

Some Adaptive Control Problems which Convert to a “Classical” Problem in Several Complex Variables

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Abstract

In this note we discuss the equivalence of what we refer to as bi- H^∞ control problems to certain problems of approximation and interpolation by analytic functions in several complex variables. We now introduce what we mean by bi- H^∞ control. The goal is to perform an H^∞ control design for a plant where part of it is known and a stable subsystem δ is not known, that is, the response of the plant at “frequency” s is $P(s, \delta(s))$. We assume that once our control (closed loop) system is running, we can identify the subsystem δ on line. Thus the problem is to design a function K offline that uses this information to produce a H^∞ controller via the formula $K(s, \delta(s))$. The challenge is to pick K so that the controller yields a closed loop system with H^∞ gain at most γ no matter which δ occurs. This is entirely a frequency domain problem, which has the flavor of some types of gain scheduling or control which adapts to slow variations. The bulk of this article is devoted to showing how several bi H^∞ control problems convert to two complex variable interpolation problems. These precisely generalize the classical (one complex variable) interpolation (AAK -commutant lifting) problems which lay at the core of H^∞ control. These problems are hard, but the last decade has seen substantial success on them in the operator theory community. In the most ideal of bi H^∞ cases these lead to a necessary and sufficient treatment of the control problem.

1 Introduction

The key mathematical optimization problem which underlies the beginnings of H^∞ control was solved by mathematicians working on the complex variables side of operator theory, c.f. [H87]. They produced extremely powerful and useful Nevanlinna -Pick, analytic function approximation, interpolation and commutant lifting formulas. H^∞ control theory soon had a life of its own and while the mathematicians in operator theory participated for about a decade, a large body of them and their students ultimately moved into Several Complex Variables, generalizations of Nevanlinna-Pick and related problems which were completely removed from systems engineering. Much of this effort in SCV focuses on approximation with functions analytic in several complex variables. This is an extremely hard subject and the results are not nearly so satisfying as in the case of one complex variable. However, there is some impressive progress.

This article presents an idea for applying some of these approximation results in SCV to bi H^∞ control problems. The hope is that specialists in the two rather widely separated topics arising here will pursue this area further. The author is a specialist in neither area.

2 The Mathematical Ingredients

2.1 Interpolation in SCV

Of particular interest is a theorem due to Jim Agler [A] and extensions of it [AM] [BT]. Let $Disk_n$ denote the unit polydisk $\{(v_1, \dots, v_n) : \|v_1\| \leq 1, \dots, \|v_n\| \leq 1\}$ in \mathbf{C}^n the space of n complex variables. Let $BH^\infty(n)$ denote the functions analytic on $Disk_n$ and bounded in supremum norm by 1. Also $M_{k \times \ell}$ denotes the $k \times \ell$ matrices, $BM_{k \times \ell}$ denotes the unit ball in $M_{k \times \ell}$. The *Nevanlinna -Pick type interpolation problem on $Disk_2$, the bi-disk, is:*

Given a finite set of points $\{(s_k, w_k)\}$ in the bi-disk and numbers z_k in the complex plane \mathbf{C} , and a positive number γ , find a function f in $\gamma BH^\infty(2)$ which takes the values z_k at the points $\{(s_k, w_k)\}$.

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Theorem 2.1 *The Nevanlinna-Pick interpolation problem on the bi-disk has a solution f in $\gamma BH^\infty(2)$ if and only if the Linear Matrix Inequality*

$$(\gamma^2 - \bar{z}_k z_\ell) = (1 - \bar{s}_k s_\ell)\Lambda_{k\ell} + (1 - \bar{w}_k w_\ell)\Omega_{k\ell} \quad (1)$$

Λ and Ω positive semidefinite matrices

has a solution. The solution is a rational function $f(\cdot, \cdot)$ in both variables s, w .

We point out a second form of the Agler inequality (1) which has fewer unknowns and so is probably more efficient for LMI software solution. This is:

Find Ω positive semidefinite, so that the selfadjoint matrix with k, ℓ th entry

$$\frac{(\gamma^2 - \bar{z}_k z_\ell)}{(1 - \bar{s}_k s_\ell)} - \frac{(1 - \bar{w}_k w_\ell)}{(1 - \bar{s}_k s_\ell)} \Omega_{k\ell} \quad (2)$$

is positive semidefinite. A condensed notation for this is

$$0 \leq S - P \circ_s \Omega, \quad (3)$$

where \circ_s denotes the Schur product (entry by entry) product of two matrices.

The proof of Agler's Theorem is in principle constructive, see [AM] [BT], though considerable effort would probably be required to refine it for computer implementation. While Theorem 2.1 is only stated for f a scalar valued function, it holds in a very similar form when f is a matrix valued function, see [BT]. However, this article involves only SISO systems, primarily for ease of pedagogy.

To derive the second form of the Agler inequality (2) from (1), just divide (1) by $(1 - \bar{s}_k s_\ell)$ and solve for $\Lambda_{k,\ell}$ to obtain that it equals (2).

2.2 Bi-disk model matching problem

Such interpolation problems arise naturally when one is trying to find a function f analytic on the bi-disk which has a supremum norm $\leq \gamma$ and which has the form

$$f = A + B^1 h_1 + B^2 h_2 \quad (4)$$

for some h_1, h_2 analytic on the bi-disk. This is because in the typical situation on the bi-disk the zero set $Z_g := \{(v, w) : g(v, w) = 0\}$ of a function g in $BH^\infty(2)$ is empty or is the union of finitely many one complex dimensional surfaces. Thus B^1 and B^2 have zero sets intersecting (transversally) in only a finite number of points $\{s_k, w_k\}$ and at these points f is constrained to take values

$$f(s_k, w_k) = z_k := A(s_k, w_k) \quad \forall k = 1, \dots, v. \quad (5)$$

regardless of what h_1, h_2 are. An expert in H^∞ control will recall that this problem on the ordinary disk (as opposed to the bi-disk) will have $B^2 = 0$ and is the model matching problem. Consequently, we shall call problem (4) the bi-disk model matching problem.

