in particular

$$L_m\,(f\,g) \;=\; (L_m f)\, g \;+\; 2\sum_{i=1}^{n}(\partial_{x_i} f)(\partial_{x_i} g) \;+\; f\,L_m g\,. \qquad 3.17$$

Proof

Note that for any index i we have

$$\partial_{x_i}^2(fg) \;=\; (\partial_{x_i}^2 f)\, g \;+\; 2(\partial_{x_i} f)(\partial_{x_i} g) \;+\; f\partial_{x_i}^2 g$$

and 3.16 is obtained by summing on i. Since the operators $\partial_{x_i} - \partial_{x_j}$ satisfy the Leibnitz rule, from 3.15 we derive that

$$\mathcal{F}_n(f\,g) \;=\; (\mathcal{F}_n f)\, g \;+\; f\,\mathcal{F}_n g$$

Thus 3.14 gives

$$\begin{aligned} L_m\,(f\,g) \;&=\; \Delta_2(f\,g) \,-\, 2m(\mathcal{F}_n f)\, g \,-\, 2m\, f\,\mathcal{F}_n g \\ \text{(by 3.16)} \;&=\; (\Delta_2 f)\, g \;+\; 2\sum_{i=1}^{n}(\partial_{x_i} f)(\partial_{x_i} g) \;+\; f\,\Delta_2\, g \,-\, 2m(\mathcal{F}_n f)\, g \,-\, 2m\, f\,\mathcal{F}_n g \end{aligned}$$

and this is 3.17.

We shall make multiple use of the following identity

Proposition 3.7

For any exponent $-\infty < a < +\infty$ we have

$$L_m(x_r - x_s)^a \;=\; 2a(a-1-2m)(x_r - x_s)^{a-2} - 2m\, a(x_r - x_s)^a \sum_{j=1}^{n}{}^{(r,s)} \tfrac{1}{(x_r - x_j)(x_s - x_j)} \qquad 3.18$$

this implies that for any $R \in \mathcal{R}^{rs}$ we have

$$L_m(x_r - x_s)^a R \;=\; 2a(a-1-2m)(x_r - x_s)^{a-2} R \;+\; (x_r - x_s)^{a-1} A \qquad 3.19$$

with $A \in \mathcal{R}^{rs}$

Proof

To begin note that

$$\Delta_2(x_r-x_s)^a \;=\; \sum_{i=1}^n \partial_{x_i}^2(x_r-x_s)^a \;=\; 2a(a-1)(x_r-x_s)^{a-2} \tag{3.20}$$

moreover from 3.15 we get

$$\begin{aligned}\mathcal{F}_n\,(x_r-x_s)^a &= a(x_r-x_s)^{a-1}\sum_{1\le i<j\le n}\frac{1}{x_i-x_j}(\partial_{x_i}-\partial_{x_j})(x_r-x_s)\\ &= a(x_r-x_s)^{a-1}\sum_{1\le i<j\le n}\frac{1}{x_i-x_j}\big(\chi(i=r)-(\chi(i=s)-(\chi(j=r)+(\chi(j=s)\big)\end{aligned} \tag{3.21}$$

But

$$\begin{aligned}\sum_{1\le i<j\le n}\frac{1}{x_i-x_j}&\big(\chi(i=r)-(\chi(i=s)-(\chi(j=r)+(\chi(j=s)\big)\\ &= \sum_{r<j\le n}\frac{1}{x_r-x_j}-\sum_{s<j\le n}\frac{1}{x_s-x_j}-\sum_{1\le i<r}\frac{1}{x_i-x_r}+\sum_{1\le i<s}\frac{1}{x_i-x_s}\\ &= \sum_{r<j\le n}\frac{1}{x_r-x_j}-\sum_{s<j\le n}\frac{1}{x_s-x_j}+\sum_{1\le j<r}\frac{1}{x_r-x_j}-\sum_{1\le j<s}\frac{1}{x_s-x_j}\\ &= \sum_{j=1}^{n}{}^{(r)}\frac{1}{x_r-x_j}-\sum_{j=1}^{n}{}^{(s)}\frac{1}{x_s-x_j}\\ &= \frac{2}{x_r-x_s}\;+\;\sum_{j=1}^{n}{}^{(r,s)}\Big(\frac{1}{x_r-x_j}-\frac{1}{x_s-x_j}\Big)\\ &= \frac{2}{x_r-x_s}-\sum_{j=1}^{n}{}^{(r,s)}\frac{x_r-x_s}{(x_r-x_j)(x_s-x_j)}\end{aligned}$$

Using this in 3.21 we get

$$\mathcal{F}_n\,(x_r-x_s)^a \;=\; 2a(x_r-x_s)^{a-2}\;-\;a(x_r-x_s)^{a-1}\sum_{j=1}^{n}{}^{(r,s)}\tfrac{x_r-x_s}{(x_r-x_j)(x_s-x_j)} \tag{3.22}$$

Thus combining 3.14, 3.20 and 3.22 we finally obtain

$$L_m\,(x_r-x_s)^a = 2a(a-1)(x_r-x_s)^{a-2}\;-\;2m2a(x_r-x_s)^{a-2}\;+\;2ma(x_r-x_s)^{a-1}\sum_{j=1}^{n}{}^{(r,s)}\tfrac{x_r-x_s}{(x_r-x_j)(x_s-x_j)}$$

This proves 3.18.

Now 3.17 gives

$$\begin{aligned}L_m(x_r-x_s)^aR &= \big(L_m(x_r-x_s)^a\big)R\;+\;2\sum_{i=1}^n\big(\partial_{x_i}(x_r-x_s)^a\big)\big(\partial_{x_i}R\big)\;+\;(x_r-x_s)^aL_mR\\ &= 2a(a-1-2m)(x_r-x_s)^{a-2}R-2m\,a(x_r-x_s)^{a-1}\sum_{j=1}^{n}{}^{(r,s)}\tfrac{(x_r-x_s)}{(x_r-x_j)(x_s-x_j)}R\\ &\qquad+\;2a(x_r-x_s)^{a-1}(\partial_{x_r}R-\partial_{x_s}R)\;+\;(x_r-x_s)^{a-1}(x_r-x_s)L_mR\end{aligned}$$

This proves 3.19 with

$$A \;=\; -2ma\sum_{j=1}^{n}{}^{(r,s)}\tfrac{x_r-x_s}{(x_r-x_j)(x_s-x_j)}\,R \;+\; 2a(\partial_{x_r}R-\partial_{x_s}R) \;+\; (x_r-x_s)L_mR$$

Since it is easily seen that $A\in\mathcal{SR}^{rs}$ our proof is now complete.

Proposition 3.8

If for some $P\in\mathcal{R}$ we have

$$L_m\,P \;=\; Q\in\mathbb{Q}[X_n] \tag{3.23}$$

then

$$P\in\mathbb{Q}[X_n]$$

Proof

Let it be possible that we can write

$$P \;=\; \frac{R}{(x_r-x_s)^k} \tag{3.24}$$

with $k\geq 1$ minimal and $R\in\mathcal{R}^{rs}$ then from Proposition 3.7 with $a{\rightarrow}-k$ we get for some $A\in\mathcal{R}^{rs}$

$$\begin{aligned} L_mP \;&=\; L_m(x_r-x_s)^{-k}R \\ &=\; 2k(k+1+2m)(x_r-x_s)^{-k-2}R \;+\; (x_r-x_s)^{-k-1}A \end{aligned}$$

Multiplying both sides by $(x_r-x_s)^{k+2}$ and using 3.23 we derive that

$$(x_r-x_s)^{k+2}Q \;=\; 2k(k+1+2m)R \;+\; (x_r-x_s)A\,. \tag{3.25}$$

Extracting R from 3.25 and placing it in 3.24 we derive that P can be rewritten in the form

$$P \;=\; \frac{R'}{(x_r-x_s)^{k-1}}$$

with $R'\in\mathcal{R}^{rs}$ contraddicting the minimality of k. Thus P can't be as in 3.26 with $k\geq 1$. Since this must be so for any pair (r,s) the only remaining possibility is that $P\in\mathbb{Q}[X_n]$.

We are now finally in a position to establish the following truly remarkable result

Theorem 3.1

If $P\in\mathcal{SR}_n(x)$ and

$$L_m\,P \;=\; Q \tag{3.26}$$

with Q a polynomial in $\mathcal{QI}_m[X_n]$ then

$$P\in\mathcal{QI}_m[X_n] \tag{3.27}$$

Proof

Since L_m is a symmetric operator then

$$(1-s_{rs})L_mP \;=\; L_m(1-s_{rs})P$$

Since we know from Proposition 3.8 that P must be a polynomial then for some integer $k\geq 0$ we must have

$$(1-s_{rs})P \;=\; (x_r-x_s)^{2k+1}P' \tag{3.28}$$

with a suitable polynomial P'. We claim that we must have $k\geq m$. So suppose if possible that $k<m$ and that k is maximal. This given, from 3.26 we get that

$$(1-s_{rs})Q \;=\; L_m\,(x_i-x_j)^{2k+1}P'$$

But then the m-quasi-invariance of Q gives that for some polynomial Q' we have

$$(x_r-x_s)^{2m+1}Q' \;=\; L_m\,(x_i-x_j)^{2k+1}P'$$

Now using 3.19 with $a=2k+1$, for a suitable $A\in\mathcal{SR}_n(x)^{rs}$ we get

$$(x_r-x_s)^{2m+1}Q' \;=\; 2(2k+1)(2k-2m)(x_r-x_s)^{2k-1}P' \;+\; (x_r-x_s)^{2k}A$$

This gives

$$P' \;=\; \frac{x_r-x_s}{2(2k+1)(2k-2m)}\Big((x_r-x_s)^{2m-2k+1}Q' \,-\, A\Big)$$

Since $A\in\mathcal{SR}_n(x)^{rs}$ we can clear denominators in this relation upon multiplication by a factor of the form

$$\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq(r,s)}}(x_i-x_j)^{p_{ij}} \tag{3.29}$$

We thus obtain the totally polynomial equation

$$P'\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq(r,s)}}(x_i-x_j)^{p_{ij}} \;=\; \frac{x_r-x_s}{2(2k+1)(2k-2m)}\Big((x_r-x_s)^{2m-2k+1}Q'\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq(r,s)}}(x_i-x_j)^{p_{ij}} - A\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq(r,s)}}(x_i-x_j)^{p_{ij}}\Big)$$

Since the factor in 3.29 is prime with x_r-x_s, we derive from this that the polynomial P' must be divisible by x_r-x_s. Setting $P'=(x_r-x_s)P''$ the identity in 3.28 becomes

$$(1-s_{rs})P \;=\; (x_r-x_s)^{2k+2}P''$$

but we have seen that when we apply $(1-s_{rs})$ to a polynomial, an odd power of (x_r-x_s) must factor out. This forces P'' itself to be divisible by (x_r-x_s). Thus we must have yet a third polynomial P''' giving

$$(1-s_{rs})P \;=\; (x_r-x_s)^{2k+3}P'''$$

But this contraddicts the maximality of k. So we must have $k \geq m$ as desired. The arbitrarity of r, s in this argument shows P must be m-quasi-invariant and our proof is complete.

But there is yet another surprising fact: Theorem 3.1 has a converse.

Theorem 3.2

If $P \in \mathcal{QI}_m[X_n]$ and

$$L_m P \;=\; Q \tag{3.30}$$

then

$$Q \in \mathcal{QI}_m[X_n] \tag{3.31}$$

Proof

We will closely follow the argument given in []. We first show that Q is a polynomial and then show that Q is m-quasi-invariant. Now, for a pair $1 \leq r < s \leq n$, the m-quasi-invariance of P yields the factorization

$$(1-s_{rs})P \;=\; (x_r - x_s)^{2m+1} P'$$

with P' a suitable polynomial. This given, using the symmetry of L_m, we derive from 3.30 that

$$\begin{aligned}
(1-s_{rs})Q \;&=\; L_m\big((x_r-x_s)^{2m+1}P'\big)\\
\text{(by 3.17)} \;&=\; \big(L_m(x_r-x_s)^{2m+1}\big)P'\\
&\qquad + 2\sum_{i=1}^{n}\big(\partial_{x_i}(x_r-x_s)^{2m+1}\big)\big(\partial_{x_i}P'\big) + (x_r-x_s)^{2m+1}L_mP'\\
&=\; \big(L_m(x_r-x_s)^{2m+1}\big)P'\\
&\qquad + 2(2m+1)(x_r-x_s)^{2m}\big(\partial_{x_r}P' - \partial_{x_s}P'\big) + (x_r-x_s)^{2m+1}L_mP'
\end{aligned} \tag{3.32}$$

Now recall from 3.18 that we have

$$L_m(x_r-x_s)^a \;=\; 2a(a-1-2m)(x_r-x_s)^{a-2} - 2m\,a(x_r-x_s)^a \sum_{j=1}^{n}{}^{(r,s)} \tfrac{1}{(x_r-x_j)(x_s-x_j)}$$

and this, for $a = 2m+1$, reduces to

$$L_m(x_r-x_s)^{2m+1} \;=\; -2m\,(2m+1)(x_r-x_s)^{2m+1} \sum_{j=1}^{n}{}^{(r,s)} \tfrac{1}{(x_r-x_j)(x_s-x_j)}$$

Thus 3.32 may be rewritten as

$$\begin{aligned}
(1-s_{rs})Q \;&=\; -2m\,(2m+1)(x_r-x_s)^{2m+1} \sum_{j=1}^{n}{}^{(r,s)} \tfrac{1}{(x_r-x_j)(x_s-x_j)} P'\\
&\qquad +\;\; 2(2m+1)(x_r-x_s)^{2m}\big(\partial_{x_r}P' - \partial_{x_s}P'\big) + (x_r-x_s)^{2m+1}L_mP'
\end{aligned} \tag{3.33}$$

It is evident that all denominators which occur in the right-hand side of this identity are cleared upon multiplication by the Vandermonde determinant $\Pi(x)$. In fact, since both

$$\sum_{j=1}^{n}{}^{(r,s)} \tfrac{\Pi(x)}{(x_r-x_j)(x_s-x_j)} P' \qquad \text{and} \qquad \Pi(x)L_mP'$$

are polynomials we derive from 3.33 that we have

$$\Pi(x)Q \;+\; s_{rs}\,\Pi(x)Q \;=\; \Pi(x)(1-s_{rs})Q \;=\; (x_r-x_s)^{2m+1}R \tag{3.34}$$

with R a polynomial. On the other hand 3.30 implies that also

$$\Pi(x)Q \;=\; \Pi(x)L_mP$$

is polynomial. We are thus assured that for some polynomial R' we have

$$\Pi(x)Q \,-\, s_{rs}\,\Pi(x)Q \;=\; (1-s_{rs})\Pi(x)Q \;=\; (x_r-x_s)R' \tag{3.35}$$

Now 3.34 and 3.35 yield the identity

$$\Pi(x)Q \;=\; \tfrac{1}{2}(x_r-x_s)^{2m+1}R \;+\; \tfrac{1}{2}(x_r-x_s)R'$$

which plainly shows that the polynomial $\Pi(x)Q$ is divisible by (x_r-x_s). Since this must hold true for all pairs $1\le r<s\le n$ and the factors of $\Pi(x)$ are relatively prime, we are forced to conclude that Q itself must be a polynomial. In particular the factor

$$(1-s_{rs})Q$$

in 3.34 must itself a polynomial. But then the second equality in 3.34 forces $(1-s_{rs})Q$ to be divisible by $(x_r-x_s)^{2m}$. Since for a polynomial Q the maximal power of (x_r-x_s) that divides $(1-s_{rs})Q$ must be odd, we see that $(1-s_{rs})Q$ must be divisible by $(x_r-x_s)^{2m+1}$ as well. Since this must hold true for all pairs $1\le r<s\le n$ we have thus established that Q is m-quasi-invariant and completed our proof.

We are now ready to study the action of Ω_m on the polynomial ring $\mathbb{Q}[X_n]$. We find here another quite surprising development.

Theorem 3.3

For any monomial x^p we have

$$\Omega_m x^p \in \mathcal{QI}_m[X_n]\,. \tag{3.36}$$

In particular the image by Ω_m of any polynomial in $\mathbb{Q}[X_n]$ is necessarily $m-$quasi-invariant.

Proof

The identity in 3.11 gives that

$$L_m\Omega_m x^p \;=\; \Omega_m\Delta_2 x^p$$

This may also be written as

$$L_m\Omega_m x^p \;=\; \sum_{p_i\ge 2} p_i(p_i-1)\,\Omega_m\, x^p/x_i^2 \tag{3.37}$$

Now when $|p|<2$ we clearly have

$$L_m\Omega_m x^p \;=\; 0$$

since 0 is obviously in $\mathcal{QI}_m[X_n]$ Theorem 3.1 assures us that the assertion is true for all monomials of degree less than 2. This given we can proceed by induction and assume that 3.36 is true for all $|p| < d$. Now let $|p| = d$ and note from 3.37 that the inductive hypothesis assures that

$$L_m \Omega_m x^p \in \mathcal{QI}_m[X_n]$$

but then again from Theorem 3.1 we derive that $\Omega_m x^p$ is a polynomial and moreover that

$$\Omega_m x^p \in \mathcal{QI}_m[X_n]\,.$$

This completes the induction and the proof.

