140A Midterm 1 Solutions - Fall 2009

November 20, 2009

Problem 1. *Prove that the cube root of* 12 *is an irrational number.*

Proof. Suppose that $12^{1/3}$ is rational, i.e. there exist relatively prime integers a and b such that

$$12^{1/3} = \frac{a}{b}.$$

Then

$$a^3 = 12b^3 = 2^2 3b^3.$$

So 3 divides a^3. Since 3 is prime, 3 divides a and so 3^3 divides a^3. Then 3^3 divides $2^2 3b^3$. Since 3 is prime, 3 divides b contradicting that a and b are relatively prime. □

Problem 2. *Describe an explicit method for constructing a bijection between the set of rational numbers and the set of positive integers.*

Proof. The key here is to define some function $f : \mathbb{N} \to \mathbb{Q}$ that hits every element of $\mathbb{Q}$ exactly once. We construct a diagram similar to the one on page 29 of Rudin in the proof of theorem 2.12.

$$\begin{array}{cccccccc}
0 & -1 & 1 & -2 & 2 & -3 & 3 & \dots \\
\frac{0}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{2}{2} & \frac{2}{2} & -\frac{3}{2} & \frac{3}{2} & \dots \\
\frac{0}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} & -\frac{3}{3} & \frac{3}{3} & \dots \\
\frac{0}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{2}{4} & \frac{2}{4} & -\frac{3}{4} & \frac{3}{4} & \dots \\
\vdots & & & & & & &
\end{array}$$

Then we define our map to go diagonally as in Rudin's diagram, skipping repeated elements of $\mathbb{Q}$. Then we have $f(1) = 0$, $f(2) = -1$, $f(3) = -1/2$, $f(4) = 1$, $f(5) = -1/3$, ... This gives a bijection from $\mathbb{N} \to \mathbb{Q}$ □

Problem 3. *Jane claims that she has found a pair of real numbers $a < b$ such that the interval $(a, b) \subset \mathbb{R}$ contains no irrational numbers. Prove that Jane is mistaken.*

Proof. Suppose there exist a pair of real numbers $a < b$ such that the interval (a, b) contains no irrational numbers. Then $(a, b) \subset \mathbb{Q}$ and hence is countable. This is a contradiction since any interval in $\mathbb{R}$ is uncountable. □

Problem 4. *Let L denote the x-axis in the usual Cartesian plane $\mathbb{R}^2$. Give an example of a closed set E in the plane which has points arbitrarily close to L, but such that E is disjoin from L. Does such an example exist if L were the circle $x^2 + y^2 = 1$ instead of the x-axis?*

Proof. The graph of the function $f(x) = 1/x$ on the interval $(0, \infty)$ does the trick, i.e. let

$$E = \left\{ \left(x, \frac{1}{x} \right) : x > 0 \right\}.$$

If L were the circle $x^2 + y^2 = 1$, no such example exists. The reason here is because the unit circle is a compact subset of $\mathbb{R}^2$, and then the following theorem applies:

Theorem 1. *Let X be a metric space. $L \subseteq X$ be compact and let $E \subseteq X$ be closed. Then $d(L, E) > 0$.*

□

Problem 5. *Let E be the set of those real numbers in the interval $(0, 1)$ with infinite decimal expansions $.p_1p_2p_3\ldots$ such that at least one of the digits p_i is 0 or 9. Is E open in $\mathbb{R}$? Justify.*

Proof. Yes, E is open in $\mathbb{R}$. To do this we show that if $p \in E$ then there is a neighborhood of p contained in E.

Let $p = 0.p_1p_2p_3\ldots \in E$. Then there exists an integer k such that p_k is 0 or 9. Let $r = 10^{-(k+1)}$. Now we have to consider several cases.

If p_{k+1} is not 0 or 9, it is clear that $N_r(p) \subset E$ since every element of $N_r(p)$ has the kth digit equal to p_k.

If $p_k = 0$ and $p_{k+1} = 0$, then any element in $N_r(p)$ has the kth digit equal to 0 or 9.

If $p_k = 0$ and $p_{k+1} = 9$, then any element in $N_r(p)$ has either the kth digit equal to 0 or the $(k+1)$th digit equal to 0.

If $p_k = 9$ and $p_{k+1} = 0$, then any element in $N_r(p)$ has either the kth digit equal to 9 or the $(k+1)$th digit equal to 9.

If $p_k = 9$ and $p_{k+1} = 9$, then any element in $N_r(p)$ has the kth digit equal to 0 or 9.

In any of the above cases, $N_r(p) \subseteq E$ and hence E is open. □

Problem 6. *Let E be a bounded open subset of $\mathbb{R}$ such that $0 \in E$. Let $M = \{x \in E : [0, x] \subset E\}$. Let α denote the least upper bound of M in $\mathbb{R}$. Prove that $\alpha \notin M$*

Proof. Suppose $\alpha \in M$. Then by definition of M, $[0, \alpha] \subset E$, in particular, $\alpha \in E$. Since E is open, there exists $r > 0$ such that $(\alpha - r, \alpha + r) \subset E$. Then we have that

$$[0, \alpha + r/2] \subset [0, \alpha + r) = [0, \alpha] \cup (\alpha - r, \alpha + r) \subset E.$$

Then by definition of M, $\alpha + r/2 \in M$ and $\alpha + r/2 > \alpha$, contradicting that $\alpha = \sup M$.

Hence $\alpha \notin M$ by contradiction. □