# MATH 140B - HW 4 SOLUTIONS

**Problem 1** (WR Ch 7 #3)**.** Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set $E$, but such that $\{f_n g_n\}$ does not converge uniformly on $E$.

*Solution.* Let $E = \mathbb{R}$ and let

$$f_n(x) = g_n(x) = x + \tfrac{1}{n}, \qquad \text{so that } f(x) = g(x) = x.$$

Then for every $\epsilon > 0$ we choose $N$ to be the smallest integer greater than $1/\epsilon$, so that for $n \geq N$ we have

$$|f_n(x) - f(x)| = |x + \tfrac{1}{n} - x| = \tfrac{1}{n} < \epsilon$$

for all $x \in \mathbb{R}$, so $\{f_n\}$ and $\{g_n\}$ converge uniformly. However, $\{f_n g_n\}$ does not converge uniformly by the following proof. For any $\epsilon > 0$ and any $n$, we have

$$|f_n g_n(x) - f(x)g(x)| = \left|\left(x^2 + \tfrac{2}{n}x + \tfrac{1}{n^2}\right) - x^2\right| = \tfrac{2}{n}|x| + \tfrac{1}{n^2} > \epsilon$$

for $|x| > \frac{n\epsilon}{2}$. Therefore we cannot choose an $N$ such that $|f_n(x)g_n(x) - f(x)g(x)| < \epsilon$ for all $x \in \mathbb{R}$ for $n \geq N$.

**Problem 2** (WR Ch 7 #4)**.** Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}.$$

For what values of $x$ does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is $f$ continuous wherever the series converges? Is $f$ bounded?

*Solution.* First of all, we know that the series doesn't converge absolutely when any of its terms are infinite. This happens when the denominator $1 + n^2 x$ is zero. So right away we know that the series does not converge absolutely when $x = -\frac{1}{n^2}$ for some integer $n \geq 1$. Also, in the case $x = 0$, our series becomes an infinite sum of 1's, and thus also diverges. That leaves a couple cases left to check.

Case 1: $\boxed{x > 0.}$

$$\sum_{n=1}^{\infty} \left|\frac{1}{1 + n^2 x}\right| = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x} \leq \sum_{n=1}^{\infty} \frac{1}{n^2 x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

so $f(x)$ converges absolutely by the comparison test.

Case 2: $\boxed{x<0.\ x\neq -\frac{1}{n^2} \text{ for } n\in\mathbb{N}.}$ After a certain number of terms in the series, we're going to have $1+n^2x<-n^2$. To find this point, choose $N$ to be the smallest integer such that $N^2 \geq \frac{1}{|1+x|}$. Then for $n\geq N$ we have

$$1+n^2x \leq -n^2 \qquad \Longrightarrow \qquad \frac{-1}{1+n^2x} \leq \frac{1}{n^2} \qquad \Longrightarrow \qquad \left|\frac{1}{1+n^2x}\right| \leq \frac{1}{n^2}.$$

Therefore, $\sum_{n=N}^{\infty}\left|\frac{1}{1+n^2x}\right|$ converges by the comparison test. For the full series to converge, we only need to add on the finitely many terms at the beginning of this sequence, so it converges.

Next we address what the question of which intervals it converges uniformly on. We should note that any interval $[a,b]$ that contains a point $\frac{-1}{n^2}$ for some $n$ (or 0) will not have convergence at that point, so it can't have uniform convergence. From Case 1 above, if we have $x\in[a,b]$ where $a>0$, then

$$\left|\frac{1}{1+n^2x}\right| \leq \frac{1}{a}\ \frac{1}{n^2},$$

so we have uniform convergence by the Weierstrass $M$-test on intervals of that type. If $x\in[a,b]$ with $b<0$ and $\frac{-1}{n^2}\notin[a,b]$ for all $n\in\mathbb{N}$, then from Case 2 there is some $N\in\mathbb{N}$ such that for $k\geq N$,

$$\left|\sum_{n=k}^{\infty}\frac{1}{1+n^2x}\right| \leq \sum_{n=k}^{\infty}\frac{1}{n^2}\to 0 \qquad \text{as} \quad k\to\infty,$$

so we once again have uniform convergence. The only case left is where one of the endpoints of the interval $(a,b)$ is $\frac{-1}{n^2}$ for some $n$, or 0. In this case we do not have uniform convergence. In the case where $a=0$, if we assume we do have uniform convergence and set $\epsilon=1/2$ then there is some $k$ such that

$$\left|\sum_{n=k}^{\infty}\frac{1}{1+n^2x}\right| < \frac{1}{2},$$

but if we let $x=\frac{1}{n^2}$ for some $n\geq k$ then the left side is at least $1/2$, generating a contradiction. The other cases follow similarly.

By Theorem 7.12, $f$ is continuous on intervals where it converges uniformly, which in this case happens to include the set of sets where it converges pointwise.

