

# EVERY CONVEX FREE BASIC SEMI-ALGEBRAIC SET HAS AN LMI REPRESENTATION.

J. WILLIAM HELTON<sup>1</sup> AND SCOTT MCCULLOUGH<sup>2</sup>

ABSTRACT. The (matricial) solution set of a Linear Matrix Inequality (LMI) is a convex non-commutative basic open semi-algebraic set (defined below). The main theorem of this paper is a converse, a result which has implications for both semidefinite programming and systems engineering.

A non-commutative basic open semi-algebraic set is defined in terms of a non-commutative  $\ell \times \ell$ -matrix polynomial  $p(x_1 \cdots, x_g)$ . Such a polynomial is a linear combinations of words in non-commuting free variables  $\{x_1, \dots, x_g\}$  with coefficients from  $M_\ell$ , the  $\ell \times \ell$  matrices (for some  $\ell$ ). The involution  $T$  on words given by sending a concatenation of letters to the same letters, but in the reverse order (for instance  $(x_j x_\ell)^T = x_\ell x_j$ ), extends naturally to such polynomials and  $p$  is itself symmetric if  $p^T = p$ . Let  $\mathbb{S}_n(\mathbb{R}^g)$  denote the set of  $g$ -tuples  $X = (X_1, \dots, X_g)$  of symmetric  $n \times n$  matrices. A polynomial can naturally be evaluated on a tuple  $X \in \mathbb{S}_n(\mathbb{R}^g)$  yielding a value  $p(X)$  which is an  $\ell \times \ell$  block matrix with  $n \times n$  matrix entries. Evaluation at  $X$  is compatible with the involution since  $p^T(X) = p(X)^T$  and if  $p$  is symmetric, then  $p(X)$  is a symmetric matrix.

Assuming that  $p(0)$  is invertible, the invertibility set  $\mathcal{D}_p(n)$  of a non-commutative symmetric polynomial  $p$  in dimension  $n$  is the component of 0 of the set

$$\{X \in \mathbb{S}_n(\mathbb{R}^g) : p(X) \text{ is invertible}\}.$$

The invertibility set,  $\mathcal{D}_p$ , is the sequence of sets  $(\mathcal{D}_p(n))$ , which is the type of set we call a nc basic open semi-algebraic set. The non-commutative set  $\mathcal{D}_p$  is called convex if, for each  $n$ ,  $\mathcal{D}_p(n)$  is convex. A linear matrix inequality is the special case where  $p = L$  is an affine linear symmetric polynomial with  $L(0) = I$ . In this case,  $\mathcal{D}_L$  is clearly convex. A set is said to have a Linear Matrix Inequality Representation if it is the set of all solutions to some LMI, that is, it has the form  $\mathcal{D}_L$  for some  $L$ .

The main theorem says: *if  $p(0)$  is invertible and  $\mathcal{D}_p$  is bounded, then  $\mathcal{D}_p$  has an LMI representation if and only if  $\mathcal{D}_p$  is convex.*

---

*Date:* January 5, 2010.

*1991 Mathematics Subject Classification.* 47Axx (Primary). 47A63, 47L07, 47L30, 14P10 (Secondary).

*Key words and phrases.* Linear Matrix Inequalities, Matrix convex set, Convex sets of matrices, Non-commutative semi-algebraic geometry.

<sup>1</sup>Research supported by NSF grants DMS-0700758, DMS-0757212, and the Ford Motor Co.

<sup>2</sup>Research supported by the NSF grant DMS-0758306.

## 1. INTRODUCTION

The main result of this article is that a convex non-commutative basic open semi-algebraic set which is bounded has a monic Linear Matrix Inequality representation. Applications and connections to semidefinite programming and linear systems engineering are discussed in Subsection 1.10 near the end of this introduction. The work is also of interest in understanding a non-commutative (free) analog of convex semi-algebraic sets [BCR98]. Often we abbreviate non-commutative by nc.

Our result is a free algebra analog of the preposterous statement:

*A bounded open convex set  $\mathcal{C}$  in  $\mathbb{R}^n$  with algebraic boundary is a simplex.*

In other words,  $\mathcal{C}$  is defined by a finite number of linear functionals. For a free algebra "this" is actually true; that is Theorem 1.3.

A recurring theme in the non-commutative setting, such as that of a subspace of C-star algebra [Ar69, Ar72, Ar08] or in free probability [Vo04, Vo05] to give two of many examples, is the need to consider the *complete matrix structure* afforded by tensoring with  $n \times n$  matrices (over positive integers  $n$ ). The resulting theory of operator algebras, systems, spaces and matrix convex sets (matrix convex set is defined below in Section 1.9.2) has matured to the point that there are now several excellent books on the subject including [BL04] [Pa02] [Pi]. Since we are dealing with matrix convex sets, it is not surprising that the starting point for our analysis is the matricial version of the Hahn-Banach Separation Theorem of Effros and Winkler [EW97] which says that given a point  $x$  not inside a matrix convex set there is a (finite) LMI which separates  $x$  from the set. For a general matrix convex set  $\mathcal{C}$ , the conclusion is then that there is a collection, likely infinite, of finite LMIs which cut out  $\mathcal{C}$ .

In the case  $\mathcal{C}$  is matrix convex and also semi-algebraic, the challenge, successfully dealt with in this paper, is to prove that there is actually a finite collection of (finite) LMIs which define  $\mathcal{C}$ . The techniques introduced here involve methods for cutting down key matrices to sizes determined by the defining polynomials for  $\mathcal{C}$ . They have little relation to previous work on convex non-commutative basic semi-algebraic sets. In particular, they do not involve non-commutative calculus and positivity of non-commutative Hessians or non-commutative second fundamental forms.

The remainder of this introduction contains a precise statement of the main result, Theorem 1.3, a refinement, Theorem 1.5, as well as the preliminaries necessary for their statement. It also contains a discussion of consequences for nc real algebraic geometry, and a broadly illustrative example. The first subsection contains the basic definitions of non-commutative polynomials in formally symmetric nc variables; the second introduces evaluation of polynomials on tuples of symmetric matrices; and the third discusses matrix-valued nc polynomials. The initiated reader may choose to proceed directly to Subsections 1.4 and 1.5 where definitions of non-commutative basic open semi-algebraic set and convex nc basic open semi-algebraic set

respectively reside. As a special class of matrix-valued nc polynomials, Linear Matrix Inequalities and monic affine linear pencils are reviewed in Subsection 1.6. The main result given in the abstract is formally stated in the subsequent subsection. The proof technique generates refined results under additional hypotheses; see Subsection 1.8. In particular, a substantial improvement on the main result of [DHM07] is obtained. The following subsection, Subsection 1.9 gives immediate consequences, for real algebraic geometry in a free algebra, of Theorem 1.3. For one (in strong distinction to the classical commutative case) the projection of a nc semi-algebraic set (even a non-commutative LMI representable set) need not be non-commutative semi-algebraic. The previously mentioned applications which motivated this work are discussed in Subsection 1.10. The introduction concludes with a road map to the remainder of the paper, Subsection 1.11.

**1.1. Non-commutative polynomials.** Let  $\mathcal{P}$  denote the real algebra of polynomials in the non-commuting indeterminates  $x = (x_1, \dots, x_g)$ . Elements of  $\mathcal{P}$  are **non-commutative polynomials**, abbreviated to **nc polynomials** or often just **polynomials**. Thus, an nc polynomial  $p$  is a finite sum,

$$(1.1) \quad p = \sum p_w w,$$

where each  $w$  is a word in  $(x_1, \dots, x_g)$  and the coefficients  $p_w \in \mathbb{R}$ . For example, with  $g = 3$ ,

$$(1.2) \quad p_1 = 2x_1x_2^3 + 5x_2 - 3x_3x_1x_2 \quad \text{and} \quad p_2 = x_1x_2^3 + x_2^3x_1 + x_3x_1x_2 + x_2x_1x_3$$

are polynomials of degree four.

There is a natural involution  $T$  on  $\mathcal{P}$  given by

$$(1.3) \quad p^T = \sum p_w w^T,$$

where, for a word  $w$ ,

$$(1.4) \quad w = x_{j_1}x_{j_2} \cdots x_{j_n} \mapsto w^T = x_{j_n} \cdots x_{j_2}x_{j_1}.$$

A polynomial  $p$  is symmetric if  $p^T = p$ . For example, of the polynomials in equation (1.2),  $p_2$  is symmetric and  $p_1$  is not. In particular,  $x_j^T = x_j$  and for this reason the variables are sometimes referred to as symmetric non-commuting variables.

Denote, by  $\mathcal{P}_d$ , the polynomials in  $\mathcal{P}$  of (total) degree  $d$  or less.

**1.2. Substituting Matrices for Indeterminates.** Let  $\mathbb{S}_n(\mathbb{R}^g)$  denote the set of  $g$ -tuples  $X = (X_1, \dots, X_g)$  of real symmetric  $n \times n$  matrices. A polynomial  $p(x) = p(x_1, \dots, x_g) \in \mathcal{P}$  can naturally be evaluated at a tuple  $X \in \mathbb{S}_n(\mathbb{R}^g)$  resulting in an  $n \times n$  matrix. This process goes as follows. When  $X \in \mathbb{S}_n(\mathbb{R}^g)$  is substituted into  $p$  the constant term  $p_\emptyset$  of  $p(x)$  becomes  $p_\emptyset I_n$ ; i.e., the empty word evaluates to  $I_n$ . Often we write  $p(0)$  for  $p_\emptyset$  interpreting the 0 as  $0 \in \mathbb{R}^g$ . For a non-empty word  $w$  as in equation (1.4),

$$(1.5) \quad w(X) = X_{j_1}X_{j_2} \cdots X_{j_n}.$$

For a general polynomial  $p$  as in equation (1.3),

$$p(X) = \sum p_w w(X).$$

Thus, for example, for the polynomial  $p_1$  from equation (1.2),

$$p_1(X) = p_1(X_1, X_2, X_3) = 2X_1X_2^3 + 5X_2 - 3X_3X_1X_2.$$

The involution on  $\mathcal{P}$  that was introduced earlier is compatible with evaluation at  $X$  and matrix transposition, i.e.,

$$p^T(X) = p(X)^T,$$

where  $p(X)^T$  denotes the transpose of the matrix  $p(X)$ . Note, if  $p$  is symmetric, then so is  $p(X)$ .

**1.3. Matrix-Valued Polynomials.** Let  $\mathcal{P}^{\delta \times \delta'}$  denote the  $\delta \times \delta'$  matrices with entries from  $\mathcal{P}$ . Sometimes we abbreviate  $\mathcal{P}^{1 \times \delta}$  to  $\mathcal{P}^\delta$ , since we use row vectors of polynomials often. Denote, by  $\mathcal{P}_d^{\delta \times \delta'}$ , the subset of  $\mathcal{P}^{\delta \times \delta'}$  whose polynomial entries have degree  $d$  or less.

Evaluation at  $X \in \mathbb{S}_n(\mathbb{R}^g)$  naturally extends to  $p \in \mathcal{P}^{\delta \times \delta'}$  with the result,  $p(X)$ , a  $\delta \times \delta'$  block matrix with  $n \times n$  entries. Up to unitary equivalence, evaluation at  $X$  is conveniently described using tensor product notation by writing  $p$  as

$$(1.6) \quad p = \sum_{|w| \leq d} p_w w,$$

where each  $p_w$  is a  $\delta \times \delta'$  matrix (with real entries) and  $|w|$  is the length of the word  $w$ , and observing

$$p(X) = \sum p_w \otimes w(X),$$

where  $w(X)$  is given by equation (1.5).

The involution  $T$  naturally extends to  $\mathcal{P}^{\delta \times \delta}$  by

$$p^T = \sum_{|w| \leq d} p_w^T w^T,$$

for  $p$  given by equation (1.6). A polynomial  $p \in \mathcal{P}^{\delta \times \delta}$  is symmetric if  $p^T = p$  and in this case  $p(X) = p(X)^T$ .

A simple method of constructing new matrix-valued polynomials from old ones is by direct sum. For instance, if  $p_j \in \mathcal{P}^{\delta_j \times \delta_j}$  for  $j = 1, 2$ , then

$$p_1 \oplus p_2 = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \in \mathcal{P}^{(\delta_1 + \delta_2) \times (\delta_1 + \delta_2)}.$$

**1.4. NC Basic Open Semi-Algebraic Sets.** Suppose  $p \in \mathcal{P}^{\delta \times \delta}$  is symmetric. In particular,  $p(0)$  is a  $\delta \times \delta$  symmetric matrix. Assume that  $p(0)$  is invertible. For each positive integer  $n$ , let

$$\mathfrak{I}_p(n) = \{X \in \mathbb{S}_n(\mathbb{R}^g) : p(X) \text{ is invertible}\},$$

and define  $\mathfrak{I}_p$  to be the sequence (graded set)  $(\mathfrak{I}_p(n))_{n=1}^{\infty}$ . Let  $\mathcal{D}_p(n)$  denote the connected component of 0 of  $\mathfrak{I}_p(n)$  and  $\mathcal{D}_p$  the sequence (graded set)  $(\mathcal{D}_p(n))_{n=1}^{\infty}$ . We call  $\mathcal{D}_p$  **the invertibility set** of  $p$ .

In analogy with classical real algebraic geometry we call sets of the form  $\mathcal{D}_p$  **nc basic open semi-algebraic sets**. (Note that it is not necessary to explicitly consider intersections of nc basic open semi-algebraic sets since the intersection  $\mathcal{D}_p \cap \mathcal{D}_q$  equals  $\mathcal{D}_{p \oplus q}$ .)

Given an invertible symmetric matrix  $Y$ , let  $\sigma_+(Y)$  and  $\sigma_-(Y)$  denote the number of positive and negative eigenvalues respectively of  $Y$ . Let  $\sigma(Y) = (\sigma_+(Y), \sigma_-(Y))$ , the **signature(s)** of  $Y$ . Note that  $\mathcal{D}_p(n)$  can alternately be described as the component of 0 of the set

$$\{X \in \mathbb{S}_n(\mathbb{R}^g) : \sigma(p(X)) = n\sigma(p_0)\}.$$

In the special case that  $p(0) = p_0$  is positive definite, so that  $\sigma = (\delta, 0)$ , we call  $\mathcal{D}_p$  the **positivity set** of  $p$ . Usually in this case we normalize and assume that  $p(0) = I_\delta$ . (In general it is possible to normalize so that  $p(0) = J$  where  $J$  is a symmetry,  $J = J^T = J^{-1}$ .)

**Remark 1.1.** By a simple affine linear change of variable the point 0 can be replaced by  $\lambda \in \mathbb{R}^g$ . Replacing 0 by a fixed  $\Lambda \in \mathbb{S}_m(\mathbb{R}^g)$  will require an extension of the theory.  $\square$

**1.5. Convex Semi-Algebraic Sets.** To say that  $\mathcal{D}_p$  is **convex** means that each  $\mathcal{D}_p(n)$  is convex (in the usual sense) and in this case we say  $\mathcal{D}_p$  is a **convex non-commutative basic open semi-algebraic set**. In addition, we generally assume that  $\mathcal{D}_p$  is bounded; i.e., there is a constant  $K$  such for each  $n$  and each  $X \in \mathcal{D}_p(n)$ , we have  $\|X\| = \sum \|X_j\| \leq K$ . Thus the following list of conditions summarizes our usual assumptions on  $p$ .

