Math 142B
Midterm Exam 1 Solution

1. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ -1 & \text{if } x \text{ is irrational.} \end{cases}$$

   (a) Show that f is not integrable on $[0, 1]$.
   Given a partition $P = \{x_0, x_1, \ldots, x_n\}$,

$$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} = -1, \text{ and}$$
$$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} = 1$$

   for every partition interval $[x_{i-1}, x_i]$. Thus,

$$U(f, P) - L(f, P) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) = 2\sum_{i=1}^{n}(x_i - x_{i-1}) = 2$$

   for every partition P. It follows that f is not integrable.

   (b) Show that $|f|$ *is* integrable on $[0, 1]$.
   $|f(x)| = 1$ for every $x \in [0, 1]$. Thus, $|f|$ is integrable (since $|f|$ is a constant function).

2. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{2}, \\ 2 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

   (a) Let P_n be the n^{th} regular partition of $[0, 1]$ into n partition intervals. Show that $\{P_n\}$ is Archimedean for f on $[0, 1]$.
   Let P_n be the n^{th} regular partition of $[0, 1]$. Observe that

$$M_i - m_i = \begin{cases} 1 & \text{if } \frac{1}{2} \in [x_{i-1}, x_i], \\ 0 & \text{otherwise.} \end{cases}$$

   Since $\frac{1}{2}$ is in at most two partition intervals, it follows that

$$U(f, P_n) - L(f, P_n) = \sum_{i=1}^{n}(M_i - m_i)\frac{1}{n} \le \frac{2}{n} \to 0.$$

   Hence, $\{P_n\}$ is Archimedean for f.

   (b) Determine the value of $\int_0^1 f$.

$$\int_0^1 f = \int_0^{\frac{1}{2}} 1 + \int_{\frac{1}{2}}^1 2 = 1\left(\frac{1}{2} - 0\right) + 2\left(1 - \frac{1}{2}\right) = \frac{3}{2}.$$

3. Let $f : (-1, 1) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{1-x^2}$ and $g : (-1, 1) \to \mathbb{R}$ be defined by $g(x) = \sin\left(\frac{1}{1-x^2}\right)$.

   (a) Extend f to $f : [-1, 1] \to \mathbb{R}$ by defining $f(-1) = f(1) = 0$. Is f integrable on $[-1, 1]$? Explain.

   No, f is not integrable on $[-1, 1]$ because f is not bounded on $[-1, 1]$.

   (b) Extend g to $g : [-1, 1] \to \mathbb{R}$ by defining $g(-1) = g(1) = 0$. Is g integrable on $[-1, 1]$? Explain.

   Yes, g is integrable on $[-1, 1]$ because it is bounded on $[-1, 1]$ and continuous on $(-1, 2)$. (Theorem 6.19)

4. Let $f : [a, b] \to \mathbb{R}$ be monotonically *decreasing*.

   (a) Show that f is bounded on $[a, b]$.

   Since f is montonically decreasing, $f(b) \leq f(x) \leq f(a)$ for every $x \in [a, b]$. Thus, f is bounded.

   (b) Let P_n be the n^{th} regular partition of $[a, b]$ into n partition intervals. Show that $\{P_n\}$ is Archimedean for f on $[a, b]$.

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\begin{aligned}
U(f, P_n) - L(f, P_n) &= \sum_{i=1}^{n} (M_i - m_i) \frac{(b-a)}{n} && \text{since } P_n \text{ is the } n^{\text{th}} \text{ regular partition of } [a, b], \\
&= \sum_{i=1}^{n} \left[f(x_{i-1}) - f(x_i)\right] \frac{(b-a)}{n} && \text{since } f \text{ is monotonically decreasing,} \\
&= \left[f(a) - f(b)\right] \frac{(b-a)}{n} \to 0.
\end{aligned}
$$

   Hence, $\{P_n\}$ is Archimedean for f.