Lastly, we prove that $f$ is not bounded. Let $x_k=\frac{1}{k}$. Then

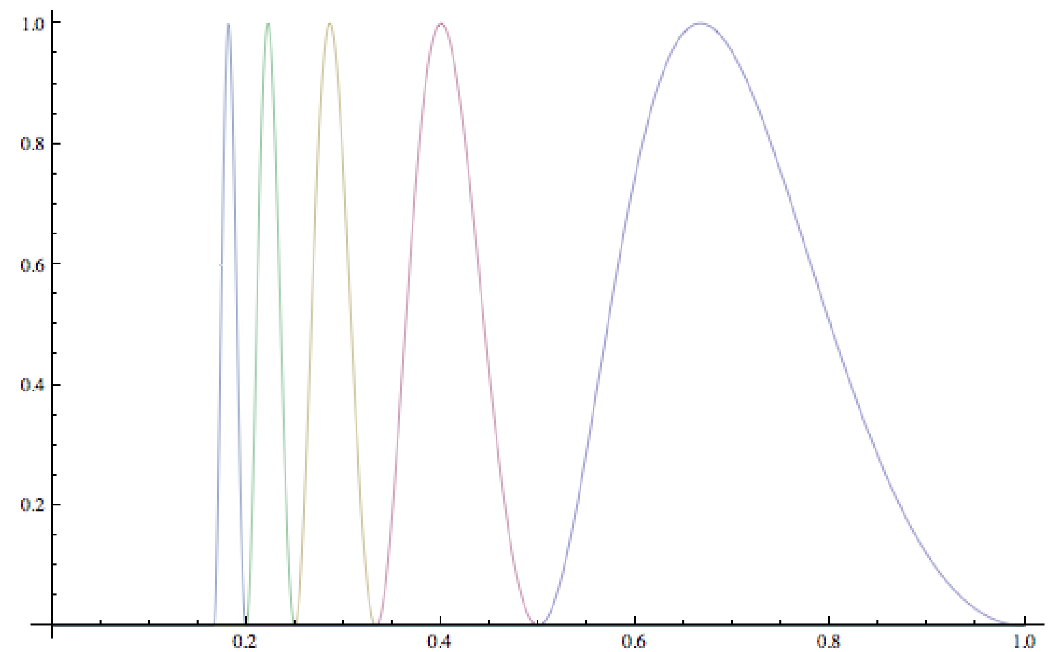
$$f(x_k)=\sum_{n=1}^{\infty}\frac{1}{1+n^2\frac{1}{k}} = \frac{1}{\frac{1}{k}}\sum_{n=1}^{\infty}\frac{1}{\frac{1}{\frac{1}{k}}+n^2} = k\sum_{n=1}^{\infty}\frac{1}{k+n^2}\to\infty \qquad \text{as} \quad k\to\infty.$$

**Problem 3** (WR Ch 7 #5)**.** Let

$$f_n(x) = \begin{cases} 0 & \left(x < \frac{1}{n+1}\right), \\ \sin^2 \frac{\pi}{x} & \left(\frac{1}{n+1} \le x \le \frac{1}{n}\right), \\ 0 & \left(\frac{1}{n} < x\right). \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all $x$, does not imply uniform convergence.

*Solution.* We claim that $f_n(x)$ converges to 0 (which is clearly a continuous function). First notice that $f_n(x) = 0$ for $x \le 0$. For $x > 0$, we can see that $f_n(x) = 0$ for $n \ge \frac{1}{x}$ (because then $x > \frac{1}{n}$). Below is a graph of $f_1, f_2, f_3, f_4$ and $f_5$, from right to left.




To prove that $f_n$ does not converge uniformly to 0 we set $\epsilon = \frac{1}{2}$ and assume we are given some $N \in \mathbb{N}$ such that $|f_n(x) - 0| < \epsilon$ for all $n \ge N$. But if we just take $x = \frac{1}{N+\frac{1}{2}}$, then $|f_N(x) - 0| = \sin^2((N + \frac{1}{2})\pi) = 1 > \epsilon$, so $f_n$ does not converge uniformly to 0.

Notice that if $f_n(x) > 0$, then $f_m(x) = 0$ for $m \ne n$. This means that $\sum f_n$ converges absolutely since at each $x$ there is only one function that is nonzero. But we proved that $f_n$ does not converges uniformly, so neither does the sequence of partial sums of $\sum f_n$.

**Problem 4** (WR Ch 7 #6)**.** Prove that the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of $x$.

*Solution.* Since $\sum_{n=1}^{\infty}(-1)^n\frac{x^2}{n^2}$ and $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ both converge uniformly on every bounded interval, so does their sum, which is exactly the series we want. Perhaps we should say why they converge uniformly on every bounded interval. Clearly $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ converges uniformly in $x$ because it doesn't even depend on $x$. To show that $\sum_{n=1}^{\infty}(-1)^n\frac{x^2}{n^2}$ converges uniformly in $x$ on bounded intervals, let $[-M,M]$ be some interval, so that $x\in[-M,M]$ implies

$$\left|(-1)^n\frac{x^2}{n^2}\right|\le M^2\frac{1}{n^2},$$

so uniform convergence follows from the Weierstrass $M$-test.

