COMPLETE POSITIVITY IN OPERATOR ALGEBRAS

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By

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ABSTRACT

COMPLETE POSITIVITY IN OPERATOR ALGEBRAS

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In this thesis we survey positive and completely positive maps defined on operator systems. In Chapter 3 we study the properties of positive maps as well as construction of positive maps under certain conditions. In Chapter 4 we focus on completely positive maps. We give some conditions on domain and range under which positivity implies complete positivity. The last chapter consists of Stinespring's dilation theorem and its applications to various areas.

Keywords: C^*-Algebras , Operator systems, Completely positive maps, Stinespring representation.

ÖZET

OPERATÖR CEBİRLERİ VE TAMAMEN POZİTİF OPERATÖRLER

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Bu tezde operatör sistemleri üzerinde tanımlı pozitif ve tamamen pozitif operatörleri inceledik. 3. bölümde pozitif operatörlerin özelliklerini ve belli koşullar altında bunların nasıl elde edilebileceğini çalıştık. 4. bölümde tamamen pozitif operatörleri inceledik. Pozitifliğin tamamen pozitifliği verebilmesi için tanım ve görüntü kümesi üzerindeki bazı koşulları verdik. Son kısımda Stinespring genleşme (dilation) teoremini sunduk ve bu teoremi çesitleri alanlara uyguladık.

Anahtar sözcükler: C^*-Cebirleri, Operatör sistemleri, Tamamen pozitif operatörler, Stinespring temsili.

Contents

Preface

In 1943, M.A. Naimark published two apparently unrelated results: the first was concerning the possibility of dilation of a positive operator valued measure to a spectral measure [14] while the second was concerning a characterization of certain operator valued positive functions on groups in terms of representations on a larger space [15]. A few years later, B. Sz.-Nagy obtained a theorem of unitary dilations of contractions on a Hilbert space [18], whose importance turned out to open a new and vast field of investigations of models of linear operators on Hilbert space in terms of a generalized Fourier analysis [19]. In addition, Sz.-Nagy Dilation Theorem turned out to be intimately connected with a celebrated inequality of J. von Nuemann [16], in this way revealing its spectral character. Later on, it turned out that the Sz.-Nagy Dilation Theorem was only a particular case of the Naimark Dilation Theorem for groups.

In 1955, W.F. Stinespring [17] obtained a theorem characterizing certain operator valued positive maps on C^*-algebras in terms of representations of those C^*-algebras, what nowadays is called *Stinespring Representation*, which was recognized as a dilation theorem as well that contains as particular cases both of the Naimark Dilation Theorems and, of course, the Sz.-Nagy Dilation Theorem. The *Stinespring Dilation Theorem* opened a large field of investigations on a new concept in operator algebra that is now called *complete positivity*, mainly due to the pioneering work of M.D. Choi [9, 10, 11]. An exposition of the most recent developments in this theory can be found in the monograph of E.G. Effros and Z.J. Ruan [12].

The aim of this work is to present in modern terms the above mentioned dilations theorems, starting from the Stinespring Dilation Theorem. In this enterprise we follow closely our weekly expositions in the Graduate Seminar on Functional Analysis and Operator Theory at the Department of Mathematics of Bilkent University, under the supervision of Aurelian Gheondea. For these presentations we have used mainly the monograph of V.I. Paulsen [4], while for the prerequisites on C^*-algebras we have used the textbooks of W.B. Arveson [1, 2].

In this presentation we tried to be as accurate and complete as possible, working out many examples and proving auxiliary results that have been left out by V.I. Paulsen as exercises. Therefore, for a few technicalities in operator theory we used the textbook of J.B. Conway [3] and the monograph of K.E. Gustafson and D.K.M. Rao [8], as well as the monograph of P. Koosis [6] on Hardy spaces.

We now briefly describe the contents of this work. The first chapter is a review of basic definitions and results on C^*-algebras, spectrum, positiveness in C^*-algebras, adjoining a unit to a nonunital C^*-algebras, as well as tensor products (for which we have used the monograph of A.Ya. Helemeskii [7]).

In the second chapter, we present the basics on the C^*-algebra structures on the algebra of complex $n \times n$ matrices, the tensor products of C^*-algebras and in particular, C^*-algebras of matrices with entries in a C^*-algebra, and a certain technical aspect related to the so-called *canonical shuffle*.

The core of our work starts with the third chapter which is dedicated to operator systems and positive maps on operator systems. Roughly speaking, an operator system is a subspace of a unital C^*-algebra, that is stable under the involution and contains the unit. The main interest here is in connection with estimations of the norms for positive maps on operator systems, a proof of the von Neumann Inequality based on the technique of positive maps and the Fejer-Riesz Lemma of representation of positive trigonometric polynomials.

Chapter four can be viewed as a preparation for the Stinespring Dilation Theorem, due to the fact that it provides the background for the understanding of completely positive maps on operator systems. The idea of complete positivity in operator algebras comes from the positivity on the tensor products of a C^*-algebras with the chain of C^*-algebras of square complex matrices of larger and larger size. This notion is closely connected with that of *complete boundedness*, but here we only present a few aspects related to our goal; this subject is vast by itself and under rapid development during the last twenty years, as reflected in the monograph [12]. In this respect, we first clarify the connection between positivity and complete positivity: completely positivity always implies positivity, while the converse holds only in special cases, related mainly with the commutativity of the domain or of the range.

In the last chapter we prove the Stinespring Dilation Theorem and show how many other dilation theorems can be obtained from here; we get from it the Sz.-Nagy Dilation Theorem and the von Neumann Inequality, we indicate the connection with the more general concept of SPECTRAL SET (due to C. Foiaş [13]), and finally prove the two Naimark Dilation Theorems, for operator valued measures and for operator valued positive definite maps on groups, as applications of Stinespring Dilation Theorem.

Chapter 1

C^*-Algebras

C^*-algebras are closely related with operators on a Hilbert space. As a concrete model, $B(\mathcal{H})$ is a C^*-algebra for any Hilbert space $\mathcal{H}$. One first defines abstract C^*-algebras and then, by a celebrated theorem of Gelfand-Naimark-Segal, it can be proven that any abstract C^*-algebra is isometric $*$-isomorphic with a norm-closed, selfadjoint subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, which can be defined as *concrete C^*-algebra*. Defining abstract C^*-algebras has the advantage of allowing many operations like quotient, direct sum and product, as well as tensor products.

1.1 Definitions and Examples

Definition 1.1. A *complex algebra* $\mathcal{A}$ is a vector space $\mathcal{A}$ over $\mathbb{C}$ with a vector multiplication $a, b \in \mathcal{A} \mapsto ab \in \mathcal{A}$ satisfying

(1) $(\alpha a + \beta b)c = \alpha\, ac + \beta\, bc$ and $c(\alpha a + \beta b) = \alpha\, ca + \beta\, cb$;

(2) $a(bc) = (ab)c$

for all a, b, c in $\mathcal{A}$ and α, β in $\mathbb{C}$.

Definition 1.2. A *Banach algebra* $\mathcal{A}$ is a Banach space $(\mathcal{A}, \|\cdot\|)$ where $\mathcal{A}$ is also a complex algebra and norm $\|\cdot\|$ satisfies $\|ab\| \leq \|a\|\|b\|$ for all a and b in $\mathcal{A}$.

Definition 1.3. Let $\mathcal{A}$ be a complex algebra. A map $a \mapsto a^*$ is called an *involution* on $\mathcal{A}$ if it satisfies

(1) $(a^*)^* = a$;

(2) $(ab)^* = b^* a^*$;

(3) $(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$

for all a and b in $\mathcal{A}$, and all α, β in $\mathbb{C}$. A complex algebra with an involution $*$ on it is called $*$-*algebra*.

Definition 1.4. A C^*-*algebra* $\mathcal{A}$ is a Banach algebra $\mathcal{A}$ with an involution $*$ satisfying $\|a^*a\| = \|a\|^2$ for all a in $\mathcal{A}$.

If $\mathcal{A}$ is a C^*-algebra then we have $\|a^*\| = \|a\|$ for all a in $\mathcal{A}$.

A complex algebra $\mathcal{A}$ is said to have a unit if it has an element, denoted by 1, satisfying $1a = a1 = a$ for all a in $\mathcal{A}$. Existence of such unit leads to the notion of unital $*$-algebra, unital Banach algebra and unital C^* algebra. A complex algebra $\mathcal{A}$ is said to be commutative if $ab = ba$ for all a and b in $\mathcal{A}$. In the following we recall some related definitions.

Definition 1.5. A C^*-algebra $\mathcal{A}$ is said to be *unital* or *have unit* 1 if it has an element, denoted by 1, satisfying $1a = a1 = a$ for all a in $\mathcal{A}$.

If $\mathcal{A}$ is a nontrivial C^*-algebra with unit 1, then $1^* = 1$ and $\|1\| = 1$.

Definition 1.6. A C^*-algebra $\mathcal{A}$ is said to be *commutative* if $ab = ba$ for all a and b in $\mathcal{A}$.

We briefly recall basic examples of C^*-algebras.

Example 1.7. Let $\mathcal{H}$ be a Hilbert space. Then $B(\mathcal{H})$ is a C^*-algebra with its usual operator norm and adjoint operation. Indeed, it is easy to show that adjoint operation $T \mapsto T^*$ is an involution. We will use the usual notation, I for the unit. $B(\mathcal{H})$ is not commutative when $\dim(\mathcal{H}) > 1$.

Example 1.8. Let $\mathcal{H}$ be Hilbert space. A subalgebra of $B(\mathcal{H})$ which is closed under norm and under adjoint operation is a C^*-algebra. We will see that such

C^*-algebras are universal. For example $K(\mathcal{H})$, the set of all compact operators on $\mathcal{H}$, is a C^*-algebra and it has no unit when $\mathcal{H}$ is infinite dimensional.

Example 1.9. Let X be a compact Hausdorff space. Then $C(X)$, the space of continuous functions from X to $\mathbb{C}$, is a commutative unital C^*-algebra with sup-norm and involution $f^*(x) = \overline{f(x)}$. We will see that this type of C^*-algebras are universal for commutative unital C^*-algebras.

Example 1.10. Let X be a locally compact Hausdorff space which is not compact. Then $C_0(X)$, the space of continuous functions vanishing at infinity, is a commutative non-unital C^*-algebra with sup-norm and involution $f^*(x) = \overline{f(x)}$. Such C^*-algebras are universal for commutative non-unital C^*-algebras, e.g. see [2]).

Definition 1.11. Let $\mathcal{H}$ be a Hilbert space. A subalgebra of $B(\mathcal{H})$ which is closed under norm and under adjoint is called a *concrete C^*-algebra*.

As we see in Example 1.8 any concrete C^*-algebra is a C^*-algebra. In section 1.3 we can see that the converse is also true by the theorem of Gelfand-Naimark-Segal.

Definition 1.12. Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras. A mapping $\pi : \mathcal{A} \to \mathcal{B}$ is called *$*$-homomorphism* if π is an algebra homomorphism and $\pi(a^*) = \pi(a)^*$ for all a in $\mathcal{A}$. A mapping $\varphi : \mathcal{A} \to \mathcal{B}$ is called *isometric $*$-isomorphism* if φ is a bijective $*$-homomorphism and preserves norms. In this case $\mathcal{A}$ and $\mathcal{B}$ are said to be *isometric $*$-isomorphic.*

Two isometric $*$-isomorphic C^*-algebras can be considered as the same C^*-algebra, since the isometric $*$-isomorphism preserves every possible operations bijectively.

Definition 1.13. Let $\mathcal{A}$ be a C^*-algebra with unit 1. An element a of $\mathcal{A}$ is said to be *invertible* if there exists an element b such that $ab = ba = 1$. Such b (necessarily unique) is said to be the *inverse of* a and denoted by a^{-1}. The set of all invertible elements of $\mathcal{A}$ is denoted by $\mathcal{A}^{-1}$.

Definition 1.14. Let $\mathcal{A}$ be a C^*-algebra. An element a of $\mathcal{A}$ is said to be *selfadjoint* if $a = a^*$, and *normal* if $aa^* = a^*a$. If $\mathcal{A}$ has unit 1, then a is called *unitary* if $aa^* = a^*a = 1$.

1.2 Spectrum

In this section we recall the notion of spectrum of an element and state basic theorems about this. Finding spectrum of an element of a C^*-algebra (or $B(\mathcal{H})$) is still a continuing part of researches. For proofs we have used [1]).

Definition 1.15. Let $\mathcal{A}$ be a C^*-algebra with unit 1 and $a \in \mathcal{A}$. We define the *spectrum of a* by

$$\sigma(a) = \{\lambda \in \mathbb{C} : \quad a - \lambda 1 \notin \mathcal{A}^{-1}\}.$$

Theorem 1.16 (Spectrum). *Let $\mathcal{A}$ be a C^*-algebra with unit and $a \in \mathcal{A}$. Then $\sigma(a)$ is a nonempty compact subset of $\{z : |z| \leq \|a\|\}$.*

Definition 1.17. Let $\mathcal{A}$ be a C^*-algebra with unit and $a \in \mathcal{A}$. We define the *spectral radius of a* by

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Theorem 1.18 (Spectral radius). *Let $\mathcal{A}$ be a C^*-algebra with unit and $a \in \mathcal{A}$, then*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

A subset of a C^*-algebra is called *C^*-subalgebra* if it is C^*-algebra with inherited operations, involution and norm. The following theorem states that the spectrum of an operator does not change by considering the spectrum in a C^*-subalgebra.

Theorem 1.19 (Spectral permanence for C^*-algebras). *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$ with $1_{\mathcal{A}} = 1_{\mathcal{B}}$. Then for any $b \in \mathcal{B}$*

$$\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{A}}(b).$$

Now we recall one of main result in the theory, the spectral theorem for normal operators. First we need the following remarks. By $C^*\{1,a\}$ we mean the smallest C^*-subalgebra containing 1 and a. It can be characterized as the closure of the set of all polynomials in 1, a and a^*. Also notice that if X is a compact subset of $\mathbb{C}$ then polynomials in z and $\bar{z}$ are dense in $C(X)$, by the Stone-Weierstrass Theorem.

Theorem 1.20 (Spectral theorem for normal operators). *Let $\mathcal{A}$ be a C^*-algebra with unit 1 and $a \in \mathcal{A}$ be normal. Then $C(\sigma(a))$ and $C^*\{1,a\}$ are isometric $*$-isomorphic via the map uniquely determined by*

$$\sum_{n,k=0}^{N} c_{nk} z^n \bar{z}^k \longmapsto \sum_{n,k=0}^{N} c_{nk} a^n (a^*)^k.$$

Another result about C^*-algebras is the uniqueness of norm, that is:

Remark 1.21 (Uniqueness of the norm of a C^*-algebra). Given a $*$-algebra there exists at most one norm on it so that it is a C^*-algebra. The proof of this result can be seen in [1]). We should also notice that $C(\mathbb{R})$ the $*$-algebra of continuous functions from $\mathbb{R}$ to $\mathbb{C}$ cannot be a C^*-algebra with a norm. Indeed if $f(x) = e^x$ than $\sigma(f) = (0, \infty)$ which is not possible in a C^*-algebra.

1.3 Fundamental Results, Positiveness

In this section we recall some basic results on positive elements. The first result states that commutative unital C^*-algebras have a special shape and the next result (GNS) shows that concrete C^*-algebras are universal.

Theorem 1.22. *Let $\mathcal{A}$ be a commutative unital C^*-algebra. Then $\mathcal{A}$ is isometric $*$-isomorphic to a $C(X)$ for some compact Hausdorff space X.*

Theorem 1.23 (Gelfand-Naimark-Segal). *Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}$ is isometric $*$-isomorphic to a concrete C^*-algebra.*

This simply means that a C^*-algebra $\mathcal{A}$ is a C^*-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. We will write $\mathcal{A} \hookrightarrow B(\mathcal{H})$ if this representation is necessary.

Definition 1.24. Let $\mathcal{A}$ be a unital C^*-algebra and $a \in \mathcal{A}$. We say a is *positive* if a is selfadjoint and $\sigma(a) \subset [0, \infty)$. We will write $a \geq 0$ when a is positive.

Remark 1.25 (Partial order on selfadjoints). Let $\mathcal{A}$ be a unital C^*-algebra. We write $a \geq b$ when a and b are selfadjoint and $a - b \geq 0$. Then $\geq$ is a partial order on selfadjoint elements of $\mathcal{A}$. Also we will use notation $a \geq b \geq 0$ to emphasize a and b are also positive.

Definition 1.26. Let $\mathcal{A}$ be a unital C^*-algebra. Then the set of all positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}^+$.

Theorem 1.27. *$\mathcal{A}^+$ is a closed cone in $\mathcal{A}$. That is, for any a, b in $\mathcal{A}^+$ and nonnegative real numbers α, β, $\alpha a + \beta b \in \mathcal{A}^+$ and $\mathcal{A}^+$ is closed.*

The following theorem gives other characterizations of positive elements.

Theorem 1.28 (Positiveness criteria). *Let $\mathcal{A}$ be a C^*-algebra with unit 1 and $a \in \mathcal{A}$. The following assertions are equivalent.*

(1) $a \geq 0$.

(2) $a = c^*c$ *for some* $c \in \mathcal{A}$.

(3) $\langle ax, x \rangle \geq 0$ *for all* $x \in H$ *(if* $\mathcal{A} \hookrightarrow B(H)$ *).*

Theorem 1.29 (n^{th} root). *Let $\mathcal{A}$ be a C^*-algebra with 1 and $a \in \mathcal{A}^+$. Then for any positive integer n there exists unique $c \in \mathcal{A}^+$ such that $a = c^n$.*

Let $T \in B(\mathcal{H})$. Then *numerical radius of* T is defined by

$$w(T) = \sup_{\|x\|=1} \{|\langle Tx, x \rangle|\}.$$

If T is normal then $\|T\| = w(T)$ ([8])). By using this result and Theorem 1.28 we can obtain the following,

Remark 1.30. Let $\mathcal{A}$ be a C^*-algebra with unit 1 and let $a, b \in \mathcal{A}$.

(1) If a is selfadjoint then $a \leq \|a\| \cdot 1$.

(2) If $0 \leq a \leq b$ then $\|a\| \leq \|b\|$.

(3) If $a, b \in \mathcal{A}^+$ then $\|a - b\| \leq \max(\|a\|, \|b\|)$.

