The book has NOT made it into reserves in the library, so here are the HW exercises FROM Chapter 1

The additive identity $0 = (0, 0)$ and the multiplicative identity $1 = (1, 0)$ for real numbers carry over to the entire complex number system. That is,

$$z + 0 = z \quad \text{and} \quad z \cdot 1 = z \tag{4}$$

for every complex number z. Furthermore, 0 and 1 are the only complex numbers with such properties (see Exercise 8).

There is associated with each complex number $z = (x, y)$ an additive inverse

$$-z = (-x, -y), \tag{5}$$

satisfying the equation $z + (-z) = 0$. Moreover, there is only one additive inverse for any given z, since the equation

$$(x, y) + (u, v) = (0, 0)$$

implies that

$$u = -x \quad \text{and} \quad v = -y.$$

For any *nonzero* complex number $z = (x, y)$, there is a number z^{-1} such that $zz^{-1} = 1$. This multiplicative inverse is less obvious than the additive one. To find it, we seek real numbers u and v, expressed in terms of x and y, such that

$$(x, y)(u, v) = (1, 0).$$

According to equation (4), Sec. 1, which defines the product of two complex numbers, u and v must satisfy the pair

$$xu - yv = 1, \quad yu + xv = 0$$

of linear simultaneous equations; and simple computation yields the unique solution

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}.$$

So *the* multiplicative inverse of $z = (x, y)$ is

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) \quad (z \neq 0). \tag{6}$$

The inverse z^{-1} is not defined when $z = 0$. In fact, $z = 0$ means that $x^2 + y^2 = 0$; and this is not permitted in expression (6).

EXERCISES

1. Verify that

   (a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$; *(b)* $(2, -3)(-2, 1) = (-1, 8)$;

   (c) $(3, 1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1)$.

2. Show that

   (a) $\operatorname{Re}(iz) = -\operatorname{Im} z$; *(b)* $\operatorname{Im}(iz) = \operatorname{Re} z$.

3. Show that $(1 + z)^2 = 1 + 2z + z^2$.

4. Verify that each of the two numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.

5. Prove that multiplication of complex numbers is commutative, as stated at the beginning of Sec. 2.

6. Verify

   (a) the associative law for addition of complex numbers, stated at the beginning of Sec. 2;

   (b) the distributive law (3), Sec. 2.

7. Use the associative law for addition and the distributive law to show that

$$z(z_1 + z_2 + z_3) = zz_1 + zz_2 + zz_3.$$

8. *(a)* Write $(x, y) + (u, v) = (x, y)$ and point out how it follows that the complex number $0 = (0, 0)$ is unique as an additive identity.

   (b) Likewise, write $(x, y)(u, v) = (x, y)$ and show that the number $1 = (1, 0)$ is a unique multiplicative identity.

9. Use $-1 = (-1, 0)$ and $z = (x, y)$ to show that $(-1)z = -z$.

10. Use $i = (0, 1)$ and $y = (y, 0)$ to verify that $-(iy) = (-i)y$. Thus show that the additive inverse of a complex number $z = x + iy$ can be written $-z = -x - iy$ without ambiguity.

11. Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$$

and then solving a pair of simultaneous equations in x and y.

Suggestion: Use the fact that no real number x satisfies the given equation to show that $y \neq 0$.

Ans. $z = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$.

3. FURTHER PROPERTIES

In this section, we mention a number of other algebraic properties of addition and multiplication of complex numbers that follow from the ones already described

EXERCISES

1. Reduce each of these quantities to a real number:

$$(a)\ \frac{1+2i}{3-4i}+\frac{2-i}{5i};\qquad (b)\ \frac{5i}{(1-i)(2-i)(3-i)};\qquad (c)\ (1-i)^4.$$

Ans. (a) $-2/5$; (b) $-1/2$; (c) -4.

2. Show that

$$\frac{1}{1/z}=z \qquad (z\neq 0).$$

3. Use the associative and commutative laws for multiplication to show that

$$(z_1z_2)(z_3z_4)=(z_1z_3)(z_2z_4).$$

4. Prove that if $z_1z_2z_3=0$, then at least one of the three factors is zero.
Suggestion: Write $(z_1z_2)z_3=0$ and use a similar result (Sec. 3) involving two factors.

5. Derive expression (6), Sec. 3, for the quotient z_1/z_2 by the method described just after it.

6. With the aid of relations (10) and (11) in Sec. 3, derive the identity

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right)=\frac{z_1z_2}{z_3z_4} \qquad (z_3\neq 0,\ z_4\neq 0).$$

7. Use the identity obtained in Exercise 6 to derive the cancellation law

$$\frac{z_1z}{z_2z}=\frac{z_1}{z_2} \qquad (z_2\neq 0,\ z\neq 0).$$

8. Use mathematical induction to verify the binomial formula (13) in Sec. 3. More precisely, note that the formula is true when $n=1$. Then, assuming that it is valid when $n=m$ where m denotes any positive integer, show that it must hold when $n=m+1$.
Suggestion: When $n=m+1$, write

$$(z_1+z_2)^{m+1}=(z_1+z_2)(z_1+z_2)^m=(z_2+z_1)\sum_{k=0}^{m}\binom{m}{k}z_1^kz_2^{m-k}$$

$$=\sum_{k=0}^{m}\binom{m}{k}z_1^kz_2^{m-k+1}+\sum_{k=0}^{m}\binom{m}{k}z_1^{k+1}z_2^{m-k}$$

and replace k by $k-1$ in the last sum here to obtain

$$(z_1+z_2)^{m+1}=z_2^{m+1}+\sum_{k=1}^{m}\left[\binom{m}{k}+\binom{m}{k-1}\right]z_1^kz_2^{m+1-k}+z_1^{m+1}.$$