A symmetry which occurs in engineering functions, which we shall call **bi-real**, is

$$g : Disk_2 \rightarrow \mathbf{C} \quad \text{satisfies} \quad g(\bar{s}, \bar{w}) = \overline{g(s, w)} \quad \forall s, w \in \mathbf{C}.$$

Corollary 2.2 *Given functions A, B^1, B^2 analytic on the bi-disk having B^1, B^2 whose zero sets intersect transversally in a finite number of points $\{(s_k, w_k)\}$ and meet a technical (generically true general position) hypothesis on how they cross the boundary of $Disk_2$. Let $z_k := A(s_k, w_k)$. The bi-disk model matching problem has a solution f in $BH^\infty(2)$ (and bi-rational) if and only if the Linear Matrix Inequality (2) has a solution. If A, B^1, B^2 are bi-real, then we may take f to be bi-real.*

Proof: This is an immediate consequence of Agler's Theorem up to the part involving the bi-real property. If A, B^1, B^2 are bi-real, then the set $\mathcal{I} := \{(s_k, w_k, z_k) : \text{all } k\}$ of interpolating data is unchanged by applying the conjugation symmetry

$$(s, w, z) \rightarrow (\bar{s}, \bar{w}, \bar{z}).$$

Thus if $f \in BH^\infty(z)$ satisfies the interpolation constraints given by \mathcal{I} , then $\tilde{f}(s, w) := \frac{1}{2}[f(\bar{s}, \bar{w}) + f(s, w)]$ does also and solves the bi-disk interpolation problem from the same \mathcal{I} .

3 The Bi-variable Control Problem

Consider the rather standard picture of a controlled discrete time system with an uncertainty δ . In our approach the uncertainty is presented in terms of its discrete time frequency response function (FRF), denoted $\delta(s)$.

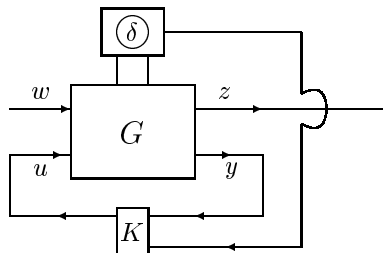


Figure 1: bi H^∞ controller

We shall take δ to be an uncertainty which is a *stable SISO system whose supremum norm over the unit disk is less than or equal to d_0* . We wish to design a controller K offline and the ground rules presume that **once the system is running the controller can identify δ** , that is, we take K to have the form

$$K(s, \delta(s)).$$

We call this a **bi-variable controller**. We shall always take w and z to be 1 dimensional signals and δ to be SISO, but the controller K can take values which are matrices of any dimension. We shall take the convention that the FRF of a stable system is analytic inside the disk, in order to be consistent with the mathematical part.

Our approach, under the assumption

$$\text{(MANY } \delta\text{'s) } \{ (s, \delta(s)) : \text{all } \delta \text{ which arise and all } |s| \leq 1 \} \text{ contains the boundary of } d_0 \text{ Disk}_2,$$

in principle gives a *necessary and sufficient theory for control design*. In practice there often will be compromises in getting particular problems to meet the (MANY δ 's) condition, but these compromises occur later in the design rather than at the core of the theory, as happens with other approaches to adapting to plant variations. This may offer some advantages. For example, if (MANY δ 's) fails, then a graphical display (frequency by frequency) of the set above would tell how much it fails and at what frequencies. This might be informative.

It is a simple variation of our theory to allow d_0 to be a smooth non-negative function of s (which has a spectral factorization); this allows a little flexibility. More elaborate variations might involve rescaling by a class of frequency dependent conformal maps (tricky business), c.f. Thanos Sideris's unpublished thesis or [SS86].

3.1 Physical set up

We mention a few physical situations where the bi H^∞ setup might arise. Our discussion has the tone of gain scheduling or adaptive control which adapts to a slowly varying subsystem or environment. More detailed studies and richer classes of examples are best left to engineers.

Suppose we build a machine G and wish to sell it to many different customers who will be docking or loading G with many different loads δ . When we sell G we must also provide a controller along with it which works for a broad range of loads. The customer before he attaches a load δ to G must determine its response function δ by modeling or with a spectrum analyser. For such situations a bi-variable controller $K(\cdot, \cdot)$ is a reasonable type of controller for our company to provide along with G .

Many variations on implementation are possible and we sketch one now primarily for tutorial reasons. Once the customer determines δ , either by finding model parameters δ_i or as spectrum analyser data, the (discrete time) function δ could be stored in many ways, for example, as data $\{e^{i\theta_k}, \delta(e^{i\theta_k})\}$ on a grid on $[0, 2\pi]$. Our company's controller algorithm is software which contains a function $K(s, w)$, say stored as a two variable rational function. The controller C which is implemented for δ loaded onto G has frequency response function on the $\{e^{i\theta_k}\}$ grid equal to

$$C(e^{i\theta_k}) = K(e^{i\theta_k}, \delta(e^{i\theta_k}))$$

which can be numerically discrete Laplace transformed to produce the impulse response $c(t)$. The controller can be implemented by convolving c with the measurement y .

Moreover, if loads are passive, they meet the assumption $\delta \in BH^\infty(1)$, and if assumption (MANY δ 's) below holds, the theory is in principle necessary and sufficient. A situation which gives some conceptual perspective is that of a turning mill; the turning mill (together with weights the designer chooses to achieve performance) corresponds to G . Work of different masses and elasticities is placed on the turning mill. The load is passive, so its FRF lies in $BH^\infty(1)$.

