Math 142B
Midterm Exam 2 Solution

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Define

$$G(x) = \int_0^x (x-t)\, f(t)\, dt \quad \text{for all } x.$$

Use the Second Fundamental Theorem to show that $G''(x) = f(x)$ for all x. (Hint: Use the linearity property of the integral to rewrite it in a more convenient form.)

By linearity of the integral, $G(x) = x \int_0^x f(t)\, dt - \int_0^x t f(t)\, dt$. Then, by the Second Fundamental Theorem,

$$\begin{aligned} G'(x) &= \int_0^x f(t)\, dt + x f(x) - x f(x) = \int_0^x f(t)\, dt \\ G''(x) &= f(x) \end{aligned}$$

2. Let $f(x) = e^x$. We have seen that the n^{th} Taylor polynomial for f at $x = 0$ is given by

$$p_n(x) = \sum_{k=0}^{n} \frac{1}{k!}\, x^k = 1 + x + \frac{1}{2!}\, x^2 + \cdots + \frac{1}{n!}\, x^n\, .$$

Prove that for every real number x, $f(x)$ is equal to its Taylor series at $x = 0$, that is,

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\, x^k = 1 + x + \frac{1}{2!}\, x^2 + \cdots + \frac{1}{n!}\, x^n + \cdots\, .$$

Let x be arbitrary. By the Lagrange Remainder Theorem, there is a c strictly between 0 and x at which $e^x - p_n(x) = \frac{e^c}{(n+1)!} x^{n+1}$. Since $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$ for every x (Lemma 8.13), it follows that $\lim_{n\to\infty} p_n(x) = e^x$.

3. Use the Lagrange Remainder Theorem to show that

$$0 < x - \ln(1+x) < \frac{1}{2}\, x^2 \quad \text{for all } x > 0.$$

$p_0(x) = 0$ and $p_1(x) = x$ are the 0^{th} and 1^{st} Taylor polynomials for $\ln(1+x)$ at 0. Let $x > 0$.

- Since $p_0(x) = 0$, there is a c_0 with $0 < c_0 < x$ at which $\ln(1+x) - 0 = \frac{1}{1+c_0} x < x$ by the Lagrange Remainder Theorem. Thus, $x - \ln(1+x) > 0$.
- Since $p_1(x) = x$, there is a c_1 with $0 < c_1 < x$ at which $\ln(1+x) - x = -\frac{1}{2(1+c_1)^2} x^2 > -\frac{1}{2} x^2$ by the Lagrange Remainder Theorem. Thus, $x - \ln(1+x) < \frac{1}{2} x^2$

The result follows.

4. Let $f : \mathbb{R} \to \mathbb{R}$ have derivatives of all orders and satisfy:

$$\begin{cases} f'(x) = f(x) & \text{for all } x, \\ f(0) = 2. \end{cases}$$

(a) Find a formula for the coefficients of the n^{th} Taylor polynomial for f at $x = 0$.

By induction on n, $f^{(n)}(0) = 2$ for every n. Thus, $p_n(x) = 2\sum_{k=0}^{n} \frac{1}{k!}x^k$ is the n^{th} Taylor polynomial for f at 0.

(b) Show that the Taylor series for f at $x = 0$ converges for all x.

Since $\left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \frac{|x|}{n+1} \to 0$ as $n \to \infty$ for every x, $p_n(x)$ converges for every x by the Ratio Test (Corollary 9.21). *Note: Corollary 9.21 is not part of the material covered on Friday's exam.*

5. For each $n \in \mathbb{N}$, define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = \dfrac{1}{1 + x^n}$. Determine the function $f : [0, \infty) \to \mathbb{R}$ to which the sequence of functions $\{f_n\}$ converges pointwise.
(Note: Use the fact that $\lim_{n\to\infty} x^n = 0$ for $|x| < 1$, and $\lim_{n\to\infty} x^n = \infty$ for $|x| > 1$.)

(i) For $0 \le x < 1$, $\lim_{n\to\infty} \dfrac{1}{1 + x^n} = 1$; (ii) For $x = 1$, $\lim_{n\to\infty} \dfrac{1}{1 + x^n} = \dfrac{1}{2}$; (iii) For $x > 1$, $\lim_{n\to\infty} \dfrac{1}{1 + x^n} = 0$.

Thus,

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ \frac{1}{2} & \text{if } x = 1, \\ 0 & \text{if } x > 1. \end{cases}$$

Note: Pointwise convergence is not part of the material covered on Friday's exam.