EXAMPLE 3. If a point z lies on the unit circle $|z| = 1$ about the origin, it follows from inequalities (7) and (8) that

$$|z - 2| \leq |z| + 2 = 3$$

and

$$|z - 2| \geq ||z| - 2| = 1.$$

The triangle inequality (4) can be generalized by means of mathematical induction to sums involving any finite number of terms:

$$(10) \qquad |z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n| \qquad (n = 2, 3, \ldots).$$

To give details of the induction proof here, we note that when $n = 2$, inequality (10) is just inequality (4). Furthermore, if inequality (10) is assumed to be valid when $n = m$, it must also hold when $n = m + 1$ since, by inequality (4),

$$\begin{aligned} |(z_1 + z_2 + \cdots + z_m) + z_{m+1}| &\leq |z_1 + z_2 + \cdots + z_m| + |z_{m+1}| \\ &\leq (|z_1| + |z_2| + \cdots + |z_m|) + |z_{m+1}|. \end{aligned}$$

EXERCISES

1. Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially when

   (a) $z_1 = 2i, \quad z_2 = \dfrac{2}{3} - i$; (b) $z_1 = (-\sqrt{3}, 1), \quad z_2 = (\sqrt{3}, 0)$;

   (c) $z_1 = (-3, 1), \quad z_2 = (1, 4)$; (d) $z_1 = x_1 + iy_1, \quad z_2 = x_1 - iy_1$.

2. Verify inequalities (3), Sec. 4, involving $\operatorname{Re} z$, $\operatorname{Im} z$, and $|z|$.

3. Use established properties of moduli to show that when $|z_3| \neq |z_4|$,

$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

4. Verify that $\sqrt{2}\,|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$.

   Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

5. In each case, sketch the set of points determined by the given condition:

   (a) $|z - 1 + i| = 1$; (b) $|z + i| \leq 3$; (c) $|z - 4i| \geq 4$.

6. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that

   (a) $|z - 4i| + |z + 4i| = 10$ represents an ellipse whose foci are $(0, \pm 4)$;

   (b) $|z - 1| = |z + i|$ represents the line through the origin whose slope is -1.

5. COMPLEX CONJUGATES

The *complex conjugate*, or simply the conjugate,
is defined as the complex number $x - iy$ and is

(1) $\bar{z} = x - iy.$

The number $\bar{z}$ is represented by the point $(x, -y)$
axis of the point (x, y) representing z (Fig. 5). N

$\bar{\bar{z}} = z$ and $|\bar{z}| =$

for all z.

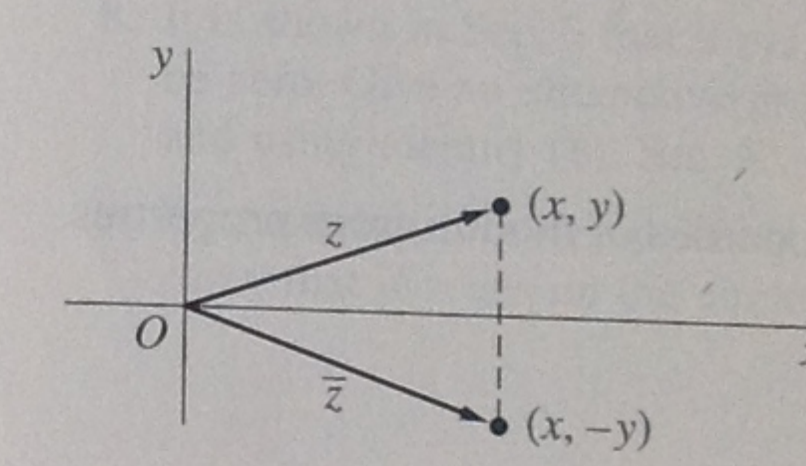

FIGURE 5

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2) =$

So the conjugate of the sum is the sum of the c

(2) $\overline{z_1 + z_2} = \bar{z}_1 +$

In like manner, it is easy to show that

(3) $\overline{z_1 - z_2} = \bar{z}_1 -$

(4) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2,$

and

(5) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ $(z_2$

The sum $z + \bar{z}$ of a complex number $z = x$
is the real number $2x$, and the difference $z - \bar{z}$
Hence

(6) $\operatorname{Re} z = \dfrac{z + \bar{z}}{2}$ and Im

An important identity relating the conjugate of a complex number $z = x + iy$ to its modulus is

$$z\bar{z} = |z|^2, \tag{7}$$

where each side is equal to $x^2 + y^2$. It suggests the method for determining a quotient z_1/z_2 that begins with expression (7), Sec. 3. That method is, of course, based on multiplying both the numerator and the denominator of z_1/z_2 by $\overline{z_2}$, so that the denominator becomes the real number $|z_2|^2$.

EXAMPLE 1. As an illustration,

$$\frac{-1+3i}{2-i} = \frac{(-1+3i)(2+i)}{(2-i)(2+i)} = \frac{-5+5i}{|2-i|^2} = \frac{-5+5i}{5} = -1+i.$$

See also the example in Sec. 3.

Identity (7) is especially useful in obtaining properties of moduli from properties of conjugates noted above. We mention that

$$|z_1 z_2| = |z_1||z_2| \tag{8}$$

and

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad (z_2 \neq 0). \tag{9}$$

Property (8) can be established by writing

$$|z_1z_2|^2 = (z_1z_2)(\overline{z_1z_2}) = (z_1z_2)(\overline{z_1}\,\overline{z_2}) = (z_1\overline{z_1})(z_2\overline{z_2}) = |z_1|^2|z_2|^2 = (|z_1||z_2|)^2$$

and recalling that a modulus is never negative. Property (9) can be verified in a similar way.