There is considerable freedom in how we use δ in Figure 1. We could have a plant which is linear and completely known except for a SISO subsystem g which is uncertain at the time of the controller design. If all uncertain subsystems g which occur will be close to a nominal g_0 in the sense

$$|g(e^{i\theta}) - g_0(e^{i\theta})| \leq d_0(e^{i\theta}) \quad \text{for all } \theta \quad (6)$$

for a predetermined function d_0 , then take $\delta := g - g_0$ to put this situation into the form of Figure 1. Bi-variable controllers seem reasonable for this type of control.

To fix a more concrete image in the reader's mind consider two carts joined by a spring:

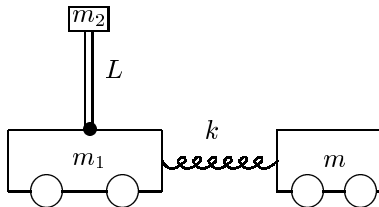


Figure 2:

Cart₁ is activated by torques on the wheels (this is the control input u to the system), and it supports an inverted pendulum of length L which we wish to keep vertical. Let us say we build cart₁, so everything about it is known; L, m_1, m_2 etc. Now cart₂ is made by someone else, and the controller we sell with cart₁ must be prepared to handle many spring sizes, masses as well as dampings. Our assumption about the cart₂s which can occur is that there is a nominal one, with FRF equal to g_0 , and that in the co-ordinates we are using each possible cart₂ has FRF g whose distance from g_0 is less than some function d_0 , that is,

$$|g_0(e^{i\theta}) - g(e^{i\theta})| \leq d_0(e^{i\theta}) < \infty \quad s \in \text{Disk}. \quad (7)$$

Note that since all carts are stable $\delta := g_0 - \tilde{g}$ is stable. This set up fits the mold of Figure 1 with δ as above and with G denoting the linearization of cart₁ connected with nominal cart₂, together with a port for the additive uncertainty δ , and together with weights emphasizing performance the designer desires.

The key sets for testing the (MANY δ 's) condition are:

$$\Delta_{d_0} := \{(s, w) : |s| \leq 1, |w| \leq |d_0(s)|\} \subset \mathbf{C} \quad \text{and}$$

$$V_{\text{cart}_2} := \{(s, \delta_{\text{cart}_2}(s)) : |s| \leq 1, \text{cart}_2 \in \text{Admissible}\} \subset \Delta_{d_0}.$$

Here d_0 denotes the minimum phase spectral factor of d_0^2 . The fact $V_{\text{cart}_2} \subset \Delta_{d_0}$ follows from (7) cart₂ being stable, and this means that the problem fits in our bi H^∞ setup. On the other hand if the reverse condition

$$V_{\text{cart}_2} \supset \text{Boundary} \Delta_{d_0} \quad (8)$$

holds, the bi H^∞ methods of this paper are (up to computational difficulties and some issues of approximation) necessary and sufficient.

When condition (8) fails our bi H^∞ approach becomes conservative, but it is easy, for each s in the right half plane, to graph cross sections

$$V_{\text{cart}_2}|_s \quad v.s. \quad \Delta_{d_0}|_s$$

This compares the uncertain loads which actually arise to the uncertain loads (of possibly more general configuration) which our controller handles.

3.2 Manufacturing Variability

In manufacturing processes there is great variability. Thus a mathematical model of the ideal product is marred by parasitics which actually occur. Suppose for the moment we manufacture and control devices which ideally consist of two *decoupled* linear systems G_1 and G_2 which in practice contain a parasitic influence δ of subsystem 1 on subsystem 2. Over the course of the devices commercial production δ may vary greatly. If after each device is built we can determine δ , then this problem would be a reasonable candidate for bi H^∞ control. By contrast if subsystem 2 also influences subsystem 1, then this adds a second uncertain system δ_2 , which if unrelated to δ , leads to a tri-disk H^∞ interpolation problem¹, namely, one based on the tri-disk $\{(s, w_1, w_2)\}$. This is not a fully solved problem. Note that one might treat δ_2, δ differently. Namely, study constraints of the form $|\delta_2|^2 + |\delta|^2 \leq d_0^2$. This has the advantage of producing a solvable H^∞ interpolation problem. Such extensions of bi H^∞ theory are indicated in Section 6.

4 Converting bi H^∞ Control to Bi-disk Model Matching and Interpolation

This paper focuses on the offline design problem of finding a suitable bi H^∞ controller $K(\cdot, \cdot)$. Henceforth, set the uncertainty size parameter d_0 equal to 1. The **standard bi H^∞ control problem** is:

Find a bi H^∞ controller

$$K : \mathbf{C} \times \mathbf{C} \rightarrow M_{pq}$$

so that for each choice of $\delta \in BH^\infty(1)$, the controller $c(s) = K(s, \delta(s))$ makes the closed loop system in Figure 1

- (1) *internally stable*
- (2) *have gain less than or equal to γ .*

4.1 Stability

The next lemma shows how one can routinely go from stability for each $\delta \in H^\infty(1)$ to "stability" on $Disk_2$. Bounded analyticity on $Disk_2$ we often call **bi-stability**.

Lemma 4.1 *If $f(s, \delta(s))$ is analytic and uniformly bounded for all $s \in Disk$ and all ¹ $\delta \in BH^\infty(1)$, then $f(s, w)$ is analytic and uniformly bounded on the bi-disk.*

One consequence of this lemma is that stability of the closed loop system for each δ is equivalent to the two variable transfer function $T(\cdot, \cdot)$ lying in $H^\infty(2)$.

