MATH 140A - HW 3 SOLUTIONS

Problem 1 (WR Ch 2 #12)**.** Let $K \subset \mathbb{R}^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Solution. Let $\{G_\alpha\}$ be any open cover of K, which means each G_α is an open set and together their union $\bigcup_\alpha G_\alpha$ contains K. In order to prove K is compact, we must show there is a finite subcover. Since $0 \in K \subset \bigcup_\alpha G_\alpha$, there is some α_0 such that $0 \in G_{\alpha_0}$. Now we know G_{α_0} is an open set, and an open set must contain a neighborhood of each of its points. Since $0 \in G_{\alpha_0}$, there is some $r > 0$ such that $N_r(0) \subset G_{\alpha_0}$. Let N be the smallest integer that is greater than $1/r$, which means $N > \frac{1}{r}$, or equivalently, $r > \frac{1}{N}$. Now notice that

$$r > \frac{1}{N} > \frac{1}{N+1} > \frac{1}{N+2} > \cdots,$$

or more formally,

$$r > d(0, \tfrac{1}{N}) > d(0, \tfrac{1}{N+1}) > d(0, \tfrac{1}{N+2}) > \cdots,$$

which means that the points $\frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \ldots$ are all in $N_r(0)$, which is contained in G_{α_0}. So we have one of the open sets that contains all but a finite number of points of K. The only points left to cover are $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N-1}$. Since 1 is in $K \subset \bigcup_\alpha G_\alpha$, there must be some α_1 such that $1 \in G_{\alpha_1}$. Repeating this for the rest of the points left, we have sets $G_{\alpha_2}, G_{\alpha_3}, \ldots, G_{\alpha_{N-1}}$ containing the points $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N-1}$ respectively. Therefore,

$$K \subset \bigcup_{n=0}^{N-1} G_{\alpha_n},$$

and we have found a finite subcover. Hence, K is compact.

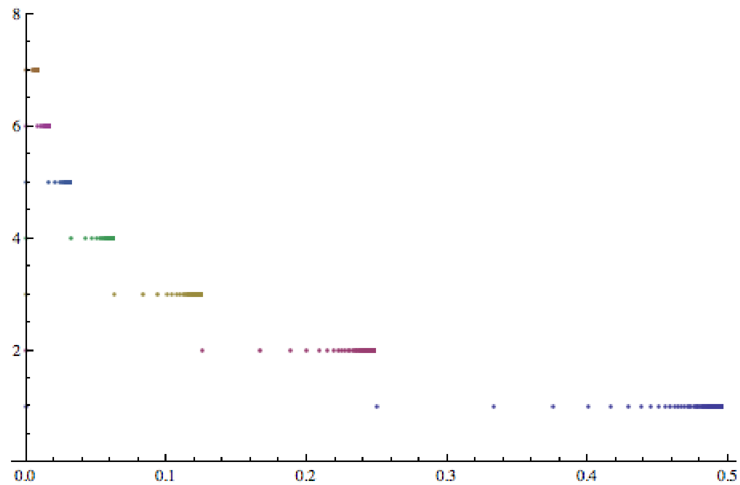
Problem 2 (WR Ch 2 #13)**.** Construct a compact set of real numbers whose limit points form a countable set.

Solution. Let

$$E = \left\{ \frac{1}{2^m}\left(1 - \frac{1}{n}\right) \,\middle|\, m, n \in \mathbb{N} \right\}.$$

This is plotted below,

A more illustrative plot follows, with the x-axis representing points of E and the y-axis representing different values of m to visually separate out different groups of points.



Consider the points of the form $p = \frac{1}{2^m}$ with $m \in \mathbb{N}$. Any neighborhood of one of these points of radius $r > 0$ will also contain the point $q = \frac{1}{2^m}(1 - \frac{1}{n})$ where we choose the positive integer n such that $\frac{1}{n} < 2^m r$, so that $|p-q| = |\frac{1}{2^m} - \frac{1}{2^m}(1-\frac{1}{n})| = |\frac{1}{2^m n}| < r$. Since $q \neq p$ and $q \in E$, that means p is a limit point, and thus E has at least a countably infinite number of limit points.

The fact that E is compact comes from its being closed (since it contains its limit points) and bounded (since each point of E is contained in $[0, \frac{1}{2}]$).

Problem 3 (WR Ch 2 #22)**.** A metric space is called separable if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable.

Solution. We claim that $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. To show this, let $p = (p_1, \ldots, p_k) \in \mathbb{R}^k$. In order to show $\mathbb{Q}^k$ is dense, we need to show that $p \in \mathbb{Q}^k$ or that p is a limit point of $\mathbb{Q}^k$. So if $p \in \mathbb{Q}^k$, we're done. If $p \notin \mathbb{Q}^k$, we want to show that p is a limit point. Let $N_r(p)$ with $r > 0$ be a neighborhood of p. Let $\delta = r/\sqrt{n}$, and since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find some $q_i \neq p_i$ in $N_\delta(p_i)$ for $i = 1, \ldots, k$. Then for $q = (q_1, \ldots, q_k)$ we have

$$d(p,q) = \sqrt{(p_1 - q_1)^2 + \cdots + (p_k - q_k)^2} < \sqrt{\frac{r^2}{n} + \cdots + \frac{r^2}{n}} = r,$$

so that $q \in N_r(p)$, and thus p is a limit point.