**Assumption 1.2.** Fix  $p$  a  $\delta \times \delta$  symmetric matrix of polynomials in  $g$  nc variables of degree  $d$ . Our standard assumptions are:

- (i)  $p(0)$  is invertible;
- (ii)  $\mathcal{D}_p$  is bounded; and
- (iii)  $\mathcal{D}_p$  is convex.

**1.6. Linear Matrix Inequalities.** Our concern in this paper is representing a convex nc basic open semi-algebraic set in a form suitable for semidefinite programming. A (affine) **linear pencil**  $L$  is an expression of the form

$$(1.7) \quad L(x) := A_0 + A_1x_1 + \cdots + A_gx_g$$

where, for some positive integer  $\ell$ , each  $A_j$  is an  $\ell \times \ell$  symmetric matrix with real entries. The pencil is monic if  $A_0 = I$  in which case we say  $L$  is a **monic affine linear pencil**.

Since  $L \in \mathcal{P}^{\ell \times \ell}$  it evaluates at a tuple  $X \in \mathbb{S}_n(\mathbb{R}^g)$  as

$$L(X) := I \otimes I_n + A_1 \otimes X_1 + \cdots + A_g \otimes X_g.$$

Because  $L$  is monic and linear, it is straightforward to verify that the positivity set of  $L$  is the sequence

$$\mathcal{D}_L = (\{X \in \mathbb{S}_n(\mathbb{R}^g) : L(X) \text{ is positive definite}\})$$

and that  $\mathcal{D}_L$  is convex (and of course nc basic open semi-algebraic). Moreover,

$$\overline{\mathcal{D}_L} = (\{X \in \mathbb{S}_n(\mathbb{R}^g) : L(X) \text{ is positive semi-definite}\}).$$

A convenient notation for  $M$  being positive (resp. semi-definite) is  $M \succ 0$  (resp.  $\succeq 0$ ). An expression of the form  $L(X) \succ 0$  or  $L(X) \succeq 0$  is a **Linear Matrix Inequality** or **LMI** for short, and one sees LMIs in many branches of engineering and science. Both the case  $n = 1$ , that is,  $x_j$  being scalar and the matrix case  $n > 1$  are common, but our focus in this article is on matrix variables.

A non-commutative set  $\mathcal{C}$  is a sequence  $\mathcal{C} = (\mathcal{C}(n))_{n=1}^{\infty}$  where  $\mathcal{C}(n) \subset \mathbb{S}_n(\mathbb{R}^g)$  and we write  $\mathcal{C} \subset \mathbb{S}(\mathbb{R}^g)$ . A set  $\mathcal{C} \subset \mathbb{S}(\mathbb{R}^g)$  has an **LMI representation** if there is a monic affine linear pencil  $L$  such that

$$\mathcal{C} = \mathcal{D}_L.$$

Of course, if  $\mathcal{C} = \mathcal{D}_L$ , then the closure  $\overline{\mathcal{C}}$  of  $\mathcal{C}$  has the representation  $\{X : L(X) \succeq 0\}$  and so we could also refer to  $\overline{\mathcal{C}}$  as having an LMI representation too.

Clearly, if  $\mathcal{C}$  has an LMI representation, then  $\mathcal{C}$  is a convex nc basic open semi-algebraic set. The main result of this paper is the converse, under the additional assumption that  $\mathcal{C}$  is bounded.

**1.7. Main Result.** Our main theorem is

**Theorem 1.3.** *Every convex non-commutative bounded basic open semi-algebraic set (as in Assumption 1.2) has an LMI representation.*

*Proof.* The proof consumes much of the paper. Ultimately, this result follows from Theorem 6.1.  $\square$

The proofs of Theorem 1.3 and the forthcoming Theorem 1.5 yield estimates on the size of the representing LMI in Theorem 1.3.

**Theorem 1.4.** *Suppose  $p$  satisfies the conditions of Assumption 1.2. Thus  $p$  is a symmetric  $\delta \times \delta$ -matrix polynomial of degree  $d$  in  $g$  variables. Let  $\nu = \delta \sum_0^d g^j$ .*

There is a  $\mu \leq \frac{\nu(\nu+1)}{2}$  and  $\mu \times \mu$  symmetric matrices  $A_1, \dots, A_g$  such that  $\mathcal{D}_p = \mathcal{D}_L$  where  $L$  is the monic affine linear pencil

$$L = I - \sum A_j x_j.$$

In the case that  $p(0) = I_\delta$  the estimate on the size of the matrices  $A_j$  in  $L$  reduces to  $\frac{\check{\nu}(\check{\nu}+1)}{2}$ , where  $\check{\nu} = \delta \sum_0^{\lfloor \frac{d}{2} \rfloor_+} g^j$ .

As usual  $\lfloor \frac{d}{2} \rfloor_+$  stands for the smallest integer greater than  $\frac{d}{2}$ . Of course

$$\lfloor \frac{d}{2} \rfloor_+ = \frac{d}{2} \text{ when } d \text{ is even} \quad \text{and} \quad \lfloor \frac{d}{2} \rfloor_+ = \frac{d-1}{2} \text{ when } d \text{ is odd.}$$

**1.8. Further Results.** As we just saw the main theorem says that a convex nc bounded basic open semi-algebraic set has a degree one matrix defining polynomial. But, in the case that  $p(0)$  is positive definite, more is true in that any “minimum degree” defining polynomial itself has degree at most two. To present this result we start by describing a refinement of the notion of the boundary of  $\mathcal{D}_p$ , a refinement that also plays an important role in the proof of Theorem 1.3.

Let  $\partial \mathcal{D}_p$  denote the boundary of  $\mathcal{D}_p$ ; i.e.,  $\partial \mathcal{D}_p$  is the sequence whose  $n$ -th term is  $\partial \mathcal{D}_p(n)$ . If  $X \in \partial \mathcal{D}_p$ , then  $p(X)$  has a non-trivial kernel. Let  $\widehat{\partial \mathcal{D}_p}$  denote the set of pairs  $(X, v)$  such that  $X \in \partial \mathcal{D}_p$  and  $p(X)v = 0$ . Thus,  $v$  is assumed compatible with the sizes of  $X$  and  $p$ ; i.e., if  $X \in \mathbb{S}_n(\mathbb{R}^g)$  and  $p \in \mathcal{P}^{\delta \times \delta}$ , then  $v \in \mathbb{R}^\delta \otimes \mathbb{R}^n$ . Often it will be implicit that we are assuming  $v \neq 0$ .

Assume  $p$  in  $\mathcal{P}_d^{\delta \times \delta}$  is as in Assumption 1.2 and moreover  $p(0) = I_\delta$ . In particular,  $\sigma = (\delta, 0)$ . The polynomial  $p$  is called **minimum degree irreducible**, or a **minimum degree defining polynomial for  $\mathcal{D}_p$** , if every (row) vector of polynomials  $q = (q_1 \ \cdots \ q_\delta)$  in  $\mathcal{P}^\delta$  of degree strictly less than  $d$  satisfying  $q(X)v = 0$  for every  $(X, v) \in \widehat{\partial \mathcal{D}_p}$  is zero. We emphasize that while  $p$  is restricted by Assumption 1.2 to be symmetric, the polynomials  $q_j$  need not be symmetric.

**Theorem 1.5.** *Suppose  $p \in \mathcal{P}_d^{\delta \times \delta}$  satisfies the conditions of Assumption 1.2 and further that  $p(0) = I_\delta$ . If  $p$  is a minimum degree defining polynomial for  $\mathcal{D}_p$ , then  $p$  has degree at most two.*

*Moreover, in the case that  $\delta = 1$ , there exists a  $1 \times 1$  monic affine linear pencil  $L_0$ , an integer  $m \leq g$  and an  $m \times 1$  linear pencil  $\hat{L}$  with  $\hat{L}(0) = 0$  such that  $\mathcal{D}_p = \mathcal{D}_L$ , where*

$$L = \begin{pmatrix} I_m & \hat{L} \\ \hat{L}^T & L_0 \end{pmatrix}.$$

*In fact,  $p$  is the Schur complement of the  $(1, 1)$  entry of  $L$ ; i.e.,*

$$p = L_0 - \hat{L}^T \hat{L}.$$

See Section 8 for a more general statement and proof. Theorem 1.5 is, for the most part, an improvement over the main result of [DHM07]. In particular, the result here removes numerous hypotheses found in [DHM07], while reaching a stronger conclusion, though here we assume that  $\mathcal{D}_p$  is convex, rather than the weaker condition that  $\overline{\mathcal{D}_p}$  is convex. The techniques here are completely different than those in [DHM07].

**Remark 1.6.** We anticipate that the results of this paper remain valid if symmetric nc variables are replaced by free nc variables. That is, with variables  $(x_1, \dots, x_g, y_1, \dots, y_g)$  with the involution  $T$  on polynomials determined by  $x_j^T = y_j$ ,  $y_j^T = x_j$ , and, for polynomials  $f$  and  $g$  in these variables,  $(fg)^T = g^T f^T$ . These polynomials are evaluated at tuples  $X = (X_1, \dots, X_g) \in M_n(\mathbb{R}^g)$  of  $n \times n$  matrices with real entries. We do not see an obstruction to the free variable case using the arguments here, indeed arguments for them are often easier than for symmetric variables.  $\square$

**1.9. NC Open Semi-algebraic Sets and Convex Examples.** In this section we introduce nc open semi-algebraic sets. Under natural convexity hypotheses such sets turn out to be basic, an observation which, combined with Theorem 1.3, allows us to give examples showing that projections in the non-commutative semi-algebraic setting behave poorly.

**1.9.1. Semi-algebraic Sets and Direct Sums.** Recall, given a symmetric  $\delta \times \delta$  matrix nc polynomial  $p$  with  $p(0)$  invertible,

$$\mathfrak{I}_p = \{X \in \mathbb{S}_n(\mathbb{R}^g) : p(X) \text{ is invertible}\}$$

and  $\mathcal{D}_p(n)$  is the component of 0 of  $\mathfrak{I}_p$ . We define an **nc open semi-algebraic set** to be the union of finitely many nc basic open semi-algebraic sets. Thus, a nc open semi-algebraic set has the form

$$\cup_j^N \mathcal{D}_{p_j}.$$

A key property of an nc basic open semi-algebraic set  $\mathcal{C}$  is it **respects direct sums** : if  $Y_1 \in \mathcal{C}(m_1)$  and  $Y_2 \in \mathcal{C}(m_2)$ , then

$$Y_1 \oplus Y_2 = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \in \mathcal{C}(m_1 + m_2).$$

One of our main concerns in this section is the projection of an nc open semi-algebraic set  $\mathcal{D}$  in  $g + h$  variables. Let  $\rho[\mathcal{D}]$  denote the projection of  $\mathcal{D}$  onto the first  $g$  coordinates, that is

$$\rho[\mathcal{D}](n) = \{X \in \mathbb{S}_n(\mathbb{R}^g) : \text{there is a } Y \in \mathbb{S}_n(\mathbb{R}^h) \text{ such that } (X, Y) \in \mathcal{D}\}.$$

It is readily seen that

$$(1.8) \quad \text{if } \mathcal{D} \text{ respects direct sums, then so does its projection } \rho[\mathcal{D}].$$

This observation motivates the next proposition.



**Proposition 1.7.** *Given symmetric polynomials  $p_1, \dots, p_N$ , with  $p_j \in \mathcal{P}^{\delta_j \times \delta_j}$ , let  $\mathcal{W}(n)$  be a sequence of subsets of  $\cup \mathfrak{I}_{p_j}(n)$ . If  $\mathcal{W} = (\mathcal{W}(n))$  respects direct sums and each  $\mathcal{W}(n)$  contains 0 and is open and connected, then there is a  $k$  such that  $\mathcal{W} \subset \mathcal{D}_{p_k}$ .*

*Proof.* We begin by proving if  $X \in \mathcal{W}$  and if  $X(t)$  is a (continuous) path for  $0 \leq t \leq 1$  such that  $X(0) = 0$ ,  $X(1) = X$ , and  $X(t)$  lies in  $\mathcal{W}$ , then there is a  $j$  such that  $p_j(X(t))$  is invertible for every  $0 \leq t \leq 1$ .

Arguing by contradiction, suppose no such  $j$  exists. Then for every  $\ell$  there exists a  $0 \leq t_\ell \leq 1$  such that  $p_\ell(X(t_\ell))$  is not invertible. Since  $\mathcal{W}$  is closed with respect to direct sums,  $Z = \oplus X(t_\ell) \in \mathcal{W}$ . It follows that there is some  $j$  such that  $Z \in \mathfrak{I}_{p_j}$  and in particular,  $p_j(Z)$  is invertible, contradicting  $p_j(X(t_j))$  not invertible. We conclude that there is some  $j$  such that  $p_j(X(t))$  is invertible for  $0 \leq t \leq 1$  and hence  $X(t) \in \mathcal{D}_{p_j}$  for all  $0 \leq t \leq 1$ .

Now suppose there is a  $Y \in \mathcal{W}$  such that  $Y \notin \mathfrak{I}_{p_N}$ . In particular,  $p_N(Y)$  is not invertible. Since  $Y$  is in  $\mathcal{W}$ , there is a continuous path  $Y(t) \in \mathcal{W}$  such that  $Y(0) = 0$  and  $Y(1) = Y$ . Consider any  $X \in \mathcal{W}$ . There is a continuous path  $X(t) \in \mathcal{W}$  with  $X(0) = 0$  and  $X(1) = 1$ . Let  $Z(t) = X(t) \oplus Y(t)$ ; which is in  $\mathcal{W}$  since  $\mathcal{W}$  respects direct sums. Thus  $Z(t) \in \mathcal{W}$  is a continuous path ( $0 \leq t \leq 1$ ) with  $Z(0) = 0$ . From what has already been proved, there is a  $j$  such that  $p_j(Z(t))$  is invertible for each  $0 \leq t \leq 1$ . Thus  $p_j(Y)$  is invertible and we conclude that  $j \neq N$ , thus  $j < N$ . At the same time  $p_j(X(t))$  is invertible for  $0 \leq t \leq 1$  and thus  $X \in \mathcal{D}_{p_j}$ . Hence  $X \in \cup_1^{N-1} \mathcal{D}_{p_j}$ . We have proved: either  $\mathcal{W} \subset \mathfrak{I}_{p_N}$  or  $\mathcal{W} \subset \cup_1^{N-1} \mathcal{D}_{p_j} \subset \cup_1^{N-1} \mathfrak{I}_{p_j}$ . Since  $\mathcal{W}$  is connected and contains 0, the first alternative becomes  $\mathcal{W}$  is a subset of  $\mathcal{D}_{p_N}$ . Induction now finishes the proof.  $\square$

**Corollary 1.8.** *Let  $\mathcal{W} = \cup_j^N \mathcal{D}_{p_j}$ . If  $\mathcal{W}$  is closed with respect to direct sums, then there is a  $k$  such that  $\mathcal{W} = \mathcal{D}_{p_k}$ .*

*Proof.* Since, for each  $n$ , each  $\mathcal{D}_{p_j}(n)$  is open and connected and contains 0, the union  $\cup \mathcal{D}_{p_j}(n)$  is also open and connected. An application of Proposition 1.7 completes the proof.  $\square$

**Corollary 1.9.** *The projection  $\rho[\mathcal{D}]$  of a nc basic open semi-algebraic set  $\mathcal{D}$ , if nc (open) semi-algebraic, is nc basic open semi-algebraic.*

*Proof.* Since  $\mathcal{D}$  is basic, it respects direct sums. Thus the set  $\rho[\mathcal{D}]$  respects direct sums as noted in (1.8). It is also connected, since it is the continuous image of the connected set  $\mathcal{D}$ . Hence, by Corollary 1.8, if  $\rho[\mathcal{D}]$  is nc open semi-algebraic, then it is nc basic open semi-algebraic.  $\square$

The next corollary says that analogous results hold if instead, we were to define a nc open semi-algebraic set to be the component of 0 of  $\cup \mathfrak{I}_{p_j}$ .