4.2 H^∞ gain

Another objective of control is to achieve desired performance. This means find K in Figure 1 which makes

$$|T(s, \delta(s))| \leq \gamma \quad \text{for all } s \in Disk_1 \text{ and } \delta \in BH^\infty(1).$$

This is equivalent to

$$|T(s, w)| \leq \gamma \quad \text{for all } (s, w) \in Disk_2.$$

γ is called the **gain** of the closed loop system.

4.3 Internal stability

Recall that internal stability of the system in Figure 1 means that when input u is added to the u loop and input i_y to the y loop, the nine closed loop transfer functions from w, i_u, i_y to z, u, y are stable (see Chapter 4.2, Lemma 1 [F87]). Bi- H^∞ control requires that all of these be stable for all $\delta \in BH^\infty(1)$. Lemma 4.1 then gives that each of these are bi-stable. The conversion of this to interpolation conditions is involved, so we merely show what happens for stable plants (an easy situation). In Section 5 we describe bi- H^∞ classical control for plants which may or may not be stable.

¹When $f(\cdot, \cdot)$ is rational the set of δ 's for which this must hold need not be the entirety of $BH^\infty(1)$, but merely be so big as to satisfy condition (MANY δ 's). To prove this as well as the lemma use Hartog's Theorem (see [Kr82]).

4.4 Model matching for stable G

Denote the transfer function G of the plant by

$$G := \begin{pmatrix} A(s, \delta(s)) & B(s, \delta(s)) \\ C(s, \delta(s)) & D(s, \delta(s)) \end{pmatrix} \quad (9)$$

Here the functions A, B, C, D are rational in two variables and take values as follows:

$$\begin{aligned} A(s, w) &\in \mathbf{C} & B(s, w) &\in M_{1p} \\ C(s, w) &\in M_{q1} & D(s, w) &\in M_{qp} \end{aligned}$$

and the controller satisfies $K(s, w) \in M_{pq}$. Now we assume that G is bi-stable to give a simple illustration of the role of model matching.

Lemma 4.2 *Suppose G is in $H^\infty(2)$, that is, G is bi-stable. The bi H^∞ controllers $K(\cdot, \cdot)$ making the closed loop system in Figure 1 internally stable for all $\delta \in BH^\infty(1)$ correspond precisely to the closed loop transfer functions T of general model matching form*

$$T = A + BQC \quad (10)$$

for some M_{pq} valued function Q with entries in $H^\infty(2)$.

Idea of Proof: For each fixed δ the closed loop transfer function in Figure 1 is

$$T = \mathcal{F}_G(K) := A + BK(1 - DK)^{-1}C. \quad (11)$$

This gives representation (10) with Q , often called the closed loop controller, taken to be

$$Q = K(1 - DK)^{-1}. \quad (12)$$

For this we need stability of Q , which follows from internal stability, since Q is the transfer function from i_y to u . Conversely, given $Q \in H^\infty(1)$ construct K by $K = (1 + QD)^{-1}Q$. Because of bi-stability of G we have internal stability of the closed loop system as is standard in H^∞ control. Since these stability facts hold for all $\delta \in H^\infty(1)$. We have by Lemma 4.1 that they hold with bi-stability replacing stability.

4.5 Converting Model Matching to Interpolation

In the previous section we reduced control problems to bi-disk model matching. In this and in the next section we convert several model matching problems to interpolation, thereby showing how Agler's Theorem can be used to solve them. These examples hopefully illustrate our main idea. Here we shall always take $\gamma = 1$.

4.5.1 Two controls – one measurement

When the plant has 1 measurement output but m control inputs, we take Q to have a single input and m outputs

$$Q = \begin{pmatrix} Q_1 \\ \vdots \\ Q_m \end{pmatrix}. \quad (13)$$

This special structure of the plant makes T have the form $T = A + BQC$ which when written out in co-ordinates is

$$T = A + B_1Q_1C + \cdots + B_mQ_mC. \quad (14)$$

For the standard problem to have a solution it is necessary that $A, B_1, \dots, B_m, C \in H^\infty(2)$.

When $m = 2$ and when C has no zeros on the bi-disk, this is precisely what was called a bi-disk model matching problem in Section 2.2, so it fits the mold of Agler's Theorem, since having a finite number of interpolating constraints is the typical situation. Thus we can apply LMI (2) it to obtain *a test to determine the limitations of control* for the model matching problem. Construction of the controller requires implementing Theorem 2.1 constructively, which was discussed in Section 2.1

4.5.2 One measurement—one control

At last we mention the traditional H^∞ control problem. Here we have $T = A + BQC$. Thus the interpolating condition becomes

$$T = A \quad \text{on } Z_{BC}$$

where $Z_{BC} := \{(s, w) : B(s, w)C(s, w) = 0\}$. This typically is a one complex dimensional curve in \mathbf{C}^2 or a finite set of such curves in \mathbf{C}^2 . Thus we do not have a finite set of interpolating constraints, a situation to be discussed in Section 5.1

5 Classical control

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One can possibly follow the calculation in this section without having read Sections 4 and 4.5. We treat a special case of the standard problem described by the familiar figure below. Assume P has poles and zeros of degree 1.

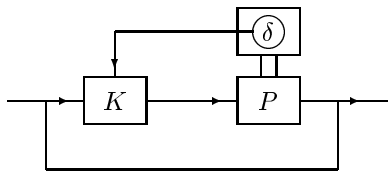


Figure 3:

5.1 The method

Here we layout the procedure; later we give more details.

