MATH 140A - HW 4 SOLUTIONS

Problem 1 (WR Ch 3 #1)**.** Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true? (Assume we are working in $\mathbb{R}^k$)

Solution. Let $a, b \in \mathbb{R}^k$. Then by the triangle inequality,

$$|a| = |a-b+b| \le |a-b| + |b| \quad \Longrightarrow \quad |a| - |b| \le |a-b|$$
$$|b| = |b-a+a| \le |b-a| + |a| \quad \Longrightarrow \quad |b| - |a| \le |b-a| = |a-b|.$$

Putting these two inequalities together, we get that

$$\big||a| - |b|\big| \le |a-b|. \qquad (*)$$

Now we are given that $\{s_n\}$ converges, say to some $s \in \mathbb{R}^k$. This means that for every $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|s_n - s| < \epsilon$ for all $n \ge N$.

We now claim that $\{|s_n|\}$ converges to $|s|$, so we need to show that for every $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that $\big||s_n| - |s|\big| < \epsilon$ for all $n \ge N$. Given any $\epsilon > 0$, we use $(*)$ and the convergence of $\{s_n\}$ to get some $N \in \mathbb{N}$ such that

$$\big||s_n| - |s|\big| \le |s_n - s| < \epsilon \qquad \text{for all } n \ge N,$$

so $\{|s_n|\}$ converges.

The converse is not true because of the following counterexample. Let $k = 1$, so we are working in $\mathbb{R}$. Let $s_n = (-1)^n$. Then the sequence $\{|s_n|\}$ is just $1, 1, 1, \ldots$, which clearly converges to 1, but the sequence $\{s_n\}$ is $-1, 1, -1, 1, \ldots$, which does not converge. It does not converge because, letting $\epsilon = 1$, for any $N \in \mathbb{N}$ we know there is an odd number $n \ge N$ and an even number $m \ge N$, so that $d(s_n, s_m) = |1 - (-1)| = 2 > \epsilon$, so that the sequence is not Cauchy. If a sequence is not Cauchy, it does not converge.

Problem 2 (WR Ch 3 #3)**.** If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \qquad (n = 1, 2, 3, \ldots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \ldots$.

Solution.
Claim: $s_n < 2$ for all $n \in \mathbb{N}$. We prove this by induction. The base case is $n = 1$, and we see that $s_1 = \sqrt{2} < 2$. The next step is to assume for our induction hypothesis that $s_n < 2$ for some $n \in \mathbb{N}$ and prove that $s_{n+1} < 2$. This follows from the fact that

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} < \sqrt{2 + \sqrt{2}} < \sqrt{2 + 2} = 2.$$

Claim: $s_n \le s_{n+1}$ for all $n \in \mathbb{N}$. We prove this by induction as well. The base case is $n = 1$, and we see that $s_1 = \sqrt{2} \le \sqrt{2+\sqrt{\sqrt{2}}} = s_2$ since $\sqrt{\sqrt{2}} \ge 0$. The next step is to assume for our induction hypothesis that $s_{n-1} \le s_n$ for some $n \in \mathbb{N}$ and prove that $s_{n-1} \le s_n$. This follows from the fact that

$$s_n = \sqrt{2+\sqrt{s_{n-1}}} \le \sqrt{2+\sqrt{s_n}} = s_{n+1}.$$

Therefore, we have proved that the sequence $\{s_n\}$ is monotonic increasing and bounded, so it converges by theorem 3.14.

Problem 3 (WR Ch 3 #4)**.** Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \qquad s_{2m} = \frac{s_{2m-1}}{2}; \qquad s_{2m+1} = \frac{1}{2} + s_{sm}.$$

Solution.
Claim: $s_{2m} = \frac{2^m-1}{2^{m+1}}$. We prove this by induction. The base case is $m = 0$, and we see that $s_0 = 0 = \frac{2^0-1}{2^1}$. The next step is to assume for our induction hypothesis that $s_{2m} = \frac{2^m-1}{2^{m+1}}$ and try to prove this formula when substituting in $m+1$ for m. This follows from the fact that

$$s_{2(m+1)} = \frac{s_{2(m+1)-1}}{2} = \frac{s_{2m+1}}{2} = \frac{\frac{1}{2}+s_{2m}}{2} = \frac{\frac{1}{2}+\frac{2^m-1}{2^{m+1}}}{2} = \frac{\frac{2^m}{2^{m+1}}+\frac{2^m-1}{2^{m+1}}}{2} = \frac{\frac{2^{m+1}-1}{2^{m+1}}}{2} = \frac{2^{m+1}-1}{2^{(m+1)+1}}.$$

Claim: $s_{2m+1} = \frac{2^{m+1}-1}{2^{m+1}}$. We prove this by induction. The base case is $m = 0$, and we see that $s_1 = \frac{1}{2} = \frac{2^1-1}{2^1}$. The next step is to assume for our induction hypothesis that $s_{2m+1} = \frac{2^{m+1}-1}{2^{m+1}}$ and try to prove this formula when substituting in $m+1$ for m. This follows from the fact that

$$s_{2(m+1)+1} = \frac{1}{2} + s_{2(m+1)} = \frac{1}{2} + \frac{s_{2m+1}}{2} = \frac{1}{2} + \frac{\frac{2^{m+1}-1}{2^{m+1}}}{2} = \frac{\frac{2^{m+1}}{2^{m+1}}+\frac{2^{m+1}-1}{2^{m+1}}}{2} = \frac{\frac{2^{m+2}-1}{2^{m+1}}}{2} = \frac{2^{(m+1)+1}-1}{2^{(m+1)+1}}.$$

Now, using these two results we have that

$$\lim_{m\to\infty} s_{2m} = \lim_{m\to\infty} \frac{2^m-1}{2^{m+1}} = \frac{1}{2} \lim_{m\to\infty} \frac{2^m-1}{2^m} = \frac{1}{2},$$

and

$$\lim_{m\to\infty} s_{2m+1} = \lim_{m\to\infty} \frac{2^{m+1}-1}{2^{m+1}} = 1.$$

Therefore, the upper limit of $\{s_n\}$ is 1 and the lower one is $\frac{1}{2}$.