Proof. By GNS we may assume that $\mathcal{A}$ is a concrete C^*-algebra in $B(\mathcal{H})$. It is easy to see that a is selfadjoint if and only if $\langle ax, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$. Since $\langle (\|a\| \cdot 1 - a)x, x\rangle \geq 0$ we obtain (1). To see (2), by Theorem 1.28, we have $0 \leq \langle ax, x\rangle \leq \langle bx, x\rangle$ for all $x \in \mathcal{H}$. This means that $w(a) \leq w(b)$ and so $\|a\| \leq \|b\|$. For (3), notice that $a - b$ is selfadjoint. For any $\|x\| = 1$,

$$|\langle (a-b)x, x\rangle| = |\underbrace{\langle ax, x\rangle}_{\geq 0} - \underbrace{\langle bx, x\rangle}_{\geq 0}| \leq \max(\langle ax, x\rangle, \langle bx, x\rangle) \leq \max(\|a\|, \|b\|).$$

So the result follows if take supremum over such x. □

1.4 Adjoining a Unit to a C^*-Algebra

Assume that the C^*-algebra $\mathcal{A}$ does not have a unit. It is possible to add a unit to $\mathcal{A}$, which is denoted by $\mathcal{A}_1$ and $\mathcal{A}$ is a two sided ideal in $\mathcal{A}_1$ with $\dim\mathcal{A}_1/\mathcal{A}=1$. For a in $\mathcal{A}$, define $L_a : \mathcal{A} \to \mathcal{A}$ by $b \mapsto ab$. Clearly L_a is a bounded linear operator on $\mathcal{A}$. We define

$$\mathcal{A}_1 = \{L_a + \lambda 1 : \ a \in \mathcal{A}, \ \lambda \in \mathbb{C}\}$$

where 1 is identity operator on $\mathcal{A}$. Then $\mathcal{A}_1$ becomes a unital complex algebra. If we define involution by $(L_a + \lambda 1)^* = L_{a^*} + \bar{\lambda} 1$ and norm by

$$||L_a + \lambda 1||_1 = \sup\{\|ab + \lambda b\| : \ b \in \mathcal{A}, \ \|b\| \leq 1\}$$

(the usual operator norm) then A_1 becomes a C^*-algebra ([1] pg. 75). It is easy to see that $\{L_a : a \in \mathcal{A}\}$ is a selfadjoint two sided ideal in $\mathcal{A}_1$ of codimension 1. $\pi : \mathcal{A} \to \mathcal{A}_1$ by $a \mapsto Ł_a$ is an isometry so its image $\{L_a : a \in \mathcal{A}\}$ is closed in $\mathcal{A}_1$. This means that $\{L_a : a \in \mathcal{A}\}$ is a C^*-subalgebra of $\mathcal{A}_1$. It is easy to see that π is isometric $*$-isomorphism. So $\mathcal{A}$ and $\{L_a : a \in \mathcal{A}\}$ are isometric $*$-isomorphic. Notice also that if $L_a + \lambda 1 = L_b + \alpha 1$ then we necessarily have $a = b$ and $\lambda = \alpha$.

1.5 Tensor Products

In this section we recall tensor products of vector spaces, algebras, $*$-algebras, Hilbert spaces and C^*-algebras. We used [7]) for the proofs.

Let $\mathcal{A}$ and $\mathcal{B}$ be two vector spaces over $\mathbb{C}$. Define $\mathcal{A} \circ \mathcal{B}$ as the vector space spanned by elements of $\mathcal{A} \times \mathcal{B}$. Consider the subspace N of $\mathcal{A} \circ \mathcal{B}$ spanned by the elements of the form

$$(a + a', b) - (a, b) - (a', b), \qquad (a, b + b') - (a, b) - (a, b'),$$

$$(\lambda a, b) - \lambda(a, b) \quad \text{and} \quad (a, \lambda b) - \lambda(a, b).$$

We define the tensor product of $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \otimes \mathcal{B}$, as the quotient space $\mathcal{A} \circ \mathcal{B}/N$ and define elementary tensors by

$$a \otimes b = (a, b) + N.$$

It is easy to show that tensors satisfy the following relations.

$$\begin{aligned} (a + a') \otimes b &= a \otimes b + a' \otimes b \\ a \otimes (b + b') &= a \otimes b + a \otimes b' \\ (\lambda a) \otimes b &= a \otimes (\lambda b) = \lambda(a \otimes b). \end{aligned} \tag{1.1}$$

So we obtain the following definition.

Definition 1.31 (Tensor products of vector spaces). Let $\mathcal{A}$ and $\mathcal{B}$ be two vector spaces over $\mathbb{C}$. The tensor product of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \otimes \mathcal{B}$, is the vector space spanned by the elemetary tensors $a \otimes b$ satisfying the equations (1.1).

Third relation implies that $0 \otimes b = a \otimes 0 = 0$.

Remark 1.32 (Tensor products of complex algebras). Let $\mathcal{A}$ and $\mathcal{B}$ be two complex algebras. Then the vector space $\mathcal{A} \otimes \mathcal{B}$ becomes a complex algebra if we define

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$$

and extend linearly to $\mathcal{A} \otimes \mathcal{B}$.

Remark 1.33 (Tensor products of $*$-algebras). Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras. Then the complex algebra $\mathcal{A} \otimes \mathcal{B}$ becomes a $*$-algebra if we define

$$\Big(\sum_i a_i \otimes b_i\Big)^* \;=\; \sum_i a_i^* \otimes b_i^* \;.$$

Remark 1.34 (Tensor products of Hilbert spaces). Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then the vector space $\mathcal{H} \otimes \mathcal{K}$ becomes an inner product space if we define

$$\left\langle \sum_i x_i \otimes z_i, \sum_j y_j \otimes w_j \right\rangle = \sum_{i,j} \langle x_i, y_j \rangle \langle z_i, w_j \rangle.$$

By tensor products of Hilbert spaces we mean the completion of this space.

Remark 1.35 (Tensor products of C^*-algebras). Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Let $\mathcal{A} \hookrightarrow B(\mathcal{H})$ and $\mathcal{A} \hookrightarrow B(\mathcal{K})$. Then an element $\sum_i a_i \otimes b_i$ of the $*$-algebra $\mathcal{A} \otimes \mathcal{B}$ can be viewed as an operator on the inner product space $\mathcal{H} \otimes \mathcal{K}$ if we set

$$(\sum_i a_i \otimes b_i)(\sum_j x_j \otimes y_j) = \sum_{i,j} a_i x_j \otimes b_i y_j.$$

With respect to the operator norm on $B(\mathcal{H} \otimes \mathcal{K})$, $\mathcal{A} \otimes \mathcal{B}$ becomes a $*$-algebra with norm satisfying

$$\|uv\| \leq \|u\|\|v\| \text{ and } \|u^* u\| = \|u\|^2.$$

Hence the completion of $\mathcal{A} \otimes \mathcal{B}$ becomes a C^*-algebra.

Chapter 2

Introduction

2.1 Matrices of C^*-algebras

Let $\mathcal{A}$ be a C^*-algebra (with or without unit). For a positive integer n we define $M_n(\mathcal{A})$ as follows

$$M_n(\mathcal{A}) = \{ \left[a_{ij} \right]_{i,j=1}^{n} : a_{ij} \in \mathcal{A} \text{ for } 1 \leq i, j \leq n \}.$$

Sometimes we will use the following notations for the elements of $M_n(\mathcal{A})$

$$\left[a_{ij} \right]_{i,j=1}^{n} = [a_{ij}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}.$$

It is easy to show that $M_n(\mathcal{A})$ is a vector space over $\mathbb{C}$ if we define

$$\alpha[a_{ij}] = [\alpha a_{ij}] \text{ and } [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}].$$

Also by defining vector multiplication and involution by

$$[a_{ij}][b_{ij}] = \left[\sum_{k=1}^{n} a_{ik} b_{kj} \right] \text{ and } [a_{ij}]^* = [a_{ji}^*]$$

we obtain a $*$-algebra. From the previous chapter we know that the $*$-algebra $M_n(\mathcal{A})$ can have at most one norm on it in order to be a C^*-algebra. Now we will show that such a norm always exists.

Let $\mathcal{H}, \langle \cdot, \cdot \rangle$ be a Hilbert space. By $\mathcal{H}^{(n)}$ we mean the direct sum of n copies of $\mathcal{H}$ (with elements in column matrices) with inner product defined by

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = \langle x_1, y_1 \rangle + \cdots + \langle x_n, y_n \rangle.$$

It is easy to show that $\mathcal{H}^{(n)}$ is also a Hilbert space. Notice that the norm of an element of $\mathcal{H}^{(n)}$ is given by

$$\left\| \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\| = (\ \|x_1\|^2 + \cdots + \|x_n\|^2\ )^{1/2}.$$

Let T_{ij} be bounded linear operators on $\mathcal{H}$ for $1 \leq i, j \leq n$. We define $(T_{ij}) = (T_{ij})_{i,j=1}^n : \mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$ by

$$(T_{ij})_{i,j=1}^n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n T_{1k} x_k \\ \vdots \\ \sum_{k=1}^n T_{nk} x_k \end{bmatrix}.$$

Clearly (T_{ij}) is also linear. We show that it is bounded. Let $x = (x_1, ..., x_n)^\tau$, where τ means the matrix transpose, then

$$\begin{aligned} \|(T_{ij})x\|^2 &= \|\sum_{k=1}^n T_{1k}x_k\|^2 + \cdots + \|\sum_{k=1}^n T_{nk}x_k\|^2 \\ &\leq \big(\sum_{k=1}^n \|T_{1k}\|^2\big)\big(\sum_{k=1}^n \|x_k\|^2\big) + \cdots + \big(\sum_{k=1}^n \|T_{nk}\|^2\big)\big(\sum_{k=1}^n \|x_k\|^2\big) \\ &= \big(\sum_{i,j=1}^n \|T_{ij}\|^2\big)\|x\|^2. \end{aligned}$$

So we obtain $\|(T_{ij})\| \leq (\ \sum_{i,j=1}^n \|T_{ij}\|^2\ )^{1/2}$ which simply means that (T_{ij}) is bounded. Conversely, we can show that any bounded linear operator on $\mathcal{H}^n$ is of this form. Let $T \in B(\mathcal{H}^{(n)})$. Define, for $j = 1, ..., n$, $P_i : \mathcal{H} \to \mathcal{H}^{(n)}$ by $P_i x$ is the column where i^{th} row is x and 0 elsewhere. So $P_i^* : \mathcal{H}^{(n)} \to \mathcal{H}$ is the map

$(x_1, ..., x_n)^\tau \mapsto x_i$. Set $T_{ij} : \mathcal{H} \to \mathcal{H}$ by $T_{ij} = P_i^* T P_j$. Clearly $T_{ij} \in B(\mathcal{H})$. We claim that $T = (T_{ij})$. Letting $x = (x_1, ..., x_n)^\tau$ and $y = (y_1, ..., y_n)^\tau$ we obtain

$$\begin{aligned}
\langle Tx, y\rangle &= \langle T(P_1x_1 + \cdots P_nx_n), P_1y_1 + \cdots P_ny_n\rangle \\
&= \sum_{i,j=1}^{n} \langle TP_jx_j, P_iy_i\rangle \\
&= \sum_{i,j=1}^{n} \langle P_i^*TP_jx_j, y_i\rangle \\
&= \langle (T_{ij})x, y\rangle.
\end{aligned}$$

We also have the inequality $\|T_{ij}\| \leq \|(T_{ij})\|$ for any $1 \leq i, j \leq n$. This is easy to show if we consider the elements of the form $x = (0, .., x_j, ..0)^\tau$ and $y = (0, .., y_i, ..0)^\tau$.

Finally, it can be easily verified that $(T_{ij})(U_{ij}) = (\sum_k T_{ik}U_{kj})$ and $(T_{ij})^* = (T_{ji}^*)$. Hence $M_n(B(\mathcal{H}))$ and $B(\mathcal{H}^{(n)})$ are $*$-isomorphic $*$-algebras via $[T_{ij}] \leftrightarrow (T_{ij})$. This means that $M_n(B(\mathcal{H}))$ is a C^*-algebra if we define the norm on it by considering the elements as operators on $\mathcal{H}^{(n)}$.

Given an arbitrary C^*-algebra $\mathcal{A}$, by GNS, we know that $\mathcal{A}$ is a closed selfadjoint subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. This means that $M_n(\mathcal{A})$ is a closed selfadjoint subalgebra of C^*-algebra $M_n(B(\mathcal{H}))$, and hence a C^*-algebra.

Notation We will use notation diag(a) in $M_n(\mathcal{A})$ for

$$\begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix}.$$

We should remark that if $\mathcal{A}$ is a C^*-algebra with unit 1 then $M_n(\mathcal{A})$ is unital with unit diag(1). Also the inequality $\|a_{ij}\| \leq \|[a_{ij}]\| \leq (\ \sum_{i,j=1}^n \|a_{ij}\|^2\ )^{1/2}$ holds for any $[a_{ij}] \in M_n(\mathcal{A})$. $[a_{ij}]$ is called *diagonal* when $a_{ij} = 0$ for $i \neq j$. If $[a_{ij}]$ is diagonal then $\|[a_{ij}]\| = \max_k \|a_{kk}\|$. To see this, set $A = [a_{ij}]$ then it can be shown that $\sigma(A^*A) = \sigma(a_{11}^*a_{11}) \cup \cdots \cup \sigma(a_{nn}^*a_{nn})$. So $\|A\|^2 = \|A^*A\| = r(A^*A) = \max_k \|a_{kk}\|^2$. We recall some examples with description of the norms.

Example 2.1. We use the notation M_n for the C^*-algebra $M_n(\mathbb{C}) = M_n(B(\mathbb{C})) = B(\mathbb{C}^n)$. The norm here is called M_n-norm and we will use $\|\cdot\|_{M_n}$ if necessary.

Remark 2.2. $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in M_2$ is positive if and only if $a, d \geq 0$, $c = \bar{b}$ and its determinant is nonnegative.

Proof. Since any positive element is of the form

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix}^* = \begin{bmatrix} |x|^2 + |y|^2 & x\bar{z} + y\bar{w} \\ z\bar{x} + w\bar{y} & |z|^2 + |w|^2 \end{bmatrix}$$

we have $a, d \geq 0$, $c = \bar{b}$ and determinant is nonnegative. Conversely let such a, b, c and d are given. If $a = 0$ then necessarily $b = c = 0$ and clearly $\left[\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right]$ is positive. If $a > 0$ then choosing

$$x = \sqrt{a}, \quad y = 0, \quad z = \frac{\bar{b}}{\sqrt{a}} \text{ and } w = \frac{\sqrt{ad - bc}}{\sqrt{a}}$$

implies that the above multiplication is $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. $\square$

Example 2.3. Let X be a compact Hausdorff space. We know that $C(X)$ is a C^*-algebra. We claim that the norm (which is unique) of the C^*-algebra $M_n(C(X))$ is

$$\|\ [f_{ij}]\ \| \quad = \quad \sup_{x \in X} \|\ [f_{ij}(x)]\ \|_{M_n}.$$

It is easy to verify that $\|\cdot\|$ is a complete norm on $M_n(C(X))$. We see that $M_n(C(X))$ is a Banach algebra with this norm as follows:

$$\begin{aligned} \|\ [f_{ij}][g_{ij}]\ \| = \|\ [\sum_k f_{ik} g_{kj}]\ \| &= \sup_{x \in X} \|\ [\sum_k f_{ik}(x) g_{kj}(x)]\ \| \\ &= \sup_{x \in X} \|\ [f_{ij}(x)][g_{ij}(x)]\ \| \\ &= \sup_{x \in X} \|\ [f_{ij}(x)]\ \| \sup_{x \in X} \|\ [g_{ij}(x)]\ \| \\ &= \|\ [f_{ij}]\ \|\ \|\ [g_{ij}]\ \|. \end{aligned}$$

Similarly we can show that $\|[f_{ij}][f_{ij}]^*\| = \|[f_{ij}]\|^2$. Hence $M_n(C(X))$ is a C^*-algebra with the norm above.

Remark 2.4. Let $[f_{ij}]$ in $M_n\big(C(X)\big)$. Then $[f_{ij}]$ is selfadjoint if and only if $[f_{ij}(x)]$ is selfadjoint for all x and we have

$$\sigma([f_{ij}]) = \bigcup_{x\in X} \sigma([f_{ij}(x)]).$$

Consequently, $[f_{ij}]$ is positive if and only if $\big[f_{ij}(x)\big]$ is positive for all $x \in X$.

Proof. Clearly we have that $[f_{ij}] = [g_{ij}]$ if and only if $[f_{ij}(x)] = [g_{ij}(x)]$ for all $x \in X$. This means that

$$[f_{ij}] = [f_{ji}^*] \text{ if and only if } [f_{ij}(x)] = [\overline{f_{ji}(x)}] \quad \forall x \in X.$$

This proves first part. For the second part it is enough to show that $[f_{ij}]$ is invertible if and only if $[f_{ij}(x)]$ is invertible for all $x \in X$. Observe that $[f_{ij}][g_{ij}] = [h_{ij}]$ if and only if $[f_{ij}(x)][g_{ij}(x)] = [h_{ij}(x)]$ for all x. This means that if $[f_{ij}]$ is invertible, with inverse $[g_{ij}]$, then

$$[f_{ij}(x)][g_{ij}(x)] = [g_{ij}(x)][f_{ij}(x)] = I$$

for all $x \in X$. This shows one part. Conversely let $[f_{ij}(x)]$ be invertible for all x. Let $[g_{ij}(x)]$ be its unique inverse. Define $g_{rs} : X \to \mathbb{C}$ by $x \mapsto g_{rs}(x)$, the rs entry of $[g_{ij}(x)]$, for $1 \leq r, s \leq n$. It is enough to show that g_{rs} is continuous since this implies $[g_{ij}] \in M_n(C(X))$ and certainly it is inverse of $[f_{ij}]$. We will use the following fact (see [1] pg. 15). If a_λ and a are invertible elements of a C^*-algebra such that $a_\lambda \to a$ then $a_\lambda^{-1} \to a^{-1}$. We have

$$|g_{rs}(x)-g_{rs}(y)| \leq \|[g_{ij}(x)-g_{ij}(y)]\| = \|[g_{ij}(x)]-[g_{ij}(y)]\| = \|[f_{ij}(x)]^{-1}-[f_{ij}(y)]^{-1}\|.$$

So when $x \to y$, we know that $f_{rs}(x) \to f_{rs}(y)$ for all $1 \leq r, s \leq n$, so $[f_{ij}(x)] \to [f_{ij}(y)]$. Hence the last term of the above inequality tends to 0 by the previous argument and so g_{rs} is continuous. $\square$

2.2 Tensor Products of C^*-Algebras

Let $\mathcal{A}$ be a C^*-algebra. In the previous section we defined the $*$-algebra $M_n(\mathcal{A})$. This $*$-algebra can be expressed by tensor products.

