Math 142B
Midterm Exam 2
August 25, 2011
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Instructions

1. **You may use any type of calculator, but no other electronic devices during this exam.**
2. **You may use one page of notes, but no books or other assistance during this exam.**
3. **Write your *Name*, *PID*, and *Section* on the front of your Blue Book.**
4. **Write your solutions clearly in your Blue Book**
   (a) **Carefully indicate the number and letter of each question and question part.**
   (b) **Present your answers in the same order they appear in the exam.**
   (c) **Start each question on a new page.**
5. **Read each question carefully, and answer each question completely.**
6. **Show all of your work; no credit will be given for unsupported answers.**

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Define
$$G(x) = \int_0^x (x - t)\, f(t)\, dt \quad \text{for all } x.$$
Use the Second Fundamental Theorem to show that $G''(x) = f(x)$ for all x. (Hint: Use the linearity property of the integral to rewrite it in a more convenient form.)

2. Let $f(x) = e^x$. We have seen that the n^{th} Taylor polynomial for f at $x = 0$ is given by
$$p_n(x) = \sum_{k=0}^{n} \frac{1}{k!}\, x^k = 1 + x + \frac{1}{2!}\, x^2 + \cdots + \frac{1}{n!}\, x^n\, .$$
Prove that for every real number x, $f(x)$ is equal to its Taylor series at $x = 0$, that is,
$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\, x^k = 1 + x + \frac{1}{2!}\, x^2 + \cdots + \frac{1}{n!}\, x^n + \cdots\, .$$

3. Use the Lagrange Remainder Theorem to show that
$$0 < x - \ln(1 + x) < \frac{1}{2}\, x^2 \quad \text{for all } x > 0.$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ have derivatives of all orders and satisfy:
$$\begin{cases} f'(x) = f(x) & \text{for all } x, \\ f(0) = 2. \end{cases}$$
   (a) Find a formula for the coefficients of the n^{th} Taylor polynomial for f at $x = 0$.
   (b) Show that the Taylor series for f at $x = 0$ converges for all x.

5. For each $n \in \mathbb{N}$, define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = \dfrac{1}{1 + x^n}$. Determine the function $f : [0, \infty) \to \mathbb{R}$ to which the sequence of functions $\{f_n\}$ converges pointwise.
(Note: Use the fact that $\lim_{n\to\infty} x^n = 0$ for $|x| < 1$, and $\lim_{n\to\infty} x^n = \infty$ for $|x| > 1$.)