The closed loop transfer function is $T(s, \delta(s)) = P(s, \delta(s))Q(s, \delta(s))$ where

$$Q = K(1 + PK)^{-1}$$

and the controller has the form $K(s, \delta(s))$. The usual performance function is the mixed sensitivity function which has performance level $\tilde{\gamma}$ for all δ in $BH^\infty(1)$ if and only if

$$\sup_{(s,w) \in Disk_2} [|\rho_1(s)(1 - T(s, w))|^2 + |\rho_2(s)T(s, w)|^2] \quad (15)$$

is less than or equal to $\tilde{\gamma}^2$. Here ρ_1, ρ_2 are minimum phase (outer) weight functions, i.e., they have no poles or zeroes in the disk. Some algebra (see Section 5.1.2 for formulas) and factoring convert this inequality to one of the form

$$\mathcal{T}(s, w) := \alpha(s) + \beta(s)T(s, w) \quad \alpha, \beta \in H^\infty(1) \quad (16)$$

$$|\mathcal{T}(s, w)| \leq \gamma \quad \text{all } (s, w) \in Disk_2$$

where T in $H^\infty(2)$ satisfies

$$\begin{aligned} T &= 0 \quad \text{on } Z_P \quad \text{the zero set of } P(\cdot, \cdot), \text{ and} \\ T &= 1 \quad \text{on } Z_{P^{-1}} \quad \text{the pole set of } P(\cdot, \cdot). \end{aligned} \quad (17)$$

See Section 5.1.1 for the proof. Thus the interpolation problem associated with the classical control problem is:

Find $\mathcal{T} \in \gamma BH^\infty(2)$ satisfying the interpolation constraints

$$\mathcal{T}(s, w) = \alpha(s) \quad \text{for } (s, w) \in Z_P \cup Z_\beta$$

$$\mathcal{T}(s, w) = \alpha(s) + \beta(s) \quad \text{for } (s, w) \in Z_{P^{-1}}.$$

For convenience set $\mu := \alpha + \beta$.

The sets Z_P and $Z_{P^{-1}}$, being the zero sets of rational functions on \mathbf{C}^2 , typically are one complex dimensional curves in \mathbf{C}^2 or a finite set of such curves in \mathbf{C}^2 . Thus we do not have a finite set of interpolating constraints. Agler's Theorem says that for a solution to the H^∞ control problem to exist Agler's inequality applied to every finite set of points satisfying this interpolation constraint must have a solution. That is, Agler's inequality is a necessary

condition for H^∞ control. How close this is to sufficient is not proved. However, it is quite likely that as one takes larger and larger finite sets F of interpolating conditions, as in (5), then the performance bounds γ_F obtained from Agler's inequality approach being sufficient. What size F is required for a specified accuracy is an issue for computer experiment.

Next we give more detail, but after that in Section 5.2 specialize our results to an important case.

5.1.1 Proof of (17)

The derivation is a trivial extension of the common one variable H^∞ control fact (see Lemma 7.2.1 [HMer 98]) which in this context says that internal stability is equivalent to: for each $w \in Disk_1$ the functions of s, w

$$T, Q = \frac{PK}{1+PK}, \quad S = T - 1, \quad PS = \frac{1}{1+PK}P$$

are in $H^\infty(1)$ with respect to the variable s . This by Lemma 4.1 implies T, Q, S, PS as functions of the two variables s, w are in $H^\infty(2)$. Now $P(T-1) = PS \in H^\infty(2)$ and $PQ = T$ both being in $H^\infty(2)$ are equivalent to the interpolation condition (17).

5.1.2 Formulas for α, β in equation (16)

To implement the method one needs formulas for α and β in the representation (16) for \mathcal{T} . These formulas are as follows. Select rational α, β, c_0 in $H^\infty(1)$ with $c_0^{-1} \in H^\infty(1)$ which satisfy spectral factorization conditions

$$|c_0|^2 = \left[\tilde{\gamma}^2 - \frac{|\rho_1|^4}{|\rho_1|^2 + |\rho_2|^2} \right] \frac{1}{\gamma^2} \quad (18)$$

$$|\alpha|^2 = \frac{|\rho_1|^4}{|\rho_1|^2 + |\rho_2|^2} \frac{1}{|c_0|^2} \quad (19)$$

$$|\beta|^2 = \left[|\rho_1|^2 + |\rho_2|^2 \right] \frac{1}{|c_0|^2} \quad (20)$$

$$\bar{\alpha}\beta \text{ is real valued} \quad (21)$$

on the $j\omega$ axis. That c_0^{-1} has no poles on the $j\omega$ -axis is guaranteed by picking $\tilde{\gamma}$ to satisfy

$$\tilde{\gamma}^2 > \sup_w \frac{|\rho_1|^4}{|\rho_1|^2 + |\rho_2|^2}.$$

Once this constraint is satisfied one can always find c_0, α, β meeting all of the conditions. To see this we only need to accomodate (21), since the rest are routine spectral factorizations. If rational $\alpha_0, \beta_0 \in H^\infty(1)$ meet conditions (18), (19), (20), then so do $\psi\alpha_0$ and $\phi\beta_0$ for any all pass functions $\psi, \phi \in H^\infty$. Choose such ψ, ϕ to make

$$\text{phase } \bar{\psi}\phi = -\text{phase } \bar{\alpha}_0\beta_0,$$

which is always possible and set $\alpha = \bar{\psi}\alpha_0, \beta = \phi\beta_0$.

That our formulas for α, β do yield the equivalence of (15) and (16) is easy to check by straightforward algebra.