We are now able to provide a remarkable improvement upon the assertion in 3.12 regarding the nature of Ω_m as a differential operator. The basic idea rests on a "*Generating Function Argument*" and the generating function happens to be the Baker-Akhiezer function. This is the formal power series $\Psi_m(x,y)$ in two sets of variables $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ defined by setting

$$\Psi_m(x,y) \;=\; \Omega_m\, e^{(x,y)} \tag{3.38}$$

with

$$(x,y) \;=\; x_1y_1 + x_2y_2 + \cdots + x_ny_n\,.$$

Note that applying Ω_m to the power expansion

$$e^{(x,y)} \;=\; \sum_p \tfrac{1}{p!}\, x^p y^p \tag{3.39}$$

we derive that $\Psi_m(x,y)$ has the expansion

$$\Psi_m(x,y) \;=\; \sum_p \tfrac{y^p}{p!}\; \Omega_m x^p \tag{3.40}$$

Thus $\Psi_m(x,y)$ may be viewed as the generating function of the polynomials $\Omega_m x^p$.

The importance of $\Psi_m(x,y)$ in the study of m-quasi-invariants will appear be quite clear in the next section. For the moment we use it to prove the following remarkable result.

Theorem 3.4

The operator Ω_m has the form

$$\Omega_m \;=\; \Pi(x)^m \Pi(\partial_x)^m \;+\; \sum_{|q|<m\binom{n}{2}} P_q(x;m)\partial_x^q \tag{3.41}$$

with $P_q(x;m)$ a polynomial in $x_1, x_2, \ldots, x_n$ and m, homogeneous in $x_1, x_2, \ldots, x_n$ of x-degree

$$deg(P_q) \;=\; |q| \qquad\qquad (\textit{for all } q) \tag{3.42}$$

In particular it follows that

$$\Psi_m(x,y) \;=\; P_m(x,y)e^{(x,y)} \tag{3.43}$$

with

$$P_m(x,y) \;=\; \Pi(x)^m\Pi(y)^m \;+\; \sum_{|q|<m\binom{n}{2}} P_q(x;m)\, y^q \tag{3.44}$$

Proof

The polynomiality in m is quite evident from the definitions of Ω_m and O_m. What is not obvious from the arguments used in the proof of Proposition 3.4 is the x-polynomiality of the coefficients $a_q(x;m)$ occurring in 3.7 and ultimately in 3.12. Yet this polynomiality is an immediate consequence of Theorem 3.3. To see this we apply Ω_m to the expansion

$$e^{(x,y)} = \sum_{k\geq 0} \tfrac{1}{k!}(x,y)^k \qquad 3.45$$

and obtain the identity

$$\Psi_m(x,y) = \sum_{k\geq 0} \tfrac{1}{k!}\,\Omega_m(x,y)^k \qquad 3.46$$

Theorem 3.3 assures that the terms $\Omega_m(x,y)^k$ are polynomials and infact the degree preserving property of Ω_m assures that $\Omega_m(x,y)^k$ is doubly homogeneous and of degree k in both $x_1,x_2,\ldots,x_n$ and $y_1,y_2,\ldots,y_n$. Now from 3.12 it follows that

$$\begin{aligned}\Psi_m(x,y) &= \Big(\Pi(x)^m\Pi(\partial_x)^m + \sum_{|q|<m\binom{n}{2}} a_q(x;m)\,\partial_x^q\Big)e^{(x,y)} \\ &= P_m(x,y)e^{(x,y)}\end{aligned} \qquad 3.47$$

with

$$P_m(x,y) = \Pi(x)^m\Pi(y)^m + \sum_{|q|<m\binom{n}{2}} a_q(x;m)\,y^q \qquad 3.48$$

We may thus combine 3.47, 3.45 an 3.46 and obtain another expression for $P_m(x,y)$. Namely we have

$$\begin{aligned}P_m(x,y) &= e^{-(x,y)}\sum_{k\geq 0}\tfrac{1}{k!}\,\Omega_m(x,y)^k \\ &= \Big(\sum_{h\geq 0}\tfrac{(-1)^h}{h!}(x,y)^h\Big)\Big(\sum_{k\geq 0}\tfrac{1}{k!}\,\Omega_m(x,y)^k\Big)\end{aligned}$$

Since we know that $P_m(x,y)$ is a polynomial in $y_1,y_2,\ldots,y_n$ of degree $d_m = m\binom{n}{2}$ all terms of y-degree larger than d_m, on the right hand side of this identity, must necessarily cancel out. Thus we must have that

$$P_m(x,y) = \sum_{r=0}^{d_m}\sum_{h+k=r}\tfrac{(-1)^h}{h!k!}(x,y)^h\Omega_m(x,y)^k \qquad 3.49$$

This remarkable formula makes quite obvious all the stated properties of $P_m(x,y)$ and completes the proof of the Theorem.

Remark 3.1

We should point out that comparing 3.50 with 3.48 we derive the surprising identity

$$\Pi(x)^m\Pi(y)^m = \sum_{h+k=m\binom{n}{2}}\tfrac{(-1)^h}{h!k!}(x,y)^h\Omega_m(x,y)^k \qquad 3.50$$

The operator L_m has an alternate expression which will play a crucial role in the study of the Baker-Akhiezer function. It is worth deriving it here before continuing with our developments.

Theorem 3.5

For any function $F(x)$ we have

$$L_m\, F(x) \;=\; \Pi[X_n]^m \Delta_2\, \Pi(x)^{-m}\, F(x) \;-\; 2(m^2+m) \sum_{1\le i<j\le n} \frac{1}{(x_i-x_j)^2}\, F(x)\,. \qquad 3.51$$

Proof

Note that 3.16 gives

$$\Delta_2\, \Pi(x)^{-m}\, F(x) \;=\; \big(\Delta_2\, \Pi(x)^{-m}\big) Q(x) + 2\sum_{i=1}^{n} \big(\partial_{x_i} \Pi(x)^{-m}\big) \partial_{x_i} F(x) \;+\; \Pi(x)^{-m} \Delta_2 Q(x)\,.$$

Thus

$$\Pi(x)^m \Delta_2\, \Pi(x)^{-m}\, F(x) = \big(\Pi(x)^m \Delta_2\, \Pi(x)^{-m}\big) Q(x) + 2\sum_{i=1}^{n} \big(\Pi(x)^m \partial_{x_i} \Pi(x)^{-m}\big) \partial_{x_i} Q(x) + \Delta_2 Q(x)\,. \qquad 3.52$$

Now for any $1\le r\le n$ we get

$$\begin{aligned}
\Pi(x)^m \partial_{x_r} \Pi(x)^{-m} \;&=\; \partial_{x_r} log\Big(\Pi(x)^{-m}\Big)\\
&=\; -m \sum_{1\le i<j\le n} \partial_{x_r} log(x_i-x_j)\\
&=\; -m \sum_{1\le i<j\le n} \frac{\partial_{x_r}(x_i-x_j)}{(x_i-x_j)}\\
&=\; -m \Big(\sum_{1\le i<r} \frac{-1}{(x_i-x_r)} \;+\; \sum_{r<j\le n} \frac{1}{(x_r-x_j)} \Big)\,,
\end{aligned}$$

In conclusion we have shown that

$$\Pi(x)^m \partial_{x_r} \Pi(x)^{-m} \;=\; -m\, U_r \qquad 3.53$$

where for convenience we have set

$$U_r \;=\; \sum_{i=1}^{n} \frac{1}{(x_r-x_i)}\, \chi(i\ne r)\,. \qquad 3.54$$

Thus we have

$$\begin{aligned}
\Pi(x)^m \Delta_2 \Pi(x)^{-m} \;&=\; \sum_{r=1}^{n} \Big(\Pi(x)^m \partial_{x_r} \Pi(x)^{-m}\Big)\Big(\Pi(x)^m \partial_{x_r} \Pi(x)^{-m}\Big)\\
&=\; -m \sum_{r=1}^{n} \Pi(x)^m \partial_{x_r} \Pi(x)^{-m}\, U_r\\
&=\; -m \sum_{r=1}^{n} \Big(\Pi(x)^m \Big(\big(\partial_{x_r} \Pi(x)^{-m}\big)\, U_r \;+\; \Pi(x)^{-m} \big(\partial_{x_r} U_r\big)\Big)\\
&=\; -m \sum_{r=1}^{n} \Big(-m\, U_r^2 \;+\; \partial_{x_r} U_r \Big) \;=\; m^2 \sum_{r=1}^{n} U_r^2 - m \sum_{r=1}^{n} \partial_{x_r} U_r\,.
\end{aligned} \qquad 3.55$$

But from 3.54 we derive that

$$\partial_{x_r} U_r \;=\; -\sum_{i=1}^n \frac{1}{(x_r - x_i)^2}\, \chi(i \neq r)\,.$$

from which it immediately follows that

$$\sum_{r=1}^n \partial_{x_r} U_r \;=\; -2 \sum_{1\le i<j\le n} \frac{1}{(x_i - x_j)^2}\,. \tag{3.56}$$

On the other hand, again from 3.54 we get

$$\begin{aligned}\sum_{r=1}^n U_r^2 &= \sum_{r=1}^n \sum_{i=1}^n \sum_{j=1}^n \frac{1}{(x_r - x_i)(x_r - x_j)} \chi(i\neq r)\chi(j \neq r) \\ &= 2\sum_{1\le i<j\le n} \frac{1}{(x_i-x_j)^2} \;+\; \sum_{r=1}^n \sum_{i=1}^n \sum_{j=1}^n \frac{\chi(i\neq j)}{(x_r - x_i)(x_r - x_j)} \chi(i\neq r)\chi(j \neq r)\,.\end{aligned}$$

Clearly the left-hand side of this equality is a symmetric function of $x_1, x_2, \ldots, x_n$, and so is the first term on the right-hand side. Thus also the second term on right hand side is necessarily symmetric. This fact immediately implies that this term must vanish identically. The reason is simple, if it didn't vanish, multiplication of this term by $\Pi(x)$ would yield an alternating polynomial of degree less than $\binom{n}{2}$ and we know that there aren't any. In conclusion we derive that

$$\sum_{r=1}^n U_r^2 \;=\; 2 \sum_{1\le i<j\le n} \frac{1}{(x_i - x_j)^2}\,. \tag{3.57}$$

Substituting 3.56 and 3.55 in 3.54 gives

$$\Pi(x)^m \Delta_2 \Pi(x)^{-m} \;=\; 2(m^2+m) \sum_{1\le i<j\le n} \frac{1}{(x_i - x_j)^2}\,. \tag{3.58}$$

To complete our proof we need to evaluate the second term on the right-hand side of 3.52. To this end note that we have (using 3.53)

$$\begin{aligned} 2\sum_{r=1}^n \big(\Pi(x)^m \partial_{x_r} \Pi(x)^{-m}\big)\partial_{x_r} F(x) &= -2m \sum_{r=1}^n \Big(\sum_{i=1}^n \frac{1}{x_r - x_i}\, \chi(i\neq r)\Big) \partial_{x_r} F(x) \\ &= -2m \Big(\sum_{r=1}^n \sum_{i=1}^{r-1} \frac{1}{x_r - x_i} \partial_{x_r} Q(x) \;+\; \sum_{r=1}^n \sum_{i=r+1}^n \frac{1}{x_r - x_i} \partial_{x_r} F(x)\Big) \\ &= -2m \Big(\sum_{j=1}^n \sum_{i=1}^{j-1} \frac{-1}{x_i - x_j} \partial_{x_j} Q(x) \;+\; \sum_{i=1}^n \sum_{j=i+1}^n \frac{1}{x_i - x_j} \partial_{x_i} F(x)\Big) \\ &= -2m \Big(\sum_{1\le i<j\le n} \frac{1}{x_i - x_j} (\partial_{x_i} - \partial_{x_j}) Q(x)\Big). \end{aligned}$$

Using this and 3.58 in 3.52 we finally obtain the identity

$$\Pi(x)^m \Delta_2[\Pi(x)^{-m} F(x) = 2(m^2+m) \sum_{1\leq i<j\leq n} \frac{F(x)}{(x_i-x_j)^2} \; - \; 2m \sum_{1\leq i<j\leq n} \frac{1}{x_i-x_j} (\partial_{x_i} - \partial_{x_j}) F(x) + \Delta_2 F(x)$$

which (given the definition in 3.14) is simply another way of writing 3.51. This completes our proof.

4. The Baker-Akhieser functions for $\mathbf{S_n}$

The goal of this section is to obtain the basic properties of $\Psi_m(x,y)$ which intimately connect it to the theory of m-quasi-invariants. Recall that we have defined it by setting

$$\Psi_m(x,y) = \Omega_m e^{(x,y)} \quad = \quad O_m O_{m-1} \cdots O_1 e^{(x,y)} \tag{4.1}$$

with

$$(x,y) \quad = \quad x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \,. \tag{4.2}$$

We begin with two fundamental properties of the Baker-Akhieser function which, as we will see, completely characterize it

Theorem 4.1

$$a) \qquad \text{Ł}_m \Psi_m(x,y) \quad = \quad (y,y)\Psi_m(x,y) \tag{4.3}$$

and

$$\Psi_m(x,y) \quad = \quad P_m(x,y) e^{(x,y)} \tag{4.4}$$

with

$$b) \qquad P_m(x,y) \quad = \quad \Pi(x)^m \Pi(y)^m + \sum_{0 \le k < m\binom{n}{2}} a_k(x,y) \tag{4.5}$$

where $a_k(x,y)$ is homogeneous in $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$, and of degree k in both variables. Moreover, $\Psi_m(x,y)$ is m-quasi-invariant as a function of x.

Proof

The definition in 4.11 and 3.11 give

$$L_m \Psi_m(x,y) = L_m \Omega_m e^{(x,y)} = \Omega_m \Delta_2 e^{(x,y)} = \Omega_m (y,y) e^{(x,y)} = (y,y)\Omega_m e^{(x,y)} = (y,y)\Psi_m(x,y)\,.$$

This proves 4.3. Property b) is essentially the contents of Theorem 3.4. and it is best expressed by the identities in 3.49 and 3.50. The quasi-invariance of $\Phi(x,y)$ is an immediate consequence of 3.36 and the definition in 3.38

For a deeper understanding of the Baker-Akhiezer function we need to enlarge the algebra of functions we work with. To this end we shall here and after call $\mathcal{SR}_n(x,y)$ the ring generated by the variables $x_1, x_2, \ldots, x_n\,;\, y_1, y_2, \ldots, y_n$ and all the fractions $\frac{1}{x_i - x_j}$, $\frac{1}{y_i - y_j}$. In symbols

$$\mathcal{SR}_n(x,y) \quad = \quad \mathbb{Q}\Big[x_i, y_i, i = 1, \ldots, n\,;\, \tfrac{1}{x_i - x_j}, \tfrac{1}{y_i - y_j}\,;\, 1 \le i < j \le n\Big]$$

This ring has a natural bigrading which decomposes it into bihomogeneous subspaces yielding the direct sum decomposition

$$\mathcal{SR}_n(x,y) \quad = \quad \bigoplus_{r=-\infty}^{+\infty} \bigoplus_{s=-\infty}^{+\infty} \mathcal{H}_{r,s}\Big(\mathcal{SR}_n(x,y)\Big)$$

Where $\mathcal{H}_{r,s}\Big(\mathcal{SR}_n(x,y)\Big)$ is spanned by rational function of the form

$$f_{r,s}(x,y) \quad = \quad \frac{x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} y_1^{q_1} y_2^{q_2} \cdots y_n^{q_n}}{\prod_{1 \le i < j \le n} (x_i - x_j)^{p_{ij}} \prod_{1 \le i < j \le n} (y_i - y_j)^{q_{ij}}}$$

with

$$\sum_{i=1}^{n} p_i \; - \sum_{1\leq i<j\leq n} p_{ij} = r \qquad \text{and} \qquad \sum_{i=1}^{n} q_i \; - \sum_{1\leq i<j\leq n} q_{ij} = s\,.$$

We call the elements of $\mathcal{H}_{r,s}\Big(\mathcal{SR}_n(x,y)\Big)$ bihomogeneous of bidegree (r,s). Clearly, each $f \in \mathcal{SR}_n(x,y)$ has a unique decomposition of the form

$$f \;\; = \;\; \sum_{r=-r_1}^{r_2} \sum_{s=-s_1}^{s_2} f_{r,s} \qquad \text{(with } f_{r,s} \in \mathcal{H}_{r,s}\Big(\mathcal{SR}_n(x,y)\Big)\text{)}\,.$$

We call the summands $f_{r,s}(x,y)$ the "*bihomogeneous components*" of f .

This given, we also need to deal with the family of special functions $\Phi(x,y)$ of the form

$$\Phi(x,y) \;\; = \;\; F(x,y)e^{(x,y)} \qquad \text{with } F(x,y) \in \mathcal{SR}_n(x,y)$$

It will be convenient here and after to denote this family by $\mathcal{SFR}(x,y)$.