Claim: $M_n(\mathcal{A})$ and $\mathcal{A} \otimes M_n$ are $*$-isomorphic $*$-algebras via

$$[a_{ij}] \longmapsto \sum_{i,j=1}^{n} a_{ij} \otimes E_{ij}$$

where E_{ij}'s are the matrix units of M_n.
Clearly the map is linear and it is multiplicative since

$$\begin{aligned}
[a_{ij}][b_{ij}] = [\sum_k a_{ik}b_{kj}] \quad \mapsto & \quad \sum_{i,j}(\sum_k a_{ik}b_{kj}) \otimes E_{ij} \\
= & \quad \sum_{i,j}(\sum_k a_{ik}b_{kj}) \otimes E_{ik}E_{kj} \\
= & \quad \sum_{i,j,k,s}(a_{ik}b_{sj}) \otimes E_{ik}E_{sj} \\
= & \quad (\sum_{i,j} a_{ij} \otimes E_{ij})(\sum_{i,j} b_{ij} \otimes E_{ij}).
\end{aligned}$$

We also have

$$[a_{ij}]^* = [a_{ji}^*] \mapsto \sum_{i,j} a_{ji}^* \otimes E_{ij} = \sum_{i,j} a_{ji}^* \otimes E_{ji}^* = (\sum_{i,j} a_{ij} \otimes E_{ij})^*.$$

This means that the map is a $*$-homomorphism. Surjectivity follows from the fact that any element of $\mathcal{A} \otimes M_n$ is of the form $\sum_{i,j} a_{ij} \otimes E_{ij}$ for some $a_{ij} \in \mathcal{A}$. To see the injectivity let $\sum_{i,j} a_{ij} \otimes E_{ij} = 0$. Then

$$(b \otimes E_{kr})(\sum_{i,j} a_{ij} \otimes E_{ij})\,(c \otimes E_{sm}) = ba_{km}c \otimes E_{km} = 0$$

that is $ba_{km}c = 0$ for all $b, c \in \mathcal{A}$ and $1 \leq k, m \leq n$. Hence $a_{km} = 0$ and so $[a_{ij}] = 0$.

Particularly, $M_n(B(\mathcal{H}))$, $B(\mathcal{H}^{(n)})$ and $B(\mathcal{H}) \otimes M_n$ are all the same $*$-algebras via the quite natural mappings that we introduced in previous and this section.

2.3 Canonical Shuffle

For any C^*-algebra $\mathcal{A}$, $M_k\big(M_n(\mathcal{A})\big)$ is isometric $*$-isomorphic to $M_{kn}(\mathcal{A})$ via *removing the additional brackets* (See [4] pg.4). It follows that $M_k\big(M_n(\mathcal{A})\big) \cong M_n\big(M_k(\mathcal{A})\big)$ by just changing the brackets without touching the elements.

There is another identification of $M_k\big(M_n(\mathcal{A})\big)$ and $M_n\big(M_k(\mathcal{A})\big)$ which is called *canonical shift*. We first deal with the case $M_k(M_n)$ and $M_n(M_k)$

Let $E_{ij}^{(n)}$, $i, j = 1, ..., n$, denote the elementary unit matrix of M_n. Then

$$\{E_{ij}^{(n)} \otimes E_{rs}^{(m)} : i, j = 1, ..., n \;\; r, s = 1, ..., m\}$$

is a basis for the $*$-algebra $M_n \otimes M_k$. It is easy to show that $M_n \otimes M_k$ and $M_k \otimes M_n$ are $*$-isomorphic via

$$\sum_{i,j,r,s} a_{ijrs} E_{ij}^{(n)} \otimes E_{rs}^{(m)} \longleftrightarrow \sum_{i,j,r,s} a_{ijrs} E_{rs}^{(m)} \otimes E_{ij}^{(n)}.$$

Now the result follows from the fact that $M_k(M_n)$ and $M_n \otimes M_k$ $*$-isomorphic and the norm on a C^*-algebra is unique.

By this observation we conclude $M_k(M_n(\mathcal{A})) \cong M_n(M_k(\mathcal{A}))$. In fact,

$$\begin{aligned} M_k(M_n(\mathcal{A})) \cong M_k(\mathcal{A} \otimes M_n) &\cong (\mathcal{A} \otimes M_n) \otimes M_k \\ &\cong \mathcal{A} \otimes (M_n \otimes M_k) \\ &\cong \mathcal{A} \otimes (M_k \otimes M_n) \cong ... \cong M_n(M_k(\mathcal{A})). \end{aligned}$$

This process (an isometric $*$-isomorphism) is called *canonical shuffle*. As an example consider $M_3(M_2(\mathcal{A}))$ and $M_2(M_3(\mathcal{A}))$. The correspondence is

$$\begin{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} & \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \\ \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} & \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} & \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \\ \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} & \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} & \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix} \end{bmatrix} \leftrightarrow \begin{bmatrix} \begin{bmatrix} a_{11} & b_{11} & c_{11} \\ d_{11} & e_{11} & f_{11} \\ g_{11} & h_{11} & j_{11} \end{bmatrix} & \begin{bmatrix} a_{12} & b_{12} & c_{12} \\ d_{12} & e_{12} & f_{12} \\ g_{12} & h_{12} & j_{12} \end{bmatrix} \\ \begin{bmatrix} a_{21} & b_{21} & c_{21} \\ d_{21} & e_{21} & f_{21} \\ g_{21} & h_{21} & j_{21} \end{bmatrix} & \begin{bmatrix} a_{22} & b_{22} & c_{22} \\ d_{22} & e_{22} & f_{22} \\ g_{22} & h_{22} & j_{22} \end{bmatrix} \end{bmatrix}.$$

Chapter 3

Operator Systems and Positive Maps

In this chapter we consider *operator systems* and *positive maps*. If $\mathcal{S}$ is a subset of a C^*-algebra then we define $\mathcal{S}^* = \{a^* : a \in \mathcal{S}\}$, and $\mathcal{S}$ is said to be selfadjoint if $\mathcal{S} = \mathcal{S}^*$.

Definition 3.1. Let $\mathcal{A}$ be a C^*-algebra with unit. A subspace $\mathcal{S}$ of $\mathcal{A}$ which is selfadjoint and containing the unit of $\mathcal{A}$ is called an *operator system*.

If $\mathcal{S}$ is an operator system in a C^*-algebra $\mathcal{A}$ then an element of $\mathcal{S}$ is called *positive (selfadjoint)* if it is positive (selfadjoint) in $\mathcal{A}$. Notice that any selfadjoint element a of $\mathcal{S}$ is the difference of two positive elements in $\mathcal{S}$ since

$$a = \frac{\|a\| \cdot 1 + a}{2} - \frac{\|a\| \cdot 1 - a}{2} \,.$$

Definition 3.2. Let $\mathcal{S}$ be an operator system and $\mathcal{B}$ be a C^*-algebra with unit then a linear map $\phi : \mathcal{S} \to \mathcal{B}$ is called *positive* if it matches positive elements of $\mathcal{S}$ to positive elements of $\mathcal{B}$, that is, $\phi(\mathcal{S}^+) \subset \mathcal{B}^+$.

We should remark that we did not assume the continuity of the map but the following proposition shows that a positive map must be continuous.

Proposition 3.3. *Let $\mathcal{S}$ be an operator system and $\mathcal{B}$ be a C^*-algebra with unit. If $\phi : \mathcal{S} \to \mathcal{B}$ is positive then*

$$\|\phi(1)\| \leq \|\phi\| \leq 2\|\phi(1)\|.$$

Before the proof recall by Remark 1.30 that if p is positive then $p \leq \|p\| \cdot 1$, if $0 \leq a \leq b$ then $\|a\| \leq \|b\|$ and if p_1 and p_2 are two positive elements then $\|p_1 - p_2\| \leq \max(\|p_1\|, \|p_2\|)$.

Proof. Let p be positive in $\mathcal{S}$. Then $0 \leq p \leq \|p\| \cdot 1$. By using linearity of ϕ we obtain that $0 \leq \phi(p) \leq \|p\|.\phi(1)$. So by the above remark $\|\phi(p)\| \leq \|p\| \; \|\phi(1)\|$.

Now let $a \in \mathcal{S}$ be selfadjoint. Again by linearity

$$\phi(a) = \phi(\frac{\|a\|.1 + a}{2}) - \phi(\frac{\|a\|.1 - a}{2}).$$

So $\phi(a)$ is the difference of two positive elements. By the above discussion and first part of the proof we see that

$$\|\phi(a)\| \leq \max(\|\phi(\frac{\|a\| \cdot 1 + a}{2})\|, \|\phi(\frac{\|a\| \cdot 1 - a}{2})\|) \leq \|a\|\|\phi(1)\|.$$

Finally let a be an arbitrary element in $\mathcal{S}$. We can write $a = b + ic$ where b and c are selfadjoint with $\|b\|, \|c\| \leq \|a\|$. Hence

$$\|\phi(a)\| \leq \|\phi(b)\| + \|\phi(c)\| \leq \|b\|\|\phi(1)\| + \|c\|\|\phi(1)\| \leq 2\|a\|\|\phi(1)\|.$$

This shows that $\|\phi\| \leq 2\|\phi(1)\|$. Since the other inequality is trivial we are done. □

The following example is due to Arveson and it shows that the latter inequality in Proposition 3.3 is strict. As usually we set $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Example 3.4. Consider the operator system $\mathcal{S}$ in $C(\mathbb{T})$ defined by $\mathcal{S}$ =span$(1, z, \overline{z})$. Define $\phi : \mathcal{S} \to M_2$ by

$$\phi(a1 + bz + c\overline{z}) = \begin{bmatrix} a & 2b \\ 2c & a \end{bmatrix}.$$

It is easy to show $a1 + bz + c\overline{z} \geq 0$ in $\mathcal{S}$ if and only if $c = \bar{b}$ and $a \geq 2|b|$. And we know that an element $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is positive in M_2 if and only if $a, d \geq 0$, $c = \bar{b}$ and its determinant is nonnegative, by Remark 2.2. So clearly ϕ is positive. But

$$2\|\phi(1)\| = 2 = \|\phi(z)\| \leq \|\phi\| \leq 2\|\phi(1)\|.$$

So $\|\phi\| = 2\|\phi(1)\|$.

Let $\phi : \mathcal{S} \to \mathcal{B}$ be positive. Clearly $\overline{\mathcal{S}}$ is also an operator system. Since ϕ is bounded it has a natural linear extension to $\overline{\mathcal{S}}$ which we still denote by ϕ. We claim that this linear extension is also positive. Let $p \in \overline{\mathcal{S}}$ be positive. It is enough to find a positive sequence $\{p_n\}$ in $\mathcal{S}$ converging to p because positiveness of $\phi(p_n)$ and $\lim \phi(p_n) = \phi(p)$ together imply that $\phi(p)$ is positive. Let $\{a_n\}$ be a sequence in $\mathcal{S}$ converging to p. We may assume that $\{a_n\}$ is a selfadjoint sequence because otherwise we can replace it by $\{\frac{a_n + a_n^*}{2}\}$. Now let $p_n = a_n + \|p - a_n\| \cdot 1$. Clearly $\{p_n\}$ is a selfadjoint sequence in $\mathcal{S}$ converging to p. To see the positivity of the sequence, by GNS, we may assume that elements of $\mathcal{S}$ are operators on a Hilbert space $\mathcal{H}$. If $x \in \mathcal{H}$ then

$$\langle a_n x, x \rangle = \langle px, x \rangle - \langle (p - a_n)x, x \rangle \geq -\|p - a_n\| \|x\|^2.$$

So $\langle p_n x, x \rangle = \langle (a_n + \|p - a_n\| \cdot 1)x, x \rangle \geq 0$ which proves the claim.

A positive map ϕ is selfadjoint in the sense that $\phi(a^*) = \phi(a)^*$ for all a in $\mathcal{S}$. This is easy to see if we write $a = p_1 - p_2 + i(p_3 - p_4)$. We now focus on the domains of positive maps which guaranty that $\|\phi\| = \|\phi(1)\|$.

Lemma 3.5. *Let $\mathcal{A}$ be a C^*-algebra with unit 1 and $p_1, ..., p_n$ be positive elements of $\mathcal{A}$ such that*

$$\sum_{i=1}^{n} p_i \leq 1.$$

If $\lambda_1, ..., \lambda_n$ are complex numbers with $|\lambda_i| \leq 1$, then

$$\left\| \sum_{i=1}^{n} \lambda_i p_i \right\| \leq 1.$$

Proof. Consider the following multiplication in $M_n(\mathcal{A})$.

$$\begin{bmatrix} \sum_i \lambda_i p_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} p_1^{1/2} & \cdots & p_n^{1/2} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} p_1^{1/2} & 0 & \cdots & 0 \\ p_2^{1/2} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ p_n^{1/2} & 0 & \cdots & 0 \end{bmatrix}$$

The norm of the first matrix is $\|\sum_i \lambda_i p_i\|$ and the norm of each matrix on right hand side is less than 1. Indeed if A denotes the leftmost matrix on the right side then the third matrix is A^* and $\|AA^*\| \leq 1$. □

Theorem 3.6. *Let $\mathcal{B}$ be a unital C^*-algebra and X be a compact Hausdorff space. If $\phi : C(X) \to \mathcal{B}$ is positive then $\|\phi\| = \|\phi(1)\|$.*

Proof. By dividing ϕ by a positive constant we may assume that $\phi(1) \leq 1$. Let $f \in C(X)$ such that $\|f\| \leq 1$. We will show $\|\phi(f)\| \leq 1$. Let $\epsilon > 0$. Since $\{B(f(x), \epsilon)\}_{x \in X}$ is an open covering for the compact set $f(X)$, there exists finite points $x_1, ..., x_n$ in X such that $\{B(f(x_i), \epsilon)\}_{i=1}^n$ is a finite subcover of $f(X)$. Let $U_i = f^{-1}(B(f(x_i), \epsilon))$. Clearly $\{U_i\}_{i=1}^n$ is an open covering for X such that if $x \in U_i$ then $|f(x) - f(x_i)| < \epsilon$. Let $\{p_i\}$ be nonnegative real valued continuous functions such that $\sum_i p_i = 1$ and $p_i(x) = 0$ for $x \notin U_i$ for $i = 1, ..., n$. Note that for any $x \in X$, $|f(x) - f(x_i)|p_i(x) \leq \epsilon p_i(x)$ because, if $x \in U_i$ then $|f(x) - f(x_i)| < \epsilon$, while if not $p_i(x) = 0$. So, if $x \in X$ then

$$\begin{aligned} \left| f(x) - \sum f(x_i) p_i(x) \right| &= \left| f(x) \sum p_i(x) - \sum f(x_i) p_i(x) \right| \\ &= \left| \sum (f(x) - f(x_i)) p_i(x) \right| \\ &\leq \epsilon \left| \sum p_i(x) \right| = \epsilon. \end{aligned}$$

Since $\|f\| \leq 1$, $|f(x_i)| \leq 1$. So $\|\sum f(x_i)\phi(p_i)\| \leq 1$ by the previous lemma. Hence

$$\|\phi(f)\| \leq \left\| \phi\left(f - \sum f(x_i) p_i\right) \right\| + \left\| \phi(\sum f(x_i) p_i) \right\| \leq \|\phi\|\epsilon + 1.$$

Since ϵ is arbitrary we obtained $\|\phi(f)\| \leq 1$. So $\|\phi\| \leq 1$. □

We know that any commutative unital C^*-algebra is isometric $*$-isomorphic to a C^*-algebra of continuous functions on a compact set X. So Theorem 3.6 is valid for any commutative unital C^*-algebra. By using this result one can obtain some further results. Indeed, whenever the operator system is a C^*-algebra then $\|\phi\| = \|\phi(1)\|$ for a positive map.

Lemma 3.7. *If p is a polynomial such that $\mathrm{Im}p(e^{i\theta}) = 0$ for all real θ then p is a real constant.*

Proof. Poisson'a formula states that if f is a harmonic function on $\{z : |z| < R\}$ for some $R > 1$, then for any $0 \leq r < 1$,

$$f(re^{i\theta}) = \int_{-\pi}^{\pi} \frac{(1-r^2)f(e^{i\theta})}{1+r^2-2r\cos(\theta-t)}\, dt$$

(See [6]). We know that $\mathrm{Im}p$ is harmonic on $\mathbb{C}$. The above formula implies that imaginary part of p is 0 in unit disk. By Cauchy-Riemann equalities, real part of p must be a real constant in the unit disc. So p is real constant in the unit disc and consequently it must be a real constant on $\mathbb{C}$ by the uniqueness of power series. $\square$

Lemma 3.8 (Fejer-Riesz). *Let p, q be polynomials such that $p(e^{i\theta}) + \overline{q(e^{i\theta})} > 0$ for all real θ. Then there exists a polynomial r such that*

$$p(e^{i\theta}) + \overline{q(e^{i\theta})} = |r(e^{i\theta})|^2 \text{ for all } \theta \in \mathbb{R}.$$

Proof. Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ and $q(z) = b_0 + b_1 z + \cdots + b_m z^m$. First we claim that $n = m$, $a_0 - b_0$ is real and $a_i = b_i$ for $i = 1, 2, ..., n(= m)$. In fact, if $p + \overline{q} > 0$ on unit circle then $p + \overline{q} = \overline{p} + q$ on unit circle and hence $p - q = \overline{p - q}$ on unit circle. So $\mathrm{Im}\{p - q\} = 0$ on the unit circle which means that $p - q$ is a real constant by the previous lemma. This proves the assertion that we claimed. Hence we see that

$$p(e^{i\theta}) + \overline{q(e^{i\theta})} = \alpha + a_1 e^{i\theta} + \cdots + a_n e^{in\theta} + \overline{a}_1 e^{-i\theta} + \cdots + \overline{a}_n e^{-in\theta} \text{ with } \alpha \in \mathbb{R}.$$

We may assume $a_n \neq 0$. Let

$$f(z) = \overline{a}_n + \overline{a}_{n-1} z + \cdots + \overline{a}_1 z^{n-1} + \alpha z^n + a_1 z^{n+1} + a_n z^{2n}.$$

Clearly $f(0) \neq 0$ and $f(e^{i\theta}) = [p(e^{i\theta}) + \overline{q(e^{i\theta})}]e^{in\theta} \neq 0$. By the antisymmetry of the coefficients of f we have

$$\overline{f(1/\overline{z})} = z^{-2n} f(z).$$

So the $2n$ zeros of f can be written as $z_1, ..., z_n, 1/\overline{z}_1, ..., 1/\overline{z}_n$.