Thus $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$, and it is countable by Theorem 2.13, so $\mathbb{R}^k$ is separable.

Problem 4 (WR Ch 2 #24)**.** Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

Solution. Fix $\delta > 0$ and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. This is essentially covering X by disjoint $delta$-balls centered at the points $x_1, x_2, \ldots$. If we can do this forever without having any overlap, then the set $\{x_j \mid j \in \mathbb{N}\}$ is an infinite set without a limit point (since we have neighborhoods around each point that don't contain any other point in the set). But his is a contradiction, so this process must stop after a finite number of steps, which means X can be covered by finitely many neighborhoods of radius δ. Since we have proved this for any $\delta > 0$, we can assign δ to be anything greater than zero and still have the same result.

Let $\delta = 1$. Then there are finitely many points $x_1^{(1)}, x_2^{(1)}, \ldots, x_{N_1}^{(1)}$ such that X is covered by δ-neighborhoods centered at those points. Now let $\delta = \frac{1}{2}$. Then there are finitely many points $x_1^{(2)}, x_2^{(2)}, \ldots, x_{N_2}^{(2)}$ such that X is covered by δ-neighborhoods centered at those points. Repeating this for any $n \in \mathbb{N}$ we let $\delta = \frac{1}{n}$ and then there are finitely many points $x_1^{(n)}, x_2^{(n)}, \ldots, x_{N_n}^{(n)}$ such that X is covered by δ-neighborhoods centered at those points. We now claim that the set

$$\begin{aligned} E = \{ & x_1^{(1)}, x_2^{(1)}, \ldots, x_{N_1}^{(1)}, \\ & x_1^{(2)}, x_2^{(2)}, \ldots, x_{N_2}^{(2)}, \\ & x_1^{(3)}, x_2^{(3)}, \ldots, x_{N_3}^{(3)}, \\ & \ldots \} \end{aligned}$$

is a countable dense subset of X. It's countable by Theorem 2.12. It's dense because for any point $x \in X$ and any neighborhood $N_r(x)$ for $r > 0$, we can choose a positive integer n such that $\frac{1}{n} < r$, and then either $x = x_i^{(n)}$ for some $1 \le i \le N_n$, or x is in some neighborhood $N_{\frac{1}{n}}(x_i)$ for some $1 \le i \le N_n$ because we have an open cover. That means that $x_i \in N_r(x)$ and $x_i \ne x$, meaning x is a limit point of E.

Problem 5 (WR Ch 2 #25)**.** Prove that every compact metric space K has a countable base, and that K is therefore separable.

Solution. For each $n \in \mathbb{N}$, make an open cover of K by neighborhoods of radius $\frac{1}{n}$, and we have a finite subcover by compactness, i.e.

$$K \subset \bigcup_{x \in K} N_{\frac{1}{n}}(x) \qquad \Longrightarrow \qquad \exists\, x_1, \ldots, x_N \in K \text{ such that } K \subset \bigcup_{i=1}^{N} N_{\frac{1}{n}}(x_i)$$

Doing this for every $n \in \mathbb{N}$, we get a countable union of finite collections of sets, so that by Theorem 2.12, the collection of these sets, call it S, is countable.

We claim that S is a countable base for K, which is defined as a countable collection of open sets such that for any $x \in K$ and any open set G with $x \in G$, there is some $V \in S$ such that $x \in V \subset G$. Let $x \in K$ and let G be any open set with $x \in G$. Then since G is open, there is some $r > 0$ such that $N_r(x) \subset G$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < r/2$, so that the maximal distance between points in a neighborhood of radius $\frac{1}{n}$ is r. Then there must be some i such that $x \in N_{\frac{1}{n}}(x_i) \subset N_r(x)$ because any neighborhood of radius $1/n$ containing x cannot contain points a distance more than r away. This shows that S is a countable base.

The second part of the question asks us to show that K is separable. Let $\{V_n\}$ be our countable base for K. For each $n \in \mathbb{N}$, choose $x_n \in V_n$, and let $E = \{x_n | n \in \mathbb{N}\}$. We claim that E is a countable dense set, which would show that K is separable. First, note that E is clearly countable. To show that it's dense, we need to show that $\bar{E} = K$. This is equivalent to showing that $(\bar{E})^c = \emptyset$. Now $(\bar{E})^c$ is an open set because it's the complement of a closed set, $\bar{E}$. If $(\bar{E})^c$ is nonempty, then there is some $x \in (\bar{E})^c$, which is open, so since $\{V_n\}$ is a base, there is some n such that $x \in V_n \subset (\bar{E})^c$, which implies that $x_n \in (\bar{E})^c$, a contradiction, because $x_n \in E \quad \Rightarrow \quad x_n \in \bar{E} \quad \Rightarrow \quad x_n \notin (\bar{E})^c$. Therefore, $(\bar{E})^c = \emptyset$, so that $\bar{E} = K$.