Proposition 4.1

For any special function $\Phi(x,y) = F(x,y)e^{(x,y)} \in \mathcal{SFR}(x,y)$ we have

$$e^{-(x,y)}\Pi(x)^{-m}\Pi(y)^{-m}\big(L_m\Phi(x,y) - (y,y)\Phi(x,y)\big) \;\; =$$
$$= \;\; 2\sum_{i=1}^{n} y_i\,\partial_{x_i} f(x,y) + \Delta_2 f(x,y) \; - \; 2(m+m^2)f(x,y) \sum_{1\leq i<j\leq n} \tfrac{1}{(x_i-x_j)^2}\,. \qquad 4.6$$

where for convenience we have set

$$f(x,y) = \frac{F(x,y)}{\Pi(x)^m\Pi(y)^m} \qquad 4.7$$

It then follows that $\Phi(x,y)$ satisfies the identity

$$L_m\Phi(x,y) \;\; = \;\; (y,y)\,\Phi(x,y) \qquad 4.8$$

if and only if the function $f(x,y)$ is a solution of the differential equation

$$2\sum_{i=1}^{n} y_i\,\partial_{x_i} f(x,y) \;\; = \;\; -\Delta_2 f(x,y) \;\; + \;\; 2(m+m^2)f(x,y) \sum_{1\leq i<j\leq n} \tfrac{1}{(x_i-x_j)^2}\,. \qquad 4.9$$

Proof

Using the identity in 3.51 we derive that

$$L_m\,\Phi(x,y) \;\; = \;\; \Pi(x)^m\Delta_2\Pi(x)^{-m}F(x,y)e^{(x,y)} \; - \; 2(m+m^2)F(x,y)e^{(x,y)} \sum_{1\leq i<j\leq n} \tfrac{1}{(x_i-x_j)^2}$$

dividing by $\Pi(x)^m\Pi(y)^m$ and using 4.7 we get

$$\Pi(x)^{-m}\Pi(y)^{-m}L_m\,\Phi(x,y) \;\; = \;\; \Delta_2 f(x,y)e^{(x,y)} \; - \; 2(m+m^2)f(x,y)e^{(x,y)} \sum_{1\leq i<j\leq n} \tfrac{1}{(x_i-x_j)^2}$$

Now the identity in 3.16 gives

$$\Pi(x)^{-m}\Pi(y)^{-m}L_m\,\Phi(x,y) = (\Delta_2 f(x,y))e^{(x,y)} \;+\; 2\sum_{i=1}^{n}\partial_{x_i}f(x,y)\partial_{x_i}e^{(x,y)} \;+\; f(x,y)(y,y)e^{(x,y)} \;-\; 2(m+m^2)f(x,y)e^{(x,y)}\sum_{1\le i<j\le n}\tfrac{1}{(x_i-x_j)^2}$$

or better

$$\Pi(x)^{-m}\Pi(y)^{-m}\Big(L_m\,\Phi(x,y)-(y,y)\Phi(x,y)\Big) = e^{(x,y)}\Delta_2 f(x,y) \;+\; 2e^{(x,y)}\sum_{i=1}^{n}y_i\partial_{x_i}f(x,y) \;-\; 2(m+m^2)f(x,y)e^{(x,y)}\sum_{1\le i<j\le n}\tfrac{1}{(x_i-x_j)^2}$$

dividing both sides by $e^{(x,y)}$ gives 4.6 precisely as asserted. Thus 4.8 is equivalent to 4.9, and our argument is complete.

To proceed we need a technical result concerning the polarization operator $\sum_{i=1}^{n} y_i\partial_{x_i}$.

Proposition 4.2

Let $f(x,y)$ be a rational function of the form

$$f(x,y) \;=\; \sum_{|p|=d} a_p(x)y^p \tag{4.14}$$

with $d>0$ and $a_p(x)\in\mathcal{SR}_n(x)$. Then the equation

$$\sum_{i=1}^{n} y_i\partial_{x_i}\phi(x,y) \;=\; 0 \tag{4.15}$$

forces all the coefficients $a_p(x)$ to be polynomials.

Proof

Denote by

$$m_1(y), m_2(y), \cdots, m_N(y), \tag{4.16}$$

the monomials y^p with $|p|=d$ arranged in the lex order corresponding to the total order $y_1<y_2<\cdots<y_n$. This given we can rewrite 4.14 in the form

$$f(x,y) \;=\; \sum_{r=1}^{N} a_r(x)\,m_r(y)$$

and 4.15 becomes

$$\sum_{r=1}^{N}\partial_{x_1}a_r(x)\,y_1m_r(y)+\sum_{r=1}^{N}\partial_{x_2}a_r(x)\,y_2m_r(y)+\cdots+\sum_{r=1}^{N}\partial_{x_n}a_r(x)\,y_nm_r(y) \;=\; 0 \tag{4.17}$$

Since, when $i > 1$ or $r > 1$, all the monomials $y_i m_r(y)$ are lexicographically larger than $y_1 m_1(y) = y_1^{d+1}$, the latter monomial can only occur once in this sum. Thus 4.14 forces

$$\partial_{x_1} a_1(x) = 0.$$

We can thus proceed by induction and suppose that we have shown that

$$\partial_{x_1} a_1(x) = \partial_{x_1}^2 a_2(x) = \cdots = \partial_{x_1}^{s-1} a_{s-1}(x) = 0$$

This given, applying $\partial_{x_1}^{s-1}$ to 4.17 we derive that

$$\sum_{r=s}^{N} \partial_{x_1}^s a_r(x)\, y_1 m_r(y) + \sum_{r=s}^{N} \partial_{x_2} \partial_{x_1}^{s-1} a_r(x)\, y_2 m_r(y) + \cdots + \sum_{r=s}^{N} \partial_{x_n} \partial_{x_1}^{s-1} a_r(x)\, y_n m_r(y) \;=\; 0 \qquad 4.18$$

but now, for the same reason as before, the monomial $y_1 m_s(y)$ can only occur once in 4.18, this forces

$$\partial_{x_1}^s a_s(x) \;=\; 0$$

and completes the induction, proving that

$$\partial_{x_1} a_1(x) = \partial_{x_1}^2 a_2(x) = \cdots = \partial_{x_1}^N a_N(x) = 0\,.$$

Now the same argument based on an order of $y_1, y_2, \ldots, y_n$, where the variable y_i is first, would prove

$$\partial_{x_i} a_{\sigma_1}(x) = \partial_{x_i}^2 a_{\sigma_2}(x) = \cdots = \partial_{x_i}^N a_{\sigma_N}(x) = 0\,.$$

when

$$m_{\sigma_1}(y), m_{\sigma_2}(y), \ldots, m_{\sigma_N}(y)\,.$$

are in the corresponding lex order. We can then be assured that for all i we have

$$\partial_{x_i}^N a_1(x) = \partial_{x_i}^N a_2(x) = \cdots = \partial_{x_i}^N a_N(x) = 0\,.$$

Thus $a_1(x), a_2(x), \ldots, a_N(x)$ must be polynomials precisely as asserted.

To identify the family of special fuctions $\Phi(x,y) = F(x,y)e^{x,y} \in \mathcal{SFR}(x,y)$ which satisfy the equation

$$L_m \Phi(x,y) \;=\; (y,y)\Phi(x,y)$$

we need the following corollary of Proposition 4.1.

Proposition 4.3

Let $f \in \mathcal{SR}_n(x,y)$ and suppose that it has a bigraded decomposition of the form

$$f(x,y) \;=\; \sum_{r_1\le r\le r_2}\ \sum_{s_1\le s\le s_2} f_{r,s} \tag{4.19}$$

Then, for $r_2<0$, the equation

$$\sum_{i=1}^{n} y_i\partial_{x_i} f(x,y) \;=\; 0 \tag{4.20}$$

forces

$$f(x,y) \;=\; 0 \tag{4.21}$$

in particular, when $r_2<0$, the equation

$$\sum_{i=1}^{n} y_i\partial_{x_i} f(x,y) \;=\; g(x,y) \tag{4.22}$$

has at most one solution, of the form given in 4.19.

On the other hand, for $r_2=0$, the equation in 4.22 forces $f(x,y)$ to be of the form

$$f(x,y) \;=\; g(y) \qquad \text{with} \qquad g(y)\in\mathcal{SR}(y)\,. \tag{4.23}$$

Proof

The bihomogeneity of the equation in 4.20 forces all the components of $f(x,y)$ to satisfy the same equation. So it is sufficient to prove 4.21 for a bihomogeneous f. So suppose that $f=f_{rs}$ is bihomogeneous of bidegree (r,s). This given, construct a factor $D(y)=\prod_{1\le i<j\le n}(y_i-y_j)^{q_{ij}}$ which clears all the y denominators occurring in f_{rs}. We shall then have an expansion of the form

$$D(y)f_{rs} \;=\; \sum_{|q|=d} a_q(x)y^q \qquad \big(\text{with } d=s+\textstyle\sum_{1\le i<j\le n} p_{ij}\big)$$

and each $a_q(x)\in\mathcal{SR}(x)$ homogeneous of degree r. Thus we can apply Proposition 4.2 to the rational function $g(x,y)=D(y)f_{rs}(x,y)$ and conclude that the coefficients $a_q(x)$ must be polynomials in x. But for $r<0$ this can only hold true when each of them vanishes identically, this proves $f_{rs}=0$ and 4.21 necessarily follows. Now when $r=0$ the only possibility is that each $a_p(x)$ is a homogeneous polynomial of degree 0 in x, i.e. a constant. Thus in this case we must have scalars a_q giving the expansion

$$g(x,y) \;=\; \sum_q a_q y^q$$

in particular we derive that

$$f_{rs}(x,y) \;=\; \sum_q a_q \frac{y^q}{\prod_{1\le r<s\le n}(y_r-y_s)^{q_{rs}}} \in \mathcal{SR}_n(y)$$

In conclusion, we see that when $r_2 = 0$, the equation in 4.20 forces $f_{r,s} = 0$ when $r < 0$ yielding the equality

$$f(x,y) \;=\; \sum_{s_1 \le s \le s_2} f_{o,s}(y)$$

with each $f_{o,s}(y) \in \mathcal{SR}_n(y)$. This proves 4.23. Finally if 4.22 had two solutions $f'(x,y)$ and $f''(x,y)$ then the difference $f(x,y) = f'(x,y) - f''(x,y)$ would be a solution of 4.20 and for $r_2 < 0$ it would necessarily vanish identically.

Theorem 4.2

Let $\Phi(x,y) = F(x,y)e^{(x,y)} \in \mathcal{SFR}(x,y)$ and suppose that $\Phi(x,y)$ satisfies the equation

$$L_m\Phi(x,y) \,-\, (y,y)\Phi(x,y) \;=\; 0. \tag{4.24}$$

Suppose further that we have the bigraded decomposition

$$F \;=\; \sum_{r_1 \le r \le r_2} \sum_{s_1 \le s \le s_2} F_{r,s} \tag{4.25}$$

Then

$$r_2 < d_m = m\binom{m}{2} \qquad \Longrightarrow \qquad \Phi(x,y) \;=\; 0\,. \tag{4.26}$$

In particular the Baker-Akhiezer function $\Psi_m(x,y)$ is the unique element of $\mathcal{SFR}(x,y)$ which satisfies 4.24 and whose multiplier $F(x,y)$ has the form

$$F(x,y) \;=\; \Pi(x)^m\Pi(y)^m \;+\; \{\textit{terms of degree less than } d_m \textit{ in } x_1, x_2, \ldots, x_n\} \tag{4.27}$$

Proof

Proposition 4.1 gives that the function

$$f \;=\; \sum_{r_1 \le r \le r_2} \sum_{s_1 \le s \le s_2} \frac{F_{r,s}(x,y)}{\Pi(x)^m\Pi(y)^m} \tag{4.28}$$

satisfies the equation

$$2\sum_{i=1}^{n} y_i\, \partial_{x_i} f(x,y) \;=\; -\Delta_2 f(x,y) \;+\; 2(m+m^2) f(x,y) \sum_{1 \le i < j \le n} \tfrac{1}{(x_i - x_j)^2}\,. \tag{4.29}$$

Setting

$$\overline{r}_i = r_i - d_m \qquad \text{and} \qquad \overline{s}_i = s_i - d_m \qquad (\text{for } i = 1,2\ )$$

we can rewrite 4.28 in the form

$$f(x,y) \;=\; \sum_{\overline{r}_1 \le r \le \overline{r}_2} \sum_{\overline{s}_1 \le s \le \overline{s}_2} f_{r,s}(x,y) \qquad \text{with } f_{r,s} \in \mathcal{H}_{r,s}\big(SR_n(x,y)\big). \tag{4.30}$$

and then the bihomogeneity of the equation in 4.29 forces the recursions

$$2\sum_{i=1}^{n} y_i\,\partial_{x_i} f_{r,s}(x,y) \;=\; -\Delta_2 f_{r+1,s+1}(x,y) \;+\; 2(m+m^2) f_{r+1,s+1}(x,y) \sum_{1\le i<j\le n} \tfrac{1}{(x_i-x_j)^2} \qquad (\forall \quad r<\overline{r}_2) \qquad 4.31$$

as well as the equality

$$2\sum_{i=1}^{n} y_i\,\partial_{x_i} f_{\overline{r}_2,s}(x,y) \;=\; 0\,. \qquad (\forall \quad \overline{s}_1 \le s \le \overline{s}_2) \qquad 4.32$$

However, note that for $r_2 < d_m$, we have $\overline{r}_2 < 0$, so we can apply Proposition 4.3 and conclude that

$$f_{\overline{r}_2,s}(x,y) \;=\; 0\,. \qquad (\forall \quad \overline{s}_1 \le s \le \overline{s}_2)$$

But now 4.31 for $r = \overline{r}_2 - 1$ reduces to

$$2\sum_{i=1}^{n} y_i\,\partial_{x_i} f_{\overline{r}_2-1,s}(x,y) \;=\; 0 \qquad (\forall \quad \overline{s}_1 \le s \le \overline{s}_2) \qquad 4.33$$

and here again Proposition 4.2 can be applied to yield

$$f_{r_x-1,s}(x,y) \;=\; 0\,. \qquad (\forall \quad \overline{s}_1 \le s \le \overline{s}_2)$$

Obviously this argument can be repeated and recursively obtain that 4.32 forces the vanishing of all the components $f_{r,s}(x,y)$ and ultimately the vanishing of $F(x,y)$ as well as $\Phi(x,y)$. This proves the implication in 4.26.

Finally, note that if $F(x,y)$ is as given in 4.27 then the difference

$$\Phi_1(x,y) \;=\; \Phi(x,y) - \Psi_m(x,y)$$

will be an element of $\mathcal{SFR}(x,y)$ of the form

$$\Phi_1(x,y) \;=\; \Big(\sum_{r_1\le r\le r_2}\;\sum_{s_1\le s\le s_2} a_{r,s}(x,y)\Big) e^{(x,y)}$$

with $r_2 < d_m$. Thus we can apply 4.26 to it and derive that it must identically vanish. This proves the equality

$$\Phi(x,y) \;=\; \Psi_m(x,y)$$

and completes our argument.