Let $g(z) = (z - z_1)...(z - z_n)$ and $h(z) = (z - 1/\overline{z}_1)...(z - 1/\overline{z}_n)$. So

$$f(z) = a_n g(z)h(z).$$

It is easy to show

$$\overline{h(z)} = \frac{(-1)^n \overline{z}^n g(1/\overline{z})}{z_1 ... z_n} \qquad (z \neq 0).$$

Hence

$$p(e^{i\theta}) + \overline{q(e^{i\theta})} = f(e^{i\theta})e^{-in\theta} = |f(e^{i\theta})| = |a_n|\,|g(e^{i\theta})|\,|\overline{h(e^{i\theta})}| = \left|\frac{a_n}{z_1 ... z_n}\right| |g(e^{i\theta})|^2.$$

So if we define the polynomial $r = |\frac{a_n}{z_1 ... z_n}|^{1/2} g$ then $p + \overline{q} = |r|^2$ on unit circle. $\square$

Theorem 3.9. *Let T be a linear operator on a Hilbert space $\mathcal{H}$ with $\|T\| \leq 1$ and let $\mathcal{S}$ be the operator system in $C(\mathbb{T})$ given by*

$$\mathcal{S} = \{p + \overline{q} : p \textit{ and } q \textit{ are plynomials}\}.$$

Then the map $\phi : \mathcal{S} \to B(\mathcal{H})$ given by $\phi(p + \overline{q}) = p(T) + q(T)^*$ is positive.

Proof. It is enough to show that $\phi(p + \overline{q}) \geq 0$ when $p + \overline{q} > 0$. Indeed, if $p + \overline{q}$ is only positive then $p + \overline{q} + \epsilon > 0$ for all $\epsilon > 0$ and so $\phi(p + \overline{q} + \epsilon) = \phi(p + \overline{q}) + \epsilon 1 \geq 0$ for all $\epsilon > 0$ which implies that $\phi(p + \overline{q}) \geq 0$. So let $p + \overline{q}$ be strictly positive. So by Fejer-Riesz Lemma there exists a polynomial r such that $p(e^{i\theta}) + \overline{q(e^{i\theta})} = |r(e^{i\theta})|^2$. Let $r(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n$. Then

$$p(e^{i\theta}) + \overline{q(e^{i\theta})} = |r(e^{i\theta})|^2 = \sum_{j,k=0}^{n} \alpha_j\ \alpha_k\ e^{i(j-k)\theta}.$$

So we must show

$$\phi(p + \overline{q}) = \sum_{j,k=0}^{n} \alpha_j\ \alpha_k\ T_{j-k} \text{ where } T_{j-k} = \begin{cases} T^{j-k} & j - k \geq 0 \\ T^{*k-j} & j - k < 0 \end{cases}$$

is positive. Let $x \in \mathcal{H}$. Then

$$\langle \phi(p+\overline{q})x, x\rangle = \left\langle \begin{bmatrix} I & T^* & \cdots & T^{*n} \\ T & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T^* \\ T^n & \cdots & T & I \end{bmatrix} \begin{bmatrix} \alpha_0 x \\ \vdots \\ \alpha_n x \end{bmatrix}, \begin{bmatrix} \alpha_0 x \\ \vdots \\ \alpha_n x \end{bmatrix} \right\rangle,$$

where the inner product on the right side is taken in $\mathcal{H}^{(n+1)}$. It will be enough to show that the matrix operator on the right hand side is positive. If we set

$$R = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ T & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & T & 0 \end{bmatrix}$$

then $I + R + R^2 + \cdots + R^n + R^* + R^{*2} + \cdots + R^{*n}$ is exactly the matrix operator. Since $R^{n+1} = 0$ it is easy to show

$$I + R + R^2 + \cdots + R^n + R^* + R^{*2} + \cdots + R^{*n} = (I-R)^{-1} + (I-R^*)^{-1} - I.$$

Also notice that $\|R\| \leq 1$ which can be shown easily when RR^* is considered. Now let $h \in \mathcal{H}^{n+1}$. There exists $y \in \mathcal{H}^{n+1}$ such that $h = (I-R)y$. Hence

$$\begin{aligned} &\langle ((I-R)^{-1} + (I-R^*)^{-1} - I)h, h\rangle \\ &= \langle y, (I-R)y\rangle + \langle (I-R)y, y\rangle - \langle (I-R)y, (I-R)y\rangle \\ &= \|y\|^2 - \|Ry\|^2 \geq 0. \quad \square \end{aligned}$$

This theorem has many corollaries.

Corollary 3.10 (von Neumann's Inequality). *Let T be a linear operator on a Hilbert space such that $\|T\| \leq 1$. Then for any polynomial p,*

$$\|p(T)\| \leq \|p\| = \sup_{|z|\leq 1} |p(z)|.$$

Proof. The operator system $\mathcal{S}$ in previous theorem separates the points of $\mathbb{T}$ so by Stone-Weierstrass theorem $\mathcal{S}$ is dense in $C(\mathbb{T})$. This means that the positive map ϕ as in Theorem 3.9 has a positive extension to $C(\mathbb{T})$. Since the domain is commutative $\|\phi\| = \|\phi(1)\| = 1$ which proves the claim. $\square$

Corollary 3.11. *Let $\mathcal{B}$ and $\mathcal{C}$ be unital C^*-algebras and let $\mathcal{A}$ be a subalgebra of $\mathcal{B}$ such that $1 \in \mathcal{A}$. If $\phi : \mathcal{A} + \mathcal{A}^* \to \mathcal{C}$ is positive then $\|\phi\| = \|\phi(1)\|$.*

Proof. Set $\mathcal{S} = \mathcal{A} + \mathcal{A}^*$. ϕ extends to a positive map on $\overline{\mathcal{S}}$. Fix $a \in \overline{\mathcal{S}}$ with $\|a\| \leq 1$. Theorem 3.9 tells us that $\psi : C(\mathbb{T}) \to \mathcal{B}$ given by $\psi(p) = p(a)$ is positive. Since $\overline{\mathcal{S}}$ is itself a C^*-algebra, the range of ψ is contained in $\overline{\mathcal{S}}$ so the map $\phi \circ \psi$ is well-defined. Clearly it is positive. So

$$\|\phi(a)\| = \|\phi \circ \psi(e^{i\theta})\| \leq \|\phi \circ \psi(1)\| \, \|e^{i\theta}\| = \|\phi(1)\|.$$

□

This corollary implies the following important fact whose proof is now trivial.

Theorem 3.12. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. If $\phi : \mathcal{A} \to \mathcal{B}$ is positive then $\|\phi\| = \|\phi(1)\|$.*

Up to here we obtained basic properties of positive maps. We now look for relevant examples. First, positive maps. For example any unital contraction is necessarily positive. Moreover, a unital contraction defined from a subspace $\mathcal{M}$ has a unique positive extension to $\mathcal{M} + \mathcal{M}^*$.

Lemma 3.13. *If $f : \mathcal{S} \to \mathbb{C}$ is a unital contraction then f is positive.*

Proof. Let $a \geq 0$. It is enough to show $f(a) \in$co$(\sigma(a))$. Since $\sigma(a)$ is compact, co$(\sigma(a))$ is the intersection of all closed discs containing $\sigma(a)$. Let $K = \{z : |z - \lambda| \leq r\}$ contain $\sigma(a)$. Then $\sigma(a - \lambda 1) \subseteq \{z : |z| \leq r\}$. Since $a - \lambda 1$ is normal $\|a - \lambda 1\| = r(a - \lambda 1) \leq r$, and consequently $|f(a - \lambda 1)| = |f(a) - \lambda| \leq \|f\| r = r$. So $f(a)$ in K. □

Proposition 3.14. *Let $\mathcal{B}$ be a unital C^*-algebra and $\phi : \mathcal{S} \to \mathcal{B}$ be a unital contraction. Then ϕ is positive.*

Proof. By GNS we may assume that $\mathcal{B}$ is a concrete C^*-algebra in $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Fix x in $\mathcal{H}$ satisfying $\|x\|$=1. Then $f : \mathcal{S} \to \mathbb{C}$ defined by $f(a) = \langle \phi(a)x, x \rangle$ is a unital contraction and so positive by the lemma above. Hence $a \geq 0$ implies $f(a) = \langle \phi(a)x, x \rangle \geq 0$. And so ϕ is positive. □

Proposition 3.15. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and $\mathcal{M}$ be a subspace of $\mathcal{A}$ containing the unit. If $\phi : \mathcal{M} \to \mathcal{B}$ is a unital contraction then $\tilde{\phi} : \mathcal{M} + \mathcal{M}^* \to \mathcal{B}$ defined by*

$$\tilde{\phi}(a + b^*) = \phi(a) + \phi(b)^*, \quad a, b \in M,$$

is well defined and is the unique positive extension of ϕ to $\mathcal{M} + \mathcal{M}^*$.

Proof. If a positive extension of ϕ exists then it must satisfy the above equation since a positive map must be selfadjoint. Thus, such an extension is unique. To see that it is well defined we must show that if $a + b^* = c + d^*$ with $a, b, c, d \in \mathcal{M}$ then $\tilde{\phi}(a + b^*) = \tilde{\phi}(c + d^*)$. This equivalent to the following: if $a, a^* \in \mathcal{M}$ then $\phi(a^*) = \phi(a)^*$, i.e. ϕ is selfadjoint. Let $\mathcal{S}_1 = \{a \in \mathcal{M} : a^* \in \mathcal{M}\}$. Then $\mathcal{S}_1$ is an operator system and $\phi|_{\mathcal{S}_1}$ is a unital contraction. By the above proposition, $\phi|_{\mathcal{S}_1}$ is positive. So ϕ is selfadjoint.

To show that $\tilde{\phi}$ is positive, by GNS, we may assume that $\mathcal{B} = B(\mathcal{H})$. Fix $x \in \mathcal{H}$ with $\|x\| = 1$. Set $\tilde{f}(a) = \langle \tilde{\phi}(a)x, x \rangle$ from $\mathcal{M} + \mathcal{M}^*$ to $\mathbb{C}$. It is enough to show that $\tilde{f}$ is positive. Define $f(a) = \langle \phi(a)x, x \rangle$ from $\mathcal{M}$ to $\mathbb{C}$. Since $\|f\| = 1$, by the Hahn-Banach Theorem f extends to a map f_1 to $\mathcal{M} + \mathcal{M}^*$ satisfying $\|f_1\| = 1$. So f_1 must be positive by Lemma 3.13. This means that for any a, b in $\mathcal{M}$, $f_1(a+b^*) = f_1(a) + f_1(b)^* = \tilde{f}(a) + \tilde{f}(b)^* = \tilde{f}(a+b^*)$. That is, $f_1 = \tilde{f}$ and so $\tilde{f}$ is positive. $\square$

Chapter 4

Completely Positive Maps

In Chapter 3 we introduced operator systems and positive operators. Recall that an operator system is a selfadjoint subspace of a unital C^*-algebra that contains the unit of the C^*-algebra and a positive map is a linear operator defined from an operator system to a C^*-algebra, which maps positive elements to positive elements. In this chapter we will consider completely positive and completely bounded maps.

As we saw in Chapter 2, by $M_n(\mathcal{A})$ we denote the set of all $n \times n$ matrices with entries from the unital C^*-algebra $\mathcal{A}$. By GNS we know that $\mathcal{A}$ is isomorphic $*$-isometric to a concrete C^*-algebra, that is, $\mathcal{A}$ can be thought as a C^*-subalgebra of a $B(\mathcal{H})$. By using this fact we obtained the unique norm of the C^*-algebra $M_n(\mathcal{A})$ by a quite natural map to $B(\mathcal{H}^{(n)})$.

Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{S}$ be an operator system in $\mathcal{A}$. By $M_n(\mathcal{S})$ we mean the subset of $M_n(\mathcal{A})$ with entries only from $\mathcal{S}$. It is easy to see that $M_n(\mathcal{S})$ is an operator system in $M_n(\mathcal{A})$. The norm on $M_n(\mathcal{S})$ is taken from $M_n(\mathcal{A})$ and, as usually, an element of $M_n(\mathcal{S})$ is called positive or selfadjoint if it is positive or selfadjoint in $M_n(\mathcal{A})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and let $\mathcal{S}$ be an operator system in $\mathcal{A}$. If

$\phi : \mathcal{S} \to \mathcal{B}$ is a linear map then for any positive integer n we define

$$\phi_n : M_n(\mathcal{S}) \to M_n(\mathcal{B}) \text{ by } \phi_n([a_{ij}]) = [\phi(a_{ij})].$$

It is easy to see that ϕ_n is also linear for all n. ϕ is called *n-positive* if ϕ_n is positive and called *completely positive* if ϕ is n-positive for all n. We define the *complete bound* of ϕ by $\|\phi\|_{cb} = \sup_n \|\phi_n\|$ and ϕ is called *completely bounded* if this supremum is finite. Similarly, ϕ is called *n-contractive* if ϕ_n is contractive. The following proposition shows that if ϕ is n-positive, that is, if ϕ_n is positive then $\phi = \phi_1, ..., \phi_{n-1}$ are all positive and if ϕ is n-contractive then $\phi_1, ..., \phi_{n-1}$ are all contractive.

Proposition 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and let $\mathcal{S}$ be an operator system in $\mathcal{A}$. If $\phi : \mathcal{S} \to \mathcal{B}$ is a linear map then:*

(1) $\|\phi_n\| \leq \|\phi_{n+1}\|$ *for all n.*
(2) $\|\phi_n\| \leq n\|\phi\|$ *for all n.*
(3) *If ϕ_n is positive then ϕ_1, ϕ_2,...,ϕ_{n-1} are all positive.*

Proof. Consider the following subspaces of $M_n(\mathcal{S})$ defined for $k \geq 1$ by

$$M_n^{(k)}(\mathcal{S}) = \{ \begin{bmatrix} a_{11} & \cdots & a_{1k} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \in M_n(\mathcal{S}) \}.$$

It is easy to see that $M_n^{(k)}(\mathcal{S})$ and $M_k(\mathcal{S})$ are isometric $*$-isomorphic. So (1) and (3) come from this identification. To see (2), recall that we showed in Section 2.1 $\max_{ij} \|a_{ij}\| \leq \|[a_{ij}]\| \leq (\sum_{i,j} \|a_{ij}\|^2)^{1/2}$, so

$$\begin{aligned} \|\phi_n([a_{ij}])\| = \|[\phi(a_{ij})]\| &\leq (\sum_{i,j=1}^{n} \|\phi(a_{ij})\|^2)^{1/2} \\ &\leq \|\phi\|(\sum_{i,j=1}^{n} \|a_{ij}\|^2)^{1/2} \\ &\leq \|\phi\| n \max_{ij} \|a_{ij}\| \leq n\|\phi\|\|[a_{ij}]\|. \end{aligned}$$

$\square$

If ϕ is positive then this does not imply that ϕ is completely positive. Indeed the following example shows that there exists a positive map which is not 2-positive.

Example 4.2. Define $\phi : M_2 \to M_2$ by $A \mapsto A^\tau$, the transpose of A. Recall that an element $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ of M_2 is positive if and only if $a, d \geq 0$, $\bar{b} = c$ and its determinant is nonnegative by Remark 2.2. So clearly ϕ is positive. But $\phi_2 : M_2(M_2) \to M_2(M_2)$ is not positive. We have $M_2(M_2) = M_4$ with a very natural identification namely removing the additional brackets. So

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

is positive since it is selfadjoint and the spectrum is $\{0, 1\}$ but

$$\phi_2\left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}\right) = \begin{bmatrix} \phi(E_{11}) & \phi(E_{12}) \\ \phi(E_{21}) & \phi(E_{22}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is not positive since its spectrum contains -1. So ϕ is not 2-positive.

The above example also shows that even if we allow the operator system to be whole C^*-algebra then this still does not imply that a positive map is 2-positive. In Proposition 4.1 we have an estimation $\|\phi_n\| \leq n\|\phi\|$. In the following example we see that this estimation is sharp for all n. Of course, this is also an example of a bounded map which is not completely bounded.

Example 4.3. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Define $J : \mathcal{H} \to \mathcal{H}$ by $J(\sum \alpha_n e_n) = \sum \bar{\alpha}_n e_n$. Clearly J is conjugate linear and $J^2 = I$. J also satisfies $\|Jx\| = \|x\|$ and $\langle Jx, y\rangle = \overline{\langle x, Jy\rangle}$ for all x and y in $\mathcal{H}$. We claim that for any T in $B(\mathcal{H})$, JTJ is also in $B(\mathcal{H})$ such that $\|T\| = \|JTJ\|$ and $T \geq 0$ if and only if $JTJ \geq 0$. Let $x = \sum \alpha_n e_n$ and $y = \sum \beta_n e_n$ in $\mathcal{H}$ and α in $\mathbb{C}$. Write $\bar{x} = \sum \bar{\alpha}_n e_n$ and $\bar{y} = \sum \bar{\beta}_n e_n$. Then

$$JTJ(\alpha x + y) = JT(\bar{\alpha}\bar{x} + \bar{y}) = J(\bar{\alpha}T\bar{x} + T\bar{y}) = \alpha JT\bar{x} + JT\bar{y} = \alpha JTJx + JTJy.$$

For any x in $\mathcal{H}$,

$$\|JTJx\| = \|TJx\| \leq \|T\|\|Jx\| = \|T\|\|x\|.$$

So $\|JTJ\| \leq \|T\|$. This also means that

$$\|T\| = \|J^2TJ^2\| = \|J(JTJ)J\| \leq \|JTJ\|$$

and consequently $\|T\| = \|JTJ\|$. Finally if $T \geq 0$ then

$$\langle JTJx, x\rangle = \overline{\langle TJx, Jx\rangle} \geq 0 \text{ for all } x \in \mathcal{H}$$

and so JTJ is positive. Similarly, if $JTJ \geq 0$ then $J(JTJ)J \geq 0$, that is, T is positive.