**Corollary 1.10.** *Let  $\mathcal{W}(n)$  denote the connected component of 0 of  $\cup_j^N \mathfrak{J}_{p_j}(n)$ . If  $\mathcal{W}$  is closed with respect to direct sums, then there is a  $k$  such that  $\mathcal{W} = \mathcal{D}_{p_k}$ .*

*Proof.* Immediate from Proposition 1.7.  $\square$

1.9.2. *Matrix Convex NC Semi-algebraic Sets.* The analysis of projected nc sets depends upon the notion of a matrix convex set [EW97, WW99]. It turns out that this a priori stronger notion of convexity is in fact equivalent to convexity for a nc basic open semi-algebraic set  $\mathcal{D}_p$ . See Theorem 4.5.

For our purposes  $\mathcal{C} = (\mathcal{C}(n))$ , where each  $\mathcal{C}(n) \subset \mathbb{S}_n(\mathbb{R}^g)$ , is an open **matrix convex** set if

- (i) each  $\mathcal{C}(m)$  is open and contains  $0 = (0, \dots, 0) \in \mathbb{S}_m(\mathbb{R}^g)$ ;
- (ii)  $\mathcal{C}$  respects direct sums;
- (iii)  $\mathcal{C}$  **respects simultaneous conjugation with contractions:** if  $Y \in \mathcal{C}(m)$  and  $F$  is an  $m \times k$  contraction, then

$$F^T Y F = (F^T Y_1 F, \dots, F^T Y_g F) \in \mathcal{C}(k); \text{ and}$$

- (iv) each  $\mathcal{C}(m)$  is convex and bounded.

It is easy to see that the property  $\mathcal{C}(m)$  is convex in item (iv) actually follows from items (ii) and (iii). Indeed, given  $X, Y \in \mathcal{C}(n)$ , choose  $F$  to be the  $2n \times n$  matrix

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n \\ I_n \end{pmatrix}$$

and note that

$$\frac{X_j + Y_j}{2} = F^* \begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} F.$$

An immediate consequence of item (iii) is if  $X \in \mathbb{S}_n(\mathbb{R}^g)$ ,  $Y \in \mathbb{S}_m(\mathbb{R}^g)$  and  $X \oplus Y \in \mathcal{C}(n+m)$ , then  $Y \in \mathcal{C}(m)$ .

1.9.3. *Projections of NC Basic Open Semi-algebraic Sets May Not be NC Open Semi-algebraic.* A key fact in classical real algebraic geometry is the projection property. Namely, the projection of a semi-algebraic set is necessarily semi-algebraic. A consequence of Theorem 1.3 is that a similar projection property does not (necessarily) hold for nc semi-algebraic sets. Moreover, the next proposition suggests that typically the projection of a convex non-commutative open semi-algebraic set is no longer semi-algebraic. Indeed, combining this proposition with the Helton-Vinnikov line test [HV07], as we do in Subsection 1.12, produces an illustrative and explicit example of this phenomena.

**Proposition 1.11.** *Let  $\tilde{L}$  denote a monic affine linear pencil in  $g+h$  variables, let  $\mathcal{D}_{\tilde{L}}$  denote the corresponding (matrix) convex non-commutative basic open semi-algebraic set. The non-commutative set  $\rho[\mathcal{D}_{\tilde{L}}]$  is an open matrix convex set. Moreover, if  $\rho[\mathcal{D}_{\tilde{L}}]$  is bounded, either there is a monic affine linear pencil  $L$  in  $g$  variables such that  $\rho[\mathcal{D}_{\tilde{L}}] = \mathcal{D}_L$  or  $\rho[\mathcal{D}_{\tilde{L}}]$  fails to be nc open semi-algebraic.*

*Proof.* It is straightforward to check that  $\rho[\mathcal{D}_{\tilde{L}}]$  is an open matrix convex set. Indeed, that  $\rho[\mathcal{D}_{\tilde{L}}]$  is closed with respect to direct sums has already been noted. Suppose  $X \in \rho[\mathcal{D}_{\tilde{L}}](n)$  and  $F$  is an  $n \times m$  contraction. There is a  $Y$  such that  $(X, Y) \in \mathcal{D}_{\tilde{L}}$ . Because  $\mathcal{D}_{\tilde{L}}$  is an open matrix convex set,  $F^*(X, Y)F = (F^*XF, F^*YF) \in \mathcal{D}_{\tilde{L}}$ . Thus  $F^*XF \in \rho[\mathcal{D}_{\tilde{L}}]$ .

By Corollary 1.8, if  $\rho[\mathcal{D}_{\tilde{L}}]$  is nc open semi-algebraic, it must be nc basic open semi-algebraic and convex. Hence, by Theorem 1.4  $\rho[\mathcal{D}_{\tilde{L}}]$  is LMI representable.  $\square$

In the commutative setting a set  $C \subset \mathbb{R}^g$  is called SDP (semi-definite program) representable if there exists an LMI representable set  $\tilde{C} \subset \mathbb{R}^{g+h}$  such that  $C$  is the projection of  $\tilde{C}$  onto the first  $g$  coordinates. Commutative SDP representable sets are semi-algebraic.

Clearly, Proposition 1.11 bears on “nc SDP representations” and tells us that if a (bounded) nc open semi-algebraic set is nc SDP representable, then it is LMI representable. Example 1.12 below shows that an SDP representable set (the projection of an LMI representable set) need not be nc semi-algebraic.

**Example 1.12.** Consider the set

$$S = \{(x, y) \in \mathbb{R}^2 : 1 - x^4 - y^4 > 0\},$$

often called the TV screen. This set is evidently convex. One non-commutative analog of the TV screen is the non-commutative set  $\mathcal{T} = (\mathcal{T}(n))$  where

$$\mathcal{T}(n) = \{(X, Y) \in \mathbb{S}_n(\mathbb{R}^2) : I - X^4 - Y^4 \succ 0\}.$$

In particular,  $S = \mathcal{T}(1)$ .

*$\mathcal{T}$  is not an open matrix convex set.*

This is established by a simple test which might apply widely: Since  $\mathcal{T}$  is a nc basic open semi-algebraic set,  $\mathcal{T}$  is an open matrix convex set if and only if each  $\mathcal{T}(n)$  is convex. It turns out that  $S = \mathcal{T}(1)$  does not pass the line test of Helton-Vinnikov [HV07], and thus is not (commutatively) LMI representable. It follows that  $\mathcal{T}$  is not LMI representable and therefore, by Theorem 1.4, it is not an open matrix convex set.

*The nc semi-algebraic property is not preserved under projection.*

Next we turn to the nc semi-algebraic property and show, using Proposition 1.11, that projections do not preserve it. The usual SDP representation of  $S \subset \mathbb{R}^2$  is the following. Let

$$L_0(x, y) = \begin{pmatrix} 1 & 0 & y_1 \\ 0 & 1 & y_2 \\ y_1 & y_2 & 1 \end{pmatrix} \quad \text{and} \quad L_j(x, y) = \begin{pmatrix} 1 & x_j \\ x_j & y_j \end{pmatrix} \quad \text{for } j = 1, 2.$$

The set  $S$  can be written

$$S = \{(x_1, x_2) \in \mathbb{R}^2 : \text{there exists } (y_1, y_2) \text{ such that } L_j(x, y) \succ 0, \quad j = 0, 1, 2\},$$

an assertion easily checked using Schur complements.

Now we give a family of nc SDP representations for  $\mathcal{T}$ . Given  $\alpha$  a positive real number, choose  $\gamma^4 = 1 + 2\alpha^2$  and let

$$L_0^\alpha = \begin{pmatrix} 1 & 0 & y_1 \\ 0 & 1 & y_2 \\ y_1 & y_2 & 1 - 2\alpha(y_1 + y_2) \end{pmatrix}$$

and

$$L_j^\alpha = \begin{pmatrix} 1 & \gamma x_j \\ \gamma x_j & \alpha + y_j \end{pmatrix}.$$

The formulas are a bit different than for the classical  $L_0, L_1, L_2$ , since we desire monic LMIs, but note that letting  $\alpha$  tend to zero produces the classical version. While the  $L_j^\alpha$  are not monic for  $j = 1, 2$ , a simple normalization produces an equivalent monic LMI.

For positive integers  $n$ , let

$$\mathcal{S}_\alpha(n) = \{(X_1, X_2) \in \mathbb{S}_n(\mathbb{R}^2) : \exists(Y_1, Y_2) \text{ with } L_j^\alpha(X, Y) \succ 0, \quad j = 0, 1, 2\}.$$

The non-commutative set  $\mathcal{S}_\alpha = (\mathcal{S}_\alpha(n))$  is of course the projection of the set  $\{(X, Y) : L_j^\alpha(X, Y) \succ 0 \text{ for } j = 0, 1, 2\}$ . It is an open matrix convex set and, as is readily checked, the set  $\mathcal{S}_\alpha(1)$  is  $S = \mathcal{T}(1)$ . Moreover, for each  $n$ ,  $\mathcal{T}(n) \subset \mathcal{S}^\alpha(n)$ .

By Proposition 1.11 the set  $\mathcal{S}_\alpha$  is not nc basic open semi-algebraic as otherwise  $S = \mathcal{S}_\alpha(1)$  would be (commutatively) LMI representable. Indeed, SDP representations of  $S$  when  $n = 1$  are not unique and any one of them projects to a matrix convex set (for all  $n$ ) containing  $S$  which fails to be nc open semi-algebraic.  $\square$

We thank Jiawang Nie for raising the issue of projected matrix convex sets and we thank Igor Klep and Victor Vinnikov for fruitful discussions of the TV screen example above.

**1.10. Motivation.** One of the main advances in systems engineering in the 1990's was the conversion of a set of problems to LMIs, since LMIs, up to modest size, can be solved numerically by semidefinite programs [SIG97]. A large class of linear systems problems are described in terms of a signal flow diagram  $\Sigma$  plus  $L^2$  constraints (such as energy dissipation). Routine methods convert such problems into a non-commutative polynomial inequalities of the form  $p(X) \succeq 0$  or  $p(X) \succ 0$ .

Instantiating specific systems of linear differential equations for the "boxes" in the system flow diagram amounts to substituting their coefficient matrices for variables in the polynomial  $p$ . Any property asserted to be true must hold when matrices of any size are substituted into  $p$ . Such problems are referred to as dimension free. We emphasize, the polynomial  $p$  itself is determined by the signal flow diagram  $\Sigma$ .

Engineers vigorously seek convexity, since optima are global and convexity lends itself to numerics. Indeed, there are over a thousand papers trying

to convert linear systems problems to convex ones and the only known technique is the rather blunt trial and error instrument of trying to guess an LMI. Since having an LMI is seemingly more restrictive than convexity, there has been the hope, indeed expectation, that some practical class of convex situations has been missed. The problem solved here (though not operating at full engineering generality, see [HLLM08]) is a paradigm for the type of algebra occurring in systems problems governed by signal-flow diagrams; such physical problems directly present nc semi-algebraic sets. Theorem 1.3 gives compelling evidence that all such convex situations are associated to some LMI. Thus we think the implications of our results here are negative for linear systems engineering; for dimension free problems there is no convexity beyond LMIs.

It is informative to view this paper in the context of semidefinite programming, SDP. Semidefinite programming, which solves LMIs up to modest size, was one of the main developments in optimization over the previous two decades. Introduced about 15 years ago [NN94] it has had a substantial effect in many areas of science and mathematics; e.g., statistics, game theory, structural design and computational real algebraic geometry, with its largest impact likely being in control systems and combinatorial optimization. For a general survey, see Nemirovskii’s Plenary Lecture at the 2006 ICM, [Ne06]. An introduction of SDP techniques into a variety of areas being pursued today was first given (and is well explained in) [P00]. The numerics of semidefinite programming is well developed and there are numerous packages; e.g., [St99] [GNLC95] and comparisons [Mi03] which apply when the constraint is input as the solution to a Linear Matrix Inequality.

A basic question regarding the range of applicability of SDP is: which sets have an LMI representation? Theorem 1.3 settles, to a reasonable extent, the case where the variables are non-commutative (effectively dimension free matrices).

For perspective, in the commutative case of a basic semi-algebraic subset  $\mathcal{C}$  of  $\mathbb{R}^g$ , as we have already mentioned, there is a stringent condition, called the “line test”, which, in addition to convexity, is necessary for  $\mathcal{C}$  to have an LMI representation. In two dimensions the line test is necessary and sufficient, [HV07]. This was seen by Lewis-Parrilo-Ramana [LPR05] to settle a 1958 conjecture of Peter Lax on hyperbolic polynomials and indeed LMI representations are closely tied to properties of hyperbolic polynomials.

In summary, a (commutative) bounded basic open semi-algebraic convex set has an LMI representation, then it must pass the highly restrictive line test; whereas a nc bounded basic open semi-algebraic set has an LMI representation if and only if it is convex.

**1.11. Layout.** The layout of the body of the paper is as follows. Sections 2 and 3 collect basic facts about the boundary of  $\mathcal{D}_p$  and zero sets of nc polynomials respectively. Such zero sets are a nc analog of a variety and the set  $\widehat{\partial\mathcal{D}}_p$  is a subset of the zero set of  $p$ . Facts about non-commutative

(matrix) convex sets generally and those for convex nc open semi-algebraic sets,  $\mathcal{D}_p$ , in particular, are presented in Section 4. Section 5 begins with a version of the non-commutative Hahn-Banach separation theorem of Effros-Winkler [EW97] and concludes with a refinement tailored to the present needs. Theorem 1.3 is proved in Section 6.

Section 8 is devoted to a discussion and proof of Theorem 1.5. Certain refined estimates available when  $p(0)$  is invertible and needed in Section 8 are collected in Section 7. Section 9, gives a proof of the needed variant of the Effros-Winkler Theorem used in the first part of Section 5. The paper concludes with final remarks and examples in Section 10.

## 2. FACTS ABOUT $\mathcal{D}_p$ AND ITS BOUNDARY

In this section we layout simple facts we need for the main proofs later.

**2.1. Life on the boundary.** We begin by recalling, from Subsection 1.8, that  $\partial\mathcal{D}_p$  denotes the boundary of  $\mathcal{D}_p$ ; i.e.,  $\partial\mathcal{D}_p$  is the sequence whose  $n$ -th term is  $\partial\mathcal{D}_p(n)$ . If  $X \in \partial\mathcal{D}_p$ , then  $p(X)$  fails to be invertible and thus there is a non-zero vector  $v$  such that  $p(X)v = 0$ . Recall,  $\widehat{\partial\mathcal{D}_p}$  denotes the pairs  $(X, v)$  such that  $X \in \partial\mathcal{D}_p$  and  $p(X)v = 0$ .