To continue with our developents we need further notation. We will use operators which act on $y_1, y_2, \ldots, y_n$ in the same manner as some of the operators introduced in previous sections acted on $x_1, x_2, \ldots, x_n$. To do this we will use the same symbols as before and simply append a superscript "x" or "y" to indicate whether they act on $x_1, x_2, \ldots, x_n$ or $y_1, y_2, \ldots, y_n$. For instance, with this convention we have

$$L_m^x \;=\; \Delta_2^x \,-\, 2\sum_{1\le i<j\le n} \tfrac{1}{x_i-x_j}(\partial_{x_i}-\partial_{x_j}) \qquad \text{with} \qquad \Delta_2^x \;=\; \sum_{i=1}^{n} \partial_{x_i}^2$$

and

$$L_m^y \;=\; \Delta_2^y \,-\, 2 \sum_{1\le i<j\le n} \tfrac{1}{y_i-y_j}(\partial_{y_i}-\partial_{y_j}) \qquad \text{with} \qquad \Delta_2^y \;=\; \sum_{i=1}^n \partial_{y_i}^2\,.$$

Similarly, for the Euler operators in $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ we set

$$E^x \;=\; \sum_{i=1}^n x_i \partial_{x_i} \qquad \text{and} \qquad E^y \;=\; \sum_{i=1}^n y_i \partial_{y_i}$$

We should also note that a simple calculation, based on 3.17, yields the commutator relations

$$a)\quad L_m^x\,(\underline{x},\underline{x}) \,-\, (\underline{x},\underline{x})\,L_m^x \;=\; c\,I + 4E^x \qquad \text{and} \qquad b)\quad L_m^y\,(\underline{y},\underline{y}) \,-\, (\underline{y},\underline{y})\,L_m^y \;=\; c\,I + 4E^y \tag{4.34}$$

with I the identity operator and

$$c \;=\; 2n - 4m\tbinom{n}{2}\,. \tag{4.35}$$

This given, we are in a position to state and prove the following remarkable fact

Proposition 4.4

Setting

$$\Phi_m(x,y) \;=\; L_m^y \Psi_m(x,y) - (x,x)\Psi_m(x,y) \tag{4.36}$$

we have

$$L_m^x \Phi_m(x,y) \;=\; (y,y)\Phi_m(x,y) \tag{4.37}$$

Proof

To begin note that since Ω_m^x preserves homogeneity and x-degree it will necessarily commute with the Euler operator E^x. On the other hand E^y and Ω^x commute as well since they act on different sets of variables. This gives

$$\begin{aligned} E^y \Psi_m(x,y) &= E^y \Omega_m^x e^{(x,y)} \\ &= \Omega_m^x E^y e^{(x,y)} \\ &= \Omega_m^x (x,y) e^{(x,y)} \\ &= \Omega_m^x E^x e^{(x,y)} \\ &= E^x \Omega_m^x e^{(x,y)} \;=\; E^x \Psi_m(x,y) \end{aligned}$$

and 4.34 a) immediately gives that we also have

$$\Big(L_m^x(\underline{x},\underline{x}) - (\underline{x},\underline{x})L_m^x\Big)\Psi_m(x,y) \;=\; \Big(L_m^y(\underline{y},\underline{y}) - (\underline{y},\underline{y})L_m^y\Big)\Psi_m(x,y)\,. \tag{4.38}$$

Now, from 4.36 we derive that the left-hand side of 4.37 can be rewritten as

$$\begin{aligned} LHS &= L_m^x\big(L_m^y \Psi_m(x,y) - (x,x)\Psi_m(x,y)\big) \\ &= L_m^y L_m^x \Psi_m(x,y) - L_m^x (x,x)\Psi_m(x,y) \\ \text{(by 4.3)} &= L_m^y (y,y)\Psi_m(x,y) - L_m^x (x,x)\Psi_m(x,y) \end{aligned}$$

and for the right-hand side we have

$$\begin{aligned} RHS &= (y,y)\big(L_m^y \Psi_m(x,y) - (x,x)\Psi_m(x,y)\big) \\ &= (y,y)L_m^y \Psi_m(x,y) - (y,y)(x,x)\Psi_m(x,y) \\ \text{(by 4.3 again)} &= (y,y)L_m^y \Psi_m(x,y) - (x,x)L_m^x \Psi_m(x,y) \end{aligned}$$

and the two are equal if and only if

$$L_m^y(y,y)\Psi_m(x,y) - L_m^x(x,x)\Psi_m(x,y) \;=\; (y,y)L_m^y\Psi_m(x,y) - (x,x)L_m^x\Psi_m(x,y)$$

but this simply another way of writing the identity in 4.38. This proves 4.37 and completes our argument.

An immediate corollary of Proposition 4.3 is the following basic identity satisfied by the Baker-Akhiezer function

Theorem 4.3

$$L_m^y \Psi_m(x,y) \;=\; (x,x)\Psi_m(x,y) \qquad 4.39$$

Proof

Since $\Psi_m(x,y) \;=\; P_m(x,y)e^{(x,y)}$ with $P_m(x,y)$ a polynomial in x and y we can apply the identity in 4.6 with the role of x and y reversed and obtain that the function $\Phi_m(x,y)$ defined by 4.36 is of the form

$$\Phi_m(x,y) = G_m(x,y)e^{(x,y)} \qquad 4.40$$

with

$$G_m(x,y) = \Pi(y)^m\Pi(x)^m\Big(2\sum_{i=1}^n x_i\,\partial_{y_i} f_m(x,y) + \Delta_2^y f_m(x,y) \;-\; 2(m+m^2)f_m(x,y)\sum_{1\le i<j\le n}\tfrac{1}{(y_i-y_j)^2}\Big) \qquad 4.41$$

and

$$f_m(x,y) \;=\; \frac{P_m(x,y)}{\Pi(y)^m\Pi(x)^m} \qquad 4.42$$

It is easily seen that the rational function defined by 4.41 lies in $\mathcal{SR}_n(x,y)$ thus 4.40 gives that $\Phi_m(x,y) \in \mathcal{SFR}(x,y)$. Since Proposition 4.4 assures that we have

$$L_m^x\Phi_m(x,y) \;=\; (y,y)\Phi_m(x,y)$$

we shall be able to derive that $\Phi_m(x,y)$ vanishes identically and prove 4.39, as soon as we derive that $G_m(x,y)$ has a bigraded expansion of the form

$$G_m(x,y) \;=\; \sum_{r_1\le r\le r_2}\sum_{s_1\le s\le s_2} G_{r,s}(x,y)$$

with $r_2 < d_m$. Or, equivalently that the rational function

$$g_m(x,y) \;=\; 2\sum_{i=1}^n x_i\,\partial_{y_i} f_m(x,y) + \Delta_2^y f_m(x,y) \;-\; 2(m+m^2)f_m(x,y)\sum_{1\le i<j\le n}\tfrac{1}{(y_i-y_j)^2} \qquad 4.43$$

has a a bigraded expansion of the form

$$g_m(x,y) \;=\; \sum_{\overline{r}_1\le r\le \overline{r}_2}\;\sum_{\overline{s}_1\le s\le \overline{s}_2} g_{r,s}(x,y) \qquad\qquad \text{with } \overline{r}_2<0. \tag{4.44}$$

Now note that from 4.5 and 4.42 it follows that $f_m(x,y)$ has a bigraded expansion of the form

$$f_m(x,y) \;=\; \sum_{-d_m\le r\le 0} f_{r,r}(x,y) \tag{4.45}$$

Thus using this in 4.43 we immediately derive that we must have

$$g_{r,s} \;=\; 2\sum_{i=1}^n x_i\,\partial_{y_i} f_m\big|_{r-1,s+1} + \Delta_2^y f_m\big|_{r,s+2} \,-\, 2(m+m^2) f_m\big|_{r,s+2} \sum_{1\le i<j\le n} \tfrac{1}{(y_i-y_j)^2}$$

where we have used the symbol "$\big|_{a,b}$" for the operation of extracting a bigraded component of bidegree(a,b). Now, in view of 4.45, in order for the right hand side not to vanish we must have the equality $s=r-2$. This gives

$$g_{r,s} \;=\; 0 \qquad\qquad \text{if } s\neq r-2$$

and

$$g_{r+1,r-1} \;=\; 2\sum_{i=1}^n x_i\,\partial_{y_i} f_{r,r} + \Delta_2^y f_{r+1,r+1} \,-\, 2(m+m^2) f_{r+1,r+1} \sum_{1\le i<j\le n} \tfrac{1}{(y_i-y_j)^2} \tag{4.46}$$

This means that the bihomogeneous component of $g_m(x,y)$ of highest x-degrees are

$$g_{1,-1} \;=\; 2\sum_{i=1}^n x_i\,\partial_{y_i} f_{0,0} \tag{4.47}$$

and

$$g_{0,-2} \;=\; 2\sum_{i=1}^n x_i\,\partial_{y_i} f_{-1,-1} + \Delta_2^y f_{0,0} \,-\, 2(m+m^2) f_{0,0} \sum_{1\le i<j\le n} \tfrac{1}{(y_i-y_j)^2} \tag{4.48}$$

all the other components have negative x degrees. Thus to prove 4.44 we need only show that both $g_{1,-1}$ and $g_{0,-2}$ do vanish identically. However, from 4.5 it follows that $f_{0,0}=1$ and 4.47 gives

$$g_{1,-1} \;=\; 0$$

as desired and from 4.48 we get

$$g_{0,-2} \;=\; 2\sum_{i=1}^n x_i\,\partial_{y_i} f_{-1,-1} \,-\, 2(m+m^2) \sum_{1\le i<j\le n} \tfrac{1}{(y_i-y_j)^2}\,. \tag{4.49}$$

So we are left to show that we have

$$\sum_{i=1}^n x_i\,\partial_{y_i} f_{-1,-1} \;=\; (m+m^2) \sum_{1\le i<j\le n} \tfrac{1}{(y_i-y_j)^2}\,. \tag{4.50}$$

To determine $f_{-1,-1}$ we apply Proposition 4.1 to the Baker-Akhiezer function and derive from 4.3 that the function in 4.45 must satisfy the differential equation

$$2\sum_{i=1}^{n} y_i\,\partial_{x_i} f_m(x,y) \;\;=\;\; -\Delta_2^x f_m(x,y) \;\;+\;\; 2(m+m^2) f_m(x,y) \sum_{1\le i<j\le n} \tfrac{1}{(x_i-x_j)^2}\,. \qquad 4.51$$

Now using 4.45 we deduce that the bihomoheneous components of $f_m(x,y)$ must satisfy the recursions

$$2\sum_{i=1}^{n} y_i\,\partial_{x_i} f_{r,r} \;\;=\;\; -\Delta_2^x f_{r+1,r+1} \;\;+\;\; 2(m+m^2) f_{r+1,r+1} \sum_{1\le i<j\le n} \tfrac{1}{(x_i-x_j)^2}\,. \qquad 4.52$$

In particular setting $r=-1$ and using $f_{0,0}=1$ we get that $f_{-1,-1}$ satisfies the equation

$$\sum_{i=1}^{n} y_i\,\partial_{x_i} f_{-1,-1} \;\;=\;\; (m+m^2) \sum_{1\le i<j\le n} \tfrac{1}{(x_i-x_j)^2}\,. \qquad 4.53$$

Now it is easily seen that we have

$$\sum_{i=1}^{n} y_i\,\partial_{x_i} \sum_{1\le r<s\le n} \tfrac{1}{(x_r-x_s)(y_r-y_s)} \;\;=\;\; -\sum_{1\le i<j\le n} \tfrac{1}{(x_i-x_j)^2}\,.$$

Thus a solution of 4.53 is given by the function

$$f_{-1,-1} \;\;=\;\; -(m+m^2) \sum_{1\le r<s\le n} \tfrac{1}{(x_r-x_s)(y_r-y_s)} \qquad 4.54$$

However, since $f_{-1,-1}$ is of negative x degree, we can use the uniqueness part of Proposition 4.3 and conclude that this is the only solution. This given we derive that

$$\begin{aligned} \sum_{i=1}^{n} x_i\,\partial_{y_i} f_{-1,-1} \;\;&=\;\; -(m+m^2) \sum_{i=1}^{n} x_i\,\partial_{y_i} \sum_{1\le r<s\le n} \tfrac{1}{(x_r-x_s)(y_r-y_s)} \\ &=\;\; -(m+m^2) \sum_{1\le r<s\le n} \tfrac{-(x_r-x_s)}{(x_r-x_s)(y_r-y_s)^2}\,. \end{aligned}$$

and this is simply another way of writing 4.50. This proves that $\Phi_m(x,y)$ identically vanishes and 4.39 must hold true precisely as asserted.

This brings us to one of the crucial properties of the Baker-Akhiezer function.

Theorem 4.4

$$\Psi_m(x,y) \;\;=\;\; \Psi_m(y,x) \qquad 4.55$$

Proof

Note that from 4.5 we derive that

$$\Psi_m(y,x) \;\;=\;\; P_m(y,x) e^{(x,y)}$$

with

$$P_m(y,x) \;=\; \Pi(x)^m\Pi(y)^m + \sum_{0\le k<m\binom{n}{2}} a_k(y,x) \qquad 4.56$$

where $a_k(y,x)$ is homogeneous in $x_1,x_2,\ldots,x_n$ and $y_1,y_2,\ldots,y_n$, and of degree k in both variables.

Now interchanging x and y in 4.39 we derive that

$$L_m^x\Psi_m(y,x) \;=\; (y,y)\Psi_m(y,x)$$

and this together with 4.56 puts us in a position to use the uniqueness part of Theorem 4.2 and thereby conclude that $\Psi_m(y,x)$ and the Baker-Akhiezer function must be one and the same, proving 4.55.

Theorem 4.2 has a windfall of consequences as we shall soon see. In particular we may deduce from it an extensive generalisation of the identity in 4.3.

Theorem 4.5

For any symmetric polynomial $P(x)$ set

$$\gamma_p(m) \;=\; \Gamma p\big(\nabla_1(m),\nabla_1(m),\ldots,\nabla_n(m)\big) \qquad 4.57$$

This given, we have

$$\begin{array}{lrcl} a) & \gamma_p^x(m)\Psi_m(x,y) &=& p(y)\Psi_m(x,y) \\ b) & \gamma_p^y(m)\Psi_m(x,y) &=& p(x)\Psi_m(x,y) \end{array} \qquad 4.58$$

Proof

Clearly 4.58 b) follows from 4.58 a) by interchanging x and y and using 4.55. Moreover, it is sufficient to establish 4.47 for p homogeous. So assuming that p is of degree $d>0$ it follows from Proposition 3.4 that $\gamma_p^x(m)$ is of the form

$$\gamma_p^x(m) \;=\; p(\partial_x) \;+\; \sum_{|q|<d} f_q(x)\,\partial_x^q \qquad 4.59$$

where for some scalar coefficients c_{pq} we have

$$f_q(x) \;=\; \sum_{|p|=d-|q|} \frac{c_{pq}}{\prod_{rs}(x_r-x_s)^{p_{rs}}}\,. \qquad 4.60$$

Recall that we also have $\Psi_m(x,y)=P_m(x,y)e^{(x,y)}$ with

$$P_m(x,y) \;=\; \Pi(x)^m\Pi(y)^m + \sum_{0\le k<m\binom{n}{2}} a_k(x,y) \qquad 4.61$$

This given set

$$\Phi(x,y) \;=\; \gamma_p^x(m)\Psi_m(x,y) \,-\, p(y)\Psi_m(x,y) \qquad 4.62$$

Now it is easily seen from 4.59,4.60 and 4.61 that

$$\Phi \in \mathcal{SFR}(x,y)\,, \qquad 4.63$$

and it follows from 2.23 and 4.3 that we have

$$\begin{aligned} L_m^x \Phi(x,y) &= L_m^x \gamma_p^x(m)\Psi_m(x,y) \,-\, L_m^x p(y)\Psi_m(x,y) \\ &= \gamma_p^x(m) L_m^x \Psi_m(x,y) \,-\, p(y) L_m^x \Psi_m(x,y) \\ &= \gamma_p^x(m)(y,y)\Psi_m(x,y) \,-\, p(y)(y,y)\Psi_m(x,y) \\ &= (y,y)\big(\gamma_p^x(m)\Psi_m(x,y) \,-\, p(y)\Psi_m(x,y)\big) \;=\; (y,y)\Phi(x,y)\,. \end{aligned}$$

We are thus again in the realm of Theorem 4.2. So to prove the vanishing of Φ and thereby establishing 4.58 a) we need only check that $\Phi(x,y) = F(x,y)e^{(x,y)}$ with F having a bigraded decomposition of the form

$$F(x,y) \;=\; \sum_{r_1 \le r < d_m} \sum_{s_1 \le s \le s_2} F_{r,s}\,. \tag{4.64}$$

Briefly we only need to show that $F(x,y)$ has no terms of degree $d_m = m\binom{n}{2}$ in x. Now from 4.59.4.61 and 4.62 it follows that

$$\begin{aligned} F(x,y) \;=\; e^{-(x,y)}\Big(& p(\partial_x)\Pi(x)^m\Pi(y)^m e^{(x,y)} \,-\, p(y)\Pi(x)^m\Pi(y)^m - \sum_{0\le k< m\binom{n}{2}} p(y)a_k(x,y)e^{(x,y)} \\ &+ \sum_{0\le k< m\binom{n}{2}} p(\partial_x)a_k(x,y)e^{(x,y)} \\ &+ \sum_{|q|<d} f_q(x)\,\partial_x^q\,\Pi(x)^m\Pi(y)^m e^{(x,y)} \\ &+ \sum_{|q|<d}\sum_{0\le k< m\binom{n}{2}} f_q(x)\,\partial_x^q\,a_k(x,y)e^{(x,y)}\Big) \end{aligned}$$

All the terms that involve differentiations here are easily handled by means of the identity in 3.1. This given, we immediately see that the only way that we can produce an x-degree d_m term from the first summand in the first line of the above display is to let $p(\partial_x)$ act entirely on $e^{(x,y)}$ but this produces the term $p(y)\Pi(x)^m\Pi(y)^m$ which is immediately cancelled out by the next summand. Since each summand $a_k(x,y)$ in 4.5 is of degree $k < d_m$ we see that all the remaining summands in the first line are of x-degree less than d_m. The same reasoning applies to the terms produced by the second line of the display, even if let $p(\partial_x)$ act entirely on $e^{(x,y)}$. Since we see from 4.60 that all the factors $f_q(x)$ are of negative x-degree, there is no way a term of x-degree less than d_m can be produced by any of the summands in the last two lines, even if we let ∂_x^q act entirely on $e^{(x,y)}$. This proves 4.64 and completes our argument.