Define $\phi : B(\mathcal{H}) \to B(\mathcal{H})$ by $\phi(T) = JT^*J$. Since $T \geq 0$ implies $T = T^* \geq 0$, ϕ is positive and also $\|\phi\| = 1$ by the above part. Now we will show $\|\phi_n\| = n$ for all n. We know that $\|\phi_n\| \leq n\|\phi\| = n$ by Proposition 4.1, so it is enough to show $\|\phi_n\| \geq n$. Consider $E_{ij} \in B(\mathcal{H})$ defined on the basis by $E_{ij}e_j = e_i$ and 0 elsewhere. It is easy to show $E_{ij}^* = JE_{ij}^*J = E_{ji}$ and $E_{ij}E_{rs} = \delta_{jr}E_{is}$. Recall that $\|a\| = \|aa^*\|^{1/2}$ in a C^*-algebra. So

$$\begin{aligned}
\left\| \begin{bmatrix} E_{11} & \cdots & E_{n1} \\ \vdots & & \vdots \\ E_{1n} & \cdots & E_{nn} \end{bmatrix} \right\| &= \left\| \begin{bmatrix} E_{11} & \cdots & E_{n1} \\ \vdots & & \vdots \\ E_{1n} & \cdots & E_{nn} \end{bmatrix} \begin{bmatrix} E_{11} & \cdots & E_{n1} \\ \vdots & & \vdots \\ E_{1n} & \cdots & E_{nn} \end{bmatrix}^* \right\|^{1/2} \\
&= \|\mathrm{diag}(E_{11} + \cdots + E_{nn})\|^{1/2} \\
&= (\|E_{11} + \cdots + E_{nn}\|)^{1/2} = 1
\end{aligned}$$

in $M_n(B(\mathcal{H}))$. But its image under ϕ_n has norm n. Indeed,

$$\begin{aligned}
\|\,\phi_n([E_{ji}])\,\| = \|[\phi(E_{ji})]\,\| &= \|\,[E_{ij}]\,\| \\
&= \|\,[E_{ij}][E_{ij}]^*\,\|^{1/2} \\
&= \|\,[nE_{ij}]\,\|^{1/2} = \sqrt{n}\,\|\,[E_{ij}]\,\|^{1/2}.
\end{aligned}$$

The equality of third and last terms implies $\|\,[E_{ij}]\,\| = n$ and so by the equality of first and third terms $\|\phi_n([E_{ji}])\| = n$. This shows that $\|\phi_n\| \geq n$.

Sometimes, in order to obtain more general results we define the linear map from a subspace and extend it to the smallest operator system that contains the subspace. So some of the above definitions can be extended for subspaces. If $\mathcal{M}$ is a subspace of $\mathcal{A}$ then $M_n(\mathcal{M})$, the subset of $M_n(\mathcal{A})$ with entries from $\mathcal{M}$, is also a subspace of $M_n(\mathcal{A})$. If $\mathcal{B}$ is a C^*-algebra and $\phi : \mathcal{M} \to \mathcal{B}$ is linear then we define

$$\phi_n : M_n(\mathcal{M}) \to M_n(\mathcal{B}) \text{ by } \phi_n([a_{ij}]) = [\phi(a_{ij})].$$

Similarly, ϕ is called *completely bounded* if $\|\phi\|_{cb} = \sup_n \|\phi_n\| < \infty$ and n-*contractive* if $\|\phi_n\| \leq 1$. By a similar argument in the proof of Proposition 4.1, one can show that $\{\|\phi_n\|\}$ is an increasing sequence such that $\|\phi_n\| \leq n\|\phi\|$. But in this case we do not have a notion of positivity because $\mathcal{M}$ and $M_n(\mathcal{M})$ may not be operator systems. However, we should remark that if $\mathcal{M}$ is a subspace of $\mathcal{A}$ containing the unit of $\mathcal{A}$ then $\mathcal{S} = \mathcal{M} + \mathcal{M}^*$ is an operator system in $\mathcal{A}$. Moreover, $M_n(\mathcal{S}) = M_n(\mathcal{M}) + M_n(\mathcal{M})^*$.

Lemma 4.4. *Let $\mathcal{A}$ be a C^*-algebra with unit 1 and let $a, b \in \mathcal{A}$. Then*

$$a^*a \leq b \Longleftrightarrow \begin{bmatrix} 1 & a \\ a^* & b \end{bmatrix} \geq 0. \textit{ Particularly, } \|a\| \leq 1 \Longleftrightarrow \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix} \geq 0.$$

Proof. By GNS we may assume that $\mathcal{A}$ is a concrete C^*-algebra in $B(\mathcal{H})$. Let $\left[\begin{smallmatrix} 1 & a \\ a^* & b \end{smallmatrix}\right]$ be positive. Then for any $x \in \mathcal{H}$,

$$\left\langle \begin{bmatrix} 1 & a \\ a^* & b \end{bmatrix} \begin{bmatrix} -ax \\ x \end{bmatrix}, \begin{bmatrix} -ax \\ x \end{bmatrix} \right\rangle \geq 0 \Rightarrow \langle (b - a^*a)x, x\rangle \geq 0 \Rightarrow b - a^*a \geq 0.$$

Conversely, if $b - a^*a \geq 0$ then $\left[\begin{smallmatrix} 0 & 0 \\ 0 & b-a^*a \end{smallmatrix}\right] \geq 0$. Also $\left[\begin{smallmatrix} 1 & a \\ 0 & 0 \end{smallmatrix}\right]^* \left[\begin{smallmatrix} 1 & a \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & a \\ a^* & a^*a \end{smallmatrix}\right] \geq 0$. So their sum must be positive. The second part now follows from the first part and the fact that $\|a\| \leq 1$ iff $a^*a \leq 1$. □

Proposition 4.5. *Let $\mathcal{S}$ be an operator system and let $\mathcal{B}$ be a unital C^*-algebra. If $\phi : \mathcal{S} \to \mathcal{B}$ is unital and 2-positive then ϕ is a contraction.*

Proof. Let $a \in \mathcal{S}$ with $\|a\| \leq 1$. Then

$$\phi_2\left(\begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & \phi(a) \\ \phi(a^*) & 1 \end{bmatrix} = \begin{bmatrix} 1 & \phi(a) \\ \phi(a)^* & 1 \end{bmatrix} \geq 0.$$

So $\|\phi(a)\| \leq 1$ by the previous lemma. $\square$

Proposition 4.6 (Schwarz inequality for 2-positive maps). *Let $\mathcal{S}$ be an operator system and let $\mathcal{B}$ be a unital C^*-algebra. If $\phi : \mathcal{S} \to \mathcal{B}$ is unital and 2-positive then $\phi(a)^*\phi(a) \leq \phi(a^*a)$ for all a in $\mathcal{S}$.*

Proof. Since $\left[\begin{smallmatrix} 1 & a \\ 0 & 0 \end{smallmatrix}\right]^* \left[\begin{smallmatrix} 1 & a \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & a \\ a^* & a^*a \end{smallmatrix}\right] \geq 0$ and ϕ is unital 2-positive,

$$\phi_2\left(\begin{bmatrix} 1 & a \\ a^* & a^*a \end{bmatrix}\right) = \begin{bmatrix} 1 & \phi(a) \\ \phi(a)^* & \phi(a^*a) \end{bmatrix} \geq 0.$$

So $\phi(a)^*\phi(a) \leq \phi(a^*a)$ by Lemma 4.4. $\square$

Proposition 4.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and let $\mathcal{M}$ be a subspace of $\mathcal{A}$ with $1 \in \mathcal{M}$. If $\phi : \mathcal{M} \to \mathcal{B}$ is unital and 2-contractive then the map $\tilde{\phi} : \mathcal{M} + \mathcal{M}^* = \mathcal{S} \to \mathcal{B}$ defined by $\tilde{\phi}(a + b^*) = \phi(a) + \phi(b)^*$ is 2-positive and contractive.*

Proof. Both ϕ and ϕ_2 are unital contractions. So both $\tilde{\phi}$ and $\tilde{\phi}_2$ are positive by Proposition 3.14. Clearly $(\tilde{\phi})_2 = \tilde{\phi}_2$ since $M_2(\mathcal{S}) = M_2(\mathcal{M}) + M_2(\mathcal{M})^*$. So $\tilde{\phi}$ is 2-positive. Since it is also unital, ϕ is contractive by Proposition 4.5. $\square$

Proposition 4.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and let $\mathcal{M}$ be a subspace of $\mathcal{A}$ with $1 \in \mathcal{M}$. If $\phi : \mathcal{M} \to \mathcal{B}$ is unital and completely contractive then the map $\tilde{\phi} : \mathcal{M} + \mathcal{M}^* = \mathcal{S} \to \mathcal{B}$ defined by $\tilde{\phi}(a + b^*) = \phi(a) + \phi(b)^*$ is completely positive and completely contractive.*

Proof. Since ϕ_n is unital and 2-contractive, $\tilde{\phi_n}$ is 2-positive and contractive, particularly it is positive, by Proposition 4.7. Clearly $(\tilde{\phi})_n = \tilde{\phi}_n$, so we are done. $\square$

The following proposition states that a completely positive map must be completely bounded. In its proof we need the following

Lemma 4.9. *If $\left[\begin{smallmatrix} p & a \\ a^* & p \end{smallmatrix}\right]$ is positive then $\|a\| \leq \|p\|$.*

Proof. If $p = 0$ then necessarily $a = 0$, indeed $\left[\begin{smallmatrix} 0 & a \\ a^* & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & a \\ a^* & 0 \end{smallmatrix}\right] \geq 0$, so $a^*a \leq 0$, that is, $a = 0$ by Lemma 4.4. Let $p \neq 0$. Firstly notice that p must be selfadjoint. So $\|p\| \cdot 1 - p$ is positive. This means that $\left[\begin{smallmatrix} \|p\|1-p & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is also positive hence their sum $\left[\begin{smallmatrix} \|p\|1 & a \\ a^* & p \end{smallmatrix}\right] \geq 0$. If we multiply this vector by $1/\|p\|$ and apply Lemma 4.4, then we obtain $a^*a \leq \|p\|p$ and so $\|a\| \leq \|p\|$. $\square$

Lemma 4.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and let $\mathcal{S}$ be an operator system in $\mathcal{A}$. If $\phi : \mathcal{S} \to \mathcal{B}$ is a completely positive map then ϕ is completely bounded with $\|\phi(1)\| = \|\phi\| = \|\phi\|_{cb}$.*

Proof. Clearly $\|\phi(1)\| \leq \|\phi\| \leq \|\phi\|_{cb}$. So it is enough to show $\|\phi\|_{cb} \leq \|\phi(1)\|$. Let $A = [a_{ij}]$ be in $M_n(\mathcal{S})$ with $\|A\| \leq 1$. And let I be the unit of $M_n(\mathcal{S})$. By Lemma 4.4 we know $\left[\begin{smallmatrix} I & A \\ A^* & I \end{smallmatrix}\right]$ is positive in $M_2(M_n(\mathcal{S})) = M_{2n}(\mathcal{S})$. So

$$\phi_{2n}\left(\begin{bmatrix} I & A \\ A^* & I \end{bmatrix}\right) = \begin{bmatrix} \phi_n(I) & \phi_n(A) \\ \phi_n(A)^* & \phi_n(I) \end{bmatrix} \geq 0.$$

Hence by the above discussion $\|\phi_n(A)\| \leq \|\phi_n(I)\| = \|\phi(1)\|$. $\square$

By an *operator space* we mean a subspace of a C^*-algebra. It may not contain the unit of the C^*-algebra.

Proposition 4.11. *Let $\mathcal{S}$ be an operator space and let $f : \mathcal{S} \to \mathbb{C}$ be a bounded linear functional. Then $\|f\|_{cb} = \|f\|$. Moreover, if $\mathcal{S}$ is an operator system and f is positive then f is completely positive.*

Proof. It is enough to show $\|f_n\| \leq \|f\|$ for all n. Fix $[a_{ij}]$ in $M_n(\mathcal{S})$. Let $x = (x_1, ..., x_n)^\tau$ and $y = (y_1, ..., y_n)^\tau$. Then

$$\begin{aligned} |\langle f_n([a_{ij}]x, y)\rangle| &= \left|\sum_{ij} f(a_{ij})x_j\bar{y}_i\right| = \left|f\left(\sum_{ij} a_{ij}x_j\bar{y}_i\right)\right| \\ &\leq \|f\| \left\|\sum_{ij} a_{ij}x_j\bar{y}_i\right\| \leq \|f\|\|x\|\|y\|\|[a_{ij}]\|. \end{aligned}$$

To see the last inequality notice that $\sum a_{ij}x_j\bar{y}_i$ appears in the 11-entry of the following product

$$\begin{bmatrix} \bar{y}_1 1 & \cdots & \bar{y}_n 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_n 1 & 0 & \cdots & 0 \end{bmatrix}$$

and clearly the norms of the first and third matrices are $\|x\|$ and $\|y\|$.

Now let $\mathcal{S}$ be an operator system and let f be positive. Fix $[a_{ij}] \geq 0$ in $M_n(\mathcal{S})$. We must show $\langle f_n([a_{ij}]x, x)\rangle = f\left(\sum a_{ij}x_j\bar{x}_i\right) \geq 0$. Notice that the above matrix multiplication in $M_n(\mathcal{S})$ is positive when $x = y$. Since $\sum a_{ij}x_j\bar{x}_i$ appears as its 11-entry $\sum a_{ij}x_j\bar{x}_i \geq 0$ and, since f is positive we are done. □

The above proposition is valid whenever the range is a commutative unital C^*-algebra. Remind that such a C^*-algebra has a special shape, they are of the form $C(X)$ for some compact Hausdorff space X. And also remind that if $[f_{ij}]$ is in $M_n(C(X))$ then $\|[f_{ij}]\| = \sup\{\|f_{ij}(x)\| : x \in X\}$ and $[f_{ij}] \geq 0$ if and only if $[f_{ij}(x)] \geq 0$ in M_n for all x in X by Example 2.3 and Remark 2.4.

Proposition 4.12. *Let $\mathcal{S}$ be an operator space and let $f : \mathcal{S} \to C(X)$ be a bounded linear map. Then $\|f\|_{cb} = \|f\|$. Moreover, if $\mathcal{S}$ is an operator system and f is positive then f is completely positive.*

Proof. Let $x \in X$ and set $\phi^x : \mathcal{S} \to \mathbb{C}$ by $\phi^x(a) = \phi(a)(x)$. Clearly $\|\phi^x\| \leq \|\phi\|$ and so $\|\phi^x_n\| \leq \|\phi\|$ for all n by the previous proposition. This means that

$$\begin{aligned} \|\phi_n([a_{ij}])\| = \|[\phi(a_{ij})]\| &= \sup_{x\in X} \|[\phi(a_{ij})(x)]\| \\ &= \sup_{x\in X} \|[\phi^x(a_{ij})]\| \\ &= \sup_{x\in X} \|\phi^x_n([a_{ij}])\| \leq \|[a_{ij}]\| \sup_{x\in X} \|\phi^x_n\| \leq \|[a_{ij}]\|\|\phi\|. \end{aligned}$$

To see the second part, notice that positivity of ϕ implies that ϕ^x is positive for all x in X. So by the previous proposition ϕ^x is completely positive. By a similar argument as before the result follows. □

Now we will see that if the domain is commutative then positivity implies complete positivity. In the proof of the following theorem we need the following.

Remark 4.13. If $[a_{ij}]$ is positive in M_n and p is positive in a C^*-algebra $\mathcal{B}$ then $[a_{ij}p]$ is positive in $M_n(\mathcal{B})$.

Proof. Let $[b_{ij}] = [a_{ij}]^{1/2}$. Then $[b_{ij}p^{1/2}][b_{ij}p^{1/2}]^* = [a_{ij}p]$. □

Theorem 4.14 (Stinespring). *Let $\mathcal{B}$ be a unital C^*-algebra. If $\phi : C(X) \to \mathcal{B}$ is positive then ϕ is completely positive.*

Proof. Let $[f_{ij}]$ be positive in $M_n(C(X))$. We must show that $\phi_n([f_{ij}])$ is positive. We first claim that given $\epsilon > 0$ there exists an open covering $U_1, ..., U_m$ of X and $\lambda_1, ..., \lambda_m$ in X with $\lambda_i \in U_i$ such that

$$\|f_{ij}(x) - f_{ij}(\lambda_k)\| \leq \epsilon \text{ for all } x \in U_k \text{ and for all } k = 1, ..., m \quad i, j = 1, ..., n.$$

This is easy to see if we consider $[f_{ij}] : X \longrightarrow M_n$ by $x \mapsto [f_{ij}(x)]$.

Let $p_1, ..., p_m$ be a partition of unity subordinate to $\{U_i\}$. Then

$$\begin{aligned}
\left\| [f_{ij}(x)] - \sum_{k=1}^{m} p_k(x) [f_{ij}(\lambda_k)] \right\| &= \left\| (\sum_{k=1}^{m} p_k(x))[f_{ij}(x)] - \sum_{k=1}^{m} p_k(x)[f_{ij}(\lambda_k)] \right\| \\
&= \left\| \sum_{k=1}^{m} p_k(x)(\, [f_{ij}(x)] - [f_{ij}(\lambda_k)] \,) \right\| \\
&\leq \sum_{k=1}^{m} |p_k(x)| n\epsilon = n\epsilon.
\end{aligned}$$

From this we deduce the following

$$\left\| [f_{ij}] - \sum_{k=1}^{m} [f_{ij}(\lambda_k) p_k] \right\| \leq n\epsilon.$$

Also we have

$$\phi_n\left(\sum_{k=1}^{m} [f_{ij}(\lambda_k) p_k]\right) = [f_{ij}(\lambda_1)\phi(p_1)] + \cdots + [f_{ij}(\lambda_m)\phi(p_m)] \geq 0$$

since each term on the right hand side is positive by the previous remark. Hence

$$\begin{aligned}\|\phi_n\big([f_{ij}]\big) - \underbrace{\phi_n\big(\sum_k [f_{ij}(\lambda_k)p_k]\big)}_{\text{positive}}\| &\leq \|\phi_n\| \left\|[f_{ij}] - \sum_{k=1}^{m} [f_{ij}(\lambda_k)p_k]\right\| \\ &\leq \underbrace{\|\phi_n\|}_{\leq n\|\phi\|} n\epsilon \leq \|\phi\| n^2\epsilon.\end{aligned}$$

We know that the set of all positive elements constitute a closed set so we are done. □

Chapter 5

Stinespring Representation

Stinespring's Dilation Theorem is one of the main theorem that characterizes the completely positive maps in terms of unital $*$-homomorphisms. In Section 2 we will apply this result to obtain some other dilation theorems in various areas. We also have Naimark's dilation theorem for groups and some of its applications.

Let $\mathcal{K}$ be a Hilbert space and $\mathcal{H}$ be a Hilbert subspace of $\mathcal{K}$. If U is in $B(\mathcal{K})$ then $P_{\mathcal{H}}U|_{\mathcal{H}}$, where $P_{\mathcal{H}}$ is the projection onto $\mathcal{H}$, is in $B(\mathcal{H})$. Set $T = P_{\mathcal{H}}U|_{\mathcal{H}}$. Then U is said to be a *dilation* of T and T is said to be *compression* of U. Certainly, any $T \in B(\mathcal{H})$ has many dilations in $B(K)$. For example it can be shown that a contraction has an isometric dilation and a isometry has a unitary dilation. A constructive proof for these can be found in [4]. Combining these results we obtain the Sz.-Nagy Dilation Theorem which states that a contraction has a unitary dilation. In section 2 we will prove this by using Stinespring's Dilation Theorem.