Problem 4 (WR Ch 3 #6)**.** Investigate the behavior (convergence or divergence) of $\sum a_n$ if

(a) $a_n = \sqrt{n+1} - \sqrt{n}$;

(b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$;

(c) $a_n = (\sqrt[n]{n} - 1)^n$;

(d) $a_n = \frac{1}{1+z^n}$, for complex values of z.

Solution.

(a) $a_n = \sqrt{n+1} - \sqrt{n}$;

Notice that

$$a_n = \sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n}) \left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right) = \frac{1}{\sqrt{n+1} + \sqrt{n}} \geq \frac{1}{\sqrt{n+1} + \sqrt{n+1}} = \frac{1}{2\sqrt{n+1}},$$

and

$$\sum_{n=0}^{\infty} \frac{1}{2\sqrt{n+1}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$

diverges by theorem 3.28, so by the comparison test, since $a_n \geq \frac{1}{2\sqrt{n+1}} \geq 0$, we know that $\sum a_n$ diverges.

(b) $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$;

From the previous part, we have that

$$a_n = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} \leq \frac{1}{n(\sqrt{n} + \sqrt{n})} = \frac{1}{2n^{\frac{3}{2}}},$$

and

$$\sum_{n=0}^{\infty} \frac{1}{2n^{\frac{3}{2}}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

converges by theorem 3.28, so by the comparison test, since $|a_n| = a_n \leq \frac{1}{2n^{\frac{3}{2}}}$, we know that $\sum a_n$ converges.

(c) $a_n = (\sqrt[n]{n} - 1)^n$;

Notice that

$$\lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \sqrt[n]{n} - 1 = 0 < 1,$$

(since $\lim_{n \to \infty} \sqrt[n]{n} = 1$ by theorem 3.20(b)) so by the root test, $\sum a_n$ converges.

(d) $a_n = \frac{1}{1+z^n}$, for complex values of z.

Case 1: $|z| \leq 1$.

$$|1 + z^n| \leq |1| + |z^n| = 1 + |z|^n \leq 1 + 1 = 2.$$

Therefore, $|1 + z^n| \leq 2$. Taking reciprocals of both sides of the inequality, we have

$$\frac{1}{|1 + z^n|} \geq \frac{1}{2},$$

so $\sum \frac{1}{|1+z^n|}$ diverges by the divergence test (theorem 3.23) because $\lim_{n \to \infty} a_n \neq 0$.

Case 2: $|z| > 1$.

$$|z^n| = |1 + z^n - 1| \leq |1 + z^n| + |-1| = |1 + z^n| + 1 \qquad \Longrightarrow \qquad |1 + z^n| \geq |z|^n - 1.$$

Now choose $N_0 \in \mathbb{N}$ such that if $n \geq N_0$ then $|z|^n > 2$, which we can do because $|z| > 1$. This also means that $\frac{|z|^n}{2} > 1$. Thus for $n \geq N_0$, we have

$$|1 + z|^n \geq |z|^n - 1 = \frac{|z|^n}{2} + \left(\frac{|z|^n}{2} - 1 \right) > \frac{|z|^n}{2}.$$

Taking reciprocals, we have

$$\left|\frac{1}{1+z^n}\right| \le \frac{2}{|z|^n} \qquad \text{for all } n \ge N_0.$$

Since

$$\sum_{n=N_0}^{\infty} \frac{2}{|z|^n} = 2\sum_{n=N_0}^{\infty} \left(\frac{1}{|z|}\right)^n$$

is a geometric series with $\frac{1}{|z|} < 1$, it converges, and since $|a_n| \le \frac{2}{|z|^n}$ for $n \ge N_0$, then $\sum a_n$ converges by the comparison test (theorem 3.24(a)).

Problem 5 (WR Ch 3 #7)**.** Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \ge 0$.

Solution. By the Cauchy-Schwarz inequality,

$$\left|\sum_{n=1}^{k} (\sqrt{a_n})\,(\tfrac{1}{n})\right|^2 \le \left(\sum_{n=1}^{k} |\sqrt{a_n}|^2\right)\left(\sum_{n=1}^{k} |\tfrac{1}{n}|^2\right) = \left(\sum_{n=1}^{k} a_n\right)\left(\sum_{n=1}^{k} \tfrac{1}{n^2}\right).$$

Now by theorem 3.28, $\sum \frac{1}{n^2}$ converges, say to some positive real number B, and if $\sum a_n$ converges, say to some positive real number A, then if we set $s_k = \sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$ and we have

$$|s_k| = \left|\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}\right| \le \sqrt{\left(\sum_{n=1}^{k} a_n\right)\left(\sum_{n=1}^{k} \tfrac{1}{n^2}\right)} \le \sqrt{AB},$$

so s_k is bounded, and s_k is monotonically increasing because $\frac{\sqrt{a_n}}{n} \ge 0$. Therefore, by theorem 3.14, s_k converges, which means that $\sum \frac{\sqrt{a_n}}{n}$ converges.