Remarkably, it was shown by Chalykh and Veselov in [] that an operator $\gamma_p(m)$ satisfying the identities in 4.58 can be constructed also for every m-quasi-invariant. Our next task here is to give this construction. But before proceeding with it we need to establish another truly surprising auxiliary result also due to Chalykh and Veselov.

Theorem 4.6

Let $\Phi(x,y)$ be a formal power series of the form

$$\Phi(x,y) \;=\; P(x,y)e^{(x,y)} \tag{4.65}$$

with $P(x,y)$ a polynomial in y of degree d, with coefficients in $\mathcal{SR}_n(x)$. Suppose further that, as a function of y, $\Phi(x,y)$ is y-m-quasi-invariant. In symbols

$$\Phi(x,y) \in \mathcal{QI}_m[Y_n]\,.$$

Then the homogeneous component of y-degree $d-i$ in $P(x,y)$ is necessarily divisible by $\Pi(y)^{m-i}$. More precisely if

$$P(x,y) \;=\; P^{(0)}(x,y) + P^{(1)}(x,y) + \cdots + P^{(d)}(x,y) \tag{4.66}$$

with $P^{(r)}(x,y)$ y-homogeneous of degree r then $d \geq m\binom{n}{2}$ and

$$P^{(d-i)}(x,y) \;=\; \Pi(y)^{m-i}Q_i(x,y) \qquad\qquad (\text{for } i = 0,1,\ldots,m-1) \tag{4.67}$$

with $Q_i(x,y)$ a y-homogeneous polynomial of degree $d-i-(m-i)\binom{n}{2}$.

In particular $\Phi(x,y)$ must identically vanish if $d < m\binom{n}{2}$.

Proof

We shall begin with some observations. Combining 4.66 with the expansion of the exponential the relation in 4.65 can be rewritten as

$$\Phi(x,y) \;=\; \sum_{k\geq 0} \Phi^{(k)}(x,y) \tag{4.68}$$

with

$$\Phi^{(k)}(x,y) \;=\; \sum_{\substack{r+s=k \\ r\leq d}} P_r(x,y)(x,y)^s/s! \tag{4.69}$$

Note that this is a polynomial in x,y and it gives the y-homogeneous component of y-degree k in $\Phi(x,y)$.

The assumption $\Phi(x,y) \in \mathcal{QI}_m[Y_n]$ is to be interpreted as saying that we have

$$\Phi^{(k)}(x,y) \in \mathcal{QI}_m[Y_n] \qquad\qquad (\text{for all } k \geq 0)$$

Here and after we shall use the abreviations

$$u = (x_1 - x_2)/2\ , \quad \overline{u} = (x_1 + x_2)/2\ , \quad v = y_1 - y_2\ , \quad \overline{v} = (y_1 + y_2) \tag{4.70}$$

Since

$$x_1y_1 + x_2y_2 \;=\; 2u\,v \;+\; 2\overline{u}\,\overline{v}$$

we may write

$$e^{(x,y)} \;=\; e^{2u\,v}e^{2\overline{u}\,\overline{v}+\sum_{i=3}^n x_iy_i} \tag{4.71}$$

here and after we shall set

$$e^{2\overline{u}\,\overline{v}+\sum_{i=3}^n x_iy_i} \;=\; e^{(\overline{x},\overline{y})}$$

so 4.71 may be written as

$$e^{(x,y)} \;=\; e^{2u\,v} e^{(\overline{x},\overline{y})} . \qquad 4.72$$

Observe next that the m y-quasi-invariance of $\Phi^{(k)}(x,y)$ may be simply expressed by stating that

$$\Phi^{(k)}(x,y) \;=\; \Phi_0^{(k)}(u,\overline{x},\overline{y}) + \Phi_2^{(k)}(u,\overline{x},\overline{y})v^2 + \cdots + \Phi_{2m}^{(k)}(u,\overline{x},\overline{y})v^{2m} + v^{2m+1}\Psi^{(k)}(u,v,\overline{x},\overline{y})$$

where $\overline{x} = (\overline{u}, x_3, \ldots x_n)$ and $\overline{y} = (\overline{v}, y_3, \ldots y_n)$, we thus obtain the decomposition

$$\Phi(x,y) \;=\; \Phi_0(u,\overline{x},\overline{y}) + \Phi_2(u,\overline{x},\overline{y})v^2 + \cdots + \Phi_{2m}(u,\overline{x},\overline{y})v^{2m} + v^{2m+1}\Psi(u,v,\overline{x},\overline{y}) \qquad 4.73$$

with

$$\Phi_{2s}(u,\overline{x},\overline{y}) \;=\; \sum_{k\geq 0} \Phi_{2s}^{(k)}(u,\overline{x},\overline{y}) \,, \qquad \Psi(u,v,\overline{x},\overline{y}) \;=\; \sum_{k\geq 0} \Psi^{(k)}(u,v,\overline{x},\overline{y}) \,. \qquad 4.74$$

At the cost of adding denominators containing difffernces $x_i - x_j$ we may also decompose $P^{(s)}(x,y)$ in the form

$$P^{(s)}(x,y) \;=\; P_0^{(s)}(u,\overline{x},\overline{y}) \;+\; P_1^{(s)}(u,\overline{x},\overline{y})2u\,v \;+\; \cdots + P_d^{(s)}(u,\overline{x},\overline{y})(2u\,v)^d$$

where $P_r^{(s)}(u,\overline{x},\overline{y})$ is a y-homogeneous polynomial of y-degree $s-r$. Of course some of the terms in this sum could very well vanish (certainly $P_r^{(s)}(u,\overline{x},\overline{y})$ for $r > s$) but we write it this way for notational convenience.

Setting

$$P_r(u,\overline{x},\overline{y}) \;=\; \sum_{s=0}^{d} P_r^{(s)}(u,\overline{x},\overline{y})$$

we may write

$$P(x,y) \;=\; P_0(u,\overline{x},\overline{y}) \;+\; P_1(u,\overline{x},\overline{y})2u\,v \;+\; \cdots + P_d(u,\overline{x},\overline{y})(2u\,v)^d$$

Using this and 4.72 in 4.65 we obtain te expansion

$$\Phi(x,y) \;=\; e^{(\overline{x},\overline{y})}\Big(\sum_{r=0}^{d}\sum_{s\geq 0} \tfrac{1}{s!}(2u\,v)^{r+s} P_r(u,\overline{x},\overline{y})\Big) \,.$$

comparing with 4.73 gives the equations

$$\sum_{s+r=2k+1} \tfrac{1}{s!} P_r(u,\overline{x},\overline{y}) \;=\; 0 \qquad (\text{for } k = 0,1,\cdots,m)$$

Since each of the y-homogeneous components of this polynomial must separately vanish, and $P_r^{(e+r)}(u,\overline{x},\overline{y})$ is a y-homogeneous polynomial of y-degree e,we get for any e

$$\sum_{s+r=2k+1} \tfrac{1}{s!} P_r^{(e+r)}(u,\overline{x},\overline{y}) \;=\; 0 \qquad (\text{for } k = 0,1,\cdots,m) \qquad 4.75$$

Now for $k = 0$ this gives (omitting the dependence on $u,\overline{x},\overline{y}$)

$$\tfrac{1}{1!}P_0^{(e)} + \tfrac{1}{0!}P_1^{(e+r)} \;=\; 0 \qquad 4.76$$

Setting $e = d$ reduces it to

$$P_0^{(d)} \;=\; 0\,. \tag{4.77}$$

The equation in 4.75 for $k = 1$ gives

$$\tfrac{1}{3!}P_0^{(e)} + \tfrac{1}{2!}P_1^{(e+1)} + \tfrac{1}{1!}P_2^{(e+2)} + \tfrac{1}{0!}P_3^{(e+3)} \;=\; 0 \tag{4.78}$$

and setting $e = d-1$ in 4.76 and 4.78 gives the system

$$\begin{aligned} \tfrac{1}{1!}P_0^{(d-1)} + \tfrac{1}{0!}P_1^{(d)} &= 0\\ \tfrac{1}{3!}P_0^{(d-1)} + \tfrac{1}{2!}P_1^{(d)} &= 0 \end{aligned}$$

which forces

$$P_0^{(d-1)} \;=\; 0\,, \qquad P_1^{(d)} \;=\; 0.$$

Now for $k = 2$ we get from 4.75

$$\tfrac{1}{5!}P_0^{(e)} + \tfrac{1}{4!}P_1^{(e+1)} + \tfrac{1}{3!}P_2^{(e+2)} + \tfrac{1}{2!}P_3^{(e+3)} + \tfrac{1}{1!}P_4^{(e+4)} + \tfrac{1}{0!}P_5^{(e+5)} \;=\; 0 \tag{4.79}$$

and setting $e = d-2$ in 4.76, 4.78 and 4.79 we get

$$\begin{aligned} &\tfrac{1}{1!}P_0^{(d-2)} + \tfrac{1}{0!}P_1^{(d-1)} \;=\; 0\\ &\tfrac{1}{3!}P_0^{(d-2)} + \tfrac{1}{2!}P_1^{(d-1)} + \tfrac{1}{1!}P_2^{(d)}\\ &\tfrac{1}{5!}P_0^{(d-2)} + \tfrac{1}{4!}P_1^{(d-1)} + \tfrac{1}{3!}P_2^{(d)} \end{aligned}$$

and since

$$det \begin{bmatrix} \frac{1}{1!} & \frac{1}{0!} & 0\\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!}\\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} \end{bmatrix} \neq 0$$

It follows that we must have

$$P_0^{(d-2)} \;=\; P_1^{(d-1)} \;=\; P_2^{(d)} \;=\; 0$$

Clearly we can continue in this manner, where at the k^{th} step we have the system of $k+1$ homogeneous linear equations

$$\sum_{s=0}^{k} \tfrac{1}{(2i+1-s)!}P_s^{(d-k+s)} \;=\; 0 \qquad\qquad (\text{for } i = 0, 1, \cdots, k-1)$$

with determinant

$$D_k \;=\; \det\left\| \tfrac{1}{(2i+1-s)!} \right\|_{i,s=0}^{k}.$$

Now it is well known see Macdonald [] that D_k equals 1 over the product of the hooks of the $k+1$-staircase partition. This gives

$$D_k \;=\; \frac{1}{(2k+1)!!(2k-1)!!\cdots 3!!\,1!!}$$

where $(2k+1)!! = (2k+1)\cdot(2k-1)\cdots 3\cdot 1$. At any rate what is important is that $D_k \neq 0$. To derive that at the k^{th} step we obtain the equalitites

$$P_0^{(d-k)} \;=\; P_1^{(d-k+1)} = \cdots = P_k^{(d)} \;=\; 0\,.$$

Since we can carry this out up to $k = m-1$, at the $(m-1)^{st}$ step we get

$$P_0^{(d-m+1)} \;=\; P_1^{(d-m+2)} = \cdots = P_{i-1}^{(d-m+i)} = \cdots = \; P_{m-1}^{(d)} \;=\; 0\,.$$

The result is that, for P^{d-m+i} we obtain

$$P_0^{(d-m+i)} \;=\; P_1^{(d-m+i)} \;=\; \cdots P_{i-1}^{(d-m+i)} \;=\; 0.$$

This implies that $P^{(d-m+i)}$ is divisible by $(x_1-x_2)^i$. Since the same argument applies to any pair of variables x_i, x_j and the differences $x_i - x_j$ are relatively prime, the inevitable conclusion is that $P^{(d-m+i)}$ is divisible by $\Pi(x)^i$. Or equivalently that $P^{(d-j)}$ is divisible by $\Pi(x)^{m-j}$ for $j = 0, 1, \ldots, m-1$. This proves 4.67. Note that if $d < m\binom{n}{2}$ the divisibility of $P^{(d)}$ by $\Pi(x)^m$ forces $P^{(d)} = 0$ contraddicting the hypothesis that $P(x,y)$ is a polynomial of degree d in y. So the only way to avoid contraddiction is that $\Phi(x,y)$ vanishes identically. This completes our proof.

Before we can proceed to establish some remarkable consequences of Theorem 4.6 we need an auxiliary result which is of intrinsic interest.

Proposition 4.5

Let $\gamma(x,\partial_x)$ be an operator of the form

$$\gamma(x,\partial_x) \;=\; \sum_{|p|\leq d} a_p(x)\partial_x^p \qquad\qquad (\textit{with } a_p(x) \in \mathcal{SR}_n(x)) \tag{4.80}$$

and suppose further that we have

$$\gamma(x,\partial_x)\Phi(x,y) \;=\; 0 \tag{4.81}$$

where $\Phi(x,y) = P(x,y)e^{(x,y)}$ with $P(x,y)$ a polynomial in $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. Then

$$\gamma(x,y) = \sum_{|p|\leq d} a_p(x)y^p \;=\; 0\,. \tag{4.82}$$

Proof

Assume, if possible that

$$\sum_{|p|=d} a_p(x)y^p \neq 0\,, \tag{4.83}$$

and let

$$P(x,y) \;=\; \sum_{r=0}^{k} P^{(r)}(x,y) \tag{4.84}$$

with $P^{(r)}(x,y)$ y-homogeneous of y-degree r and

$$P^{(k)}(x,y) \neq 0 \qquad 4.85$$

. This given, from 4.80 and Proposition 3.1 we derive

$$\begin{aligned}\gamma(x,\partial_x)\Phi(x,y) &= \sum_{r=0}^{k}\sum_{|p|\le d} a_p(x) \sum_{\alpha+\beta=p} \frac{p!}{\alpha!\beta!}\partial_x^\alpha e^{(x,y)}\partial_x^\beta P^{(r)}(x,y) \\ &= e^{(x,y)}\sum_{r=0}^{k}\sum_{|p|\le d} a_p(x) \sum_{\alpha+\beta=p} \frac{p!}{\alpha!\beta!} y^\alpha \partial_x^\beta P^{(r)}(x,y)\end{aligned}$$

Thus 4.81 gives

$$\sum_{r=0}^{k}\sum_{|p|\le d} a_p(x) \sum_{\alpha+\beta=p} \frac{p!}{\alpha!\beta!} y^\alpha \partial_x^\beta P^{(r)}(x,y) \;=\; 0. \qquad 4.86$$

Clearly, the y-homogeneous component of highest y-degree in the left-hand side of this equation is obtained by taking r and $|p|$ as large as possible and $|\beta|$ as small as possible. But these choices give the polynomial

$$\sum_{|p|=d} a_p(x) y^p P^{(k)}(x,y)$$

Since this term cannot be cancelled out by any other term in 4.86, it must separately vanish. But then 4.85 forces

$$\sum_{|p|=d} a_p(x) y^p \;=\; 0\,,$$

which is plain contraddiction with 4.83. But the only way to avoid such a contraddiction is to accept that the polynomial in 4.82 vanishes identically.

We are now ready to draw the consequences of Theorem 4.6.

Theorem 4.7

If $\Phi(x,y)$ is a formal power series of the form

$$\Phi(x,y) \;=\; P(x,y)e^{(x,y)} \qquad 4.87$$

with $P(x,y)$ a polynomial in y of degree $d \geq m\binom{n}{2}$, with coefficients in $\mathcal{SR}_n(x)$ and suppose further that

$$\Phi(x,y) \in \mathcal{QI}_m[Y_n]\,.$$

Then there is a unique differential operator

$$\gamma(x,\partial_x) \;=\; \sum_{|p|\le d-m\binom{n}{2}} a_p(x)\partial_y^p \qquad \big(\text{with } a_p(x) \in \mathcal{SR}_n(x)\big) \qquad 4.88$$

giving

$$\Phi(x,y) \;=\; \gamma(x,\partial_x)\Psi_m(x,y) \qquad 4.89$$

Proof

If $d = m\binom{n}{2}$ then Theorem 4.6 yields that $P(x,y)$ must be of the form

$$P(x,y) \;=\; a(x)\Pi(y)^m \;+\; \cdots(\text{terms of } y\text{-degree} < m\tbinom{n}{2})$$

with $a(x) \in \mathcal{SR}_n(x)$. But in this case the difference

$$\Phi'(x,y) \;=\; \Phi(x,y) - \tfrac{a(x)}{\Pi(x)^m}\,\Psi_m(x,y) \qquad 4.90$$

will satisfy the hypotheses of Theorem 4.6 as well. Infact, we have seen in Theorem 4.1 that $\Psi_m(x,y) \in \mathcal{QI}_m[X_n]$. But then the symmetry in 4.55 implies that $\Psi_m(x,y) \in \mathcal{QI}_m[Y_n]$. On the other hand by constructing the difference in 4.90 we have managed to cancel the term of y-degree $m\binom{n}{2}$ in $P(x,y)$ so that the polynomial $P'(x,y)$ giving $\Phi'(x,y) \;=\; P'(x,y)e^{(x,y)}$, will necessarily have y-degree $< m\binom{n}{2}$. Thus Theorem 4.6 will force the identity

$$\Phi(x,y) \;=\; \tfrac{a(x)}{\Pi(x)^m}\,\Psi_m(x,y)\,.$$

This proves 4.89 with γ the trivial multiplication operator

$$\gamma(x) \;=\; \frac{a(\underline{x})}{\Pi(\underline{x})^m}\,.$$

We can thus proceed by induction on d. So suppose the theorem true up to a $d-1 \geq m\binom{n}{2}$. And suppose that 4.87 holds true with $P(x,y)$ of y-degree d. Then our hypotheses combined with Theorem 4.6 imply that

$$P(x,y) \;=\; a(x,y)\Pi(y)^m \;+\; \cdots(\text{terms of } y\text{-degree } d)\,.$$

with $a(x,y)$ of y-degree $d - m\binom{n}{2}$. This given, note that from Theorem 4.1 it follows that

$$\begin{aligned} a(x,\partial_x)\tfrac{1}{\Pi(x)^m}\Psi_m(x,y) \;&=\; a(x,\partial_x)\Big(\Pi(y)^m + \cdots \text{terms of } y\text{-degree} < m\tbinom{n}{2}\Big)e^{(x,y)} \\ &=\; \Big(a(x,y)\Pi(y)^m + \cdots \text{terms of } y\text{-degree} < d\Big)e^{(x,y)} \end{aligned}$$

Thus the difference

$$\Phi'(x,y) \;=\; \Phi(x,y) \;-\; a(x,\partial_x)\tfrac{1}{\Pi(x)^m}\Psi_m(x,y) \qquad 4.91$$

will satisfy the hypothesis of Theorem 4.6 with the multiplier $P'(x,y)$ giving $\Phi'(x,y) = P'(x,y)e^{(x,y)}$ of y degree $< d$ so by induction we know there is an operator $\beta(x,\partial_x)$ giving

$$\Phi'(x,y) = \beta(x,\partial_x)\Psi_m(x,y)$$

Combining this with 4.91 proves 4.89 with

$$\gamma(x,\partial_x) \;=\; \beta(x,\partial_x) + a(x,\partial_x)\tfrac{1}{\Pi(x)^m}\,.$$

The uniqueness of the desired operator is an immediate consequence of Proposition 4.5. In fact, if there were two operators giving 4.89 than their difference would kill $\Psi_m(x,y)$ and propositio 4.5 would assure that it vanishe identically. This completes our proof.