The following Lemma gives a useful criteria for containment in  $\partial\mathcal{D}_p$  and  $\widehat{\partial\mathcal{D}_p}$ .

**Lemma 2.1.** *Suppose  $p \in \mathcal{P}^{\delta \times \delta}$  satisfies the conditions of Assumption 1.2 and  $(X, v) \in \mathbb{S}_n(\mathbb{R}^g) \times (\mathbb{R}^\delta \otimes \mathbb{R}^n)$  with  $v \neq 0$ . The pair  $(X, v) \in \widehat{\partial\mathcal{D}_p}$  if and only if  $tX \in \mathcal{D}_p$  for  $0 \leq t < 1$  and  $p(X)v = 0$ .*

*Proof.* First suppose that  $(X, v) \in \widehat{\partial\mathcal{D}_p}$ . In this case,  $X \in \partial\mathcal{D}_p$  and  $p(X)v = 0$ . Since  $\mathcal{D}_p$  is convex, so is  $\overline{\mathcal{D}_p}$ . Thus,  $tX \in \overline{\mathcal{D}_p}$  for  $0 \leq t \leq 1$ . Moreover, there are only finitely many  $0 \leq s \leq 1$  such that  $p(sX)$  is not invertible because  $p(0)$  is invertible and  $p$  is a polynomial. If  $0 \leq t < 1$  and  $p(tX)$  is invertible, then  $tX \in \mathfrak{I}_p(n)$ . To see that in fact  $tX$  is in  $\mathcal{D}_p$ , we argue by contradiction. Accordingly, suppose  $tX \notin \mathcal{D}_p$ . In this case, since  $\mathfrak{I}_p(n)$  is both open and the disjoint union of its connected components,  $tX$  is contained in some open set which does not meet  $\mathcal{D}_p$ . Thus, we have reached the contradiction  $tX \notin \overline{\mathcal{D}_p}$ . Since  $\mathcal{D}_p$  is convex, if  $tX \in \mathcal{D}_p$ , then  $sX \in \mathcal{D}_p$  for  $0 \leq s \leq t$ . Choosing a sequence  $0 < t_n < 1$  converging to 1 such that  $p(t_n X)$  is invertible it now follows that  $sX \in \mathcal{D}_p$  for  $0 \leq s < 1$ .

Conversely, if  $tX \in \mathcal{D}_p$  for  $0 \leq t < 1$ , then  $X \in \overline{\mathcal{D}_p}$ . On the other hand, if  $p(X)v = 0$ , then  $X \notin \mathcal{D}_p$  and thus  $X \in \partial\mathcal{D}_p$ .  $\square$

We close this subsection by recording the following simple useful fact.

**Lemma 2.2.** *Let  $\mathcal{C} = (\mathcal{C}(n))$  be a given non-commutative set. Suppose each  $\mathcal{C}(n) \subset \mathbb{S}_n(\mathbb{R}^g)$  is open. If  $L$  is a monic affine linear pencil, then  $L$  is positive definite on  $\mathcal{C}$  if and only if  $L$  is positive semi-definite on  $\mathcal{C}$ .*

*Proof.* Suppose  $L$  is positive semi-definite on  $\mathcal{C}$ . If  $L$  is not positive definite on  $\mathcal{C}$ , then there is an  $n$  and an  $X \in \mathcal{C}(n)$  such that  $L(X) \succeq 0$  and  $L(X)$  has a kernel. In particular, there is a unit vector  $v$  such that  $L(X)v = 0$ . Let  $q(t) = \langle L(tX)v, v \rangle$ . Thus  $q$  is affine linear in  $t$  and  $q(0) = 1$  whereas  $q(1) = 0$ . Hence  $q(t) < 0$  for  $t > 1$  and thus  $L(tX) \not\succeq 0$  for  $t > 1$ . On the other hand, since  $\mathcal{C}(n)$  is open and  $X \in \mathcal{C}(n)$ , there is  $t > 1$  such that  $tX \in \mathcal{C}(n)$  which gives the contradiction  $L(tX) \succeq 0$ .  $\square$

**2.2. Dominating Points.** There is a certain class of points where the matricial Hahn-Banach separation theorem we later employ behaves particularly well. The details follow. Given  $(X^j, v^j) \in \mathbb{S}_{n_j}(\mathbb{R}^g) \times (\mathbb{R}^\delta \otimes \mathbb{R}^{n_j})$ , for  $j = 1, 2$ , let

$$\oplus_{j=1}^2 (X^j, v^j) = \left( \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right).$$

This notion of direct sum clearly extends to a finite list  $(X^j, v^j)$ ,  $j = 1, 2, \dots, s$ . Note that if  $(X^j, v^j) \in \widehat{\partial\mathcal{D}}_p$  for  $j = 1, 2, \dots, s$ , then  $\oplus (X^j, v^j) \in \widehat{\partial\mathcal{D}}_p$ ; i.e.,  $\widehat{\partial\mathcal{D}}_p$  **respects direct sums**. Likewise, a subset  $S = (S_n)_{n=1}^\infty$  of  $\widehat{\partial\mathcal{D}}_p$  **respects direct sums** if  $(X^j, v^j) \in S$  for  $j = 1, 2, \dots, s$  implies  $\oplus (X^j, v^j) \in S$ .

Let  $S \subset \widehat{\partial\mathcal{D}}_p$  denote a non-empty set which respects direct sums. A **dominating point**  $(X, v) \in \widehat{\partial\mathcal{D}}_p$  **of**  $S$  is a point with the property that if  $q \in \mathcal{P}_d^\delta$  vanishes at  $(X, v)$ , that is  $q(X)v = 0$ , then  $q$  vanishes on all of  $S$ ; i.e.,  $(X, v)$  is dominating if  $q(X)v = 0$  and  $(Y, w) \in S$ , then  $q(Y)w = 0$ . Note that the dimension of the spaces that  $X$  and  $Y$  act on are independent of one another. Denote the **dominating points of**  $S$  **by**  $S_*$ . Note  $S_*$  need not be contained in  $S$ . On the other hand and importantly,  $S \cap S_*$  is non-empty. See Lemma 2.3 below.

Given a subset  $S = (S_n)_{n=1}^\infty$  of  $\widehat{\partial\mathcal{D}}_p$  let

$$\mathcal{I}(S) = \{q \in \mathcal{P}_d^\delta : q(X)v = 0, \text{ for all } (X, v) \in S\}.$$

In the special case that  $S$  is a singleton,  $S = \{(X, v)\}$ , we usually write  $\mathcal{I}(X, v)$  in place of the more cumbersome  $\mathcal{I}(\{(X, v)\})$ . Observe that  $\mathcal{I}(S)$  is a subspace of the  $\delta$ -tuples (row vectors) of polynomials of degree at most  $d$  (when  $\delta = 1$ , and if not for the degree restriction, the subspace  $\mathcal{I}(S)$  would be a left ideal in  $\mathcal{P}$ ).

In terms of  $\mathcal{I}(S)$ , the point  $(X, v) \in \widehat{\partial\mathcal{D}}_p$  is dominating for  $S$  if and only if

$$\mathcal{I}(X, v) \subset \mathcal{I}(S).$$

On the other hand, if  $(X, v) \in S$ , then

$$\mathcal{I}(S) \subset \mathcal{I}(X, v).$$

Thus, if  $(X, v) \in S \cap S_*$ , then

$$\mathcal{I}(X, v) = \mathcal{I}(S).$$

**Lemma 2.3.** *Suppose  $S$  is a non-empty subset of the graded set  $(\mathbb{S}_n(\mathbb{R}^d) \times (\mathbb{R}^d \otimes \mathbb{R}^n))_{n=1}^\infty$ . If  $S$  respects direct sums, then there is an  $(X, v) \in S$  such that*

$$(2.1) \quad \mathcal{I}(S) = \mathcal{I}(X, v).$$

*That is,  $S \cap S_*$  is non-empty.*

*Proof.* First note that

$$\mathcal{I}(S) = \bigcap \{ \mathcal{I}(Y, w) : (Y, w) \in S \}.$$

Thus, since each  $\mathcal{I}(Y, w)$  is a subspace of the finite dimensional vector space  $\mathcal{P}_d^\delta$ , there exists an  $s$  and  $(Y_j, w_j) \in S$  for  $j = 1, \dots, s$  such that

$$\mathcal{I}(S) = \bigcap_{j=1}^s \mathcal{I}(Y_j, w_j).$$

Let  $(X, v) = \bigoplus (Y_j, w_j)$ . Then  $(X, v) \in S$  and

$$(2.2) \quad \mathcal{I}(X, v) = \bigcap_{j=1}^s \mathcal{I}(Y_j, w_j) = \mathcal{I}(S).$$

□

We record the following property of  $S \cap S_*$  for later use.

**Lemma 2.4.** *Suppose  $S \subset \widehat{\partial\mathcal{D}}_p$  respects direct sums and  $q \in \mathcal{P}_d^\delta$ . If both  $(X, v)$  and  $(Y, w)$  are in  $S \cap S_*$ , then  $q(X)v = 0$  if and only if  $q(Y)w = 0$ ; i.e.,  $q$  either vanishes on all of  $S \cap S_*$  or none of  $S \cap S_*$ .*

*Proof.* Suppose  $q(X)v = 0$ . Then, since  $(X, v)$  is dominating for  $S$  and  $(Y, w) \in S$ , it follows that  $q(Y)w = 0$ . By symmetry, if  $q(Y)w = 0$ , then  $q(X)v = 0$  and the proof is complete. □

### 3. CLOSURE WITH RESPECT TO A SUBSPACE OF POLYNOMIALS

In this section we introduce and develop properties of a canonical closure operation on subsets  $W \subset \widehat{\partial\mathcal{D}}_p$ . While it resembles the Zariski closure, because of the degree restrictions it is not a true nc analog.

The  $\mathcal{P}_d^\delta$ -**closure** of a non-empty set  $W \subset \widehat{\partial\mathcal{D}}_p$  which respects direct sums is defined to be

$$W_z := \{ (X, v) \in \widehat{\partial\mathcal{D}}_p : f(X)v = 0 \text{ for every } f \in \mathcal{I}(W) \}.$$

Equivalently  $\mathcal{I}(W) = \mathcal{I}(W_z)$  and  $W_z \subset \widehat{\partial\mathcal{D}}_p$  is the largest set with this property. In particular, to say  $W$  is  $\mathcal{P}_d^\delta$ -**closed** means  $W_z = W$ . We emphasize these definitions only apply to non-empty sets  $W$  of  $\widehat{\partial\mathcal{D}}_p$  which respect direct sums.

**Lemma 3.1.** *If  $(X, v) \in \widehat{\partial\mathcal{D}}_p$ , then  $(X, v) \in W_z$  if and only if  $\mathcal{I}(X, v) \supset \mathcal{I}(W)$ .*

*Moreover,  $\mathcal{I}(W) = \mathcal{I}(W_z)$  and if  $U \subset \widehat{\partial\mathcal{D}}_p$  and  $\mathcal{I}(U) = \mathcal{I}(W)$ , then  $U \subset W_z$ .*



*Proof.* Let  $(X, v) \in \widehat{\partial\mathcal{D}}_p$  be given. Suppose  $(X, v) \in W_z$ . If  $q \in \mathcal{I}(W)$ , then  $q(X)v = 0$  and hence  $q \in \mathcal{I}(X, v)$ . Thus,  $\mathcal{I}(W) \subset \mathcal{I}(X, v)$ . Conversely, suppose  $\mathcal{I}(X, v) \supset \mathcal{I}(W)$ . If  $q \in \mathcal{I}(W)$ , then  $q \in \mathcal{I}(X, v)$  and hence  $q(X)v = 0$ . Hence  $(X, v) \in W_z$ . This completes the proof of the first part of the lemma.

Since  $(X, v) \in W_z$  implies  $\mathcal{I}(X, v) \supset \mathcal{I}(W)$ , it follows that  $\mathcal{I}(W_z) \supset \mathcal{I}(W)$ . On the other hand, since  $W \subset W_z$ , the inclusion  $\mathcal{I}(W) \supset \mathcal{I}(W_z)$  and the equality  $\mathcal{I}(W) = \mathcal{I}(W_z)$  follows.

Finally, suppose  $\mathcal{I}(U) = \mathcal{I}(W)$  and let  $(X, v) \in U$  be given. If  $q \in \mathcal{I}(W)$ , then  $q \in \mathcal{I}(U)$  and hence  $q(X)v = 0$ . Thus,  $(X, v) \in W_z$  and hence  $U \subset W_z$ .  $\square$

The following Lemma collects basic facts about the  $\mathcal{P}_d^\delta$ -closure operation.

**Lemma 3.2.** *Suppose  $\widehat{\partial\mathcal{D}}_p \supset A, B$  are non-empty sets which respects direct sums.*

- (1)  $A \subset A_z$ ;
- (2) If  $A \supset B$ , then  $\mathcal{I}(A) \subset \mathcal{I}(B)$ ;
- (3) If  $\mathcal{I}(A) \subset \mathcal{I}(B)$ , then  $A_z \supset B_z \supset B$ ;
- (4) If  $B \subset A$ , then  $B_z \subset A_z$ ;
- (5) If  $B$  is  $\mathcal{P}_d^\delta$ -closed and  $B \subsetneq A$ , then  $\mathcal{I}(A) \subsetneq \mathcal{I}(B)$ ;
- (6) If  $A_1 \supsetneq A_2 \supsetneq \dots$  is a strictly decreasing sequence of non-empty  $\mathcal{P}_d^\delta$ -closed sets, then it is finite; and
- (7) A non-empty collection  $\mathfrak{T}$  of non-empty  $\mathcal{P}_d^\delta$ -closed subsets of  $\widehat{\partial\mathcal{D}}_p$  contains a minimal element; i.e., there exists a set  $T \in \mathfrak{T}$  such that if  $A \subset T$  and  $A \in \mathfrak{T}$ , then  $A = T$ .

*Proof.* The first four items are obvious.

To prove (5), note that by (2),  $\mathcal{I}(A) \subset \mathcal{I}(B)$ . On the other hand, if  $\mathcal{I}(A) = \mathcal{I}(B)$ , then by (3),  $A_z \subset B_z$ . But then,

$$B_z = B \subsetneq A \subset A_z \subset B_z,$$

a contradiction.

Item (6) holds because  $\mathcal{I}(A_1) \subsetneq \mathcal{I}(A_2) \subsetneq \dots$  is, by (5), a strictly increasing nest of subspaces of the finite dimensional vector space  $\mathcal{P}_d^\delta$ . Thus there is an  $m$  such that  $\mathcal{I}(A_\ell) = \mathcal{I}(A_m)$  for all  $\ell \geq m$ . Using (3) twice and the fact that each  $A_\ell$  is  $\mathcal{P}_d^\delta$ -closed, it follows that  $A_\ell = A_m$  for  $\ell \geq m$ .

To prove (7), choose  $A_1 \in \mathfrak{T}$ . If  $A_1$  is not minimal, then there exists  $A_2 \in \mathfrak{T}$  such that  $A_1 \supsetneq A_2$ . Continuing in this fashion, we eventually find a minimal set  $T$  as the alternative is a nested strictly decreasing sequence

$$A_1 \supsetneq A_2 \supsetneq A_3 \supsetneq \dots$$

from  $\mathfrak{T}$  which contradicts (6).  $\square$

Facts about the relation between dominating points and  $\mathcal{P}_d$ -closures are collected in the next lemma.