Lastly, the series does not converge absolutely because

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{x^2+n}{n^2}\right|=\sum_{n=1}^{\infty}\frac{x^2+n}{n^2}=\sum_{n=1}^{\infty}\frac{x^2}{n^2}+\sum_{n=1}^{\infty}\frac{1}{n},$$

where the series on the left converges but the one on the right does not.

**Problem 5** (WR Ch 7 #7)**.** For $n=1,2,3,\dots$, $x$ real, put

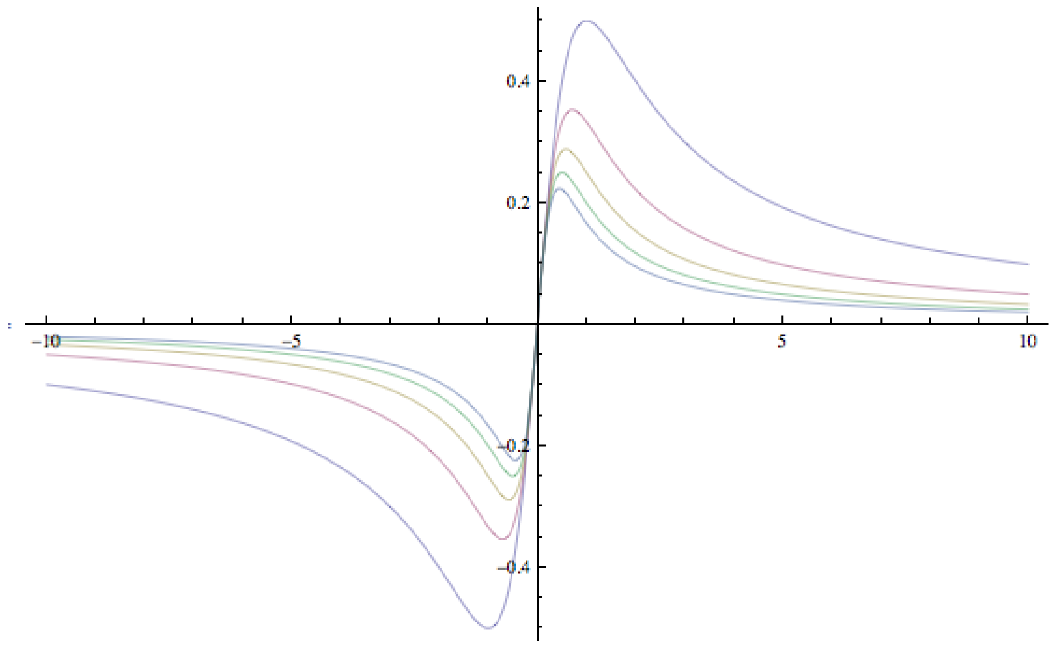
$$f_n(x)=\frac{x}{1+nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function $f$, and that the equation

$$f'(x)=\lim_{n\to\infty}f_n'(x)$$

is correct if $x\neq 0$, but false if $x=0$.

*Solution.* Here is the graph for $f_1, f_2, f_3, f_4$, and $f_5$:




(where $f_1$ has the highest and lowest peaks and $f_5$ is the closest to 0). By the quotient rule,

$$f_n'(x) = \frac{(1+nx^2) - 2nx^2}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}. \qquad (*)$$

From this we know that $f_n'(x)$ is positive for $0 < x < \frac{1}{\sqrt{n}}$ and negative for $x > \frac{1}{\sqrt{n}}$. Therefore the maximum of $f_n(x)$ on $[0,\infty)$ is at $x_0 = \frac{1}{\sqrt{n}}$, for which we have $f_n(x_0) = \frac{1}{2\sqrt{n}}$. And since $f_n(-x) = -f_n(x)$, this is also an upper bound for $x$ in $(-\infty, 0)$. We are going to prove that $f_n(x)$ converges uniformly to 0. Let $f(x) = 0$. Then

$$|f_n(x) - f(x)| = \left|\frac{x}{1+nx^2}\right| \le \frac{1}{2\sqrt{n}} \qquad \text{for all } x \in \mathbb{R}.$$

So if we are given some $\epsilon > 0$, we can choose $N$ to be the smallest integer such that $N > \frac{1}{4\epsilon^2}$, and then for $n \ge N$ we have

$$|f_n(x) - f(x)| < \frac{1}{2\sqrt{n}} \le \frac{1}{2\sqrt{N}} < \epsilon \qquad \text{for all } x \in \mathbb{R}.$$

This proves the uniform convergence. Next, if $x \neq 0$, then by $(*)$ we have

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{1-nx^2}{(1+nx^2)^2} = \lim_{n\to\infty} \frac{1-nx^2}{n^2x^4 + 2nx^2 + 1} = \lim_{n\to\infty} \frac{\frac{1}{n^2} - \frac{x^2}{n}}{x^4 + \frac{2x^2}{n} + \frac{1}{n^2}} = 0 = f'(x).$$

Also, $f_n'(0) = 1$, so that $\lim\limits_{n\to\infty} f_n'(0) = 1 \neq 0 = f'(0)$.