EXAMPLE 2. Property (8) tells us that $|z^2| = |z|^2$ and $|z^3| = |z|^3$. Hence if z is a point inside the circle centered at the origin with radius 2, so that $|z| < 2$, it follows from the generalized triangle inequality (10) in Sec. 4 that

$$|z^3 + 3z^2 - 2z + 1| \le |z|^3 + 3|z|^2 + 2|z| + 1 < 25.$$

EXERCISES

1. Use properties of conjugates and moduli established in Sec. 5 to show that

   (a) $\overline{\bar{z} + 3i} = z - 3i$; (b) $\overline{iz} = -i\bar{z}$;

   (c) $\overline{(2+i)^2} = 3 - 4i$; (d) $|(2\bar{z}+5)(\sqrt{2}-i)| = \sqrt{3}\,|2z+5|$.

2. Sketch the set of points determined by the condition

   (a) $\text{Re}(\bar{z} - i) = 2$; (b) $|2\bar{z} + i| = 4$.

3. Verify properties (3) and (4) of conjugates in Sec. 5.

4. Use property (4) of conjugates in Sec. 5 to show that

   (a) $\overline{z_1z_2z_3} = \overline{z_1}\,\overline{z_2}\,\overline{z_3}$; (b) $\overline{z^4} = \bar{z}^4$.

5. Verify property (9) of moduli in Sec. 5.

6. Use results in Sec. 5 to show that when z_2 and z_3 are nonzero,

   (a) $\overline{\left(\dfrac{z_1}{z_2z_3}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}\,\overline{z_3}}$; (b) $\left|\dfrac{z_1}{z_2z_3}\right| = \dfrac{|z_1|}{|z_2||z_3|}$.

7. Show that

$$|\text{Re}(2 + \bar{z} + z^3)| \le 4 \quad \text{when } |z| \le 1.$$

8. It is shown in Sec. 3 that if $z_1z_2 = 0$, then at least one of the numbers z_1 and z_2 must be zero. Give an alternative proof based on the corresponding result for real numbers and using identity (8), Sec. 5.

9. By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using inequality (8), Sec. 4, show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \le \frac{1}{3}.$$

10. Prove that

    (a) z is real if and only if $\bar{z} = z$;

    (b) z is either real or pure imaginary if and only if $\bar{z}^2 = z^2$.

11. Use mathematical induction to show that when $n = 2, 3, \ldots$,

    (a) $\overline{z_1 + z_2 + \cdots + z_n} = \overline{z_1} + \overline{z_2} + \cdots + \overline{z_n}$; (b) $\overline{z_1z_2\cdots z_n} = \overline{z_1}\,\overline{z_2}\cdots\overline{z_n}$.

12. Let $a_0, a_1, a_2, \ldots, a_n$ $(n \ge 1)$ denote *real* numbers, and let z be any complex number. With the aid of the results in Exercise 11, show that

$$\overline{a_0 + a_1z + a_2z^2 + \cdots + a_nz^n} = a_0 + a_1\bar{z} + a_2\bar{z}^2 + \cdots + a_n\bar{z}^n.$$

13. Show that the equation $|z - z_0| = R$ of a circle, centered at z_0 with radius R, can be written

$$|z|^2 - 2\,\text{Re}(z\overline{z_0}) + |z_0|^2 = R^2.$$

14. Using expressions (6), Sec. 5, for $\text{Re}\, z$ and $\text{Im}\, z$, show that the hyperbola $x^2 - y^2 = 1$ can be written

$$z^2 + \bar{z}^2 = 2.$$

15. Follow the steps below to give an algebraic derivation of the triangle inequality (Sec. 4)

$$|z_1 + z_2| \le |z_1| + |z_2|.$$

    (a) Show that

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2}.$$

and, since (Sec. 7)

$$z_2^{-1} = \frac{1}{r_2} e^{-i\theta_2},$$

one can see that

$$\text{(3)} \qquad \arg(z_2^{-1}) = -\arg z_2.$$

Hence

$$\text{(4)} \qquad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2.$$

Statement (3) is, of course, to be interpreted as saying that the set of all values on the left-hand side is the same as the set of all values on the right-hand side. Statement (4) is, then, to be interpreted in the same way that statement (2) is.

EXAMPLE 2. In order to find the principal argument Arg z when

$$z = \frac{-2}{1+\sqrt{3}i},$$

observe that

$$\arg z = \arg(-2) - \arg(1+\sqrt{3}i).$$

Since

$$\text{Arg}(-2) = \pi \quad \text{and} \quad \text{Arg}(1+\sqrt{3}i) = \frac{\pi}{3},$$

one value of $\arg z$ is $2\pi/3$; and, because $2\pi/3$ is between $-\pi$ and π, we find that $\text{Arg}\, z = 2\pi/3$.

EXERCISES

1. Find the principal argument $\text{Arg}\, z$ when

   (a) $z = \dfrac{i}{-2-2i}$; (b) $z = (\sqrt{3} - i)^6$.

   Ans. (a) $-3\pi/4$; (b) π.

