MATH 140B - HW 5 SOLUTIONS

Problem 1 (WR Ch 7 #8)**.** If

$$I(x) = \begin{cases} 0 & (x \le 0), \\ 1 & (x > 0), \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \qquad (a \le x \le b)$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

Solution. Let

$$f_k(x) = \sum_{n=1}^{k} c_n I(x - x_n).$$

By the Weierstrass M-test (Theorem 7.10) with $M_n = |c_n|$, $\{f_k(x)\}$ converges uniformly to $f(x)$. Let $E = [a, b] \setminus \{x_n : n \in \mathbb{N}\}$. Since each $f_k(x)$ is continuous on E, then by Theorem 7.12 we know that f is continuous on E.

Problem 2 (WR Ch 7 #9)**.** Let $\{f_n\}$ be a sequence of continuous functions which converge uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

Solution. Since f is the uniform limit of continuous functions, it is continuous (Theorem 7.12). Since f is continuous and $x_n \to x$, we know that $f(x_n) \to f(x)$ (Theorem 4.2). Set $\epsilon > 0$. Then there is some $N_1 \in \mathbb{N}$ such that

$$|f(x_n) - f(x)| < \frac{\epsilon}{2} \qquad \text{for } n \ge N_1.$$

Since each $f_n \rightrightarrows f$, there exists some $N_2 \in \mathbb{N}$ such that

$$|f_n(t) - f_n(t)| < \frac{\epsilon}{2} \qquad \text{for } n \ge N_2 \text{ and for all } t \in E.$$

Putting this together, for $n \ge N = \max(N_1, N_2)$ we have

$$|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

The converse is "If $\{f_n\}$ is a sequence of continuous functions for which $f_n(x_n) \to f(x)$ for every sequence $x_n \to x$ in E, then $f_n \rightrightarrows f$ on E." This is not true by the following counterexample. Let $f_n(x) = \frac{x}{n}$. This sequence of functions converges pointwise to 0 but not uniformly, since $|f_n(x) - f(x)| = |\frac{x}{n}| > \epsilon$ for $x > \frac{\epsilon}{n}$. The other property we need to check is that $f_n(x_n) \to f(x)$ for every sequence $x_n \to x$. Since $\{x_n\}$ is a convergent sequence, it is bounded, so $|x_n| < M$. Then given any $\epsilon > 0$, we choose $N > \frac{M}{\epsilon}$, so that for $n \geq N$ we have

$$|f_n(x_n) - f(x)| = |\tfrac{x_n}{n} - 0| = \tfrac{|x_n|}{n} \leq \tfrac{M}{N} < \epsilon.$$

This proves that $f_n(x_n) \to f(x)$.

Problem 3 (WR Ch 7 #10)**.** Letting (x) denote the fractional part of the real number x, consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} \qquad (x \text{ real }).$$

Find all discontinuities of f, and show that they form a countable dense set. Show that f is nevertheless Riemann-integrable on every bounded interval.

Solution. First notice that the function $g(x) = (x)$ is discontinuous on $\mathbb{Z}$ and continuous on $\mathbb{R} \setminus \mathbb{Z}$. This means that $g_n(x) = (nx)$ is discontinuous at all x such that $nx \in \mathbb{Z}$, which means only at rational numbers of the form $\frac{m}{n}$ (where $m, n \in \mathbb{Z}$ and $\gcd(m, n) = 1$). Let $E = \mathbb{R} \setminus \mathbb{Q}$. As n ranges over all positive integers, we see that $g_n(x)$ only has discontinuities at points which lie outside E, so that if we let

$$f_k(x) = \sum_{n=1}^{k} \frac{g_n(x)}{n^2} = \sum_{n=1}^{k} \frac{(nx)}{n^2},$$

we see that each $f_k(x)$ is continuous on E. By the Weierstrass M-test (Theorem 7.10) with $M_n = \frac{1}{n^2}$, we know that $\{f_k(x)\}$ converges uniformly to $f(x)$, and thus by Theorem 7.12 we know that f is continuous on E since each $f_k(x)$ is continuous on E.