5.1 Stinespring's Dilation Theorem

Theorem 5.1. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{H}$ be a Hilbert space. If $\phi : \mathcal{A} \to B(\mathcal{H})$ is completely positive then there exists a Hilbert space $\mathcal{K}$, a unital $*$-homomorphism $\pi : \mathcal{A} \to B(\mathcal{K})$, and a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$ with $\|\phi(1)\| = \|V\|^2$, such that*

$$\phi(a) = V^* \pi(a) V \quad \text{for all } a \in \mathcal{A}.$$

Proof. Consider the vector space $\mathcal{A} \otimes \mathcal{H}$. Define the sesquilinear form $[\cdot, \cdot]$ on $\mathcal{A} \otimes \mathcal{H}$ by

$$[a \otimes x, b \otimes y] = \langle \phi(b^* a)x, y \rangle_{\mathcal{H}} \quad a, b \in \mathcal{A}, \quad x, y \in \mathcal{H}.$$

and extend it linearly, where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is the inner product on $\mathcal{H}$.

Since ϕ is completely positive it follows that $[\cdot, \cdot]$ is positive semidefinite. Indeed for any $n \geq 1$, $a_1, ..., a_n \in \mathcal{A}$ and $x_1, ..., x_n \in \mathcal{H}$ we have

$$\begin{aligned} \left[\sum_{j=1}^{n} a_j \otimes x_j \, , \, \sum_{i=1}^{n} a_i \otimes x_i \right] &= \sum_{i,j=1}^{n} \langle \phi(a_i^* \, a_j) x_j, x_i \rangle_{\mathcal{H}} \\ &= \left\langle \phi_n([a_i^* \, a_j]) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\rangle_{\mathcal{H}^{(n)}} \geq 0. \end{aligned}$$

Positive semidefinite sesquilinear forms satisfy the Cauchy-Schwarz inequality, hence

$$\mathcal{N} := \{u \in \mathcal{A} \otimes \mathcal{H} : [u, u] = 0\} = \{u \in \mathcal{A} \otimes \mathcal{H} : [u, v] = 0 \; \forall \, v \in \mathcal{A} \otimes \mathcal{H}\}$$

is a subspace of $\mathcal{A} \otimes \mathcal{H}$. This means that

$$\langle u + \mathcal{N}, v + \mathcal{N} \rangle := [u, v]$$

is an inner product on the quotient space $\mathcal{A} \otimes \mathcal{H}/\mathcal{N}$. Let $\mathcal{K}$ be the completion of this space to a Hilbert space.

For any element a in $\mathcal{A}$, define $\pi(a) : \mathcal{A} \otimes \mathcal{H} \to \mathcal{A} \otimes \mathcal{H}$ by

$$\pi(a)\left(\sum a_i \otimes x_i\right) = \sum (aa_i) \otimes x_i.$$

Linearity of $\pi(a)$ is clear. $\pi(a)$ also satisfies the following inequality

$$[\pi(a)u, \pi(a)u] \quad \leq \quad \|a\|^2[u,u] \text{ for all } u \in \mathcal{A} \otimes \mathcal{H}. \tag{5.1}$$

To see this, observe that $a^*b^*ba \leq \|b\|^2 a^*a$ in any C^*-algebra. It follows that

$$[a_i^* a^* a a_j] \leq \|a\|^2 [a_i^* a_j] \quad (\text{in } M_n(\mathcal{A})).$$

Therefore,

$$\begin{aligned}
\left[\pi(a)\left(\sum a_j \otimes x_j\right), \pi(a)\left(\sum a_i \otimes x_i\right)\right] &= \sum_{i,j} \langle \phi(a_i^* a^* a a_j) x_j, x_i \rangle_{\mathcal{H}} \\
&\leq \|a\|^2 \sum_{i,j} \langle \phi(a_i^* a_j) x_j, x_i \rangle_{\mathcal{H}} \\
&= \|a\|^2 \left[\sum a_j \otimes x_j \,,\, \sum a_i \otimes x_i\right].
\end{aligned}$$

The inequality (5.1) shows that the null space of $\pi(a)$ contains $\mathcal{N}$ and, consequently, $\pi(a)$ can be viewed as a linear operator on $\mathcal{A} \otimes \mathcal{H}/\mathcal{N}$, which we will still denote by $\pi(a)$. Again by the inequality (5.1) it is easy to see that the quotient linear operator $\pi(a)$ is bounded, actually $\|\pi(a)\| \leq \|a\|$. Therefore it extents to a bounded linear operator on $\mathcal{K}$ and we will denote it again by $\pi(a)$.

Let us define $\pi : \mathcal{A} \to B(\mathcal{K})$ by $a \mapsto \pi(a)$. It is easy to verify that π is a unital $*$-homomorphism.

Also, define $V : \mathcal{H} \to \mathcal{K}$ by $Vx = 1 \otimes x + \mathcal{N}$. Clearly V is linear and we have

$$\begin{aligned}
\|Vx\|^2 = \langle 1 \otimes x, 1 \otimes x \rangle = \langle \phi(1)x, x \rangle_{\mathcal{H}} &= \langle \phi(1)x, x \rangle_{\mathcal{H}} \\
&= \langle \phi(1)^{1/2}x, \phi(1)^{1/2}x \rangle_{\mathcal{H}} \\
&= \|\phi(1)^{1/2}x\|^2, \quad x \in \mathcal{H},
\end{aligned}$$

so $\|V\|^2 = \|\phi(1)^{1/2}\|^2 = \|\phi(1)\|$.

Finally

$$\langle V^*\pi(a)Vx, y \rangle_{\mathcal{H}} = \langle \pi(a) 1 \otimes x, 1 \otimes y \rangle_{\mathcal{K}} = \langle \phi(a)x, y \rangle_{\mathcal{H}}$$

for all x and y in $\mathcal{H}$ and hence $V^*\pi(a)V = \phi(a)$, which completes the proof. $\square$

Several remarks have to be made.

Remark 5.2. If $\pi : \mathcal{A} \to B(\mathcal{K})$ is a unital $*$-homomorphism and $V \in B(\mathcal{H}, \mathcal{K})$ then the map $\phi : \mathcal{A} \to B(\mathcal{H})$ defined by $\phi(a) = V^*\pi(a)V$ is completely positive. So Stinespring's Dilation Theorem characterizes the completely positive maps.

Remark 5.3. When ϕ is unital we may assume that $\mathcal{K}$ contains $\mathcal{H}$ as a sub-Hilbert space. Indeed,

$$I = \phi(1) = V^*\pi(1)V = V^*V$$

implies that V is an isometry. So, instead of $\mathcal{K} = V(\mathcal{H}) \oplus V(\mathcal{H})^{\perp}$ we may consider $\mathcal{K}' = \mathcal{H} \oplus V(\mathcal{H})^{\perp}$. Thus we have

$$\phi(a) = P_{\mathcal{H}}\pi(a)|_{\mathcal{H}} \text{ for all } a \in \mathcal{A}.$$

In other words, any completely positive unital map is a *compression* of a unital $*$-homomorphism.

Remark 5.4. When $\mathcal{A}$ and $\mathcal{H}$ are separable then we may assume that $\mathcal{K}$ is separable. Similarly, when $\mathcal{A}$ and $\mathcal{H}$ are finite dimensional then $\mathcal{K}$ may be taken finite dimensional.

Definition 5.5. The triple $(\pi, V, \mathcal{K})$ obtained in the Stinespring's Dilation Theorem is called a *Stinespring representation for ϕ*. If

$$\pi(\mathcal{A})V\mathcal{H} = \{\pi(a)Vx : a \in \mathcal{A} \text{ and } x \in \mathcal{H}\}$$

has dense span in $\mathcal{K}$ then the triple $(\pi, V, \mathcal{K})$ is called a *minimal Stinespring representation for ϕ*.

Remark 5.6. Given a Stinespring representation $(\pi, V, \mathcal{K})$ for $\phi : \mathcal{A} \to B(\mathcal{H})$, it is possible to make it minimal. Let $\mathcal{K}_1$ be the closed linear span of $\pi(\mathcal{A})V\mathcal{H}$ in $\mathcal{K}$. Since π is unital, $V\mathcal{H}$ lies in $\mathcal{K}_1$ so we may assume that $V : \mathcal{H} \to \mathcal{K}_1$. Also $\pi(a)(\mathcal{K}_1)$ lies in $\mathcal{K}_1$ for all $a \in \mathcal{A}$ since π is multiplicative and continuous. So $\pi_1 : \mathcal{A} \to B(\mathcal{K}_1)$ defined by $\pi_1(a) = \pi(a)|_{\mathcal{K}_1}$ is well defined and still a unital $*$-homomorphism. It is easy to see that $(\pi_1, V, \mathcal{K}_1)$ is a minimal Stinespring representation for ϕ.

The following proposition shows that minimal Stinespring representations are unique up to unitary equivalence.

Proposition 5.7. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\phi : \mathcal{A} \to B(\mathcal{H})$ be completely positive. If $(\pi_1, V_1, \mathcal{K}_1)$ and $(\pi_2, V_2, \mathcal{K}_2)$ are two minimal Stinespring representations for ϕ, then there exists a unitary operator $U : \mathcal{K}_1 \to \mathcal{K}_2$ such that $UV_1 = V_2$ and $U\pi_1(\cdot)U^* = \pi_2$.*

Proof. We know that $\mathrm{span}\pi_1(\mathcal{A})V_1\mathcal{H}$ and $\mathrm{span}\pi_2(\mathcal{A})V_2\mathcal{H}$ are dense in $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. First define

$$U : \mathrm{span}\pi_1(\mathcal{A})V_1\mathcal{H} \to \mathrm{span}\pi_2(\mathcal{A})V_2\mathcal{H} \text{ by } \sum_i \pi_1(a_i)V_1x_i \mapsto \sum_i \pi_2(a_i)V_2x_i.$$

The following calculation shows that U is an isometry (which also implies that U is well-defined):

$$\begin{aligned} \left\| \sum_i \pi_1(a_i)V_1x_i \right\|^2 &= \sum_{i,j} \langle V_1^*\pi_1(a_i^*a_j)V_1x_j, x_i\rangle \\ &= \sum_{i,j} \langle \phi_1(a_i^*a_j)x_j, x_i\rangle = \left\| \sum_i \pi_2(a_i)V_2x_i \right\|^2. \end{aligned}$$

Clearly U is onto. We may extend it linearly from $\mathcal{K}_1$ to $\mathcal{K}_2$, and the extension is still an onto isometry, and so a unitary operator. The remaining part of the proof just follows from the definition of U. □

5.2 Applications of Stinespring Representation

5.2.1 Unitary dilation of a contraction

Theorem 5.8 (Sz.-Nagy's Dilation Theorem). *Let $T \in B(\mathcal{H})$ with $\|T\| \leq 1$. Then there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a Hilbert subspace and a unitary operator $U \in B(\mathcal{K})$ such that $\{U^k\mathcal{H} : k \in \mathbb{Z}\}$ has dense span in $\mathcal{K}$ and*

$$T^n = P_{\mathcal{H}}U^n|_{\mathcal{H}} \textit{ for all nonnegative integers } n.$$

Moreover, if $(\mathcal{K}', U')$ is another pair satisfying the above properties, then there exists a unitary operator $V : \mathcal{K} \to \mathcal{K}'$ such that $Vh = h$ for all $h \in \mathcal{H}$ and $VUV^* = U'$.

Proof. By Theorem 3.9 $\phi : C(\mathbb{T}) \to B(\mathcal{H})$ defined by $p + \overline{q} \mapsto p(T) + q(T)^*$ is positive. So by Theorem 4.14 ϕ is completely positive. Let $(\pi, V, \mathcal{K})$ be minimal a Stinespring representation for ϕ. Since ϕ is unital, by Remark 5.3 we may assume that $\mathcal{K}$ contains $\mathcal{H}$ as a Hilbert subspace and V is the imbedding $\mathcal{H} \hookrightarrow \mathcal{K}$ so

$$\phi(f) = P_{\mathcal{H}}\pi(f)|_{\mathcal{H}} \text{ for all } f \in C(\mathbb{T}).$$

Set $U = \pi(z)$, where z is the coordinate function. Since π is a unital $*$-homomorphism, U is unitary and we have

$$T^n = \phi(z^n) = P_{\mathcal{H}}\pi(z^n)|_{\mathcal{H}} = P_{\mathcal{H}}\pi(z)^n|_{\mathcal{H}} = P_{\mathcal{H}}U^n|_{\mathcal{H}} \text{ for all } n \geq 0.$$

The minimality condition of $(\pi, V, \mathcal{K})$ means that span of $\pi(C(\mathbb{T}))V\mathcal{H}$ is dense in $\mathcal{K}$. Equivalently, span of $\{\pi(z^k)V\mathcal{H} : k \in \mathbb{Z}\}$ is dense in $\mathcal{K}$. Since $V\mathcal{H} = \mathcal{H}$ and $\pi(z^k) = U^k$, we obtain that $\{U^k\mathcal{H} : k \in \mathbb{Z}\}$ has dense span in $\mathcal{K}$.

Let $(\mathcal{K}', U')$ be another pair satisfying the properties in the theorem. Set $\pi'(p + \bar{q}) = p(U') + q(U')^*$ for the polynomials $p, q \in C(\mathbb{T})$ into $B(\mathcal{K}')$. It is easy to show that π' is a unital $*$-homomorphism (so it must be bounded since it is positive). Hence π' extends to a unital $*$-homomorphism on $C(\mathbb{T})$. Notice that

$$\phi(p + \bar{q}) = p(T) + q(T)^* = P_{\mathcal{H}}\big(p(U') + q(U')^*\big)|_{\mathcal{H}} = P_{\mathcal{H}}\pi'(p + \bar{q})|_{\mathcal{H}}.$$

So, necessarily, $\phi(f) = P_{\mathcal{H}}\pi'(f)|_{\mathcal{H}}$ for all $f \in C(\mathbb{T})$. This means that (π', V', K'), where V' is the imbedding of $\mathcal{H}$ into $\mathcal{K}'$, is a Stinespring representation for ϕ. Moreover, the condition $\{U'^k\mathcal{H} : k \in \mathbb{Z}\}$ ensures that it is minimal. Now the result follows from Proposition 5.7, which states that the minimal Stinespring representations are unitarily equivalent. $\square$

5.2.2 Spectral Sets

In this part we have an application of the Stinespring representation to spectral sets. We first recall some definitions.

Let X be a compact subset of $\mathbb{C}$. Let $R(X)$ be the subalgebra of $C(X)$ containing the rational functions, that is,

$$\mathcal{R}(X) = \{f : X \to \mathbb{C} : f = \frac{p}{q} \text{ for some polynomials } p, q \text{ and } q(z) \neq 0 \; \forall z \in X\}.$$

Note that the above representation of a function $f \in \mathcal{R}(X)$ may not be unique, for example if $X = \{\lambda\}$ then $0/1$ and $z - \lambda/1$ are the same functions in $\mathcal{R}(X)$.

Let $T \in B(\mathcal{H})$ with $\sigma(T) \subseteq X$. We may define the homomorphism

$$\rho : \mathcal{R}(X) \to B(\mathcal{H}) \text{ by } \rho(p/q) = p(T)q(T)^{-1}.$$

However this definition may not be correct since the representation by polynomials may not be unique. As an example if $\sigma(T) = \{\lambda\}$ and $X = \{\lambda\}$ then ρ is well defined if and only if $T = \lambda I$. When X is infinite compact set then ρ is well-defined, indeed if $p/q = r/s$ on X then $ps = qr$ on X and so on the complex plane. Hence we obtain $p(T)q(T)^{-1} = r(T)s(T)^{-1}$. When X is a finite set, say $X = \{\lambda_1, ..., \lambda_n\}$, then ρ is well defined if and only if $(T - \lambda_1 I) \cdot ... \cdot (T - \lambda_n I) = 0$. Note that $\|\rho\| \geq 1$ since it is unital. When $\|\rho\| = 1$ X is called a *spectral set for* T, and when ρ satisfies $\|\rho\| \leq K$, then X is said to be a K-*spectral set for* T.

We can consider $\mathcal{R}(X)$ as a subalgebra of $C(\partial X)$ since the maximum modulus principle implies that the norms of a rational function on X and on ∂X are same.

Let $T \in B(\mathcal{H})$ and assume that $\sigma(T)$ lies in a compact set X. T is said to have a *normal* ∂X-*dilation* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a normal operator N in $B(\mathcal{K})$ with $\sigma(N) \subseteq \partial X$, such that

$$r(T) = P_{\mathcal{H}} r(N)|_{\mathcal{H}}$$

for any rational function r in $\mathcal{R}(X)$. N is called a *minimal normal* ∂X-*dilation for* T when $\{r(N)\mathcal{H} : r \in \mathcal{R}(X)\}$ has dense span in $\mathcal{K}$. When T has a normal ∂X-dilation, X must be necessarily a spectral set for T since

$$\|r(T)\| \leq \|r(N)\| = \sup_{z \in \sigma(N)} |r(z)| \leq \sup_{z \in \partial X} |r(z)| = \|r\|.$$

However, the converse is not correct. That is, there exists T and a spectral set X for T, but T has no normal dilation with spectrum contained in ∂X, cf. [5].

Let $\mathcal{S} = R(X) + \overline{R(X)}$. Then $\mathcal{S}$ is an operator system in $C(\partial X)$. $R(X)$ is called a *Dirichlet algebra on* ∂X if $\mathcal{S}$ is dense in $C(\partial X)$.

Theorem 5.9 (Berger-Foias-Lebow). *Let X be a spectral set for T such that $R(X)$ is a Dirichlet algebra on ∂X. Then T has a minimal normal ∂X-dilation. Furthermore, two minimal normal ∂X-dilations are unitarily equivalent in such a way that the unitary map leaves invariant $\mathcal{H}$.*

Proof. Since $\rho : \mathcal{R}(X) \to B(\mathcal{H})$ defined by $\rho(r) = r(T)$ is a unital contraction, it extends to a positive map $\tilde{\rho}$ to $\mathcal{S} = R(X) + \overline{R(X)}$. We are given that $\mathcal{S}$ is dense in $C(\partial X)$ so $\tilde{\rho}$ extends to a positive map on $C(\partial X)$. This extension is completely positive by Theorem 4.14. The rest of the proof is similar to the proof of the previous theorem. The only difference is that if $(\mathcal{K}, V, \pi)$ is minimal Stinespring representation for ρ then $\pi(z) = N$ is a normal dilation of T. □

Sz.-Nagy's Dilation Theorem is a very special case of the theorem above. In fact, when $\|T\| \leq 1$ then the closed unit disc is a spectral set for T and it is a Dirichlet algebra on $\mathbb{T}$.