2. Show that (a) $|e^{i\theta}| = 1$; (b) $\overline{e^{i\theta}} = e^{-i\theta}$.

3. Use mathematical induction to show that

$$e^{i\theta_1} e^{i\theta_2} \cdots e^{i\theta_n} = e^{i(\theta_1+\theta_2+\cdots+\theta_n)} \qquad (n = 2, 3, \ldots).$$

4. Using the fact that the modulus $|e^{i\theta} - 1|$ is the distance between the points $e^{i\theta}$ and 1 (see Sec. 4), give a geometric argument to find a value of θ in the interval $0 \le \theta < 2\pi$ that satisfies the equation $|e^{i\theta} - 1| = 2$.

   Ans. π.

5. By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

   (a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$; (b) $5i/(2+i) = 1+2i$;

   (c) $(-1+i)^7 = -8(1+i)$; (d) $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$.

6. Show that if $\text{Re}\, z_1 > 0$ and $\text{Re}\, z_2 > 0$, then

$$\text{Arg}(z_1 z_2) = \text{Arg}\, z_1 + \text{Arg}\, z_2,$$

   where principal arguments are used.

7. Let z be a nonzero complex number and n a negative integer ($n = -1, -2, \ldots$). Also, write $z = re^{i\theta}$ and $m = -n = 1, 2, \ldots$. Using the expressions

$$z^m = r^m e^{im\theta} \quad \text{and} \quad z^{-1} = \left(\frac{1}{r}\right) e^{i(-\theta)},$$

   verify that $(z^m)^{-1} = (z^{-1})^m$ and hence that the definition $z^n = (z^{-1})^m$ in Sec. 7 could have been written alternatively as $z^n = (z^m)^{-1}$.

8. Prove that two nonzero complex numbers z_1 and z_2 have the same moduli if and only if there are complex numbers c_1 and c_2 such that $z_1 = c_1 c_2$ and $z_2 = c_1 \overline{c_2}$.

   Suggestion: Note that

$$\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\exp\left(i\frac{\theta_1-\theta_2}{2}\right) = \exp(i\theta_1)$$

   and [see Exercise 2(*b*)]

$$\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\overline{\exp\left(i\frac{\theta_1-\theta_2}{2}\right)} = \exp(i\theta_2).$$

9. Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z} \qquad (z \ne 1)$$

   and then use it to derive *Lagrange's trigonometric identity:*

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$

   Suggestion: As for the first identity, write $S = 1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

10. Use de Moivre's formula (Sec. 7) to derive the following trigonometric identities:

    (a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$; (b) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

11. *(a)* Use the binomial formula (Sec. 3) and de Moivre's formula (Sec. 7) to write

$$\cos n\theta + i \sin n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta\,(i\sin\theta)^k \qquad (n = 0, 1, 2, \ldots).$$

Then define the integer m by means of the equations

$$m = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

and use the above summation to show that [compare with Exercise 10*(a)*]

$$\cos n\theta = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k \cos^{n-2k}\theta \sin^{2k}\theta \qquad (n = 0, 1, 2, \ldots).$$

(b) Write $x = \cos\theta$ in the final summation in part *(a)* to show that it becomes a polynomial

$$T_n(x) = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k x^{n-2k}(1 - x^2)^k$$

of degree n $(n = 0, 1, 2, \ldots)$ in the variable x.*

9. ROOTS OF COMPLEX NUMBERS

Consider now a point $z = re^{i\theta}$, lying on a circle centered at the origin with radius r (Fig. 10). As θ is increased, z moves around the circle in the counterclockwise direction. In particular, when θ is increased by 2π, we arrive at the original point; and the same is true when θ is decreased by 2π. It is, therefore, evident from Fig. 10 that *two nonzero complex numbers*

$$z_1 = r_1 e^{i\theta_1} \quad \textit{and} \quad z_2 = r_2 e^{i\theta_2}$$

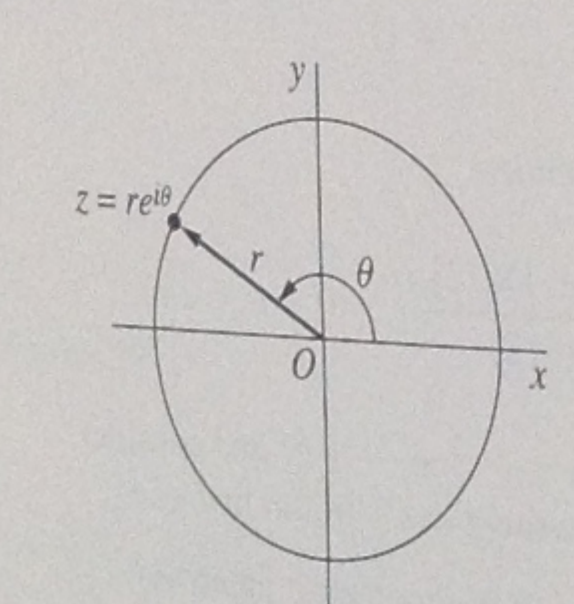

FIGURE 10

*These are called Chebyshev polynomials and are prominent in approximation theory.

are equal if and only if

$$r_1 = r_2 \quad \textit{and} \quad \theta_1 = \theta_2 + 2k\pi,$$

where k is some integer $(k = 0, \pm 1, \pm 2, \ldots)$.