**Lemma 3.3.** *Suppose  $\widehat{\partial\mathcal{D}}_p \supset A, B$  are non-empty sets which respects direct sums.*

- (1) *If  $A \supset B$ , then  $A_* \subset B_*$ ;*
- (2)  *$A_* = (A_z)_*$ ;*
- (3)  *$B \cap B_*$  is non-empty;*
- (4)

$$(3.1) \quad B \cap B_* \subset \{(X, v) \in \widehat{\partial\mathcal{D}}_p : \mathcal{I}(X, v) = \mathcal{I}(B)\} \text{ and};$$

- (5) *If  $A$  is  $\mathcal{P}_d^\delta$  closed, then*

$$A \cap A_* = \{(X, v) \in \widehat{\partial\mathcal{D}}_p : \mathcal{I}(X, v) = \mathcal{I}(A)\}.$$

Hence for any  $B$ ,

$$B_z \cap B_* = \{(X, v) \in \widehat{\partial\mathcal{D}}_p : \mathcal{I}(X, v) = \mathcal{I}(B)\}.$$

**Remark 3.4.** *Note that item (3) is Lemma 2.3 and (4) ((3.1)) follows from the remarks preceding Lemma 2.3. Item (4) is also related to Lemma 2.4 which, says if  $(X, v), (Y, w) \in B \cap B_*$ , then  $\mathcal{I}(X, v) = \mathcal{I}(Y, w)$ .  $\square$*

*Proof.* We prove the items in order.

- (1) If  $(X, v) \in A_*$ , then  $\mathcal{I}(X, v) \subset \mathcal{I}(A) \subset \mathcal{I}(B)$ , so  $(X, v) \in B_*$ .
- (2) By Lemma 3.2(1),  $A \subset A_z$ . Thus, by part (1) of this lemma,  $A_* \supset (A_z)_*$ . On the other hand, if  $(X, v) \in A_*$ , then

$$\mathcal{I}(X, v) \subset \mathcal{I}(A) = \mathcal{I}(A_z)$$

and thus  $(X, v) \in (A_z)_*$ . Hence  $A_* \subset (A_z)_*$ .

- (5) One inclusion follows from the previous item. To prove the other inclusion, suppose  $A$  is  $\mathcal{P}_d^\delta$ -closed,  $(X, v) \in \widehat{\partial\mathcal{D}}_p$ , and  $\mathcal{I}(X, v) = \mathcal{I}(A)$ . Since  $\mathcal{I}(X, v) \supset \mathcal{I}(A)$  and  $A$  is  $\mathcal{P}_d^\delta$ -closed,  $(X, v) \in A$ . On the other hand,  $(X, v) \in A_*$  since  $\mathcal{I}(X, v) \subset \mathcal{I}(A)$ . Thus the reverse inclusion holds and the proof is complete.  $\square$

For a monic affine linear pencil  $L$  let  $i(L)$  denote

$$i(L) := \{(Y, w) \in \widehat{\partial\mathcal{D}}_p : L(Y) \text{ is invertible}\}.$$

**Proposition 3.5.** *Suppose  $S \subset \widehat{\partial\mathcal{D}}_p$  is a non-empty set which respects direct sums and  $L$  is a monic affine linear pencil. If*

- (i)  *$L$  is singular on  $S_*$ ; and*
- (ii)  *$i(L) \subset S$ ,*

*then  $i(L)_z$  is properly contained in  $S_z$ :*

$$i(L)_z \subsetneq S_z.$$

*Proof.* By (ii) and Lemma 3.2(4) we have  $i(L)_z \subset S_z$ . Arguing by contradiction, suppose that  $i(L)_z = S_z$ . Then, from Lemma 3.3 parts (2) and (3) (twice)

$$\emptyset \neq i(L) \cap i(L)_* = i(L) \cap (i(L)_z)_* = i(L) \cap (S_z)_* = i(L) \cap S_*.$$

Hence there is an  $(X, v) \in i(L) \cap S_*$ . But then  $L(X) \succ 0$  since  $(X, v) \in i(L)$  and on the other hand, by (i),  $L(X)$  is singular because  $(X, v) \in S_*$ . This contradiction proves the indicated inclusion is proper.  $\square$

#### 4. CONVEX BASIC NON-COMMUTATIVE SEMI-ALGEBRAIC SETS

This section contains proofs of two facts about a convex non-commutative basic open semi-algebraic set  $\mathcal{D}_p$ . First, it is in fact an open matrix convex set; and second, if  $p \in \mathcal{P}^{\delta \times \delta}$ , then membership in  $\mathcal{D}_p$  and its boundary is determined by compressions to subspaces of dimension at most  $\nu = \delta \sum_0^d g^j$ .

**4.1. Matrix Convexity.** The following lemma applies to any nc basic open semi-algebraic set.

**Lemma 4.1.** *Suppose  $p \in \mathcal{P}^{\delta \times \delta}$  is symmetric and  $p(0)$  is invertible.*

- (i) *The set  $\mathcal{D}_p$  is closed under unitary similarity; i.e., if  $X \in \mathcal{D}_p(n)$  and  $U$  is  $(n \times n)$  unitary, then*

$$U^* X U = (U^* X_1 U, \dots, U^* X_g U) \in \mathcal{D}_p(n).$$

- (ii) *The set  $\mathcal{D}_p$  is closed with respect to direct sums; i.e., if  $X, Y \in \mathcal{D}_p$ , then so is  $X \oplus Y$ .*

*Proof.* The first item follows from the fact that  $p(U^* X U) = U^* p(X) U$ .

The second item is readily verified.  $\square$

Recall the definition of an open matrix convex set from Section 1.9.2 and that  $\mathcal{D}_p$  is convex means each  $\mathcal{D}_p(n)$  is convex.

**Lemma 4.2.** *Suppose  $p \in \mathcal{P}^{\delta \times \delta}$  is symmetric and  $p(0)$  is invertible. If  $\mathcal{D}_p$  is convex,  $X \in \mathbb{S}_n(\mathbb{R}^g)$ ,  $Y \in \mathbb{S}_m(\mathbb{R}^g)$ , and  $X \oplus Y \in \mathcal{D}_p(n+m)$ , then  $X \in \mathcal{D}_p(n)$  and  $Y \in \mathcal{D}_p(m)$ .*

*Proof.* Let  $Z = X \oplus Y$ . By convexity,  $tZ \in \mathcal{D}_p(n+m)$  for  $0 \leq t \leq 1$ . It follows that  $p(tX)$  is invertible for  $0 \leq t \leq 1$  and so there is a path from 0 to  $X$  lying in  $\mathcal{D}_p(n)$ . Thus  $X \in \mathcal{D}_p(n)$ . Likewise for  $Y$ .  $\square$

**Remark 4.3.** Similar conclusions hold, in both lemmas, if instead it is assumed that  $p(0) = I$ , and the sets  $\{X \in \mathbb{S}_n(\mathbb{R}^g) : p(X) \succ 0\}$  or the sets  $\{X \in \mathbb{S}_n(\mathbb{R}^g) : p(X) \succeq 0\}$  are convex.  $\square$

**Remark 4.4.** In Lemma 4.2 if we used the weaker hypothesis that the closure of  $\mathcal{D}_p$  is convex, then the proof breaks down. This is the main reason we use open sets in this paper.  $\square$

**Theorem 4.5.** *If  $p \in \mathcal{P}^{\delta \times \delta}$  satisfies the conditions of Assumption 1.2, then  $\mathcal{D}_p$  is an open matrix convex set.*

*Proof.* Since  $p(0)$  is invertible,  $\mathcal{D}_p$  contains a neighborhood of 0.

That  $\mathcal{D}_p$  is closed with respect to direct sums is part of Lemma 4.1 (and does not depend upon convexity or boundedness).

To prove that  $\mathcal{D}_p$  is closed with respect to simultaneous conjugation by contractions, suppose that  $X \in \mathcal{D}_p(n)$  and  $C$  is a given  $n \times n$  contraction. Let  $U$  denote the Julia matrix (of  $C$ ),

$$U = \begin{pmatrix} C & (I - CC^*)^{\frac{1}{2}} \\ -(I - C^*C)^{\frac{1}{2}} & C^* \end{pmatrix}.$$

Routine calculations show  $U$  is unitary.

Let  $0$  denote the  $g$ -tuple of zero matrices of size  $n \times n$ . Then, since both  $X$  and  $0$  are in  $\mathcal{D}_p$ , the direct sum  $X \oplus 0$  is also in  $\mathcal{D}_p$ . Since  $\mathcal{D}_p$  is closed with respect to unitary conjugation both the matrices

$$Y = U^* \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} U$$

$$Z = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} Y \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

are in  $\mathcal{D}_p(2n)$ . Using the convexity assumption on  $\mathcal{D}_p(2n)$ ,

$$\frac{1}{2}(Y + Z) = \begin{pmatrix} C^*XC & 0 \\ 0 & (I - CC^*)^{\frac{1}{2}}X(I - CC^*)^{\frac{1}{2}} \end{pmatrix}$$

is in  $\mathcal{D}_p(2n)$ . An application of the Lemma 4.2 implies  $C^*XC \in \mathcal{D}_p(n)$ .

By hypothesis  $\mathcal{D}_p$  is bounded.  $\square$

**4.2. Compressions.** Given  $(X, v) \in \mathbb{S}_n(\mathbb{R}^g) \times (\mathbb{R}^\delta \otimes \mathbb{R}^n)$  define a subspace  $\mathcal{M}$  of  $\mathbb{R}^n$  by

$$(4.1) \quad \mathcal{M} := \{q(X)v : q \in \mathcal{P}_d^\delta\} \subset \mathbb{R}^n$$

where

$$q(X)v = (q_1(X) \dots q_\delta(X)) \begin{pmatrix} v_1 \\ \vdots \\ v_\delta \end{pmatrix} = \sum q_j(X)v_j,$$

with  $v_j \in \mathbb{R}^n$ .

Let  $\nu = \delta \sum_{j=0}^d g^j$ . It is both the dimension of the vector space  $\mathcal{P}_d^\delta$  and, importantly, an upper bound for the dimension of the vector space  $\mathcal{M}$  of equation (4.1).

**Lemma 4.6.** *Suppose  $p$  in  $\mathcal{P}_d^{\delta \times \delta}$  satisfies the hypotheses of Assumption 1.2. If  $(X, v) \in \widehat{\partial \mathcal{D}}_p$ , then  $(P_{\mathcal{M}}X|_{\mathcal{M}}, v) \in \widehat{\partial \mathcal{D}}_p$ ; indeed,  $tP_{\mathcal{M}}X|_{\mathcal{M}} \in \mathcal{D}_p$  for  $0 \leq t < 1$  and  $p(P_{\mathcal{M}}X|_{\mathcal{M}})v = 0$ .*

*Proof.* From Lemma 2.1,  $tX \in \mathcal{D}_p$  for  $0 \leq t < 1$ . Let  $V$  denote the inclusion of  $\mathcal{M}$  into  $\mathbb{R}^n$ . Since  $V$  is a contraction and, by Theorem 4.5,  $\mathcal{D}_p$  is a (open)

matrix convex set, we obtain  $tP_{\mathcal{M}}X|_{\mathcal{M}} = V^*tXV \in \mathcal{D}_p$ . On the other hand, from the definition of  $\mathcal{M}$ , for any word  $w$  of length at most  $d$ ,

$$w(P_{\mathcal{M}}X|_{\mathcal{M}})v = P_{\mathcal{M}}w(X)|_{\mathcal{M}}v = P_{\mathcal{M}}w(X)v.$$

Hence,

$$p(P_{\mathcal{M}}X|_{\mathcal{M}})v = P_{\mathcal{M}}p(X)v = 0.$$

□

## 5. SEPARATING LINEAR PENCILS

In this section we develop a Hahn-Banach separation theorem for the (matrix) convex bounded nc basic semi-algebraic set  $\mathcal{D}_p$ . See Theorem 5.5 in Subsection 5.2. A version of the Effros-Winkler separation Theorem is the topic of the first subsection.

**5.1. The Effros-Winkler Separation Theorem.** The following Lemma is both a refinement and specialization of the non-commutative Hahn-Banach separation theorem of Effros and Winkler [EW97]. It is specialized to convex bounded nc basic open semi-algebraic sets  $\mathcal{D}_p$ ; and refined in that it isolates a point on the boundary of  $\mathcal{D}_p$  from  $\mathcal{D}_p$ .

**Lemma 5.1.** *Suppose  $p$  satisfies the conditions of Assumption 1.2. If  $X \in \partial\mathcal{D}_p(n)$ , then there exists a monic affine linear pencil  $L$  of size  $n$  such that  $L$  is positive definite  $\mathcal{D}_p$  and  $L(X)$  is singular.*

The proof of Lemma 5.1 is in Section 9. See Proposition 9.1. A subtlety is that while  $X \in \partial\mathcal{D}_p(n)$  (size  $n$ ), for every  $m$  and  $Y \in \mathcal{D}_p(m)$ ,  $L(Y) \succ 0$ . We give a more quantitative versions of this lemma. Recall the definition of  $\nu = \delta \sum_0^d g^j$  from the previous section.

**Lemma 5.2.** *Suppose  $p$  satisfies Assumption 1.2. If  $(X, v) \in \widehat{\partial\mathcal{D}_p}$ , then there exists a monic affine linear pencil  $L$  of size  $\ell \leq \nu$  and a non-zero vector  $w \in \mathbb{C}^\ell \otimes \mathcal{M}$  such that  $L$  is positive definite  $\mathcal{D}_p$  and  $L(X)w = 0$ . Here*

$$\mathcal{M} = \{q(X)v : q \in \mathcal{P}_d^\delta\}.$$

**Remark 5.3.** In terms of  $\{e_1, \dots, e_\ell\}$ , the standard basis for  $\mathbb{C}^\ell$ , there exists  $m_1, \dots, m_\ell \in \mathcal{M}$  such that  $w = \sum e_\alpha \otimes m_\alpha$ . From the definition of  $\mathcal{M}$ , there thus exists  $q_j \in \mathcal{P}_d^\delta$  such that  $m_\alpha = q_\alpha(X)v$  and hence,

$$w = \sum e_\alpha \otimes q_\alpha(X)v.$$

□

**Remark 5.4.** From the proof of Lemma 5.2 it will follow that  $\ell$  can be chosen at most the dimension of

$$\mathcal{M} = \{q(X)v : q \in \mathcal{P}_d^\delta\}.$$

□

*Proof.* Let  $Y = P_{\mathcal{M}}X|_{\mathcal{M}}$ . By Lemma 4.6, we have  $(Y, v) \in \widehat{\partial\mathcal{D}}_p$ . By Lemma 5.1, there exists an  $\ell$  at most the dimension of  $\mathcal{M}$  and a monic affine linear pencil of size  $\ell$  such that  $L$  is positive definite on  $\mathcal{D}_p$  and  $L(Y)$  is singular. Hence, there is a non-zero  $w \in \mathbb{C}^\ell \otimes \mathcal{M}$  such that  $L(Y)w = 0$ . Hence,

$$\begin{aligned} \langle L(X)w, w \rangle &= \langle (I_\ell \otimes P_{\mathcal{M}}) L(X) (I_\ell \otimes P_{\mathcal{M}})w, w \rangle \\ &= \langle L(Y)w, w \rangle \\ &= 0. \end{aligned}$$

Since also  $L(X) \succeq 0$ , the conclusion  $L(X)w = 0$  follows.  $\square$

In the next subsection we use Lemma 5.2 to obtain one of the key tools we shall need for our proofs.