What remains is to prove that f is discontinuous at all rational points. Let $x = \frac{a}{b}$ for $a, b \in \mathbb{Z}$, $\gcd(a, b) = 1$. Then $g_b(x)$ is discontinuous at x, or more specifically $\lim_{y \to x-} g_b(x) = 1$ and $\lim_{y \to x+} = 0$. More generally, at any discontinuity of g_n, we will have that the limit from the left will be larger than the limit from the right, meaning that

$$\lim_{y \to x-} f_k(y) - \lim_{y \to x+} f_k(y) \geq \frac{1}{b^2} \qquad \text{for } k \geq b.$$

Since $f_k(x) \to f(x)$ pointwise (the series is bounded by $\sum_{n=1}^{\infty} 1/n^2$), there exists some $N \in \mathbb{N}$ ($N > b$) such that $\sum_{n=N+1}^{\infty} \frac{(nx)}{n^2} < \frac{1}{3b^2}$, so that

$$\begin{aligned}\lim_{y \to x-} f(y) - \lim_{y \to x+} f(y) &= \left(\lim_{y \to x-} f_N(y) - \lim_{y \to x+} f_N(y)\right) + \left(\lim_{y \to x-} f \sum_{n=N+1}^{\infty} \tfrac{(nx)}{n^2} - \lim_{y \to x+} f \sum_{n=N+1}^{\infty} \tfrac{(nx)}{n^2}\right) \\ &> \frac{1}{b^2} - 2\frac{1}{3b^2} = \frac{1}{3b^2} > 0.\end{aligned}$$

Therefore f is discontinuous at every rational point. The fact that f is Riemann integrable follows directly from Theorem 7.16.

Problem 4 (WR Ch 7 #11)**.** Suppose $\{f_n\}$, $\{g_n\}$ are defined on E, and

- **(a)** $\sum f_n$ has uniformly bounded partial sums;
- **(b)** $g_n \to 0$ uniformly on E;
- **(c)** $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$.

Prove that $\sum f_n g_n$ converges uniformly on E.

Solution. Let $A_n(x) = \sum_{k=1}^{n} f_n$. Choose M such that $|A_n(x)| \leq M$ for all n. Given $\epsilon > 0$, by uniform continuity there is an integer N such that $g_N(x) \leq (\epsilon/2M)$ for all $x \in E$. For $N \leq p \leq q$, we have

$$\begin{aligned}
\left|\sum_{n=p}^{q} f_n(x) g_n(x)\right| &= \left|\sum_{n=p}^{q-1} A_n(x)(g_n(x) - g_{n+1}(x)) + A_q(x) g_q(x) - A_{p-1}(x) g_p(x)\right| \\
&\leq M \left|\sum_{n=p}^{q-1} (g_n(x) - g_{n+1}(x)) + g_q(x) + g_p(x)\right| \\
&= 2M g_p(x) \leq 2M g_N(x) \leq \epsilon.
\end{aligned}$$

Convergence follows from the Cauchy criterion for uniform convergence.

Problem 5 (WR Ch 7 #15)**.** Suppose f is a real continuous function on $\mathbb{R}^1$, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \ldots$, and $\{f_n\}$ is equicontinuous on $[0,1]$. What conclusion can you draw about f?

Solution. We can conclude f is constant on $[0,\infty)$. The fact that $\{f_n\}$ is equicontinuous means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|s - t| < \delta \qquad \Longrightarrow \qquad |f_n(s) - f_n(t)| < \epsilon \qquad \text{for all } n \in \mathbb{N}.$$

For any $x \in [0,\infty)$, set $\epsilon > 0$ and find a $\delta > 0$ so that the above inequality holds. Then choose N to be the smallest integer such that $N > x/\delta$ (so that $x/N < \delta$). Then if we set $s = 0$ and $t = x/N$, we have $|s - t| = x/N < \delta$, so by the inequality above we have

$$|f(0) - f(x)| = |f_n(s) - f_n(t)| < \epsilon.$$

But our choice of ϵ was arbitrary, so that means $f(0) = f(x)$ for all $x \in [0,\infty)$, proving our claim.