This observation, together with the expression $z^n = r^n e^{in\theta}$ in Sec. 7 for integral powers of complex numbers $z = re^{i\theta}$, is useful in finding the nth roots of any nonzero complex number $z_0 = r_0 e^{i\theta_0}$, where n has one of the values $n = 2, 3, \ldots$. The method starts with the fact that an nth root of z_0 is a nonzero number $z = re^{i\theta}$ such that $z^n = z_0$, or

$$r^n e^{in\theta} = r_0 e^{i\theta_0}.$$

According to the statement in italics just above, then,

$$r^n = r_0 \quad \text{and} \quad n\theta = \theta_0 + 2k\pi,$$

where k is any integer $(k = 0, \pm 1, \pm 2, \ldots)$. So $r = \sqrt[n]{r_0}$, where this radical denotes the unique *positive* nth root of the positive real number r_0, and

$$\theta = \frac{\theta_0 + 2k\pi}{n} = \frac{\theta_0}{n} + \frac{2k\pi}{n} \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Consequently, the complex numbers

$$z = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

are the nth roots of z_0. We are able to see immediately from this exponential form of the roots that they all lie on the circle $|z| = \sqrt[n]{r_0}$ about the origin and are equally spaced every $2\pi/n$ radians, starting with argument θ_0/n. Evidently, then, all of the *distinct* roots are obtained when $k = 0, 1, 2, \ldots, n-1$, and no further roots arise with other values of k. We let c_k $(k = 0, 1, 2, \ldots, n-1)$ denote these distinct roots and write

(1) $$c_k = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \qquad (k = 0, 1, 2, \ldots, n-1).$$

(See Fig. 11.)

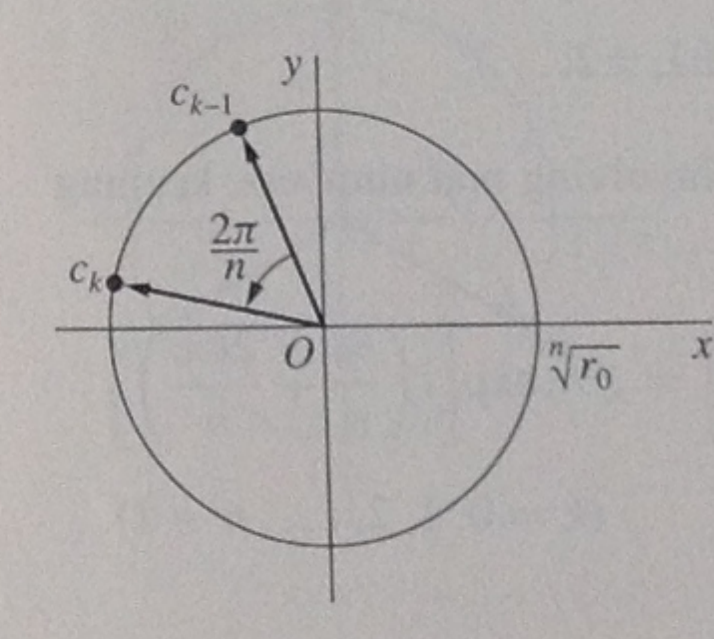

FIGURE 11

EXAMPLE 2. In order to determine the nth roots of unity, we start with

$$1 = 1\exp[i(0+2k\pi)] \qquad (k=0,\pm1,\pm2\ldots)$$

and find that

(2) $$1^{1/n} = \sqrt[n]{1}\exp\left[i\left(\frac{0}{n}+\frac{2k\pi}{n}\right)\right] = \exp\left(i\frac{2k\pi}{n}\right) \qquad (k=0,1,2,\ldots,n-1).$$

When $n=2$, these roots are, of course, ±1. When $n\geq 3$, the regular polygon at whose vertices the roots lie is inscribed in the unit circle $|z|=1$, with one vertex corresponding to the principal root $z=1$ $(k=0)$. In view of expression (3), Sec. 9, these roots are simply

$$1,\omega_n,\omega_n^2,\ldots,\omega_n^{n-1} \quad \text{where} \quad \omega_n = \exp\left(i\frac{2\pi}{n}\right).$$

See Fig. 13, where the cases $n=3, 4$, and 6 are illustrated. Note that $\omega_n^n = 1$.

