Math 140a - HW 1 Solutions

Problem 1 (WR Ch 1 #1)**.** If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Solution. Given that r is rational, we can write $r = \frac{a}{b}$ for some integers a and b. We are also given that x is irrational. From here, we proceed with a proof by contradiction. We first assume that $r + x$ is rational, and then we use this fact in some way to show that x is rational, contradicting one of the facts we were given. This will prove that $r + x$ is instead irrational.

So if $r + x$ is rational, we can write $r + x = \frac{c}{d}$ for some relatively prime integers c and d. But then

$$x = \frac{c}{d} - r = \frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd},$$

and thus x is rational, which is a contradiction. Therefore, $r + x$ is irrational.

Next, we prove that rx is irrational using a similar contradiction proof. Assume that rx is rational. Then we can write $rx = \frac{c}{d}$ for some integers c and d. But then

$$x = \frac{c}{rd} = \frac{c}{\frac{a}{b}d} = \frac{bc}{ad},$$

and thus x is rational, which is a contradiction. Therefore, rx is irrational.

Problem 2 (WR Ch 1 #2)**.** Prove that there is no rational number whose square is 12.

Solution. Let x be a rational number such that $x^2 = 12$. Then we can write $x = \frac{a}{b}$, and furthermore, we can choose a and b to be relatively prime (which means there is no prime number dividing both a and b), so that the fraction $\frac{a}{b}$ is written in lowest terms. With a little algebraic manipulation,

$$12 = x^2 = \frac{a^2}{b^2} \qquad \Longrightarrow \qquad 12b^2 = a^2.$$

Now, the prime factorization of 12 is $2^2 \cdot 3^1$, so since there is an odd number of factors of 3 (just 1), we'll concentrate on 3 and how it divides both sides of the equation $12b^2 = a^2$. Notice that 3 divides the left side since it has a factor of 12. Therefore, 3 must divide the right side of the equation, a^2. From here, the crucial step is realizing that if 3 divides a^2, then it must also divide a. This is because if we factor a^2 into its prime factors, saying that 3 divides a^2 is equivalent to saying that 3 is one of those prime factors, but a square of an integer must have an even number of each factor (it can't have just 1), so that means 3^2 must divide a^2, and 3 must divide a.

Since we have shown that 3^2 divides the right side, 3^2 must divide the left side, but there is only one factor of 3 in 12, so that means 3 divides b^2. Using the same logic as before, this means that 3 must divide b.

Therefore, we have shown that 3 divides both a and b, but this contradicts the fact that we already chose a and b to be relatively prime (so that $\frac{a}{b}$ would be expressed in lowest terms). Since our initial assumption leads to a contradiction, we have instead that there is no rational number whose square is 12.

Problem 3 (WR Ch 1 #7)**.** Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This is called the *logarithm of* y *to the base* b.)

(a) For any positive integer n, $b^n - 1 \geq n(b-1)$.

Solution. First, we factorize the left hand side:

$$b^n - 1 = (b-1)(b^{n-1} + b^{n-2} + \cdots + b^2 + b + 1).$$

Then, since $b > 1$, we know that $b^{n-1} + b^{n-2} + \cdots + b^2 + b + 1 \geq n$. So

$$b^n - 1 = (b-1)(b^{n-1} + b^{n-2} + \cdots + b^2 + b + 1) \geq (b-1)n.$$

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

Solution. If $b > 1$ then $b^{1/n} > 1$, so since we proved that $b^n - 1 \geq n(b-1)$ for any $b > 1$, we can substitute $b^{1/n}$ for b in that equation to get that $b - 1 \geq n(b^{1/n} - 1)$.

(c) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

Solution. $n > (b-1)/(t-1)$ implies that $n(t-1) > (b-1)$. Using the previous result for the second inequality,

$$n(t-1) > (b-1) \geq n(b^{1/n} - 1).$$

Therefore, $n(t-1) > n(b^{1/n} - 1)$. Dividing by $n \neq 0$ we have that

$$t - 1 > b^{1/n} - 1.$$

Then we add 1 to both sides to get the result.

(d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n; to see this, apply part (c) with $t = y \cdot b^{-w}$.