We are now ready to prove the following fundamental fact

Theorem 4.8

For every m-quasi-invariant $q(x)$ of degree d there is a unique operator $\gamma_q(x,\partial_x)$ of the form

$$\gamma_q(x,\partial_x) \;=\; q(\partial_x) \;+\; \sum_{|p|<d} a_p(x)\partial_x^p \qquad \big(\textit{with } a_p(x)\in \mathcal{SR}_n(x)\big) \tag{4.92}$$

such that

$$\begin{aligned} &a) \quad \gamma_q(x,\partial_x)\Psi_m(x,y) = q(y)\Psi_m(x,y)\,. \\ &b) \quad \gamma_q(y,\partial_y)\Psi_m(x,y) = q(x)\Psi_m(x,y)\,. \end{aligned} \tag{4.93}$$

Moreover we have the commutativity relations.

$$\begin{aligned} &a) \quad L_m^x\gamma_q(x,\partial_x) \;=\; \gamma_q(x,\partial_x)L_m^x\,, \\ &b) \quad L_m^y\gamma_q(y,\partial_y) \;=\; \gamma_q(y,\partial_y)L_m^y\,. \end{aligned} \tag{4.94}$$

Proof

Clearly 4.93 b) follows from 4.93 a) by means of the symmetry in 4.55. We can also assume without loss that q is homogeneous. This given we see that, because of the m-y-quasi-invariance of both $q(y)$ and $\Psi_m(x,y)$, the difference

$$\Phi'(x,y) \;=\; q(y)\Psi_m(x,y) \,-\, q(\partial_x)\Psi_m(x,y) \tag{4.95}$$

satisfies the hypotheses of Theorem 4.6. Now from 4.5 we derive that

$$\begin{aligned} q(\partial_x)\Psi_m(x,y) \;&=\; q(\partial_x)\Big(\Pi(x)^m\Pi(y)^m \;+\; \cdots \text{terms of } y\text{-degree} < m\tbinom{n}{2}\Big)e^{(x,y)} \\ &=\; \Big(q(y)\Pi(x)^m\Pi(y)^m \;+\; \cdots \text{terms of } y\text{-degree} < d+m\tbinom{n}{2}\Big)e^{(x,y)} \end{aligned}$$

It then follows from 4.5 again that the multiplier $P'(x,y)$ giving $\Phi'(x,y)=P'(x,y)e^{(x,y)}$ of y-degree less than $d+m\binom{n}{2}$. This implies that the operator $\beta(x,\partial_x)$ which, according to Theorem 4.6, gives

$$\Phi'(x,y) \;=\; \beta(x,\partial_x)\Psi_m(x,y)$$

is of the form

$$\beta(x,\partial_x) = \sum_{|p|<d} a_p(x)\partial_y^p \qquad \big(\text{with } a_p(x)\in\mathcal{SR}_n(x)\big)$$

But then from 4.95 we derive that we have

$$\gamma(x,\partial_x)\Psi_m(x,y) \;=\; q(y)\Psi_m(x,y)$$

with

$$\gamma(x,\partial_x) \;=\; q(\partial_x) \;+\; \sum_{|p|<d} a_p(x)\partial_y^p\,.$$

This proves 4.93 a) with 4.92.

Finally, note that now 4.3 and 4.93 a) give

$$L_m^x\gamma_q(x,\partial_x)\Psi_m(x,y) \;=\; q(y)(y,y)\Psi_m(x,y) \;=\; (y,y)q(y)\Psi_m(x,y) \;=\; \gamma_q(x,\partial_x)L_m^x\Psi_m(x,y)\,,$$

that is

$$\big(L_m^x\gamma_q(x,\partial_x)\,-\,\gamma_q(x,\partial_x)\,L_m^x\big)\Psi_m(x,y) \;=\; 0\,.$$

and Proposition 4.5 imediately gives the equality

$$L_m^x\gamma_q(x,\partial_x) \;=\; \gamma_q(x,\partial_x)\,L_m^x$$

proving 4.94 a). This given, 4.94 b) follows from the symmetry in 4.55. The uniqueness of $\gamma_q(x,\partial_x)$ follows again from Proposition 4.5. This completes our proof.

We terminate this section with one final rather curious application of Theorem 4.7.

Theorem 4.9

There is a unique operator $U_m(x,\partial_x)$ of the form

$$U_m(x,\partial_x) \;=\; \Pi(x)^{-1}\Pi(\partial_x) \;+\; \sum_{|p|<\binom{n}{2}} u_p(x)\partial_x^p \qquad 4.96$$

with $u_p(x)\in\mathcal{SR}_n(x)$ giving

$$U_m(x,\partial_x)\Psi_m(x,y) \;=\; \Pi(y)^2\Psi_{m-1}(x,y) \qquad 4.97$$

In particular we must also have

$$U_m(x,\partial_x)\Omega_m^x \;=\; \Omega_{m-1}^x\Pi(\partial_x)^2 \;=\; \Gamma\Pi(\nabla(m))^2\Omega_{m-1}^x \qquad 4.98$$

Proof

Note that if $Q(x)\in\mathcal{QI}_{m-1}[X_n]$ then $\Pi(x)^2Q(x)\in\mathcal{QI}_m[X_n]$. Infact, for any pair $1\le i<j\le n$ we have

$$(1-s_{ij})\Pi(x)^2Q(x) \;=\; \Pi(x)^2\,(1-s_{ij})Q(x)$$

and the $m-1$-quasi-invariance of $Q(x)$ gives (for a suitable Q')

$$(1-s_{ij})\Pi(x)^2Q(x) \;=\; \Pi(x)^2(x_i-x_j)^{2m-1}\,Q'(x)$$

since $\Pi(x)^2$ contains the factor $(x_i-x_j)^2$ the m-quasi-invariance of $\Pi(x)^2Q(x)$ necessarily follows. Of course all of this remains true with $x_1,x_2,\ldots,x_n$ replaced by $y_1,y_2,\ldots,y_n$. In particular it follows that

$$\Pi(y)^2\Psi_{m-1}(x,y)\in\mathcal{QI}_m[Y_n]\,.$$

But, in view of 4.5, we see that this special function is of the form

$$\Phi(x,y) \;=\; \Big(\Pi(x)^{m-1}\Pi(y)^{m+1} \;+\; \sum_{0\le k<(m-1)\binom{n}{2}} a_p(x,y)\Pi(y)^2\Big)e^{(x,y)}$$

Thus the existence of the desired operator $U_m(x,\partial_x)$ immeditely follows by applying Theorem 4,7 to the difference

$$\Pi(y)^2\Psi_{m-1}(x,y)\,-\,\Pi(x)^{-1}\Pi(\partial_x)\Psi_m(x,y)\,.$$

Finally, both the identities in 4.98 are easily derived from 4.97 by means of Proposition 4.5. We leave this last verification to the reader.

5. Some remarkable actions of the Laplace operator.

This section is dedicated to the derivation of a number of useful of consequences the following surprising operator identity

Theorem 5.1

For any polynomial $P \in \mathbb{Q}[X_n]$ homogeneous of degree d we have

$$\frac{1}{2^d d!} \sum_{r=0}^{d} \binom{d}{r} (-1)^r \Delta_2^{d-r} P(\underline{x})\, \Delta_2^r \;\; = \;\; P(\partial_x)\,. \tag{5.1}$$

Proof

Note that we have the expansion

$$e^{t(a,x)} \;\; = \;\; \sum_{p_1 \geq 0} \sum_{p_2 \geq 0} \cdots \sum_{p_n \geq 0} t^{p_1+p_2+\cdots p_n} x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \, \frac{a_1^{p_1}}{p_1!} \frac{a_2^{p_2}}{p_2!} \cdots \frac{a_n^{p_n}}{p_n!}\,.$$

Using the abreviation

$$\frac{x_1^{p_1}}{p_1!} \frac{x_2^{p_2}}{p_2!} \cdots \frac{x_n^{p_n}}{p_n!} \;\; = \;\; \frac{x^p}{p!}$$

we may simply write this as

$$e^{t(a,x)} \;\; = \;\; \sum_{p} t^{|p|}\, x^p\, \frac{a^p}{p!}\,. \tag{5.2}$$

Thus we may view the exponential $e^{t(a,x)}$ as the generating function of the monomials in $x_1, x_2, \ldots, x_n$. In this vein, since

$$\partial_x^p e^{(x,y)} \;\; = \;\; y^p\, e^{(x,y)}$$

we can prove the identity in 5.1 by showing that for any exponent vector p, whose components add up to d, we have

$$2^d d!\, y^p\, e^{(x,y)} = \sum_{r=0}^{d} \binom{d}{r} (-1)^r \Delta_2^{d-r} x^p\, \Delta_2^r\, e^{(x,y)}\,.$$

Using 5.2 we may also rewrite this as

$$2^d d!\, y^p\, e^{(x,y)} \;\; = \;\; \sum_{r=0}^{d} \binom{d}{r} (-1)^r \Delta_2^{d-r} e^{t(x,a)} \Big|_{a^p/p!} \Big|_{t^d} \Delta_2^r\, e^{(x,y)}\,. \tag{5.3}$$

Since we have

$$\Delta_2\, e^{(x,y)} \;\; = \;\; (y,y) e^{(x,y)}$$

the identity in 5.13 is none other than

$$\begin{aligned} 2^d d!\, y^p\, e^{(x,y)} &= \sum_{r=0}^{d}\binom{d}{r}(-1)^r \Delta_2^{d-r} e^{t(x,a)}(y,y)^r\, e^{(x,y)}\ \Big|_{a^p/p!}\Big|_{t^d} \\ &= \sum_{r=0}^{d}\binom{d}{r}(-1)^r \Delta_2^{d-r} e^{(x,t\,a+y)}(y,y)^r\ \Big|_{a^p/p!}\Big|_{t^d} \\ &= \sum_{r=0}^{d}\binom{d}{r}(-1)^r \Big((t\,a+y,t\,a+y)\Big)^{d-r}(y,y)^r\, e^{(x,t\,a+y)}\ \Big|_{a^p/p!}\Big|_{t^d} \\ &= \Big((t\,a+y,t\,a+y)-(y,y)\Big)^d e^{(x,t\,a+y)}\ \Big|_{a^p/p!}\Big|_{t^d} \\ &= t^d\Big((t\,(a,a)+2(a,y)\Big)^d e^{(x,t\,a+y)}\ \Big|_{a^p/p!}\Big|_{t^d} \\ &= 2^d(a,y)^d\, e^{(x,y)}\ \Big|_{a^p/p!} \end{aligned}$$

which reduces to a tautology because of the multinomial expansion

$$(a_1y_1+a_2y_2+\cdots+a_n\,y_n)^d \;=\; \sum_{|q|=d}\frac{d!}{q!}\,a^q\,y^q\,.$$

This completes our proof.

The first consequence of Theorem 5.1 is an identity which is in some sense “ *dual* ” to 5.1

Theorem 5.2

For any polynomial $P(x)\in\mathbb{Q}[X_n]$ homogeneous of degree d we have

$$\frac{(-1)^d}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r p_2(\underline{x})^{d-r}P(\partial_x)\,p_2(\underline{x})^r \;=\; P(\underline{x})\,. \qquad 5.4$$

where

$$p_2(x) \;=\; x_1^2+x_2^2+\cdots+x_n^2$$

Proof

The identity in 5.1 gives

$$\frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r\Delta_2^{d-r}P(\underline{x})\,\Delta_2^r\,e^{(x,y)} \;=\; P(y)e^{(x,y)}\,. \qquad 5.5$$

However we also have

$$\begin{aligned} \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r\Delta_2^{d-r}P(\underline{x})\,\Delta_2^r\,e^{(x,y)} &= \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r p_2(y)^r\Delta_2^{d-r}P(\underline{x})\ e^{(x,y)} \\ (x \text{ and } y \text{ derivatives commute}) &= \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r p_2(y)^r P(\partial_y)\,\Delta_2^{d-r}\ e^{(x,y)} \\ &= \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r p_2(y)^r P(\partial_y)\,p_2(y)^{d-r}\ e^{(x,y)} \end{aligned}$$

Combining this identity with 5.5 gives

$$\frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r p_2(y)^r P(\partial_y)\, p_2(y)^{d-r}\, e^{(x,y)} \;=\; P(y)e^{(x,y)}$$

and Proposition 4.5 gives the operator equality

$$\frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r p_2(\underline{y})^r P(\partial_y)\, p_2(\underline{y})^{d-r} \;=\; P(\underline{y})$$

which is 5.4 with $x_1, x_2, \ldots, x_n$ replaced by $y_1, y_2, \ldots, y_n$.

To see where the left hand hand side of 5.1 comes from we need a few preliminary observations. Given a vector space V let us denote by $L[V]$ the vector space of all linear operators on V. Note that for $A, B \in L[V]$ it is customary to set

$$[A, B\,] \;=\; AB - BA\,.$$

For a given $C \in L[V]$ we define D_C to be the linear operator on $L[V]$ defined by setting for any $A \in L[V]$

$$D_C\, A \;=\; [C, A]\,. \qquad 5.6$$

It develops that D_C acts as a differentiation on $L[V]$.

Proposition 5.1

For any $C \in L[V]$ we have the Leibnitz formula

$$D_C\,(A\,B) \;=\; (D_C A)\,B \;+\; A\,D_C B \qquad \big(\,\forall\; A, B \in L[V]\,\big)\,. \qquad 5.7$$

In particular it follows that

$$D_C^m\,(A\,B) \;=\; \sum_{r=0}^{m}\binom{m}{r}\big(D_C^{m-r}A\big)\big(D_C^r B\big)\,. \qquad 5.8$$

Proof

Note that by definition

$$D_C\,(A\,B) \;=\; C\,A\,B \;-\; A\,B\,C$$

and this can be rewritten as

$$D_C\,(A\,B) \;=\; (C\,A - A\,C)\,B \;+\; A\,(CB \,-\, B\,C)\,.$$

This proves 5.7. Note further that 5.8 reduces to 5.7 for $m = 1$. So to prove 5.8 we may proceed by induction. This given, assuming 5.4 to be true for m we derive from 5.7

$$\begin{aligned} D_C^{m+1}(AB) &= \sum_{r=0}^{m}\binom{m}{r}\Big(\big(D_C^{m-r+1}A\big)\big(D_C^r B\big) + \big(D_C^{m-r}A\big)\big(D_C^{r+1}B\big)\Big) \\ &= \sum_{r=0}^{m}\binom{m}{r}\,\big(D_C^{m+1-r}A\big)\big(D_C^r B\big) \;+\; \sum_{r=1}^{m+1}\binom{m}{r-1}\big(D_C^{m+1-r}A\big)\big(D_C^r B\big)\Big) \\ &= \sum_{r=0}^{m+1}\binom{m+1}{r}\,\big(D_C^{m+1-r}A\big)\big(D_C^r B\big) \end{aligned}$$

where the last equality follows from the binomial identities

$$\binom{m+1}{r} \;=\; \binom{m}{r} \;+\; \binom{m}{r-1}. \qquad 5.9$$

This completes the induction and our proof.