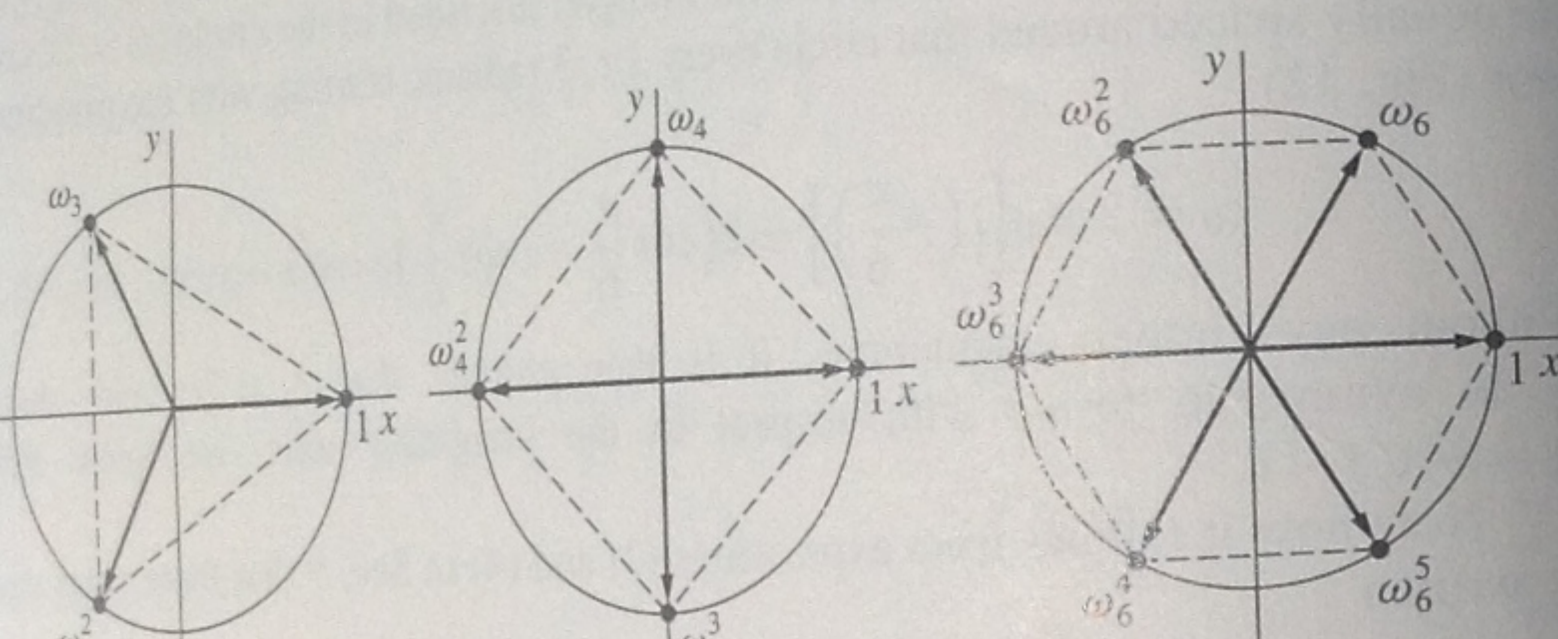

FIGURE 13

EXAMPLE 3. The two values c_k $(k=0,1)$ of $(\sqrt{3}+i)^{1/2}$, which are the square roots of $\sqrt{3}+i$, are found by writing

$$\sqrt{3}+i = 2\exp\left[i\left(\frac{\pi}{6}+2k\pi\right)\right] \qquad (k=0,\pm1,\pm2,\ldots)$$

and (see Fig. 14)

(3) $$c_k = \sqrt{2}\exp\left[i\left(\frac{\pi}{12}+k\pi\right)\right] \qquad (k=0,1).$$

Euler's formula tells us that

$$c_0 = \sqrt{2}\exp\left(i\frac{\pi}{12}\right) = \sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right),$$

and the trigonometric identities

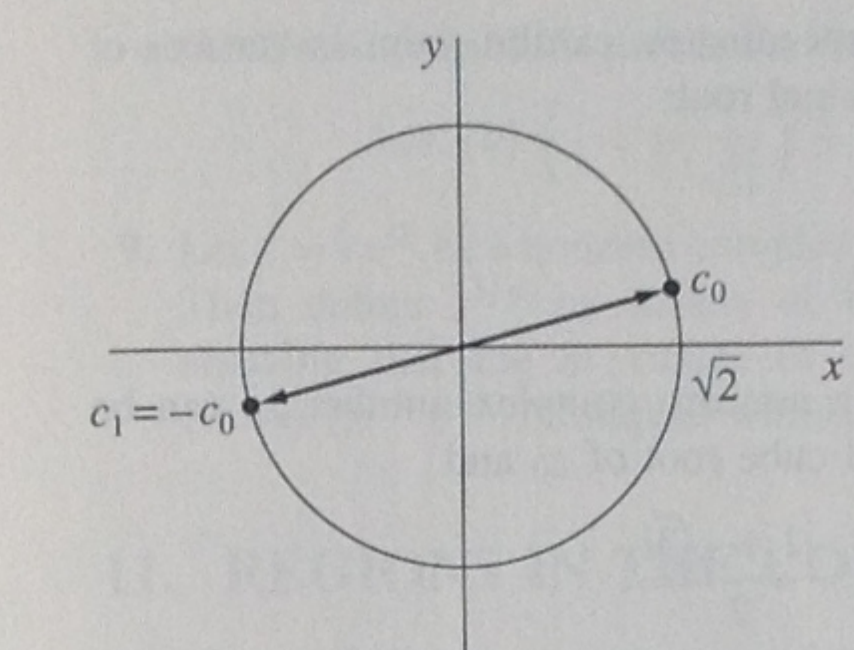

FIGURE 14

(4) $$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}, \qquad \sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$$

enable us to write

$$\cos^2\frac{\pi}{12} = \frac{1}{2}\left(1+\cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right) = \frac{2+\sqrt{3}}{4},$$

$$\sin^2\frac{\pi}{12} = \frac{1}{2}\left(1-\cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right) = \frac{2-\sqrt{3}}{4}.$$

Consequently,

$$c_0 = \sqrt{2}\left(\sqrt{\frac{2+\sqrt{3}}{4}}+i\sqrt{\frac{2-\sqrt{3}}{4}}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}}+i\sqrt{2-\sqrt{3}}\right).$$

Since $c_1 = -c_0$, the two square roots of $\sqrt{3}+i$ are, then,

(5) $$\pm\frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}}+i\sqrt{2-\sqrt{3}}\right).$$

EXERCISES

1. Find the square roots of *(a)* $2i$; *(b)* $1-\sqrt{3}i$ and express them in rectangular coordinates.

   Ans. *(a)* $\pm(1+i)$; *(b)* $\pm\dfrac{\sqrt{3}-i}{\sqrt{2}}$.

2. In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain squares, and point out which is the principal root:

   (a) $(-16)^{1/4}$; *(b)* $(-8-8\sqrt{3}i)^{1/4}$.

   Ans. *(a)* $\pm\sqrt{2}(1+i)$, $\pm\sqrt{2}(1-i)$; *(b)* $\pm(\sqrt{3}-i)$, $\pm(1+\sqrt{3}i)$.