Solution. First of all, $b^w < y$ implies that $yb^{-w} > 1$. Therefore, for $t = yb^{-w}$, we have $t > 1$, so if we also choose some $n > \frac{b-1}{yb^{-w}-1}$ we can then use part (c) to get that

$$b^{1/n} < t \implies b^{1/n} < yb^{-w} \implies b^{w+(1/n)} < y.$$

(e) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

Solution. First of all, $b^w > y$ implies that $y^{-1}b^w > 1$. Therefore, for $t = y^{-1}b^w$, we have $t > 1$, so if we also choose some $n > \frac{b-1}{y^{-1}b^w-1}$ we can then use part (c) to get that

$$b^{1/n} < t \quad \Longrightarrow \quad b^{1/n} < y^{-1}b^w \quad \Longrightarrow \quad y < b^{w-(1/n)}.$$

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

Solution. To show that $b^x = y$, we will first show that b^x is not greater than y and then show that it is not less than y.

Assume (by way of contradiction) that $b^x > y$. Then by part (e), there is some integer n such that $b^{x-(1/n)} > y$. However, this means that $x - (1/n)$ is an upper bound for A, but since $x - (1/n) < x$, this means that x is not the least upper bound, a contradiction of the definition of x as the supremum of A. Therefore, instead we have that b^x is not greater than y, or equivalently, $b^x \leq y$.

Next assume (by way of contradiction) that $b^x < y$. Then by part (d), there is some integer n such that $b^{x+(1/n)} < y$. However, this means that $x + (1/n) \in A$, but $x < x + (1/n)$, which means that x is not an upper bound, a contradiction of the definition of x as the supremum of A. Therefore, instead we have that b^x is not less than y, or equivalently, $b^x \geq y$.

(g) Prove that this x is unique.

Solution. Assume there is some other $z \in \mathbb{R}$ such that $y = b^z$. Then

$$b^x = y = b^z.$$

If we assume that $x \neq z$, then either $x > z$ or $z > x$. In the first case, we divide both sides of the above equation by b^z to get $b^{x-z} = 1$ (and note that $x - z$ is a positive real number). However, $b > 1$, so that $b^w > 1$ for any positive real number w, contradicting the fact that $b^{x-z} = 1$. In the second case, we divide both sides of the above equation by b^x to get $b^{z-x} = 1$ (and note that $z - x$ is a positive real number). However, $b > 1$, so that $b^w > 1$ for any positive real number w, contradicting the fact that $b^{x-z} = 1$.

Problem 4 (WR Ch 1 #9)**.** Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?

Solution. To prove that "$<$" is an order on the set of all complex numbers, we need to check the two axioms of an ordered set.

The first is that for any $z, w \in \mathbb{C}$, one and only one of the statements $z < w$, $z = w$, or $z > w$ is true. To show that this is true, we need to break the proof up into cases using the first axiom of the usual ordering for real numbers:

Case 1 : $a < c$. This results in $z < w$.

Case 2 : $a = c$. Here we are again essentially using the first axiom of the usual ordering for real numbers. We must have exactly one of the following cases: $b < d$, $b = d$, or $b > d$. From here, $b < d$ if and only if $z < w$; $b = d$ if and only if $z = w$; and $b > d$ if and only if $z > w$.

Case 3 : $a > c$. This results in $z > w$.

Next we prove the second axiom, which is that for any $z, y, w \in \mathbb{C}$,

$$z < y \text{ and } y < w \implies z < w.$$

For notation, let $y = e + fi$. Then if $z < y$ we know that either $a < e$ or $a = e$ and $b < f$. Likewise, if $y < w$ we know that either $e < c$ or $e = c$ and $f < d$. This give us four possible cases to check:

Case 1 : $a < e$ and $e < c$. By the transitive property of the usual ordering for the real numbers, we then know that $a < c$, which implies that $z < w$.

Case 2 : $a < e$ and $e = c$. Then $a < c$, which implies that $z < w$.

Case 3 : $a = e$ and $e < c$. Then $a < c$, which implies that $z < w$.

Case 4 : $a = e$ and $e = c$, with $b < f$ and $f < d$. Then by the transitive property of the usual ordering for the real numbers, we then know that $b < d$, which means that $a = c$ and $b < d$, so that $z < w$.