In the same manner we derive.

Proposition 5.2

For any $C, A \in L[V]$

$$D_C^m\, A \;=\; \sum_{r=0}^{m} \binom{m}{r} (-1)^r\, C^{m-r} A\, C^r\,. \qquad 5.10$$

Proof

Note that the case $m = 1$ of 5.10 is simply the definition of D_C. So we may again proceed by induction and assume 5.10 to be true for m. This given, from 5.7 we derive that

$$\begin{aligned} D_C^{m+1}\, A \;&=\; \sum_{r=0}^{m} \binom{m}{r} (-1)^r \left(C^{n+1-r} A\, C^r \;-\; C^{n-r} A\, C^{r+1} \right) \\ &=\; \sum_{r=0}^{m} \binom{m}{r} (-1)^r\, C^{m+1-r} A\, C^r \;+\; \sum_{r=1}^{m+1} \binom{m}{r-1} (-1)^r\, C^{m+1-r} A\, C^r) \end{aligned}$$

and 5.10 again follows from the binomial identities in 5.9.

To set formula 5.1 in the present context, we should take $V = \mathbb{Q}[X_n]$, $C = \Delta_2$, and $A = P(\underline{x})$. Thus in this notation formula 5.1 is none other than

$$\frac{1}{2^d d!} D_{\Delta_2}^d\, P(\underline{x}) \;=\; P(\partial_x)\,. \qquad 5.11$$

In the same vein formula 5.4 becomes

$$\frac{(-1)^d}{2^d d!} D_{p_2(\underline{x})}^d\, P(\partial_x) \;=\; P(\underline{x})\,. \qquad 5.12$$

This view point allows us to extend Theorem 5.2 to an even more surprising result

Theorem 5.3

For any polynomial $P(x) \in \mathbb{Q}[X_n]$ homogeneous of degree d_P we have

$$\sum_{r=0}^{d} \binom{d}{r} (-1)^r p_2(\underline{x})^{d-r}\, P(\partial_x)\, p_2(\underline{x})^r \;=\; \begin{cases} (-1)^d 2^d d! P(\underline{x}) & \text{if } d_P = d \\ 0 & \text{if } d_P < d \end{cases} \qquad 5.13$$

Proof

The first case of 5.13 is formula 5.5. Now if $d_P < d$ we can still use 5.12 with $d = d_P$ and get

$$D_{p_2(\underline{x})}^{d_P}\, P(\partial_x) \;=\; (-1)^{d_P} 2^{d_P} d_P!\; P(\underline{x})\,. \qquad 5.14$$

But since the two multiplication operators $p_2(\underline{x})$ and $P(\underline{x})$ commute we necessarily have

$$D_{\underline{p}_2}\, P(\underline{x}) \;=\; 0$$

and 5.14 gives

$$D^d_{p_2(\underline{x})}\, P(\partial_x) \;=\; (-1)^{d_P} 2^{d_P} d_P!\; D^{d-d_P}_{p_2(\underline{x})} P(\underline{x}) \;=\; 0\,.$$

as desired.

We are now ready to obtain some truly surprising new expressions (due to Berest []) for the operators $\gamma_q(x,\partial_x)$ whose existence was established in the last section.

Theorem 5.4

For any polynomial $q(x) \in \mathcal{QI}[X_n]$ homogeneous of degree d we have

$$\gamma_q(x,\partial_x) \;=\; \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r L_m^{d-r} q(\underline{x}) L_m^r \tag{5.15}$$

Proof

Note that from 4.3 we derive the sequence of equalities

$$\begin{aligned}
\frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r L_m^{d-r} q(\underline{x}) L_m^r \Psi_m(x,y) \;&=\; \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r L_m^{d-r} q(\underline{x}) p_2(y)^r \Psi_m(x,y) \\
\text{(by 4.93)} \;&=\; \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r L_m^{d-r} p_2(y)^r \gamma(y,\partial_y)\Psi_m(x,y) \\
\text{(x and y operators commute)} \;&=\; \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r p_2(y)^r \gamma(y,\partial_y) L_m^{d-r}\Psi_m(x,y) \\
&=\; \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r p_2(y)^r \gamma(y,\partial_y) p_2(y)^{d-r}\Psi_m(x,y)
\end{aligned} \tag{5.16}$$

Next using 4.92 with $x_1, x_2, \ldots, x_n$ replaced by $y_1, y_2, \ldots, y_n$ gives the operator identity

$$\begin{aligned}
\frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r p_2(y)^r \gamma(y,\partial_y) p_2(y)^{d-r} \;&=\; \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r p_2(y)^r q(\partial_y) p_2(y)^{d-r} \\
&\quad + \sum_{|p|<d} a_p(y)\, \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r p_2(y)^r \partial_y^p p_2(y)^{d-r}
\end{aligned}$$

But now a double application of the identities in 5.13 with $x_1, x_2, \ldots, x_n$ replaced by $y_1, y_2, \ldots, y_n$ assures that the first sum reduces to $q(y)$ and the second sum vanishes identically. Using this in 5.15 yields

$$\frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r L_m^{d-r} q(\underline{x}) L_m^r \Psi_m(x,y) \;=\; q(y)\Psi_m(x,y) \;=\; \gamma(x,\partial_x)\Psi_m(x,y)\,,$$

and the operator identity in 5.14 immediately follows by another application of Proposition 4.5.

All this may appear quite mysterious at this point. Now it develops that the true nature of the identities we have derived in this section can be best understood within their natural $sl(2)$ setting. This will be the primary goal of the next section.

6. $\mathbf{sl[2]}$ Theory as it applies to $\mathbb{Q}[X_n]$ and $\mathcal{QI}_m[X_n]$

Let V be a vector space and let as before $L(V)$ denote the space of linear operators on V. Let us recall that three operators $E, F, H \in L(V)$ generate a representation of $sl(2)$ if and only if

$$a)\ [E,F]=H\,, \qquad b)\ [H,E]=2E\,, \qquad c)\ [H,F]=-2F\,. \tag{6.1}$$

These relations imply the following identities that will be needed in our arguments.

Proposition 6.1

$$\begin{aligned} i)_1 \quad HE^m &= E^m(H+2m\,I) \\ i)_2 \quad HF^m &= F^m(H-2m\,I) \\ ii)_1 \quad FE^{m+1} &= E^{m+1}F \,-\, (m+1)E^m(H+m\,I) \\ ii)_2 \quad EF^{m+1} &= F^{m+1}E \,+\, (m+1)F^m(H-m\,I) \end{aligned} \tag{6.2}$$

Proofs may be found in our $sl(2)$ lecture notes [].

The next result is also well known but since it will play a crucial role in our development it will be good give a proof here.

Proposition 6.2

For any $u \in V$ we have

$$iii)_1 \quad \begin{cases} a) \quad Fu=0 \\ b) \quad Hu=-du \end{cases} \implies F^m\,E^m\,u=\frac{m!d!}{(d-m)!}\,u \qquad \textit{for all } 0\le m\le d \tag{6.3_1}$$

$$iii)_2 \quad \begin{cases} a) \quad Eu=0 \\ b) \quad Hu=du \end{cases} \implies E^m\,F^m\,u=\frac{m!d!}{(d-m)!}\,u \qquad \textit{for all } 0\le m\le d \tag{6.3_2}$$

Proof

Note that both $iii)_1$ and $iii)_2$ are trivial for $m=0$. So we can proceed by induction on m. Now to show $iii)_1$ we use $ii)_1$ and a) and b) of $iii)_1$ and get

$$FE^{m+1}u \;=\; -(m+1)E^m(-d+m)u\,,$$

multiplying both sides by F^m then gives the recursion

$$F^{m+1}E^{m+1}u \;=\; (m+1)(d-m)F^mE^mu$$

assuming $iii)_1$ to be true for m, this gives

$$F^{m+1}E^{m+1}u \;=\; (m+1)(d-m)\frac{m!d!}{(d-m)!}\,u \;=\; \frac{(m+1)!d!}{(d-m-1)!}\,u$$

and this completes the induction. Note next that using $ii)_2$ with $iii)_2$ a) and b) we get

$$EF^{m+1}u = (m+1)F^m(d-m)u$$

and multiplication by E^m gives the recursion

$$E^mF^{m+1}u = (m+1)(d-m)E^mF^mu\,,$$

from which $iii)_2$ readily follows as in the previous argument.

We will also make use of the following basic fact.

Theorem 6.1

If $E, F, H \in L(V)$ satisfy $a), b)$ and $c)$ of 6.2 then we also have

$$a)\ [D_E, D_F] \;=\; D_H\ , \qquad b)\ [D_H, D_E] \;=\; 2D_E\ , \qquad a)\ [D_H, D_F] \;=\; -2D_F\ , \tag{6.4}$$

Thus D_E, D_F, D_H generate an $sl(2)$ representation on $L(V)$. In particular they will also satisfy the identities $i)_1, i)_2$, $ii)_1, ii)_2$, and $iii)_1$, $iii)_2$ with E, F, H replaced by D_E, D_F, D_H respectively.

Proof

It is sufficient to show that if A and B are any two elements of $L(V)$ then

$$[D_A, D_B] \;=\; D_{[A,B\,]}\,. \tag{6.5}$$

Let $Q \in L(V)$, then the definition in 6.6 gives

$$\begin{aligned}
[D_A, D_B]Q \;&=\; D_AD_B\,Q \;-\; D_BD_A\,Q \\
&=\; D_A(BQ-QB) \;-\; D_B(AQ-QA) \\
&=\; A(BQ-QB)-(BQ-QB)A \;-\; B(AQ-QA)+(AQ-QA)B \\
&=\; ABQ+QBA \;-\; BAQ-QAB\,.
\end{aligned} \tag{6.6}$$

On the other hand we have

$$\begin{aligned}
D_{[A,B]}\,Q \;&=\; [A,B\,]\,Q \;-\; Q\,[A,B\,] \\
&=\; (AB-BA)Q \;-\; Q\,(AB-BA) \\
&=\; ABQ-BAQ \;-\; QAB+QBA
\end{aligned}$$

Comparing this with 6.6 proves 6.5 and completes our proof of the theorem.

Our study of the operators $L_p(M)$ requires some $sl(2)$-results in the case that V is infinite dimensional. This case is not considered in [] and we shall have to treat it in detail here. However, since our developments only require $V=\mathcal{QI}_m[X_n]$ or $V=\mathbb{Q}[X_n]$. We will carry out this extension in the presence of some additional restrictions, which as we shall see are well satisfied in these two cases.

To begin we shall assume that V is a graded algebra over $\mathbb{Q}$

$$\mathcal{A} \;=\; \mathbb{Q}\oplus\mathcal{H}_1(\mathcal{A})\oplus\mathcal{H}_2(\mathcal{A})\oplus\cdots\oplus\mathcal{H}_k(\mathcal{A})\oplus\cdots \tag{6.7}$$

with the usual requirement that

$$\mathcal{H}_r(\mathcal{A})\times\mathcal{H}_s(\mathcal{A})\;\subseteq\;\mathcal{H}_{r+s}(\mathcal{A})\,. \tag{6.8}$$

We shall call the elemements of $\mathcal{H}_d(\mathcal{A})$ "*homogeneous of degree d*" and we set $deg(u)=d$ for all $u\in\mathcal{H}_d(\mathcal{A})$. The operators E,F and H are also heavily restricted. To begin we shall require that H acts by a constant on each $\mathcal{H}_d(\mathcal{A})$. More precisely we assume that for some scalar c we have for any homogeneous element of $\mathcal{A}$:

$$H\,u \;=\; \big(c-deg(u)\big)\,u\,. \tag{6.9}$$

Since $\mathcal{A}$ is an algebra, for any $q\in\mathcal{A}$, the operator "*multiplication by q*", which we denote by "$\underline{q}$", may also be considered to be an element of $L(\mathcal{A})$. This given we require that

$$F\,\underline{q} \;=\; \underline{q}\,F \qquad\qquad (\text{for all } q\in\mathcal{A})\,. \tag{6.10}$$

The final condition that E itself must satisfy is now forced by 6.9. More precisely we have

Proposition 6.3

The operator E sends a homogeneous element of degree d into a homogeneous element of degree $d-2$

Proof

Note first that 6.8 and 6.2 $i)_1$ for $m=1$ give, for a homogeneous q of degree d

$$HE\,q=\;(c-d+2)E\,q \tag{6.11}$$

on the other hand the direct sum decomposition in 6.7 gives

$$E\,q \;=\; \sum_{r\geq 0}E\,q\,\big|_r$$

where the symbol "$E\,q\,\big|_r$" denotes the homogeneous component of degree r in $E\,q$. Applying 6.9 again we now get

$$HE\,q \;=\; \sum_{r\geq 0}(c-r)E\,q\,\big|_r\,.$$

Comparing with 6.11, the uniqueness of decomposition forces

$$E\,q\,\big|_r \;=\; 0 \qquad\qquad \text{for } c-r\;\neq\;c-d+2$$

This leaves $r=d-2$ which gives

$$E\,q \;=\; E\,q\,\big|_{d-2}\,.$$

This proves our assertion.

Our next results concern the representation induced by the action of D_E, D_F and D_H on $L(\mathcal{A})$. To begin we have

Proposition 6.4

For any homogeneous $q \in \mathcal{A}$

$$D_H \underline{q} \;=\; -deg(q)\;\underline{q} \tag{6.12}$$

Proof

Let $P, q \in \mathcal{A}$ be homogeneous . Then, by definition

$$\begin{aligned} D_H \underline{q}\, P &= (H\underline{q} - \underline{q}H)P \\ &= HqP - q\,HP \\ \text{(by 6.8 and 6.9)} &= \big((c - deg(q) - deg(P)\big)qP - \big(c - deg(P)\big)qP \;=\; -deg(q)\;\underline{q}\,P\,. \end{aligned}$$

The validity of this for all homogeneous P yields 6.16.

We are now in a position to derive following basic results.

Theorem 6.2

Let $q \in \mathcal{A}$ be homogeneous of degree d then, either the sequence

$$\underline{q} \to D_E\,\underline{q} \to \;\; D_E^2\,\underline{q} \to \;\; D_E^3\,\underline{q} \to \;\ldots\; \to D_E^k\,\underline{q} \to \;\cdots\;. \tag{6.13}$$

has infinite length, or it terminates at $k = d$. That is we have

$$D_E^{d+1}\,\underline{q} \;=\; 0 \tag{6.14}$$

and $\underline{q}$ is the head of an $sl(2)$-string of length $d+1$ which consists of $\underline{q}$ and the operators

$$D_E\,\underline{q}\,, \quad D_E^2\,\underline{q}\,, \quad D_E^3\,\underline{q}\,,\;\ldots\;, D_E^d\,\underline{q}\;. \tag{6.15}$$

which are all non vanishing. In any case, we have at least the commutativity relation

$$D_E^{d+1}\underline{q}\;F \;=\; F\;D_E^{d+1}\underline{q} \tag{6.16}$$

Proof

Note that from 6.12 and property 6.10 we get

$$a)\quad D_F\,\underline{q} \;=\; 0 \qquad \text{and} \qquad b)\quad D_H\underline{q} = -d\underline{q}\,.$$

Thus we may apply $iii)_1$ of Proposition 6.1 and obtain that

$$D_F^m D_E^m\,\underline{q} \;=\; \frac{m!d!}{(d-m)!}\;\underline{q}\,. \qquad \big(\text{for } 1 \le m \le d\,\big) \tag{6.17}$$

This shows that none of the elements in 6.15 vanishes. Next we are to show that if $D_E^k\,\underline{q}=0$ for some $k>d$ then 6.14 must also hold true. This given let k_o be such that

$$a)\quad D_E^{k_o}\,\underline{q}\;\neq 0 \qquad \text{and} \qquad b)\quad D_E^{k_o+1}\,\underline{q}\;=0\,. \tag{6.18}$$

To this end note that from 6.2 $ii)_1$ with $m=k_o$ we get

$$\begin{aligned} D_F D_E^{k_o+1}\underline{q} &= D_E^{k_o+1} D_F\,\underline{q} \;+\; (k_o+1)D_E^{k_o}\big(D_H+k_o\,I\big)\underline{q} \\ \text{(by 6.15)} &= (k_o+1)D_E^{k_o}\big(D_H+k_o\,I\big)\underline{q} \\ \text{(by 6.12)} &= (k_o+1)\big(-d+k_o\big)D_E^{k_o}\underline{q}\,. \end{aligned}$$

This given 6.18 b) gives

$$0 \;=\; D_F D_E^{k_o+1}\underline{q} \;=\; (k_o+1)\big(-d+k_o\big)D_E^{k_o}\underline{q}$$

and 6.18 a) forces

$$k_0 \;=\; d\,,$$

as desired. In any case, we can certainly use the relation in $ii)_1$ and derive (using properties a) and b) above)

$$D_F D_E^d\,\underline{q} \;=\; -(d+1)D_E^m(-d+d)\underline{q} \;=\; 0\,.$$

But this is precisely the identity on 6.16. The proof is thus complete.