Problem 6 (WR Ch 2 #26)**.** Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact.

Solution. By exercises 23 and 24, X has a countable base. Thus for any open cover $\{G_\alpha\}$, we can write each G_α as a union of sets from the countable base, meaning that every open cover has a countable subcover $\{G_n\}$. Assume by way of contradiction that no finite subcollection of $\{G_n\}$ covers X. By setting

$$F_n = \left(\bigcup_{i=1}^{n} G_i\right)^c,$$

we have that $F_n \neq \emptyset$ for each n since if that were the case, $X = F_n^c = \bigcup_{i=1}^n G_i$, which means $\{G_1, \ldots, G_n\}$ would be a finite subcover. Notice also that $F_{n+1} \subset F_n$, since we are only removing points as n goes up. Now we also have

$$\bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} \left(\bigcup_{i=1}^{n} G_i\right)^c = \left(\bigcup_{n=1}^{\infty} \left(\bigcup_{i=1}^{n} G_i\right)\right)^c = \left(\bigcup_{i=1}^{\infty} G_i\right)^c = X^c = \emptyset,$$

since $\{G_n\}$ is an open cover of X. Thus $\bigcap F_n$ is empty. We now create an infinite set E by choosing some $x_n \in F_n$ (which is nonempty) for each $n \in \mathbb{N}$ and letting $E = \{x_n | n \in \mathbb{N}\}$. The only way that E could not be infinite is if some $x \in E$ were in an infinite number of sets F_n, but that would make $\bigcap F_n$ nonempty. Since E is an infinite set, E has a limit point p (which is given in the beginning of the problem statement). For each n, all but finitely many points of E lie in F_n, so P must be a limit point of F_n for all n. But the F_n's are closed, so $p \in F_n$ for all n, meaning that $\bigcap F_n \neq \emptyset$, a contradiction. Therefore, any open cover of X has a finite subcover, and thus X is compact.

Problem 7 (WR Ch 2 #29)**.** Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments.

Solution. By exercise 22, $\mathbb{R}^1$ is separable, and thus has a countable dense set, namely $\mathbb{Q}$.

Let $G \subset \mathbb{R}$ be any open set. Then $\mathbb{Q} \cap G$ is a countable dense set in G by the Archimedean property, and since G is open we can choose an open interval around every rational in G. Then G is the union of that countable collection of intervals. However, we need to find a countable collection of **disjoint** intervals. Notice that the union of any intervals which contain the same point is an interval with a lower endpoint equal to the infimum of the lower endpoints of the intervals (possibly $-\infty$) and with an upper endpoint equal to the supremum of the upper endpoints of the intervals (possibly ∞). We create a new countable collection of intervals whose union is G by the following procedure.

Take any point in $G \cap \mathbb{Q}$ and take the union of all intervals in G that contain it. Call this interval I_1. Now take some point in $(G \setminus I_1) \cap \mathbb{Q}$ and take the union of all intervals in $G \setminus I_1$ that contain it. Repeating this process we get a countable collection of disjoint intervals $I_1, I_2, I_3, \ldots$, each of which is in G and which together cover G.

Problem 8 (WR Ch 2 #30)**.** If $\mathbb{R}^k = \bigcup_1^\infty F_n$, where F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior.

Solution. Assume by way of contradiction that each F_n has an empty interior. Let $V_n = \bigcup_{i=1}^n F_i$. Since F_1 is closed, F_1^c is open. If F_1^c were empty, then $F_1 = \mathbb{R}^k$, but then $F_1^\circ \neq \emptyset$, so instead F_1^c must be nonempty. Let K_1 be some neighborhood in F_1^c such that $\overline{K_1} \cap V_1 = \emptyset$ (which we can do by shrinking the neighborhood if necessary). Now if we have defined K_n so that $\overline{K_n} \cap V_n = \emptyset$, we define K_{n+1} by taking a neighborhood in $K_n \setminus F_{n+1}$, which is nonempty because F_{n+1} has a nonempty interior. By shrinking if necessary, we can ensure that $\overline{K_{n+1}} \subset K_n$. Notice again that $\overline{K_{n+1}} \cap V_{n+1} = \emptyset$. This last property gives us that $\bigcap_1^\infty \overline{K_n}$ is disjoint from every F_n. Also, since each $\overline{K_n}$ is compact and $\overline{K_{n+1}} \subset \overline{K_n}$ then by Theorem 2.39 we know that $\bigcap_1^\infty \overline{K_n}$ is nonempty. Thus, there is some point

$$x \in \bigcap_1^\infty \overline{K_n} \subset \left(\bigcup_1^\infty F_n \right)^c = (\mathbb{R}^k)^c = \emptyset,$$

a contradiction, since the empty set cannot have a point in it.