5.2 Plants with linear fractional uncertainty

Suppose that the SISO plant $P(s, \delta(s))$ has a linear fractional dependence on δ , namely there are rational functions a, b, c, d meeting the non-degeneracy condition $a(s)b(s) - c(s)d(s) \neq 0$ so that

$$P(s, w) = \frac{a(s) + b(s)w}{c(s) + d(s)w}$$

Suppose the uncertainty $\delta \in d_0BH^\infty(2)$. Then

$$\begin{aligned} Z_{P^{-1}} &= \left\{ \left(s, \frac{-c(s)}{d(s)} \right) : |s| \leq 1 \text{ and } \left| \frac{c(s)}{d(s)} \right| \leq d_0 \right\} \\ Z_P &= \left\{ \left(s, \frac{-a(s)}{b(s)} \right) : |s| \leq 1 \text{ and } \left| \frac{a(s)}{b(s)} \right| \leq d_0 \right\} \end{aligned}$$

and non-degeneracy ensures the two sets do not intersect. Let

$$v_{pole}(s) := \frac{-c(s)}{d(s)} \quad \text{and} \quad v_{zero}(s) := \frac{-a(s)}{d(s)}.$$

The cleanest way to express the Agler inequality (2) is in terms of $M_{2 \times 2}$ valued functions on \mathbf{C}^4 , and their restriction to a certain subset \mathcal{R} of \mathbf{C}^4 . Define the set \mathcal{R} to be all $(p, r) \in \mathbf{C}^2$ satisfying

$$|p| \leq 1 \text{ and } |v_{pole}(p)| \leq d_0, \quad |r| \leq 1 \text{ and } |v_{zero}(r)| \leq d_0.$$

Note \mathcal{R} is completely determined by the plant model.

The unknown is the restriction to $\mathcal{R} \times \mathcal{R}$ of the $M_{2 \times 2}$ valued Ω defined on \mathbf{C}^4

$$\Omega(\{p, r\}, \{q, s\}) := \begin{pmatrix} \Omega^{11}(p, q) & \Omega^{12}(p, s) \\ \overline{\Omega^{12}(q, r)} & \Omega^{22}(r, s) \end{pmatrix}$$

It is positive semidefinite in the following sense

$$\int_{\mathcal{R} \times \mathcal{R}} \Omega(\{p, r\}, \{q, s\}) \overline{f(p, r)} f(q, s) dp dq dr ds \geq 0 \quad (22)$$

for all integrable $f : \mathcal{R} \rightarrow \mathbf{C}^2$. Obviously, a necessary condition which is practical to check is: select any finite subset of \mathcal{R} which is not too large, plug it into the formulas above and thereby get a finite dimensional LMI.

The coefficients in the Agler Inequality are based on the two $M_{2 \times 2}$ valued functions on \mathbf{C}^4

$$S(\{p, r\}, \{q, s\}) := \begin{pmatrix} \frac{\gamma^2 - \overline{\mu(p)}\mu(q)}{1 - \overline{p}q} & \frac{\gamma^2 - \overline{\mu(p)}\alpha(s)}{1 - \overline{p}s} \\ \frac{\gamma^2 - \overline{\alpha(r)}\mu(q)}{1 - \overline{r}q} & \frac{\gamma^2 - \overline{\alpha(r)}\alpha(s)}{1 - \overline{r}s} \end{pmatrix} \quad (23)$$

which is completely determined by the performance specs μ and α and $P(\{p, r\}, \{q, s\})$ defined to be

$$\begin{pmatrix} \frac{d_0^2 - \overline{v_{pole}(p)}v_{pole}(q)}{1 - \overline{p}q} & \frac{d_0^2 - \overline{v_{pole}(p)}v_{zero}(s)}{1 - \overline{p}s} \\ \frac{d_0^2 - \overline{v_{zero}(r)}v_{pole}(q)}{1 - \overline{r}q} & \frac{d_0^2 - \overline{v_{zero}(r)}v_{zero}(s)}{1 - \overline{r}s} \end{pmatrix} \quad (24)$$

which is completely determined by the plant model. The Agler Inequality ² is

$$0 \leq S - P \circ_s \Omega \quad (25)$$

restricted to \mathcal{R} , where recall that \circ_s denotes the Schur product (entry by entry product) of two matrices and Ω must be positive semidefinite in the sense (22). It can be proved using a normal families argument, that the function $\Omega(\{p, r\}, \{q, s\})$ may be taken to be anti-analytic in $\{p, r\}$ and analytic in $\{q, s\}$. That is if a nonnegative solution Ω to (25) exists, then one with these analyticity properties exists.

It might be helpful to note how the classical control problem with no uncertainty fits in formally. This is the case where $d_0 = 0$. In this case the terms involving Ω drop out, so what remains is $S = \Lambda \geq 0$. Since $d_0 = 0$ the set \mathcal{R} is finite, namely, the plant's pole zero set,

$$\mathcal{R} = \mathcal{R} := \{(p, r) \in Disk_2 : P(p) = \infty \text{ and } P(r) = 0\}. \quad (26)$$

By inspection one sees that S restricted to \mathcal{R} is a Pick matrix; indeed we have the old fashion H^∞ control Nevanlinna-Pick solution.

The inequalities (25) suggest an interpretation when $d_0 > 0$. If $\Omega = 0$ is a solution, then

$$0 \leq S \quad \text{on } \mathcal{R} \quad (27)$$

Each choice of δ gives $\mathcal{R}(s, \delta(s))$ with a finite zero pole set \mathcal{R}_{F_δ} and the statement $0 \leq S$ on \mathcal{R}_{F_δ} is equivalent to the statement that an H^∞ controller meeting the specs exists (for this choice of δ). When $d_0 > 0$ many δ exist, so (27) implies much stronger conditions like $0 \leq S$ on $\mathcal{R}_{F_{\delta_1} \cup F_{\delta_2} \cup F_{\delta_3}}$, conditions so strong as to be likely unmeetable (a statement which can actually be formalized into a theorem). Fortunately, (25) is more lenient than (27) because $-\mathcal{P} \circ_s \Omega$ can be used to lessen the $0 \leq S$ constraint.