Problem 6 (Supp. HW2 #4)**.** Given an example of a metric space X and a sequence of functions $\{f_n\}$ on X such that $\{f_n\}$ is equicontinuous but not uniformly bounded.

Solution. Let $X = \mathbb{R}$ and $f_n(x) = n$. Then for any $\epsilon > 0$, choose any $\delta > 0$ and we have

$$|f_n(x) - f_n(y)| = |n - n| = 0 < \epsilon$$

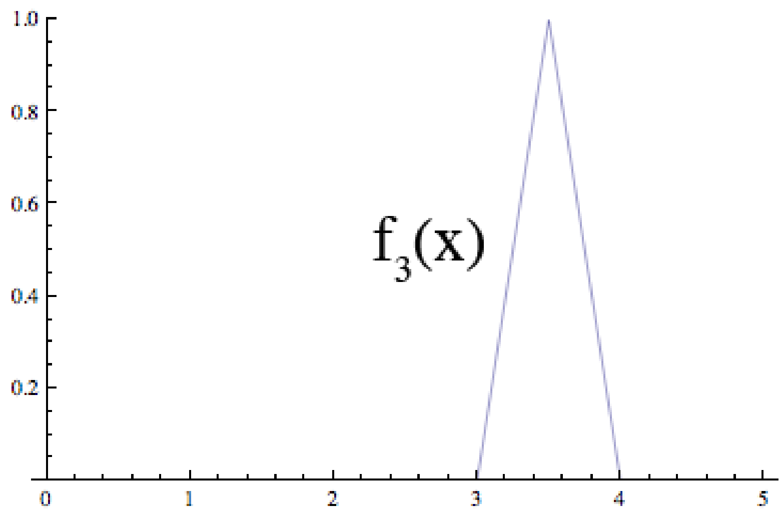
whenever $|x - y| < \delta$, so $\{f_n\}$ is equicontinuous. If it were uniformly bounded then there would be some $M > 0$ such that $|f_n(x)| < M$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, but this is clearly not possible by taking $n > M$.

Problem 7 (Supp. HW2 #5)**.** Give an example of a uniformly bounded and equicontinuous sequence of functions on $\mathbb{R}$ which does not have any uniformly convergent subsequences.

Solution. Let

$$f_n(x) = \begin{cases} 2(x-n) & n \le x \le n + \frac{1}{2} \\ 2(n+1-x) & n + \frac{1}{2} < x \le n+1 \\ 0 & \text{otherwise} \end{cases}.$$

For example, we graph $f_3(x)$ below. In loose terms, $f_n(x)$ is zero everywhere except for a "triangle" of height 1 on the interval $[n, n+1]$.



From this definition it's clear that $|f_n(x)| \le 1$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, so the sequence is uniformly bounded. To prove equicontinuity, set some $1 > \epsilon > 0$ and choose $\delta = \epsilon/2$, so that if

$|x-y|<\delta$ and $x<y$ we have

$$|f_n(x)-f_n(y)| \le \max\begin{pmatrix} \left|0-2(y-n)\right|, \\ \left|2(x-n)-2(y-n)\right|, \\ \left|2(x-n)-2(n+1-y)\right|, \\ \left|2(n+1-x)-2(n+1-y)\right|, \\ |2(n+1-x)-0| \end{pmatrix} = 2|x-y| < 2\delta < \epsilon.$$

The reason this sequence doesn't have any uniformly convergent subsequences is that the sequence converges pointwise to 0, so any subsequence must converge pointwise to 0, but $f_n(n+\frac{1}{2})=1$, so if we have some subsequence $\{f_{n_k}\}$ and we set $\epsilon<1$, then

$$\sup_{x\in\mathbb{R}}|f_{n_k}(x)-f(x)| \ge 1 > \epsilon \qquad \text{for all } k.$$