Lastly, in order to show that $\mathbb{C}$ doesn't have the least upper bound property, we need to find a subset of $\mathbb{C}$ that has no least upper bound. There are plenty, but let's consider the set

$$A = \{z \in \mathbb{C} | z = a + bi \text{ for some } a, b \in \mathbb{R}, \text{ and } a < 0\}.$$

Assume there is a least upper bound $w = c + di$. Then if $c > 0$, we know that $0 + di < w$, and $0 + di$ is an upper bound for A, contradicting the fact that w is a least upper bound. If $c < 0$, then $c/2 + di$ is a complex number with a negative real part, so $c/2 + di$ is in A and $c + di < c/2 + di$, contradicting the fact that w is an upper bound. Therefore $c = 0$, and thus w is of the form di for some $d \in \mathbb{R}$.

However, for any choice of $d \in \mathbb{R}$, the complex number $(d - 1)i$ is an upper bound for A which is less than di under the order, contradicting the fact that w is a a least upper bound once again. Hence, A has no least upper bound, and more generally $\mathbb{C}$ does not have the least upper bound property with the order described above.

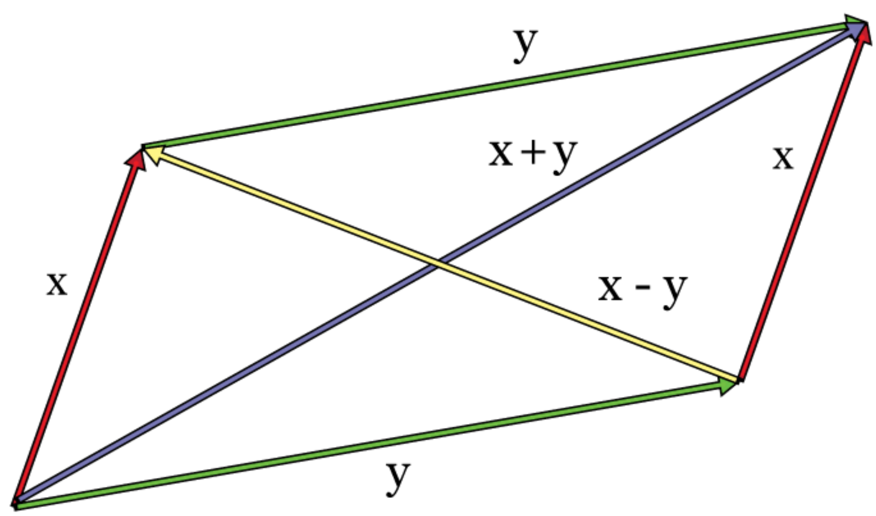
Problem 5 (WR Ch 1 #17)**.** Prove that

$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$$

if $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

Solution. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Then

$$\begin{aligned}
|x+y|^2 + |x-y|^2 &= \left(\sqrt{(x_1+y_1)^2 + \cdots + (x_n+y_n)^2}\right)^2 + \left(\sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}\right)^2 \\
&= (x_1+y_1)^2 + \cdots + (x_n+y_n)^2 + (x_1-y_1)^2 + \cdots + (x_n-y_n)^2 \\
&= (x_1^2 + 2x_1y_1 + y_1^2) + \cdots + (x_n^2 + 2x_ny_n + y_n^2) + (x_1^2 - 2x_1y_1 + y_1^2) + \cdots + (x_n^2 - 2x_ny_n + y_n^2) \\
&= (x_1^2 + \cancel{2x_1y_1} + y_1^2) + \cdots + (x_n^2 + \cancel{2x_ny_n} + y_n^2) + (x_1^2 - \cancel{2x_1y_1} + y_1^2) + \cdots + (x_n^2 - \cancel{2x_ny_n} + y_n^2) \\
&= 2(x_1^2 + \cdots + x_n^2) + 2(y_1^2 + \cdots + y_n^2) \\
&= 2\left(\sqrt{x_1^2 + \cdots + x_n^2}\right)^2 + 2\left(\sqrt{y_1^2 + \cdots + y_n^2}\right)^2 \\
&= 2|x|^2 + 2|y|^2.
\end{aligned}$$



Interpreted geometrically, the statement simply says that:

> The sum of the squares of the diagonals of a parallelogram is equal to twice the sum of the squares of the sides.