²To help one see how this is derived note that the "first form" of Agler's inequality (1) has component

$$(\gamma^2 - \overline{\mu(p)}\mu(q)) = (1 - \overline{p}q)\Lambda^{11}(p, q) + (d_0^2 - \overline{v_{pole}(p)}v_{pole}(q))\Omega^{11}(p, q).$$

due entirely to interpolation constraints at poles. This clearly gives rise to the upper diagonal block of inequality (25). The lower diagonal block corresponds to interpolation constraints at zeroes and the off diagonal block accounts for interactions.

6 More General Math Theorems

This article has focused on model matching problems, on the bi-disk. That is because this is one situation for which there is an extremely clean solution. There are other results which have potential and we direct the curious reader to them.

Of direct engineering interest is the case where we do H^∞ interpolation on domains in \mathbf{C}^{1+U} of the form $\Omega = Disk_1 \times \Delta$, where Δ is a domain in \mathbf{C}^U . For example, this occurred in Section 3.2 with Δ equal to the ball in \mathbf{C}^2 , denoted $Ball_2$. Fortunately, exactly which such Ω give problems having an LMI solution of the "nicest possible form" is determined in A. T. Tomerlin's thesis under S. McCullough, cf. [T prep]. In fact the Section 3.2 case, $\Omega = Disk_1 \times Ball_2$, has a very nice LMI solution.

An example of a Domain which fails the "niceness test" is $\Delta = Disk_2$ the bi-disk, then $\Omega = Disk_1 \times Disk_2 =: Disk_3$, the tri-disk. However, Jim Agler's [A] results on the bi-disk generalize to any polydisk by formulating the interpolation problem with exactly the same constraints (5) as before, but with a different norm. Agler's norm $Norm_n$ on H^∞ of the $Disk_n$ is stronger than the supremum norm, and is not classical (unless $n = 1$ or 2 in which case it is the supremum norm). However, $Norm_n$ has the property that the solution to the associated interpolation problem reduces to an LMI of dimension comparable to the number of interpolation constraints, and so is potentially solvable. The potential value of this to control theorists is that we have solvable problems and an engineer who studies $Norm_n$ might find it approximates something physical. Also of possible interest is work by Alpay, J. Ball, V. Bolotnikov, and T. Trent (see [BT], [ABB]) which puts Agler's results and other SCV results in statespace co-ordinates.

Secondly, results of Cotlar and Sadosky [CS94] [CS96] on interpolation and approximation problems on the polydisk use a performance measure called the "Bounded Mean Oscillation" norm (**BMO norm**). For a survey see [S98]. They show that various BMO norms give elegant results. Their formulas are quite possibly computable, so the main issue is whether or not BMO norms are close to something physical.

7 Thanks

I am most grateful to Mihai Putinar for assistance with technical issues in SCV connected with the general position assumption and to Joe Ball for help with several mathematical points. Thanks are certainly due to Cora Sadosky for constantly urging me to find an application for this realm of SCV hotly pursued these days by operator theorists. The author is grateful to Andy Packard and Hiroshi Ito for his explanations of LPV control and LFT based control. Also thanks to Hidenori Kimura for pointing out typos in the original manuscript and for discussions. Thanks are due to Bob Bitmead for suggesting the example in Section 3.2

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9 Other Plant Models

9.1 Many uncertainties

We start with an example. Suppose the uncertainty structure is

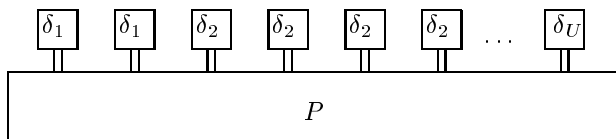


Figure 4: Lots of uncertainty

where the large system acts in a linear fractional way on each individual δ . Then the plant is described by a rational function

$$P(s, w_1, \dots, w_U)$$

of $1 + U$ complex variables w_ℓ , in that its response to “frequency” s is

$$P(s, \delta_1(s), \dots, \delta_U(s)).$$

The dependence of P on w is not linear fractional when there are repeating δ in Figure ??, since this repetition makes $P(s, w)$ depend on w in a complicated (though rational) way. If δ_ℓ is restricted only by $\delta_\ell \in BH^\infty(1)$, then bi H^∞ control for this circumstance leads to, model matching and interpolation problems on $Disk_{1+U}$.

More generally if there is a set $\Delta \subset \mathbf{C}^U$, so that $(\delta_1(\cdot), \delta_2(\cdot), \dots, \delta_U(\cdot))$ is a function analytic on $Disk_1$ with values in Δ , then bi H^∞ control leads to model matching and interpolation problems on the domain

$$\Omega := Disk_1 \times \Delta \quad \text{in } \mathbf{C}^{1+U}.$$

As was said in Section 6 exactly which Δ s give problems having an LMI solution of the “nicest possible form” is determined in A. T. Tomerlin’s thesis under S. McCullough, cf. [T prep]. For example, the problem which occurred in Section 3.2 with Δ equal to the ball in \mathbf{C}^2 , denoted $Ball_2$ has a very nice LMI solution, easily derivable by specialists in the area.

We leave this as an exercise to the reader.

9.2 When D is not in $H^\infty(2)$

Now we move to the standard problem when D is not assumed to be in $H^\infty(2)$. Then the form $Q = K(1 + DK)^{-1}$ of Q with poles and zeroes of D and K not being allowed to cancel, forces $Q(s, w)$ to be 0 on the zero set

$$Z_{D^{-1}} = \{(s, w) : D(s, w) = \infty\}.$$

Typically this is a continuum of points. Thus when D has a pole we must add an interpolation condition

$$T = A \quad \text{on } Z_{D^{-1}} \tag{28}$$

to the ones which already treated in the model matching section. Henceforth, in our discussion we shall treat only the case where D has poles of degree one.