3. In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:
(a) $(-1)^{1/3}$; (b) $8^{1/6}$.

Ans. (b) $\pm\sqrt{2}, \pm\dfrac{1+\sqrt{3}i}{\sqrt{2}}, \pm\dfrac{1-\sqrt{3}i}{\sqrt{2}}$.

4. According to Sec. 9, the three cube roots of a nonzero complex number z_0 can be written $c_0, c_0\omega_3, c_0\omega_3^2$ where c_0 is the principal cube root of z_0 and

$$\omega_3 = \exp\left(i\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}i}{2}.$$

Show that if $z_0 = -4\sqrt{2}+4\sqrt{2}i$, then $c_0 = \sqrt{2}(1+i)$ and the other two cube roots are, in rectangular form, the numbers

$$c_0\omega_3 = \frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}}, \quad c_0\omega_3^2 = \frac{(\sqrt{3}-1)-(\sqrt{3}+1)i}{\sqrt{2}}.$$

5. (a) Let a denote any fixed real number and show that the two square roots of $a+i$ are

$$\pm\sqrt{A}\exp\left(i\frac{\alpha}{2}\right)$$

where $A = \sqrt{a^2+1}$ and $\alpha = \operatorname{Arg}(a+i)$.

(b) With the aid of the trigonometric identities (4) in Example 3 of Sec. 10, show that the square roots obtained in part (a) can be written

$$\pm\frac{1}{\sqrt{2}}\left(\sqrt{A+a}+i\sqrt{A-a}\right).$$

(Note that this becomes the final result in Example 3, Sec. 10, when $a = \sqrt{3}$.)

6. Find the four zeros of the polynomial z^4+4, one of them being

$$z_0 = \sqrt{2}e^{i\pi/4} = 1+i.$$

Then use those zeros to factor z^2+4 into quadratic factors with real coefficients.

Ans. $(z^2+2z+2)(z^2-2z+2)$.

7. Show that if c is any nth root of unity other than unity itself, then

$$1+c+c^2+\cdots+c^{n-1} = 0.$$

Suggestion: Use the first identity in Exercise 9, Sec. 8.

8. (a) Prove that the usual formula solves the quadratic equation

$$az^2+bz+c = 0 \qquad (a \neq 0)$$

when the coefficients a, b, and c are complex numbers. Specifically, by completing the square on the left-hand side, derive the *quadratic formula*

$$z = \frac{-b+(b^2-4ac)^{1/2}}{2a},$$

where both square roots are to be considered when $b^2-4ac \neq 0$,

(b) Use the result in part (a) to find the roots of the equation $z^2+2z+(1-i) = 0$

Ans. (b) $\left(-1+\dfrac{1}{\sqrt{2}}\right)+\dfrac{i}{\sqrt{2}}, \quad \left(-1-\dfrac{1}{\sqrt{2}}\right)-\dfrac{i}{\sqrt{2}}$.

9. Let $z = re^{i\theta}$ be a nonzero complex number and n a negative integer ($n = -1, -2, \ldots$). Then define $z^{1/n}$ by means of the equation $z^{1/n} = (z^{-1})^{1/m}$ where $m = -n$. B[y] showing that the m values of $(z^{1/m})^{-1}$ and $(z^{-1})^{1/m}$ are the same, verify tha[t] $z^{1/n} = (z^{1/m})^{-1}$. (Compare with Exercise 7, Sec. 8.)

11. REGIONS IN THE COMPLEX PLANE

In this section, we are concerned with sets of complex numbers, or points in th[e] z plane, and their closeness to one another. Our basic tool is the concept of a[n] ε *neighborhood*

$$(1) \qquad |z-z_0| < \varepsilon$$

of a given point z_0. It consists of all points z lying inside but not on a circl[e] centered at z_0 and with a specified positive radius ε (Fig. 15). When the value of [ε] is understood or is immaterial in the discussion, the set (1) is often referred to as jus[t] a neighborhood. Occasionally, it is convenient to speak of a *deleted neighborhoo[d]* or punctured disk,

$$(2) \qquad 0 < |z-z_0| < \varepsilon$$

consisting of all points z in an ε neighborhood of z_0 except for the point z_0 itself.

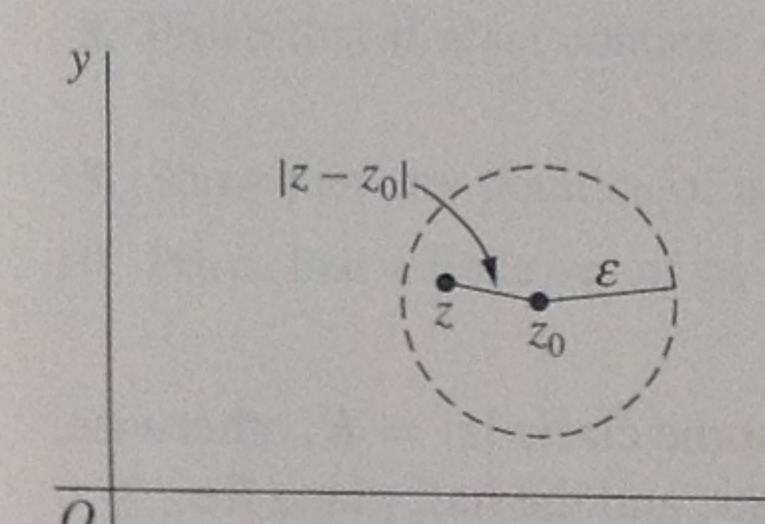

FIGURE 15

A point z_0 is said to be an *interior point* of a set S whenever there is so[me] neighborhood of z_0 that contains only points of S; it is called an *exterior point* [of] S when there exists a neighborhood of it containing no points of S. If z_0 is neith[er] of these, it is a *boundary point* of S. A boundary point is, therefore, a point all whose neighborhoods contain at least one point in S and at least one point not in [S]. The totality of all boundary points is called the *boundary* of S. The circle $|z| =$ [1,] for instance, is the boundary of each of the sets

$$(3) \qquad |z| < 1 \quad \text{and} \quad |z| \leq 1.$$

A set is *open* if it contains none of its boundary points. It is left as an exercise to show that a set is open if and only if each of its points is an interior point. A set is *closed* if it contains all of its boundary points, and the *closure* of a set S is the closed set consisting of all points in S together with the boundary of S. Note that the first of the sets (3) is open and that the second is its closure.