**5.2. Dominating Points and Separation.** The following proposition relates dominating points to the separating LMIs produced by Lemma 5.2. It is the main result of this subsection.

**Proposition 5.5.** *Suppose  $p$  in  $\mathcal{P}_d^{\delta \times \delta}$  satisfies Assumption 1.2. If  $S \subset \widehat{\partial\mathcal{D}}_p$  is non-empty and respects direct sums, then there exists a monic affine linear pencil  $L$  which is positive definite on  $\mathcal{D}_p$  and singular on  $S \cap S_*$ . Further, the size of  $L$  can be chosen to be at most the maximum of the dimensions of  $\{q(Y)w : q \in \mathcal{P}_d^\delta\}$  over  $(Y, w) \in S$ .*

We begin the proof with a lemma. Given  $\epsilon > 0$ , the nc  $\epsilon$ -neighborhood of 0, denoted  $\mathcal{N}_\epsilon$  is the sequence of sets  $(\mathcal{N}_\epsilon(n))_{n=1}^\infty$  where

$$\mathcal{N}_\epsilon(n) = \{X \in \mathbb{S}_n(\mathbb{R}^g) : \sum \|X_j\| \prec \epsilon\}.$$

**Lemma 5.6.** *If  $p$  satisfies the conditions of Assumption 1.2, then  $\mathcal{D}_p$  contains an  $\epsilon > 0$  neighborhood of 0. Moreover, if  $L$  is a monic affine linear pencil with  $\ell \times \ell$  self-adjoint matrix coefficients  $A_j$  and if  $L$  is positive definite on  $\mathcal{D}_p$ , then  $\|A_j\| \leq \frac{1}{\epsilon}$  for each  $j$ .*

*Proof.* Write  $p$  as in equation (1.6). Thus each  $p_w$  is a  $\delta \times \delta$  matrix. Let  $M$  denote the maximum of  $\{\|p_w\| : 1 \leq |w| \leq d\}$ . Let  $\tau = \sum_1^d g^j$ . Thus  $\tau$  is the number of words  $w$  with  $1 \leq |w| \leq d$ .

Let  $0 < \Delta$  denote the minimum of  $\{|\lambda| : \lambda \text{ is an eigenvalue of } p(0)\}$ . Choose  $\epsilon = \min\{1, \frac{\Delta}{\tau(M+1)}\}$ .

If  $\|X_j\| < \epsilon$  for  $1 \leq j \leq g$ , then  $\|w(tX)\| \leq \frac{\Delta}{\tau(M+1)}$  for non-empty words  $w$  and  $0 \leq t \leq 1$ . Hence,

$$\left\| \sum_{1 \leq |w| \leq d} p_w \otimes w(tX) \right\| \leq \sum_{1 \leq |w| \leq d} \|p_w\| \|w(tX)\| < \Delta.$$

It follows that  $p(tX)$  is invertible for  $0 \leq t \leq 1$  and thus  $X \in \mathcal{D}_p$ . Consequently  $\mathcal{D}_p$ , contains the non-commutative set  $\mathcal{N}_\epsilon$ .

Now suppose  $L$  is a monic affine linear pencil which is positive definite on  $\mathcal{D}_p$  and thus on  $\mathcal{N}_\epsilon$ . For  $0 \leq t < \epsilon$ , the points  $\pm te_j$  are in  $\mathcal{N}_\epsilon$  and hence  $L(\pm te_j) = I \pm tA_j \succeq 0$ . It follows that  $\pm A_j \preceq \frac{1}{\epsilon}I$  and thus  $\|A_j\| \leq \frac{1}{\epsilon}$ .  $\square$

*Proof of Proposition 5.5.* Let  $\mu$  denote the maximum of the dimensions of  $\{q(Y)w : q \in \mathcal{P}_d^\delta\}$  for  $(Y, w) \in S$ .

Given  $(X, v) \in S$ , let  $\Lambda_X$  denote the set of monic affine linear pencils  $L$  of size  $\mu$  which are both positive definite on  $\mathcal{D}_p$  and for which  $L(X)$  is singular. By identifying  $L = I + \sum A_j x_j$  with the tuple  $A = (A_1, \dots, A_g) \in \mathbb{S}_\mu(\mathbb{R}^g)$ , we view  $\Lambda_X$  as a subset of a finite dimensional vector space.

By Lemma 5.2, each  $\Lambda_X$  is non-empty. By Lemma 5.6 each  $\Lambda_X$  is bounded. If a sequence from  $\Lambda_X$  converges to the monic affine linear pencil  $L$ , then  $L(X) \succeq 0$  for all  $X \in \mathcal{D}_p$ . By an application of Lemma 2.2, it follows that  $L$  is in fact positive definite on  $\mathcal{D}_p$ . Hence  $\Lambda_X$  is closed and thus compact.

Given an  $s$  and  $(X^1, v^1), \dots, (X^s, v^s) \in S \subset \widehat{\partial\mathcal{D}}_p$ , let  $(W, u) = \oplus(X^j, v^j)$ . Since  $S$  is closed with respect to direct sums,  $(W, u) \in S$ .

Define

$$\mathcal{N} := \{q(W)u : q \in \mathcal{P}_d^\delta\}.$$

By Lemma 5.2 there is a non-zero monic affine linear pencil  $L = I + \sum A_j x_j$  of size  $\mu$  such that  $L$  is positive definite on  $\mathcal{D}_p$  and a non-zero vector  $\gamma \in \mathbb{C}^\mu \otimes \mathcal{N}$  such that  $L(W)\gamma = 0$ . From the definitions of  $\mathcal{N}$  and  $\mathbb{C}^\mu \otimes \mathcal{N}$ , there exists  $q_\alpha \in \mathcal{P}_d^\delta$  for  $1 \leq \alpha \leq \mu$ , such that

$$\gamma = \sum_{\alpha=1}^{\mu} e_\alpha \otimes q_\alpha(W)u.$$

Let

$$q = \sum_{\alpha=1}^{\mu} e_\alpha \otimes q_\alpha = \begin{pmatrix} q_1 \\ \vdots \\ q_\mu \end{pmatrix}.$$

Thus  $q$  is a  $\mu \times \delta$  matrix of polynomials of degree at most  $d$ ; i.e.,  $q \in \mathcal{P}_d^{\mu \times \delta}$ . Further,

$$\gamma = q(W)u.$$

Up to unitary equivalence (the canonical shuffle),

$$L(W)\gamma = L(W)q(W)u = \begin{pmatrix} L(X^1)q(X^1)v^1 \\ \vdots \\ L(X^s)q(X^s)v^s \end{pmatrix}.$$

Let

$$\gamma_j = q(X^j)v^j = \begin{pmatrix} q_1(X^j)v^j \\ q_2(X^j)v^j \\ \vdots \\ q_\mu(X^j)v^j \end{pmatrix}.$$

Since  $L(W)\gamma = 0$ ,

$$(5.1) \quad L(X^j)\gamma_j = 0$$

for each  $1 \leq j \leq s$ .

To prove that each  $\gamma_j \neq 0$  we now invoke the hypothesis that each  $(X^j, v^j) \in S \cap S_*$ . If  $\gamma_k = 0$  (for some  $k$ ), then  $q_\alpha(X^k)v^k = 0$  for each  $\alpha$ . By Lemma 2.4, for a fixed  $\alpha$ , either  $q_\alpha(X^j)v^j = 0$  for every  $j$  or  $q_\alpha(X^j)v^j \neq 0$  for every  $j$ . Since  $q_\alpha(X^k)v^k = 0$  we thus conclude that  $q_\alpha(X^j)v^j = 0$  for every  $j$  and every  $\alpha$ . Thus each  $\gamma_j = 0$  and hence  $\gamma = 0$ , a contradiction.

Since, for each  $j$ , we have  $\gamma_j \neq 0$ , but  $L(X^j)\gamma_j = 0$ , it follows that  $L \in \Lambda_{X^j}$ . This proves

$$\bigcap_{j=1}^s \Lambda_{X^j} \neq \emptyset.$$

Consequently, the collection of compact sets  $\{\Lambda_X : (X, v) \in S \cap S_*\}$  has the finite intersection property. Hence the full intersection is non-empty and any  $L$  in this intersection is positive definite on  $\mathcal{D}_p$  and singular on all of  $S \cap S_*$  (meaning, if  $(X, v) \in S \cap S_*$ , then  $L(X)$  is singular).  $\square$

**Corollary 5.7.** *Suppose  $p$  in  $\mathcal{P}_d^{\delta \times \delta}$  satisfies Assumption 1.2. The set  $(\partial\mathcal{D}_p)_*$  is non-empty and there is a monic affine linear pencil  $L$  which is positive definite on  $\mathcal{D}_p$  and singular on  $(\partial\mathcal{D}_p)_*$ .*

*Proof.* Apply Proposition 5.5 to the set  $\partial\mathcal{D}_p$  and note  $\partial\mathcal{D}_p \cap (\partial\mathcal{D}_p)_* = (\partial\mathcal{D}_p)_*$ .  $\square$

## 6. PROOF OF THE MAIN THEOREM

Theorem 1.3 follows quickly from

**Theorem 6.1.** *Given a symmetric non-commutative  $p$  satisfying Assumption 1.2, there exists a monic affine linear pencil  $L$  such that  $L$  is positive definite on  $\mathcal{D}_p$  and  $L(X)$  has a kernel for every  $X \in \partial\mathcal{D}_p$ . Hence,  $\mathcal{D}_p = \mathcal{D}_L = \{X : L(X) \succ 0\}$  and thus  $\mathcal{D}_p$  has an LMI representation.*

*Proof.* Recall

$$i(L) := \{(Y, w) \in \widehat{\partial\mathcal{D}_p} : L(Y) \text{ is invertible}\}.$$

We argue by contradiction. Accordingly, suppose for each monic affine linear pencil  $L$  which is positive definite on  $\mathcal{D}_p$  the set  $i(L)$  is non-empty.

Let  $\mathfrak{S}$  denote pairs  $(S, L)$  with  $S$  a  $\mathcal{P}_d^\delta$ -closed set and  $L$  a monic affine linear pencil satisfying:

- (i)  $L$  is positive definite on  $\mathcal{D}_p$ ;
- (ii)  $L$  is singular on  $S_*$ ; and
- (iii)  $i(L) \subset S$ .

The assumption in the previous paragraph which we wish to contradict implies if  $(S, L) \in \mathfrak{S}$ , then  $S$  is non-empty.

Note that  $\mathfrak{S}$  itself is not empty since, by Corollary 5.7, there is an  $L$  such that  $(\mathcal{D}_p, L) \in \mathfrak{S}$ . Let  $\mathfrak{S}_1$  denote the collection of sets  $S$  occurring in the pairs  $(S, L)$  belonging to  $\mathfrak{S}$ . Choose a minimal (with respect to set inclusion) set  $S$  in  $\mathfrak{S}_1$  using Lemma 3.2 part (7). We will show that  $S$  is not minimal, a contradiction which will complete the proof.



Since  $S \in \mathfrak{S}_1$ , there exists an  $L$  satisfying the conditions (i)(ii)(iii) with respect to this  $S$ ; that is,  $(S, L) \in \mathfrak{S}$ .

By assumption,  $i(L) \neq \emptyset$ . By Proposition 3.5,  $i(L)_z \subsetneq S_z$ . Since also  $S$  is  $\mathcal{P}_d^\delta$  closed ( $S = S_z$ ), we have

$$(6.1) \quad i(L)_z \subsetneq S.$$

Using the fact that  $i(L)$  is non-empty and respects direct sums, Proposition 5.5, produces a monic affine linear pencil  $M$  which is positive definite on  $\mathcal{D}_p$  and singular on  $i(L) \cap i(L)_*$ . The proof now proceeds by showing  $(i(L)_z, L \oplus M) \in \mathfrak{S}$ , which, by the strict inclusion in equation (6.1), contradicts the minimality of  $S$ .

From the construction,  $L \oplus M$  is positive definite on  $\mathcal{D}_p$ ; that is,  $L \oplus M$  satisfies condition (i).

By Lemma 2.3 the set  $i(L)_*$  is not empty. Suppose now that  $(X, v) \in (i(L)_z)_* = i(L)_*$ . If  $(X, v) \in i(L)$ , then  $M(X)$ , and hence  $(L \oplus M)(X)$  is singular. On the other hand, if  $(X, v) \notin i(L)$ , then  $L(X)$ , and hence  $(L \oplus M)(X)$  is singular. Thus, if  $(X, v) \in (i(L)_z)_*$ , then  $(L \oplus M)(X)$  is singular. Hence  $L \oplus M$  satisfies condition (ii) with respect to  $i(L)_z$ .

Finally,  $i(L \oplus M) \subset i(L) \subset i(L)_z$  and thus  $(i(L)_z, L \oplus M)$  satisfies condition (iii) with respect to  $i(L)_z$ . Hence  $(i(L)_z, L \oplus M) \in \mathfrak{S}$  and the proof is complete.  $\square$

## 7. THE CASE OF SIGNATURE $(\delta, 0)$

When  $p(0)$  is positive definite (wlog we can normalize to take  $p(0) = I_\delta$ ), it is possible to refine the estimates on the size of  $L$  occurring in Lemma 5.2. In the following section this refined estimate is used to prove Theorem 1.5.

Recall that  $[\frac{d}{2}]_+$  denotes the largest integer less than or equal to  $\frac{d}{2}$ . Let  $\check{\nu} = \delta \sum_{j=0}^{[\frac{d}{2}]_+} g^j$ . Notice that  $\check{\nu}$  is the dimension of the vector space  $\mathcal{P}_{[\frac{d}{2}]_+}^\delta$  and, given  $(X, v) \in \widehat{\partial\mathcal{D}}_p$ , it is thus an upper bound for the dimension of

$$\check{M} = \{q(X)v : q \in \mathcal{P}_{[\frac{d}{2}]_+}^\delta\}.$$

Compare the following lemma about  $\check{M}$  to Lemma 4.6 about  $\mathcal{M}$ .

**Lemma 7.1.** *Suppose  $p \in \mathcal{P}_d^{\delta \times \delta}$  satisfies the conditions of Assumption 1.2 and moreover that  $p(0) = I_\delta$ . If  $(X, v) \in \widehat{\partial\mathcal{D}}_p$ , then  $(P_{\check{M}} X|_{\check{M}}, v) \in \widehat{\partial\mathcal{D}}_p$ ; indeed,  $tP_{\check{M}} X|_{\check{M}} \in \mathcal{D}_p$  for  $0 \leq t < 1$  and  $p(P_{\check{M}} X|_{\check{M}})v = 0$ .*

*Proof.* Just as in Lemma 4.6, for  $0 \leq t < 1$ , we have  $tP_{\check{M}} X|_{\check{M}} \in \mathcal{D}_p$ . Since  $p(0) = I_\delta$ , it follows that  $p(tP_{\check{M}} X|_{\check{M}}) \succ 0$  and hence  $p(P_{\check{M}} X|_{\check{M}}) \succeq 0$ .

