**Math 142B**
**Midterm Exam 2 Solution**

1. Given $f : [a,b] \to \mathbb{R}$ continuous. Use the Second Fundamental Theorem and additivity of the integral to show that
$$\frac{d}{dx}\left[\int_x^b f\right] = -f(x) \quad \text{for all } x \text{ in } (a,b).$$
(Note: The *definition* that $\int_b^a f = -\int_a^b f$ is based on this result. Thus, your proof should not appeal to this definition.)

   Given $x \in (a,b)$, $\int_x^b f = \int_a^b f - \int_a^x f$. Thus, $\frac{d}{dx}\left[\int_x^b f\right] = -\frac{d}{dx}\left[\int_a^x f\right]$ by the Second Fundamental Theorem (and since $\frac{d}{dx}\left[\int_a^b f\right] = 0$).

2. Given a polynomial $p$ of degree at most $n$ and $x_0$ any point. Show that the $n^{th}$ Taylor polynomial for $p$ at $x_0$ is $p$ itself. You may assume that $p$ can be written in the form $p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n$.

   Let $p_n$ denote the the $n^{\text{th}}$ Taylor polynomial for $p$ at $x_0$. By definition, $p_n(x_0) = p(x_0)$ and, given $x \neq x_0$, the Lagrange Remainder Theorem asserts the existence of a number $c$ strictly between $x$ and $x_0$ at which
$$p(x) - p_n(x) = \frac{p^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$
   Since $p$ is a polynomial of degree at most $n$, $p^{(n+1)}(c) = 0$. It follows that $p_n(x) = p(x)$ for all $x$. (Note: This is Corollary 8.9.)

3. Given the function $f : [0,1] \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$
As we've seen, $f$ is integrable and $\int_0^1 f = \frac{3}{2}$. Show that the Mean Value Theorem for Integrals does not hold for $f$ and explain why $f$ does not satisfy the hypotheses for the Mean Value Theorem for Integrals.

   The mean value of $f$ is $\frac{1}{(1-0)}\int_0^1 f = \frac{3}{2}$, but there is no point $x_0$ in $[0,1]$ at which $f(x_0) = \frac{3}{2}$. Thus, the Mean Value Theorem does not hold for $f$. $f$ does not satisfy the hypotheses for the Mean Value Theorem because it is not continuous at $x = \frac{1}{2}$.

4. Given a number $r$ with $0 < r < 1$, let $f : [-r,r] \to \mathbb{R}$ be defined by $f(x) = (1-x)^{-1}$.

   (a) Find a formula for the $n^{\text{th}}$ Taylor polynomial for $f$ at 0.

   By induction, $f^{(n)}(x) = n!(1-x)^{-(n+1)}$. Thus, $\frac{f^{(n)}(0)}{n!} = 1$ for every index $n$. It follows that the $n^{\text{th}}$ Taylor polynomial for $f$ at 0 is $p_n(x) = \sum_{k=0}^n x^k = 1 + x + x^2 + \cdots + x^n$.

   (b) Use the Lagrange Remainder Theorem to show that the Taylor series for $f$ at 0 converges to $f(x)$ for all $x \in [-r,r]$.

   Let $x$ be in $[-r,r]$ with $x \neq 0$. By the Lagrange Remainder Theorem, there is a number $c$ strictly between 0 and $x$ at which
$$f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{(n+1)!(1-c)^{-(n+2)}}{(n+1)!}x^{n+1} = \frac{1}{1-c}\left(\frac{x}{1-c}\right)^{n+1}.$$
   Unfortunately, it is not obvious that $\left(\frac{x}{1-c}\right)^{n+1} \to 0$ as $n \to \infty$ since for $0 < c < x$, $\frac{x}{1-c} < 1$ if and only if $c < 1-x$. Of course, $\sum_{k=0}^n x^k = \frac{1-x^{n+1}}{1-x} \to \frac{1}{1-x}$ as $n \to \infty$ for all $|x| < 1$, using the Geometric Sum Formula.