Actually, the interpolation condition on $Z_{D^{-1}}$ is stronger than this obvious one, as can be read off from the identity

$$\begin{aligned} T &= A + BD^{-1}DK(1 - DK)^{-1}C \\ &= A - BD^{-1}C + BD^{-1}(1 - DK)^{-1}C. \end{aligned} \tag{29}$$

Now observe that $K \neq 0$ near $Z_{D^{-1}}$, since $D^{-1}(1 - DK)^{-1} \cong D^{-2}$ on $Z_{D^{-1}}$ and so vanishes to second order there. Thus (29) implies $T = A - BD^{-1}C$ up to second order on $Z_{D^{-1}}$.

Since this produces a second order interpolation problem Agler’s result as it stands does not apply. Often one can obtain a second order formula by interpolating at two nearby points, rescaling and taking a limit. Also this produces an infinite interpolation problem. The use of Agler’s Theorem for these was already discussed in Section 4.5.2.

Now we reiterate the interpolation problems associated with various cases of the standard problem we studied in Section 4.5, but when D may not be in $H^\infty(2)$.

Case	$T = A$ on
2- controls, 1 - measurement	$(Z_{B_1C} \cap Z_{B_2C})$
1- controls, 1 - measurement	Z_{BC}

in addition $T = A - BD^{-1}C$ to second order on $Z_{D^{-1}}$.

9.3 Two measurements and two controls

In this case the controller K and consequently the closed loop controller Q have values in $M_{2 \times 2}$ and the model matching equation becomes

$$T = A + B_1C_1Q_{11} + B_1C_2Q_{12} + B_2C_1Q_{21} + B_2C_2Q_{22}$$

with A, B_1, B_2, C_1, C_2 in $H^\infty(2)$.

Suppose the sets

$$\begin{aligned} Z_B^2 &:= \{\rho_B : B^1(\rho_B) = 0 = B^2(\rho_B)\} \\ Z_C^2 &:= \{\rho_C : C_1(\rho_C) = 0 = C_2(\rho_C)\} \end{aligned}$$

are finite and the zero sets of all functions involved are in general position.

As the $Q_{k\ell}$ sweep all of $H^\infty(2)$, T sweeps all functions of the form

$$T = A + g \tag{30}$$

with g in $H^\infty(2)$ satisfying

$$g(\rho_B) = 0 \quad \text{and} \quad g(\rho_C) = 0 \tag{31}$$

for all ρ_B and ρ_C in $Disk_2$ in the sets Z_B^2 and Z_C^2 . To verify this note that

$$H_1 := C_1Q_{11} + C_2Q_{12}$$

sweeps all functions in $H^\infty(2)$ which equal zero on Z_C^2 as Q_{11}, Q_{12} sweep $H^\infty(2)$. Likewise, $H_2 := C_1Q_{21} + C_2Q_{22}$ sweeps functions in $H^\infty(2)$ which equal zero on Z_C^2 . Since g in (30) is

$$g = B_1H_1 + B_2H_2$$

we have $g(\rho_B) = 0$ and $g(\rho_C) = 0$. These are the only constraints.

We have seen that the 2 measurement and 2 controller model matching problem gives a bi-disk model matching problem and so the Recipe in Section 9.3 gives a solution to it.

10 Some Comparisons

10.1 Parameterized uncertainties an LPV

In examples above and in some types of gain scheduling the uncertain system δ or its linearization is determined by parameters $\alpha \in \mathbf{R}^q$ and so would have the form δ_α . Then a usual set up has a controller $\tilde{K}(x, \alpha)$ depending only on the state x and the parameters α . When at each instant of running the closed loop system α is known, therefore δ_α is known, this form of controller is natural. We reemphasize that it also makes sense to design a controller based on a function $K(\cdot, \cdot)$ of two complex variables, via the formula

$$K(s, \delta_\alpha(s)). \tag{32}$$

Now we mention some possible benefits of using the form (32). Often gain scheduled controllers are validated only by simulation, since no definitive theory is available in that subject. On the other hand Linear Parameter Varying control LPV³ is appealing, since it presents a design technique and automatic validation of the controller. However, LPV techniques give inherently *conservative designs*, since they usually have one quadratic storage function for all systems in the class they are controlling. Possibly bi H^∞ control might offer some ideas on how to judge or adjust the conservativeness of an LPV design. Another point about bi H^∞ (especially for adaptive situations) is that we *do not need to model* δ in order to control the system we merely need to obtain its FRF experimentally. Also bi H^∞ control accommodates measurement feedback as opposed to just state feedback.

³See [BP94], [AG95] for LPV. Also of interest are LFT based methods (cf Workshop ACC 98). One of the best references for current information is [P].

10.2 Relationship to LFT approaches

Another comparison is to the time domain **Linear Fractional Transform LFT approach** taken by Packard and collaborators where the plant P and controller K are both linear fractional in δ . In LFT based control δ would be determined as a time varying system and the controller would change as δ changes.

We could do a frequency domain analog here and assume a linear fractional dependence of $P(s, \delta(s))$ on δ , see Section 5.2. This is reasonable but, requiring K to be linear fractional in δ seems very restrictive and probably leads to a very conservative answer. The advantage of this frequency domain LFT method is that the setup leads to a complex μ synthesis problem (not convex but biconvex).

Our setup on the other hand leads to an LMI, but possibly an infinite one.

10.3 Bibliography on LPV

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