Some sets are, of course, neither open nor closed. For a set to be not open, there must be a boundary point that is contained in the set; and if a set is not closed, there exists a boundary point not contained in the set. Observe that the punctured disk $0 < |z| \le 1$ is neither open nor closed. The set of all complex numbers is, on the other hand, both open and closed since it has no boundary points.

An open set S is *connected* if each pair of points z_1 and z_2 in it can be joined by a *polygonal line*, consisting of a finite number of line segments joined end to end, that lies entirely in S. The open set $|z| < 1$ is connected. The annulus $1 < |z| < 2$ is, of course, open and it is also connected (see Fig. 16). A nonempty open set that is connected is called a *domain*. Note that any neighborhood is a domain. A domain together with some, none, or all of its boundary points is referred to as a *region*.

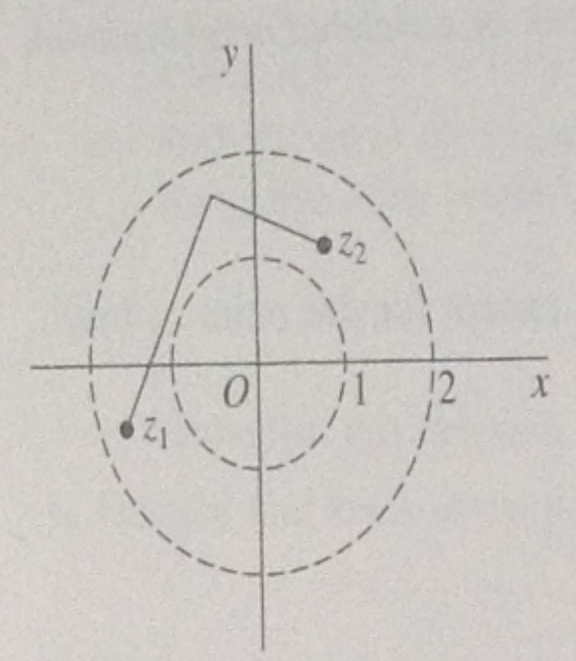

FIGURE 16

A set S is *bounded* if every point of S lies inside some circle $|z| = R$; otherwise, it is *unbounded*. Both of the sets (3) are bounded regions, and the half plane $\text{Re}\, z \ge 0$ is unbounded.

A point z_0 is said to be an *accumulation point* of a set S if each deleted neighborhood of z_0 contains at least one point of S. It follows that if a set S is closed, then it contains each of its accumulation points. For if an accumulation point z_0 were not in S, it would be a boundary point of S; but this contradicts the fact that a closed set contains all of its boundary points. It is left as an exercise to show that the converse is, in fact, true. Thus a set is closed if and only if it contains all of its accumulation points.

Evidently, a point z_0 is *not* an accumulation point of a set S whenever there exists some deleted neighborhood of z_0 that does not contain at least one point of S. Note that the origin is the only accumulation point of the set $z_n = i/n$ $(n = 1, 2, \ldots)$.

EXERCISES

1. Sketch the following sets and determine which are domains:
(a) $|z - 2 + i| \le 1$; (b) $|2z + 3| > 4$;
(c) $\text{Im}\, z > 1$; (d) $\text{Im}\, z = 1$;
(e) $0 \le \arg z \le \pi/4$ $(z \ne 0)$; (f) $|z - 4| \ge |z|$.

Ans. (b), (c) are domains.

2. Which sets in Exercise 1 are neither open nor closed?
Ans. (e).

3. Which sets in Exercise 1 are bounded?
Ans. (a).

4. In each case, sketch the closure of the set:
(a) $-\pi < \arg z < \pi$ $(z \ne 0)$; (b) $|\text{Re}\, z| < |z|$;
(c) $\text{Re}\left(\frac{1}{z}\right) \le \frac{1}{2}$; (d) $\text{Re}(z^2) > 0$.

5. Let S be the open set consisting of all points z such that $|z| < 1$ or $|z - 2| < 1$. State why S is not connected.

6. Show that a set S is open if and only if each point in S is an interior point.

7. Determine the accumulation points of each of the following sets:
(a) $z_n = i^n$ $(n = 1, 2, \ldots)$; (b) $z_n = i^n/n$ $(n = 1, 2, \ldots)$;
(c) $0 \le \arg z < \pi/2$ $(z \ne 0)$; (d) $z_n = (-1)^n(1 + i)\frac{n-1}{n}$ $(n = 1, 2, \ldots)$.

Ans. (a) None; (b) 0; (d) $\pm(1 + i)$.

8. Prove that if a set contains each of its accumulation points, then it must be a closed set.

9. Show that any point z_0 of a domain is an accumulation point of that domain.

10. Prove that a finite set of points $z_1, z_2, \ldots, z_n$ cannot have any accumulation points.