5.2.3 $B(\mathcal{H})$-valued measures

Another application of the Stinespring Representation Theorem deals with operator valued measures on a compact set X. We first give some related definitions.

Let X be a compact Hausdorff space and let $\mathcal{B}$ be the σ-algebra of Borel sets of X, that is, $\mathcal{B}$ is the smallest σ-algebra containing all open subsets of X. A map $E : \mathcal{B} \to B(\mathcal{H})$ is called a $B(\mathcal{H})$-*valued measure* if E is weakly countably additive, that is, for any disjoint sequence $\{B_i\}$ in $\mathcal{B}$ with union B

$$\langle E(B)x, y\rangle = \sum_{i=1}^{\infty} \langle E(B_i)x, y\rangle \text{ for all } x, y \in \mathcal{H}.$$

The $B(\mathcal{H})$-valued measure E is called *bounded* if $\sup\{\|E(B)\| : B \in \mathcal{B}\} < \infty$. We put $\|E\|$ for this supremum. Note that if we fix x and y in $\mathcal{H}$ then $E_{x,y} : \mathcal{B} \to \mathbb{C}$

defined by

$$E_{x,y}(B) = \langle E(B)x, y\rangle$$

is a complex measure. E is said to be *regular* if $E_{x,y}$ is regular for all $x, y \in \mathcal{H}$.

Remark 5.10. There is a bijective correspondence between regular bounded $B(\mathcal{H})$-valued measures and bounded linear maps $\phi : C(X) \to B(\mathcal{H})$. Indeed, let E be a regular bounded $B(\mathcal{H})$-valued measure. Fix $f \in C(X)$. Define the sequi-linear form $[\cdot, \cdot]$ on $\mathcal{H}$ by

$$[x, y] = \int f \, dE_{x,y}.$$

It is easy to show that $[\cdot, \cdot]$ is bounded, in fact,

$$|[x, y]| \leq \|f\| \, |\langle E(X)x, y\rangle| \leq \|f\| \, \|E\| \, \|x\| \, \|y\|.$$

So by Riesz Representation Theorem there exists unique $T_f \in B(H)$ such that

$$\langle T_f x, y\rangle \;\; = \;\; [x, y] = \int f \, dE_{x,y}. \tag{5.2}$$

Let $\phi_E : C(X) \to B(\mathcal{H})$ be defined by $f \mapsto T_f$. It is easy to show ϕ_E is linear and bounded.

Conversely, let a bounded linear map $\phi : C(X) \to B(\mathcal{H})$ be given. Fix x and y in $\mathcal{H}$. The map

$$f \mapsto \langle \phi(f)x, y\rangle \quad f \in C(X)$$

is in the dual space of $C(X)$. So by Riesz-Markov Theorem there is a complex valued finite Borel measure $\mu_{x,y}$ such that

$$\langle \phi(f)x, y\rangle = \int f \, d\mu_{x,y} \text{ for all } f \in C(X).$$

For any $B \in \mathcal{B}$, define a sequi-linear form $[\cdot, \cdot]$ on $\mathcal{H}$ by $[x, y] = \mu_{x,y}(B)$. By a similar argument above, there exists unique $E(B) \in B(\mathcal{H})$ such that

$$\langle E(B)x, y\rangle \;\; = \;\; [x, y] = \mu_{x,y}(B). \tag{5.3}$$

It is easy to check that $E : \mathcal{B} \to B(\mathcal{H})$ given by $B \mapsto E(B)$ is a regular bounded $B(\mathcal{H})$-valued measure.

To sum up, the following equation gives this correspondence

$$\langle \phi(f)x, y\rangle = \int f \, d\langle E(\cdot)x, y.$$

We also remark that the above correspondence (5.2) and (5.3) are inverse one to each other.

Definition 5.11. A regular bounded $B(\mathcal{H})$-valued measure E is said to be

(i) *spectral* if $E(M \cap N) = E(M)E(N)$;

(ii) *positive* if $E(M) \geq 0$;

(iii) *selfadjoint* if $E(M)^* = E(M)$;

for all $M, N \in \mathcal{B}$.

Note that (i) and (iii) together imply (ii), since $E(M)^2 = E(M) = E(M)^*$ means that $E(M) \geq 0$ for all M.

The following proposition shows the connection between this kind of measures and the linear map obtained by them.

Proposition 5.12. *Let E be a bounded regular $B(H)$-valued measure and let $\phi : C(X) \to B(\mathcal{H})$ be the corresponding linear map. Then:*

(i) *ϕ is a homomorphism if and only if E is spectral;*

(ii) *ϕ is positive if and only if E is positive;*

(iii) *ϕ is selfadjoint if and only if E is selfadjoint;*

(iv) *ϕ is a $*$-homomorphism if and only if E is selfadjoint and spectral.*

Proof. We first remark that if M is a Borel set, then $\langle E(M \cap \cdot)x, y\rangle$ is a measure such that for any measurable function g,

$$\int_M g \, d\langle E(\cdot)x, y\rangle = \int g \, d\langle E(M \cap \cdot)x, y\rangle. \tag{5.4}$$

(i) ($\Leftarrow$) Let E be spectral. It is enough to show that for any simple function φ and $g \in C(X)$,

$$\int \varphi g \, d\langle E(\cdot)x, y\rangle = \int \varphi \, d\langle E(\cdot)\phi(g)x, y\rangle. \tag{5.5}$$

Indeed, if the equality holds for simple functions then it holds for any continuous function f and we have $\langle\phi(fg)x, y\rangle = \langle\phi(f)\phi(g)x, y\rangle$, that is, ϕ is multiplicative. Showing equation (5.5) is equivalent to show

$$\int_M g\, d\langle E(\cdot)x, y\rangle = \int_M d\langle E(\cdot)\phi(g)x, y\rangle.$$

for any Borel set M. But the right hand side is $\langle E(M)\phi(g)x, y\rangle$ and, by (5.4), the left hand side is

$$\int g\, d\langle E(M\cap\cdot)x, y\rangle = \int g\, d\langle E(\cdot)x, E(M)^*y\rangle = \langle\phi(g)x, E(M)^*y\rangle.$$

($\Rightarrow$) Let ϕ be multiplicative. Fix x, y in $\mathcal{H}$ and $N, M \in \mathcal{B}$. For any continuous function g,

$$\begin{aligned}
& |\langle E(N\cap M)x, y\rangle - \langle E(N)E(M)x, y\rangle| \\
= & \left|\int_N \chi_M d\langle E(\cdot)x, y\rangle - \langle E(N)E(M)x, y\rangle\right| \\
\leq & \left|\int_N \chi_M d\langle E(\cdot)x, y\rangle - \int_N g d\langle E(\cdot)x, y\rangle\right| + \left|\int_N g d\langle E(\cdot)x, y\rangle - \langle E(N)E(M)x, y\rangle\right| \\
= & \left|\int_N (\chi_M - g) d\langle E(\cdot)x, y\rangle\right| + \left|\int_N d\langle E(\cdot)\phi(g)x, y\rangle - \langle E(N)E(M)x, y\rangle\right| \\
= & \left|\int_N (\chi_M - g) d\langle E(\cdot)x, y\rangle\right| + |\langle E(N)\phi(g)x, y\rangle - \langle E(N)E(M)x, y\rangle| \\
= & \left|\int_N (\chi_M - g) d\langle E(\cdot)x, y\rangle\right| + |\langle\phi(g)x, E(N)^*y\rangle - \langle E(M)x, E(N)^*y\rangle| \\
= & \left|\int_N (\chi_M - g) d\langle E(\cdot)x, y\rangle\right| + \left|\int g d\langle E(\cdot)x, E(N)^*y\rangle - \int \chi_M d\langle E(\cdot)x, E(N)^*y\rangle\right| \\
= & \left|\int_N (\chi_M - g) d\langle E(\cdot)x, y\rangle\right| + \left|\int (g - \chi_M) d\langle E(\cdot)x, E(N)^*y\rangle\right|.
\end{aligned}$$

So we can choose a continuous function g such that the last sum is arbitrarily small.

(ii)($\Leftarrow$) Let E be positive. Then $\langle E(\cdot)x, x\rangle$ is a positive measure. This means that for any $f \geq 0$,

$$\int f\, d\langle E(\cdot)x, x\rangle \geq 0$$

since f can be approximated by simple functions having nonnegative real coefficients. So $\langle\phi(f)x, x\rangle \geq 0$, equivalently ϕ is positive.

($\Rightarrow$) Let ϕ be positive. Fix $M \in \mathcal{B}$. For any $f \in C(X)$ and $x \in \mathcal{H}$,

$$\begin{aligned} |\langle \phi(f)x, x\rangle - \langle E(M)x, x\rangle| &= \left| \int f \, d\langle E(\cdot)x, x\rangle - \int \chi_{_M} \, d\langle E(\cdot)x, x\rangle \right| \\ &\leq \int \|f - \chi_{_M}\| \, d|\langle E(\cdot)x, x\rangle|. \end{aligned}$$

It is possible to make the last integral arbitrarily small by a choice of a continuous function f. But since $\|\mathrm{Re} f^+ - \chi_{_M}\| \leq \|f - \chi_{_M}\|$ and $|\langle E(\cdot)x, x\rangle|$ is a positive measure, we may also assume $f \geq 0$. So $\langle E(M)x, x\rangle \geq 0$, equivalently E is positive.

(iii) If μ is a bounded regular measure then $\bar{\mu}$ is also a bounded regular measure and for any measurable function f we have

$$\overline{\int f \, d\mu} = \int \bar{f} \, d\bar{\mu}.$$

It is easy to see that $\langle y, E(\cdot)x\rangle$ is also a measure such that $\overline{\langle E(\cdot)x, y\rangle} = \langle y, E(\cdot)x\rangle$ for all x, y. This means that

$$\begin{aligned} \langle \big(\phi(f^*) - \phi(f)^*\big)x, y\rangle &= \langle \phi(f^*)x, y\rangle - \overline{\langle \phi(f)y, x\rangle} \\ &= \int \bar{f} \, d\langle E(\cdot)x, y\rangle - \overline{\int f \, d\langle E(\cdot)y, x\rangle} \\ &= \int \bar{f} \, d\langle E(\cdot)x, y\rangle - \int \bar{f} \, d\overline{\langle E(\cdot)y, x\rangle} \\ &= \int \bar{f} \, d\big(\langle E(\cdot)x, y\rangle - \langle x, E(\cdot)y\rangle\big). \end{aligned}$$

From here it follows that ϕ is selfadjoint if and only if E is selfadjoint.

(iv) By (i) and (iii). $\square$

The following theorem states that a positive measure has selfadjoint spectral dilation. We give a proof based on Stinespring Dilation Theorem.

Theorem 5.13 (Naimark). *Let E be a positive $B(\mathcal{H})$-valued measure on X. Then there exists a Hilbert space $\mathcal{K}$, a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$ and a selfadjoint spectral $B(\mathcal{K})$-valued measure F on X such that*

$$E(B) = V^* F(B) V \text{ for every Borel set } B.$$

Proof. Let $\phi : C(X) \to B(\mathcal{H})$ be the linear map corresponding to E. ϕ is positive by the previous proposition and so ϕ is completely positive by Theorem 4.14. Let $(\pi, V, \mathcal{K})$ be a Stinespring representation for ϕ. So $\pi : C(X) \to B(\mathcal{K})$ is a unital $*$-homomorphism and $V \in B(\mathcal{H}, \mathcal{K})$ such that $\phi(f) = V^*\pi(f)V$ for all $f \in C(X)$. Let F be the $B(\mathcal{H})$-valued measure corresponding to π. By the previous proposition F is regular selfadjoint and spectral. It is clear now that F has the desired property. □

5.2.4 Completely positive maps between complex matrices

If $\phi : M_n \to M_k$ is completely positive then ϕ has a special shape, more precisely, $\phi(A) = V_1^*AV_1 + \cdots + V_j^*AV_j$ for some $n \times k$ matrices $V_1, ..., V_j$ with $j \leq nk$. We will prove this result by use of the Stinespring Representation Theorem.

Lemma 5.14. *Let $\pi : M_n \to B(\mathcal{K})$ be a unital $*$-homomorphism. Then there exists a Hilbert space $\mathcal{H}$ such that*

$$\mathcal{K} \cong \mathcal{H} \oplus \cdots \oplus \mathcal{H} \text{ (n copies)}$$

and $\pi : M_n \to B(\mathcal{K}) \cong M_n(B(\mathcal{H}))$ satisfies $\pi(E_{ij}) = \widetilde{E}_{ij}$ for all $i, j = 1, ..., n$ where E_{ij} and $\widetilde{E}_{ij}$ are the standard matrix units for M_n and $M_n(B(\mathcal{H}))$, respectively.

Proof. Set $\mathcal{H}_i = \pi(E_{ii})\mathcal{K}$ for $i = 1, ..., n$. Then $\mathcal{K} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ as direct sum of Hilbert spaces. (Indeed, it is easy to see that $\mathcal{K} = \mathcal{H}_1 + \cdots + \mathcal{H}_n$ and $\mathcal{H}_i \perp \mathcal{H}_j$ for $i \neq j$ and consequently the sum is direct and all subspaces are complete.)

We claim that $\mathcal{H}_1, ..., \mathcal{H}_n$ are isometric isomorphic. Since the range of $\pi(E_{ji})$ lies in $\mathcal{H}_j$, $U_{ji} = \pi(E_{ji})|_{\mathcal{H}_i}$ is well-defined as an operator from $\mathcal{H}_i$ to $\mathcal{H}_j$. We claim that U_{ji} is an isometric isomorphism with $U_{ji}^{-1} = U_{ij}$ for all i, j. Linearity is clear. Since $\pi(E_{ij})\mathcal{K}$ lies in $\mathcal{H}_i$ and $U_{ji}\pi(E_{ij})x = \pi(E_{jj})x$, U_{ji} is surjective. U_{ji} preserves inner-product and so it is one-to-one. A typical element of $\mathcal{H}_i$ is of the form $\pi(E_{ii})x$. So

$$\langle U_{ji}\pi(E_{ii})x, U_{ji}\pi(E_{ii})y\rangle = \langle \pi(E_{ji})x, \underbrace{\pi(E_{ji})}_{\pi(E_{ij})^*} y\rangle = \langle \pi(E_{ii})x, y\rangle = \langle \pi(E_{ii})x, \pi(E_{ii})y\rangle$$

where the last equality follows from $\pi(E_{ii}) = \pi(E_{ii})\pi(E_{ii}) = \pi(E_{ii})^*\pi(E_{ii})$. Finally, it is easy to see that $U_{ji}^{-1} = U_{ij}$:

$$\langle \pi(E_{ii})x, \pi(E_{ii})y\rangle = \langle \pi(E_{ii})x, y\rangle = \langle \pi(E_{ij}E_{ji})x, y\rangle = \langle \pi(E_{ji})x, \pi(E_{ji})y\rangle.$$

We can represent any operator on $\mathcal{K} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ by an $n \times n$ matrix with operator entries. It is easy to see that $\pi(E_{ij})$ corresponds to the matrix where ij^{th} entry is U_{ij} and 0 elsewhere. Since

$$\mathcal{K} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_n = \mathcal{H}_1 \oplus U_{21}\mathcal{H}_1 \oplus \cdots \oplus U_{n1}\mathcal{H}_1 \cong \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_1,$$

$\pi(E_{ij})$, as an operator on the last summand, corresponds to $\widetilde{E}_{ij}$. $\square$

When $\mathcal{K}$ is finite dimensional we can say more.

Lemma 5.15. *Let $\mathcal{K}$ be a finite dimensional Hilbert space and $\pi : M_n \to B(\mathcal{K})$ be a unital $*$-homomorphism. Then*

$$\mathcal{K} \cong \mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n \quad \left(r = \frac{\dim \mathcal{K}}{n} \text{ copies}\right)$$

and $\pi : M_n \to B(\mathcal{K}) \cong B(\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n) \cong M_r(B(\mathbb{C}^n)) \cong M_r(M_n)$ satisfies

$$\pi(A) = \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & A \end{bmatrix}.$$

Proof. By the above lemma $\mathcal{K} \cong \mathbb{C}^r \oplus \cdots \oplus \mathbb{C}^r \quad$ (n copies) such that

$$\pi : M_n \to B(\mathcal{K}) \cong B(\mathbb{C}^r \oplus \cdots \oplus \mathbb{C}^r) \cong M_n(B(\mathbb{C}^r)) = M_n(M_r)$$

satisfies

$$\pi([a_{ij}]) = \begin{bmatrix} a_{11}I & \cdots & a_{1n}I \\ \vdots & \ddots & \vdots \\ a_{n1}I & \cdots & a_{nn}I \end{bmatrix},$$

where I is identity in $\mathbb{C}^r$. So if we apply the canonical shuffle between $M_r(M_n)$ and $M_n(M_r)$ (Section 2.3) then we obtain the desired result. $\square$

Theorem 5.16. *Let $\phi : M_n \to M_k$ be completely positive. Then there exists at most nk linear maps $V_i : \mathbb{C}^k \to \mathbb{C}^n$ such that*

$$\phi(A) = \sum_i V_i^* A V_i \quad \textit{for all } A \in M_n.$$

Proof. We are given that $\phi : M_n \to M_k = B(\mathbb{C}^k)$ is completely positive. Let $(\phi, V, \mathcal{K})$ be Stinespring representation for ϕ. By the proof of Theorem 5.1 we know that $\dim\mathcal{K} \leq \dim(M_n \otimes \mathbb{C}^k) = n^2k$. Since $\pi : M_n \to B(\mathcal{K})$ is a unital $*$-homomorphism, by the above lemma, we can write

$$\mathcal{K} \cong \mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n \quad (r \text{ copies})$$

with $r \leq nk$ such that $\pi : M_n \to B(\mathcal{K}) \cong B(\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n) \cong M_r(B(\mathbb{C}^n)) \cong M_r(M_n)$ satisfies

$$\phi(A) = \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & A \end{bmatrix}.$$

$V : \mathbb{C}^k \to \mathcal{K} \cong \mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n$ can be represented as a column operator matrix

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_r \end{bmatrix}$$

for some $V_i : \mathbb{C}^k \to \mathbb{C}^n$. And so $V^* = [V_1^*, ..., V_r^*]$. Therefore

$$\phi(A) = V^*\pi(A)V = [V_1^*, ..., V_r^*] \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & A \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_r \end{bmatrix} = \sum_{i=1}^{r} V_i^* A V_i.$$

$\square$

5.3 Naimark's Dilation Theorem

Let G be a group and $\phi : G \to B(\mathcal{H})$ be a map. ϕ is said to be *completely positive definite* if for any finite number of elements $g_1, ..., g_n$ of G the matrix $\left[\phi(g_i^{-1}g_j)\right]$ is a positive operator in $M_n(B(\mathcal{H}))$. If G is a topological group then $\phi : G \to B(\mathcal{H})$ is called *weakly continuous* if

$$\langle \phi(g_\lambda)x, y\rangle \to \langle \phi(g)x, y\rangle \text{ for all nets } g_\lambda \to g \text{ in } G \text{ for all } x, y \text{ in } \mathcal{H}.$$

Similarly ϕ is called *strongly continuous* if

$$\|\phi(g_\lambda)x - \phi(g)x\| \to 0 \text{ for all net } g_\lambda \to g \text{ in } G \text{ for all } x, y \text{ in } \mathcal{H}.$$

and $*$-*strongly continuous* if ϕ is strongly continuous and

$$\|\phi(g_\lambda)^*x - \phi(g)^*x\| \to 0 \text{ for all net } g_\lambda \to g \text{ in } G \text{ and } x \text{ in } \mathcal{H}.$$

Theorem 5.17. *Let G be a topological group and let $\phi : G \to B(\mathcal{H})$ be weakly continuous and completely positive definite. Then there exists a Hilbert space $\mathcal{K}$, a bounded operator $V : \mathcal{H} \to \mathcal{K}$ and a unitary representation $\rho : G \to B(\mathcal{H})$ such that*

$$\phi(g) = V^*\rho(g)V \textit{ for all } g \in G.$$

In particular, ϕ is $*$-strongly continuous.