On the other hand, for any word  $w$  of length at most  $d$ , we can write  $w = w_1 x_j w_2$  where both words  $w_1$  and  $w_2$  have length at most  $[\frac{d}{2}]_+$ . Write  $v \in \mathbb{R}^n \otimes \mathbb{R}^\delta$  as  $v = \sum_{\alpha=1}^\delta e_\alpha \otimes v_\alpha$ . Since both  $w_2(X)v_\alpha$  and  $w_1^T(X)v_\beta$  are

in  $\check{M}$  we find

$$\begin{aligned} \langle w(P_{\check{M}}X|_{\check{M}})v_\alpha, v_\beta \rangle &= \langle P_{\check{M}}X_j w_2(X)v_\alpha, w_1(X)^T v_\beta \rangle \\ &= \langle X_j w_2(X)v_\alpha, w_1^T(X)v_\beta \rangle \\ &= \langle w(X)v_\alpha, v_\beta \rangle. \end{aligned}$$

Consequently,

$$\langle p(P_{\check{M}}X|_{\check{M}})v, v \rangle = \langle p(X)v, v \rangle = 0.$$

Since also  $p(P_{\check{M}}X|_{\check{M}}) \succeq 0$ , it follows that  $p(P_{\check{M}}X|_{\check{M}})v = 0$ .  $\square$

An application of Lemma 7.1 produces the following improvement on Lemma 5.2.

**Proposition 7.2.** *Suppose for  $p$  in  $\mathcal{P}_d^{\delta \times \delta}$  the set  $\mathcal{D}_p$  is bounded and convex and  $p(0) = I_\delta$ . If  $(X, v) \in \widehat{\partial \mathcal{D}}_p$ , then there exists a monic affine linear pencil  $L$  of size  $\ell \leq \check{\nu}$  and a non-zero vector  $w \in \mathbb{C}^\ell \otimes \check{M}$  such that  $L$  is positive definite on  $\mathcal{D}_p$  and  $L(X)w = 0$ . Here*

$$\check{M} = \{q(X)v : q \in \mathcal{P}_{[\frac{d}{2}]_+}^\delta\}.$$

## 8. THE CASE OF IRREDUCIBLE $p$

In this section we show, under the conditions of Assumption 1.2 plus  $p(0) = I_\delta$ , if  $p$  is, in an appropriate sense, irreducible, then it has degree at most two. Then we prove Theorem 1.5 from the introduction.

**8.1. A polynomial which vanishes on  $\widehat{\partial \mathcal{D}}_p$ .** The main result of this subsection is Theorem 8.3 below. We begin with a lemma.

**Lemma 8.1.** *Suppose  $p \in \mathcal{P}_d^{\delta \times \delta}$  satisfies the conditions of Assumption 1.2. Suppose further that  $p(0) = I_\delta$ . If*

- (i)  $(X, v) \in \widehat{\partial \mathcal{D}}_p$ , (with  $v \neq 0$ );
- (ii)  $L$  is a monic affine linear pencil of size  $\ell$  which is positive definite on  $\mathcal{D}_p$ ; and
- (iii) there is a vector  $0 \neq w \in \mathbb{C}^\ell \otimes \check{M}$ , where

$$\check{M} = \{q(X)v : q \in \mathcal{P}_{[\frac{d}{2}]_+}^\delta\},$$

such that  $L(X)w = 0$ ,

then there exists a non-zero  $q \in \mathcal{P}_{[\frac{d}{2}]_+ + 1}^\delta$  such that  $q(X)v = 0$ . (Note: it is not assumed that  $L$  is the “master LMI” from Theorem 6.1.)

*Proof.* Write the monic affine linear pencil  $L$  as

$$L = I + \sum A_j x_j,$$

where the  $A_j$  are  $\ell \times \ell$  symmetric matrices. The tuple  $X$  acts on  $\mathbb{C}^n$  for some  $n$ . Hence  $A_j \otimes X$  acts upon  $\mathbb{C}^\ell \otimes \mathbb{C}^n$ . With respect to this tensor product decomposition,  $w = \sum e_j \otimes h_j$  where  $\{e_1, \dots, e_\ell\}$  is the standard

orthonormal basis for  $\mathbb{C}^\ell$  and  $h_j \in \check{M}$ . From the definition of  $\check{M}$ , there exists polynomials  $r_j \in \mathcal{P}_{[\frac{d}{2}]_+}^\delta$  such that  $h_j = r_j(X)v$ .

Since  $L(X)w = 0$ , for each  $m$  we have  $0 = [e_m^T \otimes I]L(X)w$ . Thus,

$$\begin{aligned} 0 &= [e_m^T \otimes I][w + \sum_k \sum_j A_k e_j \otimes X_k r_j(X)v] \\ &= [r_m + \sum_{k,j} (e_m^T A_k e_j) x_k r_j](X)v. \end{aligned}$$

Now we argue, by contradiction, that the elements  $q_m$  of  $\mathcal{P}_{[\frac{d}{2}]_++1}^\delta$  given by

$$q_m(x) = r_m(x) + \sum_{k,j} (e_m^T A_k e_j) x_k r_j(x)$$

are not all 0. If they were all 0, then each  $r_m$  satisfies  $r_m(0) = 0$ ; i.e.,  $r_m$  has no constant term. But, then, by the same reasoning, each  $r_m$  has no linear terms and continuing along these lines we ultimately conclude that all the  $r_m$  are 0. On the other hand, since  $w \neq 0$ , there is an  $m$  such that  $h_m = r_m(X)v \neq 0$ ; a contradiction. Thus we conclude there is an  $m$  such that  $q_m \neq 0$  and at the same time  $q_m(X)v = 0$ . To complete the proof, observe that the degree of this  $q_m$  is at most  $[\frac{d}{2}]_+ + 1$ .  $\square$

**Remark 8.2.** Let  $R \in \mathcal{P}^{\delta \times \delta}$  denote the matrix-valued nc polynomial whose  $m$ -th row is the  $r_m$  produced in the proof of Lemma 8.1. The lemma says that  $R$  is not zero. On the other hand,  $R(X)v = w$  and  $L(X)R(X)v = L(X)w = 0$ . Hence the symmetric polynomial  $R^T L R$  is non-zero, but vanishes at  $(X, v)$ .  $\square$

**Theorem 8.3.** *If polynomial  $p \in \mathcal{P}_d^{\delta \times \delta}$  satisfies Assumption 1.2 and if also  $p(0) = I_\delta$ , then there exists a non-zero  $q \in \mathcal{P}_{[\frac{d}{2}]_++1}^\delta$  such that  $q(X)v = 0$  for every  $(X, v) \in \widehat{\partial \mathcal{D}}_p$ .*

*In particular, if  $\mathcal{D}_p$  is bounded and convex and  $p(0) = I_\delta$  and if  $p$  is a minimum degree defining polynomial for  $\mathcal{D}_p$ , then the degree of  $p$  is at most two.*

*Proof.* Given  $(X, v) \in \widehat{\partial \mathcal{D}}_p$ , let

$$C_{(X,v)} = \{q \in \mathcal{P}_{[\frac{d}{2}]_++1}^\delta : q(X)v = 0\}.$$

Note that  $C_{(X,v)}$  is a subspace of  $\mathcal{P}_{[\frac{d}{2}]_++1}^\delta$ .

Let  $\check{M} = \{r(X)v : r \in \mathcal{P}_{[\frac{d}{2}]_+}^\delta\}$ . By Proposition 7.2 there is a monic affine linear pencil  $L$  of some size  $\ell \leq \check{\nu}$  ( $\check{\nu}$  is defined at the outset of Section 7) such that  $L$  is positive definite on  $\mathcal{D}_p$  and a non-zero vector  $w \in \mathbb{C}^\ell \otimes \check{M}$  such that  $L(X)w = 0$ . Thus Lemma 8.1 applies to produce a non-zero  $q \in \mathcal{P}_{[\frac{d}{2}]_++1}^\delta$  such that  $q(X)v = 0$ . Hence  $C_{(X,v)}$  is non-trivial (not  $(0)$ ).

Given  $(X^1, v^1), (X^2, v^2), \dots, (X^s, v^s) \in \widehat{\partial\mathcal{D}}_p$ , let  $(W, u) = \oplus(X^j, v^j)$ . Then  $(W, u) \in \widehat{\partial\mathcal{D}}_p$  also and thus, by what has already been proved, there exists a non-zero  $q \in \mathcal{P}_{[\frac{d}{2}]_++1}^\delta$  such that  $q(W)u = 0$ . But then  $q(X^j)v^j = 0$  for each  $j$ . Hence  $q \in \bigcap_{j=1}^\ell C_{(X^j, v^j)}$ . It follows that the collection of subspaces  $C_{(X, v)}$  is closed with respect to finite intersections. Since also each  $C_{(X, v)}$  is a non-trivial subspace of the finite dimensional space  $\mathcal{P}_{[\frac{d}{2}]_++1}^\delta$ , there is a smallest (and non-trivial) subspace  $C_{(Y, w)}$  uniquely determined by the condition that it has minimum dimension. Note that any (non-zero)  $q \in C_{(Y, w)}$  must vanish on all of  $\widehat{\partial\mathcal{D}}_p$ , since if  $(X, v) \in \widehat{\partial\mathcal{D}}_p$  and  $q(X)v \neq 0$ , then  $C_{(X, v)} \cap C_{(Y, w)} \subsetneq C_{(Y, w)}$ .

The second part of the Theorem follows immediately from the first part and the definition of minimum degree defining polynomial.  $\square$

*Proof of Theorem 1.5.* The first part of Theorem 1.5 is covered by Theorem 8.3. It remains to prove if  $p$  is a symmetric nc polynomial in  $\mathcal{P}_2^{1 \times 1}$ , if  $p(0) = 1$  and if  $\mathcal{D}_p$  is both bounded and convex, then  $p$  has the form

$$p = 1 + \ell(x) - \sum_{j=1}^g \lambda_j(x)^2,$$

where  $\ell$  and each  $\lambda_j$  are linear.

Since  $p$  has degree two and is symmetric, there is a uniquely determined symmetric  $g \times g$  matrix  $\Lambda$  such that

$$p(x) = 1 + \ell(x) - \langle \Lambda x, x \rangle,$$

where  $x$  is the vector with entries  $x_j$ . If  $\Lambda$  is not positive semi-definite, then there is a  $t \in \mathbb{R}^g$  such that  $\langle \Lambda t, t \rangle < 0$  and hence, for  $s \in \mathbb{R}$ ,

$$p(st) = 1 + s\ell(t) - s^2\langle \Lambda t, t \rangle$$

is either positive for all  $s \geq 0$  or is positive for all  $s \leq 0$  depending upon the sign of  $\ell(t)$ . In either case,  $\mathcal{D}_p(1)$  is not bounded. Hence we conclude that  $\Lambda$  is positive semi-definite. Hence there is an  $0 \leq m \leq g$  and an orthogonal set of vectors  $u_1, \dots, u_g$  such that

$$\Lambda = \sum_{\ell=1}^m u_\ell u_\ell^T.$$

Letting  $\lambda_\ell = \sum_j (u_\ell)_j x_j$ ,

$$\hat{L} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}$$

and  $L_0 = 1 + \ell$  the conclusion of Theorem 1.5 follows.  $\square$

**Remark 8.4.** A variation on the proof of Theorem 8.3 shows: given an  $L$  such that  $L$  is positive definite on  $\mathcal{D}_p$  and singular on  $\partial\mathcal{D}_p$ , there exists a non-zero  $R \in \mathcal{P}^{\ell \times \delta}$  of degree at most  $\lfloor \frac{d}{2} \rfloor_+$  such that  $\hat{p} = R^T L R$  vanishes on  $\widehat{\partial\mathcal{D}_p}$ . The degree of  $\hat{p}$  is  $2\lfloor \frac{d}{2} \rfloor_+ + 1$ , which is either  $d$  or  $d + 1$  depending upon  $d$  odd or even. In particular, the degree of  $\hat{p}$  is close to that of  $p$ .

To prove this variation of Theorem 8.3, observe that for each  $(X, v) \in \widehat{\partial\mathcal{D}_p}$  the vector space

$$C'_{(X,v)} = \{R \in \mathcal{P}^{\ell \times \delta} : R^T L R(X)v = 0\}$$

is non-trivial (not the 0 subspace) by Remark 8.2. Thus, arguing as in the proof of Theorem 8.3, the intersection of all such subspaces is non-trivial and the conclusion follows.  $\square$

## 9. A REFINEMENT OF THE EFFROS-WINKLER SEPARATION THEOREM

This section contains a proof of the separation Theorem of Effros and Winkler [EW97] in the special case of certain matrix convex subsets of  $\mathbb{S}(\mathbb{R}^g) = (\mathbb{S}_n(\mathbb{R}^g))_{n=1}^\infty$ . The specialization makes the proof of Proposition 9.1 immediately below decidedly simpler than that of the strictly more general version in [EW97]. On the other hand Proposition 9.1 is not explicitly covered by the results in [EW97]. Thus we have included a proof.

**Proposition 9.1.** *Let  $\mathcal{C} = (\mathcal{C}(n))_{n=1}^\infty$  denote a sequence of sets where  $\mathcal{C}(n) \subset \mathbb{S}_n(\mathbb{R}^g)$ . If  $\mathcal{C}$  is an open matrix convex (see Section 1.9.2 for the definition) and if  $X^b \in \partial\mathcal{C}(n)$ , the boundary of  $\mathcal{C}(n)$ , then there exists a monic affine linear pencil  $L$  of size  $n$  such that  $L$  is positive definite on  $\mathcal{C}$ , but  $L(X^b)$  is singular.*

*Proof.* By the conventional Hahn-Banach Separating Hyper-plane Theorem for  $\mathbb{R}^\ell$  (see [Ru73] Chapter 3, exercise 1 for instance) and item (iv) in the definition of open matrix convex set, there exists a linear functional  $\lambda : \mathbb{S}_n(\mathbb{R}^g) \rightarrow \mathbb{R}$  and a  $\rho \in \mathbb{R}$  such that  $\lambda(X) \leq \rho = \lambda(X^b)$  for  $X \in \mathcal{C}(n)$ . Since  $\mathcal{C}(n)$  contains a neighborhood of 0, we conclude  $\rho > 0$  and thus may assume  $\rho = 1$ .

Let  $E_{\alpha,\beta}$  denote the elementary matrices for  $M_n(\mathbb{R})$ . Hence  $e_j \otimes \frac{1}{2}(E_{\alpha,\beta} + E_{\beta,\alpha})$  is a basis for  $\mathbb{S}_n(\mathbb{R}^g)$ . (Here  $e_j$  is the usual basis for  $\mathbb{R}^g$  and  $1 \leq j \leq g$ ,  $1 \leq \alpha, \beta \leq n$ .)