From here on we shall work under the additional assumption that our algebra $\mathcal{A}$ satisfies the extra condition

$$\textit{For every } q\in\mathcal{A} \textit{ the sequence } \{D_E^k\,\underline{q}\}_{k\geq 0} \textit{ terminates.} \tag{6.19}$$

The following result shows that we are not dealing with a vacuous notion and at the same time throws some further light onto the nature of the identity in 6.14.

Theorem 6.3

Let $\mathcal{A}=\mathbb{Q}[X_n]$, and take

$$E=-\Delta_2/2\,,\qquad F=\underline{p}_2/2\,,\qquad H=[E,F\,]\,, \tag{6.20}$$

with

$$p_2=x_1^2+x_2^2+\cdots+x_n^2\,. \tag{6.21}$$

Then

$$a)\quad [H,E]=2E\,,\qquad\qquad b)\quad [H,F]=-2F\,. \tag{6.22}$$

Moreover conditions 6.9, 6.10 and 6.19 old true as well.

Proof

We will begin by computing H. To this end note that for any polynomial $Q(x)\in\mathbb{Q}[X_n]$ we have

$$\Delta_2\,p_2\,Q \;=\; \big(\Delta_2\,p_2\big)Q \;+\; 2\sum_{i=1}^n(\partial_{x_i}p_2)(\partial_{x_i}Q) \;+\; p_2\Delta_2 Q\,. \tag{6.23}$$

Since a simple computation shows that

$$\Delta_2\, p_2 = 2n \qquad \text{and} \qquad \sum_{i=1}^{n} (\partial_{x_i} p_2)(\partial_{x_i} Q) = 2 \sum_{i=1}^{n} x_i \partial_{x_i} Q\,,$$

substituting these two identities in 6.23 gives

$$(\Delta_2\, \underline{p}_2 - \underline{p}_2 \Delta_2)\, Q \;=\; 2n\, Q \;+\; 4 \sum_{i=1}^{n} x_i \partial_{x_i} Q\,,$$

or, equivalently

$$[E, F] \;=\; -n\, I/2 - \sum_{i=1}^{n} x_i \partial_{x_i}\,. \tag{6.24}$$

This gives

$$H \;=\; -n\, I/2 \,-\, \sum_{i=1}^{n} x_i \partial_{x_i}\,. \tag{6.25}$$

Since for any homogeneous polynomial Q we have

$$\sum_{i=1}^{n} x_i \partial_{x_i}\, Q \;=\; deg(Q) Q\,, \tag{6.26}$$

the identity in 6.25 proves 6.9 with $c = -n/2$.

Note that 6.10 is trivially satisfied since for any $q \in \mathbb{Q}[X_n]$ we have

$$F\underline{q} - \underline{q}F \;=\; \underline{p}_2\, \underline{q} \,-\, \underline{q}\, \underline{p}_2 \;=\; 0\,.$$

Note next that for any homogeneous $P \in \mathbb{Q}[X_n]$ we have

$$\begin{aligned}
[H, E]P \;&=\; HE\,P - EH\,P \\
&=\; -\tfrac{1}{2}\Big(H\Delta_2 P \,-\, \Delta_2 H P\Big) \\
&=\; -\tfrac{1}{2}\big(-deg(\Delta_2 P) + deg(P)\Delta_2 P\big) \\
&=\; -\tfrac{1}{2}\big(-deg(P) + 2 + deg(P)\big)\Delta_2 P \;=\; -\Delta_2 P \;=\; 2EP\,.
\end{aligned}$$

this proves 6.22 a). Moreover, using 6.20 we get

$$[H, F]P \;=\; \tfrac{1}{2}\big(Hp_2 P - p_2 H\big)Q = \tfrac{1}{2}\big(-deg(P) - 2 + deg(P)\big)p_2 Q = -2F\, Q$$

and this proves 6.22 b).

Finally, note that the identity in 6.1, for $q \in \mathcal{H}_d\big(\mathbb{Q}[X_n]\big)$, gives

$$D_E^d\, \underline{q} \;=\; \frac{(-1)^d}{2^d} \sum_{r=0}^{d} \binom{d}{2} (-1)^r \Delta_2^{d-r} \underline{q} \Delta_2^r \;=\; (-1)^d d!\, q(\partial_x)\,.$$

In particular this shows that $D_E^d\,\underline{q}$ commutes with Δ_2. But this means that

$$D_E^{d+1}\,\underline{q} \;=\; E\,D_E^d\,\underline{q} - D_E^d\underline{q}\,E \;=\; 0\,.$$

This proves the validity of 6.19 and completes the proof of the Theorem.

We have seen that in this particular case the operators $L_q = \frac{1}{d!}D_E^d\underline{q}$ reduce to $q(\partial_x)$ when $q \in \mathcal{H}_d(\mathbb{Q}[X_n])$. Thus all these operators commute with each other and with Δ_2. It develops that, in the presence of condition 6.19, this commutativity property, holds true in general. To be precise we have

Theorem 6.4

Setting for a homogeneous q

$$L_q = \frac{1}{deg(q)!}D_E^{deg(q)}\,\underline{q}\,, \tag{6.27}$$

under 6.19 we have the identities

$$E\,L_p \;=\; L_p\,E \qquad \textit{(for all homogeneous } p \in \mathcal{A}) \tag{6.28}$$

and

$$L_p\,L_q \;=\; L_{p\,q} \qquad \text{(for all homogeneous } p, q \in \mathcal{A}) \tag{6.29}$$

Proof

The identity in 6.28 is simply

$$\frac{1}{deg(q)!}D_E^{deg(q)+1}\underline{q} \;=\; 0\,,$$

and we have seen that this is implied by 6.19. To show 6.29 let $p \in \mathcal{H}_{d_1}(\mathcal{A})$, $q \in \mathcal{H}_{d_2}(\mathcal{A})$ and set $d = d_1 + d_2$. Then using Proposition 6.1 we derive that

$$\frac{1}{d!}D_E^d\underline{pq} \;=\; \frac{1}{d!}\sum_{r=0}^{d}\binom{d}{r}\big(D_E^{d-r}\,\underline{p}\big)\big(D_E^r\underline{q}\big)\,. \tag{6.30}$$

However, 6.19 implies that $D_E^k\underline{p} = 0$ for $k > d_1$ and $D_E^k\underline{q} = 0$ for $k > d_2$ thus the summand $\big(D_E^{d-r}\,\underline{p}\big)\big(D_E^r\underline{q}\big)$ vanishes unless $d_1 + d_2 - r \leq d_1$ and $r \leq d_2$. This forces $r = d_2$ and reduces 6.30 to the identity

$$\frac{1}{d!}D_E^d\underline{pq} \;=\; \frac{1}{d_1!d_2!}\big(D_E^{d_1}\,\underline{p}\big)\big(D_E^{d_2}\,\underline{q}\big)\,. \tag{6.31}$$

which is simply another way of writing 6.29.

Our next goal is to show that all these results hold for $\mathcal{A} = \mathcal{QI}_m[X_n]$. To be precise we have

Theorem 6.5

Let $\mathcal{A} = \mathcal{QI}_m[X_n]$ and set

$$E \;=\; -L_m/2\,, \qquad F \;=\; \underline{p}/2\,, \qquad H \;=\; [E,F] \tag{6.32}$$

Then

$$a)\quad [H,F] = 2E\,, \qquad\qquad b)\quad [H,F] = -2F\,. \tag{6.33}$$

Moreover conditions 6.9, 6.10 and 6.19 hold true as well.

Proof

We will begin by computing H. To this end we use Proposition 3.6 and get, for any $Q \in \mathcal{QI}_m[X_n]$

$$L(m)\,(p_2 Q) \;=\; \big(L(m)\,p_2\big)Q \;+\; 2\sum_{i=1}^{n}(\partial_{x_i}p_2)(\partial_{x_i}Q) \;+\; p_2\,\big(L(m)\,Q\big)\,. \qquad 6.34$$

Now note that we have

$$\frac{1}{x_i-x_j}(\partial_{x_i}-\partial_{x_j})\sum_{r=1}^{n}x_r^2 \;=\; \frac{1}{x_i-x_j}(2x_i-2x_j) \;=\; 2\,.$$

Since we have seen in the proof of Theorem 6.3 that

$$\Delta_2\,p_2 = 2n \qquad \text{and} \qquad \sum_{i=1}^{n}(\partial_{x_i}p_2)(\partial_{x_i}Q) = 2\sum_{i=1}^{n}x_i\partial_{x_i}Q\,,$$

we immediately derive from 6.34 that

$$\big(L(m)\underline{p}_2 - \underline{p}_2 L(m)\big)Q \;=\; 2n\,Q \;-\; 4m\binom{n}{2}Q \;+\; 4\sum_{i=1}^{n}x_i\partial_{x_i}Q\,,$$

consequently

$$[E,F]\,Q \;=\; -n\,Q/2 \;+\; m\binom{n}{2}Q \;-\; \sum_{i=1}^{n}x_i\partial_{x_i}Q\,.$$

This gives

$$H \;=\; -n/2 \;+\; m\binom{n}{2} \;-\; \sum_{i=1}^{n}x_i\partial_{x_i}\,,$$

which is property 6.9 with

$$c \;=\; -n/2 \;+\; m\binom{n}{2}\,. \qquad 6.35$$

Property 6.10 is again immediate, as in the proof of Theorem 6.3, since $F = \underline{p}_2/2$. We shall next verify 6.33 a) and b). To this end note that for a homogeneous $Q \in \mathcal{QI}_m[X_n]$ we have (with c as in 6.35)

$$\begin{aligned} [H,E]\,Q \;&=\; -\tfrac{1}{2}\big(HL(m) - L(m)H\big)Q \\ \text{(since } L(m) \text{ lowers degrees by 2)} \;&=\; -\tfrac{1}{2}\Big((c-deg(Q)+2)L(m)Q \;-\; L(m)(c-deg(Q))Q\Big) \\ &=\; -L(m)\,Q \;=\; 2E\,Q\,. \end{aligned}$$

This proves 6.33 a). The identity in 6.33 b) is proved exactly as we did in the proof of Theorem 6.3 and we need not repeat it here. We are left to verify that property 19 hold true as well. To this end let $q \in \mathcal{QI}_m[X_n]$ be homogeneous of degree d and let $\gamma_p(x,\partial_x)$ be the operator constructed from q according to the algorithm given by Theorem 4.8. Now we have shown (Theorem 5.4) that $\gamma_p(x,\partial_x)$ may be also obtained from the identity

$$\gamma_p(x,\partial_x) \;=\; \frac{1}{2^d d!}\sum_{i=1}(-1)^d\binom{d}{r}(-1)^r L_m^{d-r}\,\underline{q}\,L_m^r\,.$$

In the present notation this may be rewritten as

$$\gamma_p(x,\partial_x) \;=\; \frac{(-1)^d}{d!} D_E^d\, \underline{q}\,.$$

However, one of the assertions of Theorem 4.8 (see 4.94 a)) is that $\gamma_p(x,\partial_x)$ commutes with L_m. But in the present notation this is simply

$$D_E^{d+1}\, \underline{q} \;=\; 0\,.$$

This proves that condition 5.19 holds true also in this case and completes our proof.

We can thus state.

Theorem 6.6

Let $\mathcal{A} = \mathcal{QI}_m[X_n]$ and E,F,H be in as in 6.32 and set for a homogeneous $q \in \mathcal{QI}_m[X_n]$

$$L_q = \frac{1}{deg(q)!} D_E^{deg(q)}\, \underline{q}\,. \tag{6.37}$$

Then we have

$$L(m)\,L_p \;=\; L_p\,L(m) \qquad \textit{(for all homogeneous } p \in \mathcal{QI}_m[X_n]) \tag{6.38}$$

and

$$L_p\,L_q \;=\; L_{p\,q} \qquad \text{(for all homogeneous } p,q \in \mathcal{QI}_m[X_n]) \tag{6.39}$$

Proof

This is an immediate corollary of Theorem 6.4.

By combining the results of the last two sections with some of the properties of the Bakerf-Akhiezer functions $\Psi_m(x,y)$, we can put together a remarkable tool kit for working with quasi-invariants. However, for our presentation to be complete we need to show that $\Psi_m(x,y)$ has a non vanishing constant term. More precisely we have the following beautiful identity

Theorem 6.6 (Opdam [])

$$\Pi(\nabla(m))\Pi(\overline{x})\,1 \;=\; (-1)^{\binom{n}{2}} n! \prod_{1\le i<j\le n} (jm-i)\,. \tag{6.40}$$

In particular, the constant term of the polynomial $P_m(x,y)$ in 4.5 is given by

$$\prod_{r=1}^{m} (-1)^{\binom{r}{2}} r! \prod_{1\le i<j\le n} (jr-i) \tag{6.41}$$

Proof

For the moment there no elementary simple proof of 6.40. An elementary but rather intricate argument proving 6.40 was given by Dunkl and Hanlon in []. Thus we shall have to accept 6.40 as given at this time and proceed with the rest of the argument. It is easily seen from 3.38, 4.4 and the definition of Ω_m that

$$\Phi_m(0,0) \;=\; P_m(0,0) \;=\; \Omega_m\,1 \;=\; \Big(\prod_{r=1}^{n} O_r\Big)\,1$$

But since 1 is certainly symmetric we see that we have

$$O_r\, 1 \;=\; \Big(\Gamma\Pi(\nabla(r)\Pi(\overline{x})\Big)\cdot 1 \;=\; \Pi(\nabla(r)\Pi(x))\cdot 1 \;=\; \Pi(\nabla(r)\Pi(x))$$

and since each of these quantities is a scalar it follows that we must also have

$$\Phi_m(0,0) \;=\; P_m(0,0) \;=\; \prod_{r=1}^{n}\Pi(\nabla(r)\Pi(x))$$

Thus the assertion in 6.41 follows from 6.40.

We can now collect a windfall of consequences of this result.

Theorem 6.7

For any $m \geq 1$ we have

(1) The bilinear form defined by setting for any two polynomials in $p, q \in \mathcal{QI}[X_n]$

$$(p\ ,\ q)_m \;=\; \tfrac{1}{c_0}\,\gamma_p(x,\partial_x)q(x)\,\big|_{x=0} \qquad 6.42$$

is non-degenerate.

(2) If $\big\{\phi_k^{(d)}(x)\big\}_{k=1}^{N_d}$ is any complete orthonormal system for the homogeneous m-quasi-invariants of degree d with respect to the form $(\ ,\ )_m$, then

$$\Omega_m \tfrac{(x,y)^d}{d!} \;=\; \sum_{i=1}^{N_d}\phi_k^{(d)}(x)\phi_k^{(d)}(y) \qquad 6.43$$

as well as

$$\Psi_m(x,y) \;=\; c_0 \;+\; \sum_{d\geq 1}\sum_{i=1}^{N_d}\phi_k^{(d)}(x)\phi_k^{(d)}(y) \qquad 6.44$$

where c_0 is the constant given by 6.41.

(2) In particular $\Psi_m(x,y)$ is the reproducing kernel for the form $(\ ,\ )_m$.

Proof

Form 4.93 a) we derive that for any homogeneous m-quasi-invariant $p(x)$ we get

$$\sum_{k\geq 0}\gamma_p(x,\partial_x)\Omega_m \tfrac{(x,y)^k}{k!} \;=\; q(y)\sum_{k\geq 0}\Omega_m \tfrac{(x,y)^k}{k!}$$

Now if p is of degree d, since the operator $\gamma_p(x,\partial_x)$ will then decrease x-degrees by d, then by equating homgeneous components of equal degrees we get for $k \geq d$

$$\gamma_p(x,\partial_x)\Omega_m \tfrac{(x,y)^k}{k!} \;=\; q(y)\Omega_m \tfrac{(x,y)^{k-d}}{(k-d)!}$$

and setting $k = d$ we obtain

$$\gamma_p(x,\partial_x)\Omega_m \tfrac{(x,y)^d}{d!}\Big|_{x=0} \;=\; c_o q(y)$$

In other words, we have shown that

$$(p(x)\ ,\ \Omega_m \tfrac{(x,y)^d}{d!})_m \;\; = \;\; q(y) \tag{6.45}$$

This proves the non degeneracy of the form $(\ ,\ )_m$. Replacing $x_1, x_2, \ldots, x_n$ by $z_1, z_2, \ldots, z_n$ and $p(x)$ by $\phi_k^{(d)}(z)$ in 6.45 gives

$$(\phi_k(z)^{(d)}\ ,\ \Omega_m \tfrac{(z,y)^d}{d!})_m \;\; = \;\; \phi_k^{(d)}(y) \tag{6.46}$$

multiplying both sides by $\phi_k^{(d)}(x)$, the completeness and orthonormality of the set $\left\{\phi_k^{(d)}(x)\right\}_{k=1}^{N_d}$ gives

$$\sum_{k=1}^{N_d} \phi_k^{(d)}(x)\phi_k^{(d)}(y) \;\; = \;\; \Omega_m \tfrac{(x,y)^d}{d!}$$

this proves 6.43 and 6.44 immediately follows.

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