Proof. Let $F(G, \mathcal{H})$ be the vector space of finitely supported functions from G to $\mathcal{H}$. Define a sesquilinear form $[\cdot,\cdot]$ on $F(G, \mathcal{H})$ by

$$[f_1, f_2] = \sum_{g,g' \in G} \left\langle \phi(g^{-1}g')f_1(g'), f_2(g)\right\rangle_{\mathcal{H}}.$$

By a very similar argument used in the proof of Stinespring's Dilation Theorem $[\cdot,\cdot]$ is positive definite and so $\mathcal{N} = \{f : [f, f] = 0\}$ is a subspace of $F(G, \mathcal{H})$. Hence

$$\langle\cdot,\cdot\rangle = [\cdot + \mathcal{N}, \cdot + \mathcal{N}]$$

is an inner product on the quotient space $F(G, \mathcal{H})/\mathcal{N}$. Let $\mathcal{K}$ be the completion of this space.

Define $V : \mathcal{H} \to \mathcal{K}$ as follows. First, consider $V : \mathcal{H} \to F(G, \mathcal{H})$ defined by Vx as the function

$$Vx(g) = \begin{cases} x & g = e \\ 0 & g \neq e \end{cases}$$

where e is the unit of G. Clearly V is linear. We can define V from $\mathcal{H}$ to $F(G, \mathcal{H})/\mathcal{N}$ by $x \mapsto Vx + \mathcal{N}$. V is still linear and it is easy to show $\|V\| \leq \|\phi(e)\|^{1/2}$. And finally we may assume $V : \mathcal{H} \to \mathcal{K}$, since $F(G, \mathcal{H})/\mathcal{N}$ is contained in $\mathcal{K}$.

Let L_g be the left translation on $F(G, \mathcal{H})$ by g, that is, $L_g f$ is the function satisfying $L_g f(g') = f(g^{-1}g')$. Since the null space of L_g is contained in $\mathcal{N}$, we may assume L_g is defined on $F(G, \mathcal{H})/\mathcal{N}$ via $f + \mathcal{N} \mapsto L_g f + \mathcal{N}$. It can be shown that the quotient operator satisfies $\|L_g\| \leq 1$ so we may extend it linearly on $\mathcal{K}$, which we will still denote by L_g. We have that $L_e = I$ and $L_{gh} = L_g L_h$. Define

$$\rho : G \to B(K); \quad \rho(g) = L_g.$$

Clearly ρ is a unitary representation. Indeed, $\rho(g)^* = \rho(g^{-1})$. We must show that $\phi(g) = V^* \rho(g) V$ for all g in G. Fix $g \in G$ then for all x, y in $\mathcal{H}$,

$$\begin{aligned} \langle V^* \rho(g) V x, y \rangle_{\mathcal{H}} = \langle \rho(g) V x, V y \rangle_{\mathcal{K}} &= \langle L_g V x, V y \rangle_{\mathcal{K}} \\ &= \sum_{a, b \in G} \left\langle \phi(a^{-1}b) L_g V x(b), V x(a) \right\rangle_{\mathcal{H}} \\ &= \sum_{a, b \in G} \left\langle \phi(a^{-1}b) V x(g^{-1}b), V x(a) \right\rangle_{\mathcal{H}} \\ &= \langle \phi(g) x, y \rangle_{\mathcal{H}}. \end{aligned}$$

To show that ϕ is $*$-strongly continuous, it is enough to show that ρ is $*$-strongly continuous, since

$$\|\phi(g_\lambda)x - \phi(g)x\| \leq \|\rho(g_\lambda)(Vx) - \rho(g)(Vx)\| \, \|V^*\|,$$

and

$$\|\phi(g_\lambda)^* x - \phi(g)^* x\| \leq \|\rho(g_\lambda)^*(Vx) - \rho(g)^*(Vx)\| \, \|V^*\|.$$

We know that if a unitary net converges weakly to a unitary operator then it converges $*$-strongly. So we must show that ρ is weakly continuous. Since

$F(G,\mathcal{H})/\mathcal{N}$ is dense in $\mathcal{K}$ it is enough to show that ρ is weakly continuous on $F(G,\mathcal{H})/\mathcal{N}$. Let $g_\lambda \to g$ in G, then $f_1, f_2 \in F(G,\mathcal{H})/\mathcal{N}$ we have

$$\begin{aligned}
\langle \rho(g_\lambda) f_1, f_2 \rangle &= \sum_{a,b\in G} \big\langle \phi(a^{-1}b)\rho(g_\lambda) f_1(b), f_2(a) \big\rangle_{\mathcal{H}} \\
&= \sum_{a,b\in G} \big\langle \phi(a^{-1}b) f_1(g_\lambda^{-1}b), f_2(a) \big\rangle_{\mathcal{H}} \\
&= \sum_{a,b\in G} \big\langle \phi(a^{-1}g_\lambda^{-1}b) f_1(b), f_2(a) \big\rangle_{\mathcal{H}}.
\end{aligned}$$

The sum is finite and ϕ is weakly continuous. So the net converges to

$$\sum_{a,b\in G} \big\langle \phi(a^{-1}gb) f_1(b), f_2(a) \big\rangle_{\mathcal{H}} = \langle \rho(g) f_1, f_2 \rangle.$$

□

Remark 5.18. When $\phi(e) = I$ in Theorem 5.17, we may also assume that $\mathcal{K}$ contains $\mathcal{H}$ as a subspace and V turns out to be an imbedding. Indeed, this holds because $V^*V = \phi(e)$ So in this case we can write

$$\phi(g) = P_{\mathcal{H}}\rho(g)|_{\mathcal{H}} \qquad \text{for all } g \in G.$$

The triple $(\rho, V, \mathcal{K})$ is called a *Naimark representation for* ϕ. It is said to be a *minimal Naimark representation* when $\rho(G)V(\mathcal{H})$ has dense span in $\mathcal{K}$. Given a Naimark representation $(\rho, V, \mathcal{K})$, it is possible to make it minimal as we did for Stinespring representation. In the following proposition we show that two minimal Naimark representations are unitarily equivalent.

Proposition 5.19. *Let $(\rho_1, V_1, \mathcal{K}_1)$ and $(\rho_2, V_2, \mathcal{K}_2)$ be two minimal Naimark representations for $\phi : G \to \mathcal{H}$. Then there exists a unitary $U : \mathcal{K}_1 \to \mathcal{K}_2$ such that $UV_1 = V_2$ and $U\rho_1(\cdot)U^* = \rho_2$.*

Proof. The proof is similar to the proof of Proposition 5.7. If we define

$$U : \text{span}\rho_1(G)V_1(\mathcal{H}) \to \text{span}\rho_2(G)V_2 \text{ by } \sum_i \rho_1(g_i)V_1(h_i) \mapsto \sum_i \rho_2(g_i)V_2(h_i)$$

then U is well-defined and a surjective isometry. So it can be extended to a unitary map from $\mathcal{K}_1$ to $\mathcal{K}_2$. The required equalities are satisfied since they are satisfied on dense subsets of $\mathcal{K}_1$ and $\mathcal{K}_2$. □

A map $\phi : G \to B(\mathcal{H})$ is said to be *positive definite* if for every finite number of elements $g_1, ..., g_n$ in G and $\alpha_1, ..., \alpha_n$ in $\mathbb{C}$, the operator

$$\sum_{i,j=1}^{n} \bar{\alpha}_i \, \alpha_j \, \phi(g_i^{-1} g_j)$$

is positive. It is easy to show that a completely positive definite map is positive definite.

Remark 5.20. We note that if $\phi : G \to B(\mathcal{H})$ is positive definite and $x \in \mathcal{H}$ then the map $\phi_x : G \to \mathbb{C}$ defined by $\phi_x(g) = \langle \phi(g)x, x \rangle$ is completely positive definite. Indeed, if $g_1, ..., g_n$ in G and $\alpha_1, ..., \alpha_n$ in $\mathbb{C}$ are given then

$$\begin{aligned}
\left\langle \left[\phi_x(g_i^{-1} g_j)\right] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \right\rangle &= \sum_{i,j=1}^{n} \phi_x(g_i^{-1} g_j) \alpha_j \, \bar{\alpha}_i \\
&= \sum_{i,j=1}^{n} \langle \phi(g_i^{-1} g_j) x, x \rangle \alpha_j \, \bar{\alpha}_i \\
&= \left\langle \Big(\sum_{i,j=1}^{n} \bar{\alpha}_i \, \alpha_j \, \phi(g_i^{-1} g_j) \Big) x, x \right\rangle .
\end{aligned}$$

As an application of Naimark's Dilation Theorem we consider the special case $G = \mathbb{Z}^n$. Notice that any mapping from $\mathbb{Z}^n$ to $B(\mathcal{H})$ is weakly continuous since $\mathbb{Z}^n$ has discrete topology. We will show that there is a bijective correspondence between (completely) positive definite maps from $\mathbb{Z}^n$ to $B(\mathcal{H})$ and (completely) positive maps defined from $C(\mathbb{T}^n)$ to $B(\mathcal{H})$.

Let $\mathbb{Z}^n$ be the group defined as Cartesian product of n copies of $\mathbb{Z}$ and let $\mathbb{T}^n = \{(\alpha_1, ..., \alpha_n) : \alpha_i \in \mathbb{T}\}$. For $J = (j_1, ..., j_n) \in \mathbb{Z}^n$, we define $z^J : \mathbb{T}^n \to \mathbb{C}$ by $(\alpha_1, ..., \alpha_n) \mapsto \alpha_1^{j_1} \cdots \alpha_n^{j_n}$. It is easy to see that $z^J \in C(\mathbb{T}^n)$ for all $J \in \mathbb{Z}^n$ and we have $(z^J)^* = z^{-J}$ and $z^J z^K = z^{J+K}$.

Lemma 5.21. *There is a one to one correspondence between unital $*$-homomorphisms $\pi : C(\mathbb{T}^n) \to B(\mathcal{H})$ and (unital) unitary representations $\rho : \mathbb{Z}^n \to B(\mathcal{H})$ determined by $\pi(z^J) = \rho(J)$.*

Proof. Let $\pi : C(\mathbb{T}^n) \to B(\mathcal{H})$ be a unital $*$-homomorphism. Define $\rho : \mathbb{T}^n \to B(\mathcal{H})$ by $\rho(J) = \pi(z^J)$. Then clearly ρ is unital and for any J in $\mathbb{Z}^n$,

$$\rho(J)\rho(J)^* = \pi(z^J)\pi(z^J)^* = \pi(z^J)\pi(z^{-J})^* = \phi(1) = I.$$

Similarly $\rho(J)^*\rho(J) = I$, so ρ is unitary. Finally it is easy to show $\rho(J + K) = \rho(J)\rho(K)$. Thus, ρ is a unitary representation.

Conversely, let $\rho : \mathbb{T}^n \to B(\mathcal{H})$ be a (unital) unitary representation. By the Stone-Weierstrass Theorem the subalgebra span$\{z^J : J \in \mathbb{Z}^n\}$ is dense in $C(\mathbb{T}^n)$. We define π :span$\{z^J : J \in \mathbb{Z}^n\} \to B(\mathcal{H})$ by

$$\pi(a_1 z^{J_1} + \cdots + a_k z^{J_k}) = a_1\rho(J_1) + \cdots + a_1\rho(J_1).$$

Clearly the domain of π is selfadjoint. It is easy to show that ϕ is unital $*$-homomorphism. So it extends to a unital $*$-homomorphism on $C(\mathbb{T}^n)$ and we have $\pi(z^J) = \rho(J)$. □

Let $\mathcal{H}$ be a Hilbert space. A function $\gamma : \mathcal{H} \to \mathbb{C}$ is said to be *bounded quadratic* if it satisfies $\gamma(\alpha x) = |\alpha|^2\gamma(x)$ and $\gamma(x+y)+\gamma(x-y) = 2(\gamma(x)+\gamma(y))$ and there exists a constant M such that $|\gamma(x)| \leq M\|x\|^2$ for all $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{C}$.

In the proof of the following proposition we will use:

Lemma 5.22. *Let $\gamma : \mathcal{H} \to \mathbb{C}$ be bounded quadratic. Then there exists unique $T \in B(\mathcal{H})$ such that $\gamma(x) = \langle Tx, x\rangle$. Conversely, for any $T \in B(\mathcal{H})$, $x \mapsto \langle Tx, x\rangle$ is bounded quadratic.*

Proof. Define $[\cdot,\cdot] : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by

$$[x, y] = \frac{1}{2}(\gamma(x + y) - \gamma(x) - \gamma(y) - i\gamma(x - iy) + i\gamma(x) + i\gamma(y)).$$

It is not difficult to show that $[\cdot,\cdot]$ is a sesquilinear form on $\mathcal{H}$. So there exists $T \in B(\mathcal{H})$ such that $\langle Tx, y\rangle = [x, y]$. This means that $\langle Tx, x\rangle = \gamma(x)$ for all x (put $y = x$ in above equality). Uniquenes of T and the converse implication of the claim are easy to show. □

Proposition 5.23. *Let $\phi : \mathbb{Z}^n \to B(\mathcal{H})$ be a (completely) positive definite map. Then there exists a unique (completely) positive map $\psi : C(\mathbb{T}^n) \to B(\mathcal{H})$ satisfying $\psi(z^J) = \phi(J)$. Conversely, let $\psi : C(\mathbb{T}^n) \to B(\mathcal{H})$ be a (completely) positive map. Then $\phi : \mathbb{Z}^n \to B(\mathcal{H})$ defined by $\phi(J) = \psi(z^J)$ is a (completely) positive definite map.*

Proof. Let $\phi : \mathbb{Z}^n \to B(\mathcal{H})$ be completely positive definite. Let $(\rho, V, \mathcal{K})$ be a Naimark representation for ϕ. So $\rho : \mathbb{Z}^n \to B(\mathcal{K})$ is a unitary representation. Let $\pi : C(\mathbb{T}^n) \to B(\mathcal{K})$ be the unital $*$-homomorphism satisfying $\pi(z^J) = \rho(J)$. Set $\psi : C(\mathbb{T}^n) \to B(\mathcal{H})$ by $\psi(f) = V^*\pi(f)V$. Then ψ is completely positive by Remark 5.2 and it satisfies $\psi(z^J) = V^*\pi(z^J)V = V^*\rho(J)V = \phi(J)$. Uniqueness of ψ is easy to see.

The proof of the converse for the completely positive is similar.

Let $\phi : \mathbb{Z}^n \to B(\mathcal{H})$ be positive definite. If we fix $x \in \mathcal{H}$ then $\phi_x : \mathbb{Z}^n \to \mathbb{C}$ is completely positive definite. By the above part there exists unique completely positive map $\psi_x : C(\mathbb{T}^n) \to \mathbb{C}$ such that $\psi_x(z^J) = \phi_x(J)$ for all $J \in \mathbb{Z}^n$. Let $\mathcal{S} = \{z^J : J \in \mathbb{Z}^n\}$. We claim that for all $f \in \mathcal{S}$, $\gamma_f : \mathcal{H} \to \mathbb{C}$ defined by $\gamma_f(x) = \psi_x(f)$ is a bounded quadratic form. Indeed, if we write $f = a_1 z^{J_1} + \cdots + a_k z^{J_k}$ then

$$\begin{aligned} \gamma_f(x) = \psi_x(f) = \psi_x(a_1 z^{J_1} + \cdots + a_k z^{J_k}) &= a_1\rho_x(J_1) + \cdots + a_k\rho_x(J_k) \\ &= \langle \big(a_1\rho(J_1) + \cdots + a_k\rho(J_k)\big)x, x\rangle. \end{aligned}$$

So there exists unique $T_f \in B(\mathcal{H})$ such that $\langle T_f x, x\rangle = \gamma_f(x)$. It is easy to show $\psi : \mathcal{S} \to B(\mathcal{H})$ is linear. We also claim that ψ is positive ($\mathcal{S}$ is an operator system in $C(\mathbb{T}^n)$). If $f \geq 0$ and $x \in \mathcal{H}$ then

$$\langle \psi(f)x, x\rangle = \langle T_f x, x\rangle = \gamma_f(x) = \psi_x(f) \geq 0.$$

ψ also satisfies

$$\langle \psi(z^J)x, x\rangle = \psi_x(z^J) = \phi_x(J) = \langle \phi(J)x, x\rangle$$

for all $x \in \mathcal{H}$. So $\psi(z^J) = \phi(J)$ for all $J \in \mathbb{Z}^n$. The positive extension of ψ on $C(\mathbb{T}^n)$ is the desired map.

The proof for the converse of positive case follows from Theorem 4.14. $\square$

Corollary 5.24. *Let $\phi : \mathbb{Z}^n \to B(\mathcal{H})$ be a map. Then ϕ is positive definite if and only if it is completely positive definite.*

Proof. It is enough to show ($\Rightarrow$). Let $\psi : C(\mathbb{T}^n) \to B(\mathcal{H})$ be the positive map satisfying $\psi(z^J) = \phi(J)$. By Theorem 4.14, ψ is completely positive. This means that ϕ is completely positive definite. $\square$

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