Define  $n \times n$  matrices  $A_j$  by

$$(A_j)_{\alpha,\beta} = g \lambda( e_j \otimes \frac{1}{2}(E_{\alpha,\beta} + E_{\beta,\alpha}) ).$$

Note  $A = (A_1, \dots, A_g) \in \mathbb{S}_n(\mathbb{R}^g)$ . Let  $\mathcal{L} = \sum_{j=1}^g A_j x_j$  and set  $L = I - \mathcal{L}$ .

$$\begin{aligned}
\text{With } \mathbf{e} &= \sum e_\alpha \otimes e_\alpha, \\
\frac{1}{g} \langle \mathbf{e}, \mathbf{e} \rangle &= 1 \\
&= \lambda(X^{\mathbf{b}}) \\
&= \sum_{\alpha, \beta, j} \lambda(e_j \otimes \frac{1}{2}(E_{\alpha, \beta} + E_{\beta, \alpha}))(X_j^{\mathbf{b}})_{\alpha, \beta} \\
&= \frac{1}{g} \sum_{\alpha, \beta, j} (A_j)_{\alpha, \beta} (X_j^{\mathbf{b}})_{\alpha, \beta} \\
&= \frac{1}{g} \langle \mathcal{L}(X^{\mathbf{b}}) \mathbf{e}, \mathbf{e} \rangle.
\end{aligned}$$

Hence

$$(9.1) \quad \langle L(X^{\mathbf{b}}) \mathbf{e}, \mathbf{e} \rangle = \langle (I - \mathcal{L}(X^{\mathbf{b}})) \mathbf{e}, \mathbf{e} \rangle = 0.$$

Next, we show that  $L(X) = I - \mathcal{L}(X) \succeq 0$  on  $\mathcal{C}(n)$ . Given  $f_1, \dots, f_n \in \mathbb{R}^n$ , let

$$F = (f_1 \quad f_2 \quad \dots \quad f_n) \in M_n.$$

Note that

$$\|F^* F\| \leq \text{tr}(F^* F) = \sum_j f_j^T f_j.$$

Thus, if we assume  $\sum_j f_j^T f_j = 1$ , then  $F$  is a contraction. Hence  $F^T X F \in \mathcal{C}(n)$  if  $X \in \mathcal{C}(n)$  by item (iii) in the definition of open matrix convex set. Consequently,

$$\begin{aligned}
1 &\geq \lambda(F^T X F) \\
&= \sum_{\alpha, \beta, j} (A_j)_{\alpha, \beta} (F^T X_j F)_{\alpha, \beta} \\
&= \sum_{\alpha, \beta, j} (A_j)_{\alpha, \beta} \langle X f_\alpha, f_\beta \rangle \\
&= \langle \mathcal{L}(X) u, u \rangle.
\end{aligned}$$

with  $u$  defined by  $u = \sum_{\alpha=1}^n e_\alpha \otimes f_\alpha$ . Since  $1 = \langle u, u \rangle$ , we get  $\langle [I - \mathcal{L}(X)] u, u \rangle \geq 0$ . The  $f_j$  were arbitrary (except for the condition  $\sum f_j^T f_j = 1$  which can be achieved by scaling), so this applies to any vector  $u \in \mathbb{R}^{n^2}$ , thereby giving  $L(X) \succeq 0$  for  $X \in \mathcal{C}(n)$ .

Returning to  $X^{\mathbf{b}}$ , since  $L(X^{\mathbf{b}}) \succeq 0$ , equation (9.1) now implies that  $L(X^{\mathbf{b}}) \mathbf{e} = 0$  and thus  $L(X^{\mathbf{b}})$  is singular.

Finally, we show, for any positive integer  $m$ , that  $L$  is positive semi-definite on  $\mathcal{C}(m)$ . If  $m < n$  and  $X \in \mathcal{C}(m)$ , then  $X \oplus 0 \in \mathcal{C}(n)$  by item (ii), where  $0$  is the  $g$ -tuple of zeros of size  $(n - m) \times (n - m)$ . Hence

$$0 \preceq L(X \oplus 0) = L(X) \oplus I$$

and therefore  $L(X) \succeq 0$ .

Now suppose  $m > n$  and let  $u \in \mathbb{C}^n \otimes \mathbb{C}^m$  be given. Thus  $u = \sum_{\alpha=1}^n e_\alpha \otimes f_\alpha$  for some vectors  $f_\alpha \in \mathbb{C}^m$ . Let  $\mathcal{M}$  denote the span of  $\{f_1, \dots, f_n\}$ . Then  $P_{\mathcal{M}}X|_{\mathcal{M}} \in \mathcal{C}(m')$  by item (iii), where  $m' \leq n$  is the dimension of  $\mathcal{M}$ . Hence,

$$\begin{aligned} 0 &\leq \langle L(P_{\mathcal{M}}X|_{\mathcal{M}})u, u \rangle \\ &= \langle L(X)u, u \rangle. \end{aligned}$$

Since  $u$  was arbitrary,  $L(X) \succeq 0$  and the proof is nearly complete. An application of Lemma 2.2 gives  $L(X) \succ 0$  for all  $X$  in  $\mathcal{D}_p$ .  $\square$

Note, if  $p$  satisfies the conditions of Assumption 1.2, then, by Theorem 4.5 and the remarks preceding,  $\mathcal{D}_p$  satisfies the hypotheses of Proposition 9.1. Lemma 5.1 thus follows from Proposition 9.1.

## 10. A FINAL REMARK AND EXAMPLE

This section contains a final example and a remark about the proof of Theorem 1.3

**10.1. A Not Irreducible Defining Polynomial.** The following example shows that Theorem 1.5 requires the irreducibility hypothesis. Here we work with two variables  $(x, y)$ . Let  $b(x, y) = 1 - x^2 - y^2$  and  $f(x, y) = 1 - (x - \frac{1}{4})^2 - y^2$ . The set

$$\mathcal{D} = \mathcal{D}_{b \oplus f} = \{(X, Y) : b(X, Y) \succ 0, f(X, Y) \succ 0\}$$

is convex. Let  $p_1 = fbf$  and  $p_2 = bfb$ . Then  $\mathcal{D}_{p_1} = \mathcal{D} = \mathcal{D}_{p_2}$ . Hence, neither  $p_1$  nor  $p_2$  is a minimum degree defining polynomial for  $\mathcal{D}$ .

**10.2. Convexity and Semi-algebraic Sets.** The next discussion is intended to highlight the additional structure afforded by semi-algebraic sets over general matrix convex sets as in [EW97] i.e. sets satisfying the hypotheses of Proposition 9.1. We also add the requirement of finite type in the sense of item (v) below.

- (v) there exists a positive integer  $\nu$  such that  $X \in \mathcal{C}$  if and only if  $P_{\mathcal{M}}X|_{\mathcal{M}} \in \mathcal{C}$  for every subspace  $\mathcal{M}$  of dimension at most  $\nu$ .

In this case it does follow that  $X \in \partial\mathcal{C}$  if and only if there exists a subspace  $\mathcal{M}$  of dimension at most  $\nu$  such that  $P_{\mathcal{M}}X|_{\mathcal{M}} \in \partial\mathcal{C}$ . However, one does not have the fine control, afforded by a vector  $v$  with  $p(X)v = 0$ , over the choice of  $\mathcal{M}$  needed to carry out the argument found in Proposition 5.5.

Of course, what is true is that there is a family  $\mathcal{L}$  of monic affine linear pencils of size (at most)  $\nu$  such that

$$\mathcal{C} = \{X : L(X) \succ 0 \text{ for all } L \in \mathcal{L}\}.$$

However, the family  $\mathcal{L}$  can not generally be chosen finite.

## REFERENCES

- [Ar69] Arveson, William, *On subalgebras of  $C^*$ -algebras*, Bull. Amer. Math. Soc. **75** (1969), 790–794.
- [Ar72] Arveson, William, *Subalgebras of  $C^*$ -algebras II*, Acta Math. **128** (1972), no. 3-4, 271–308.
- [Ar08] Arveson, William, *The noncommutative Choquet boundary*, J. Amer. Math. Soc. **21** (2008), no. 4, 1065–1084.
- [BCR98] By Bochnak, J.; Coste, M.; Roy, M.-F., *Real algebraic geometry*, Springer, 1998 (ISBN 3540646639, 9783540646631).
- [BL04] Blecher, David; Le Merdy, Christian, *Operator Algebras and Their Modules An operator space approach*, David P. Blecher and Christian Le Merdy, Clarendon Press, London Mathematical Society Monographs 30, 2004 (978-0-19-852659-9).
- [DHM07] Dym, Harry; Helton, William; McCullough, Scott, *Irreducible noncommutative defining polynomials for convex sets have degree four or less*, Indiana Univ. Math. J. **56** (2007), no. 3, 1189–1231.
- [EW97] Effros, Edward G.; Winkler, Soren, *Matrix convexity: operator analogues of the bipolar and Hahn-Banach theorems*, J. Funct. Anal. **144** (1997), no. 1, 117–152.
- [GNLC95] P. Gahinet, A. Nemirovskii, A.J. Laub, M. Chilali, *LMI Control Toolbox*, The Math Works, Inc., USA, 1995.
- [HHLM08] Hay, Damon M.; Helton, J. William; Lim, Adrian; McCullough, Scott Non-commutative partial matrix convexity. Indiana Univ. Math. J. **57** (2008), no. 6, 2815–2842.
- [HV07] Helton, J. William; Vinnikov, Victor, *Linear matrix inequality representation of sets*, Comm. Pure Appl. Math. **60** (2007), no. 5, 654–674.
- [LPR05] Lewis, A. S.; Parrilo, P. A.; Ramana, M. V. The Lax conjecture is true. Proc. Amer. Math. Soc. **133** (2005), no. 9, 2495–2499 (electronic).
- [Mi03] Mittelmann, H. D., *An independent benchmarking of SDP and SOCP solvers*, Journal Mathematical Programming **95** (2003), no. 2, 407–430.
- [NN94] A. Nemirovskii and Y. Nesterov, Interior-point polynomial algorithms in convex programming. SIAM Studies in Applied Mathematics, 13, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.
- [Ne06] A. Nemirovskii, Advances in convex optimization: conic programming, *Plenary Lecture, International Congress of Mathematicians (ICM)*, Madrid, Spain, 2006.
- [P00] Parrilo, Pablo, *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*, Cal Tech Thesis, 2000, <http://thesis.library.caltech.edu/1647/> or <http://www.mit.edu/~parrilo/pubs/index.html#thesis>.
- [Pa02] Paulsen, Vern I., *Completely Bounded Maps and Operator Algebras*, Cambridge University Press, 2002 (978-0521816694)
- [Pi] Pisier, Gilles, *Introduction to operator space theory*, London Mathematical Society Lecture Note Series, 294. Cambridge University Press, Cambridge, 2003.
- [Ru73] Rudin, Walter, *Functional Analysis*, McGraw Hill, 1973 (ISBN 0-07-054225-2).
- [SIG97] Skelton, R. E., Iwasaki, T., Grigoriadas, K. M., *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis, San Francisco, 1997.
- [St99] J. F. Sturm, *Using SeDuMi 1.02, a MATLAB Toolbox for Optimization over Symmetric Cones*, j-OMS, 1999, 11/12, 1-4, 625–653.
- [Vo04] Voiculescu, Dan, *Free analysis questions. I. Duality transform for the coalgebra of  $\partial_X: B$* , Int. Math. Res. Not. 2004, no. 16, 793–822.
- [Vo05] Voiculescu, Dan, *Aspects of free probability*, (English summary) XIVth International Congress on Mathematical Physics, 145–157, World Sci. Publ., Hackensack, NJ, 2005.



- [WW99] Webster, Corran; Winkler, Soren, *The Krein-Milman theorem in operator convexity*, Trans. Amer. Math. Soc. **351** (1999), no. 1, 307–322.
- [WSV00] H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors. *Handbook of semidefinite programming*, Kluwer's Publisher, 2000.

## NOT FOR PUBLICATION

## CONTENTS

|   |    |
|---|----|
| 1. Introduction   | 2  |
| 1.1. Non-commutative polynomials                                      | 3  |
| 1.2. Substituting Matrices for Indeterminates                         | 3  |
| 1.3. Matrix-Valued Polynomials  | 4  |
| 1.4. NC Basic Open Semi-Algebraic Sets                                | 5  |
| 1.5. Convex Semi-Algebraic Sets                                       | 5  |
| 1.6. Linear Matrix Inequalities                                       | 5  |
| 1.7. Main Result  | 6  |
| 1.8. Further Results  | 7  |
| 1.9. NC Open Semi-algebraic Sets and Convex Examples                  | 8  |
| 1.10. Motivation  | 12 |
| 1.11. Layout  | 13 |
| 2. Facts about $\mathcal{D}_p$ and its Boundary                       | 14 |
| 2.1. Life on the boundary   | 14 |
| 2.2. Dominating Points  | 15 |
| 3. Closure with Respect to a Subspace of Polynomials                  | 16 |
| 4. Convex Basic Non-Commutative Semi-Algebraic Sets                   | 19 |
| 4.1. Matrix Convexity   | 19 |
| 4.2. Compressions   | 20 |
| 5. Separating Linear Pencils  | 21 |
| 5.1. The Effros-Winkler Separation Theorem                            | 21 |
| 5.2. Dominating Points and Separation                                 | 22 |
| 6. Proof of the Main Theorem  | 24 |
| 7. The Case of Signature $(\delta, 0)$                                | 25 |
| 8. The Case of Irreducible $p$  | 26 |
| 8.1. A polynomial which vanishes on $\widehat{\partial\mathcal{D}_p}$ | 26 |
| 9. A Refinement of the Effros-Winkler Separation Theorem              | 29 |
| 10. A Final Remark and Example  | 31 |
| 10.1. A Not Irreducible Defining Polynomial                           | 31 |
| 10.2. Convexity and Semi-algebraic Sets                               | 31 |
| References  | 32 |
| Index   | 35 |

## INDEX

- $S_*$ , 15
- $\lfloor \frac{d}{2} \rfloor_+ = \frac{d-1}{2}$  if  $d$  odd, 7
- $\lfloor \frac{d}{2} \rfloor_+ = \frac{d}{2}$  if  $d$  even, 7
- $\partial \mathcal{D}_p$ , 7, 14
- $\mathcal{D}_p$ , 5
- $\mathcal{I}(S)$ , 15
- $\mathcal{M}$ , 20
- $\mathcal{P}$ , 3
- $\mathcal{P}^{\delta \times \delta'}$ , 4
- $\mathcal{P}_m^{\delta \times \delta'}$ , 4
- $\widehat{\partial \mathcal{D}_p}$ , 7, 14
- $\nu$ , 20
- $\oplus (X^j, v^j)$ , 15
- $\mathfrak{J}_p(n)$ , 5
- $\sigma$ , signature, 5
- $\mathbb{S}_n(\mathbb{R}^g)$ , 3
- $i(L)$ , 18
- $p^T$ , 3
- $p_1 \oplus p_2$ , 4
  
- Assumption 1.2, 5
  
- boundary, 7
- bounded, 5
  
- closed,  $\mathcal{P}_d^\delta$ , 16
- closure, 16
- convex, 5
- convex nc basic open nc semi-algebraic set, 5
  
- direct sum, 4, 15
- dominating point, 15
  
- finite type, 31
- free variables, 8
  
- invertibility set, 5
  
- Linear Matrix Inequality, 6
- linear pencil, 5
- LMI, 6
- LMI representation, 6
  
- matrix convex, 10
- minimum degree defining polynomial, 7
- minimum degree irreducible, 7
- monic affine linear pencil, 6
  
- nc polynomial, 3
- non-commutative polynomial, 3
- non-commutative set, 6
  
- positivity set, 5
- respects direct sums, 8, 15
- semi-algebraic set, basic open nc, 5
- symmetric polynomial, 3
- symmetric variables, 3, 8
  
- vanishes, 15

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO 92093  
*E-mail address:* [helton@math.ucsd.edu](mailto:helton@math.ucsd.edu)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA  
*E-mail address:* [sam@math.ufl.edu](mailto:sam@math.ufl.edu)